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**WEIGHTED FUNCTIONAL INEQUALITIES  
AND NONLINEAR DIFFUSIONS  
OF POROUS MEDIUM TYPE**

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## Abstract

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The main topic of this thesis is the study of the asymptotic behaviour of solutions to certain nonlinear diffusion equations, whose most important models are the porous medium equation and the fast diffusion equation. In the first chapter we analyse in detail the connections between  $L^p$  smoothing and decay properties of weighted versions of the porous medium equation and the validity of suitable functional inequalities involving the weights. In the second chapter we investigate the asymptotics of the solutions to the fractional porous medium equation with power-type weights: this is strictly linked with a similar fractional parabolic problem having as initial datum a positive finite measure, which we study separately. The third chapter is mostly devoted to the characterization of the optimal functions for a family of Caffarelli-Kohn-Nirenberg interpolation inequalities: it turns out that if the power of the weight that appears in the  $L^p$  norms is small enough, then such optimal functions are radial. As a consequence, solutions to the Euclidean fast diffusion equation with the same power weight converge towards special solutions of Barenblatt type with an optimal rate, at least for  $m$  larger than a suitable critical value. In the fourth and last chapter we consider the fast diffusion equation on hyperbolic space: the most important result we obtain, for  $m$  close to one, is the convergence of radial solutions, as  $t$  tends to the extinction time, to a separable solution in the uniform norm of the relative error.



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## Summary

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This thesis deals with a class of nonlinear diffusion equations whose prototype is

$$u_t = \Delta(|u|^{m-1}u), \quad (1)$$

which is known as *porous medium equation* (PME) in the range  $m > 1$  and as *fast diffusion equation* (FDE) in the range  $m < 1$ . One may see (1) as a particular nonlinear version of the heat equation whose diffusion coefficient is a power of the solution  $u$  itself. We point out that such coefficient becomes *degenerate* or *singular* at  $u = 0$  depending on whether one considers the PME or the FDE, respectively.

Evolution equations of porous medium type have a long history, both as concerns the applications which originally motivate their consideration (in this regard, see the Introduction to Chapter 1) and from the mathematical point of view. We refer in particular to the monograph [173] by J. L. Vázquez, where a very detailed and comprehensive survey of the state of the art is given.

Here we are concerned with a number of (weighted) variants and extensions of (1), for which we address several fundamental issues such as existence and uniqueness of solutions and their asymptotic behaviour. Our analysis aims at linking as far as possible the properties of the solutions with suitable functional inequalities associated with the underlying functional spaces, thus giving a certain amount of generality to the discussion. Below we outline the main topics we investigate in our work, which is divided into four chapters. A more detailed introduction to each single topic, as well as full bibliographic references, is provided at the beginning of the corresponding chapter.

**Weighted porous medium equations.** The first variant we deal with is the *weighted* PME, namely

$$\rho_\nu(x)u_t = \operatorname{div} \left[ \rho_\mu(x) \nabla(|u|^{m-1}u) \right], \quad (2)$$

where  $\rho_\nu$  and  $\rho_\mu$  are suitable positive functions, to which we refer as *weights* from here on. *A priori* we only assume that  $\rho_\nu$  and  $\rho_\mu$  are locally bounded and locally bounded away from zero. The weight  $\rho_\nu$  corresponds to an inhomogeneous mass distribution in the medium where diffusion takes place, whereas  $\rho_\mu$  describes a diffusion coefficient that depends on the spatial variable as well. It is therefore natural to introduce the measure  $d\nu := \rho_\nu(x)dx$  for the problem at hand, whose associated Lebesgue spaces we denote as  $L^p(\nu)$  for  $p \geq 1$ . Our first concern consists in considering *smoothing* and *decay* properties of solutions to (2), namely bounds of the type of

$$\|u(t)\|_{L^p(\nu)} \leq K t^{-\alpha} \|u_0\|_{L^q(\nu)}^\beta \quad \forall t > 0, \quad (3)$$

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for any given solution  $u(t)$  corresponding to an initial datum  $u_0 \in L^q(\nu)$ , where  $1 \leq q < p \leq \infty$  and  $\alpha, \beta, K$  are suitable positive constants independent of  $u_0$ . This is a classical problem in linear semigroup theory which has widely been studied in the nonlinear setting as well, see for instance the general reference [172]. The strategy followed here aims at connecting smoothing and decay estimates of the type of (3) with functional inequalities that may hold in weighted Sobolev spaces which involve the weights  $\rho_\nu$  and  $\rho_\mu$ . In fact, we prove full equivalence between suitable versions of (3) and appropriate *Poincaré*-, *Sobolev*- or *Gagliardo-Nirenberg-type inequalities* (the latter are introduced to deal with low dimensions). It is worth mentioning that (3) may not be satisfied up to  $p = \infty$  unless Sobolev- or Gagliardo-Nirenberg-type inequalities hold true (*i.e.* the validity of a Poincaré-type inequality is not enough). The details of the stated equivalence depend on the particular problem at hand: we consider Euclidean domains with both homogeneous Dirichlet and homogeneous Neumann boundary conditions. In the case of finite- $\nu$ -measure domains, estimate (3) can be improved for large times, still by means of a purely functional analytic approach. This is particularly relevant for the Neumann problem, where convergence of solutions to their  $\nu$ -mean value (which is preserved in time) is proved with *sharp* rates.

The above short- and long-time asymptotics of solutions, together with a detailed well-posedness analysis, is addressed in Chapter 1 and gathers the results of the papers [P1–P3]. In the same chapter we also consider an *inhomogeneous filtration equation* in the whole Euclidean space  $\mathbb{R}^d$ , which has been studied in [P4], namely an equation of the type of (2) where  $\rho_\mu \equiv 1$  and the nonlinearity  $|u|^{m-1}u$  is replaced by a more general function  $G(u)$ . It is known that, in the weighted case, nonuniqueness phenomena can occur even for bounded data and solutions. Uniqueness can then be restored by adding a suitable nonhomogeneous condition at infinity, usually formulated in an *integral* sense. Here we show that existence and uniqueness results can be proved, under suitable assumptions on the nonlinearity and on the weight, provided certain *pointwise* conditions at infinity are prescribed. This seems to be the first result of this kind for such a problem.

The content of Chapter 1 is based on the following papers:

- [P1] G. Grillo, M. Muratori and M. M. Porzio, *Porous media equations with two weights: smoothing and decay properties of energy solutions via Poincaré inequalities*, Discrete Contin. Dyn. Syst. 33 (2013), 3599–3640.
- [P2] G. Grillo and M. Muratori, *Sharp short and long time  $L^\infty$  bounds for solutions to porous media equations with homogeneous Neumann boundary conditions*, J. Differential Equations 254 (2013), 2261–2288.
- [P3] G. Grillo and M. Muratori, *Sharp asymptotics for the porous media equation in low dimensions via Gagliardo-Nirenberg inequalities*, Riv. Mat. Univ. Parma 5 (2014), 15–38; proceedings of the workshop “New Trends in Nonlinear Parabolic Equations”, Parma (Italy), November 12–16, 2012.
- [P4] G. Grillo, M. Muratori and F. Punzo, *Conditions at infinity for the inhomogeneous filtration equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 413–428.

**Fractional porous medium equations.** Chapter 2 is devoted to the study of nonnegative solutions to the following *fractional porous medium equation*:

$$u_t = -(-\Delta)^s(u^m), \quad (4)$$

where  $m > 1$  and for  $s \in (0, 1)$  we denote as  $(-\Delta)^s$  the fractional Laplacian operator on  $\mathbb{R}^d$ , which can be defined either spectrally or by means of the extension method of Caffarelli and Silvestre [42]. The investigation of such a class of equations has been started in the

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works [60, 61] by J. L. Vázquez et al., where in particular well-posedness of the evolution is proved for general  $L^1$  data. Recently, existence and uniqueness of Barenblatt-type solutions, namely solutions having a Dirac delta as initial datum, has been proved again by Vázquez in [174]. Motivated by these results, we have tackled the delicate problem of (existence and) uniqueness of weak solutions to (4) when the initial datum is a general *positive finite Radon measure*. This is performed by adapting a clever method first introduced by M. Pierre in [148] a long ago, whose applicability is however far from obvious in the present context and requires several new arguments. Existence and uniqueness of an initial trace for solutions to (4) without a prescribed initial condition is also shown. In fact our method, based on *Riesz potential* techniques, also works in the case of the following *weighted fractional PME*:

$$\rho(x)u_t = -(-\Delta)^s(u^m), \quad (5)$$

provided some power-type conditions on the weight (or *density*)  $\rho$  are required both at the origin, where it can be singular, and at infinity. The mentioned well-posedness analysis is carried out in the first part of Chapter 2 and relies on the results of [P5]. Being able to prove existence and uniqueness of Barenblatt-type solutions in the weighted case as well, and in particular in the case of a *singular-power* density  $\rho$ , we can study the asymptotic behaviour of general solutions to (5) in terms of Barenblatt-type solutions, at least when the decay at infinity of  $\rho$  is sufficiently *slow*. It is worth noticing that, as it happens in the local case ( $s = 1$ ), the convergence result takes advantage of scaling properties of the Barenblatt-type solution associated with a singular-power density even when the equation at hand involves a *regular* density. This gives motivation to the analysis of (5) in the case where  $\rho$  is singular at the origin. The asymptotic behaviour of solutions to (5) with *rapidly decaying* densities is quite different. In fact, still inspired by the local case, we prove that the latter is actually determined by a *separable solution* which involves the solution of a suitable *fractional sublinear elliptic equation*. This requires techniques completely different from the ones we use in the slowly decaying case. The present asymptotic analysis, both for slowly decaying and for rapidly decaying densities, collects the results of [P6].

Finally, we mention that a crucial technical result in proving uniqueness of solutions to (5) with measure data, based on [P7], is the *self-adjointness*, in suitably weighted  $L^2$  spaces, of the linear operator formally given by  $A := \rho^{-1}(-\Delta)^s$  (and defined on its maximal domain), together with the Markov property for the associated semigroup. These topics also have an independent interest and are therefore discussed separately in an appendix at the end of the chapter.

The content of Chapter 2 is based on the following papers:

- [P5] G. Grillo, M. Muratori and F. Punzo, *Weighted fractional porous media equations: existence and uniqueness of weak solutions with measure data*, submitted, preprint (2014), <http://arxiv.org/abs/1312.6076v2>.
- [P6] G. Grillo, M. Muratori and F. Punzo, *On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density*, accepted for publication in Discrete Contin. Dyn. Syst., preprint (2014), <http://arxiv.org/abs/1403.5293v2>.
- [P7] M. Muratori, *On the self-adjointness of the fractional Laplacian in some weighted  $L^2$  spaces*, in preparation.

**Caffarelli-Kohn-Nirenberg inequalities and weighted fast diffusion.** In Chapter 3 we address the problem of finding the functions which optimize the following family of Caffarelli-Kohn-Nirenberg inequalities:

$$\|w\|_{2p,\gamma} \leq C \|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta} \quad \forall w \in \mathcal{D}(\mathbb{R}^d), \quad (6)$$



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where  $C > 0$ ,  $d \geq 3$ ,  $\gamma \in (0, 2)$ ,  $p \in (1, (d - \gamma)/(d - 2))$ ,  $\vartheta$  is an explicit exponent depending on  $d$ ,  $\gamma$ ,  $p$  and  $\|w\|_{q,\gamma}^q := \int_{\mathbb{R}^d} |w(x)|^q |x|^{-\gamma} dx$  for all  $q \geq 1$ . Such inequalities (which can be extended by density to the space of functions for which the right-hand side of (6) is finite), in a more general form, were first introduced in [41]. We remark however that the particular family (6) can readily be deduced by interpolation from Hardy's inequality and the Sobolev inequality. Finding optimal functions for (6) means finding those functions which attain the *best constant*  $C$  in (6). Such problem has completely been solved by M. Del Pino and J. Dolbeault in [62] in the non-weighted case, *i.e.* for  $\gamma = 0$ . In particular, it is shown there that optimal functions coincide with the Aubin-Talenti-type functions

$$\mathfrak{b}_0(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}},$$

up to a multiplication by a constant and a scaling. Our aim here is to generalize this result to (6), namely to prove that the corresponding optimal functions are of the type of

$$\mathfrak{b}_\gamma(x) := \left(1 + |x|^{2-\gamma}\right)^{-\frac{1}{p-1}}. \quad (7)$$

Establishing that (7) optimizes (6) *among radial functions* is not particularly difficult, since one can bring back the problem to the non-weighted case. Hence, our analysis amounts to proving that optimal functions are *radial*, which is far from trivial even though the weight  $|x|^{-\gamma}$  is radial (in general, *symmetry breaking* phenomena are known in the literature). By means of a perturbation argument which involves concentration-compactness techniques, we prove that the result holds true provided  $\gamma > 0$  is sufficiently small. This has very interesting consequences for the asymptotics of nonnegative solutions to the power-weighted Euclidean fast diffusion equation

$$|x|^{-\gamma} u_t = \Delta(u^m) \quad (8)$$

in the supercritical range  $(2d - \gamma - 2)/[2(d - \gamma)] < m < 1$ . In fact, the optimality of (7) for (6) is equivalent to the validity of a suitable sharp *free energy-Fisher information* (or *entropy-entropy production*) *inequality*, which is in turn equivalent to a sharp *exponential decay* estimate for the free energy functional. The latter is actually a measure of the distance between a general solution to (8) and (7), after a suitable time-space scaling which makes (7) a stationary solution of a Fokker-Planck version of (8). Such estimates provide key ingredients to start the well-established *entropy method* (after [23]) in the power-weighted case as well, as it will be discussed in detail in [25].

Chapter 3 is based on the following paper:

[P8] J. Dolbeault, M. Muratori and B. Nazaret, *Symmetry in weighted interpolation inequalities*, submitted.

**Fast diffusion on hyperbolic space.** As a final topic of the thesis, in Chapter 4 we study the fast diffusion equation on *hyperbolic space*  $\mathbb{H}^N$  ( $N \geq 2$ ). The asymptotics of nonnegative *radial solutions* (namely solutions depending on the spatial component only through the geodesic distance  $r$  from a given pole) turns out to be, in the range  $m \in ((N - 2)/(N + 2), 1)$ , driven by a *separable solution* of the type of

$$\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V_m^{\frac{1}{m}}, \quad (9)$$

where  $V$  is the unique positive *energy* solution of the elliptic equation

$$-\Delta V = \frac{1}{(1-m)T} V_m^{\frac{1}{m}} \quad \text{on } \mathbb{H}^N, \quad (10)$$

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also known as *Lame-Emden-Fowler equation*. We point out that  $T > 0$  (depending on the initial datum) is the *extinction time*: in fact, as it happens for the fast diffusion equation on bounded Euclidean domains with homogeneous Dirichlet boundary conditions, solutions become identically zero in a finite time. Convergence to the separable solution is proved, in the spirit of [31], in a very strong sense, namely in the uniform norm of the *relative error* and at the level of all derivatives (although without explicit rates).

Notice that on general Riemannian manifolds, in local coordinates, the Laplace-Beltrami operator can be written as  $\Delta = |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij} \partial_j)$ , where the summation convention is used and  $|g|$  denotes the determinant of the metric tensor  $g$ . In particular, at least in the special case of spherically symmetric Riemannian manifolds known as *model manifolds*, we basically fall in the class of equations described by (2). In fact, in this framework, the *radial component* of the Riemannian Laplacian can always be written in the two-weight form  $\rho^{-1}(r) \partial_r (\rho(r) \partial_r)$  in geodesic polar coordinates. As mentioned above, here we perform a detailed analysis on hyperbolic space, as the topologically simplest example of noncompact, *negatively curved* model manifold. We believe however that our results can be extended at least to more general model manifolds with strictly negative curvature, following essentially the same method of proof.

Chapter 4 is based on the following paper:

- [P9] G. Grillo and M. Muratori, *Radial fast diffusion on the hyperbolic space*, Proc. Lond. Math. Soc. 109 (2014), 283–317.



Questa tesi tratta una classe di equazioni non-lineari di diffusione il cui prototipo è

$$u_t = \Delta(|u|^{m-1}u), \quad (1)$$

nota come *equazione dei mezzi porosi* (PME) nel range  $m > 1$  ed *equazione della diffusione veloce* (FDE) nel range  $m < 1$ . La (1) può essere vista come una particolare versione non-lineare dell'equazione del calore, il cui coefficiente di diffusione è una potenza della soluzione stessa  $u$ . Si noti che tale coefficiente diventa degenere o singolare dove  $u = 0$  a seconda che si consideri la PME o la FDE, rispettivamente.

Lo studio di equazioni d'evoluzione di tipo mezzi porosi ha una lunga storia, sia a livello delle applicazioni dalle quali esse nascono (a questo proposito, si veda l'Introduzione al Capitolo 1), sia da un punto di vista puramente matematico. Ci riferiamo in particolare alla monografia [173] di J. L. Vázquez, nella quale è fornita una dettagliata e comprensiva rassegna dello stato dell'arte.

Siamo qui interessati ad alcune varianti ed estensioni (pesate) della (1), per le quali affronteremo diverse questioni fondamentali quali esistenza e unicità delle soluzioni e loro comportamento asintotico. La nostra analisi ha lo scopo di legare il più possibile le proprietà delle soluzioni ad opportune disuguaglianze funzionali associate ai naturali spazi funzionali sottostanti, il che fornisce un certo grado di generalità alla trattazione. Nel seguito delineeremo gli argomenti principali affrontati nel lavoro di tesi, il quale è suddiviso in quattro capitoli. Un'introduzione maggiormente dettagliata, assieme ad una vasta bibliografia, verrà fornita all'inizio dei relativi capitoli.

**Equazione dei mezzi porosi pesata.** La prima variante che affrontiamo è la PME pesata, ovvero

$$\rho_\nu(x)u_t = \operatorname{div} \left[ \rho_\mu(x) \nabla(|u|^{m-1}u) \right], \quad (2)$$

dove  $\rho_\nu$  e  $\rho_\mu$  sono opportune funzioni positive, che d'ora in poi chiameremo *pesi*. A priori assumeremo solo che  $\rho_\nu$  e  $\rho_\mu$  siano funzioni localmente limitate e localmente lontane da zero. Il peso  $\rho_\nu$  corrisponde ad una distribuzione non-omogenea di massa nel mezzo in cui ha luogo la diffusione, mentre il peso  $\rho_\mu$  descrive un coefficiente di diffusione che dipende possibilmente anche dalla variabile spaziale. Risulta perciò naturale introdurre la misura  $d\nu := \rho_\nu(x)dx$  per il problema considerato, il cui spazio di Lebesgue associato verrà per il momento denotato come  $L^p(\nu)$  per  $p \geq 1$ . Il nostro primo obiettivo consiste nel considerare

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proprietà *regolarizzanti* e *asintotiche* per le soluzioni della (2), ovvero stime del tipo

$$\|u(t)\|_{L^p(\nu)} \leq K t^{-\alpha} \|u_0\|_{L^q(\nu)}^\beta \quad \forall t > 0, \quad (3)$$

per ogni data soluzione  $u(t)$  corrispondente ad un dato iniziale  $u_0 \in L^q(\nu)$ , dove  $1 \leq q < p \leq \infty$  e  $\alpha, \beta, K$  sono opportune costanti positive indipendenti da  $u_0$ . Questo è un problema classico nella teoria dei semigruppì lineari, il quale è stato di recente largamente studiato anche nel contesto non-lineare, si veda ad esempio la referenza generale [172]. La strategia che seguiamo qui ha lo scopo di legare stime regolarizzanti e asintotiche come la (3) a disuguaglianze funzionali che possono valere in spazi di Sobolev pesati che coinvolgono i pesi  $\rho_\nu$  e  $\rho_\mu$ . Di fatto dimostreremo una completa equivalenza tra opportune versioni della (3) e determinate disuguaglianze di tipo *Poincaré*, *Sobolev* o *Gagliardo-Nirenberg* (queste ultime vengono introdotte in particolare per gestire le dimensioni basse). Vale la pena notare che la (3) potrebbe non essere soddisfatta fino a  $p = \infty$  se non valgono disuguaglianze di tipo Sobolev o Gagliardo-Nirenberg (in altre parole, la validità di una disuguaglianza di tipo Poincaré non è di per se sufficiente). Nel dettaglio, lo specifico risultato di equivalenza dipenderà dal particolare problema: considereremo domini Euclidei sia con condizioni di Dirichlet che con condizioni di Neumann omogenee. Nel caso di domini di  $\nu$ -misura finita, la stima (3) può essere migliorata, per tempi lunghi, sempre tramite un approccio puramente analitico-funzionale. Ciò è particolarmente rilevante per il problema di Neumann, in cui la convergenza delle soluzioni alla loro  $\nu$ -media (la quale è conservata nel tempo) è dimostrata con velocità *ottimali*.

L'asintotica delle soluzioni appena discussa per tempi brevi e lunghi, assieme ad una dettagliata analisi di buona posizione, è affrontata nel Capitolo 1 e riunisce i risultati dei lavori [P1–P3]. Nello stesso capitolo considereremo anche un'equazione della filtrazione non-omogenea nell'intero spazio euclideo  $\mathbb{R}^d$ , la quale è stata studiata in [P4], ovvero un'equazione del tipo (2) in cui  $\rho_\mu \equiv 1$  e la non-linearità  $|u|^{m-1}u$  è sostituita da una generica funzione  $G(u)$ . È noto che, nel caso pesato, fenomeni di non-unicità possono verificarsi anche solo per dati e soluzioni limitate. L'unicità può allora essere recuperata a patto di aggiungere opportune condizioni non-omogenee all'infinito, tipicamente formulate in un senso integrale. Qui stabiliremo che esistenza e unicità possono essere dimostrate, richiedendo opportune ipotesi sulla non-linearità e sul peso, anche prescrivendo determinate condizioni *puntuali* all'infinito. Quest'ultimo sembra essere il primo risultato di questa tipologia per un tale problema.

Il contenuto del Capitolo 1 è basato sui seguenti lavori:

- [P1] G. Grillo, M. Muratori and M. M. Porzio, *Porous media equations with two weights: smoothing and decay properties of energy solutions via Poincaré inequalities*, Discrete Contin. Dyn. Syst. 33 (2013), 3599–3640.
- [P2] G. Grillo and M. Muratori, *Sharp short and long time  $L^\infty$  bounds for solutions to porous media equations with homogeneous Neumann boundary conditions*, J. Differential Equations 254 (2013), 2261–2288.
- [P3] G. Grillo and M. Muratori, *Sharp asymptotics for the porous media equation in low dimensions via Gagliardo-Nirenberg inequalities*, Riv. Mat. Univ. Parma 5 (2014), 15–38; proceedings del convegno “New Trends in Nonlinear Parabolic Equations”, Parma (Italia), 12–16 Novembre 2012.
- [P4] G. Grillo, M. Muratori and F. Punzo, *Conditions at infinity for the inhomogeneous filtration equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 413–428.

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**Equazione dei mezzi porosi frazionaria.** Il Capitolo 2 è dedicato allo studio di soluzioni non-negative della seguente *equazione dei mezzi porosi frazionaria*:

$$u_t = -(-\Delta)^s(u^m), \quad (4)$$

dove  $m > 1$  e per ogni  $s \in (0, 1)$  denotiamo con  $(-\Delta)^s$  l'operatore Laplaciano frazionario su  $\mathbb{R}^d$ , il quale può essere definito o a livello spettrale o per mezzo del metodo di estensione dovuto a Caffarelli e Silvestre [42]. L'analisi di tale classe di equazioni è stata avviata dai lavori [60, 61] di J. L. Vázquez e altri, dove in particolare la buona posizione dell'evoluzione è dimostrata per generici dati  $L^1$ . Recentemente, l'esistenza e l'unicità di soluzioni di tipo Barenblatt, ovvero soluzioni aventi la delta di Dirac come dato iniziale, sono state dimostrate sempre da Vázquez in [174]. Motivati da questi risultati, abbiamo quindi studiato il delicato problema dell'unicità (e dell'esistenza) di soluzioni deboli della (4) quando il dato iniziale è una generica *misura di Radon positiva finita*. Ciò verrà affrontato adattando un particolare metodo introdotto inizialmente da M. Pierre in [148], la cui applicabilità nel contesto attuale tuttavia non è ovvia e richiede molti argomenti nuovi. Sarà inoltre mostrata l'esistenza e l'unicità di una traccia iniziale per funzioni che siano soluzioni della (4) senza una particolare condizione iniziale data.

Di fatto il nostro metodo di lavoro, basato su tecniche che coinvolgono *potenziali di Riesz*, funziona anche nel caso della seguente *PME frazionaria pesata*:

$$\rho(x)u_t = -(-\Delta)^s(u^m), \quad (5)$$

posto che alcune condizioni di tipo potenza siano assunte sul peso (o *densità*) sia nell'origine, dove può essere singolare, che all'infinito. L'analisi di buona posizione appena menzionata verrà sviluppata nel corso del Capitolo 2 e si basa sui risultati di [P5]. Dimostrando esistenza e unicITÀ di soluzioni di tipo Barenblatt anche nel caso pesato, ed in particolare nel caso di una densità  $\rho$  con *potenza singolare*, possiamo studiare il comportamento asintotico di soluzioni generiche della (5) in termini delle suddette soluzioni di tipo Barenblatt, perlomeno quando il decadimento di  $\rho$  all'infinito è sufficientemente *lento*. Vale la pena notare che, così come accade nel caso locale ( $s = 1$ ), la dimostrazione del risultato di convergenza sfrutta proprietà di riscaldamento della soluzione di tipo Barenblatt associata alla potenza singolare anche quando nell'equazione considerata appare una densità *regolare*. Ciò effettivamente motiva l'analisi della (5) nel caso in cui  $\rho$  sia singolare nell'origine. Il comportamento asintotico delle soluzioni della (5) con densità che *decadono velocemente* è decisamente diverso. Infatti, sempre ispirati dal caso locale, dimostreremo che quest'ultimo è in realtà determinato da una *soluzione a separazione di variabili* che è legata alla soluzione di un'opportuna *equazione sublineare ellittica frazionaria*. Questo studio richiede tecniche completamente differenti da quelle usate nel caso di densità che decadono lentamente. L'analisi asintotica qui sviluppata, sia per densità che decadono lentamente che per densità che decadono velocemente, si riferisce ai risultati di [P6].

Infine, facciamo notare che un risultato tecnico cruciale nel dimostrare l'unicità di soluzioni della (5) con dati misura, basato su [P7], è l'auto-aggiuntezza, in opportuni spazi  $L^2$  pesati, dell'operatore lineare formalmente dato da  $A := \rho^{-1}(-\Delta)^s$  (e definito sul suo dominio massimale), assieme alla proprietà di Markov per il semigruppato associato. Tali argomenti hanno anche un interesse indipendente e perciò saranno discussi separatamente in un'appendice alla fine del capitolo.

Il contenuto del Capitolo 2 è basato sui seguenti lavori:

[P5] G. Grillo, M. Muratori and F. Punzo, *Weighted fractional porous media equations: existence and uniqueness of weak solutions with measure data*, sottomesso a una rivista scientifica, preprint (2014), <http://arxiv.org/abs/1312.6076v2>.

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- [P6] G. Grillo, M. Muratori and F. Punzo, *On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density*, accettato per la pubblicazione su Discrete Contin. Dyn. Syst., preprint (2014), <http://arxiv.org/abs/1403.5293v2>.
- [P7] M. Muratori, *On the self-adjointness of the fractional Laplacian in some weighted  $L^2$  spaces*, in preparazione.

**Disuguaglianze di Caffarelli-Kohn-Nirenberg e diffusione veloce pesata.** Nel Capitolo 3 affrontiamo il problema di trovare le funzioni che ottimizzino la seguente famiglia di disuguaglianze di Caffarelli-Kohn-Nirenberg:

$$\|w\|_{2p,\gamma} \leq C \|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta} \quad \forall w \in \mathcal{D}(\mathbb{R}^d), \quad (6)$$

dove  $C > 0$ ,  $d \geq 3$ ,  $\gamma \in (0, 2)$ ,  $p \in (1, (d - \gamma)/(d - 2))$ ,  $\vartheta$  è un esponente esplicito che dipende da  $d$ ,  $\gamma$ ,  $p$  e  $\|w\|_{q,\gamma}^q := \int_{\mathbb{R}^d} |w(x)|^q |x|^{-\gamma} dx$  per ogni  $q \geq 1$ . Tali disuguaglianze (che possono essere estese per densità allo spazio di funzioni per le quali il membro destro della (6) è finito), in forma più generale sono state introdotte in [41]. Facciamo comunque notare che la particolare famiglia (6) può essere facilmente ricavata per interpolazione dalle disuguaglianze di Hardy e di Sobolev. Trovare le funzioni ottimali per la (6) significa trovare quelle funzioni non-triviali per le quali la (6) vale come uguaglianza con *costante ottimale*  $C$ . Tale problema è stato completamente risolto da M. Del Pino e J. Dolbeault in [62] nel caso non pesato, ovvero per  $\gamma = 0$ . In particolare, viene lì dimostrato che le funzioni ottimali coincidono con le funzioni di tipo Aubin-Talenti

$$\mathbf{b}_0(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}},$$

a meno di moltiplicazioni per costanti e riscalamanti. Il nostro obiettivo qui è generalizzare questo risultato alla (6), cioè dimostrare che le corrispondenti funzioni ottimali sono del tipo

$$\mathbf{b}_\gamma(x) := \left(1 + |x|^{2-\gamma}\right)^{-\frac{1}{p-1}}. \quad (7)$$

Stabilire che (7) ottimizza (6) *tra le funzioni radiali* non è particolarmente difficile, dato che in questo caso il problema può essere facilmente ricondotto al caso non pesato. Di conseguenza, la nostra analisi si riduce a dimostrare che le funzioni ottimali sono *radiali*, il che è tutt'altro che triviale persino quando il peso  $|x|^{-\gamma}$  è radiale (in generale, in letteratura, sono noti fenomeni di *symmetry breaking*). Per mezzo di un argomento perturbativo che coinvolge tecniche di concentrazione-compattatezza, dimostreremo che il risultato vale posto che  $\gamma > 0$  sia sufficientemente piccolo. Ciò ha conseguenze molto interessanti per l'asintotica di soluzioni non-negative dell'equazione della diffusione veloce Euclidea pesata con peso di tipo potenza:

$$|x|^{-\gamma} u_t = \Delta(u^m) \quad (8)$$

nel range supercritico  $(2d - \gamma - 2)/[2(d - \gamma)] < m < 1$ . Di fatto, l'ottimalità della (7) per la (6) è equivalente alla validità di un'opportuna *disuguaglianza di energia libera-informazione di Fisher* (o *disuguaglianza di entropia-produzione di entropia*) ottimale, la quale è a sua volta equivalente a una stima ottimale di *decadimento esponenziale* per il funzionale di energia libera. Quest'ultimo rappresenta sostanzialmente una misura della distanza tra una soluzione generica della (8) e la (7), a posteriori di un opportuno riscaldamento temporale che rende la (7) una soluzione stazionaria di una versione di tipo Fokker-Planck della (8). Tali stime forniscono ingredienti chiave per avviare il ben consolidato *metodo d'entropia* (sviluppato in [23]) anche nel caso pesato, come verrà discusso in dettaglio in [25].

Il Capitolo 3 è basato sul seguente lavoro:

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[P8] J. Dolbeault, M. Muratori and B. Nazaret, *Symmetry in weighted interpolation inequalities*, sottomesso a una rivista scientifica.

**Diffusione veloce sullo spazio iperbolico.** Come argomento finale della tesi, nel Capitolo 4 studiamo la diffusione veloce sullo *spazio iperbolico*  $\mathbb{H}^N$  ( $N \geq 2$ ). L'asintotica per soluzioni non-negative *radiali* (ovvero soluzioni che dipendono solo dalla componente spaziale attraverso la distanza geodetica  $r$  da un fissato polo) risulta essere, nel range  $m \in ((N-2)/(N+2), 1)$ , determinata da una *soluzione a variabili separabili* del tipo

$$\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}, \quad (9)$$

dove  $V$  è l'unica soluzione positiva *di energia* dell'equazione ellittica

$$-\Delta V = \frac{1}{(1-m)T} V^{\frac{1}{m}} \quad \text{su } \mathbb{H}^N, \quad (10)$$

nota anche come *equazione di Lane-Emden-Fowler*. Facciamo notare che  $T > 0$  (dipendente dal dato iniziale) è il *tempo d'estinzione*: infatti, così come accade per l'equazione della diffusione veloce su domini Euclidei limitati con condizioni di Dirichlet omogenee, le soluzioni diventano identicamente nulle in un tempo finito. La convergenza alla soluzione a variabili separabili è dimostrata, sulle linee di [31], in un senso molto forte, ovvero nella norma uniforme dell'*errore relativo* e a livello di tutte le derivate (sebbene senza velocità esplicite).

Si noti che su varietà Riemanniane generali, in coordinate locali, l'operatore di Laplace-Beltrami può essere scritto come  $\Delta = |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij} \partial_j)$ , dove gli indici sono sottintesi sommati e  $|g|$  denota il determinante del tensore metrico  $g$ . In particolare, almeno nel caso di speciali varietà Riemanniane a simmetria sferica note come *varietà modello*, fondamentalmente ricadiamo nella classe di equazioni descritte dalla (2). Infatti, in questo contesto, la *componente radiale* del Laplaciano Riemanniano può sempre essere scritta nella forma pesata  $\rho^{-1}(r) \partial_r (\rho(r) \partial_r)$  in coordinate geodetiche polari. Come precedentemente menzionato, qui svolgeremo una dettagliata analisi sullo spazio iperbolico, essendo quest'ultimo l'esempio topologicamente più semplice di varietà modello non-compatta e *a curvatura negativa*. Crediamo infine che i nostri risultati possano essere estesi almeno a varietà più generali con curvatura strettamente negativa, seguendo essenzialmente lo stesso metodo di dimostrazione.

Il Capitolo 4 si basa sul seguente lavoro:

[P9] G. Grillo and M. Muratori, *Radial fast diffusion on the hyperbolic space*, Proc. Lond. Math. Soc. 109 (2014), 283–317.





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## Synthèse

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Cette thèse s'occupe d'une classe d'équations non-linéaires de diffusion dont le prototype est

$$u_t = \Delta(|u|^{m-1}u), \quad (1)$$

aussi connue sous le nom d'*équation des milieux poreux* (PME) dans le cas où  $m > 1$  et *équation de diffusion rapide* (FDE) dans le cas où  $m < 1$ . L'équation (1) peut être vue comme une version non-linéaire particulière de l'équation de la chaleur, dont le coefficient de diffusion est une puissance de la solution  $u$  elle-même. Il faut remarquer que ce coefficient devient dégénéré ou singulier selon que l'on considère respectivement la PME ou la FDE.

L'étude d'équations d'évolution de type milieux poreux a une longue histoire, tant au niveau des applications physiques dans lesquelles elles apparaissent (à ce propos, voir l'Introduction au Chapitre 1) que du point de vue purement mathématique. On pourra se référer, en particulier, à la monographie [173] de J. L. Vázquez, dans laquelle est fournie une analyse détaillée et exhaustive de l'état de l'art.

On s'intéresse ici à des variantes et des extensions (à poids) de (1), en abordant plusieurs questions fondamentales comme l'existence, l'unicité et le comportement asymptotique des solutions. Notre analyse a pour but de lier autant que possible les propriétés des solutions à certaines inégalités fonctionnelles associées dans les espaces fonctionnels naturels sous-jacents, ce qui fournit un certain degré de généralité à la dissertation. Dans la suite, on va exposer brièvement les arguments principaux traités dans ce travail de thèse, lequel est divisé en quatre chapitres. Une introduction plus détaillée, avec une large bibliographie, sera donnée au début des chapitres correspondants.

**Équation des milieux poreux à poids.** La première variante que l'on étudie est la PME à poids, c'est à dire

$$\rho_\nu(x)u_t = \operatorname{div} \left[ \rho_\mu(x) \nabla(|u|^{m-1}u) \right], \quad (2)$$

où  $\rho_\nu$  et  $\rho_\mu$  sont des fonctions positives appropriées, que dorénavant on appellera *poids*. À priori on supposera uniquement que  $\rho_\nu$  et  $\rho_\mu$  sont des fonctions localement bornées et localement minorées par une constante strictement positive. Le poids  $\rho_\nu$  correspond à une distribution non-homogène de masse dans le milieu où a lieu la diffusion, alors que le poids  $\rho_\mu$  décrit un coefficient de diffusion qui peut dépendre de la variable spatiale. Il est donc assez naturel introduire la mesure  $d\nu := \rho_\nu(x)dx$  pour le problème considéré, dont l'espace de Lebesgue associé sera pour l'instant noté  $L^p(\nu)$  pour  $p \geq 1$ . Notre premier but consiste à

considérer des propriétés de régularisation et asymptotiques pour les solutions de (2), c'est à dire des estimations de type

$$\|u(t)\|_{L^p(\nu)} \leq K t^{-\alpha} \|u_0\|_{L^q(\nu)}^\beta \quad \forall t > 0, \quad (3)$$

pour la solution  $u(t)$  correspondante à n'importe quelle donnée initiale  $u_0 \in L^q(\nu)$ , où  $1 \leq q < p \leq \infty$  et  $\alpha, \beta, K$  sont des constantes positives appropriées qui dépendent de  $u_0$ . C'est un problème classique dans la théorie des demi-groupes linéaires, lequel a été largement étudié récemment également dans un contexte non-linéaire, voir par exemple la référence générale [172]. La stratégie que l'on adopte a pour objectif de lier des estimations de régularisation et asymptotiques de type (3) avec des inégalités fonctionnelles dans des espaces de Sobolev à poids dans lesquels apparaissent les poids  $\rho_\nu$  et  $\rho_\mu$ . En fait on prouvera une complète équivalence entre des versions appropriées de (3) et certaines inégalités de type *Poincaré*, *Sobolev* ou *Gagliardo-Nirenberg* (ces dernières étant introduites en particulier pour s'occuper des dimensions basses). Il est important de remarquer que (3) pourrait ne pas être satisfaite jusqu'à  $p = \infty$  dans le cas où des inégalités de type Sobolev ou Gagliardo-Nirenberg ne sont pas satisfaites (autrement dit, la seule validité d'une inégalité de type Poincaré n'est pas suffisante). Le résultat d'équivalence dépendra aussi du problème étudié : on considérera des domaines Euclidiens soit avec conditions de Dirichlet soit avec conditions de Neumann homogènes. Dans le cas de domaines avec  $\nu$ -mesure finie, l'estimation (3) peut être améliorée, lorsque le temps  $t$  est grand, toujours avec une approche purement analytique-fonctionnelle. Cela est particulièrement significatif pour le problème de Neumann, où la convergence des solutions vers leur  $\nu$ -moyenne (laquelle est conservée le long du temps) est montrée avec un taux *optimal*.

Le comportement asymptotique des solutions dont on vient de discuter pour des temps brefs et longues, avec une analyse détaillée du caractère bien posé, sera traité dans le Chapitre 1, et réunit les résultats des travaux [P1–P3]. Dans le même chapitre on considérera aussi une *équation de filtration non-homogène* dans l'entier espace Euclidien  $\mathbb{R}^d$ , laquelle a été étudiée dans [P4], c'est à dire une équation de type (2) où  $\rho_\mu \equiv 1$  et la non-linéarité  $|u|^{m-1}u$  est remplacée par une fonction générique  $G(u)$ . Il est bien connu que, dans le cas à poids, des phénomènes de non-unicité peuvent apparaître même parmi les solutions bornées avec données bornées. L'unicité peut donc être restaurée à condition que l'on ajoute des conditions non-homogènes à l'infini, typiquement formulées au sens intégral. Ici on établira que l'existence et l'unicité peuvent être démontrées, sous certaines hypothèses sur la non-linéarité et sur le poids, en posant des conditions appropriées de type *ponctuel* à l'infini. Ce dernier résultat semble être le premier de cette typologie pour un tel problème.

Le contenu du Chapitre 1 est basé sur les travaux suivants :

- [P1] G. Grillo, M. Muratori and M. M. Porzio, *Porous media equations with two weights: smoothing and decay properties of energy solutions via Poincaré inequalities*, Discrete Contin. Dyn. Syst. 33 (2013), 3599–3640.
- [P2] G. Grillo and M. Muratori, *Sharp short and long time  $L^\infty$  bounds for solutions to porous media equations with homogeneous Neumann boundary conditions*, J. Differential Equations 254 (2013), 2261–2288.
- [P3] G. Grillo and M. Muratori, *Sharp asymptotics for the porous media equation in low dimensions via Gagliardo-Nirenberg inequalities*, Riv. Mat. Univ. Parma 5 (2014), 15–38; comptes-rendus de la conférence “New Trends in Nonlinear Parabolic Equations”, Parma (Italie), du 12 au 16 Novembre 2012.
- [P4] G. Grillo, M. Muratori and F. Punzo, *Conditions at infinity for the inhomogeneous filtration equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 413–428.

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**Équation des milieux poreux fractionnaire.** Le Chapitre 2 est dédié à l'étude de solutions positives de l'équation des milieux poreux fractionnaire suivante :

$$u_t = -(-\Delta)^s(u^m), \quad (4)$$

où  $m > 1$  et pour n'importe quel  $s \in (0, 1)$  on note par  $(-\Delta)^s$  l'opérateur Laplacien fractionnaire sur  $\mathbb{R}^d$ , lequel peut être défini soit au niveau spectral soit à travers la méthode d'extension de Caffarelli et Silvestre [42]. L'analyse d'une telle classe d'équations a été initiée par les travaux [60, 61] de J. L. Vázquez et ses collaborateurs, où en particulier le caractère bien posé du problème d'évolution est montré pour des données  $L^1$ . Récemment, l'existence et l'unicité de solutions de type Barenblatt, c'est à dire solutions ayant la masse de Dirac pour donnée initiale, ont été montrées toujours par Vázquez dans [174]. Motivé par ces résultats, on a donc étudié le délicat problème de l'unicité (et de l'existence) de solutions faibles de (4) lorsque la donnée initiale est une *mesure de Radon positive finie* générale. Pour cela, on adapte une méthode introduite par M. Pierre dans [148], dont l'applicabilité dans notre contexte n'est pas du tout évidente et réclame un certain nombre de nouveaux arguments. On a montré de plus l'existence et l'unicité d'une trace initiale pour des fonctions qui sont solutions de (4) sans une condition initiale particulière donnée.

En fait notre méthode de travail, basée sur des techniques de *potentiel de Riesz*, peut aussi s'appliquer dans le cas de la *PME fractionnaire à poids* suivante :

$$\rho(x)u_t = -(-\Delta)^s(u^m), \quad (5)$$

pourvu que certaines conditions de type puissance soient satisfaites par le poids (ou *densité*), à l'origine où ce dernier peut être singulier, et à l'infini. L'analyse du caractère bien posé dont on vient de parler sera présentée dans le Chapitre 2 et se base sur les résultats de [P5]. Étant là encore capable de montrer l'existence et l'unicité de solutions de type Barenblatt dans le cas à poids aussi, et en particulier dans le cas d'une densité  $\rho$  avec une *puissance singulière*, on peut donc étudier le comportement asymptotique de solutions génériques de (5) à travers des solutions Barenblatt ci-dessus, au moins lorsque la décroissance de  $\rho$  à l'infini est suffisamment *lente*. Notons que, comme dans le cas local ( $s = 1$ ), la démonstration du résultat de convergence exploite des propriétés de mise à l'échelle de la solution de type Barenblatt associée à la puissance singulière même lorsque dans l'équation considérée apparaît une densité *régulière*. Cela motive l'analyse de (5) dans le cas où  $\rho$  est singulier dans l'origine. Le comportement asymptotique des solutions de (5) avec densités qui *décroissent rapidement* est complètement différent. En effet, toujours inspiré par le cas local, on démontrera que ce dernier est en fait déterminé par une *solution à variables séparées*, laquelle est liée à la solution d'une *équation sous-linéaire elliptique fractionnaire* appropriée. Cette étude nécessite des techniques complètement différentes par rapport à celles utilisées dans le cas de densités qui décroissent lentement. L'analyse asymptotique ici développée, et pour des densités qui décroissent lentement et pour des densités qui décroissent rapidement, se réfère aux résultats de [P6].

Enfin, remarquons qu'un résultat technique crucial dans la démonstration de l'unicité des solutions de (5) avec donnée initiale mesure, basé sur [P7], est le fait que, dans des espaces  $L^2$  à poids appropriés, l'opérateur linéaire formellement donné par  $A := \rho^{-1}(-\Delta)^s$  (et défini sur son domaine maximal) est autoadjoint. De plus, le demi-groupe associé a la propriété de Markov. Telles arguments ont aussi un intérêt indépendant et donc ils seront abordés séparément dans une annexe à la fin du chapitre.

Le contenu du Chapitre 2 est basé sur les travaux suivants :

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- [P5] G. Grillo, M. Muratori and F. Punzo, *Weighted fractional porous media equations: existence and uniqueness of weak solutions with measure data*, soumis pour publication, prépublication (2014), <http://arxiv.org/abs/1312.6076v2>.
- [P6] G. Grillo, M. Muratori and F. Punzo, *On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density*, à paraître dans *Discrete Contin. Dyn. Syst.*, prépublication (2014), <http://arxiv.org/abs/1403.5293v2>.
- [P7] M. Muratori, *On the self-adjointness of the fractional Laplacian in some weighted  $L^2$  spaces*, en préparation.

**Inégalités de Caffarelli-Kohn-Nirenberg et diffusion rapide à poids.** Le Chapitre 3 est consacré au problème de trouver les fonctions qui optimisent la famille d'inégalités de Caffarelli-Kohn-Nirenberg suivante :

$$\|w\|_{2p,\gamma} \leq C \|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta} \quad \forall w \in \mathcal{D}(\mathbb{R}^d), \quad (6)$$

où  $C > 0$ ,  $d \geq 3$ ,  $\gamma \in (0, 2)$ ,  $p \in (1, (d - \gamma)/(d - 2))$ ,  $\vartheta$  est un exposant explicite qui dépend de  $d$ ,  $\gamma$ ,  $p$  et  $\|w\|_{q,\gamma}^q := \int_{\mathbb{R}^d} |w(x)|^q |x|^{-\gamma} dx$  pour tout  $q \geq 1$ . De telles inégalités (qui peuvent être prolongées par densité à l'espace de fonctions pour lesquelles le membre de droite dans (6) est fini), ont été introduites sous une forme plus générale dans [41]. Remarquons que la famille (6) peut être facilement obtenue par interpolation à partir des inégalités de Hardy et de Sobolev. Trouver les fonctions optimales pour (6) signifie trouver des fonctions non-triviales pour lesquelles (6) devient une égalité, lorsque  $C$  est la *constante optimale*. Ce problème a été complètement résolu par M. Del Pino et J. Dolbeault dans [62] dans le cas sans poids, c'est à dire lorsque  $\gamma = 0$ . En particulier, les auteurs démontrent que les fonctions optimales coïncident avec les fonctions de type Aubin-Talenti

$$\mathfrak{b}_0(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}},$$

à multiplications, translations et changements d'échelle près. Notre objectif ici est de généraliser ce résultat à (6), c'est à dire montrer que les fonctions optimales correspondantes sont de type

$$\mathfrak{b}_\gamma(x) := \left(1 + |x|^{2-\gamma}\right)^{-\frac{1}{p-1}}. \quad (7)$$

Établir que (7) optimise (6) *parmi les fonctions radiales* n'est pas particulièrement difficile, puisque dans ce cas le problème peut être facilement ramené au cas sans poids. Par conséquent, notre analyse se réduit à montrer que les fonctions optimales sont *radiales*, ce qui n'est pas du tout trivial même si le poids  $|x|^{-\gamma}$  est radial (en général, dans la littérature, sont connus des phénomènes de *brisure de symétrie*). Au moyen d'un argument de perturbation qui nécessite des techniques de concentration-compacité, on montrera que le résultat est vrai à condition que  $\gamma > 0$  soit suffisamment petit. Ceci a des conséquences très intéressantes pour le comportement asymptotique de solutions positives de l'équation de diffusion rapide sur l'espace Euclidien avec poids puissance :

$$|x|^{-\gamma} u_t = \Delta(u^m) \quad (8)$$

dans l'intervalle super-critique  $(2d - \gamma - 2)/[2(d - \gamma)] < m < 1$ . Il s'avère que l'optimalité de (7) dans (6) est équivalente à la validité d'une certaine *inégalité d'énergie libre-information de Fisher* (ou *inégalité d'entropie-production d'entropie*) optimale, laquelle est à son tour équivalente à une estimation optimale de *décroissance exponentiel* pour la fonctionnelle d'énergie libre. Cette dernière est en fait une mesure de la distance entre une solution générique de (8)

et (7), après un changement d'échelle qui fait apparaître une version de type Fokker-Planck de (8) dont (7) est une solution stationnaire. Ces estimations fournissent des ingrédients clés pour entamer la *méthode d'entropie* (développée dans [23]) dans le cas à poids, ce qui sera abordé en détail dans [25].

Le Chapitre 3 est basé sur le travail suivant :

[P8] J. Dolbeault, M. Muratori and B. Nazaret, *Symmetry in weighted interpolation inequalities*, soumis pour publication.

**Diffusion rapide sur l'espace hyperbolique.** La dernière partie de la thèse, analysée dans le Chapitre 4, est l'équation de diffusion rapide sur l'*espace hyperbolique*  $\mathbb{H}^N$  ( $N \geq 2$ ). Il s'avère que le comportement asymptotique des solutions positives *radiales* (c'est à dire solutions qui ne dépendent que de la composante spatiale à travers la distance géodésique d'un pôle fixé), dans l'intervalle  $m \in ((N-2)/(N+2), 1)$ , est déterminé par une *solution à variables séparées* de type

$$\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}, \quad (9)$$

où  $V$  est la seule solution positive *d'énergie* de l'équation elliptique

$$-\Delta V = \frac{1}{(1-m)T} V^{\frac{1}{m}} \quad \text{sur } \mathbb{H}^N, \quad (10)$$

aussi connue sous le nom d'*équation de Lamé-Emden-Fowler*. On remarque que  $T > 0$  (lequel dépend de la donnée initiale) est le *temps d'extinction*. En effet, comme pour l'équation de diffusion rapide dans des domaines Euclidiens bornés avec conditions de Dirichlet homogènes, les solutions deviennent identiquement nulles en temps fini. En s'inspirant de [31], la convergence vers la solution à variables séparées est démontrée dans un sens très fort, c'est à dire dans la norme uniforme de l'*erreur relative* ainsi qu'au niveau de toutes les dérivées (par contre sans taux explicites).

Il est important de remarquer que sur des variétés Riemanniennes générales, en coordonnées locales, l'opérateur de Laplace-Beltrami peut être écrit  $\Delta = |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij} \partial_j)$ , où il est entendu que les indices sont sommés et  $|g|$  désigne le déterminant du tenseur métrique  $g$ . En particulier, au moins dans le cas de variétés Riemanniennes spéciales avec symétrie sphérique aussi connues sous le nom de *variétés modèles*, on tombe essentiellement dans la classe d'équations décrites par (2). En fait, dans ce contexte, la *composante radiale* du Laplacien Riemannien peut toujours être écrite sous la forme à poids  $\rho^{-1}(r) \partial_r (\rho(r) \partial_r)$  en coordonnées géodésiques polaires. Comme déjà indiqué ci-dessus, ici on développera une analyse détaillée sur l'espace hyperbolique, ce dernier étant l'exemple topologiquement le plus simple de variété modèle non-compacte et à *courbure négative*. Pour conclure, il est raisonnable de penser que ces résultats peuvent être étendus à des variétés plus générales avec courbure strictement négative, en suivant essentiellement la même méthode de démonstration.

Le Chapitre 4 est basé sur le travail suivant :

[P9] G. Grillo and M. Muratori, *Radial fast diffusion on the hyperbolic space*, Proc. Lond. Math. Soc. 109 (2014), 283–317.



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## Basic notations and conventions

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a.e.	“almost every” or “almost everywhere”
w.r.t.	“with respect to”
l.h.s.	“left-hand side”
r.h.s.	“right-hand side”
$a \vee b$	maximum between the numbers $a$ and $b$
$a \wedge b$	minimum between the numbers $a$ and $b$
$\langle \mathbf{a}, \mathbf{b} \rangle$ or $\mathbf{a} \cdot \mathbf{b}$	Euclidean scalar product between the vectors $\mathbf{a}$ and $\mathbf{b}$
$A : B$	Frobenius scalar product between the matrices $A$ and $B$
$\mathbf{n}$	outer normal vector at the boundary of a regular domain
$\mathbb{R}^+$	the set of positive real numbers; when needed, we shall be more precise by using $(0, +\infty)$ or $[0, +\infty)$
$B_r(x_0)$	Euclidean (unless otherwise specified) ball of radius $r$ centred at $x_0$
$B_r$	$= B_r(0)$
$\chi_\Omega$	characteristic function of the set $\Omega$ , unless otherwise specified
$\Omega_0 \Subset \Omega_1$	the closure of the set $\Omega_0$ is compact in $\Omega_1$
$f_+, f_-$	positive and negative part of the function $f$ , respectively, in the sense that $f = f_+ - f_-$
$f _\Omega$	the function $f$ restricted to the set $\Omega$
$u(t)$	when dealing with a space-time function $u(x, t)$ , it means the whole function of $x$ at fixed $t$
$\delta_{x_0}$	Dirac delta distribution centred at $x_0$ , unless otherwise specified
$\delta$	$= \delta_0$
$W^{k,p}(\Omega)$	usual Sobolev space of functions whose weak derivatives up to the $k$ -th order are in $L^p(\Omega)$
$H^k(\Omega)$	$= W^{k,2}(\Omega)$
$L^p((0, T); V)$	space of $V$ -valued (let $V$ be a Banach space) functions $u: \ u(t)\ _V \in L^p((0, T))$
$W^{1,p}((0, T); V)$	functions in $L^p((0, T); V)$ with $u_t \in L^p((0, T); V)$ , where $H^1 = W^{1,2}$
$V_{\text{loc}}(\Omega)$	space of functions defined in the domain $\Omega$ belonging to a certain space $V(\Omega_0)$ for all $\Omega_0 \Subset \Omega$
$V_c$	subspace of compactly supported functions belonging to $V$
$C^{\alpha,\beta}$	depending on the context, it denotes either functions which are $C^\alpha$ in space and $C^\beta$ in time or functions with continuous derivatives up to the order $\alpha$ whose $\alpha$ -th derivatives are $C^\beta$
$\mathcal{D}(\Omega)$	$= C_c^\infty(\Omega)$



## Porous medium equations with general weights

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### 1.1 Introduction

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This chapter is devoted to the analysis of *weighted porous medium equations*. In particular, given a domain  $\Omega \subset \mathbb{R}^d$  and two weights  $\rho_\nu, \rho_\mu > 0$  independent of time and complying with suitable assumptions that will be specified below, we shall mostly deal with the homogeneous Dirichlet problem

$$\begin{cases} \rho_\nu(x)u_t = \operatorname{div} [\rho_\mu(x)\nabla(u^m)] & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u = u_0 & \text{on } \Omega \times \{0\} \end{cases} \quad (1.1.1)$$

and with the homogeneous Neumann problem

$$\begin{cases} \rho_\nu(x)u_t = \operatorname{div} [\rho_\mu(x)\nabla(u^m)] & \text{in } \Omega \times \mathbb{R}^+, \\ \rho_\mu(x)\frac{\partial(u^m)}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases} \quad (1.1.2)$$

where  $m > 1$  is a fixed parameter. We stress that the initial datum  $u_0$  (and so the corresponding solution) is not required to have a sign: for any  $q > 0$  and  $y \in \mathbb{R}$ , we therefore use the convention  $y^q := |y|^q \operatorname{sign}(y)$  (French powers). Nevertheless, solutions with positive [negative] initial data stay positive [negative]. This is a formal consequence of standard comparison principles. A precise meaning to the notions of weak solution to (1.1.1) and (1.1.2) will be given in Subsections 1.2.1 and 1.3.1, respectively. For such solutions a key (and classical) estimate that one can prove is the so-called *L<sup>1</sup> comparison principle* (see Theorems 1.2.4 and 1.3.4 below), which includes an L<sup>1</sup> contraction inequality and a comparison principle. We mention from the beginning that our techniques are general enough to permit us to handle any Euclidean domain, which can also coincide with the whole space  $\mathbb{R}^d$ ; in this regard, we point out that the boundary conditions in (1.1.1) and (1.1.2) are just formal and have to be interpreted in a suitable weak sense.



Problems (1.1.1) and (1.1.2) are generalizations of the well-known porous medium equation: we refer the reader to the recent monograph [173] by J. L. Vázquez for a fully-detailed analysis and a complete overview of the topic. The appearance of the weights  $\rho_\nu$  and  $\rho_\mu$  is related to spatial inhomogeneity of the medium as concerns mass density and diffusion coefficient, respectively: see Subsection 1.1.1 for a brief collection of possible physical frameworks where the (weighted) porous medium equation has proved to be a good model for the underlying phenomena.

There is a wide literature on porous medium equations with weights: with no claim of generality we refer the reader *e.g.* to [82, 83, 89, 90, 107, 111, 114–116, 118, 141, 147, 153, 162–164] and references therein. The one-weight case in  $\mathbb{R}^d$  (*i.e.* when  $\rho_\mu \equiv 1$ ) is by far the most studied one (also for more general nonlinearities compared with  $u^m$ ). In fact, the latter was addressed in several papers (included in the above list), some of which deal with a weight that decays as a negative power of  $|x|$  at infinity. Such property allows for a very detailed analysis of the asymptotic behaviour of solutions in terms of suitably generalized Barenblatt-type fundamental solutions (for the definition of the latter, see again Subsection 1.1.1) or of separable solutions, depending on the decay rate of the density, much in the spirit of the non-weighted case (we refer in particular to [114, 141, 164]). On the other hand, two-weight operators are of common use in linear analysis and are widely studied in several different contexts: for example, every Riemannian Laplacian can be written, locally, as a two-weight linear operator of second order. Still in the linear framework, the validity of functional inequalities for the quadratic form of the generator of an evolution is well known to be strictly connected with smoothing and asymptotic properties of the evolution itself: we refer the reader to the monograph [59] and references quoted therein for a complete discussion. In the nonlinear case, such connection is more subtle and much less investigated (see *e.g.* [27–30, 149]): as we shall explain in detail below, our main goal here is to give a further contribution to such analysis.

Our first concern is to establish a well-posedness theory for problems (1.1.1) and (1.1.2), which allows us to deal with sufficiently general weights. In fact, under the sole assumption

$$\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L_{\text{loc}}^\infty(\Omega), \quad (1.1.3)$$

we shall prove existence of solutions with initial data that belong to suitable  $L^p(\Omega; \nu)$  spaces (let  $p \geq m + 1$ ), where the latter symbol denotes the usual  $L^p$  Lebesgue space with respect to the measure  $d\nu = \rho_\nu(x)dx$  (see Subsection 1.1.2 for notations). Our method of proof is based on a careful generalization of the strategy developed in [173, Chapter 5], which turns out to work both for the Dirichlet and, with minor modifications, for the Neumann problem (see Subsections 1.2.2 and 1.3.2). The fundamental idea is to approximate the degenerate (and possibly singular, depending on the behaviour of the weights at  $\partial\Omega$ ) problems (1.1.1) and (1.1.2) with a sequence of similar non-degenerate quasilinear problems posed in a corresponding sequence of regular bounded domains  $\{\Omega_n\} \Subset \Omega$ . Along with the mentioned  $L^1$  comparison principle, key energy estimates are shown to hold for the solutions to such approximate problems, which permit us to pass to the limit as  $n \rightarrow \infty$  and finally obtain weak solutions to (1.1.1) and (1.1.2). Note that (1.1.3) is required in order to ensure well-posedness of the non-degenerate quasilinear problems. When the initial data have a worse integrability (*e.g.* they only belong to  $L^1(\Omega; \nu)$ ), taking advantage of the  $L^1$  comparison principle we construct *limit solutions*, in agreement with [173, Chapter 6]. Uniqueness, in general, is a delicate issue: as already noticed in [89], even in the one-weight case it need not hold in the class of bounded solutions (in this regard, see also Section 1.4). Nevertheless, by means of a routine argument originally introduced by O. A. Oleĭnik we can prove that it does hold in the class of *energy solutions*, namely solutions belonging to suitable weighted Sobolev spaces

(see Definitions 1.2.2 and 1.3.1 below).

Our main goal here is not a full investigation of qualitative properties of solutions for some explicit classes of weights. We aim at considering weighted porous medium equations from a more abstract point of view. In fact, we are rather interested in the comprehension of how the validity of  $L^{q_0}$ - $L^\varrho$  (let  $1 \leq q_0 < \varrho$ ) smoothing and asymptotic properties of solutions to (1.1.1) and (1.1.2) is connected with the validity of functional inequalities that involve the weights  $\rho_\nu$  and  $\rho_\mu$ . More precisely, as concerns the Dirichlet problem, we shall deal with *Poincaré-type inequalities* of the form

$$\|v\|_{2;\nu} \leq C_P \|\nabla v\|_{2;\mu} \quad \forall v \in H_0^1(\Omega; \nu, \mu), \quad (1.1.4)$$

*Sobolev-type inequalities* of the form (let  $\sigma > 1$ )

$$\|v\|_{2\sigma;\nu} \leq C_S \|\nabla v\|_{2;\mu} \quad \forall v \in H_0^1(\Omega; \nu, \mu), \quad (1.1.5)$$

*one-dimensional Gagliardo-Nirenberg-type inequalities* of the form (let  $\lambda \geq 2$ )

$$\|v\|_\infty \leq C_{G_1} \|\nabla v\|_{2;\mu}^{\frac{\lambda}{s+\lambda}} \|v\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu) \quad (1.1.6)$$

and *two-dimensional Gagliardo-Nirenberg-type inequalities* of the form

$$\|v\|_{r;\nu} \leq C_{G_2} \|\nabla v\|_{2;\mu}^{1-\frac{s}{r}} \|v\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu), \quad (1.1.7)$$

for suitable positive constants  $C_P$ ,  $C_S$ ,  $C_{G_1}$  and  $C_{G_2}(r)$ . By the symbol  $H_0^1(\Omega; \nu, \mu)$  we denote the weighted Sobolev space obtained as the closure of  $\mathcal{D}(\Omega)$  with respect to the norm

$$\|\phi\|_{H_0^1(\Omega; \nu, \mu)} := \left( \|\phi\|_{2;\nu}^2 + \|\nabla \phi\|_{2;\mu}^2 \right)^{\frac{1}{2}} \quad \forall \phi \in \mathcal{D}(\Omega),$$

where  $d\mu = \rho_\mu(x)dx$  and

$$\|f\|_{p;\eta} := \left( \int_\Omega |f(x)|^p d\eta \right)^{\frac{1}{p}} \quad \forall f \in L^p(\Omega; \eta), \quad \eta = \nu \text{ or } \mu,$$

see Subsection 1.1.2 for details. Assuming the validity of (1.1.4), we shall prove that solutions to (1.1.1) satisfy the bounds

$$\|u(t)\|_{\varrho;\nu} \leq K t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} \quad \forall t > 0 \quad (1.1.8)$$

for some  $K > 0$  independent of  $t$  and  $u_0$ , provided  $u_0 \in L^{q_0}(\Omega; \nu)$  with  $q_0 \geq 1$  and  $\varrho \in (q_0, \infty)$ . On the other hand, the validity of any of the inequalities (1.1.5), (1.1.6) or (1.1.7) entails an estimate of the form

$$\|u(t)\|_\infty \leq K t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta} \quad \forall t > 0, \quad (1.1.9)$$

where  $K > 0$  is as above and  $u_0 \in L^{q_0}(\Omega; \nu)$  with  $q_0 \geq 1$ . The constant  $\theta \in (0, 1)$  is explicit and depends on  $q_0$ ,  $m$  and on the particular inequality which holds true through  $\sigma$  or  $\lambda$ . If in addition  $\nu(\Omega) < \infty$  then the above estimates can be improved for long times, yielding a decay rate of order  $t^{-1/(m-1)}$  (*absolute bound*). Furthermore, we shall also prove *converse implications*: that is, the validity of estimate (1.1.8) for solutions to (1.1.1) implies in turn that the Poincaré-type inequality (1.1.4) holds true. The same applies to (1.1.9) as related to (1.1.5), (1.1.6) or (1.1.7). We refer the reader to Subsection 1.2.1 for the precise statements of the corresponding theorems.

As the reader might have noticed, the validity of a Poincaré-type inequality, in contrast with Sobolev- and Gagliardo-Nirenberg-type inequalities, *a priori* does not allow us to obtain

an  $L^\infty$  smoothing effect:  $\varrho$  is arbitrarily large but has to be strictly smaller than  $\infty$  (indeed the constant  $K$  in (1.1.8) blows up as  $\varrho \uparrow \infty$ ). In agreement with that, we shall provide explicit examples of couples of weights for which the Poincaré-type inequality holds true and the  $L^\infty$  regularization fails for (some) solutions to the corresponding Dirichlet problem. However, even the fact that a smoothing effect can be deduced from the validity of (1.1.4) is somehow surprising. In this regard, let us introduce the differential operator formally given by

$$\mathcal{L}_{\nu,\mu} := -\rho_\nu^{-1} \operatorname{div}(\rho_\mu \nabla),$$

which is formally self-adjoint and nonnegative in  $L^2(\Omega; \nu)$ . The Poincaré-type inequality (1.1.4) has a clear spectral interpretation: it amounts to requiring that

$$\min \mathcal{S}(\mathcal{L}_{\nu,\mu}) \geq \frac{1}{C_P^2} > 0,$$

where  $\mathcal{S}(\mathcal{L}_{\nu,\mu})$  denotes the  $L^2(\Omega; \nu)$  spectrum of  $\mathcal{L}_{\nu,\mu}$ . It would therefore seem quite unlikely, from the linear case, that such inequalities can be related to anything but a long-time bound on the  $L^2(\Omega; \nu)$  norm of the solution. Nevertheless, in the particular nonlinear framework of  $p$ -Laplacian-type operators, in [105] it has been shown that suitable Poincaré-type inequalities imply smoothing effects of the type of (1.1.8) (and they are in fact equivalent to one another). One of our contributions here consists in proving the same type of results for the porous medium equation.

Sobolev- and Gagliardo-Nirenberg-type inequalities naturally arise from the non-weighted context (*i.e.* when  $\rho_\nu = \rho_\mu \equiv 1$ ). In fact, as it is well known, if  $d \geq 3$  the Sobolev inequality (1.1.5) holds true in  $H_0^1(\Omega)$  with  $\sigma = d/(d-2)$ ; in dimension two the family of Gagliardo-Nirenberg inequalities (1.1.7) holds true in  $H_0^1(\Omega)$ , while in dimension one the family of Gagliardo-Nirenberg inequalities (1.1.6) holds true in  $H_0^1(\Omega)$  with  $\lambda = 2$ . In fact, all of these inequalities are a consequence of the Nash inequality

$$\|\phi\|_2 \leq C(d) \|\nabla \phi\|_2^{\frac{d}{d+2}} \|\phi\|_1^{\frac{2}{d+2}} \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d),$$

which is valid in any dimension: we refer the reader *e.g.* to the remarkable paper [13] for a thorough discussion on the equivalence of families of functional inequalities of Poincaré, Sobolev and Gagliardo-Nirenberg type (and not only). As we shall mention later on, the smoothing effects (1.1.9) we obtain in the Sobolev and Gagliardo-Nirenberg case allow us to recover the correct estimates in the non-weighted framework, namely (1.1.9) with

$$\theta = \frac{d(m-1)}{2q_0 + d(m-1)}.$$

Such estimates had already been proved in the monograph [172] (where smoothing and decay properties of solutions to large classes of nonlinear evolution equations are thoroughly analysed) with different techniques, and they are known to be sharp. However, our method of proof is more general and only relies on the validity of the above functional inequalities. In Subsection 1.2.3 we shall also provide explicit examples of nontrivial weights (of power-type) for which our estimates are sharp.

Let us now consider the Neumann problem (1.1.2). In this case, we assume  $\nu(\Omega) < \infty$  and investigate smoothing and asymptotic properties of solutions in connection with the validity of Poincaré-type inequalities of the form

$$\|v\|_{2;\nu} \leq W_P \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right) \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.1.10)$$

and

$$\|v - \bar{v}\|_{2;\nu} \leq M_P \|\nabla v\|_{2;\mu} \quad \forall v \in H^1(\Omega; \nu, \mu), \quad (1.1.11)$$

Sobolev-type inequalities of the form (let  $\sigma > 1$ )

$$\|v\|_{2\sigma;\nu} \leq W_S \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right) \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.1.12)$$

and

$$\|v - \bar{v}\|_{2\sigma;\nu} \leq M_S \|\nabla v\|_{2;\mu} \quad \forall v \in H^1(\Omega; \nu, \mu), \quad (1.1.13)$$

one-dimensional Gagliardo-Nirenberg-type inequalities of the form (let  $\lambda \geq 2$ )

$$\|v\|_\infty \leq W_{G_1} \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right)^{\frac{\lambda}{s+\lambda}} \|v\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.1.14)$$

and

$$\|v - \bar{v}\|_\infty \leq M_{G_1} \|\nabla v\|_{2;\mu}^{\frac{\lambda}{s+\lambda}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu), \quad (1.1.15)$$

two-dimensional Gagliardo-Nirenberg-type inequalities of the form

$$\|v\|_{r;\nu} \leq W_{G_2} \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right)^{1-\frac{s}{r}} \|v\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.1.16)$$

and

$$\|v - \bar{v}\|_{r;\nu} \leq M_{G_2} \|\nabla v\|_{2;\mu}^{1-\frac{s}{r}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu), \quad (1.1.17)$$

where  $W_P$ ,  $M_P$ ,  $W_S$ ,  $M_S$ ,  $W_{G_1}$ ,  $M_{G_1}$ ,  $W_{G_2}(r)$  and  $M_{G_2}(r)$  are suitable positive constants,

$$\bar{f} := \frac{\int_\Omega f(x) \, d\nu}{\nu(\Omega)} \quad \forall f \in L^1(\Omega; \nu)$$

and  $H^1(\Omega; \nu, \mu)$  is the Sobolev space of weakly differentiable functions  $v$  such that  $\|v\|_{2;\nu} + \|\nabla v\|_{2;\mu} < \infty$  (see again Subsection 1.1.2). If one thinks about the non-weighted case, the above inequalities are the analogues of (1.1.4)–(1.1.7) in regular bounded Euclidean domains, for functions which do not necessarily vanish on  $\partial\Omega$ . Inequalities (1.1.11), (1.1.13), (1.1.15) and (1.1.17) are stronger than (1.1.10), (1.1.12), (1.1.14) and (1.1.16), respectively; however, it is easy to show that the former are equivalent to the latter plus the Poincaré-type inequality (1.1.11). Assuming the validity of (1.1.10), we shall prove that solutions to (1.1.2) satisfy the bounds

$$\|u(t)\|_{\varrho;\nu} \leq K \left( t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0 \quad (1.1.18)$$

for some  $K > 0$  independent of  $t$  and  $u_0$ , provided  $u_0 \in L^{q_0}(\Omega; \nu)$  with  $q_0 \geq 1$  and  $\varrho \in (q_0, \infty)$ . As in the Dirichlet case, by means of an explicit counterexample we show that in general, if only the Poincaré-type inequality (1.1.10) (or (1.1.11)) holds true,  $L^\infty$  regularization fails. On the other hand, the validity of any of the inequalities (1.1.12)–(1.1.17) entails an estimate of the form

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0, \quad (1.1.19)$$

where  $K > 0$  is as above and  $u_0 \in L^{q_0}(\Omega; \nu)$  with  $q_0 \geq 1$ . The constant  $\theta \in (0, 1)$  is the same appearing in the corresponding estimate (1.1.9) for the Dirichlet problem. Converse implications, which yield equivalence results between our estimates and the above functional inequalities, continue to hold in this context as well. We refer the reader to Subsection 1.3.1 for the precise statements. The bounds (1.1.18) and (1.1.19) can be improved for long

times upon requiring the validity of (1.1.11), (1.1.13), (1.1.15) or (1.1.17). Here we need to distinguish between initial data with zero-mean and with nonzero-mean (and solutions accordingly, since the Neumann boundary condition forces the mean value to be preserved). In particular, we shall prove that if  $\bar{u}_0 = 0$  then one can recover exactly the same estimates as in the Dirichlet case, as well as a decay rate of order  $t^{-1/(m-1)}$ . On the contrary, the most important result we shall prove for solutions with nonzero mean is the *uniform convergence* to their mean value with a precise *exponential rate*:

$$\|u(t) - \bar{u}\|_\infty \leq G e^{-\frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall t \geq 1, \quad (1.1.20)$$

where  $G > 0$  is a suitable constant. We point out that (1.1.20) holds true as a sole consequence of the validity of a Sobolev- or Gagliardo-Nirenberg-type inequality like (1.1.13), (1.1.15) or (1.1.17): no result of this kind seems to have appeared in the related literature so far, at least to the best of our knowledge. If we only assume the validity of the Poincaré-type inequality (1.1.11), then we have a weaker result. That is, for initial data in  $L^\infty(\Omega)$  we shall prove that the corresponding solutions still converge to their mean value with an exponential rate; however, such rate is unknown and convergence takes place in  $L^\varrho(\Omega; \nu)$  for all  $\varrho \in [1, \infty)$ . Actually, uniform convergence in general need not hold, as we can show through a counterexample. Finally, for unbounded initial data we shall prove the same type of convergence to the mean value, though with some negative-power rate which is not necessarily sharp.

The Neumann problem for the porous medium equation, even in the non-weighted case, is much less studied in the literature with respect to the Dirichlet problem (or to the Cauchy problem in the whole space  $\mathbb{R}^d$ ). Some asymptotic estimates were first provided by the pioneering work of N. D. Alikakos and R. Rostamian [3]. In particular, they proved that if  $\bar{u}_0 = 0$  then the solution to the homogeneous Neumann problem in regular Euclidean domains converges uniformly to zero with the sharp rate  $t^{-1/(m-1)}$ . On the other hand, if  $\bar{u}_0 \neq 0$  convergence to the mean value is exponential. However, such results were established only for  $u_0 \in L^\infty(\Omega)$ , whereas a smoothing effect was proved later on in [27] (in the framework of Riemannian manifolds). In Subsections 1.3.3 and 1.3.4 we shall explain in detail how our results improve on the ones of [3] and [27]. In this regard, our contributions consist in having proved estimates which, in the non-weighted case, are sharp both for short and long times, by means of purely functional arguments. In the weighted case, as already mentioned, our short-time estimates are in fact equivalent to the functional inequalities we start from; moreover, we shall also provide explicit examples of power-law weights for which such estimates are sharp, in the sense that there exist solutions whose short-time behaviour is exactly the one given by (1.1.19). As concerns the exponential bound (1.1.20), we point out that, though we can rigorously prove its sharpness only in the non-weighted case, the rate predicted by it is precisely the one we obtain by linearising the solution  $u$  about its mean value  $\bar{u}$ .

Besides the quoted papers [3, 27], let us also mention [116], where the issue of convergence to the mean value for solutions to the porous medium equation (and more general nonlinear, degenerate diffusions) is addressed in dimension one and in the one-weight case (namely  $\rho_\mu \equiv 1$ ). The authors establish local uniform convergence, while global uniform convergence is not dealt with; actually, this is in agreement with the fact that the Sobolev-type inequalities in general may not hold under the assumptions they make on the density  $\rho = \rho_\nu$ .

Other works related to the Neumann problem for equations of porous medium type can be found for instance in [5, 6]. However, we remark that the discussions there involve domains of infinite volume, so that convergence to zero, rather than to the mean value, takes place, thus giving rise to a kind of evolution which is to some extent closer to the Dirichlet case.

Notice in addition that in also in the papers [82, 83, 176] one can find asymptotic bounds

for weighted porous medium equations in connection with functional inequalities, although no smoothing effect is dealt with there.

As concerns methods of proof, most of our short-time estimates will be obtained by means of techniques of *Moser iteration* type, named after J. Moser who introduced and developed them in a series of papers (see *e.g.* [134–137]), mainly in connection with the study of uniformly elliptic operators in  $\mathbb{R}^d$ . The interesting point in such techniques is that, basically, they only make use of functional inequalities like Sobolev’s. As concerns the porous medium equation, similar kind of methods were used in the quoted paper [3] and also in [2], even though the estimates proved there are not smoothing effects since they still depend in a nontrivial way upon the  $L^\infty$  norm of the initial datum. A Moser iteration very similar to the ones we exploit here can be found in [61], see also Chapter 2, Subsection 2.2.2.

We point out that, although the estimates for the Neumann problem (1.1.18) and (1.1.19) are equal to the corresponding estimates for the Dirichlet problem plus a constant times  $\|u_0\|_{q_0;\nu}$  in the right-hand side, they are much harder to prove compared to the latter, since additional terms to be dealt with appear in the Moser iterations.

On the other hand, long-time bounds will be proved by means of purely differential methods, which amount to find a good differential inequality for

$$\frac{d}{dt} \|u(t)\|_{q;\nu}^e \quad \text{or} \quad \frac{d}{dt} \|u(t) - \bar{u}\|_{q;\nu}^e.$$

Especially in the case of solutions with nonzero mean, obtaining such differential inequalities is not straightforward: in order to do it, we will have to prove new functional inequalities from (1.1.11) and (1.1.13). Actually, suitable differential methods, much in the spirit of L. Gross [106], are thoroughly used by the authors in [27] to prove their smoothing effects. This yields very good estimates for Dirichlet-type problems (in fact (1.1.8) will be proved by a differential argument and (1.1.9) is basically a consequence of the results in [27]), while for Neumann-type problems our Moser iteration techniques give better results.

In Section 1.4 we shall focus on a particular one-weight problem in Euclidean space:

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } \mathbb{R}^d \times (0, T] =: S_T, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (1.1.21)$$

where  $d \geq 3$ ,  $T > 0$  and the weight  $\rho$  (which here is commonly referred to as *density*), the nonlinearity  $G$  and the initial datum  $u_0$  comply with the following conditions:

$$\begin{cases} (i) & \rho \in C(\mathbb{R}^d), \text{ with } \rho > 0; \\ (ii) & G \in C^1(\mathbb{R}), \text{ with } G(0) = 0 \text{ and } G' > 0 \text{ in } \mathbb{R} \setminus \{0\}; \\ (iii) & \text{if } G'(0) = 0 \text{ then } G' \text{ is decreasing in } (-\delta, 0) \text{ and increasing in } (0, \delta) \text{ for some } \delta > 0; \\ (iv) & u_0 \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{cases} \quad (H_0)$$

A typical choice for the function  $G$  is of course the one corresponding to the porous medium equation, namely  $G(u) = |u|^{m-1}u$  for some  $m > 1$  (French power). In this case, the differential equation in (1.1.21) is usually called *inhomogeneous porous medium equation*, which arises in several contexts of physical interest (see Subsection 1.1.1). On the contrary, if one keeps a general nonlinearity  $G$  as in  $(H_0)$ -(ii, iii), then (1.1.21) is referred to as *inhomogeneous filtration equation*.

When one gives up the class of energy solutions we informally introduced above and only deals with (very weak) bounded solutions, well-posedness of problem (1.1.21) turns

out to depend strongly on the behaviour of the density  $\rho(x)$  as  $|x| \rightarrow \infty$ . As it is well known, if assumption  $(H_0)$  is satisfied, then there exists a bounded solution (see for instance [90, 116]). Moreover, if  $d = 1, 2$  and  $\rho \in L^\infty(\mathbb{R}^d)$ , such solution is unique [107]. In higher dimensions  $d \geq 3$ , we can have uniqueness or nonuniqueness of bounded solutions depending on asymptotic properties of  $\rho$  at infinity. More precisely, suppose that  $\rho$  does not decay too fast, in the sense that there exist  $R_1 > 0$  and  $\rho_1 \in C([R_1, \infty))$  such that  $\rho(x) \geq \rho_1(|x|) > 0$  for all  $x \in B_{R_1}^c$  and  $\int_{R_1}^\infty r \rho_1(r) dr = \infty$ . Then problem (1.1.21) admits *at most* one bounded solution (see [153, 156] or [154] for similar issues on bounded domains). A natural choice for  $\rho_1$  is *e.g.*  $\rho_1(r) = r^{-\alpha}$ , with  $r \in [R_1, \infty)$ , for some  $\alpha \in (-\infty, 2]$  and  $R_1 > 0$ .

On the other hand, if  $\rho$  decays fast enough at infinity, in the sense that  $\Gamma * \rho \in L^\infty(\mathbb{R}^d)$ , where  $\Gamma$  is the fundamental solution of the Laplace equation in  $\mathbb{R}^d$ , then nonuniqueness prevails (see again [153, 156] or even [113] for the linear case, namely  $G(u) = u$ ). That is, for any function  $A \in \text{Lip}([0, T])$  with  $A(0) = 0$  there exists a solution  $u$  to problem (1.1.21) which satisfies the additional condition

$$\lim_{R \rightarrow \infty} \frac{1}{|\partial B_R|} \int_{\partial B_R} |U(\sigma, t) - A(t)| d\sigma = 0 \quad (1.1.22)$$

uniformly as  $t \in [0, T]$ , where

$$U(x, t) := \int_0^t G(u(x, \tau)) d\tau \quad \forall (x, t) \in S_T.$$

Note that the hypothesis  $\Gamma * \rho \in L^\infty(\mathbb{R}^d)$  can be replaced by the following stronger (but more explicit) assumption: there exists  $R_0 > 0$  and  $\rho_0 \in C([R_0, \infty))$  such that  $\rho(x) \leq \rho_0(|x|)$  for all  $x \in B_{R_0}^c$  and  $\int_{R_0}^\infty r \rho_0(r) dr < \infty$ . Then we can in turn replace (1.1.22) with

$$\lim_{|x| \rightarrow \infty} |U(x, t) - A(t)| = 0 \quad (1.1.23)$$

uniformly as  $t \in [0, T]$ , which trivially implies (1.1.22). Again, a natural choice for  $\rho_0$  is *e.g.*  $\rho_0(r) = r^{-\alpha}$ , with  $r \in [R_0, \infty)$ , for some  $\alpha \in (2, \infty]$  and  $R_0 > 0$ .

We point out that (1.1.22) and (1.1.23) can also be considered as *nonhomogeneous Dirichlet conditions at infinity* in a suitable integral sense. In this regard, it seems therefore natural to ask whether imposing conditions at infinity *in a pointwise sense*, which resembles more closely the usual Dirichlet boundary conditions on bounded domains, still restores existence and uniqueness of solutions. In order to avoid confusion, we remark that here the term “inhomogeneous” is referred to the presence of the density  $\rho$  in the differential equation, while the term “nonhomogeneous” is referred to possible conditions at infinity with  $A \not\equiv 0$ .

As concerns uniqueness subject to (1.1.22) or (1.1.23), so far it was only known that there exists at most one solution  $u \in L^\infty(S_T)$  to problem (1.1.21) satisfying the extra conditions (1.1.22) or (1.1.23) either when  $G(u) = u$  (see the results obtained in [113]) or when  $u_0 \geq 0$  and  $A \equiv 0$  (see [90]). However, the methods used to obtain such results did not work for general  $G$  and  $A$ . Here we show that, to some extent, it is possible to deal both with nonlinearities and nonhomogeneous pointwise conditions *simultaneously*.

In Subsection 1.4.2 we address existence. More precisely, first we prove that if  $\rho$  decays fast enough at infinity, the diffusion is non-degenerate in a suitable sense,  $u_0 \in C(\mathbb{R}^d)$  and  $\lim_{|x| \rightarrow \infty} u_0(x)$  exists and is finite, then for any  $a \in C([0, T])$  with

$$a(0) = \lim_{|x| \rightarrow \infty} u_0(x) \quad (1.1.24)$$

there exists a bounded solution  $u$  to problem (1.1.21) which satisfies

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{uniformly as } t \in [0, T]. \quad (1.1.25)$$

Furthermore, we can drop the assumption of non-degeneracy for suitable classes of initial data  $u_0$  and conditions at infinity  $a$ . Indeed, if  $a = a_0 := \lim_{|x| \rightarrow \infty} u_0(x) \in \mathbb{R}$  and  $(H_0)$  holds true, then there exists a bounded solution  $u$  to problem (1.1.21) such that

$$\lim_{|x| \rightarrow \infty} u(x, t) = a_0 \quad \text{uniformly as } t \in [0, T],$$

independently of the asymptotic behaviour of  $\rho$ . Finally, if  $(H_0)$  is fulfilled and  $\rho$  decays fast enough at infinity, then for any  $a \in C([0, T])$  with  $a > 0$  in  $[0, T]$  there exists a bounded solution  $u$  to problem (1.1.21) which satisfies (1.1.25), provided  $u_0$  complies with (1.1.24). For all the precise statements, we refer the reader to Subsection 1.4.1.

Let us comment that in [153], by means of generalizations of the arguments in [113], conditions (1.1.22) and (1.1.23) for solutions to (1.1.21) are achieved by constructing *barriers at infinity*, which are sub- or supersolutions to suitable linear elliptic problems. In the present context, in order to impose Dirichlet conditions at infinity in a pointwise sense, we will construct, in a neighbourhood of each  $t_0 \geq 0$ , delicate time-dependent barriers, which are sub- or supersolutions to suitable nonlinear parabolic problems.

Uniqueness is dealt with in Subsection 1.4.3. In particular, we prove that it holds true under the weaker condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{for a.e. } t \in (0, T),$$

for a general  $G$  complying with  $(H_0)$ -*(ii, iii)* and bounded  $\rho$  and  $a$ . The arguments we exploit are modelled after [8] ( $\rho \equiv 1$ ,  $d = 1$ ) and [107] ( $d = 1, 2$ ), where uniqueness was established in low dimensions without requiring extra conditions at infinity.

### 1.1.1 Some physical motivations

Given a regular bounded domain  $\Omega \subset \mathbb{R}^d$  and  $m > 1$ , let us consider the porous medium equation in the non-weighted case:

$$u_t = \Delta(u^m) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1.26}$$

with  $u \geq 0$ . Undoing the Laplacian as the divergence of the gradient, from (1.1.26) we obtain

$$u_t = \operatorname{div} \left( m u^{m-1} \nabla u \right) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1.27}$$

whence one sees that the diffusivity coefficient appearing in the equation is in fact  $m u^{m-1}$ . If we think of  $u$  as the concentration of some fluid substance in  $\Omega$ , we find that the diffusion is *slower* where the concentration is *higher*. From the mathematical point of view, this entails the *finite speed of propagation* of solutions to (1.1.26), what mostly distinguishes them from the corresponding solutions to the heat equation ( $m = 1$ ). To fix ideas, let us examine the situation in Euclidean space  $\Omega = \mathbb{R}^d$ . As it is well known, the Gaussian function

$$u_C(x, t) := \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}} \tag{1.1.28}$$

is the *heat kernel* or *fundamental solution* of the heat equation, from which, by convolution, one obtains all solutions with integrable initial data. In fact, (1.1.28) solves the heat equation with a Dirac delta  $\delta$  (centred at zero) as initial datum, which represents *e.g.* the distribution of a substance that is ideally concentrated at a single point. Hence, the speed of propagation of such solution is clearly infinite, as after an arbitrarily small time  $t > 0$  it becomes strictly



positive in the whole of  $\mathbb{R}^d$ ; the same holds true for any solution corresponding to a compactly supported (nonnegative) datum. As concerns the porous medium equation, existence of a particular family of solutions in  $\mathbb{R}^d$ , which represents the nonlinear counterpart of the fundamental solution of the heat equation (1.1.28), was proved in the fifties by three Russian mathematicians: Zel'dovich, Kompaneets and Barenblatt. Their explicit form is the following (see [173, Chapter 4]):

$$u_{ZKB}(x, t) := t^{-\alpha} \left( C - b |x|^2 t^{-2\kappa} \right)_+^{\frac{1}{m-1}}, \quad (1.1.29)$$

where

$$\alpha := d\kappa, \quad \kappa := \frac{1}{d(m-1) + 2}, \quad b := \frac{m-1}{2m} \kappa$$

and  $C > 0$  is a free parameter associated with the total mass  $M := \int_{\mathbb{R}^d} u(x, t) dx$ , which is preserved in time (the same degree of freedom is recovered in the heat equation by multiplying the fundamental solution by any positive constant). The functions (1.1.29) are commonly referred to as *Barenblatt solutions*, or Barenblatt functions. Actually, it is not difficult to prove that the Barenblatt solution with unitary mass suitably converges, as  $m \downarrow 1$ , to the fundamental solution of the heat equation. Nevertheless, there are at least three important properties (related to each other), enjoyed by Barenblatt solutions, which entail significant differences between (1.1.28) and (1.1.29):

- a *finite* speed of propagation;
- a natural *free boundary* which separates the region where  $u > 0$  from the one where  $u \equiv 0$ , which is a time-space surface of the form

$$t = c(C, m, d) |x|^{d(m-1)+2};$$

- a *limited regularity* ( $u_C$  is  $C^\infty$  whereas  $u_{ZKB}$  is only Hölder).

It is then possible to prove that such properties, in fact, continue to hold for generic solutions with compactly supported nonnegative initial data, see [173, Chapter 14]. From the physical point of view, finite speed of propagation is more realistic, making the porous medium equation a reasonable model for some important physical phenomena, among which heat propagation can surely be included (in this regard, *infinite* speed of propagation predicted by its solutions is one of the main issues associated with the heat equation).

Consider now the weighted porous medium equation

$$\rho_\nu(x) u_t = \operatorname{div} [\rho_\mu(x) \nabla(u^m)] \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.1.30)$$

where  $\rho_\nu$  and  $\rho_\mu$  are positive regular functions in  $\Omega$ . A few recent papers, among which we quote *e.g.* [114, 163, 164], thoroughly studied the Euclidean case  $\Omega = \mathbb{R}^d$  for  $\rho_\mu \equiv 1$  and assuming suitable power-decay conditions at infinity on  $\rho_\nu$ . Basically, it is required that  $\rho(x) \approx |x|^{-\gamma}$  as  $|x| \rightarrow \infty$ , for some  $\gamma > 0$ . As  $\gamma$  increases, interesting phenomena arise. If  $\gamma \in (0, 2)$ , Barenblatt-type solutions continue to exist and rule the asymptotics of a large class of solutions (see also Paragraph 1.2.3.3 below or the Introduction to Chapter 2). If  $\gamma \in (2, \gamma_2)$ , where

$$\gamma_2 := 2 + \frac{(m-1)(d-2)}{m},$$

then solutions tend to behave more like *separable solutions* of the form

$$u_S(x, t) := t^{-\frac{1}{m-1}} S(x),$$

$S$  being the (minimal) solution to the elliptic problem

$$-\Delta(S^m) = \frac{1}{m-1} \rho_\nu S \quad \text{in } \mathbb{R}^d.$$

Actually, for such values of the parameter, there are still Barenblatt-type solutions which play a role in the asymptotics (though they become unbounded) in some space-time “outer sets”; however, for  $\gamma > \gamma_2$  they definitively cease to exist. Hence, the thresholds  $\gamma = 2$  and  $\gamma = \gamma_2$  separate classes of weights  $\rho_\nu$  which, depending on their behaviour at infinity, make Euclidean space look more like a bounded domain. In fact the value  $\gamma_2$  is deeply linked with speed of propagation: it is possible to prove that for  $\gamma < \gamma_2$  initial data with compact support stay compactly supported along the evolution, while such property is lost in a finite time for  $\gamma > \gamma_2$ .

As we mentioned above, here we shall also analyse *signed* solutions to the weighted porous medium equation, *i.e.* solutions corresponding to initial data which are possibly negative somewhere. In this case  $u^m$  is meant as the French power  $|u|^{m-1} \text{sign}(u)$ . As far as we are concerned, the interest for such kind of data is purely mathematical: below we shall see that the main physical models which lead to the porous medium equation deal with the evolution of intrinsically nonnegative quantities such as (for instance) densities, temperatures, lengths or populations of species.

As for boundary conditions, we consider homogeneous Dirichlet conditions of the type of

$$u = 0 \quad \text{on } \partial\Omega$$

or homogeneous Neumann conditions of the type of

$$F(u) := \frac{\partial(u^m)}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where  $F$  represents the outgoing flow on  $\partial\Omega$  of the substance that diffuses in  $\Omega$ . In case of equations with weights, the flow has to be multiplied by  $\rho_\mu$ . Clearly, the Dirichlet condition amounts to imposing that there is no substance on the boundary, whereas the Neumann condition entails that there is no incoming or outgoing flow of the substance to or from the domain. In agreement with that, if we think about the porous medium equation on bounded domains (see also the results we state in Subsections 1.2.1 and 1.3.1), solutions to the Dirichlet problem vanish as  $t \rightarrow \infty$ , while solutions to the Neumann problem preserve the mass and converge to their mean value.

Finally, we point out that the case  $m \in (0, 1)$  gives rise to the so-called *fast diffusion equation*. From (1.1.27) it is natural to expect that the behaviour of solutions is substantially different compared to “porous” case  $m > 1$ . In fact, now, the lower is the concentration of the substance, the higher is the diffusivity. For a short overview of this kind of equation, which we shall address in Chapters 3 and 4 in two different mathematical frameworks, we refer the reader *e.g.* to [173, Section 5.10] and references quoted therein. Let us mention that the most important phenomenon associated with fast diffusion is *extinction in finite time*, namely the fact that, at least for the Dirichlet problem in bounded domains, the diffusion is so rapid that after a finite time all the substance has gone out of the domain.

In the sequel we briefly describe some physical applications (taken from [173, Chapter 2]) where the porous medium equation has proved to be a good model for the related phenomena.

Let us first consider the distribution of the density  $\sigma$  of a polytropic gas which diffuses in a porous medium (the name *porous medium equation* was basically born from the description of this particular phenomenon). The continuity equation (or mass balance) reads

$$\varepsilon \sigma_t + \text{div}(\sigma \mathbf{V}) = 0,$$

where  $\mathbf{V}$  is the velocity of the gas and  $\varepsilon \in (0, 1)$  is the *porosity* of the medium, that is the fraction of its total volume which is indeed available for fluid circulation. A second equation is *Darcy's law*, namely an empirical relation which links the velocity  $\mathbf{V}$  with the pressure  $p$ :

$$v\mathbf{V} = -\lambda\nabla p,$$

where  $v$  is the viscosity of the gas and  $\lambda$  is the *permeability* of the medium (a parameter which is related to the ease with which a fluid under pressure can penetrate it). Finally we have the state equation for perfect gases, connecting pressure with density:

$$p = p_0 \sigma^\eta;$$

here  $\eta \geq 1$  is the so-called polytropic exponent and  $p_0$  is the reference pressure. After a few computations, we end up with the following equation for the density:

$$\sigma_t = \frac{\eta\lambda p_0}{\varepsilon v(\eta + 1)} \Delta(\sigma^{\eta+1}),$$

namely the porous medium equation with  $m = \eta + 1$  (up to a multiplicative constant of which we can get rid by means of a straightforward time scaling). The final result gives rise to the *Leibenzon-Muskat model* (in this regard, see the monograph [140]). It is apparent that if we allow for a spatial inhomogeneity of the porosity, the role of the latter is equivalent to the role of the outer weight  $\rho_\nu$  in the weighted equation (1.1.30), while a possible inhomogeneity of the permeability is associated with the presence of the inner weight  $\rho_\mu$ .

Another important application of the porous medium equation arises from heat propagation in plasmas (ionized gases) at very high temperatures. The corresponding physical model is based on a modification of Fourier's law (which leads to the heat equation), prescribing the following relation between thermal conductivity and temperature:

$$c\sigma T_t = \operatorname{div}(\phi(T) \nabla T), \tag{1.1.31}$$

where  $c$  is the specific heat at constant pressure,  $\sigma$  is the density of the gas,  $T$  is the distribution of temperature of the latter (the unknown) and  $\phi$  is the thermal conductivity, which is supposed to be a function of the temperature. The choice  $\phi(T) = aT^3$ , for a fixed constant  $a > 0$ , yields the *Zel'dovich-Raizer model*, which corresponds to the porous medium equation with  $m = 4$ . Again, we obtain weighed versions of (1.1.31) as soon as we allow the density of the gas (which here is considered as a datum) or the thermal conductivity to depend on the spatial variable as well.

A good model for the description of the phenomenon of groundwater infiltrations is *Boussinesq's equation*. Such a phenomenon consists in the penetration of some fluid (typically water) through a porous layer of ground having a given height  $H$ , which in turn lies on a solid impervious material. The aim is to establish the height  $h \leq H$  (depending on the planar coordinates  $(x, y)$ ) of the water infiltrated in the porous layer, assuming that the impervious material is at height zero. The equations one uses are essentially the same described above for the Leibenzon-Muskat model (continuity equation and Darcy's law, which here also takes into account gravity), even though in this case the unknown is not a density but part of the domain itself, namely the height  $h$  of the infiltrated water. If one assumes that water flows in the soil with a velocity which is mainly horizontal and that there are no significant variations with respect to  $y$ , after some computations (see [173, Section 2.3]) we end up with the following equation (in fact Boussinesq's equation):

$$h_t = \frac{\sigma g \lambda}{2\varepsilon v} (h^2)_{xx}, \tag{1.1.32}$$

where  $g$  is gravity,  $\varepsilon$  and  $\lambda$  are the porosity and the permeability of the ground layer, respectively, while  $\sigma$  and  $\nu$  are the density and the viscosity of water, respectively. It is then apparent that (1.1.32) is the porous medium equation with  $d = 1$  and  $m = 2$ . The possible presence of the weights  $\rho_\nu$  and  $\rho_\mu$  is still associated with inhomogeneous features of the medium.

Finally, from [108] we have a simple law which can be used, in principle, to describe the evolution of a biological species, whose density in a suitable region we denote as  $u$ :

$$u_t = \operatorname{div}(\phi(u) \nabla u) + f(u),$$

where  $\phi$  is a given increasing function (representing the diffusivity of the species) and the forcing term  $f(u)$  subsumes interactions with other species. A common choice is  $\phi(u) = au$  (for a constant  $a > 0$ ).

### 1.1.2 Preliminary tools: weighted Sobolev spaces

This subsection is devoted to a short overview of weighted Sobolev spaces, which serve as basic tools for our analysis. We are mostly interested in the case  $p = 2$  and in weights that are locally bounded and locally bounded away from zero, but for completeness we give a more general presentation.

In agreement with the above notations, we denote as  $\Omega$  a general domain of  $\mathbb{R}^d$  (possibly coinciding with the whole Euclidean space itself) and as  $\nu$  and  $\mu$  two positive measures on  $\Omega$  absolutely continuous with respect to the Lebesgue measure,  $\rho_\nu$  and  $\rho_\mu$  being the corresponding weights (or densities), respectively. Throughout, we shall always assume

$$\rho_\nu(x), \rho_\mu(x) > 0 \quad \text{for a.e. } x \in \Omega,$$

so that also the Lebesgue measure is absolutely continuous with respect to  $\nu$  and  $\mu$ .

For all  $p \in [1, \infty)$ , we denote as  $L^p(\Omega; \nu)$  the Banach space of (equivalence classes of) Lebesgue measurable functions  $f$  such that

$$\|f\|_{p;\nu}^p := \int_{\Omega} |f(x)|^p d\nu = \int_{\Omega} |f(x)|^p \rho_\nu(x) dx < \infty.$$

The same holds for  $L^p(\Omega; \mu)$ . If  $p \in (0, 1)$  we keep an analogous notation, even though in this case  $L^p(\Omega; \nu)$  is no longer a normed space. We also define the weighted Sobolev space  $W^{1,p}(\Omega; \nu, \mu)$  (see *e.g.* [119]) as the set of all (equivalence classes of) functions  $v \in W_{\text{loc}}^{1,1}(\Omega)$  such that

$$\|v\|_{p;\nu,\mu}^p := \|v\|_{p;\nu}^p + \|\nabla v\|_{p;\mu}^p < \infty.$$

With some abuse of notation, sometimes we shall write  $L^p(\Omega; \rho_\nu)$  or  $W^{1,p}(\Omega; \rho_\nu, \rho_\mu)$  instead (the same applies to the corresponding norms). In the special case  $p = 2$  we set  $H^1(\Omega; \nu, \mu) := W^{1,2}(\Omega; \nu, \mu)$ . Actually, without further assumptions on  $\rho_\nu$  and  $\rho_\mu$ , in general  $W^{1,p}(\Omega; \nu, \mu)$  is not complete.

**Definition 1.1.1.** For all  $p \in (1, \infty)$  we denote as  $\mathfrak{B}^p(\Omega)$  the class of all measurable functions  $f$  such that  $f > 0$  a.e. and

$$|f|^{-\frac{1}{p-1}} \in L_{\text{loc}}^1(\Omega).$$

One can prove [119, Theorem 2.1] that if  $p \in (1, \infty)$  and  $\rho_\mu \in \mathfrak{B}^p(\Omega)$  then  $W^{1,p}(\Omega; \nu, \mu)$  is complete. If  $p = 1$  the result is true providing that the condition  $\rho_\mu \in \mathfrak{B}^p(\Omega)$  is replaced by  $\rho_\mu^{-1} \in L_{\text{loc}}^\infty(\Omega)$ .

The fact that for any  $\phi \in C_c^\infty(\Omega) =: \mathcal{D}(\Omega)$  the quantity  $\|\phi\|_{p;\nu,\mu}$  is finite is equivalent (see [119, Lemma 4.4]) to the local finiteness of  $\nu$  and  $\mu$ , that is

$$\rho_\nu, \rho_\mu \in L_{\text{loc}}^1(\Omega). \quad (1.1.33)$$

Under the above hypothesis, we define the space  $W_0^{1,p}(\Omega; \nu, \mu)$  as the closure of  $\mathcal{D}(\Omega)$  with respect to the norm  $\|\cdot\|_{p;\nu,\mu}$ . If  $p = 2$ , we set  $H_0^1(\Omega; \nu, \mu) := W_0^{1,p}(\Omega; \nu, \mu)$ .

When dealing with  $W^{1,p}(\Omega; \nu, \mu)$  or  $W_0^{1,p}(\Omega; \nu, \mu)$  we shall assume, without further comment, that  $\rho_\mu \in \mathfrak{B}^p(\Omega)$  or that  $\rho_\mu \in \mathfrak{B}^p(\Omega)$  and (1.1.33) holds true, respectively.

**Proposition 1.1.2.** *Let  $p \in [1, \infty)$ . Then  $W_c^{1,\infty}(\Omega) \subset W_0^{1,p}(\Omega; \nu, \mu)$ .*

*Proof.* Given  $v \in W_c^{1,\infty}(\Omega)$ , thanks e.g. to [1, Lemmas 2.18 and 3.15] we know that there exists a sequence of functions  $\{v_n\} \subset \mathcal{D}(\Omega)$  (the *mollification* of  $v$ , see also Chapter 2, Appendix 2.A) such that, as  $n \rightarrow \infty$ ,

$$v_n \xrightarrow{\text{a.e.}} v, \quad \nabla v_n \xrightarrow{\text{a.e.}} \nabla v, \quad \|v_n\|_\infty \leq \|v\|_\infty, \quad \|\nabla v_n\|_\infty \leq \|\nabla v\|_\infty$$

and  $\text{supp}(v_n) \subset \Omega' \Subset \Omega$  for all  $n \in \mathbb{N}$ . Since  $\nu$  e  $\mu$  are locally finite measures, this implies that  $\|v_n - v\|_{p;\nu,\mu} \rightarrow 0$  by dominated convergence, and the assertion follows by definition of  $W_0^{1,p}(\Omega; \nu, \mu)$ .  $\square$

**Proposition 1.1.3.** *Let  $p \in [1, \infty)$ . If  $\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L_{\text{loc}}^\infty(\Omega)$ , then  $C^\infty(\Omega) \cap W^{1,p}(\Omega; \nu, \mu)$  is dense in  $W^{1,p}(\Omega; \nu, \mu)$ .*

*Proof.* The assumptions imply that the weighted norms  $\|\cdot\|_{p;\nu}$  and  $\|\cdot\|_{p;\mu}$  are *locally* equivalent to the non-weighted norm  $\|\cdot\|_p$ . This is enough in order to reproduce the proof of [1, Theorem 3.16]: the idea is to combine the result of Proposition 1.1.2 with a partition of unity argument.  $\square$

**Proposition 1.1.4.** *Let  $p \in [1, \infty)$ . Then  $L^\infty(\Omega) \cap W^{1,p}(\Omega; \nu, \mu)$  is dense in  $W^{1,p}(\Omega; \nu, \mu)$ .*

*Proof.* One proceeds just as in the non-weighted case. That is, given any  $v \in W^{1,p}(\Omega; \nu, \mu)$ , consider the approximating sequence of functions  $v_n := n \wedge (-n \vee v)$ . By construction,  $\{v_n\} \subset L^\infty(\Omega) \cap W^{1,p}(\Omega; \nu, \mu)$  and  $|v_n| \leq |v|$ ; moreover,  $\nabla v_n = \chi_{\{-n \leq v \leq n\}} \nabla v$ . The assertion is therefore a consequence of monotone convergence.  $\square$

We now introduce some other weighted Sobolev spaces, which we need throughout the discussion below.

**Definition 1.1.5.** *Let  $p \in [1, \infty)$ ,  $\rho_\nu, \rho_\mu \in L_{\text{loc}}^1(\Omega)$  and  $\rho_\mu \in \mathfrak{B}^2(\Omega)$ . We denote as  $V_0^p(\Omega; \nu, \mu)$  the closure of  $\mathcal{D}(\Omega)$  with respect to the norm*

$$\|\phi\|_{V_0^p(\Omega; \nu, \mu)} := \|\phi\|_{p;\nu} + \|\nabla \phi\|_{2;\mu} \quad \forall \phi \in \mathcal{D}(\Omega)$$

and as  $V_0(\Omega; \mu)$  the space of all functions  $v \in W_{\text{loc}}^{1,1}(\Omega)$  with  $\nabla v \in [L^2(\Omega; \mu)]^d$  and for which there exists a sequence  $\{\phi_n\} \subset \mathcal{D}(\Omega)$  such that

$$\lim_{n \rightarrow \infty} \|\nabla v - \nabla \phi_n\|_{2;\mu} = 0.$$

Clearly,  $V_0^p(\Omega; \nu, \mu)$  is a (reflexive if in addition  $p > 1$ ) Banach space.

In the special case  $\Omega = \mathbb{R}^d$  and  $\rho_\mu \equiv 1$ , we point out that  $V_0(\Omega; \mu)$  is a Hilbert space (endowed with the norm  $\|\nabla(\cdot)\|_2$ ), usually referred to as  $\dot{H}^1(\mathbb{R}^d)$ .

**Definition 1.1.6.** Let  $p \in [1, \infty)$  and  $\rho_\mu \in \mathfrak{B}^2(\Omega)$ . We denote as  $V^p(\Omega; \nu, \mu)$  the space of all functions  $v \in W_{\text{loc}}^{1,1}(\Omega)$  such that

$$\|v\|_{V^p(\Omega; \nu, \mu)} := \|v\|_{p; \nu} + \|\nabla v\|_{2; \mu} < \infty.$$

It is apparent that  $V^p(\Omega; \nu, \mu)$  is also a (reflexive if in addition  $p > 1$ ) Banach space.

### 1.1.3 Outline of the chapter

The chapter is organized as follows.

In Section 1.2 we address the homogeneous Dirichlet problem for the weighted porous medium equation. In Subsection 1.2.1 we state the corresponding main results. Subsection 1.2.2 is devoted to the proofs of existence and uniqueness of energy solutions (in Paragraph 1.2.2.1 we discuss comparisons with some previous well-posedness results). Subsection 1.2.3 deals with smoothing and decay properties of solutions in connection with functional inequalities of Poincaré, Sobolev and Gagliardo-Nirenberg type, and it is split into Paragraphs 1.2.3.1–1.2.3.4 accordingly (Paragraph 1.2.3.2 contains counterexamples in the Poincaré case).

In Section 1.3 we address the homogeneous Neumann problem. In Subsection 1.3.1 we state the corresponding main results. Existence and uniqueness of energy solutions are proved in Subsection 1.3.2. Subsection 1.3.3 deals with smoothing estimates in connection with functional inequalities of Poincaré, Sobolev and Gagliardo-Nirenberg type, and it is split into Paragraphs 1.3.3.1–1.3.3.5. In particular, Paragraph 1.3.3.2 contains a counterexample in the Poincaré case, while Paragraph 1.3.3.5 is concerned with the explanation of how our results improve on previous ones. Subsection 1.3.4 is devoted to the analysis of long-time behaviour, and it is split similarly to Subsection 1.3.3 into Paragraphs 1.3.4.1–1.3.4.4.

Section 1.4 deals with the inhomogeneous filtration equation in Euclidean space. In Subsection 1.4.1 we state our main existence and uniqueness results. In Subsection 1.4.2 we give proofs of existence of solutions with nonhomogeneous conditions at infinity, while in Subsection 1.4.3 we establish uniqueness of such solutions.

Finally, in Appendix 1.A we collect some explicit examples of admissible weights (taken from available literature) for which Poincaré-, Sobolev- and Gagliardo-Nirenberg-type inequalities hold true, so that the results of Sections 1.2 and 1.3 can be readily applied to the corresponding weighted problems. We divide it into Subsections 1.A.1 (inequalities for the Dirichlet problem) and 1.A.2 (inequalities for the Neumann problem), which are in turn divided into Paragraphs 1.A.1.1–1.A.1.3 and 1.A.2.1–1.A.2.3, respectively, depending on the particular type of inequality we discuss.

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## 1.2 The homogeneous Dirichlet problem

This section is devoted to the analysis of well-posedness issues and smoothing and decay estimates for solutions to the weighted Dirichlet problem (1.1.1), specially as concerns their connections with the validity of Poincaré-, Sobolev- and Gagliardo-Nirenberg-type inequalities for the weights involved.

### 1.2.1 Statements of the main results

We begin by providing our notion of weak solution to (1.1.1).

**Definition 1.2.1.** A function  $u$  such that:

$$u \in L^1((0, T); L_{\text{loc}}^1(\Omega; \nu)), \quad u^m(t) \in V_0(\Omega; \mu), \quad \nabla(u^m) \in L^1((0, T); [L^2(\Omega; \mu)]^d)$$

for a.e.  $t > 0$  and every  $T > 0$ ,

is a weak solution to (1.1.1) with initial datum  $u_0 \in L^1_{\text{loc}}(\Omega; \nu)$  if it satisfies

$$\int_0^T \int_{\Omega} u(x, t) \varphi_t(x, t) \, d\nu dt = - \int_{\Omega} u_0(x) \varphi(x, 0) \, d\nu + \int_0^T \int_{\Omega} \langle \nabla(u^m), \nabla \varphi \rangle(x, t) \, d\mu dt \quad (1.2.1)$$

for all  $\varphi \in C_c^\infty(\Omega \times [0, T])$  such that  $\varphi(T) \equiv 0$ .

The above definition is similar to the one given in [173, Definition 5.4] (non-weighted porous medium equation on bounded domains). Differences are due to the fact that here we have to deal with general domains and weights, on which no *a priori* regularity or boundedness properties are required.

In agreement with a common terminology often used in [173], we also provide the important notion of *energy solution*.

**Definition 1.2.2.** *We say that  $u$  is a (weak) energy solution to (1.1.1) if it is a weak solution (in the sense of Definition 1.2.1) such that*

$$u^m \in L^{1+1/m}((0, T); V_0^{1+1/m}(\Omega; \nu, \mu)) \quad \text{and} \quad \nabla(u^m) \in L^2((0, T); [L^2(\Omega; \mu)]^d) \quad (1.2.2)$$

for all  $T > 0$ .

Energy solutions enjoy a (relatively standard, see [173, Theorem 5.3]) uniqueness property.

**Proposition 1.2.3.** *There exists at most one energy solution to (1.1.1).*

As concerns existence, we shall prove the following fundamental result.

**Theorem 1.2.4.** *Let  $\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L^\infty_{\text{loc}}(\Omega)$ . Take  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , with  $q_0 \geq m + 1$ . Then there exists a weak solution  $u$  to (1.1.1), in the sense of Definition 1.2.1, which satisfies (1.2.2) and the estimates*

$$\frac{4q(q+1)m}{(m+q)^2} \int_0^T \int_{\Omega} |\nabla(u^{\frac{m+q}{2}})(x, t)|^2 \, d\mu dt + \int_{\Omega} |u(x, T)|^{q+1} \, d\nu \leq \int_{\Omega} |u_0(x)|^{q+1} \, d\nu \quad (1.2.3)$$

and

$$\int_0^T \int_{\Omega} \zeta(t) |z_t(x, t)|^2 \, d\nu dt \leq \max_{t \in [0, T]} \zeta'(t) \frac{m+1}{8m} \int_{\Omega} |u_0(x)|^{m+1} \, d\nu \quad (1.2.4)$$

for all  $q \in [0, q_0 - 1]$  and  $T > 0$ , where  $z := u^{(m+1)/2}$  and  $\zeta$  is any nonnegative  $C_c^1((0, T))$  function. In particular,  $u$  is the unique energy solution.

Moreover, if  $v$  is the energy solution corresponding to another initial datum  $v_0 \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$ , the following  $L^1(\Omega; \nu)$  comparison principle holds true:

$$\int_{\Omega} [u(x, t) - v(x, t)]_+ \, d\nu \leq \int_{\Omega} [u_0(x) - v_0(x)]_+ \, d\nu \quad \forall t > 0. \quad (1.2.5)$$

When dealing with initial data in  $L^1(\Omega; \nu)$  but with no further integrability property, we cannot provide a weak solution of (1.1.1) in the sense of Definition 1.2.1. Nevertheless, in view of (1.2.5), we infer that the map  $L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu) \mapsto L^\infty((0, \infty); L^1(\Omega; \nu))$  which associates to an initial datum  $u_0$  the corresponding energy solution  $u$ , is Lipschitz and densely defined in  $L^1(\Omega; \nu)$ . Hence, it admits a unique Lipschitz extension to the whole of  $L^1(\Omega; \nu)$ . We shall refer to the corresponding images, according to a terminology used in [173, Section 6.1], as *limit solutions*.

We now state the main results which connect the validity of suitable functional inequalities to smoothing and decay estimates for solutions to (1.1.1). In the sequel, by the term “the

solution”, we mean the weak energy solution provided by Theorem 1.2.4 if  $u_0 \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$  or the limit solution if  $u_0 \in L^1(\Omega; \nu)$ . The assumptions on the weights are also meant as in Theorem 1.2.4.

**Theorem 1.2.5** (Poincaré case). *Let  $q_0 \in [1, \infty)$  and  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ . Suppose that the Poincaré-type inequality*

$$\|v\|_{2;\nu} \leq C_P \|\nabla v\|_{2;\mu} \quad \forall v \in H_0^1(\Omega; \nu, \mu) \quad (1.2.6)$$

*holds true for some  $C_P > 0$ . Then the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies the smoothing estimate*

$$\|u(t)\|_{\varrho;\nu} \leq K_1 t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} \quad \forall t > 0 \quad (1.2.7)$$

*for all  $\varrho \in (q_0, \infty)$ , where  $K_1 > 0$  is a suitable constant depending only on  $\varrho$ ,  $m$  and  $C_P$ .*

*If moreover  $\nu(\Omega) < \infty$ , then the above estimate can be improved for large times:*

$$\|u(t)\|_{\varrho;\nu} \leq K_2 t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \left( t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{-\frac{q_0}{(m-1)\varrho}} \quad \forall t > 0, \quad (1.2.8)$$

*where  $K_2 > 0$  is a suitable constant depending only on  $\varrho$ ,  $q_0$ ,  $m$ ,  $C_P$  and  $\nu(\Omega)$ . In particular, the absolute bound*

$$\|u(t)\|_{\varrho;\nu} \leq K_2 t^{-\frac{1}{m-1}} \quad \forall t > 0$$

*holds true.*

**Theorem 1.2.6** (Poincaré case, converse implication). *Suppose that there exist a constant  $K_1 > 0$  and two given numbers  $q_0 \in [1, m+1)$  and  $\varrho \geq m+1$  such that, for all  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies estimate (1.2.7). Then the Poincaré-type inequality (1.2.6) holds true. In particular, the validity of (1.2.7) for two given  $q_0 \in [1, m+1)$  and  $\varrho \geq m+1$  is equivalent to (1.2.6) and so to the validity of (1.2.7) for any  $q_0 \in [1, \infty)$  and  $\varrho \in (q_0, \infty)$ .*

**Theorem 1.2.7** (Sobolev case). *Let  $q_0 \in [1, \infty)$  and  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ . Suppose that the Sobolev-type inequality*

$$\|v\|_{2\sigma;\nu} \leq C_S \|\nabla v\|_{2;\mu} \quad \forall v \in H_0^1(\Omega; \nu, \mu) \quad (1.2.9)$$

*holds true for some  $\sigma > 1$  and  $C_S > 0$ . Then the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies the smoothing estimate*

$$\|u(t)\|_{\infty} \leq K_1 t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0+\sigma(m-1)}} \quad \forall t > 0, \quad (1.2.10)$$

*where  $K_1 > 0$  is a suitable constant depending only on  $m$ ,  $C_S$  and  $\sigma$ .*

*If moreover  $\nu(\Omega) < \infty$ , then the above estimate can be improved for large times:*

$$\|u(t)\|_{\infty} \leq K_2 t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \left( t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{-\frac{(\sigma-1)q_0}{(m-1)[(\sigma-1)q_0+\sigma(m-1)]}} \quad \forall t > 0,$$

*where  $K_2 > 0$  is a suitable constant depending only on  $q_0$ ,  $m$ ,  $C_S$ ,  $\sigma$  and  $\nu(\Omega)$ . In particular, the absolute bound*

$$\|u(t)\|_{\infty} \leq K_2 t^{-\frac{1}{m-1}} \quad \forall t > 0 \quad (1.2.11)$$

*holds true.*



**Theorem 1.2.8** (Sobolev case, converse implication). *Suppose that there exist a constant  $K_1 > 0$  and two numbers  $q_0 \in [1, m+1)$  and  $\sigma > 1$  such that, for all  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies estimate (1.2.10). Then the Sobolev-type inequality (1.2.9) holds true. In particular, the validity of (1.2.10) for a given  $q_0 \in [1, m+1)$  is equivalent to (1.2.9) and so to the validity of (1.2.10) for any  $q_0 \in [1, \infty)$ .*

**Theorem 1.2.9** (One-dimensional case). *Let  $d = 1$  (that is,  $\Omega$  is an interval). Let  $q_0 \in [1, \infty)$  and  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ . Suppose that the family of one-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v\|_\infty \leq C_{G_1} \|\nabla v\|_{2;\mu}^{\frac{\lambda}{s+\lambda}} \|v\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu) \quad (1.2.12)$$

*holds true for some  $\lambda \geq 2$  and  $C_{G_1} > 0$ . Then the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies the estimate*

$$\|u(t)\|_\infty \leq K_1 t^{-\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{(\lambda+2)q_0+\lambda(m-1)}} \quad \forall t > 0, \quad (1.2.13)$$

*where  $K_1 > 0$  is a suitable constant depending only on  $m$ ,  $C_{G_1}$  and  $\lambda$ .*

*If moreover  $\nu(\Omega) < \infty$ , then the above estimate can be improved for large times:*

$$\|u(t)\|_\infty \leq K_2 t^{-\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{(\lambda+2)q_0}{(m-1)[(\lambda+2)q_0+\lambda(m-1)]}} \quad \forall t > 0, \quad (1.2.14)$$

*where  $K_2 > 0$  is a suitable constant depending only on  $q_0$ ,  $m$ ,  $C_{G_1}$ ,  $\lambda$  and  $\nu(\Omega)$ . In particular, the absolute bound (1.2.11) holds true.*

**Theorem 1.2.10** (One-dimensional case, converse implication). *Suppose that there exist a constant  $K_1 > 0$  and two numbers  $q_0 \in [1, m+1)$  and  $\lambda \geq 2$  such that, for all  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies estimate (1.2.13). Then the family of one-dimensional Gagliardo-Nirenberg-type inequalities (1.2.12) holds true. In particular, the validity of (1.2.13) for a given  $q_0 \in [1, m+1)$  is equivalent to (1.2.12) and so to the validity of (1.2.13) for any  $q_0 \in [1, \infty)$ .*

**Theorem 1.2.11** (Two-dimensional case). *Let  $d \leq 2$ . Let  $q_0 \in [1, \infty)$  and  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ . Suppose that the family of two-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v\|_{r;\nu} \leq C_{G_2} \|\nabla v\|_{2;\mu}^{1-\frac{s}{r}} \|v\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu) \quad (1.2.15)$$

*holds true for some constant  $C_{G_2} > 0$  which depends on  $r, s$  only through a finite upper bound on  $r$ . Then the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies the smoothing estimate*

$$\|u(t)\|_\infty \leq K_1 t^{-\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}} \quad \forall t > 0, \quad (1.2.16)$$

*where  $K_1 > 0$  is a suitable constant depending only on  $m$  and  $C_{G_2}$ .*

*If moreover  $\nu(\Omega) < \infty$ , then the above estimate can be improved for large times:*

$$\|u(t)\|_\infty \leq K_2 t^{-\frac{1}{q_0+m-1}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{q_0}{(m-1)(q_0+m-1)}} \quad \forall t > 0, \quad (1.2.17)$$

*where  $K_2 > 0$  is a suitable constant depending only on  $q_0$ ,  $m$ ,  $C_{G_2}$  and  $\nu(\Omega)$ . In particular, the absolute bound (1.2.11) holds true.*

**Theorem 1.2.12** (Two-dimensional case, converse implication). *Suppose that there exist a constant  $K_1 > 0$  and a number  $q_0 \in [1, m + 1)$  such that, for all  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , the solution  $u$  to (1.1.1) with initial datum  $u_0$  satisfies estimate (1.2.16). Then the family of two-dimensional Gagliardo-Nirenberg-type inequalities (1.2.15) holds true with some constant  $C_{G_2} > 0$  which depends on  $r, s$  only through a finite upper bound on  $r$ . In particular, the validity of (1.2.16) for a given  $q_0 \in [1, m + 1)$  is equivalent to (1.2.15) and so to the validity of (1.2.16) for any  $q_0 \in [1, \infty)$ .*

Proposition 1.2.3 and Theorem 1.2.4 will be proved in the next Subsection 1.2.2. Theorems 1.2.5 and 1.2.6 will be proved in Paragraph 1.2.3.1. In the subsequent Paragraph 1.2.3.2 we show some counterexample to the  $L^\infty$  regularization, which therefore does not occur (in general) when only a Poincaré-type inequality holds true for the weights involved. In Paragraph 1.2.3.3 we shall only give hints on the proofs of the Sobolev case: in fact, Theorem 1.2.7 is basically a consequence of the results in [27], whereas Theorem 1.2.8 follows by means of the same techniques of proof of Theorem 1.2.6. Finally, Theorems 1.2.9–1.2.12 will be proved in Paragraph 1.2.3.4.

**Remark 1.2.13.** When  $\Omega = \mathbb{R}^d$  and  $\rho_\nu = \rho_\mu \equiv 1$ , the results of Theorems 1.2.7 (with  $d \geq 3$  and  $\sigma = d/(d - 2)$ ), 1.2.9 (with  $\lambda = 2$ ) and 1.2.11 (with  $d = 2$ ) are in agreement with the sharp results obtained in [172, Chapters 2, 3], where the same smoothing estimates are proved with different techniques. We point out that such techniques are not applicable in our general framework; nevertheless, at least in the case  $q_0 = 1$ , they permit us to find the *best constant*  $K_1$  for which the estimates hold (attained by Barenblatt solutions).

As concerns bounded Euclidean domains, still in the non-weighted case  $\rho_\nu = \rho_\mu \equiv 1$ , in view of the asymptotic results of [171] (we refer the reader to [173, Chapter 20] as well) we know that the absolute bound (1.2.11) is in sharp. Note that such an estimate fails in Euclidean space, since the  $L^\infty$  norm of Barenblatt solutions decays with a rate slower than  $t^{-1/(m-1)}$  (see also Subsection 1.1.1 above). This is in agreement with the hypothesis  $\nu(\Omega) < \infty$  we required in order to prove (1.2.11).

**Remark 1.2.14.** Even though we did not mention it explicitly, it is apparent that if  $d \geq 3$  then the Sobolev-type inequality (1.2.9) can only hold for some  $\sigma \leq d/(d - 2)$ : this is just a consequence of the fact that our weights are locally equivalent to the unitary weight (recall the hypotheses of Theorem 1.2.4 above). For the same reason, an inequality like (1.2.12) can only hold in dimension one and for some  $\lambda \geq 2$ . Similarly, (1.2.15) cannot hold if  $d \geq 3$ . In this regard we refer the reader to Subsection 1.A.1 for some explicit examples.

**Remark 1.2.15.** Note that, formally, the smoothing estimates of Theorem 1.2.7 can be obtained by letting  $\lambda = -2\sigma$  in the ones of Theorem 1.2.9. On the other hand, the smoothing estimates of Theorem 1.2.11 are in fact the limit, as  $\sigma \rightarrow \infty$  or as  $\lambda \rightarrow \infty$ , of the ones of Theorem 1.2.7 or Theorem 1.2.9, respectively. This is not just a coincidence: as it is well known, and as we shall see below, Sobolev- and Gagliardo-Nirenberg-type inequalities are deeply connected with one another.

## 1.2.2 Well-posedness analysis

The uniqueness result of Proposition 1.2.3 follows by means of a quite standard argument, whose main steps we outline for the reader's convenience.

*Proof of Proposition 1.2.3.* Thanks to (1.2.2) and to a density argument, in fact it is possible to choose in (1.2.1) any test function  $\varphi$  such that

$$\varphi \in W^{1,1+1/m}((0, T); V_0^{1+1/m}(\Omega; \nu, \mu)), \quad \nabla \varphi \in L^2((0, T); [L^2(\Omega; \mu)]^d), \quad \varphi(T) \equiv 0. \quad (1.2.18)$$

We can therefore exploit the method of proof of [173, Theorem 5.3], which actually goes back to the pioneering papers [143, 144]. That is, we plug Oleïnik's test function

$$\varphi(x, t) = \int_t^T [u_1^m(x, s) - u_2^m(x, s)] ds, \quad (1.2.19)$$

which clearly complies with (1.2.18), in the weak formulation satisfied by the difference of two energy solutions  $u_1$  and  $u_2$  having the same initial datum  $u_0$ . After an integration by parts in time, we end up with

$$\begin{aligned} \int_0^T \int_{\Omega} (u_1(x, t) - u_2(x, t))(u_1^m(x, t) - u_2^m(x, t)) d\nu dt \\ + \frac{1}{2} \int_{\Omega} \left| \int_0^T \nabla(u_1^m - u_2^m)(x, s) ds \right|^2 d\mu = 0. \end{aligned}$$

Since both terms in the l.h.s. are nonnegative, the assertion follows.  $\square$

In order to prove our existence Theorem 1.2.4, we first need an intermediate result.

**Lemma 1.2.16.** *Let  $\Omega$  be a  $C^{2,\alpha}$  bounded domain of  $\mathbb{R}^d$ . Let  $\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L_{\text{loc}}^\infty(\Omega')$  for some domain  $\Omega' \ni \Omega$ . Take  $u_0 \in C_c^{2,\alpha}(\Omega)$ . Then the conclusions of Theorem 1.2.4 hold true (for all  $q \geq 0$ ).*

*Proof.* To begin with, let us assume that  $\rho_\nu \in C^{3,\alpha}(\overline{\Omega})$  and  $\rho_\mu \in C^{2,\alpha}(\overline{\Omega})$ , with  $\rho_\nu, \rho_\mu > 0$ . Under these additional hypotheses, proceeding along the lines of the proof of [173, Lemma 5.8] we can get a first existence result. The basic idea is to approximate problem (1.1.1) with suitable non-degenerate problems. More precisely, pick a sequence of smooth functions  $\Phi'_n : \mathbb{R} \mapsto \mathbb{R}^+$  such that:

- $\Phi'_n(y) \rightarrow m|y|^{m-1}$  locally uniformly as  $n \rightarrow \infty$ ;
- $\Phi'_n(y) > 0 \quad \forall y \in \mathbb{R}$ ;
- $\Phi'_n(y) = \Phi'_n(-y)$ , so that in particular  $\Phi_n(0) = 0$ , where  $\Phi_n(y) := \int_0^y \Phi'_n(r) dr$ .

Then consider the following non-degenerate quasilinear problem:

$$\begin{cases} (u_n)_t = \rho_\nu^{-1} \operatorname{div} [\rho_\mu \nabla(\Phi_n(u_n))] & \text{in } \Omega \times \mathbb{R}^+, \\ u_n = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_n = u_0 & \text{on } \Omega \times \{0\}. \end{cases} \quad (1.2.20)$$

In view of the change of variable  $w = \rho_\nu u_n$ , it is convenient to write the latter in divergence form:

$$\begin{cases} w_t = \operatorname{div} \left[ \frac{\rho_\mu}{\rho_\nu} \Phi'_n \left( \frac{w}{\rho_\nu} \right) \nabla w - \frac{\rho_\mu}{\rho_\nu^2} \nabla(\rho_\nu) \Phi'_n \left( \frac{w}{\rho_\nu} \right) w \right] & \text{in } \Omega \times \mathbb{R}^+, \\ w = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ w = \rho_\nu u_0 & \text{on } \Omega \times \{0\}. \end{cases} \quad (1.2.21)$$

With no loss of generality we shall suppose that  $\Phi'_n(y) = c$  for  $|y| \geq \|u_0\|_\infty + 1$ , where  $c > 0$  is a suitable constant, possibly depending on  $n$ . Under our assumptions, we can apply Theorem V.6.1 of [120], which provides us with a solution  $w \in C^{2,1}(\overline{\Omega} \times [0, T]) \quad \forall T > 0$  to (1.2.21). From standard parabolic regularity results (see *e.g.* [120, Theorem IV.5.2]) we also have, in particular,  $w_t \in C^{1,0}(\Omega \times (0, T))$ . It is then apparent that  $u_n$  is a solution to (1.2.20) with the same regularity as  $w$ . Moreover, thanks to the parabolic maximum principle (see *e.g.* [120, Theorem I.2.9]), there holds

$$|u_n(x, t)| \leq \|u_0\|_\infty \quad \forall (x, t) \in \Omega \times \mathbb{R}^+. \quad (1.2.22)$$

Given a regular function  $\varphi$  as in the weak formulation (1.2.1), multiplying the differential equation in (1.2.20) by  $\rho_\nu \varphi$  and integrating by parts in  $\Omega \times (0, T)$ , we obtain:

$$\int_0^T \int_\Omega u_n(x, t) \varphi_t(x, t) \, d\nu dt = - \int_\Omega u_0(x) \varphi(x, 0) \, d\nu + \int_0^T \int_\Omega \langle \nabla(\Phi_n(u_n)), \nabla \varphi \rangle(x, t) \, d\mu dt. \quad (1.2.23)$$

In order to pass to the limit in (1.2.23) as  $n \rightarrow \infty$ , we first need suitable estimates on  $u_n$ . To this end, letting

$$\Psi_n(y) := \int_0^y \Phi_n(r) \, dr, \quad \Upsilon_n^1(y) := \int_0^y \sqrt{\Phi_n'(r)} \, dr,$$

by means of computations similar to the ones performed in [173, Lemma 5.8], we get:

$$\int_\Omega \Psi_n(u_n(x, T)) \, d\nu + \int_0^T \int_\Omega |\nabla(\Phi_n(u_n))(x, t)|^2 \, d\mu dt = \int_\Omega \Psi_n(u_0(x)) \, d\nu, \quad (1.2.24)$$

$$\frac{1}{2} \int_\Omega |u_n(x, T)|^2 \, d\nu + \int_0^T \int_\Omega |\nabla(\Upsilon_n^1(u_n))(x, t)|^2 \, d\mu dt = \frac{1}{2} \int_\Omega |u_0(x)|^2 \, d\nu, \quad (1.2.25)$$

$$\int_0^T \int_\Omega \zeta(t) |(\Upsilon_n^1(u_n))_t(x, t)|^2 \, d\nu dt = \frac{1}{2} \int_0^T \int_\Omega \zeta'(t) |\nabla(\Phi_n(u_n))(x, t)|^2 \, d\mu dt, \quad (1.2.26)$$

where  $\zeta$  is any nonnegative  $C_c^1(0, T)$  function. Let us mention that (1.2.24), (1.2.25) and (1.2.26) follow by multiplying the differential equation in (1.2.20) by the functions  $\rho_\nu \Phi_n(u_n)$ ,  $\rho_\nu u_n$  and  $\rho_\nu \zeta \Phi_n'(u_n)(u_n)_t$ , respectively, and integrating by parts in  $\Omega \times (0, T)$ .

From (1.2.22), (1.2.25), (1.2.26) and the fact that  $\rho_\nu, \rho_\mu, \rho_\nu^{-1}, \rho_\mu^{-1} \in L_{\text{loc}}^\infty(\Omega')$ , we deduce that the sequence  $\{\Upsilon_n^1(u_n)\}$  is bounded in  $H_{\text{loc}}^1(\Omega \times \mathbb{R}^+)$ . Hence, up to a subsequence,  $\{\Upsilon_n^1(u_n)\}$  converges almost everywhere in  $\Omega \times \mathbb{R}^+$ . Thanks to the good properties of the approximating sequence  $\{\Phi_n'\}$ , this easily entails the existence of a function  $u$  such that:

$$u_n \rightarrow u, \quad \Psi_n(u_n) \rightarrow \frac{1}{m+1} |u|^{m+1}, \quad \Phi_n(u_n) \rightarrow u^m, \quad \Upsilon_n^1(u_n) \rightarrow \frac{2\sqrt{m}}{m+1} u^{\frac{m+1}{2}} \quad (1.2.27)$$

a.e. in  $\Omega \times \mathbb{R}^+$ . Estimates (1.2.22), (1.2.24), (1.2.26) and the pointwise limits in (1.2.27) ensure that (again, along subsequences)

$$u_n \rightarrow u \quad \text{in } L^2((0, T); L^2(\Omega)),$$

$$\Phi_n(u_n) \rightarrow u^m \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)) \quad (\text{and so in } L^{1+1/m}((0, T); V_0^{1+1/m}(\Omega; \nu, \mu))),$$

$$\Upsilon_n^1(u_n) \rightarrow \frac{2\sqrt{m}}{m+1} u^{\frac{m+1}{2}} \quad \text{weakly in } H^1((\tau, T); L^2(\Omega)) \quad \forall \tau \in (0, T).$$

We are therefore allowed to pass to the limit in (1.2.23) to conclude that  $u$  is a weak energy solution to (1.1.1) with initial datum  $u_0$ .

It remains to prove estimates (1.2.3)–(1.2.5). As concerns (1.2.3), the idea is just to multiply the differential equation in (1.2.20) by  $\rho_\nu u_n^q$ , integrate in  $\Omega \times (0, T)$  to obtain the analogue of (1.2.25) for  $\Upsilon_n^q(u_n)$ , where

$$\Upsilon_n^q(y) := \int_0^y \sqrt{q \Phi_n'(r)} |r|^{q-1} \, dr,$$

and pass to the limit as  $n \rightarrow \infty$ . This works rigorously at least for  $q \geq 1$ : if instead  $q < 1$  one can consider a smooth approximation  $f_k(y)$  of the real function  $y^q$ , deduce analogous estimates as above, first let  $n \rightarrow \infty$  and then  $k \rightarrow \infty$ . Estimate (1.2.4) is a direct consequence

of (1.2.24), (1.2.26) and the weak convergence of  $\{\Upsilon_n^1(u_n)\}$  to  $[2\sqrt{m}/(m+1)]u^{(m+1)/2}$  in  $H^1((\tau, T); L^2(\Omega))$ . The  $L^1(\Omega; \nu)$  comparison principle (1.2.5) follows exactly as in [173, Proposition 3.5]: formally, one picks the test function  $\text{sign}_+[\Phi_n(u_n) - \Phi_n(v_n)]$  in the problem solved by the difference  $u_n - v_n$ , obtains the bound for  $u_n(t) - v_n(t)$  and then lets  $n \rightarrow \infty$ .

Finally, let us remove the initial assumptions  $\rho_\nu \in C^{3,\alpha}(\bar{\Omega})$  and  $\rho_\mu \in C^{2,\alpha}(\bar{\Omega})$ . In order to do so, it is enough to notice that, in view of the hypotheses on the weights, the corresponding mollifications  $(\rho_\nu)_\varepsilon$  and  $(\rho_\mu)_\varepsilon$  (see *e.g.* Chapter 2, Subsection 2.A.2) are  $C^\infty(\bar{\Omega})$  functions, uniformly bounded from above and below in  $\bar{\Omega}$  by positive constants independent of  $\varepsilon > 0$  and converging almost everywhere to  $\rho_\nu$  and  $\rho_\mu$ , respectively. Hence, as a consequence of the first part of the proof, there exists an energy solution  $u_\varepsilon$  to (1.1.1) which satisfies (1.2.3) and (1.2.4), with  $\rho_\nu$  and  $\rho_\mu$  replaced by  $(\rho_\nu)_\varepsilon$  and  $(\rho_\mu)_\varepsilon$ . Letting  $\varepsilon \rightarrow 0$ , by means of arguments similar to the ones used above, it is straightforward to check that  $\{u_\varepsilon\}$  converges to the energy solution of (1.1.1), which still complies with (1.2.3) and (1.2.4). The validity of (1.2.5) follows likewise.  $\square$

Taking advantage of Lemma 1.2.16, we are able to prove existence of weak energy solutions to (1.1.1) when the initial datum  $u_0$  and the domain  $\Omega$  are less regular and possibly unbounded.

*Proof of Theorem 1.2.4.* First of all, notice that the results of Lemma 1.2.16 can be readily extended to general  $L^1(\Omega; \nu) \cap L^\infty(\Omega)$  data: in fact, it suffices to approximate  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$  with a sequence  $\{u_{0n}\} \subset C_c^{2,\alpha}(\Omega)$  and establish as above proper convergence of the corresponding (sub)sequence of solutions  $\{u_n\}$ .

In order to handle general domains  $\Omega$ , one can proceed similarly to the end of proof of [173, Theorem 5.7]: that is, given  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ , one picks an increasing sequence of smooth bounded domains  $\{\Omega_n\}$  such that  $\bigcup_n \Omega_n = \Omega$ ,  $\Omega_n \Subset \Omega$  and solves in each of them the homogeneous Dirichlet problem with initial datum  $u_{0n} = u_0|_{\Omega_n}$ , denoting as  $u_n$  the corresponding solution, which is set to zero outside  $\Omega_n$ . Estimates (1.2.3) and (1.2.4) then read as follows:

$$\frac{4q(q+1)m}{(m+q)^2} \int_0^T \int_\Omega \left| \nabla \left( u_n^{\frac{m+q}{2}} \right) (x, t) \right|^2 d\mu dt + \int_\Omega |u_n(x, T)|^{q+1} d\nu \leq \int_\Omega |u_0(x)|^{q+1} d\nu, \quad (1.2.28)$$

$$\int_0^T \int_\Omega \zeta(t) \left| \left( u_n^{\frac{m+1}{2}} \right)_t (x, t) \right|^2 d\nu dt \leq \max_{t \in [0, T]} \zeta'(t) \frac{m+1}{8m} \int_\Omega |u_0(x)|^{m+1} d\nu. \quad (1.2.29)$$

Hence, with minor modifications w.r.t. to the proof of Lemma 1.2.16, no major difficulty arises in showing that  $\{u_n\}$ , up to subsequences, converges to the weak energy solution  $u$  of problem (1.1.1) *e.g.* in the following way:

$$\begin{aligned} u_n &\rightarrow u \quad \text{weakly in } L^2((0, T); L^2(\Omega; \nu)), \\ u_n^m &\rightarrow u^m \quad \text{weakly in } L^2((0, T); V_0^{1+1/m}(\Omega; \nu, \mu)), \\ u_n^{\frac{m+q}{2}} &\rightarrow u^{\frac{m+q}{2}} \quad \text{weakly in } L^2((0, T); V_0^{2p/(m+q)}(\Omega; \nu, \mu)) \quad \forall p > (m+q)/2, \\ \nabla(u_n^m) &\rightarrow \nabla(u^m) \quad \text{weakly in } L^2((0, T); [L^2(\Omega; \mu)]^d), \\ u_n^{\frac{m+1}{2}} &\rightarrow u^{\frac{m+1}{2}} \quad \text{weakly in } H^1((\tau, T); L^2(\Omega; \nu)) \quad \forall \tau \in (0, T), \end{aligned} \quad (1.2.30)$$

so that estimates (1.2.3)–(1.2.5) are preserved at the limit (as for the latter, one just applies simultaneously the same approximating scheme to another initial datum  $v_0$ ).

We finally need to remove the hypothesis  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ . To this end, one picks a sequence of initial data  $\{u_{0n}\} \subset L^1(\Omega; \nu) \cap L^\infty(\Omega)$  converging to  $u_0$  in  $L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$

and considers the corresponding sequence  $\{u_n\}$  of energy solutions to (1.1.1). Thanks to the  $L^1(\Omega; \nu)$  comparison principle (1.2.5),  $\{u_n\}$  is Cauchy in  $L^\infty((0, \infty); L^1(\Omega; \nu))$  and therefore converges to a function  $u$  belonging to the same space. In order to prove that  $u$  is the energy solution to (1.1.1) corresponding to  $u_0$  and satisfying (1.2.3) and (1.2.4), one proceeds exactly as above, recalling that  $q_0 \geq m + 1$  (the validity of (1.2.5) now is trivial). The only difference is that (1.2.30) holds true for  $q \leq q_0 - 1$  and  $p \in ((m + q)/2, q_0]$ .  $\square$

**Remark 1.2.17.** In the case of a *nonnegative* initial datum  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ , the above sequence of energy solutions  $\{u_n\}$ , which solve the homogeneous Dirichlet problems in the regular bounded domains  $\Omega_n$  with initial data  $u_{0n} = u_0|_{\Omega_n}$ , is in fact *monotone increasing*. This is a consequence *e.g.* of a standard comparison principle for regular sub- and supersolutions to the approximating problems constructed in Lemma 1.2.16: the solution to (1.2.20) in  $\Omega_{n+1}$  is clearly a supersolution to the same problem in  $\Omega_n$  (it is nonnegative on  $\partial\Omega_n$ ), which implies that it lies *above* the corresponding solution in  $\Omega_n$ . It is then apparent that such property is preserved at the limit.

Notice that from estimate (1.2.4) we infer, for instance, that  $u^{(m+1)/2}$  is an absolutely continuous curve in  $C([\tau, \infty); L^2(\Omega; \nu))$  (for any  $\tau > 0$ ), which in particular implies that  $u \in C((0, \infty); L^{m+1}(\Omega; \nu))$ . In view of this property, it is not difficult to prove the validity of the so called *semigroup property*: for every  $\tau > 0$ ,  $u|_{[\tau, \infty)}$  is the (weak) energy solution of (1.1.1) with initial datum  $u(\tau)$ . Actually, if  $u_0$  is smooth enough, we can ensure that  $u^{(m+1)/2}$  is continuous down to  $t = 0$ . In fact, we have the following result (see also [173, Section 5.6]).

**Corollary 1.2.18.** *If, in addition to the hypotheses of Theorem 1.2.4, we assume that*

$$u_0^m \in V_0^{1+1/m}(\Omega; \nu, \mu),$$

*then the estimates*

$$\int_0^T \int_\Omega \left| \left( u^{\frac{m+1}{2}} \right)_t(x, t) \right|^2 d\nu dt + \frac{(m+1)^2}{8m} \int_\Omega |\nabla(u^m)(x, T)|^2 d\mu \leq \frac{(m+1)^2}{8m} \int_\Omega |\nabla(u_0^m)(x)|^2 d\mu \quad (1.2.31)$$

*and*

$$\int_\Omega |u_0(x)|^{m+1} d\nu \leq \int_\Omega |u(x, t)|^{m+1} d\nu + (m+1)t \int_\Omega |\nabla(u_0^m)(x)|^2 d\mu \quad (1.2.32)$$

*hold true for all  $T, t > 0$ . In particular,  $u^{(m+1)/2}$  is an absolutely continuous curve in  $C([0, \infty); L^2(\Omega; \nu))$ .*

*Proof.* Suppose first that  $u_0 \in C_c^{2,\alpha}(\Omega)$ . In this case, (1.2.31) and (1.2.32) are direct consequences of the estimates (1.2.24) and (1.2.26) obtained for the corresponding approximating problems (in (1.2.26) one just lets  $\zeta'(t) \rightarrow \delta_0(t) - \delta_T(t)$ ). In fact, throughout all the subsequent passages to the limit in the proofs of Lemma 1.2.16 and Theorem 1.2.4, such estimates are stable. The only nontrivial point is to prove that we can pass to the limit safely in the first term in the r.h.s. of (1.2.32) along the proof of Theorem 1.2.4. To this end, let us denote as  $u_n$  the energy solution to (1.1.1) in  $\Omega_n$  with initial datum  $u_0$  and as  $u_n^-$  and  $u_n^+$  the energy solutions to the same problem but with initial data  $-(u_0)_-$  and  $(u_0)_+$ , respectively. Since  $-(u_0)_- \leq u_0 \leq (u_0)_+$ , thanks to the  $L^1(\Omega_n; \nu)$  comparison principle we deduce that  $u_n^- \leq u_n \leq u_n^+$  in  $\Omega_n \times \mathbb{R}^+$ . Recalling Remark 1.2.17 (which holds similarly for nonpositive data), we have that  $\{u_n^-\}$  and  $\{u_n^+\}$  converge monotonically to  $u^-$  and  $u^+$ , respectively, the latter still being the corresponding energy solutions to (1.1.1), but in the whole of  $\Omega$ . In particular,  $\{u_n(t)\}$  is dominated from above and below by two  $L^{m+1}(\Omega; \nu)$  functions. Since  $\{u_n(t)\}$  converges pointwise to  $u$ , estimate (1.2.32) follows by dominated convergence.

Suppose now that

$$u_0^m \in V_{0,c}^{1+1/m}(\Omega; \nu, \mu) \quad \text{and} \quad u_0 \in L^1(\Omega; \nu).$$

Recalling the definition of  $V_0^{1+1/m}(\Omega; \nu, \mu)$ , it is easy to see that there exists a sequence  $\{u_{0n}\} \subset C_c^{2,\alpha}(\Omega)$  such that  $\text{supp}(u_{0n}) \subset \Omega_0 \Subset \Omega$  for some fixed domain  $\Omega_0$  and  $\{u_{0n}^m\}$  converges to  $u_0^m$  in  $V_0^{1+1/m}(\Omega; \nu, \mu)$  (it is just a matter of approximating the real function  $y^{1/m}$  with a sequence of smooth functions). Thanks to the hypotheses on the initial datum, we can moreover assume that  $|u_{0n}| \leq f$  for some fixed nonnegative  $f \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$ . In view of the  $L^1(\Omega; \nu)$  comparison principle, this is enough in order to exploit the first part of the proof and pass to the limit in (1.2.31) and (1.2.32) (with  $u = u_n$ , the energy solution to (1.1.1) corresponding to  $u_{0n}$ ) as  $n \rightarrow \infty$ .

Finally, let

$$u_0^m \in V_0^{1+1/m}(\Omega; \nu, \mu) \quad \text{and} \quad u_0 \in L^1(\Omega; \nu).$$

The existence of  $\{u_{0n}\} \subset C_c^{2,\alpha}(\Omega)$  such that  $\{u_{0n}^m\}$  converges to  $u_0^m$  in  $V_0^{1+1/m}(\Omega; \nu, \mu)$  can be established as above. It is straightforward to check that the sequence

$$\hat{u}_{0n}^m := [(u_0)_+^m \wedge (u_{0n})_+^m] - [(u_0)_-^m \wedge (u_{0n})_-^m]$$

belongs to  $V_{0,c}^{1+1/m}(\Omega; \nu, \mu)$  and converges to  $u_0^m$  in  $V_0^{1+1/m}(\Omega; \nu, \mu)$ . Moreover,  $|\hat{u}_{0n}| \leq |u_0| \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$ . Still by comparison, the sequence of solutions  $\{\hat{u}_n\}$  to (1.1.1), with initial data  $\{\hat{u}_{0n}\}$ , is trapped between two  $L^{m+1}(\Omega; \nu)$  functions at every time and converges pointwise to the energy solution with initial datum  $u_0$ . This permits us to pass to the limit in (1.2.31), (1.2.32) and conclude the proof.  $\square$

Below we discuss some standard properties of the *limit solutions* introduced in Subsection 1.2.1.

**Proposition 1.2.19.** *Let  $u$  and  $v$  be two limit solutions of (1.1.1), corresponding to the initial data  $u_0 \in L^1(\Omega; \nu)$  and  $v_0 \in L^1(\Omega; \nu)$ , respectively. Then:*

- *if in addition  $u_0 \in L^{m+1}(\Omega; \nu)$ ,  $u$  is the energy solution;*
- *for a.e.  $\tau > 0$ ,  $u|_{[\tau, \infty)}$  is the limit solution corresponding to the initial datum  $u(\tau)$  (semigroup property);*
- *the  $L^1(\Omega; \nu)$  comparison principle (1.2.5) holds true for a.e.  $t > 0$ .*

*Proof.* The first and the third claim are trivial consequences of the concept of limit solution and of the  $L^1(\Omega; \nu)$  comparison principle for energy solutions. As concerns the semigroup property, by definition of limit solution there exists a sequence  $\{u_n\}$  of energy solutions such that

$$\|u(\tau) - u_n(\tau)\|_{1,\nu} \rightarrow 0 \tag{1.2.33}$$

and

$$u_n|_{[\tau, \infty)} \rightarrow u|_{[\tau, \infty)} \quad \text{in } L^\infty((\tau, \infty); L^1(\Omega; \nu)) \tag{1.2.34}$$

for a.e.  $\tau > 0$ . Since  $u_n|_{[\tau, \infty)}$  is the energy solution corresponding to the initial datum  $u_n(\tau)$ , thanks to (1.2.33) and (1.2.34) we infer that  $u|_{[\tau, \infty)}$  is the limit solution in  $\Omega \times (\tau, \infty)$  with initial datum  $u(\tau)$ .  $\square$

In Subsection 1.2.3 we shall see that the validity of some functional inequalities (of Poincaré-, Sobolev- or Gagliardo-Nirenberg-type) for the weights  $\rho_\nu$  and  $\rho_\mu$  gives rise to an  $L^{q_0}(\Omega; \nu)$ - $L^\varrho(\Omega; \nu)$  regularizing effect, at least for any  $q_0 \in [1, \infty)$  and  $\varrho \in (q_0, \infty)$ : this

fact, together with Proposition 1.2.19, ensures that limit solutions are actually weak energy solutions after an arbitrarily small time  $\tau > 0$ .

Here we shall not analyse further the issue of  $L^1(\Omega; \nu)$  continuity of solutions: for a related discussion, though in a relatively different context, see for instance Chapter 2, Subsection 2.3.4. We only mention that, if  $\nu(\Omega) < \infty$ , it is a direct consequence of Corollary 1.2.18 and (1.2.5).

**Remark 1.2.20.** The statements and proofs of Lemma 1.2.16, Theorem 1.2.4 and Corollary 1.2.18 have been given for *every*  $T, t > 0$  and not only for *almost every*  $T, t > 0$ . This is a consequence of the just recalled  $C((0, \infty); L^{m+1}(\Omega; \nu))$  property of energy solutions. The same applies to Theorems 1.2.5, 1.2.7, 1.2.9 and 1.2.11.

### 1.2.2.1 Comparison with some previous results

In the particular case where  $\Omega = \mathbb{R}^d$  ( $d \geq 3$ ),  $\rho_\mu \equiv 1$  and  $\rho_\nu$  is a weight which satisfies appropriate decay conditions as  $|x| \rightarrow \infty$ , there are some works which provide existence and uniqueness results for nonnegative solutions to the corresponding weighted porous medium equation (also referred to as *inhomogeneous* porous medium equation, see *e.g.* [163] and quoted references). Let us briefly compare such results to ours.

Given a nonnegative initial datum  $u_0 \in L^1(\mathbb{R}^d; \nu)$ , according to [163, Definition 1.1] any nonnegative function  $u$  is a weak solution to (1.1.1) if it is continuous in  $\mathbb{R}^d \times (0, \infty)$  and:

- $u \in C([0, \infty); L^1(\mathbb{R}^d; \nu)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty)) \quad \forall \tau > 0$ ;
- $\nabla(u^m) \in [L^2(\mathbb{R}^d \times (\tau, \infty))]^d \quad \forall \tau > 0$ ;
- for any  $\varphi \in C_c^1(\mathbb{R}^d \times (0, \infty))$  there holds

$$\int_0^\infty \int_{\mathbb{R}^d} [\langle \nabla(u^m), \nabla \varphi \rangle(x, t) - u(x, t) \varphi_t(x, t) \rho_\nu(x)] dx dt = 0; \quad (1.2.35)$$

- $\lim_{t \rightarrow 0} u(t) = u_0$  in  $L^1(\Omega; \nu)$ .

The most important difference between our definition of weak solution and the one just given lies in the space where one looks for the solution  $u$ . In fact, note that in [163, Definition 1.1]  $u^m$  is not related to the test function space chosen in (1.2.35): in other words, it is not imposed that  $u^m(t)$  belongs to the closure of  $C_c^1(\mathbb{R}^d)$  (or  $\mathcal{D}(\mathbb{R}^d)$ ) with respect to a suitable norm. Indeed, when  $\rho_\nu(x)$  goes to zero sufficiently fast as  $|x| \rightarrow \infty$ , some non-uniqueness issues arise (see the Introduction to Chapter 2).

The two main well-posedness results proved in [163] are the following:

- if  $\rho_\nu \in C^1(\mathbb{R}^d)$  is bounded and strictly positive, then [163, Theorem 3.1] there exists a weak solution according to the above definition;
- if, in addition,  $\rho_\nu$  satisfies

$$A_0(1 + |x|)^{-d} \leq \rho_\nu(x) \quad \forall x \in \mathbb{R}^d$$

for a suitable constant  $A_0 > 0$ , then such solution is also unique [163, Theorem 4.1].

The uniqueness result, to some extent, is not improvable: if  $\rho_\nu(x)$  behaves like  $|x|^{-\gamma}$  at infinity, with  $\gamma > d$ , the finiteness of  $\nu(\mathbb{R}^d)$  implies that if the initial datum is  $u_0 \equiv 1$  then  $u \equiv 1$  is a weak solution to (1.1.1) according to the above definition. On the other hand, it is possible to prove (see [163, Section 8]) that in this case (actually for any  $\gamma > 2$ ) the solution constructed in [163, Theorem 3.1] satisfies the decay condition

$$\lim_{R \rightarrow \infty} \frac{1}{R^{d-1}} \int_{\partial B_R} \int_0^T u^m(\sigma, t) dt d\sigma = 0 \quad \forall T > 0; \quad (1.2.36)$$

since (1.2.36) is trivially *not* fulfilled by nonzero constants, this means that we have at least two solutions. When the initial datum belongs to  $L^1(\mathbb{R}^d; \nu) \cap L^\infty(\mathbb{R}^d)$  actually the



solution we get from [163, Theorem 3.1] satisfies the requirements of [163, Definition 1.1] down to  $\tau = 0$  and it is indeed an energy solution. It seems therefore natural to inquire how such non-uniqueness problem is in agreement with uniqueness of energy solutions proved in Proposition 1.2.3. The answer is that in this case nonzero constants *do not* belong to

$$V_0^{1+1/m}(\mathbb{R}^d; \nu, \lambda), \quad (1.2.37)$$

where we denote as  $\lambda$  the Lebesgue measure on  $\mathbb{R}^d$ . In fact it is direct to show that the Euclidean Sobolev inequality always holds true in the space (1.2.37) (because it is valid in  $\dot{H}^1(\mathbb{R}^d)$ ), which prevents any nonzero constant from belonging to it. Roughly speaking, the choice of test functions in the weak formulation (1.2.35) is typical of a Dirichlet problem; however, no “boundary condition” is prescribed on  $u$ . When the weight  $\rho_\nu(x)$  vanishes sufficiently fast as  $|x| \rightarrow \infty$ ,  $\mathbb{R}^d$  behaves like a bounded domain, so that one expects to have to put boundary conditions at infinity to restore uniqueness. Indeed, condition (1.2.36) turns out to be *sufficient* for uniqueness, see again [163, Section 8] and references quoted therein.

Despite these non-uniqueness issues (for  $\alpha > d$ ), it is not difficult to check that the weak solution of (1.1.1) (according to [163, Definition 1.1]) constructed in [163, Theorem 3.1] coincides with the weak energy solution of the same problem (according to Definition 1.2.1) whose existence was proved in Theorem 1.2.4, at least for any  $u_0 \in L^1(\mathbb{R}^d; \nu) \cap L^\infty(\mathbb{R}^d)$  (and so for any  $L^1(\Omega; \nu)$  datum in view of (1.2.5)). As a consequence, in this context solutions are in fact  $C([0, \infty); L^1(\mathbb{R}^d; \nu))$  functions. Still from results in [163] we know that an  $L^\infty$  smoothing effect takes place: this is consistent with the validity of the Sobolev inequality in  $H_0^1(\mathbb{R}^d; \nu, \lambda)$  (see Paragraph 1.2.3.3 below).

### 1.2.3 Smoothing and decay estimates

In this subsection we shall prove Theorems 1.2.5–1.2.12, which show how the validity of Poincaré-, Sobolev- or Gagliardo-Nirenberg-type inequalities in  $H_0^1(\Omega; \nu, \mu)$  is connected with suitable smoothing effects for solutions to (1.1.1).

#### 1.2.3.1 Poincaré-type inequalities

*Proof of Theorem 1.2.5.* We shall proceed by means of a Gross differential method. The computations follow closely to the ones performed in the proof of [105, Theorem 1.3], so we just point out the most significant differences. Upon defining the *entropy functional*

$$J(r, f) := \int_{\Omega} \frac{|f(x)|^r}{\|f\|_{r;\nu}^r} \log \left( \frac{|f(x)|}{\|f\|_{r;\nu}} \right) d\nu,$$

the validity of the family of logarithmic Sobolev inequalities

$$\left( J(r, v) + \frac{1}{2-r} \log \varepsilon \right) \frac{(2-r)\|v\|_{r;\nu}^2}{\varepsilon C_P^2} \leq \|\nabla v\|_{2;\mu}^2 \quad (1.2.38)$$

$$\forall r \in [1, 2), \quad \forall \varepsilon > 0, \quad \forall v \in L^r(\Omega; \nu) \cap H_0^1(\Omega; \nu, \mu)$$

was established in [105, Theorem 1.3]. Given  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ ,  $t > 0$ ,  $q_0 \in (1, \infty)$  and  $\varrho \in (q_0, \infty)$ , let  $u$  be the solution to (1.1.1) with initial datum  $u_0$ . Introducing an increasing, one-to-one and  $C^1([0, t])$  function  $q : [0, t] \mapsto [q_0, \varrho]$ , after explicit calculations we get:

$$\begin{aligned} & \frac{d}{ds} \log \|u(s)\|_{q(s);\nu} \\ &= \frac{q'(s)}{q(s)} J(q(s), u(s)) - \left( \frac{2}{q(s) + m - 1} \right)^2 \frac{m(q(s) - 1)}{\|u(s)\|_{q(s);\nu}^{q(s)}} \left\| \nabla \left( u^{\frac{q(s)+m-1}{2}} \right) (s) \right\|_{2;\mu}^2. \end{aligned}$$

One applies (1.2.38) to  $v = u^{(q+m-1)/2}$  in the above identity, sets  $q(s) := q_0 + \frac{s}{t}(\varrho - q_0)$ , chooses  $r$  and  $\varepsilon$  appropriately and solves the resulting differential inequality in the variable  $y(s) := \log \|u(s)\|_{q(s);\nu}$  along the lines of the proof of [105, Theorem 1.3]. With respect to the notations used therein, we point out that it is enough to replace  $q - 1$  with  $m(q - 1)$ ,  $p - 2$  with  $m - 1$ ,  $p$  with 2 and  $C$  with  $C_P^2$ . This yields estimate (1.2.7) for  $q_0 > 1$ . The case  $q_0 = 1$  follows by taking limits, since the constant  $K_1$  in (1.2.7) can be shown to be bounded as  $q_0 \downarrow 1$ .

In the case  $\nu(\Omega) < \infty$ , for any fixed  $q_0 > 1$  we have:

$$\begin{aligned} \frac{d}{ds} \|u(s)\|_{q_0;\nu}^{q_0} &= - \left( \frac{2}{q_0 + m - 1} \right)^2 m q_0 (q_0 - 1) \left\| \nabla \left( u^{\frac{q_0+m-1}{2}} \right) (s) \right\|_{2;\mu}^2 \\ &\leq - \left( \frac{2}{q_0 + m - 1} \right)^2 \frac{m q_0 (q_0 - 1)}{C_P^2} \|u(s)\|_{q_0+m-1;\nu}^{q_0+m-1} \\ &\leq - \left( \frac{2}{q_0 + m - 1} \right)^2 \frac{m q_0 (q_0 - 1)}{C_P^2 \nu(\Omega)^{\frac{m-1}{q_0}}} \|u(s)\|_{q_0;\nu}^{q_0+m-1} \\ &= -D \left( \|u(s)\|_{q_0;\nu}^{q_0} \right)^{\frac{q_0+m-1}{q_0}}, \end{aligned}$$

where  $D > 0$  is a suitable constant depending only on  $q_0$ ,  $m$ ,  $C_P$  and  $\nu(\Omega)$ . Solving the above differential inequality in the variable  $y(s) := \|u(s)\|_{q_0;\nu}^{q_0}$  we end up with the estimate

$$\|u(t)\|_{q_0;\nu} \leq \frac{1}{\left( D t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{\frac{1}{m-1}}} \quad \forall t > 0, \quad (1.2.39)$$

upon relabelling  $D$ . Hence, using (1.2.39) and (1.2.7) between  $t$  and  $t/2$  (notice that we are also exploiting the *semigroup property*, see Proposition 1.2.19), we obtain:

$$\|u(t)\|_{\varrho;\nu} \leq K_1 (t/2)^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u(t/2)\|_{q_0;\nu}^{\frac{q_0}{\varrho}} \leq K_2 t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \left( t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{-\frac{q_0}{(m-1)\varrho}} \quad \forall t > 0,$$

where  $K_2 > 0$  is another constant depending only on  $\varrho$ ,  $q_0$ ,  $m$ ,  $C_P$  and  $\nu(\Omega)$ .

It remains to prove the validity of (1.2.8) for  $q_0 = 1$ . To this end, it is convenient to use the smoothing effect (1.2.7) with, for instance,  $\varrho$  replaced by  $(\varrho + 1)/2$  and  $q_0 = 1$ :

$$\|u(t)\|_{\frac{\varrho+1}{2};\nu} \leq K_1 t^{-\frac{\varrho-1}{(\varrho+1)(m-1)}} \|u_0\|_{1;\nu}^{\frac{2}{\varrho+1}} \quad \forall t > 0. \quad (1.2.40)$$

Exploiting (1.2.40) and (1.2.8) (with  $q_0 = (\varrho + 1)/2$ ) between  $t$  and  $t/2$ , we get:

$$\begin{aligned} \|u(t)\|_{\varrho;\nu} &\leq K_2 (t/2)^{-\frac{\varrho-1}{2\varrho(m-1)}} \left( t/2 + \|u(t/2)\|_{\frac{\varrho+1}{2};\nu}^{1-m} \right)^{-\frac{\varrho+1}{2\varrho(m-1)}} \\ &\leq K_2 t^{-\frac{\varrho-1}{2\varrho(m-1)}} \left( t + \|u(t/2)\|_{\frac{\varrho+1}{2};\nu}^{1-m} \right)^{-\frac{\varrho+1}{2\varrho(m-1)}} \\ &\leq K_2 t^{-\frac{\varrho-1}{\varrho(m-1)}} \left( t^{\frac{2}{\varrho+1}} + \|u_0\|_{1;\nu}^{\frac{2(1-m)}{\varrho+1}} \right)^{-\frac{\varrho+1}{2\varrho(m-1)}} \\ &\leq K_2 t^{-\frac{\varrho-1}{\varrho(m-1)}} \left( t + \|u_0\|_{1;\nu}^{1-m} \right)^{-\frac{1}{(m-1)\varrho}} \end{aligned}$$

for all  $t > 0$ , where  $K_2 > 0$  is a constant as above, possibly changing from line to line.

Finally, in order to remove the hypothesis  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ , it is enough to take a sequence  $\{u_{0n}\} \subset L^1(\Omega; \nu) \cap L^\infty(\Omega)$  which converges to  $u_0$  in  $L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , apply to

the corresponding solutions  $\{u_n\}$  of (1.1.1) the just proved estimates and pass to the limit as  $n \rightarrow \infty$  (also exploiting the  $L^1(\Omega; \nu)$  comparison principle (1.2.5)).  $\square$

Notice that, *a priori*, our solutions are not regular enough (in time) in order to justify rigorously the calculations we performed in the above proof. Nevertheless, one can overcome this issue in several ways: for instance by performing analogous computations in the non-degenerate problems solved in the proof of Lemma 1.2.16 and then passing to the limit or by exploiting the general fact that solutions are actually *strong*. We shall not investigate further the latter property: for a short overview in this regard, we refer the reader to Chapter 2, Paragraph 2.2.2.1, where strong solutions are dealt with in a different context, but the same arguments work here as well. Alternatively, Theorem 1.2.5, at least as concerns (1.2.7), can be proved by means of a Moser iteration technique, which does not require time regularity. We shall see several applications of this method in the proofs of Theorems 1.2.9, 1.2.11 and in Subsection 1.3.3.

We now prove the converse implication of Theorem 1.2.5.

*Proof of Theorem 1.2.6.* As in the second part of the proof of [105, Theorem 1.3], we want to take advantage of [13, Theorem 3.1]. In order to do that, let us consider any initial datum  $u_0 \in W_c^{1,\infty}(\Omega)$ . It is apparent that the hypotheses of Corollary 1.2.18 are fulfilled, so that (1.2.32) holds true. Thanks to an interpolation argument and to the fact that the  $L^{q_0}(\Omega; \nu)$  norm of the solution does not increase along the evolution (trivial consequence of (1.2.3)), from the validity of (1.2.7) for some  $\varrho \geq m+1$  it is direct to deduce the validity of the latter also for  $\varrho = m+1$ , possibly with a different constant  $K_1$  which we do not relabel. Hence, by combining such a smoothing estimate with (1.2.32), we obtain:

$$\|u_0\|_{m+1;\nu}^{m+1} \leq K_1^{m+1} t^{-\frac{m+1-q_0}{m-1}} \|u_0\|_{q_0;\nu}^{q_0} + (m+1)t \|\nabla(u_0^m)\|_{2;\mu}^2 \quad \forall t > 0.$$

Minimizing the explicitly the right-hand side w.r.t.  $t > 0$ , we end up with

$$\|u_0\|_{m+1;\nu} \leq B \|\nabla(u_0^m)\|_{2;\mu}^{\frac{2(m+1-q_0)}{(m+1)(2m-q_0)}} \|u_0\|_{q_0;\nu}^{\frac{q_0(m-1)}{(m+1)(2m-q_0)}}, \quad (1.2.41)$$

where  $B = B(q_0, m, K_1) > 0$  is a suitable constant. In order to rewrite (1.2.41) in a more convenient way for our purposes, consider the following sequence  $\{\xi_n\}$  of real functions:

$$\xi_n(y) := 2 \left( y - \frac{1}{2n} \right) \chi_{[\frac{1}{2n}, \frac{1}{n})}(y) + 2 \left( y + \frac{1}{2n} \right) \chi_{(-\frac{1}{n}, -\frac{1}{2n}]}(y) + y \chi_{[\frac{1}{n}, \infty)}(|y|) \quad \forall y \in \mathbb{R}.$$

By construction, the regularized approximations

$$v_n := \xi_n \left( u_0^{1/m} \right)$$

of  $u_0^{1/m}$  still belong to  $W_c^{1,\infty}(\Omega)$ . Moreover,

$$|v_n(x)| \leq |u_0(x)|^{\frac{1}{m}}, \quad \nabla(v_n^m)(x) = \xi_n' \left( u_0^{1/m}(x) \right) |v_n(x)|^{m-1} |u_0(x)|^{\frac{1}{m}-1} \nabla u_0(x)$$

for a.e.  $x \in \Omega$ . In view of the above estimates and from the properties of  $\{\xi_n\}$ , it is straightforward to check that

$$|\nabla(v_n^m)|(x) \leq 2 |\nabla u_0(x)| \quad \text{for a.e. } x \in \Omega$$

and that  $\{v_n\}$  and  $\{\nabla(v_n^m)\}$  converge pointwise to  $u_0^{1/m}$  and  $\nabla(u_0)$ , respectively. Applying (1.2.41) to the sequence of initial data  $\{v_n\}$  and passing to the limit as  $n \rightarrow \infty$ , by dominated convergence we infer that

$$\|u_0\|_{\frac{m+1}{m};\nu} \leq B^m \|\nabla u_0\|_{2;\mu}^{\frac{2m(m+1-q_0)}{(m+1)(2m-q_0)}} \|u_0\|_{\frac{q_0}{m};\nu}^{\frac{q_0(m-1)}{(m+1)(2m-q_0)}}. \quad (1.2.42)$$

Setting

$$\vartheta := \frac{2m(m+1-q_0)}{(m+1)(2m-q_0)}, \quad r := \frac{m+1}{m}, \quad s := \frac{q_0}{m}, \quad q := 2, \quad \mathcal{W}(f) := \|\nabla f\|_{2;\mu},$$

where  $f$  is any nonnegative function belonging to  $W_c^{1,\infty}(\Omega)$ , inequality (1.2.42) reads

$$\|f\|_{r;\nu} \leq [\hat{B} \mathcal{W}(f)]^\vartheta \|f\|_{s;\nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad \hat{B} := B^{\frac{m}{\vartheta}}. \quad (1.2.43)$$

We are therefore in position to apply Theorem 3.1 of [13], which ensures the validity of (1.2.43) for  $\vartheta = 1$  and  $r = 2$  as well, that is

$$\|f\|_{2;\nu} \leq \hat{B} \|\nabla f\|_{2;\mu}, \quad (1.2.44)$$

up to a different positive constant  $\hat{B}$ . Hence, we have proved the Poincaré inequality for nonnegative functions in  $W_c^{1,\infty}(\Omega)$ . The extension of (1.2.44) to signed functions in  $W_c^{1,\infty}(\Omega)$  is easily achieved by writing  $f = f_+ - f_-$ , while the extension to the whole of  $H_0^1(\Omega; \nu, \mu)$  just follows by density (recall Proposition 1.1.2).  $\square$

### 1.2.3.2 Counterexamples to $L^\infty$ regularization

As already mentioned, estimate (1.2.7) provides us with an  $L^{q_0}(\Omega; \nu)$ – $L^\varrho(\Omega; \nu)$  regularization which *fails* at  $\varrho = \infty$  (by direct computations one checks that  $K_1$  diverges as  $\varrho \rightarrow \infty$ ). In this paragraph, by means of two explicit counterexamples, we show that in general one cannot do better, that is the sole validity of a Poincaré-type inequality prevents  $L^\infty$  regularization.

Let  $\Omega = \mathbb{R}^+$ . With respect to the weights  $\rho_\nu(x) = \rho_\mu(x) = e^{-x}$ , it is known that the Poincaré-type inequality (1.2.6) holds true (see Appendix 1.A, Paragraph 1.A.1.1). In this framework, problem (1.1.1) reads

$$\begin{cases} u_t = e^x (e^{-x} (u^m)_x)_x & \text{in } \mathbb{R}^+ \times \mathbb{R}^+, \\ u = 0 & \text{on } \{0\} \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^+ \times \{0\}. \end{cases} \quad (1.2.45)$$

We aim at proving that the solution  $u$  corresponding to the initial datum  $u_0(x) = \log(x+1)$  is unbounded for all  $t \geq 0$ . To this end, let us consider the following family of functions:

$$v_B(x, t) := \frac{\log(x+1)}{(1+B^{-1}(m-1)t)^{\frac{1}{m-1}}}, \quad B > 0.$$

One can show that  $v_B$  is a *subsolution* to (1.2.45) for a suitable choice of the constant  $B$ . In fact, after some standard computations, one gets:

$$\begin{aligned} e^x (e^{-x} ([\log(x+1)]^m)_x)_x &= -m \frac{[\log(x+1)]^{m-1}}{x+1} - m \frac{[\log(x+1)]^{m-1}}{(x+1)^2} \\ &\quad + m(m-1) \frac{[\log(x+1)]^{m-2}}{(x+1)^2}. \end{aligned}$$

It is then apparent that there exists a constant  $B > 0$  such that

$$\log(x+1) \geq -B e^x (e^{-x} ([\log(x+1)]^m)_x)_x,$$

so that

$$(v_B)_t = -\frac{\log(x+1)}{B(1+B^{-1}(m-1)t)^{\frac{m}{m-1}}} \leq \frac{e^x(e^{-x}([\log(x+1)]^m)_x)_x}{(1+B^{-1}(m-1)t)^{\frac{m}{m-1}}} = e^x(e^{-x}(v_B^m)_x)_x.$$

As a consequence,  $v_B$  is a subsolution to (1.2.45) for  $u_0(x) = \log(x+1)$ . The comparison principle for strong sub-supersolutions (see [173, Theorem 8.10]) entails  $v_B \leq u$ ; in particular,  $u(t)$  is *unbounded* for all  $t \geq 0$ . Moreover, since  $u_0 \in L^{q_0}(\mathbb{R}^+; e^{-x})$  for all  $q_0 \in [1, \infty)$ , in this case the  $L^{q_0}(\mathbb{R}^+; e^{-x})$ - $L^\infty(\mathbb{R}^+)$  regularization *does not* take place for any  $q_0 \in [1, \infty)$ .

Let  $\Omega = \mathbb{R}^d$ , with  $d \geq 3$ . Take the weights  $\rho_\nu(x) = |x|^{-2}$  and  $\rho_\mu \equiv 1$ , so that (1.1.1) here reads

$$\begin{cases} |x|^{-2} u_t = \Delta(u^m) & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \quad (1.2.46)$$

Problem (1.2.46) is also known as inhomogeneous porous medium equation *with critical density*, and has been thoroughly studied in the recent paper [141] (see also [169] for connections with the porous medium equation on hyperbolic space). It is straightforward to check that the (Barenblatt-type) function

$$U(x, t) := \left[ \frac{\log(t^\kappa/|x|)}{m(d-2)t} \right]_+^{\frac{1}{m-1}}, \quad \kappa := \frac{1}{(d-2)(m-1)}$$

is in fact an energy solution to (1.2.46) after an arbitrarily small  $t > 0$ , which belongs to  $L^\varrho(\mathbb{R}^d; |x|^{-2})$  for all  $\varrho \in [1, \infty)$  but is clearly *unbounded* in a neighbourhood of  $\{x = 0\}$ . On the other hand, the validity of the classical Hardy inequality

$$\int_{\mathbb{R}^d} v^2(x) |x|^{-2} dx \leq C_H^2 \int_{\mathbb{R}^d} |\nabla v(x)|^2 dx \quad \forall v \in \dot{H}^1(\mathbb{R}^d), \quad C_H > 0,$$

is equivalent to the validity of the weighted Poincaré inequality in  $H_0^1(\mathbb{R}^d; |x|^{-2}, 1)$ .

In the latter case, actually, standard time-space scaling arguments imply that the only possible smoothing effect of the type

$$\|u(t)\|_{\varrho; |x|^{-2}} \leq K_1 t^{-\alpha_0} \|u_0\|_{\varrho_0; |x|^{-2}}^{\beta_0} \quad \forall t > 0$$

for some  $\alpha_0 > 0$  and  $\beta_0 > 0$  is (1.2.7). Moreover, estimate (1.2.7) is *sharp* on  $U$ , in the sense that it captures the exact rate at which  $\|U(t)\|_{\varrho; |x|^{-2}}$  blows up as  $t \rightarrow 0$  or vanishes as  $t \rightarrow \infty$ , namely  $t^{-(\varrho-1)/[\varrho(m-1)]}$ .

As the reader might have noticed, strictly speaking, problem (1.2.46) does not fall within the framework of those for which we established a well-posedness theory in Subsection 1.2.2, since the weight  $|x|^{-2}$  is singular at  $x = 0$ . Nevertheless, we can overcome this issue at least in two ways. One consists in considering the same problem in  $\mathbb{R}^d \setminus \{0\}$  rather than in  $\mathbb{R}^d$ : it is easy to check that energy solutions to the two problems actually coincide. As an alternative, one can prove well-posedness results for (1.2.46) by approximating  $|x|^{-2}$  with a sequence of regular weights (see also Chapter 2, Subsection 2.2.2 for a similar procedure as related to a fractional problem of the type of (1.2.46)).

### 1.2.3.3 Sobolev-type inequalities

We only provide a short proof of Theorem 1.2.8 and make some comments on Theorem 1.2.7.

*Proof of Theorem 1.2.8.* Given an initial datum  $u_0 \in W_c^{1,\infty}(\Omega)$  and denoting as  $u$  the corresponding solution to (1.1.1), using the fact the  $L^{q_0}(\Omega; \nu)$  norm of  $u(t)$  does not increase, by interpolation we get:

$$\|u(t)\|_{m+1;\nu}^{m+1} \leq \|u(t)\|_{\infty}^{m+1-q_0} \|u_0\|_{q_0;\nu}^{q_0} \quad \forall t > 0.$$

Applying the smoothing effect (1.2.10) to the r.h.s. we obtain

$$\|u(t)\|_{m+1;\nu}^{m+1} \leq C t^{-\frac{\sigma(m+1-q_0)}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0(2\sigma m-m-1)}{(\sigma-1)q_0+\sigma(m-1)}} \quad \forall t > 0 \quad (1.2.47)$$

for a suitable  $C = C(q_0, m, \sigma, K_1) > 0$ . Thanks again to Corollary 1.2.18, (1.2.47) entails

$$\|u_0\|_{m+1;\nu}^{m+1} \leq C t^{-\frac{\sigma(m+1-q_0)}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0(2\sigma m-m-1)}{(\sigma-1)q_0+\sigma(m-1)}} + (m+1)t \|\nabla(u_0^m)\|_{2;\mu}^2 \quad \forall t > 0. \quad (1.2.48)$$

Minimizing the explicitly the right-hand side of (1.2.48) w.r.t.  $t > 0$  and proceeding exactly as in the proof of Theorem 1.2.6, we end up with the following inequality:

$$\|f\|_{r;\nu} \leq [C \|\nabla f\|_{2;\mu}]^{\vartheta} \|f\|_{s;\nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad (1.2.49)$$

where  $f$  is any nonnegative function belonging to  $W_c^{1,\infty}(\Omega)$ ,  $C$  is a positive constant as above and now

$$\vartheta := \frac{2\sigma m(m+1-q_0)}{(m+1)(2\sigma m-q_0)}, \quad r := \frac{m+1}{m}, \quad s := \frac{q_0}{m}, \quad q := 2\sigma.$$

Hence, we are again in position to apply Theorem 3.1 of [13], from which we deduce the validity of (1.2.49) for  $\vartheta = 1$  and  $r = 2\sigma$  as well, that is

$$\|f\|_{2\sigma;\nu} \leq C \|\nabla f\|_{2;\mu} \quad (1.2.50)$$

up to a different positive constant  $C$ . The extension of (1.2.50) to the whole of  $H_0^1(\Omega; \nu, \mu)$  just follows as in the end of proof of Theorem 1.2.6.  $\square$

Theorem 1.2.7, to some extent, was already proved in [27] (see Theorem 1.5 there), at least as concerns (1.2.10). In fact, in such paper, the authors establish the same smoothing effect for solutions to the porous medium equation on Riemannian manifolds of infinite volume and without boundary, in dimension  $d \geq 3$ . They proceed by means of a Gross differential method, and their approach is purely functional analytic, in the sense that they only exploit the validity of the Sobolev inequality (1.2.9) with  $\sigma = d/(d-2)$ . It is therefore immediate to check that their proof works in our context as well.

Furthermore, there are at least another two ways to prove Theorem 1.2.7. They are very similar to each other and they both rely on a Moser iteration technique. As for the first one, we refer the reader to the proof of Theorem 1.3.8 in Paragraph 1.3.3.3 (smoothing estimates for the homogeneous Neumann problem with the validity of a Sobolev-type inequality): it is apparent that the same strategy also works for the Dirichlet problem (in which case it is even much simpler). As for the second one, let us notice that, in view of the deep results in [13], the Sobolev-type inequality (1.2.9) is in fact *equivalent* to the family of Gagliardo-Nirenberg-type inequalities (1.2.49), where  $q = 2\sigma$  and  $r, s > 0, \vartheta \in (0, 1]$  are free parameters (linked by the corresponding constraint in (1.2.49)). Hence, one can repeat the proof of Theorem 1.2.11 below with minor modifications. We point out that, still as a consequence of the results in [13], the family (1.2.49) is equivalent to (1.2.12) for  $q = -\lambda$ , while for  $q = \infty$  it coincides with (1.2.15).

Let us consider problem (1.2.9) with the choices  $\Omega = \mathbb{R}^d$  ( $d \geq 3$ ),  $\rho_\mu \equiv 1$  and  $\rho_\nu(x) = |x|^{-\gamma}$  for some  $\gamma \in (0, 2)$ , that is

$$\begin{cases} |x|^{-\gamma} u_t = \Delta(u^m) & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \quad (1.2.51)$$

In this case, the Hardy-Sobolev inequality (see also Chapter 3, Subsection 3.2.2)

$$\left( \int_{\mathbb{R}^d} |v(x)|^{\frac{d-\gamma}{d-2}} |x|^{-\gamma} dx \right)^{\frac{d-2}{2(d-\gamma)}} \leq C_{HS} \left( \int_{\mathbb{R}^d} |\nabla v(x)|^2 dx \right)^{\frac{1}{2}} \quad \forall v \in \dot{H}^1(\mathbb{R}^d), \quad C_{HS} > 0,$$

holds true, namely (1.2.9) with  $\sigma = (d - \gamma)/(d - 2)$ . The Barenblatt-type function (see [164] or the the Introduction to Chapter 2)

$$U(x, t) := t^{-\alpha} \left[ 1 - b (t^{-\kappa} |x|)^{2-\gamma} \right]_+^{\frac{1}{m-1}}, \quad \alpha := (d - \gamma)\kappa, \quad \kappa := \frac{1}{d(m-1) + 2 - m\gamma},$$

where  $b$  is a fixed positive constant depending only on  $m$ ,  $d$  and  $\gamma$ , is an energy solution to (1.2.51) after an arbitrarily small  $t > 0$ . Notice that, in agreement with Theorem 1.2.7 and unlike the case  $\gamma = 2$  discussed in Paragraph 1.2.3.2,  $U(t)$  is a *bounded* function at any  $t > 0$ . Moreover, a direct computation shows that the smoothing estimate (1.2.10) (with  $\sigma = (d - \gamma)/(d - 2)$ ) is sharp on  $U$ .

As concerns the fact that the weight  $|x|^{-\gamma}$  is singular at  $x = 0$ , the same comments as in the end of Paragraph 1.2.3.2 apply.

#### 1.2.3.4 Gagliardo-Nirenberg-type inequalities

We first prove  $L^\infty$  bounds in the one-dimensional case.

*Proof of Theorem 1.2.9.* First notice that inequality (1.2.12) is equivalent to

$$\frac{\|v\|_\infty^{\frac{2(s+\lambda)}{\lambda}}}{C \|v\|_{s;\nu}^{\frac{2s}{\lambda}}} \leq \|\nabla v\|_{2;\mu}^2 \quad \forall s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu) : v \not\equiv 0, \quad (1.2.52)$$

where we set  $C := C_{G_1}^{2(s+\lambda)/\lambda}$ . Let  $q_0 > 1$ ,  $t > 0$  and take  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ . Estimate (1.2.3) with  $q = q_0 - 1$  and  $T = t$  trivially entails

$$\frac{4q_0(q_0 - 1)m}{(q_0 + m - 1)^2} \int_0^t \int_\Omega \left| \nabla \left( u^{\frac{q_0+m-1}{2}} \right) (x, \tau) \right|^2 d\mu d\tau \leq \|u_0\|_{q_0;\nu}^{q_0}.$$

Applying (1.2.52) to the function  $v = u^{(q_0+m-1)/2}(\tau)$ , we obtain:

$$\frac{4q_0(q_0 - 1)m}{C (q_0 + m - 1)^2} \int_0^t \frac{\left\| u^{\frac{q_0+m-1}{2}}(\tau) \right\|_\infty^{\frac{2(s+\lambda)}{\lambda}}}{\left\| u^{\frac{q_0+m-1}{2}}(\tau) \right\|_{s;\nu}^{\frac{2s}{\lambda}}} d\tau \leq \|u_0\|_{q_0;\nu}^{q_0}. \quad (1.2.53)$$

Since  $L^p(\Omega; \nu)$  norms are nonincreasing in time (direct consequence of (1.2.3)), from (1.2.53) we infer (provided  $s(q_0 + m - 1)/2 \geq 1$ )

$$t \frac{4q_0(q_0 - 1)m}{C (q_0 + m - 1)^2} \frac{\|u(t)\|_\infty^{\frac{(s+\lambda)(q_0+m-1)}{\lambda}}}{\|u_0\|_{\frac{s(q_0+m-1)}{\lambda};\nu}^{\frac{s(q_0+m-1)}{\lambda}}} \leq \|u_0\|_{q_0;\nu}^{q_0}. \quad (1.2.54)$$

Letting  $s = 2q_0/(q_0 + m - 1)$ , (1.2.54) reads

$$t \frac{4q_0(q_0 - 1)m}{C(q_0 + m - 1)^2} \frac{\|u(t)\|_\infty^{\frac{(\lambda+2)q_0 + \lambda(m-1)}{\lambda}}}{\|u_0\|_{q_0;\nu}^{\frac{2q_0}{\lambda}}} \leq \|u_0\|_{q_0;\nu}^{q_0},$$

whence

$$\|u(t)\|_\infty \leq K_1 t^{-\frac{\lambda}{(\lambda+2)q_0 + \lambda(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{(\lambda+2)q_0 + \lambda(m-1)}} \quad (1.2.55)$$

for a suitable constant  $K_1 > 0$  as in the statement which, *a priori*, also depends on  $q_0$  since it blows up as  $q_0 \downarrow 1$  (but it is stable as  $q_0 \uparrow \infty$ ). In order to prove the validity of (1.2.55) down to  $q_0 = 1$ , we use the same argument as in Corollary 8.1 of [61]. That is, by exploiting the interpolation inequality  $\|u(t/2)\|_{q_0;\nu}^{q_0} \leq \|u(t/2)\|_\infty^{q_0-1} \|u(t/2)\|_{1;\nu}$ , the fact that the  $L^1(\Omega; \nu)$  norm of the solution is not increasing (recall (1.2.3)) and (1.2.55) between  $t$  and  $t/2$ , we get:

$$\|u(t)\|_\infty \leq K_1 \left(\frac{2}{t}\right)^{\frac{\lambda\alpha}{\lambda+2}} \|u(t/2)\|_\infty^{\alpha(q_0-1)} \|u_0\|_{1;\nu}^\alpha, \quad \alpha := \frac{\lambda+2}{(\lambda+2)q_0 + \lambda(m-1)}. \quad (1.2.56)$$

Iterating (1.2.56)  $k-1$  times we end up with

$$\|u(t)\|_\infty \leq \frac{K_1^{\sum_{h=0}^{k-1} [\alpha(q_0-1)]^h} 2^{\frac{\lambda\alpha}{\lambda+2} \sum_{h=0}^{k-1} (h+1)[\alpha(q_0-1)]^h}}{t^{\frac{\lambda\alpha}{\lambda+2} \sum_{h=0}^{k-1} [\alpha(q_0-1)]^h}} \|u(t/2^k)\|_\infty^{[\alpha(q_0-1)]^k} \|u_0\|_{1;\nu}^{\sum_{h=0}^{k-1} [\alpha(q_0-1)]^h}. \quad (1.2.57)$$

Letting  $k \rightarrow \infty$  in (1.2.57), it is just a matter of straightforward computations to verify that the multiplicative constants above stay bounded, the powers to which  $t^{-1}$  and  $\|u_0\|_{1;\nu}$  are elevated tend to  $\lambda/(\lambda m + 2)$  and  $(\lambda + 2)/(\lambda m + 2)$ , respectively, while the one to which  $\|u(t/2^k)\|_\infty \leq \|u_0\|_\infty$  is elevated tends to zero. This yields

$$\|u(t)\|_\infty \leq \widetilde{K} t^{-\frac{\lambda}{\lambda m + 2}} \|u_0\|_{1;\nu}^{\frac{\lambda+2}{\lambda m + 2}}, \quad (1.2.58)$$

namely (1.2.13) for  $\bar{q}_0 = 1$ , where  $\widetilde{K} > 0$  is a constant that depends only on  $m$ ,  $C_{G_1}$  and  $\lambda$  (fix *e.g.*  $q_0 = 2$  in the calculations above). We can assert that the multiplicative constant  $K_1$  appearing in (1.2.13) is finally independent of  $q_0$  because it is locally bounded, it stays bounded as  $q_0 \rightarrow \infty$  and for any  $q_0 \in [1, 2]$  (for instance) it is possible to repeat the same argument we used to prove (1.2.58), which in particular ensures boundedness of such constant as  $q_0$  varies in  $[1, 2]$ .

For the proof of (1.2.14) in the case  $\nu(\Omega) < \infty$ , we can proceed similarly to the end of proof of Theorem 1.2.5. In fact, the validity of (1.2.39) for  $q_0 > 1$  follows by means of an identical argument (since  $\nu(\Omega) < \infty$ , it is apparent that (1.2.12) entails the Poincaré inequality for a constant  $C_P$  that depends on  $C_{G_1}$ ,  $\lambda$  and  $\nu(\Omega)$ ). Hence, using (1.2.39) and (1.2.13) between  $t$  and  $t/2$ , we obtain:

$$\begin{aligned} \|u(t)\|_{\infty;\nu} &\leq K_1 (t/2)^{-\frac{\lambda}{(\lambda+2)q_0 + \lambda(m-1)}} \|u(t/2)\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{(\lambda+2)q_0 + \lambda(m-1)}} \\ &\leq K_2 t^{-\frac{\lambda}{(\lambda+2)q_0 + \lambda(m-1)}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{(\lambda+2)q_0}{(m-1)[(\lambda+2)q_0 + \lambda(m-1)]}} \quad \forall t > 0, \end{aligned}$$

where  $K_2 > 0$  is another constant depending only on  $q_0$ ,  $m$ ,  $C_{G_1}$ ,  $\lambda$  and  $\nu(\Omega)$ . It therefore remains to prove the validity of (1.2.14) for  $q_0 = 1$ . To this end, note that, thanks to the inequality  $\|u(t)\|_{2;\nu}^2 \leq \|u(t)\|_\infty \|u_0\|_{1;\nu}$  and to (1.2.13) (evaluated at  $q_0 = 1$ ), we get:

$$\|u(t)\|_{2;\nu} \leq K_1^{\frac{1}{2}} t^{-\frac{\lambda}{2\lambda m + 4}} \|u_0\|_{1;\nu}^{\frac{\lambda(m+1)+4}{2\lambda m + 4}} \quad \forall t > 0. \quad (1.2.59)$$



Exploiting (1.2.59) and (1.2.14) (with  $q_0 = 2$ ) between  $t$  and  $t/2$ , we find

$$\begin{aligned} \|u(t)\|_\infty &\leq K_2 t^{-\frac{\lambda}{\lambda(m+1)+4}} \left( t + \|u(t/2)\|_{2;\nu}^{1-m} \right)^{-\frac{2\lambda+4}{(m-1)[\lambda(m+1)+4]}} \\ &\leq K_2 t^{-\frac{\lambda}{\lambda m+2}} \left( t^{\frac{\lambda(m+1)+4}{2\lambda m+4}} + \|u_0\|_{1;\nu}^{(1-m)\frac{\lambda(m+1)+4}{2\lambda m+4}} \right)^{-\frac{2\lambda+4}{(m-1)[\lambda(m+1)+4]}} \\ &\leq K_2 t^{-\frac{\lambda}{\lambda m+2}} \left( t + \|u_0\|_{1;\nu}^{1-m} \right)^{-\frac{\lambda+2}{(m-1)(\lambda m+2)}} \quad t > 0, \end{aligned}$$

where  $K_2 > 0$  is still a suitable constant depending only on  $m$ ,  $C_{G_1}$ ,  $\lambda$  and  $\nu(\Omega)$ , possibly changing from line to line.

In order to conclude the proof, we need to get rid of the assumption  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ . To this aim, given  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , pick a sequence  $\{u_{0n}\} \subset L^1(\Omega; \nu) \cap L^\infty(\Omega)$  which converges to  $u_0$  in  $L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$  and notice that the corresponding sequence of solutions  $\{u_n\}$  of (1.1.1) converges to  $u$  in  $L^\infty((0, \infty); L^1(\Omega; \nu))$  (recall (1.2.5)). Moreover, in view of (1.2.13),  $\{u_n(t)\}$  converges to  $u(t)$  for a.e.  $t > 0$  in the weak\* topology of  $L^\infty(\Omega)$ , which is enough to pass to the limit in all the above estimates thanks to the weak\* lower semi-continuity of the  $L^\infty$  norm.  $\square$

Now let us prove the converse implication of Theorem 1.2.9.

*Proof of Theorem 1.2.10.* We can repeat the same proof as Theorem 1.2.8 to obtain the following functional inequality:

$$\|f\|_{r;\nu} \leq [C \|\nabla f\|_{2;\mu}]^\vartheta \|f\|_{s;\nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad (1.2.60)$$

where  $f$  is any nonnegative function belonging to  $W_c^{1,\infty}(\Omega)$ ,  $C > 0$  is a suitable constant depending on  $q_0$ ,  $m$ ,  $\lambda$ ,  $K_1$  and

$$\vartheta := \frac{\lambda m(m+1-q_0)}{(m+1)(\lambda m+q_0)}, \quad r := \frac{m+1}{m}, \quad s := \frac{q_0}{m}, \quad q := -\lambda.$$

We are therefore in position to apply Theorem 3.2 of [13], which ensures the validity of (1.2.60) for any  $s > 0$ ,  $\vartheta = \lambda/(s+\lambda)$  and  $r = \infty$ , that is

$$\|f\|_\infty \leq C \|\nabla f\|_{2;\mu}^{\frac{\lambda}{s+\lambda}} \|f\|_{s;\nu}^{\frac{s}{s+\lambda}}, \quad (1.2.61)$$

up to a possibly different constant  $C > 0$  independent of  $s$ . For a signed  $f \in W_c^{1,\infty}(\Omega)$ , inequality (1.2.61) follows by applying the latter both to  $f_+$  and  $f_-$ . A standard mollification argument then allows us to assert its validity for all  $f \in H_{0,c}^1(\Omega; \nu, \mu)$  (recall that  $\rho_\nu$  and  $\rho_\mu$  are locally equivalent to 1). We can finally deal with any  $f \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu)$  by density: given  $\{\phi_n\} \subset \mathcal{D}(\Omega)$  such that  $\phi_n \rightarrow f$  in  $H_0^1(\Omega; \nu, \mu)$ , just notice that the sequence

$$f_n := [f_+ \wedge (\phi_n)_+] - [f_- \wedge (\phi_n)_-] \in H_{0,c}^1(\Omega; \nu, \mu)$$

still converges to  $f$  in  $H_0^1(\Omega; \nu, \mu)$  and it is such that  $\int_\Omega |f_n(x)|^s d\nu \rightarrow \int_\Omega |f(x)|^s d\nu$ , which is enough in order to pass to the limit in (1.2.61).  $\square$

We now pass to the two-dimensional case.

*Proof of Theorem 1.2.11.* Let  $q_0 > 1$ ,  $t > 0$  and take  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ . In order to prove the smoothing estimate (1.2.16), we shall proceed by means of a Moser iteration technique. That is, given two increasing sequences of positive numbers  $\{p_k\}$  (with  $p_0 = q_0$ ) and  $\{t_k\}$ , to be defined later, estimate (1.2.3) with  $(0, T)$  replaced by  $(t_k, t_{k+1})$  and  $q = p_k - 1$  yields

$$\frac{4p_k(p_k - 1)m}{(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \int_{\Omega} \left| \nabla \left( u^{\frac{p_k+m-1}{2}} \right) (x, \tau) \right|^2 d\mu d\tau \leq \|u(t_k)\|_{p_k; \nu}^{p_k}. \quad (1.2.62)$$

Let us rewrite (1.2.15) as

$$\frac{\|v\|_{r; \nu}^{\frac{2r}{r-s}}}{C \|v\|_{s; \nu}^{\frac{2s}{r-s}}} \leq \|\nabla v\|_{2; \mu}^2 \quad \forall r > s > 0, \quad \forall v \in H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu) : v \not\equiv 0, \quad (1.2.63)$$

where  $C := C_{G_2}^{2r/(r-s)}$ . Applying (1.2.63) to the function  $v = u^{(p_0+m-1)/2}(\tau)$  in order to bound the l.h.s. of (1.2.62), we get:

$$\frac{4p_k(p_k - 1)m}{C(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \frac{\left\| u^{\frac{p_k+m-1}{2}}(\tau) \right\|_{r; \nu}^{\frac{2r}{r-s}}}{\left\| u^{\frac{p_k+m-1}{2}}(\tau) \right\|_{s; \nu}^{\frac{2s}{r-s}}} d\tau \leq \|u(t_k)\|_{p_k; \nu}^{p_k}. \quad (1.2.64)$$

Now let  $t_k := (1 - 2^{-k})t$ , so that  $t_0 = 0$ ,  $t_{k+1} - t_k = t/2^{k+1}$  and  $\lim_{k \rightarrow \infty} t_k = t$ . By exploiting (1.2.64) and the fact that  $L^p(\Omega; \nu)$  norms do not increase along the evolution (let  $s(q_0 + m - 1)/2 \geq 1$ ), we infer the inequality

$$t \frac{4p_k(p_k - 1)m}{2^{k+1} C (p_k + m - 1)^2} \frac{\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1); \nu}^{\frac{r}{2}(p_k+m-1)}}{\|u(t_k)\|_{\frac{s}{2}(p_k+m-1); \nu}^{\frac{s}{2}(p_k+m-1)}} \leq \|u(t_k)\|_{p_k; \nu}^{p_k}. \quad (1.2.65)$$

If we choose  $s = s_k = 2p_k/(p_k + m - 1)$ , then (1.2.65) reads

$$\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1); \nu}^{\frac{r}{2}(p_k+m-1)} \leq 2^k C t^{-1} \|u(t_k)\|_{p_k; \nu}^{p_k + \frac{s_k}{r-s_k}(p_k+m-1)}, \quad (1.2.66)$$

where we relabelled  $C$ . It is convenient to set  $r = r_k = s_k + 2$ : in particular, we point out that in this way both  $\{s_k\}$  and  $\{r_k\}$  are bounded sequences, so that the constant  $C$  only depends on  $q_0$ ,  $m$  and on  $C_{G_2}$  through a fixed finite upper bound on  $r$  (note that  $r_k \leq 4$ ).

Assuming that  $\{p_k\}$  satisfies the recurrence relation  $p_{k+1} = 2p_k + m - 1$ , we can rewrite (1.2.66) as follows:

$$\|u(t_{k+1})\|_{p_{k+1}; \nu} \leq 2^{\frac{k}{p_{k+1}}} C^{\frac{1}{p_{k+1}}} t^{-\frac{1}{p_{k+1}}} \|u(t_k)\|_{p_k; \nu}^{\frac{2p_k}{p_{k+1}}}. \quad (1.2.67)$$

Iterating (1.2.67) and using the fact that  $\{p_{k+1}\}$  is explicit, namely

$$p_{k+1} = (q_0 + m - 1) 2^{k+1} + 1 - m,$$

we obtain:

$$\begin{aligned} \|u(t_{k+1})\|_{p_{k+1}; \nu} &\leq 2^{\frac{1}{p_{k+1}} \sum_{h=0}^k 2^h(k-h)} (C t^{-1})^{\frac{1}{p_{k+1}} \sum_{h=0}^k 2^h} \|u_0\|_{q_0; \nu}^{\frac{2^{k+1} q_0}{p_{k+1}}} \\ &\leq 2^{\frac{\sum_{h=0}^k 2^h(k-h)}{(q_0+m-1)2^{k+1}+1-m}} (C t^{-1})^{\frac{2^{k+1}-1}{(q_0+m-1)2^{k+1}+1-m}} \|u_0\|_{q_0; \nu}^{\frac{2^{k+1} q_0}{(q_0+m-1)2^{k+1}+1-m}} \\ &\leq K_1 t^{-\frac{2^{k+1}-1}{(q_0+m-1)2^{k+1}+1-m}} \|u_0\|_{q_0; \nu}^{\frac{2^{k+1} q_0}{(q_0+m-1)2^{k+1}+1-m}}, \end{aligned} \quad (1.2.68)$$

where  $K_1 > 0$  is a suitable constant as in the statement, which *a priori* also depends on  $q_0$ . Recalling that  $p_k \uparrow \infty$  and  $t_k \uparrow t$  as  $k \rightarrow \infty$  and exploiting once again the fact that  $L^p(\Omega; \nu)$  norms do not increase in time, from (1.2.68) we deduce that

$$\|u(t)\|_\infty = \lim_{k \rightarrow \infty} \|u(t)\|_{p_{k+1}; \nu} \leq \liminf_{k \rightarrow \infty} \|u(t_{k+1})\|_{p_{k+1}; \nu} \leq K_1 t^{-\frac{1}{q_0+m-1}} \|u_0\|_{\frac{q_0}{q_0+m-1}; \nu},$$

that is (1.2.16). In order to prove (1.2.17) in the case  $\nu(\Omega) < \infty$  and to remove the hypotheses  $q_0 > 1$  and  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$  one can proceed exactly as in the end of proof of Theorem 1.2.9 above, so we omit details.  $\square$

We finally give a sketch of proof of the converse implication of Theorem 1.2.11.

*Proof of Theorem 1.2.12.* Proceeding again as in the proof of Theorem 1.2.8, we end up with the following functional inequality:

$$\|f\|_{r; \nu} \leq [C \|\nabla f\|_{2; \mu}]^\vartheta \|f\|_{s; \nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s},$$

where  $f$  is any nonnegative function belonging to  $W_c^{1,\infty}(\Omega)$ ,  $C > 0$  is a suitable constant depending on  $q_0, m, K_1$  and

$$\vartheta := \frac{m+1-q_0}{m+1}, \quad r := \frac{m+1}{m}, \quad s := \frac{q_0}{m}, \quad q := \infty.$$

Theorem 3.3 of [13] ensures the validity of (1.2.60) for *all*  $r > s > 0$ ,  $\vartheta = 1 - s/r$  and a constant  $C > 0$  which depends on  $r, s$  only through a finite upper bound on  $r$ . In order to deduce the just proved inequality for the whole of  $H_0^1(\Omega; \nu, \mu) \cap L^s(\Omega; \nu)$ , one can argue as in the proof of Theorem 1.2.10 above.  $\square$

**Remark 1.2.21.** Note that, thanks to the equivalence results of [13], the conclusions of Theorem 1.2.11 (smoothing estimates, two-dimensional case) remain true if, instead of the whole family of Gagliardo-Nirenberg-type inequalities (1.2.15), we require the validity of *one* inequality of such family (for some  $r > s > 0$ ). The same comment applies to Theorem 1.2.9 (one-dimensional case).

### 1.3 The homogeneous Neumann problem

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The aim of this section is to establish the same kind of results we proved above for solutions to the weighted Neumann problem (1.1.2). As we shall see, some of them will follow by means of adaptations of the arguments we used in the Dirichlet case, but most of them (specially as concerns asymptotics) need much more elaborated strategies.

#### 1.3.1 Statements of the main results

Below we only give the notion of weak *energy* solution to (1.1.2). In fact, the natural counterpart of Definition 1.2.1 here would be the one where one chooses *e.g.* test functions in  $C^1(\bar{\Omega} \times [0, T])$ , rather than only in  $C_c^\infty(\Omega \times [0, T])$ . However, since we assume no *a priori* regularity property for  $\Omega$  and  $\rho_\nu, \rho_\mu$  up to  $\partial\Omega$ , these test functions may not be directly related to the underlying weighted Sobolev spaces where it is natural to look for a weak solution (see Subsection 1.1.2 above: in the generality of our discussion, the only available result as concerns density of regular functions is Proposition 1.1.3). It is thus convenient to introduce such spaces in the very definition of weak solution.

**Definition 1.3.1.** A function  $u$  such that:

$$u \in L^{m+1}((0, T); L^{m+1}(\Omega; \nu)), \quad \nabla(u^m) \in L^2((0, T); [L^2(\Omega; \mu)]^d)$$

for a.e.  $t > 0$  and every  $T > 0$ ,

is a weak energy solution to (1.1.2) with initial datum  $u_0 \in L^{m+1}(\Omega; \nu)$  if it satisfies

$$\int_0^T \int_{\Omega} u(x, t) \varphi_t(x, t) \, d\nu dt = - \int_{\Omega} u_0(x) \varphi(x, 0) \, d\nu + \int_0^T \int_{\Omega} \langle \nabla(u^m), \nabla \varphi \rangle(x, t) \, d\mu dt \quad (1.3.1)$$

for all  $\varphi \in W^{1+1/m}((0, T); L^{1+1/m}(\Omega; \nu))$  with  $\nabla \varphi \in L^2((0, T); [L^2(\Omega; \mu)]^d)$  and  $\varphi(T) \equiv 0$ .

It is apparent that the above properties entail

$$u^m \in L^{1+1/m}((0, T); V^{1+1/m}(\Omega; \nu, \mu)) \quad \forall T > 0.$$

Notice that now it is *by definition* admissible to pick Oleřnik's test function (1.2.19) in (1.3.1). Hence, repeating the proof of Proposition 1.2.3, we get the same result in the Neumann case.

**Proposition 1.3.2.** *There exists at most one weak energy solution to (1.1.2).*

When  $\nu$  is a finite measure, we denote as  $\bar{f}$  the weighted mean value of a function  $f \in L^1(\Omega; \nu)$ , that is

$$\bar{f} := \frac{\int_{\Omega} f(x) \, d\nu}{\nu(\Omega)}.$$

In our short- and long-time analysis of the behaviour of solutions to (1.1.2), we shall focus on the case  $\nu(\Omega) < \infty$ . In this regard, a classical key property is the *conservation of (signed) mass or mean value*.

**Proposition 1.3.3.** *Let  $\nu(\Omega) < \infty$ . If  $u$  is a weak energy solution to (1.1.2) then*

$$\overline{u(t)} = \bar{u}_0 = \bar{u} \quad \text{for a.e. } t > 0.$$

*Proof.* Thanks to the assumptions, for any fixed  $t > 0$  and  $h \in (0, 2t)$  we can plug in (1.3.1) the following test functions (independent of  $x$ ):

$$\varphi_h(s) := \chi_{[0, t-h/2)}(s) + \chi_{[t-h/2, t+h/2]}(s) \left( \frac{t-s}{h} + \frac{1}{2} \right) \quad \forall s \in \mathbb{R}^+,$$

from which we get the identity

$$\frac{1}{h} \int_{t-h/2}^{t+h/2} \int_{\Omega} u(x, s) \, d\nu ds = \int_{\Omega} u_0(x) \, d\nu.$$

In order to conclude the proof we let  $h \rightarrow 0$  and use Lebesgue's differentiation Theorem.  $\square$

As for existence, we have the analogue of Theorem 1.2.4.

**Theorem 1.3.4.** *Let  $\rho_{\nu}, \rho_{\mu}, \rho_{\nu}^{-1}, \rho_{\mu}^{-1} \in L_{\text{loc}}^{\infty}(\Omega)$ . Take  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , with  $q_0 \geq m+1$ . Then there exists the weak energy solution  $u$  to (1.1.2), in the sense of Definition 1.3.1, which satisfies the estimates*

$$\frac{4q(q+1)m}{(m+q)^2} \int_0^T \int_{\Omega} \left| \nabla \left( u^{\frac{m+q}{2}} \right) (x, t) \right|^2 \, d\mu dt + \int_{\Omega} |u(x, T)|^{q+1} \, d\nu \leq \int_{\Omega} |u_0(x)|^{q+1} \, d\nu \quad (1.3.2)$$

and

$$\int_0^T \int_{\Omega} \zeta(t) |z_t(x, t)|^2 \, d\nu dt \leq \max_{t \in [0, T]} \zeta'(t) \frac{m+1}{8m} \int_{\Omega} |u_0(x)|^{m+1} \, d\nu \quad (1.3.3)$$

for all  $q \in [0, q_0 - 1]$  and  $T > 0$ , where  $z := u^{(m+1)/2}$  and  $\zeta$  is any nonnegative  $C_c^1((0, T))$  function.

Moreover, if  $v$  is the weak energy solution corresponding to another initial datum  $v_0 \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$ , the following  $L^1(\Omega; \nu)$  comparison principle holds true:

$$\int_{\Omega} [u(x, t) - v(x, t)]_+ \, d\nu \leq \int_{\Omega} [u_0(x) - v_0(x)]_+ \, d\nu \quad \forall t > 0. \quad (1.3.4)$$

Thanks to (1.3.4), for general  $L^1(\Omega; \nu)$  data we can define limit solutions exactly as in Subsection 1.2.1.

**Remark 1.3.5.** In the case where

$$V^{1+1/m}(\Omega; \nu, \mu) = V_0^{1+1/m}(\Omega; \nu, \mu),$$

it is plain that the two notions of energy solution provided by Definitions 1.2.2 and 1.3.1 coincide. Hence, in view of the above existence and uniqueness results, the energy solution to the Dirichlet problem with initial datum  $u_0 \in L^1(\Omega; \nu) \cap L^{m+1}(\Omega; \nu)$  is also the energy solution to the Neumann problem with the same initial datum, and vice versa. We can draw an analogous conclusion if

$$H^1(\Omega; \nu, \mu) = H_0^1(\Omega; \nu, \mu). \quad (1.3.5)$$

Actually, in such case, we cannot state that Definitions 1.2.2 and 1.3.1 are equivalent. Nevertheless, thanks to estimate (1.3.2), we know that for any  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$  the corresponding energy solution to the Neumann problem is such that  $u^m(t) \in V^{1+1/m}(\Omega; \nu, \mu) \cap L^2(\Omega; \nu)$  for a.e.  $t > 0$ . It is direct to check that, in view of (1.3.5), this entails

$$u^m(t) \in V_0^{1+1/m}(\Omega; \nu, \mu).$$

Hence, the energy solution to the Neumann problem is also the energy solution to the Dirichlet problem, so that such solutions coincide in this case as well. For a general  $u_0 \in L^1(\Omega; \nu)$ , the same results hold by taking limits (recall (1.3.4)).

If the above functional equivalences occur, we can claim that the Dirichlet and the Neumann problems (1.1.1) and (1.1.2) are indistinguishable to one another. In this regard, a classical example is the Euclidean space  $\Omega = \mathbb{R}^d$  with the choices  $\rho_\mu \equiv 1$  and  $\rho_\nu(x) = |x|^{-\gamma}$  for some  $\gamma \in [0, 2)$ .

Below we state the results which connect the validity of suitable functional inequalities to smoothing estimates for solutions to (1.1.2) in the case of finite measure, namely  $\nu(\Omega) < \infty$ . By the term “the solution”, we mean either the weak energy solution provided by Theorem 1.3.4 or the limit solution, depending on whether  $u_0 \in L^{m+1}(\Omega; \nu)$  or not. The implicit assumptions on the weights are local boundedness and boundedness away from zero, just as in the statement of Theorem 1.3.4.

**Theorem 1.3.6** (Poincaré case). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the Poincaré-type inequality*

$$\|v\|_{2; \nu} \leq W_P \left( \|\nabla v\|_{2; \mu} + \|v\|_{1; \nu} \right) \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.6)$$

holds true for some  $W_P > 0$ . Then the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies the smoothing estimate

$$\|u(t)\|_{\varrho;\nu} \leq K \left( t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0 \quad (1.3.7)$$

for all  $\varrho \in (q_0, \infty)$ , where  $K > 0$  is a suitable constant depending only on  $\varrho$ ,  $m$ ,  $W_P$  and  $\nu(\Omega)$ .

**Theorem 1.3.7** (Poincaré case, converse implication). *Let  $\nu(\Omega) < \infty$ . Suppose that there exist a constant  $K > 0$  and a number  $\varrho \geq m+1$  such that, for all  $u_0 \in L^m(\Omega; \nu)$ , the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies estimate (1.3.7) for  $q_0 = m$ . Then the Poincaré-type inequality (1.3.6) holds true. In particular, the validity of (1.3.7) for  $q_0 = m$  and a given  $\varrho \geq m+1$  is equivalent to (1.3.6) and so to the validity of (1.3.7) for any  $q_0 \in [1, \infty)$  and  $\varrho \in (q_0, \infty)$ .*

**Theorem 1.3.8** (Sobolev case). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the Sobolev-type inequality*

$$\|v\|_{2\sigma;\nu} \leq W_S \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right) \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.8)$$

holds true for some  $\sigma > 1$  and  $W_S > 0$ . Then the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies the smoothing estimate

$$\|u(t)\|_{\infty} \leq K \left( t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0+\sigma(m-1)}} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0, \quad (1.3.9)$$

where  $K > 0$  is a suitable constant depending only on  $m$ ,  $W_S$ ,  $\sigma$  and  $\nu(\Omega)$ .

**Theorem 1.3.9** (Sobolev case, converse implication). *Let  $\nu(\Omega) < \infty$ . Suppose that there exist a constant  $K > 0$  and a number  $\sigma > 1$  such that, for all  $u_0 \in L^m(\Omega; \nu)$ , the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies estimate (1.3.9) for  $q_0 = m$ . Then the Sobolev-type inequality (1.3.8) holds true. In particular, the validity of (1.3.9) for  $q_0 = m$  is equivalent to (1.3.8) and so to the validity of (1.3.9) for any  $q_0 \in [1, \infty)$ .*

**Theorem 1.3.10** (One-dimensional case). *Let  $d = 1$  (that is,  $\Omega$  is an interval). Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the family of one-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v\|_{\infty} \leq W_{G_1} \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right)^{\frac{\lambda}{s+\lambda}} \|v\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.10)$$

holds true for some  $\lambda \geq 2$  and  $W_{G_1} > 0$ . Then the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies the estimate

$$\|u(t)\|_{\infty} \leq K \left( t^{-\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{(\lambda+2)q_0+\lambda(m-1)}} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0, \quad (1.3.11)$$

where  $K > 0$  is a suitable constant depending only on  $m$ ,  $W_{G_1}$ ,  $\lambda$  and  $\nu(\Omega)$ .

**Theorem 1.3.11** (One-dimensional case, converse implication). *Let  $\nu(\Omega) < \infty$ . Suppose that there exist a constant  $K > 0$  and a number  $\lambda \geq 2$  such that, for all  $u_0 \in L^m(\Omega; \nu)$ , the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies estimate (1.3.11) for  $q_0 = m$ . Then the family of one-dimensional Gagliardo-Nirenberg-type inequalities (1.3.10) holds true. In particular, the validity of (1.3.11) for  $q_0 = m$  is equivalent to (1.3.10) and so to the validity of (1.3.11) for any  $q_0 \in [1, \infty)$ .*

**Theorem 1.3.12** (Two-dimensional case). *Let  $d \leq 2$ . Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the family of two-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v\|_{r;\nu} \leq W_{G_2} \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right)^{1-\frac{s}{r}} \|v\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.12)$$

*holds true for some constant  $W_{G_2} > 0$  which depends on  $r, s$  only through a finite upper bound on  $r$ . Then the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies the smoothing estimate*

$$\|u(t)\|_{\infty} \leq K \left( t^{-\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}} + \|u_0\|_{q_0;\nu} \right) \quad \forall t > 0, \quad (1.3.13)$$

*where  $K > 0$  is a suitable constant depending only on  $m, W_{G_2}$  and  $\nu(\Omega)$ .*

**Theorem 1.3.13** (Two-dimensional case, converse implication). *Let  $\nu(\Omega) < \infty$ . Suppose that there exists a constant  $K > 0$  such that, for all  $u_0 \in L^m(\Omega; \nu)$ , the solution  $u$  to (1.1.2) with initial datum  $u_0$  satisfies estimate (1.3.13) for  $q_0 = m$ . Then the family of two-dimensional Gagliardo-Nirenberg-type inequalities (1.3.12) holds true. In particular, the validity of (1.3.13) for  $q_0 = m$  is equivalent to (1.3.12) and so to the validity of (1.3.13) for any  $q_0 \in [1, \infty)$ .*

If stronger functional inequalities hold true, then all the above estimates can be improved for large times, in the sense that we can prove convergence of  $u(t)$  to the mean value  $\bar{u}$  with rates, which are different depending on whether  $\bar{u} = 0$  or  $\bar{u} \neq 0$ .

**Theorem 1.3.14** (Poincaré case,  $\bar{u} = 0$ ). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the Poincaré-type inequality*

$$\|v - \bar{v}\|_{2;\nu} \leq M_P \|\nabla v\|_{2;\mu} \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.14)$$

*holds true for some  $M_P > 0$ . If in addition  $\bar{u}_0 = 0$ , then the smoothing estimate (1.3.7) can be improved as follows:*

$$\|u(t)\|_{q;\nu} \leq K t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \left( t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{-\frac{q_0}{(m-1)\varrho}} \quad \forall t > 0, \quad (1.3.15)$$

*where  $K > 0$  is another constant depending on  $\varrho, q_0, m, \Omega, \nu$  and  $\mu$ . In particular, the absolute bound*

$$\|u(t)\|_{q;\nu} \leq K t^{-\frac{1}{m-1}} \quad \forall t > 0$$

*holds true.*

**Theorem 1.3.15** (Sobolev case,  $\bar{u} = 0$ ). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the Sobolev-type inequality*

$$\|v - \bar{v}\|_{2\sigma;\nu} \leq M_S \|\nabla v\|_{2;\mu} \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.16)$$

*holds true for some  $\sigma > 1$  and  $M_S > 0$ . If in addition  $\bar{u}_0 = 0$ , then the smoothing estimate (1.3.9) can be improved as follows:*

$$\|u(t)\|_{\infty} \leq K t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \left( t + \|u_0\|_{q_0;\nu}^{1-m} \right)^{-\frac{(\sigma-1)q_0}{(m-1)[(\sigma-1)q_0+\sigma(m-1)]}} \quad \forall t > 0, \quad (1.3.17)$$

*where  $K > 0$  is another suitable constant depending on  $q_0, m, \Omega, \nu$  and  $\mu$ . In particular, the absolute bound*

$$\|u(t)\|_{\infty} \leq K t^{-\frac{1}{m-1}} \quad \forall t > 0 \quad (1.3.18)$$

*holds true.*

**Theorem 1.3.16** (One-dimensional case,  $\bar{u} = 0$ ). *Let  $d = 1$ ,  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the family of one-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v - \bar{v}\|_\infty \leq M_{G_1} \|\nabla v\|_{2;\mu}^{\frac{\lambda}{s+\lambda}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.19)$$

*holds true for some  $M_{G_1} > 0$ . If in addition  $\bar{u}_0 = 0$ , then the smoothing estimate (1.3.11) can be improved as follows:*

$$\|u(t)\|_\infty \leq K t^{-\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{(\lambda+2)q_0}{(m-1)[(\lambda+2)q_0+\lambda(m-1)]}} \quad \forall t > 0, \quad (1.3.20)$$

*where  $K > 0$  is another suitable constant depending on  $q_0$ ,  $m$ ,  $\Omega$ ,  $\nu$  and  $\mu$ . In particular, the absolute bound (1.3.18) holds true.*

**Theorem 1.3.17** (Two-dimensional case,  $\bar{u} = 0$ ). *Let  $d \leq 2$ ,  $\nu(\Omega) < \infty$  and  $u_0 \in L^{q_0}(\Omega; \nu)$ , with  $q_0 \in [1, \infty)$ . Suppose that the family of two-dimensional Gagliardo-Nirenberg-type inequalities*

$$\|v - \bar{v}\|_{r;\nu} \leq M_{G_2} \|\nabla v\|_{2;\mu}^{1-\frac{s}{r}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.21)$$

*holds true for some constant  $M_{G_2} > 0$  which depends on  $r, s$  only through a finite upper bound on  $r$ . If in addition  $\bar{u}_0 = 0$ , then the smoothing estimate (1.3.13) can be improved as follows:*

$$\|u(t)\|_\infty \leq K t^{-\frac{1}{q_0+m-1}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{q_0}{(m-1)(q_0+m-1)}} \quad \forall t > 0, \quad (1.3.22)$$

*where  $K > 0$  is another suitable constant depending on  $q_0$ ,  $m$ ,  $\Omega$ ,  $\nu$  and  $\mu$ . In particular, the absolute bound (1.3.18) holds true.*

**Theorem 1.3.18** (Poincaré case,  $\bar{u} \neq 0$ ). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^1(\Omega; \nu)$ , with  $\bar{u}_0 \neq 0$ . Suppose that the Poincaré-type inequality (1.3.14) holds true. Then*

$$\|u(t) - \bar{u}\|_{\varrho;\nu} \leq K_1 t^{-\frac{1}{m-1}} \quad \forall \varrho \in [1, 2], \quad \forall t > 0 \quad (1.3.23)$$

*and*

$$\|u(t) - \bar{u}\|_{\varrho;\nu} \leq K_2 t^{-\frac{2}{\varrho(m-1)}+\varepsilon} \quad \forall \varrho \in (2, \infty), \quad \forall \varepsilon \in \left(0, \frac{2}{\varrho(m-1)}\right), \quad \forall t \geq 1, \quad (1.3.24)$$

*where  $K_1 > 0$  is a constant depending only on  $\varrho$ ,  $m$ ,  $M_P$ ,  $\nu(\Omega)$ , while  $K_2 > 0$  is a constant depending on  $\varepsilon$ ,  $\varrho$ ,  $m$ ,  $M_P$ ,  $\nu(\Omega)$  and  $\|u_0\|_{1;\nu}$ .*

*If in addition  $u_0 \in L^\infty(\Omega)$ , then  $u$  converges exponentially to its mean value. More precisely:*

$$\|u(t) - \bar{u}\|_{\varrho;\nu} \leq e^{-C|\bar{u}|^{m-1}t} \|u_0 - \bar{u}\|_{\varrho;\nu} \quad \forall \varrho \in (1, \infty), \quad \forall t \geq 0, \quad (1.3.25)$$

*where  $C > 0$  is a suitable constant (unknown) that depends on  $\varrho$ ,  $m$ ,  $\Omega$ ,  $\nu$ ,  $\mu$  and on  $u_0$  only through a finite upper bound on  $\|u_0 - \bar{u}\|_\infty/|\bar{u}|$ . Furthermore, in the case where  $\nabla \rho_\nu \in L^\infty_{\text{loc}}(\Omega)$ , exponential convergence to the mean value is locally uniform. That is, for any  $\Omega_K \Subset \Omega$  there holds*

$$\|u(t) - \bar{u}\|_{L^\infty(\Omega_K)} \leq G_K e^{-C_K t} \quad \forall t \geq 1, \quad (1.3.26)$$

*where  $G_K, C_K > 0$  are suitable constants depending on  $m$ ,  $\Omega$ ,  $\Omega_K$ ,  $\rho_\nu$ ,  $\rho_\mu$ ,  $|\bar{u}|$  and  $\|u_0\|_\infty$ .*



If Sobolev- or Gagliardo-Nirenberg-type inequalities hold true, then the results of Theorem 1.3.18 can be considerably improved.

**Theorem 1.3.19** (Sobolev or Gagliardo-Nirenberg case,  $\bar{u} \neq 0$ ). *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^1(\Omega; \nu)$ , with  $\bar{u}_0 \neq 0$ . Suppose that one between the Sobolev-type inequality (1.3.16), the one-dimensional Gagliardo-Nirenberg-type inequality (1.3.19) and the two-dimensional Gagliardo-Nirenberg-type inequality (1.3.21) holds true. Then*

$$\|u(t) - \bar{u}\|_\infty \leq G e^{-\frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall t \geq 1, \quad (1.3.27)$$

where  $G > 0$  is a suitable constant depending on  $m, \Omega, \nu, \mu, |\bar{u}|$  and  $\|u_0\|_{1;\nu}$ , whereas  $M_P$  is the best constant appearing in the Poincaré-type inequality (1.3.14).

In the particular framework of the Neumann problem in regular bounded domains (with, for simplicity,  $\rho_\nu = \rho_\mu \equiv 1$ ), estimate (1.3.27) is *sharp*.

**Proposition 1.3.20.** *Let  $\Omega$  be a  $C^\infty$  bounded domain of  $\mathbb{R}^d$ . Consider the first nonzero eigenvalue  $\lambda_1 = 1/M_P^2$  of (minus) the Laplacian with homogeneous Neumann boundary conditions, and the corresponding set of eigenfunctions (possibly consisting of one single element) normalized in  $L^2(\Omega)$ . Pick any of such eigenfunctions and denote it by  $\psi_1$ . Given a constant  $\bar{u} \neq 0$ , let  $u$  be the solution to (1.1.2), with  $\rho_\nu = \rho_\mu \equiv 1$ , corresponding to the initial datum  $u_0 = \bar{u} + c_1 \psi_1$ , with  $|c_1| > 0$  small enough. Then there exists a constant  $\underline{G} > 0$ , which depends on  $c_1, m, \Omega$  and  $\bar{u}$ , such that*

$$\|u(t) - \bar{u}\|_\infty \geq \underline{G} e^{-\frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall t \geq 0. \quad (1.3.28)$$

Theorem 1.3.4 will be proved in the next Subsection 1.3.2. In Subsection 1.3.3 we deal with smoothing estimates. Theorems 1.3.6 and 1.3.7 will be proved in Paragraph 1.3.3.1. In Paragraph 1.3.3.2 we provide a counterexample to the  $L^\infty$  regularization in the Poincaré case. Theorems 1.3.8 and 1.3.9 will be proved in Paragraph 1.3.3.3, whereas Theorems 1.3.10–1.3.13 will be proved in Paragraph 1.3.3.4. In Paragraph 1.3.3.5 we discuss to what extent our results improve on previous ones, and show some contexts where they are sharp. In Subsection 1.3.4 we analyse the long-time behaviour of solutions. Theorems 1.3.14 and 1.3.18 will be proved in Paragraph 1.3.4.1. In Paragraph 1.3.4.2 we provide counterexamples, in the Poincaré case, to the uniform convergence of solutions to their mean value. Theorems 1.3.15, 1.3.16, 1.3.17 and 1.3.19 will be proved in Paragraph 1.3.4.3. Finally, in Paragraph 1.3.4.4 we shall prove Proposition 1.3.20 and discuss improvements of our results, as concerns asymptotics, with respect to previous ones.

**Remark 1.3.21.** The conclusions of Theorems 1.3.15, 1.3.16, 1.3.17 and 1.3.19 remain true if, instead of (1.3.16), (1.3.19) or (1.3.21), we require the validity of the weaker inequalities (1.3.8), (1.3.10) or (1.3.12) *plus* the (zero-mean) Poincaré-type inequality (1.3.14).

In fact, it is plain that the Sobolev-type inequality (1.3.8) plus the (zero-mean) Poincaré-type inequality (1.3.14) is *equivalent* to the (zero-mean) Sobolev-type inequality (1.3.16). Such equivalence also holds for Gagliardo-Nirenberg-type inequalities. Let us prove it in the two-dimensional context (in dimension one the proof is even easier). If the family (1.3.21) holds true, then by picking  $r = 2$  and  $s = 1$  we get:

$$\|v - \bar{v}\|_{2;\nu} \leq M_{G_2} \|\nabla v\|_{2;\mu}^{\frac{1}{2}} \quad \|v - \bar{v}\|_{1;\nu}^{\frac{1}{2}} \leq M_{G_2} \nu(\Omega)^{\frac{1}{4}} \|\nabla v\|_{2;\mu}^{\frac{1}{2}} \|v - \bar{v}\|_{2;\nu}^{\frac{1}{2}} \quad \forall v \in H^1(\Omega; \nu, \mu),$$

that is

$$\|v - \bar{v}\|_{2;\nu} \leq M_{G_2}^2 \nu(\Omega)^{\frac{1}{2}} \|\nabla v\|_{2;\mu} \quad \forall v \in H^1(\Omega; \nu, \mu).$$

Moreover, still (1.3.21) with  $r = 2$  and  $s = 1$  entails

$$\|v\|_{2;\nu} \leq \max \left\{ \sqrt{2} M_{G_2}, \nu(\Omega)^{-\frac{1}{2}} \right\} \left( \|\nabla v\|_{2;\mu} + \|v\|_{1;\nu} \right)^{\frac{1}{2}} \|v\|_{1;\nu}^{\frac{1}{2}} \quad \forall v \in H^1(\Omega; \nu, \mu),$$

from which we recover the whole family (1.3.12) thanks to [13, Theorem 3.3]. On the other hand, the validity of (1.3.12) (applied to  $v - \bar{v}$ ) plus the (zero-mean) Poincaré-type inequality (1.3.14) yields

$$\begin{aligned} \|v - \bar{v}\|_{r;\nu} &\leq W_{G_2} \left( \|\nabla v\|_{2;\mu} + \|v - \bar{v}\|_{1;\nu} \right)^{1 - \frac{s}{r}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{r}} \\ &\leq W_{G_2} \left( 1 + \nu(\Omega)^{\frac{1}{2}} M_P \right)^{1 - \frac{s}{r}} \|\nabla v\|_{2;\mu}^{1 - \frac{s}{r}} \|v - \bar{v}\|_{s;\nu}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu). \end{aligned}$$

**Remark 1.3.22.** As the reader may notice, the smoothing estimates (1.3.7), (1.3.9), (1.3.11) and (1.3.13) coincide with the corresponding estimates for the Dirichlet problem *plus* a constant times  $\|u_0\|_{q_0;\nu}$ . Let us mention that, at least when also a zero-mean Poincaré-type inequality holds true (see the above remark), such an extra term is necessary: in fact, without it, any solution would be forced to converge to zero, which is in contradiction with Theorems 1.3.18 and 1.3.19 in the case  $\bar{u} \neq 0$ .

**Remark 1.3.23.** The comments of Remarks 1.2.14 and 1.2.15 above also apply to the present context. As already mentioned, our results are new and improve on previous ones (in particular, we refer to [3] and [27]), even in the non-weighted case.

### 1.3.2 Well-posedness analysis

The proof of existence of weak energy solutions follows an approximation procedure similar to the one used in the proof of the analogous result for the Dirichlet problem (Theorem 1.2.4).

*Proof of Theorem 1.3.4.* Given a sequence of smooth functions  $\{\Phi'_n(y)\}$  which approximates  $m|y|^{m-1}$  as in Lemma 1.2.16 and a fixed  $C^{2,\alpha}$  bounded domain  $\Omega_0 \Subset \Omega$ , we solve the following Neumann problems:

$$\begin{cases} (u_n)_t = \rho_\nu^{-1} \operatorname{div} [\rho_\mu \nabla(\Phi_n(u_n))] & \text{in } \Omega_0 \times \mathbb{R}^+, \\ \frac{\partial(\Phi_n(u_n))}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega_0 \times \mathbb{R}^+, \\ u_n = u_0 & \text{on } \Omega_0 \times \{0\}, \end{cases} \quad (1.3.29)$$

where, for the moment, we assume in addition  $\rho_\nu \in C_{\text{loc}}^{3,\alpha}(\Omega)$ ,  $\rho_\mu \in C_{\text{loc}}^{2,\alpha}(\Omega)$  and  $u_0 \in C^{2,\alpha}(\bar{\Omega}_0)$  with  $\frac{\partial u_0}{\partial \mathbf{n}} = 0$  on  $\partial\Omega_0$ . Setting  $w = \rho_\nu u_n$ , let us rewrite (1.3.29) in divergence form:

$$\begin{cases} w_t = \operatorname{div} \left[ \frac{\rho_\mu}{\rho_\nu} \Phi'_n \left( \frac{w}{\rho_\nu} \right) \nabla w - \frac{\rho_\mu}{\rho_\nu^2} \nabla(\rho_\nu) \Phi'_n \left( \frac{w}{\rho_\nu} \right) w \right] & \text{in } \Omega_0 \times \mathbb{R}^+, \\ \Phi'_n \left( \frac{w}{\rho_\nu} \right) \left[ \nabla w - \frac{w}{\rho_\nu} \nabla \rho_\nu \right] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_0 \times \mathbb{R}^+, \\ w = \rho_\nu u_0 & \text{on } \Omega_0 \times \{0\}. \end{cases} \quad (1.3.30)$$

Quasilinear theory (see *e.g.* [125, Theorem 13.24]) ensures that problem (1.3.30) admits a solution  $w \in C^{2,1}(\bar{\Omega}_0 \times [0, T])$  with  $w_t \in C^{1,0}(\Omega_0 \times (0, T))$  for all  $T > 0$ . Hence,  $u_n$  is a solution to (1.3.29) with the same regularity as  $w$ . Taking advantage of the upper bound

$$|u_n(x, t)| \leq \|u_0\|_\infty \quad \forall (x, t) \in \Omega_0 \times \mathbb{R}^+$$

and performing exactly the same computations as in the Dirichlet case, we can pass to the limit as  $n \rightarrow \infty$  to get a weak energy solution  $u$  to (1.1.2) (with  $\Omega = \Omega_0$ ), in the sense of

Definition 1.3.1, which satisfies estimates (1.3.2) and (1.3.3). The validity of the  $L^1(\Omega_0; \nu)$  comparison principle (1.3.4) is ensured by repeating simultaneously the same approximation scheme for another initial datum  $v_0$  (again, the formal idea is to use the test function  $\text{sign}_+[\Phi_n(u_n) - \Phi_n(v_n)]$ ).

Now notice that, proceeding as in the end of proof of Lemma 1.2.16 and as in the beginning of proof of Theorem 1.2.4, there is no difficulty in extending the above result to weights  $\rho_\nu, \rho_\mu$  as in the statement and to data  $u_0 \in L^1(\Omega_0; \nu) \cap L^\infty(\Omega_0)$ . The crucial point is the possibility to consider general domains. To this end, take an initial datum  $u_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ , an increasing sequence of smooth bounded domains  $\{\Omega_n\}$  such that  $\bigcup_n \Omega_n = \Omega$ ,  $\Omega_n \Subset \Omega$  and solve in each of them the homogeneous Neumann problem (1.1.2) with initial datum  $u_{0n} = u_0|_{\Omega_n}$ , denoting as  $\{u_n\}$  the corresponding sequence of solutions. We point out that an admissible test function  $\varphi$  for the Neumann problem in  $\Omega_n$  is also an admissible test function for the Neumann problem in  $\Omega$  (recall Definition 1.3.1). Hence, upon setting

$$f_n := u_n \chi_{\Omega_n}, \quad \mathbf{g}_n := \nabla(u_n^m) \chi_{\Omega_n},$$

we infer that

$$\int_0^T \int_\Omega f_n(x, t) \varphi_t(x, t) \, d\nu dt = - \int_\Omega u_0(x) \varphi(x, 0) \chi_{\Omega_n}(x) \, d\nu + \int_0^T \int_\Omega \langle \mathbf{g}_n, \nabla \varphi \rangle(x, t) \, d\mu dt \quad (1.3.31)$$

for any  $T > 0$  and  $\varphi$  as in Definition 1.3.1. Exploiting estimates (1.2.28) and (1.2.29) as in the Dirichlet case (let  $\Omega = \Omega_n$  there, since  $u_n$  has a gradient only inside  $\Omega_n$ ), we can assert that  $\{f_n\}$  and  $\{\mathbf{g}_n\}$  converge pointwise and weakly in  $L^{m+1}((0, T); L^{m+1}(\Omega; \nu))$  to a function  $u$  and weakly in  $L^2((0, T); [L^2(\Omega; \mu)]^d)$  to a vector valued function  $\mathbf{g}$  (up to subsequences), respectively. This is enough in order to pass to the limit in (1.3.31) as  $n \rightarrow \infty$  to get

$$\int_0^T \int_\Omega u(x, t) \varphi_t(x, t) \, d\nu dt = - \int_\Omega u_0(x) \varphi(x, 0) \, d\nu + \int_0^T \int_\Omega \langle \mathbf{g}, \nabla \varphi \rangle(x, t) \, d\mu dt.$$

It therefore remains to prove that  $\mathbf{g} = \nabla(u^m)$ . First of all note that, given any  $\Omega_0 \Subset \Omega$ , we have  $\mathbf{g}_n|_{\Omega_0} = \nabla(u_n^m|_{\Omega_0})$  for all  $n$  large enough. Since  $u_n^m|_{\Omega_0} \rightarrow u^m|_{\Omega_0}$  weakly in  $L^2((0, T); H^1(\Omega_0))$  (recall estimate (1.2.28) and that  $u_n \rightarrow u$  pointwise) and at the same time  $\mathbf{g}_n|_{\Omega_0} \rightarrow \mathbf{g}|_{\Omega_0}$  weakly in  $L^2((0, T); [L^2(\Omega_0)]^d)$ , there holds  $\mathbf{g}|_{\Omega_0} = \nabla(u^m|_{\Omega_0})$ . The inner domain  $\Omega_0$  being arbitrary, the assertion follows.

Finally, the validity of estimates (1.3.2) and (1.3.3) is a consequence of the above convergence results, just as in the end of proof of Theorem 1.2.4. The validity of the  $L^1(\Omega; \nu)$  comparison principle can be established by performing simultaneously the same approximation procedure for another initial datum  $v_0 \in L^1(\Omega; \nu) \cap L^\infty(\Omega)$ . In order to handle general data  $u_0 \in L^1(\Omega; \nu) \cap L^{q_0}(\Omega; \nu)$ , with  $q_0 \geq m + 1$ , one argues again as in the end of proof of Theorem 1.2.4.  $\square$

For more regular initial data, we have the analogue of Corollary 1.2.18.

**Corollary 1.3.24.** *If, in addition to the hypotheses of Theorem 1.3.4, we assume that*

$$u_0^m \in V^{1+1/m}(\Omega; \nu, \mu),$$

*then the estimate*

$$\int_0^T \int_\Omega \left| \left( u^{\frac{m+1}{2}} \right)_t(x, t) \right|^2 \, d\nu dt + \frac{(m+1)^2}{8m} \int_\Omega |\nabla(u^m)(x, T)|^2 \, d\mu \leq \frac{(m+1)^2}{8m} \int_\Omega |\nabla(u_0^m)(x)|^2 \, d\mu \quad (1.3.32)$$

holds true for all  $T > 0$ . In particular,  $u^{(m+1)/2}$  is an absolutely continuous curve in  $C([0, \infty); L^2(\Omega; \nu))$ . If moreover  $\nu(\Omega) < \infty$ , then also the estimate

$$\int_{\Omega} |u_0(x)|^{m+1} d\nu \leq \int_{\Omega} |u(x, t)|^{m+1} d\nu + (m+1)t \int_{\Omega} |\nabla(u_0^m)(x)|^2 d\mu \quad (1.3.33)$$

holds true for all  $t > 0$ .

*Proof.* If  $\Omega_0 \Subset \Omega$  is a  $C^{2,\alpha}$  bounded domain and  $u_0 \in C^{2,\alpha}(\overline{\Omega}_0)$  with  $\frac{\partial u_0}{\partial \mathbf{n}} = 0$  on  $\partial\Omega_0$ , estimates (1.3.32) and (1.3.33) (with  $\Omega = \Omega_0$ ) are readily inherited from analogous estimates that we can obtain for the corresponding solutions to the approximate non-degenerate problems (1.3.29). For an initial datum  $u_0 \in L^\infty(\Omega)$  such that  $u_0^m \in V^{1+1/m}(\Omega; \nu, \mu)$ , by means of a similar argument as in the proof of Corollary 1.2.18 we can pick a sequence  $\{u_{0n}\} \subset C_c^{2,\alpha}(\Omega)$ , with  $\frac{\partial u_{0n}}{\partial \mathbf{n}} = 0$  on  $\partial\Omega_0$ , such that  $u_{0n} \rightarrow u_0$  in  $L^p(\Omega_0; \nu)$  for all  $p \in [1, \infty)$  and

$$u_{0n}^m \rightarrow u_0^m \quad \text{in } V^{1+1/m}(\Omega_0; \nu, \mu).$$

Denoting as  $\{u_n\}$  the corresponding sequence of energy solutions to (1.1.2), no difficulty arises in passing to the limit in (1.3.32) and (1.3.33) (with  $u = u_n$  and  $\Omega = \Omega_0$ ) as  $n \rightarrow \infty$  to recover the validity of such estimates for  $u$  as well, the latter being the energy solution in  $\Omega_0$  with initial datum  $u_0$ . Now we replace  $\Omega_0$  with an increasing sequence of domains  $\{\Omega_n\}$  as in the proof of Theorem 1.3.4: thanks to the latter, we know that the corresponding sequence of energy solutions  $\{u_n\}$  to (1.1.2) in  $\Omega_n$  suitably converges to the energy solution  $u$  in  $\Omega$ . In particular, convergence takes place a.e. in  $\Omega \times \mathbb{R}^+$  and  $\|u_n\|_\infty \leq \|u_0\|_\infty$ , which suffices in order to pass to the limit in (1.3.32) and (1.3.33) (on the first term in the r.h.s. of (1.3.33) we exploit the finiteness of the measure). Finally, the assumption  $u_0 \in L^\infty(\Omega)$  can be removed by considering the sequence of initial data

$$u_{0n} := [(u_0)_+ \wedge n] - [(u_0)_- \wedge n]$$

and proceeding as above (using in addition the  $L^1(\Omega; \nu)$  comparison principle).  $\square$

We point out that, in view of estimate (1.3.3), weak energy solutions to the Neumann problem are again  $C((0, \infty); L^{m+1}(\Omega; \nu))$  curves, with all related implications as in the Dirichlet case. In particular, this is the reason why we are allowed to state our main results for every  $T, t > 0$  rather than only for *almost every*  $T, t > 0$  (*a posteriori*, this remark applies to Proposition 1.3.3 as well).

### 1.3.3 Smoothing estimates

In this subsection we shall prove Theorems 1.3.6–1.3.13, which connect the validity of Poincaré-, Sobolev- or Gagliardo-Nirenberg-type inequalities in  $H^1(\Omega; \nu, \mu)$  with suitable smoothing effects for solutions to (1.1.2). Such results are primarily interesting for small times.

#### 1.3.3.1 Poincaré-type inequalities

Before proving Theorem 1.3.6, we provide an elementary numerical inequality that we shall frequently exploit in this and subsequent paragraphs.

**Lemma 1.3.25.** *Given  $\alpha, \beta \in (0, 1)$ , with  $\alpha > \beta$ , there exists a constant  $c = c(\alpha, \beta) > 0$  such that*

$$x^{-\alpha}y^{1-\alpha} + x^{-\beta}y^{1-\beta} + y \leq c(x^{-\alpha}y^{1-\alpha} + y) \quad \forall x, y \in \mathbb{R}^+. \quad (1.3.34)$$

*Proof.* We need to show that the function

$$R(x, y) := \frac{x^{-\beta} y^{1-\beta}}{x^{-\alpha} y^{1-\alpha} + y} \quad (1.3.35)$$

is bounded in  $\mathbb{R}^+ \times \mathbb{R}^+$  by a constant which depends only on  $\alpha$  and  $\beta$ . In order to do that, we can fix  $y > 0$  and find the zeros  $x^*(y)$  of  $R_x(\cdot, y)$  (in fact  $0 < \beta < \alpha$  implies  $R(0^+, y) = R(+\infty, y) = 0$ ). Through an explicit calculation we get

$$x^*(y) = \left( \frac{\alpha - \beta}{\beta} \right)^{\frac{1}{\alpha}} y^{-1}.$$

Substituting the above value in (1.3.35) we obtain (1.3.34) with

$$c(\alpha, \beta) = 1 + \left( \frac{\beta}{\alpha} \right)^{\frac{\beta}{\alpha}} \left( 1 - \frac{\beta}{\alpha} \right)^{1 - \frac{\beta}{\alpha}} \leq 2.$$

□

*Proof of Theorem 1.3.6.* Let  $q_0 \in (1, \infty) \cap [m - 1, \infty)$ ,  $t > 0$  and take  $u_0 \in L^\infty(\Omega)$ . With no loss of generality we shall also assume  $\nu(\Omega) = 1$ . First of all, note that we can rewrite the Poincaré-type inequality (1.3.6) as follows:

$$\frac{1}{2W_P^2} \|v\|_{2;\nu}^2 - \|v\|_{1;\nu}^2 \leq \|\nabla v\|_{2;\mu}^2 \quad \forall v \in H^1(\Omega; \nu, \mu). \quad (1.3.36)$$

By exploiting estimate (1.3.2) with  $q = q_0 - 1$  and (1.3.36) with  $v = u^{(q_0+m-1)/2}(\tau)$ , we carry out a single step of the Moser iteration:

$$\frac{4(q_0 - 1)q_0 m}{(m + q_0 - 1)^2} \int_0^t \left( \frac{1}{2W_P^2} \|u(\tau)\|_{q_0+m-1;\nu}^{q_0+m-1} - \|u(\tau)\|_{\frac{q_0+m-1}{2};\nu}^{q_0+m-1} \right) d\tau \leq \|u_0\|_{q_0;\nu}^{q_0}. \quad (1.3.37)$$

Since  $q_0 \geq m - 1$  and  $\nu(\Omega) = 1$ , we have  $\|u(\tau)\|_{(q_0+m-1)/2;\nu} \leq \|u(\tau)\|_{q_0;\nu}$ . Hence, taking advantage of the fact that  $L^p(\Omega; \nu)$  norms do not increase along the evolution (see again (1.3.2)), from (1.3.37) we easily obtain

$$\|u(t)\|_{q_0+m-1;\nu} \leq K \left( t^{-\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}} + \|u_0\|_{q_0;\nu} \right), \quad (1.3.38)$$

where  $K > 0$  is a suitable constant that depends on  $\varrho$ ,  $q_0$ ,  $m$  and  $W_P$ , which in the sequel may change from line to line (we do not relabel it). It is apparent that (1.3.38) only provides us with the smoothing estimate (1.3.7) from  $L^{q_0}(\Omega; \nu)$  to  $L^{q_0+m-1}(\Omega; \nu)$ . By means of a standard interpolation inequality, we can readily extend it to any  $\varrho \in (q_0, q_0 + m - 1]$ :

$$\|u(t)\|_{\varrho;\nu} \leq K \left( t^{-\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}} + \|u_0\|_{q_0;\nu} \right)^{\frac{(q_0+m-1)(\varrho-q_0)}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0(q_0+m-1-\varrho)}{\varrho(m-1)}},$$

whence

$$\|u(t)\|_{\varrho;\nu} \leq K \left( t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{\varrho}} + \|u_0\|_{q_0;\nu} \right). \quad (1.3.39)$$

In order to prove the validity of (1.3.39) for every  $\varrho \in (q_0, \infty)$ , we can proceed by induction. In fact, we already know that it holds for  $\varrho \in (q_0, q_0 + m - 1]$ . Let us assume that it also holds

for all  $\varrho \in (q_0, q_0 + n(m-1)]$  for some  $n \in \mathbb{N}$ . Given  $\varrho_1 \in (q_0 + n(m-1), q_0 + (n+1)(m-1)]$ , still from (1.3.39) (pick  $q_0 + n(m-1)$  in place of  $q_0$  and  $\varrho_1$  in place of  $\varrho$ ) we infer:

$$\|u(t)\|_{\varrho_1; \nu} \leq K \left( t^{-\frac{\varrho_1 - q_0 - n(m-1)}{\varrho_1(m-1)}} \|u_0\|_{\frac{\varrho_1}{q_0 + n(m-1); \nu} + \|u_0\|_{q_0 + n(m-1); \nu} \right). \quad (1.3.40)$$

Shifting the time origin to  $t/2$  in (1.3.40) and using the inductive hypothesis on the term  $\|u(t/2)\|_{q_0 + n(m-1)}$ , we find

$$\begin{aligned} \|u(t)\|_{\varrho_1; \nu} \leq K & \left( t^{-\frac{\varrho_1 - q_0}{\varrho_1(m-1)}} \|u_0\|_{\frac{\varrho_1}{q_0; \nu}} + t^{-\frac{\varrho_1 - q_0 - n(m-1)}{\varrho_1(m-1)}} \|u_0\|_{\frac{\varrho_1}{q_0; \nu}}^{\frac{q_0 + n(m-1)}{\varrho_1}} + \right. \\ & \left. + t^{-\frac{n(m-1)}{(m-1)[q_0 + n(m-1)]}} \|u_0\|_{\frac{q_0}{q_0; \nu}}^{\frac{q_0}{q_0 + n(m-1)}} + \|u_0\|_{q_0; \nu} \right), \end{aligned}$$

which, thanks to Lemma 1.3.25, entails

$$\|u(t)\|_{\varrho_1; \nu} \leq K \left( t^{-\frac{\varrho_1 - q_0}{\varrho_1(m-1)}} \|u_0\|_{\frac{\varrho_1}{q_0; \nu}} + \|u_0\|_{q_0; \nu} \right),$$

namely estimate (1.3.39) for all  $\varrho_1 \in (q_0 + n(m-1), q_0 + (n+1)(m-1)]$ .

We now get rid of the assumption  $q_0 \in (1, \infty) \cap [m-1, \infty)$ . Given any  $q_0 \geq 1$ , estimate (1.3.39) evaluated between  $t$  and  $t/2$ , with  $\varrho$  replaced by  $2mq_0$  and  $q_0$  replaced by  $mq_0$  (plus a standard interpolation inequality), yields

$$\|u(t)\|_{2mq_0; \nu} \leq K \left[ (t/2)^{-\frac{1}{2(m-1)}} \|u(t/2)\|_{\frac{m-1}{2mq_0; \nu}}^{\frac{m-1}{2m-1}} \|u_0\|_{\frac{1}{q_0; \nu}}^{\frac{1}{4m-2}} + \|u(t/2)\|_{\frac{2m-2}{2mq_0; \nu}}^{\frac{2m-2}{2m-1}} \|u_0\|_{\frac{1}{q_0; \nu}}^{\frac{1}{2m-1}} \right]. \quad (1.3.41)$$

Consider first an initial datum with  $\|u_0\|_{2mq_0; \nu} = 1$ , so that  $\|u_0\|_{q_0; \nu} \leq 1$  (recall that  $\nu(\Omega) = 1$ ). Since  $L^p(\Omega; \nu)$  norms do not increase along the evolution, there follows  $\|u(s)\|_{2mq_0; \nu} \leq 1$  for all  $s > 0$ . In particular, (1.3.41) implies

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( t^{-\frac{1}{2(m-1)}} + 1 \right) \|u(t/2)\|_{\frac{m-1}{2mq_0; \nu}}^{\frac{m-1}{2m-1}} \|u_0\|_{\frac{1}{q_0; \nu}}^{\frac{1}{4m-2}}. \quad (1.3.42)$$

Iterating (1.3.42) (for more detailed computations in a similar context we refer the reader to the proof of Theorem 1.3.8) we end up with the estimate

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( t^{-\frac{2m-1}{2m(m-1)}} \|u_0\|_{\frac{1}{q_0; \nu}}^{\frac{1}{2m}} + \|u_0\|_{\frac{1}{q_0; \nu}}^{\frac{1}{2m}} \right). \quad (1.3.43)$$

However, (1.3.43) only holds for initial data such that  $\|u_0\|_{2mq_0; \nu} = 1$ . By means of a standard time-scaling argument (we refer again to the proof of Theorem 1.3.8), from (1.3.43) we infer

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0; \nu}^{1-\theta} + \|u_0\|_{2mq_0; \nu}^{\theta} \|u_0\|_{q_0; \nu}^{1-\theta} \right), \quad (1.3.44)$$

which holds true for any  $L^{2mq_0}(\Omega; \nu)$  initial datum, where we set  $\theta := (2m-1)/2m$ . Taking advantage of (1.3.44) between  $t$  and  $t/2$  and performing  $n$  iterations, we obtain:

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( \sum_{i=1}^n t^{-\frac{\theta^i}{m-1}} \|u_0\|_{q_0; \nu}^{1-\theta^i} + \|u_0\|_{2mq_0; \nu}^{\theta^n} \|u_0\|_{q_0; \nu}^{1-\theta^n} \right), \quad (1.3.45)$$

where the constant  $K$  here also depends on  $n$ . Now we need a suitable smoothing estimate between  $L^{q_0}(\Omega; \nu)$  and  $L^{2mq_0}(\Omega; \nu)$ . To this end, consider first an initial datum with  $\|u_0\|_{q_0; \nu} =$

1. Performing another iteration and recalling once again that  $L^p(\Omega; \nu)$  norms do not increase, it is direct to check that one can use (1.3.41) in the simplified form

$$\|u(t/2^k)\|_{2mq_0; \nu} \leq 2^{\frac{k+1}{2(m-1)}} K \left( t^{-\frac{1}{2(m-1)}} + 1 \right) \|u(t/2^{k+1})\|_{2mq_0; \nu}^{\frac{m-1}{2m-1}}$$

up to the largest  $k \in \mathbb{N}$  for which  $\|u(t/2^{k+1})\|_{2mq_0; \nu} \leq 1$  and

$$\|u(t/2^k)\|_{2mq_0; \nu} \leq 2^{\frac{k+1}{2(m-1)}} K \left( t^{-\frac{1}{2(m-1)}} + 1 \right) \|u(t/2^{k+1})\|_{2mq_0; \nu}^{\frac{2m-2}{2m-1}} \quad (1.3.46)$$

for larger  $k$ . It is apparent that the worst scenario occurs when we are forced to use at any step (1.3.46), in which case we get, as  $k \rightarrow \infty$ ,

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( t^{-\frac{2m-1}{2(m-1)}} + 1 \right). \quad (1.3.47)$$

We point out that (1.3.47) holds true provided  $\|u_0\|_{q_0; \nu} = 1$ . Nevertheless, still by time-scaling arguments, we can recover an estimate that is valid for all initial data in  $L^{q_0}(\Omega; \nu)$ :

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( t^{-\frac{2m-1}{2(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{3-2m}{2}} + \|u_0\|_{q_0; \nu} \right). \quad (1.3.48)$$

Finally, using (1.3.45) between  $t$  and  $t/2$ , with  $n$  equal to the first integer such that

$$\theta^n(2m-1)/2 < \theta,$$

and combining it with (1.3.48) (evaluated at  $t/2$ ), we deduce:

$$\|u(t)\|_{2mq_0; \nu} \leq K \left( \sum_{i=1}^n t^{-\frac{\theta^i}{m-1}} \|u_0\|_{q_0; \nu}^{1-\theta^i} + t^{-\frac{2m-1}{2(m-1)}\theta^n} \|u_0\|_{q_0; \nu}^{1-\frac{2m-1}{2}\theta^n} + \|u_0\|_{q_0; \nu} \right).$$

Hence, applying Lemma 1.3.25  $n+1$  times, we obtain (1.3.7) for  $\varrho = 2mq_0$  and any  $q_0 \geq 1$ . Given a general  $\varrho > q_0$ , if  $\varrho > 2mq_0$  we can reason as follows:

$$\begin{aligned} \|u(t)\|_{\varrho; \nu} &\leq K \left( (t/2)^{-\frac{\varrho-2mq_0}{\varrho(m-1)}} \|u(t/2)\|_{2mq_0; \nu}^{\frac{2mq_0}{\varrho}} + \|u(t/2)\|_{2mq_0; \nu} \right) \\ &\leq K \left( t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{q_0}{\varrho}} + t^{-\frac{\varrho-2mq_0}{\varrho(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{2mq_0}{\varrho}} + t^{-\frac{2m-1}{2m(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{1}{2m}} + \|u_0\|_{q_0; \nu} \right), \end{aligned}$$

whence (1.3.7) by exploiting again Lemma 1.3.25. If instead  $\varrho \in (q_0, 2mq_0)$ , we just proceed by interpolation:

$$\begin{aligned} \|u(t)\|_{\varrho; \nu} &\leq \|u(t)\|_{2mq_0; \nu}^{\frac{2m(\varrho-q_0)}{\varrho(2m-1)}} \|u_0\|_{q_0; \nu}^{\frac{2mq_0-\varrho}{\varrho(2m-1)}} \leq K \left( t^{-\frac{2m-1}{2m(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{1}{2m}} + \|u_0\|_{q_0; \nu} \right)^{\frac{2m(\varrho-q_0)}{\varrho(2m-1)}} \|u_0\|_{q_0; \nu}^{\frac{2mq_0-\varrho}{\varrho(2m-1)}} \\ &\leq K \left( t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \|u_0\|_{q_0; \nu}^{\frac{q_0}{\varrho}} + \|u_0\|_{q_0; \nu} \right). \end{aligned}$$

It is possible to remove the assumption  $\nu(\Omega) = 1$  by spatial scaling. In fact, if  $u(x, t)$  is a solution to (1.1.2) in the domain  $\Omega$  of measure  $V = \nu(\Omega)$ , with respect to the weights  $\rho_\nu(x)$ ,  $\rho_\mu(x)$  and with initial datum  $u_0(x)$ , then

$$\tilde{u}(\tilde{x}, t) := V^{-\frac{2}{d(m-1)}} u\left(V^{1/d}\tilde{x}, t\right) \quad (1.3.49)$$

is also a solution to (1.1.2) in the domain  $\tilde{\Omega} := \Omega/V^{1/d}$  of measure 1, with respect to the weights

$$\tilde{\rho}_\nu(\tilde{x}) := \rho_\nu(V^{1/d}\tilde{x}), \quad \tilde{\rho}_\mu(\tilde{x}) := \rho_\mu(V^{1/d}\tilde{x})$$

and with initial datum

$$\tilde{u}_0(\tilde{x}) := V^{-\frac{2}{d(m-1)}} u_0(V^{1/d}\tilde{x}).$$

One then applies estimate (1.3.7) to  $\tilde{u}$  and goes back to the original solution  $u$  through (1.3.49) and the bounds

$$\|\tilde{u}\|_{p;\tilde{\nu}} = V^{-\frac{2}{d(m-1)} - \frac{1}{p}} \|u\|_{p;\nu}, \quad W_P(\tilde{\Omega}) \leq \max\{V^{-1/d}, V^{1/2}\} W_P(\Omega),$$

whence the validity of (1.3.7) for general domains, with a multiplicative constant  $K$  that depends on  $\nu(\Omega)$  and  $d$  as well. Nevertheless, as a consequence of the above computations, it is straightforward to check that  $K$  can be in turn bounded from above by a constant that is independent of  $q_0$  and  $d$ .

In order to deal with data  $u_0 \in L^{q_0}(\Omega; \nu)$  which are not necessarily bounded, one argues exactly as in the end of proof of Theorem 1.2.5.  $\square$

We now prove the converse implication of Theorem 1.3.6.

*Proof of Theorem 1.3.7.* We begin by considering a nonnegative initial datum  $u_0 \in L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ . Proceeding along the lines of the proof of Theorem 1.2.6 (notice that we are allowed to use Corollary 1.3.24), we end up with the following inequality:

$$\|u_0\|_{m+1;\nu} \leq B \left( \|\nabla(u_0^m)\|_{2;\mu}^{\frac{2}{m(m+1)}} \|u_0\|_{m;\nu}^{\frac{m-1}{m+1}} + \|u_0\|_{m;\nu} \right), \quad (1.3.50)$$

where  $B = B(\varrho, m, K) > 0$  is a suitable constant. Now take the sequence of real functions

$$\xi_n(y) := \frac{1}{n} \chi_{[0, \frac{1}{n}]}(y) + y \chi_{(\frac{1}{n}, \infty)}(y) \quad \forall y \in \mathbb{R}^+.$$

Letting

$$v_n := \xi_n(u_0^{1/m}) \in L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu),$$

it is immediate to show that  $\{v_n\}$  converges to  $u_0^{1/m}$  in  $L^\infty(\Omega)$  and  $\{\nabla(v_n^m)\}$  converges to  $\nabla u_0$  in  $[L^2(\Omega; \mu)]^d$ . Hence, replacing  $u_0$  with  $v_n$  in (1.3.50), passing to the limit as  $n \rightarrow \infty$  and raising to the power of  $m$ , we obtain:

$$\|u_0\|_{\frac{m+1}{m};\nu} \leq B^m \left( \|\nabla u_0\|_{2;\mu}^{\frac{2}{m+1}} \|u_0\|_{1;\nu}^{\frac{m-1}{m+1}} + \|u_0\|_{1;\nu} \right). \quad (1.3.51)$$

Upon defining

$$\vartheta := \frac{2}{m+1}, \quad r := \frac{m+1}{m}, \quad s := 1, \quad q := 2, \quad \mathcal{W}(f) := \|\nabla f\|_{2;\mu} + \|f\|_{1;\nu},$$

(1.3.51) entails

$$\|f\|_{r;\nu} \leq [\hat{B} \mathcal{W}(f)]^\vartheta \|f\|_{s;\nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad \hat{B} := B^{\frac{m}{\vartheta}}, \quad (1.3.52)$$

where  $f$  is any nonnegative function belonging to  $L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ . We are therefore in position to apply Theorem 3.1 of [13], from which we infer that (1.3.52) also holds for  $\vartheta = 1$  and  $r = 2$  (up to a different positive constant  $\hat{B}$ ), that is

$$\|f\|_{2;\nu} \leq \hat{B} \left( \|\nabla f\|_{2;\mu} + \|f\|_{1;\nu} \right).$$



We have then proved the Poincaré-type inequality (1.3.6) for nonnegative functions belonging to  $L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ . The validity of the latter for signed functions follows by writing  $f = f_+ - f_-$ . Finally, Proposition 1.1.4 permits us to conclude the proof.  $\square$

### 1.3.3.2 Counterexample to $L^\infty$ regularization

In this paragraph, similarly to what we did in Paragraph 1.2.3.2, we show that in general the validity of a Poincaré-type inequality like (1.3.6) is not enough in order to guarantee  $L^\infty$  regularization. In fact, consider the case  $\Omega = \mathbb{R}$  and  $\rho_\nu(x) = \rho_\mu(x) = e^{-|x|}$ . Actually, for such weights, there holds the stronger (zero-mean) Poincaré-type inequality (1.3.14) (see Appendix 1.A, Paragraph 1.A.2.1). Moreover, it is easy to prove that  $H^1(\mathbb{R}; e^{-|x|}, e^{-|x|}) = H_0^1(\mathbb{R}; e^{-|x|}, e^{-|x|})$ , so that in this case the Neumann problem coincides with the Dirichlet problem (recall Remark 1.3.5). In particular, in order to prove that a certain function is a (sub- or super-) solution to (1.1.2), one can neglect its behaviour at infinity and just test it on compactly supported functions in the corresponding weak formulation. Hence, by means of computations analogous to the ones we performed in Paragraph 1.2.3.2, it is direct to check that there exists a constant  $B > 0$  such that the function

$$v_B(x, t) := \frac{\log(x^2 + 2)}{(1 + B^{-1}(m - 1)t)^{\frac{1}{m-1}}}$$

is a *subsolution* to (1.1.2), so that  $u(t) \notin L^\infty(\mathbb{R})$  for all  $t \geq 0$ . We have therefore proved that for the initial datum  $u_0(x) = \log(x^2 + 2)$ , which clearly belongs to  $L^{q_0}(\mathbb{R}; e^{-|x|})$  for all  $q_0 \in [1, \infty)$ , there is no  $L^{q_0}(\mathbb{R}; e^{-|x|})$ - $L^\infty(\mathbb{R})$  regularization.

### 1.3.3.3 Sobolev-type inequalities

In contrast with the Poincaré case, when a Sobolev-type inequality like (1.3.8) holds true we can get regularization up to  $L^\infty(\Omega)$ .

*Proof of Theorem 1.3.8.* We proceed along the lines of a classical Moser iteration technique, with careful modifications in order to handle the additional term (w.r.t. (1.2.9)) in the r.h.s. of (1.3.8). With no loss of generality, we can assume that  $\nu(\Omega) = 1$  and  $u_0 \in L^\infty(\Omega)$  (see the end of the proofs of Theorems 1.2.9 and 1.3.6). Given  $t > 0$  and  $q_0 \in (1, \infty) \cap [m - 1, \infty)$ , take the time sequence  $t_k := (1 - 2^{-k})t$ . Also, let  $\{p_k\}$  be an increasing sequence of positive numbers starting from  $q_0$ , which will be explicitly defined below. Using estimate (1.3.2) with  $(0, T)$  replaced by  $(t_k, t_{k+1})$  and  $q = p_k - 1$ , we obtain:

$$\frac{4(p_k - 1)p_k m}{(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \int_{\Omega} \left| \nabla \left( u^{\frac{p_k + m - 1}{2}} \right) (x, \tau) \right|^2 d\mu d\tau \leq \|u(t_k)\|_{p_k; \nu}^{p_k}. \quad (1.3.53)$$

Notice that inequality (1.3.8) entails

$$\frac{1}{2W_S^2} \|v\|_{2\sigma; \nu}^2 - \|v\|_{1; \nu}^2 \leq \|\nabla v\|_{2; \mu}^2 \quad \forall v \in H^1(\Omega; \nu, \mu). \quad (1.3.54)$$

Applying (1.3.54) to the function  $u^{(p_k + m - 1)/2}(\tau)$  in (1.3.53), we get:

$$\begin{aligned} & \frac{2(p_k - 1)p_k m}{W_S^2(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \|u(\tau)\|_{\sigma(p_k + m - 1); \nu}^{p_k + m - 1} d\tau \\ & \leq \|u(t_k)\|_{p_k; \nu}^{p_k} + \frac{4(p_k - 1)p_k m}{(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \|u(\tau)\|_{\frac{p_k + m - 1}{2}; \nu}^{p_k + m - 1} d\tau. \end{aligned}$$

Since  $\nu(\Omega) = 1$ ,  $q_0 \geq m - 1$  and  $p_k$  is increasing,  $\|u(\tau)\|_{(p_k+m-1)/2;\nu} \leq \|u(\tau)\|_{p_k;\nu}$ . Moreover,  $t_{k+1} - t_k = t/2^{k+1}$ . By combining these properties with the fact that  $L^p(\Omega; \nu)$  norms do not increase along the evolution, we deduce:

$$\frac{(p_k - 1)p_k m}{W_S^2(p_k + m - 1)^2} 2^{-k} t \|u(t_{k+1})\|_{p_{k+1};\nu}^{\frac{p_{k+1}}{\sigma}} \leq \|u(t_k)\|_{p_k;\nu}^{p_k} + \frac{2(p_k - 1)p_k m}{(p_k + m - 1)^2} 2^{-k} t \|u(t_k)\|_{p_k;\nu}^{p_k+m-1}, \quad (1.3.55)$$

where we set  $p_{k+1} := \sigma(p_k + m - 1)$ . Let us assume  $\|u_0\|_\infty = 1$ . This hypothesis, together with the monotonicity of the  $L^p(\Omega; \nu)$  norms, implies

$$\|u(t_k)\|_{p_k;\nu}^{p_k+m-1} \leq \|u(t_k)\|_{p_k;\nu}^{p_k},$$

so that (1.3.55) reads

$$\|u(t_{k+1})\|_{p_{k+1};\nu}^{\frac{p_{k+1}}{\sigma}} \leq \frac{W_S^2(p_k + m - 1)^2}{(p_k - 1)p_k m} 2^k t^{-1} \|u(t_k)\|_{p_k;\nu}^{p_k} + 2W_S^2 \|u(t_k)\|_{p_k;\nu}^{p_k}. \quad (1.3.56)$$

It is apparent that there exists a suitable constant  $K > 0$  depending on  $q_0$ ,  $m$  and  $W_S$  (which may change from line to line below) such that (1.3.56) entails

$$\|u(t_{k+1})\|_{p_{k+1};\nu} \leq K^{\frac{k+1}{p_{k+1}}} (t^{-1} + 1)^{\frac{\sigma}{p_{k+1}}} \|u(t_k)\|_{p_k;\nu}^{\frac{\sigma}{p_{k+1}}}.$$

Setting  $U_k := \|u(t_k)\|_{p_k;\nu}$ , it is a matter of tedious but elementary computations to prove that the sequence  $\{U_k\}$  satisfies

$$U_{k+1} \leq K^{\frac{\sigma^{k+2} - (k+2)\sigma + k+1}{p_{k+1}(\sigma-1)^2}} (t^{-1} + 1)^{\frac{\sigma^{k+2} - \sigma}{p_{k+1}(\sigma-1)}} U_0^{q_0 \frac{\sigma^{k+1}}{p_{k+1}}}, \quad (1.3.57)$$

with

$$p_k = (q_0 - A)\sigma^k + A, \quad A := -\frac{\sigma}{\sigma - 1}(m - 1).$$

Letting  $k \rightarrow \infty$  in (1.3.57), we infer:

$$\|u(t)\|_\infty = \lim_{k \rightarrow \infty} \|u(t)\|_{p_{k+1};\nu} \leq \liminf_{k \rightarrow \infty} U_{k+1} \leq K(t^{-1} + 1)^{\frac{\sigma}{(\sigma-1)q_0 + \sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0 + \sigma(m-1)}}. \quad (1.3.58)$$

We point out that (1.3.58) is not an  $L^\infty$  smoothing estimate. In fact, in order to establish it, we assumed that  $\|u_0\|_\infty = 1$ . Through a simple time-scaling argument, we can deal with general  $L^\infty(\Omega)$  data. That is, given a solution  $u$  to (1.1.2) corresponding to the initial datum  $u_0$ , it is straightforward to check that  $\hat{u}(\cdot, t) := \lambda^{-1}u(\cdot, \lambda^{1-m}t)$  is the solution to the same problem corresponding to the initial datum  $\hat{u}_0 := \lambda^{-1}u_0$ . Choosing  $\lambda = \|u_0\|_\infty$  and applying (1.3.58) to  $\hat{u}(t)$  we find

$$\|u(t)\|_\infty = \|u_0\|_\infty \hat{u}(\|u_0\|_\infty^{m-1} t) \leq K(t^{-1} + \|u_0\|_\infty^{m-1})^{\frac{\sigma}{(\sigma-1)q_0 + \sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0 + \sigma(m-1)}}. \quad (1.3.59)$$

It is then possible to improve on the dependence of estimate (1.3.59) on  $\|u_0\|_\infty$ . To this aim, let us rewrite the latter as

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\sigma}{(\sigma-1)q_0 + \sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0 + \sigma(m-1)}} + \|u_0\|_\infty^{\frac{\sigma(m-1)}{(\sigma-1)q_0 + \sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0 + \sigma(m-1)}} \right),$$

that is, setting

$$\theta := \frac{\sigma(m-1)}{(\sigma-1)q_0 + \sigma(m-1)},$$

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta} + \|u_0\|_\infty^\theta \|u_0\|_{q_0;\nu}^{1-\theta} \right). \quad (1.3.60)$$

Shifting the time origin from 0 to  $t/2$  in (1.3.60) (and using the fact that  $\|u(t/2)\|_{q_0;\nu} \leq \|u_0\|_{q_0;\nu}$ ) we obtain

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta} + \|u(t/2)\|_\infty^\theta \|u_0\|_{q_0;\nu}^{1-\theta} \right),$$

whence, applying (1.3.60) to  $\|u(t/2)\|_\infty$ ,

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta} + t^{-\frac{\theta^2}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta^2} + \|u_0\|_\infty^\theta \|u_0\|_{q_0;\nu}^{1-\theta^2} \right).$$

It is then clear that proceeding as above along  $n + 1$  steps we end up with

$$\|u(t)\|_\infty \leq K \left( \sum_{i=1}^n t^{-\frac{\theta^i}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta^i} + \|u_0\|_\infty^{\theta^n} \|u_0\|_{q_0;\nu}^{1-\theta^n} \right), \quad (1.3.61)$$

where  $K$  now also depends on  $n$ . In order to get rid of the term  $\|u_0\|_\infty$ , we need a suitable smoothing estimate. To this end, we can reason in the following way. Suppose  $\|u_0\|_{q_0;\nu} = 1$ . From (1.3.55), letting again  $U_k := \|u(t_k)\|_{p_k;\nu}$ , we infer:

$$U_{k+1}^{\frac{p_{k+1}}{\sigma}} \leq \frac{W_S^2 (p_k + m - 1)^2}{(p_k - 1) p_k m} 2^k t^{-1} U_k^{p_k} + 2W_S^2 U_k^{p_k+m-1}. \quad (1.3.62)$$

Consider a *solution* to (1.3.62), namely a sequence  $\{V_k\}$  such that  $V_0 = U_0 = 1$  and

$$V_{k+1}^{\frac{p_{k+1}}{\sigma}} = \frac{W_S^2 (p_k + m - 1)^2}{(p_k - 1) p_k m} 2^k t^{-1} V_k^{p_k} + 2W_S^2 V_k^{p_k+m-1};$$

it is easy to show (*e.g.* by induction) that  $U_k \leq V_k$  and  $V_k \geq 1$ . Hence, the leading term in the right-hand side is  $V_k^{p_k+m-1}$ , so that  $\{V_k\}$  also satisfies the following recurrence inequality:

$$V_{k+1} \leq K^{\frac{k+1}{p_{k+1}}} (t^{-1} + 1)^{\frac{\sigma}{p_{k+1}}} V_k, \quad (1.3.63)$$

where  $K > 0$  is a suitable constant as above. Iterating (1.3.63) and exploiting the fact that  $p_k \geq q_0 \sigma^k$ , we get:

$$V_{k+1} \leq K^{\frac{(k+2)(1-\sigma)\sigma^{-k-1}-\sigma^{-k-1}+\sigma}{q_0(\sigma-1)^2}} (t^{-1} + 1)^{\frac{\sigma-\sigma^{-k}}{q_0(\sigma-1)}},$$

from which, passing to the limit as  $k \rightarrow \infty$ ,

$$\|u(t)\|_\infty = \lim_{k \rightarrow \infty} \|u(t)\|_{p_k;\nu} \leq \liminf_{k \rightarrow \infty} U_k \leq \liminf_{k \rightarrow \infty} V_k \leq K (t^{-1} + 1)^{\frac{\sigma}{(\sigma-1)q_0}}. \quad (1.3.64)$$

Due to scaling arguments, (1.3.64) entails

$$\|u(t)\|_\infty \leq K \left( t^{-1} + \|u_0\|_{q_0;\nu}^{m-1} \right)^{\frac{\sigma}{(\sigma-1)q_0}} \|u_0\|_{q_0;\nu}^{1-\frac{\sigma(m-1)}{(\sigma-1)q_0}},$$

which reads

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\delta}{m-1}} \|u_0\|_{q_0;\nu}^{1-\delta} + \|u_0\|_{q_0;\nu} \right) \quad (1.3.65)$$

upon setting

$$\delta := \frac{\sigma(m-1)}{(\sigma-1)q_0}.$$

Shifting the time origin to  $t/2$  in (1.3.61) and using (1.3.65) at time  $t/2$ , we obtain:

$$\|u(t)\|_\infty \leq K \left( \sum_{i=1}^n t^{-\frac{\theta^i}{m-1}} \|u_0\|_{q_0;\nu}^{1-\theta^i} + t^{-\frac{\delta\theta^n}{m-1}} \|u_0\|_{q_0;\nu}^{1-\delta\theta^n} + \|u_0\|_{q_0;\nu} \right).$$

Now we choose  $n$  so large that  $\delta\theta^n < \theta$  and apply iteratively Lemma 1.3.25 with  $x = t^{1/(m-1)}$ ,  $y = \|u_0\|_{q_0;\nu}$ ,  $\alpha = \theta$ ,  $\beta = \delta\theta^n$  in the first step and then  $\beta = \theta^j$  as  $j = n \dots 2$ . We finally get the estimate

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\sigma}{(\sigma-1)q_0+\sigma(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\sigma-1)q_0}{(\sigma-1)q_0+\sigma(m-1)}} + \|u_0\|_{q_0;\nu} \right). \quad (1.3.66)$$

We conclude the proof by showing that we can drop the assumption  $q_0 \in (1, \infty) \cap [m-1, \infty)$ . We shall exploit an argument similar to the one performed in the proof of Theorem 1.2.9 (see also the proof of Theorem 1.3.6 above). For notational simplicity, we shall consider only the case  $q_0 = 1$ . Using estimate (1.3.66) between  $t$  and  $t/2$  for  $q_0 = m$ , that is

$$\|u(t)\|_\infty \leq K \left( (t/2)^{-\frac{\sigma}{(\sigma-1)m+\sigma(m-1)}} \|u(t/2)\|_{m;\nu}^{\frac{(\sigma-1)m}{(\sigma-1)m+\sigma(m-1)}} + \|u(t/2)\|_{m;\nu} \right),$$

plus the inequality  $\|u(t/2)\|_{m;\nu}^m \leq \|u(t/2)\|_\infty^{m-1} \|u_0\|_{1;\nu}$ , we find:

$$\|u(t)\|_\infty \leq K \left( 2^{\frac{\gamma}{m-1}} t^{-\frac{\gamma}{m-1}} \|u(t/2)\|_\infty^{\frac{(m-1)(1-\gamma)}{m}} \|u_0\|_{1;\nu}^{\frac{1-\gamma}{m}} + \|u(t/2)\|_\infty^{\frac{m-1}{m}} \|u_0\|_{1;\nu}^{\frac{1}{m}} \right), \quad (1.3.67)$$

where

$$\gamma := \frac{\sigma(m-1)}{(\sigma-1)m + \sigma(m-1)}.$$

In order to deal with (1.3.67), we can argue as in the first part of this proof. That is, consider an initial datum  $u_0$  such that  $\|u_0\|_\infty = 1$ . This, in particular, implies that  $\|u(t/2)\|_\infty \leq 1$  and  $\|u_0\|_{1;\nu} \leq 1$ , so that (1.3.67) reads

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\gamma}{m-1}} + 1 \right) \|u(t/2)\|_\infty^{\frac{(m-1)(1-\gamma)}{m}} \|u_0\|_{1;\nu}^{\frac{1-\gamma}{m}}. \quad (1.3.68)$$

Iterating (1.3.68) along  $k-1$  steps we get

$$\begin{aligned} \|u(t)\|_\infty &\leq 2^{\frac{\gamma}{m-1}} \sum_{h=0}^{k-1} h^{\frac{(m-1)(1-\gamma)}{m}} \left[ K \left( t^{-\frac{\gamma}{m-1}} + 1 \right) \right]^{\sum_{h=0}^{k-1} \frac{(m-1)(1-\gamma)}{m}} \\ &\quad \times \|u(t/2^k)\|_\infty^{\frac{(m-1)(1-\gamma)}{m}} \|u_0\|_{1;\nu}^{\frac{1-\gamma}{m} \sum_{h=0}^{k-1} \frac{(m-1)(1-\gamma)}{m}}. \end{aligned}$$

Letting  $k \rightarrow \infty$  we end up with

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{m\gamma}{(m-1)[1+(m-1)\gamma]}} \|u_0\|_{1;\nu}^{\frac{1-\gamma}{1+(m-1)\gamma}} + \|u_0\|_{1;\nu}^{\frac{1-\gamma}{1+(m-1)\gamma}} \right). \quad (1.3.69)$$

Estimate (1.3.69) is analogous to (1.3.58). By reasoning likewise we obtain again (1.3.61) with  $q_0 = 1$  and  $\theta = m\gamma/[1 + (m-1)\gamma]$ :

$$\|u(t)\|_\infty \leq K \left( \sum_{i=1}^n t^{-\frac{\theta^i}{m-1}} \|u_0\|_{1;\nu}^{1-\theta^i} + \|u_0\|_\infty^{\theta^n} \|u_0\|_{1;\nu}^{1-\theta^n} \right). \quad (1.3.70)$$

In order to remove the dependence of the r.h.s. on  $\|u_0\|_\infty$ , it is convenient to assume first that  $\|u_0\|_{1;\nu} = 1$  and look for a suitable smoothing estimate. Arguing as in the end of proof

of Theorem 1.3.6, there is no loss of generality in supposing that  $\|u(s)\|_\infty \geq 1$  for all  $s < t$ , in which case (1.3.67) entails

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{\gamma}{m-1}} + 1 \right) \|u(t/2)\|_\infty^{\frac{m-1}{m}} \|u_0\|_{1;\nu}^{\frac{1}{m}}. \quad (1.3.71)$$

By means of another iteration, (1.3.71) yields

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{m\gamma}{m-1}} \|u_0\|_{1;\nu} + \|u_0\|_{1;\nu} \right),$$

which holds true provided  $\|u_0\|_{1;\nu} = 1$ . The usual time-scaling argument then ensures that the estimate

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{m\gamma}{m-1}} \|u_0\|_{1;\nu}^{1-m\gamma} + \|u_0\|_{1;\nu} \right) \quad (1.3.72)$$

holds true for all  $u_0 \in L^1(\Omega; \nu)$ . Choosing  $n$  in (1.3.70) so large that  $m\gamma\theta^n < \theta$  and combining the latter with (1.3.72) (through a  $t/2$ -shift of the time origin), we obtain:

$$\|u(t)\|_\infty \leq K \left( t^{-\frac{m\gamma}{(m-1)[1+(m-1)\gamma]}} \|u_0\|_{1;\nu}^{\frac{1-\gamma}{1+(m-1)\gamma}} + \|u_0\|_{1;\nu} \right),$$

which is exactly (1.3.66) with  $q_0 = 1$ . □

We close this paragraph with a short proof of the converse implication of Theorem 1.3.8.

*Proof of Theorem 1.3.9.* Given any nonnegative initial datum  $u_0 \in L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ , by proceeding along the lines of the proofs of Theorems 1.2.8 and 1.3.7 we obtain the following family of inequalities:

$$\|u_0\|_{m+1;\nu}^{m+1} \leq C \left( t^{-\frac{\sigma}{2\sigma m - m - \sigma}} \|u_0\|_{m;\nu}^{\frac{m(2\sigma m - m - 1)}{2\sigma m - m - \sigma}} + (m+1)t \|\nabla(u_0^m)\|_{2;\mu}^2 + \|u_0\|_{m;\nu}^{m+1} \right)$$

for all  $t > 0$  and a suitable constant  $C = C(m, \sigma, K) > 0$  (possibly varying from line to line). Minimizing the r.h.s. with respect to  $t > 0$  we end up with

$$\|u_0\|_{m+1;\nu}^{m+1} \leq C \left( \|\nabla(u_0^m)\|_{2;\mu}^{\frac{2\sigma}{m(2\sigma-1)}} \|u_0\|_{m;\nu}^{\frac{2\sigma m - m - 1}{2\sigma - 1}} + \|u_0\|_{m;\nu}^{m+1} \right). \quad (1.3.73)$$

Approximating the function  $u_0^{1/m}$  as in the proof of Theorem 1.3.7, we deduce that (1.3.73) is in fact equivalent to

$$\|f\|_{r;\nu} \leq [C \mathcal{W}(f)]^\vartheta \|f\|_{s;\nu}^{1-\vartheta}, \quad \frac{1}{r} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}, \quad (1.3.74)$$

where

$$\vartheta := \frac{2\sigma}{(m+1)(2\sigma-1)}, \quad r := \frac{m+1}{m}, \quad s := 1, \quad q := 2\sigma, \quad \mathcal{W}(f) := \|\nabla f\|_{2;\mu} + \|f\|_{1;\nu}$$

and  $f$  is any nonnegative function belonging to  $L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ . Theorem 3.1 of [13] then ensures that (1.3.74) also holds for  $\vartheta = 1$  and  $r = 2\sigma$ , that is

$$\|f\|_{2\sigma;\nu} \leq C \left( \|\nabla f\|_{2;\mu} + \|f\|_{1;\nu} \right). \quad (1.3.75)$$

The validity of (1.3.75) in the whole of  $H^1(\Omega; \nu, \mu)$  follows as in the end of proof of Theorem 1.3.7. □

## 1.3.3.4 Gagliardo-Nirenberg-type inequalities

We begin by proving  $L^\infty$  bounds in the one-dimensional case.

*Proof of Theorem 1.3.10.* As above, with no loss of generality, we can and shall assume that  $\nu(\Omega) = 1$  and  $u_0 \in L^\infty(\Omega)$ . First of all, let us rewrite inequality (1.3.10) in the following way:

$$\frac{\|v\|_\infty^{\frac{2(s+\lambda)}{\lambda}}}{C \|v\|_{s;\nu}^{\frac{2s}{\lambda}}} - \|v\|_{1;\nu}^2 \leq \|\nabla v\|_{2;\mu}^2 \quad \forall s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) : v \not\equiv 0, \quad (1.3.76)$$

where  $C := 2W_{G_1}^{2(s+\lambda)/2}$ . Given  $t > 0$  and  $q_0 \in (1, \infty) \cap [m-1, \infty)$ , using inequality (1.3.2) with  $q = q_0 - 1$  and (1.3.76) with

$$v = u^{\frac{q_0+m-1}{2}}(\tau),$$

we get:

$$\begin{aligned} \|u_0\|_{q_0;\nu}^{q_0} &\geq \frac{4q_0(q_0-1)m}{(q_0+m-1)^2} \int_0^t \left( \frac{\|u^{\frac{q_0+m-1}{2}}(\tau)\|_\infty^{\frac{2(s+\lambda)}{\lambda}}}{C \|u^{\frac{q_0+m-1}{2}}(\tau)\|_{s;\nu}^{\frac{2s}{\lambda}}} - \|u^{\frac{q_0+m-1}{2}}(\tau)\|_{1;\nu}^2 \right) d\tau \\ &\geq \frac{4q_0(q_0-1)m}{(q_0+m-1)^2} \int_0^t \left( \frac{\|u(\tau)\|_\infty^{\frac{(s+\lambda)(q_0+m-1)}{\lambda}}}{C \|u(\tau)\|_{\frac{s(q_0+m-1)}{\lambda};\nu}^{\frac{s(q_0+m-1)}{2}}} - \|u(\tau)\|_{\frac{q_0+m-1}{2};\nu}^{q_0+m-1} \right) d\tau \\ &\geq \frac{4q_0(q_0-1)m}{(q_0+m-1)^2} \left( t \frac{\|u(t)\|_\infty^{\frac{(s+\lambda)(q_0+m-1)}{\lambda}}}{C \|u_0\|_{\frac{s(q_0+m-1)}{\lambda};\nu}^{\frac{s(q_0+m-1)}{2}}} - \int_0^t \|u(\tau)\|_{q_0;\nu}^{q_0+m-1} d\tau \right) \\ &\geq t \frac{4q_0(q_0-1)m}{(q_0+m-1)^2} \left( \frac{\|u(t)\|_\infty^{\frac{(s+\lambda)(q_0+m-1)}{\lambda}}}{C \|u_0\|_{\frac{s(q_0+m-1)}{\lambda};\nu}^{\frac{s(q_0+m-1)}{2}}} - \|u_0\|_{q_0;\nu}^{q_0+m-1} \right). \end{aligned}$$

Picking  $s = 2q_0/(q_0+m-1)$  in the above inequality we deduce that

$$\|u_0\|_{q_0;\nu}^{q_0} \geq t \frac{4q_0(q_0-1)m}{(q_0+m-1)^2} \left( \frac{\|u(t)\|_\infty^{\frac{(\lambda+2)q_0+\lambda(m-1)}{\lambda}}}{C \|u_0\|_{q_0;\nu}^{\frac{2q_0}{\lambda}}} - \|u_0\|_{q_0;\nu}^{q_0+m-1} \right),$$

or equivalently (upon relabelling  $C$ )

$$\begin{aligned} \|u(t)\|_\infty &\leq C \left( t^{-1} \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{\lambda}} + \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0+\lambda(m-1)}{\lambda}} \right)^{\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \\ &\leq C \left( t^{-\frac{\lambda}{(\lambda+2)q_0+\lambda(m-1)}} \|u_0\|_{q_0;\nu}^{\frac{(\lambda+2)q_0}{\lambda}} + \|u_0\|_{q_0;\nu} \right), \end{aligned}$$

namely (1.3.11). The fact that  $C$  can be taken to be independent of  $q_0$  is just a consequence of the above computations. The assumption  $q_0 \in (1, \infty) \cap [m-1, \infty)$  can be removed by means of an argument completely analogous to the one performed in the end of the proof of Theorem 1.3.8, which we therefore skip.  $\square$

We now prove the analogue of Theorem 1.3.10 in dimension  $d \leq 2$ .

*Proof of Theorem 1.3.12.* Again, with no loss of generality, we assume that  $\nu(\Omega) = 1$  and  $u_0 \in L^\infty(\Omega)$ . We can rewrite inequality (1.3.12) as follows:

$$\frac{\|v\|_{r;\nu}^{\frac{2r}{r-s}}}{C \|v\|_{s;\nu}^{\frac{2s}{r-s}}} - \|v\|_{1;\nu}^2 \leq \|\nabla v\|_{2;\mu}^2 \quad \forall r > s > 0, \quad \forall v \in H^1(\Omega; \nu, \mu) : v \neq 0, \quad (1.3.77)$$

where  $C := 2W_{G_2}^{2r/(r-s)}$ . Given  $t > 0$  and  $q_0 \in (1, \infty) \cap [m-1, \infty)$ , let  $t_k := (1 - 2^{-k})t$  and  $\{p_k\}$  be an increasing sequence of positive numbers starting from  $q_0$ , which we shall define below. Using estimate (1.3.2) with  $(0, T)$  replaced by  $(t_k, t_{k+1})$  and  $q = p_k - 1$  and applying (1.3.77) to the function

$$v = u^{\frac{p_k+m-1}{2}}(\tau),$$

we obtain:

$$\frac{4p_k(p_k - 1)m}{(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \left( \frac{\|u^{\frac{p_k+m-1}{2}}(\tau)\|_{r;\nu}^{\frac{2r}{r-s}}}{C \|u^{\frac{p_k+m-1}{2}}(\tau)\|_{s;\nu}^{\frac{2s}{r-s}}} - \|u^{\frac{p_k+m-1}{2}}(\tau)\|_{1;\nu}^2 \right) d\tau \leq \|u(t_k)\|_{p_k;\nu}^{p_k},$$

that is

$$\frac{4p_k(p_k - 1)m}{(p_k + m - 1)^2} \int_{t_k}^{t_{k+1}} \left( \frac{\|u(\tau)\|_{\frac{r}{2}(p_k+m-1);\nu}^{\frac{r}{2}(p_k+m-1)}}{C \|u(\tau)\|_{\frac{s}{2}(p_k+m-1);\nu}^{\frac{s}{2}(p_k+m-1)}} - \|u(\tau)\|_{\frac{p_k+m-1}{2};\nu}^{p_k+m-1} \right) d\tau \leq \|u(t_k)\|_{p_k;\nu}^{p_k}. \quad (1.3.78)$$

Exploiting the fact that  $(p_k + m - 1)/2 \leq p_k$  and that  $L^p(\Omega; \nu)$  norms do not increase in time, from (1.3.78) we deduce

$$t \frac{2p_k(p_k - 1)m}{2^k (p_k + m - 1)^2} \left( \frac{\|u(t_{k+1})\|_{\frac{r}{2}(p_k+m-1);\nu}^{\frac{r}{2}(p_k+m-1)}}{C \|u(t_k)\|_{\frac{s}{2}(p_k+m-1);\nu}^{\frac{s}{2}(p_k+m-1)}} - \|u(t_k)\|_{p_k;\nu}^{p_k+m-1} \right) \leq \|u(t_k)\|_{p_k;\nu}^{p_k}.$$

Letting  $s = s_k = 2p_k/(p_k + m - 1)$  and  $r = r_k = s_k + 2$ , we get

$$\|u(t_{k+1})\|_{\frac{r_k}{2}(p_k+m-1);\nu}^{\frac{r_k}{2}(p_k+m-1)} \leq C \left( 2^k t^{-1} \|u(t_k)\|_{p_k;\nu}^{p_k + \frac{s_k}{r_k - s_k}(p_k+m-1)} + \|u(t_k)\|_{p_k;\nu}^{p_k + \frac{s_k}{r_k - s_k}(p_k+m-1) + m - 1} \right)$$

for another positive constant  $C = C(q_0, m, W_{G_2})$  (which we do not relabel below) independent of  $k$ , or equivalently

$$\|u(t_{k+1})\|_{p_{k+1};\nu}^{p_{k+1}} \leq C \left( 2^k t^{-1} \|u(t_k)\|_{p_k;\nu}^{2p_k} + \|u(t_k)\|_{p_k;\nu}^{2p_k+m-1} \right), \quad (1.3.79)$$

where we set  $p_{k+1} := 2p_k + m - 1$ . Estimate (1.3.79) is the analogue of (1.3.55) in the proof of Theorem 1.3.8, so we can handle it by means of similar techniques. That is, for initial data with  $\|u_0\|_\infty = 1$  one can readily show that (1.3.79) yields

$$\|u(t)\|_\infty \leq C \left( t^{-1} + 1 \right)^{\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}}; \quad (1.3.80)$$

then, through a standard time-scaling argument, from (1.3.80) one infers

$$\|u(t)\|_\infty \leq C \left( t^{-1} + \|u_0\|_\infty^{m-1} \right)^{\frac{1}{q_0+m-1}} \|u_0\|_{q_0;\nu}^{\frac{q_0}{q_0+m-1}}, \quad (1.3.81)$$

which holds true for any  $u_0 \in L^\infty(\Omega)$ . On the other hand, considering first initial data with  $\|u_0\|_{q_0; \nu} = 1$  and then general  $L^{q_0}(\Omega; \nu)$  data by time scaling, still from (1.3.79) we can get the following (non-sharp) smoothing estimate:

$$\|u(t)\|_\infty \leq C \left( t^{-\frac{1}{q_0}} \|u_0\|_{q_0; \nu}^{\frac{q_0+1-m}{q_0}} + \|u_0\|_{q_0; \nu} \right). \quad (1.3.82)$$

Finally, the proper smoothing estimate (1.3.13) can be deduced from (1.3.81) and (1.3.82) by reasoning exactly as in the proof of Theorem 1.3.8. The same holds for the arguments that allow us to get rid of the assumption  $q_0 \in (1, \infty) \cap [m-1, \infty)$ .  $\square$

We close the paragraph by giving a sketch of proof of the converse implications of Theorems 1.3.10 and 1.3.12.

*Proof of Theorems 1.3.11 and 1.3.13.* There is no significant difference with respect to the proof of Theorem 1.3.9 above. Let us just mention that, by reasoning likewise, we end up with the validity of the functional inequality (1.3.74), with the sole difference that

$$\vartheta = \frac{\lambda}{(m+1)(\lambda+1)} \quad \text{and} \quad q = -\lambda$$

in the one-dimensional case and

$$\vartheta = \frac{1}{m+1} \quad \text{and} \quad q = \infty$$

in the two-dimensional case. Hence, Theorems 3.2 and 3.3 of [13] ensure the validity of the families (1.3.10) and (1.3.12), respectively, at least for nonnegative functions in  $L^\infty(\Omega) \cap H^1(\Omega; \nu, \mu)$ . In order to extend the result to the whole of  $H^1(\Omega; \nu, \mu)$  we can again take advantage of Proposition 1.1.4.  $\square$

Finally, we point out that the same conclusions as in Remark 1.2.21 apply to Theorems 1.3.10 and 1.3.12 as well.

### 1.3.3.5 Improvements on previous results and comments on sharpness

In [27] the authors also consider the porous medium equation on smooth *compact* Riemannian manifolds  $M$  (endowed with a suitable metric) without boundary, in dimension  $d \geq 3$ . In this framework the Sobolev-type inequality

$$\|v - \bar{v}\|_{\frac{2d}{d-2}} \leq M_S \|\nabla v\|_2 \quad \forall v \in H^1(M)$$

holds true, where  $L^p$  norms are meant w.r.t. to the underlying Riemannian measure and  $\nabla$  denotes the corresponding Riemannian gradient. Even though we did not work on such Riemannian manifolds, it is readily seen that result of Theorem 1.3.8 still applies (with  $\sigma = d/(d-2)$ ), since our proofs are only based on chain rule-type arguments and on the validity of a Sobolev-type inequality. Moreover, it improves on the estimate

$$\|u(t)\|_\infty \leq K t^{-\frac{\theta}{m-1}} \|u_0\|_{q_0}^{1-\theta} e^{E_0 \|u_0\|_{1V(m-1)}^{m-1} t} \quad \forall t > 0, \quad (1.3.83)$$

where  $K$  and  $E_0$  are suitable positive constants that depend only on  $q_0, m, M_S, d, \text{Vol}(M)$  and

$$\theta := \left[ 1 - \left( \frac{q_0}{q_0 + m - 1} \right)^{\frac{d}{2}} \right], \quad (1.3.84)$$



which was proved in [27, Theorem 1.1] (by means of a Gross differential method). In fact, it is plain that

$$\theta > \frac{d(m-1)}{2q_0 + d(m-1)} \quad \forall m > 1, \quad \forall d > 2, \quad \forall q_0 \geq 1.$$

Furthermore, note that (1.3.83) holds for data in  $L^{q_0}$  with  $q_0 \geq 1 \vee (m-1)$  and blows up as  $t \rightarrow \infty$ , whereas (1.3.9) holds for all  $q_0 \geq 1$  and stays bounded as  $t \rightarrow \infty$ .

In the non-weighted Euclidean framework (*i.e.*  $\Omega$  is a regular bounded domain and  $\rho_\nu = \rho_\mu \equiv 1$ ), there is a simple argument that shows that our smoothing estimates are sharp, at least for small times. To this end, we need to introduce a suitable notion. By a *scaling-invariant estimate* for solutions to (1.1.2) (or to (1.1.1)), we mean a bound of the type

$$\|u(t)\|_\infty \leq S_{q_0}(t, \|u_0\|_{q_0; \nu}) \quad \forall t > 0, \quad \forall u_0 \in L^{q_0}(\Omega; \nu),$$

for some function  $S_{q_0} : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$  such that

$$S_{q_0}(t, \|u_0\|_{q_0; \nu}) = \|u_0\|_{q_0; \nu} S_{q_0}(\|u_0\|_{q_0; \nu}^{m-1} t, 1) \quad \forall t > 0, \quad \forall u_0 \in L^{q_0}(\Omega; \nu).$$

In particular,  $S$  is completely determined by the one-variable function  $S_{q_0}(\cdot, 1)$ . As it is well known (see *e.g.* [142, pp. 125-126]), if  $\Omega$  is a regular bounded domain of  $\mathbb{R}^d$  then:

- for  $d \geq 3$  the Sobolev-type inequality (1.3.8) holds true in  $H^1(\Omega)$  with  $\sigma = d/(d-2)$ ;
- for  $d = 2$  the family of Gagliardo-Nirenberg-type inequalities (1.3.12) holds true in  $H^1(\Omega)$ ;
- for  $d = 1$  the family of Gagliardo-Nirenberg-type inequalities (1.3.10) holds true in  $H^1(\Omega)$  with  $\lambda = 2$ .

In this case, the bounds (1.3.9), (1.3.11) and (1.3.13) can therefore be subsumed in the estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq K \left( t^{-\frac{d}{2q_0 + d(m-1)}} \|u_0\|_{L^{q_0}(\Omega)}^{\frac{2q_0}{2q_0 + d(m-1)}} + \|u_0\|_{L^{q_0}(\Omega)} \right) \quad \forall t > 0, \quad (1.3.85)$$

which is valid for all  $d \geq 1$ . It is apparent that (1.3.85) is scaling invariant (as well as (1.3.83)), with

$$S_{q_0}(\tau, 1) = \bar{S}_{q_0}(\tau, 1) := K \left( \tau^{-\frac{d}{2q_0 + d(m-1)}} + 1 \right) \quad \forall \tau > 0.$$

Consider now the classical Barenblatt functions (let  $b = b(m, d) > 0$ )

$$U(x, t) := t^{-\alpha} \left( 1 - b t^{-2\kappa} |x|^2 \right)_+^{\frac{1}{m-1}}, \quad \alpha := d\kappa, \quad \kappa := \frac{1}{d(m-1) + 2},$$

which in particular solve the Neumann problem for sufficiently small times (provided  $x = 0 \in \Omega$ , otherwise one just shifts them). It is a matter of straightforward computations to check that  $\|U(t)\|_{q_0} \sim t^{-\alpha(1-1/q_0)}$  for all  $t > 0$  and  $q_0 \in [1, \infty]$ . Moreover, using (1.3.85) on  $U$  between  $t$  and  $t/2$ , we get:

$$t^{-\alpha} \sim \|U(t)\|_\infty \leq \bar{S}_{q_0}(t/2, \|U(t/2)\|_{q_0}) = \|U(t/2)\|_{q_0} \bar{S}_{q_0}(\|U(t/2)\|_{q_0}^{m-1} t/2, 1) \sim t^{-\alpha},$$

for all  $t$  small enough. This implies that estimate (1.3.85) is *sharp* on  $U$ , in the sense that there cannot exist another scaling-invariant estimate  $S_{q_0}(\tau, 1)$  with a better rate as  $\tau \rightarrow 0$ .

We can also provide a weighted example in which the bound (1.3.9) is sharp. The setting is one dimensional, in particular we choose  $\Omega = (0, 1)$  with the weights  $\rho_\nu(x) \equiv 1$  and

$\rho_\mu(x) = x^\beta$ . The Sobolev-type inequality (1.3.8) associated to such weights is known to hold with  $\sigma = 1/(\beta - 1)$  for  $\beta \in (1, 2)$  (see Appendix 1.A, Paragraph 1.A.2.2). An explicit calculation shows that the Barenblatt-type functions

$$U_\beta(x, t) := t^{-\kappa} \left[ 1 - b(t^{-\kappa}x)^{2-\beta} \right]_+^{\frac{1}{m-1}}, \quad \kappa := \frac{1}{m+1-\beta}, \quad b := \frac{m-1}{m(2-\beta)(m+1-\beta)}$$

are (energy) solutions to the corresponding weighted Neumann problem (1.1.2) in  $\Omega \times (t/2, t)$ , provided  $t$  is so small that the support of  $U_\beta(\cdot, t)$  is bounded away from the point  $x = 1$ . By performing similar computations as in the non-weighted case above, it is straightforward to check that the smoothing estimate (1.3.9) (with  $\sigma = 1/(\beta - 1)$ ) is sharp on  $U_\beta$ , so that there cannot exist a better scaling-invariant estimate for small times.

### 1.3.4 Long-time behaviour

Estimates (1.3.7), (1.3.9), (1.3.11) and (1.3.13) do not provide us with significant information on the behaviour of solutions as  $t \rightarrow \infty$ . In this subsection we shall see that actually, under the validity of stronger inequalities like (1.3.14), (1.3.16), (1.3.19) or (1.3.21), any solution  $u$  converges to its mean value  $\bar{u}$  (which is preserved in time, recall Proposition 1.3.3), with suitable rates. In order to prove it, it will be crucial to deal with the quantity

$$\frac{d}{d\tau} \|u(\tau) - \bar{u}\|_{\varrho; \nu}^\varrho.$$

To this end, we shall establish proper functional inequalities for zero-mean functions, which are new in this framework.

#### 1.3.4.1 Poincaré-type inequalities

The following preliminary result is key to our analysis.

**Lemma 1.3.26.** *Suppose that the (zero-mean) Poincaré-type inequality (1.3.14) holds true. Let  $\Phi : \mathbb{R} \mapsto \mathbb{R}$  be a continuous and increasing function with the following properties:*

$$\lim_{y \rightarrow 0} \frac{\Phi(y)}{y^r} =: l_0, \quad \lim_{y \rightarrow -\infty} \frac{\Phi(y)}{y^r} =: l_-, \quad \lim_{y \rightarrow +\infty} \frac{\Phi(y)}{y^r} =: l_+ \quad (1.3.86)$$

for some  $r \geq 1/2$  and  $l_0, l_-, l_+ \in (0, +\infty)$ . Then there exists a constant  $M_\Phi > 0$ , which depends on  $\Phi$ ,  $\Omega$ ,  $\nu$  and  $\mu$ , such that the inequality

$$\|\Phi(\xi)\|_{2; \nu} \leq M_\Phi \|\nabla \Phi(\xi)\|_{2; \mu} \quad (1.3.87)$$

holds true for every function  $\xi \in L^1(\Omega; \nu)$  with  $\bar{\xi} = 0$  and  $\Phi(\xi) \in H^1(\Omega; \nu, \mu)$ .

*Proof.* We proceed by contradiction. Should the assertion be false, then there would exist a sequence of functions

$$\{\xi_n\} \subset \{\xi \in L^1(\Omega; \nu) : \xi \not\equiv 0, \bar{\xi} = 0, \Phi(\xi) \in H^1(\Omega; \nu, \mu)\}$$

such that

$$\|\nabla \Phi(\xi_n)\|_{2; \mu} \leq \frac{1}{n} \|\Phi(\xi_n)\|_{2; \nu} \quad \forall n \in \mathbb{N}.$$

Let  $a_n := \|\Phi(\xi_n)\|_{2; \nu}$  and

$$\Psi_n(\xi_n) := \frac{\Phi(\xi_n)}{a_n}.$$

Clearly,

$$\|\Psi_n(\xi_n)\|_{2;\nu} = 1, \quad \|\nabla\Psi_n(\xi_n)\|_{2;\mu} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}. \quad (1.3.88)$$

Applying the Poincaré-type inequality (1.3.14) to the sequence  $\{\Psi_n(\xi_n)\}$  and exploiting the second inequality in (1.3.88), we deduce that

$$\left\| \Psi_n(\xi_n) - \overline{\Psi_n(\xi_n)} \right\|_{2;\nu} \leq \frac{M_P}{n} \quad \forall n \in \mathbb{N}. \quad (1.3.89)$$

The above inequality and the normalization condition in (1.3.88) imply that the sequence of real numbers  $\{\overline{\Psi_n(\xi_n)}\}$  is bounded, so that up to subsequences it converges to some limit  $c_0$ . This fact and again (1.3.89) allow us to assert that

$$\|\Psi_n(\xi_n) - c_0\|_{2;\nu} \rightarrow 0,$$

that is, up to subsequences,

$$\Psi_n(\xi_n(x)) \rightarrow c_0 \quad \text{for a.e. } x \in \Omega. \quad (1.3.90)$$

The just mentioned normalization condition prevents  $c_0$  from being zero. Now we need to distinguish three cases according to the value of the quantity

$$a_\infty := \lim_{n \rightarrow \infty} a_n,$$

which exists upon picking another subsequence. If  $a_\infty \in (0, +\infty)$ , then the continuity of  $\Phi$  (and so of  $\Phi^{-1}$ ) entails

$$\xi_n(x) \rightarrow \Phi^{-1}(a_\infty c_0) \neq 0 \quad \text{for a.e. } x \in \Omega.$$

If  $a_\infty = 0$  or  $a_\infty = +\infty$  instead, things are more delicate. Let us begin with the case  $a_\infty = 0$ . Recalling the definition and the properties of  $\Phi$ , in view of (1.3.90) there follows

$$\xi_n(x) \rightarrow 0 \quad \text{for a.e. } x \in \Omega;$$

whence, exploiting the first identity in (1.3.86) and again (1.3.90),

$$\mathcal{Z}_n(x) := \frac{\xi_n(x)}{a_n^{1/r}} = \left[ \frac{[\xi_n(x)]^r}{\Phi(\xi_n(x))} \Psi_n(\xi_n(x)) \right]^{\frac{1}{r}} \rightarrow \left( \frac{c_0}{l_0} \right)^{\frac{1}{r}} \neq 0 \quad \text{for a.e. } x \in \Omega.$$

In the case  $a_\infty = +\infty$  one can argue likewise. In fact, suppose  $c_0 > 0$ . From the properties of  $\Phi$  and (1.3.90) we deduce that

$$\xi_n(x) \rightarrow +\infty \quad \text{for a.e. } x \in \Omega,$$

from which, thanks to the third identity in (1.3.86),

$$\mathcal{Z}_n(x) = \frac{\xi_n(x)}{a_n^{1/r}} = \left[ \frac{[\xi_n(x)]^r}{\Phi(\xi_n(x))} \Psi_n(\xi_n(x)) \right]^{\frac{1}{r}} \rightarrow \left( \frac{c_0}{l_+} \right)^{\frac{1}{r}} \neq 0 \quad \text{for a.e. } x \in \Omega.$$

Reasoning in the same way, when  $c_0 < 0$  we get

$$\mathcal{Z}_n(x) \rightarrow \left( \frac{c_0}{l_-} \right)^{\frac{1}{r}} \neq 0 \quad \text{for a.e. } x \in \Omega.$$

Hence, in any of the above cases, the sequence  $\{\mathcal{Z}_n\}$  converges pointwise to a nonzero constant. Since  $\bar{\mathcal{Z}}_n = 0$  and the mean value operator is trivially continuous in  $L^1(\Omega; \nu)$ , we come to a contradiction as soon as we prove that  $\{\mathcal{Z}_n\}$  also converges in  $L^1(\Omega; \nu)$  to such constant. To this end, in view of Egoroff's Theorem, it is enough to show that the sequence

$$\int_E |\mathcal{Z}_n(x)| \, d\nu$$

converges to zero uniformly as  $n \rightarrow \infty$  and  $\nu(E) \rightarrow 0$ . First of all, notice that (1.3.86), together with the continuity and the monotonicity of  $\Phi$ , entails the existence of a constant  $D > 0$  such that

$$D^{-1} |y|^r \leq |\Phi(y)| \leq D |y|^r \quad \forall y \in \mathbb{R}.$$

As a consequence,

$$\begin{aligned} \int_E |\mathcal{Z}_n(x)| \, d\nu &= \int_E \frac{|\xi_n(x)|}{a_n^{1/r}} \, d\nu \leq D^{\frac{1}{r}} \int_E |\Psi_n(\xi_n(x))|^{\frac{1}{r}} \, d\nu \\ &\leq D^{\frac{1}{r}} \nu(E)^{1-\frac{1}{2r}} \left( \int_E |\Psi_n(\xi_n(x))|^2 \, d\nu \right)^{\frac{1}{2r}}, \end{aligned}$$

so that the sequence

$$\int_E |\Psi_n(\xi_n(x))|^2 \, d\nu$$

does vanish uniformly as  $n \rightarrow \infty$  and  $\nu(E) \rightarrow 0$  since

$$\int_E |\Psi_n(\xi_n(x))|^2 \, d\nu \leq 2 \left( \int_\Omega |\Psi_n(\xi_n(x)) - c_0|^2 \, d\nu + \nu(E) c_0^2 \right).$$

We have therefore proved that  $\{\mathcal{Z}_n\}$  converges in  $L^1(\Omega; \nu)$  to a nonzero constant with zero mean, a contradiction.  $\square$

In the case of bounded Euclidean domains with  $\rho_\nu = \rho_\mu \equiv 1$ , the above result had already been proved in [3, Lemma 3.2] for  $\Phi(y) = y^m$  ( $m > 1$ ). However, the corresponding proof exploited in an essential way the compactness of the embedding  $H^1(\Omega) \hookrightarrow L^2(\Omega)$ . On the contrary, the proof of Lemma 1.3.26 does not require compactness (which, in the generality of our analysis, may not hold). Note that it is crucial that the behaviour of  $\Phi(y)$  as  $y \rightarrow 0$  and as  $y \rightarrow \pm\infty$  is given by the *same* power of  $y$ : if, for instance,  $\Phi(y) \approx y^{r_1}$  as  $y \rightarrow 0$  and  $\Phi(y) \approx y^{r_2}$  as  $y \rightarrow \pm\infty$  with  $r_1 \neq r_2$ , our proof does not work (we fail to bound  $\int_E |\mathcal{Z}_n(x)| \, d\nu$ ).

We are now ready to prove our asymptotic result, in the Poincaré case, for zero-mean solutions. When  $\Phi(y) = y^r$ , with a slight abuse of notation, we shall denote as  $M_r$  the constant  $M_\Phi$  appearing in (1.3.87).

*Proof of Theorem 1.3.14.* Let  $q_0 \in (1, \infty)$  and, with no loss of generality, take  $u_0 \in L^\infty(\Omega)$ , with  $\bar{u}_0 = 0$ . Consider the (formal) identity

$$\frac{d}{d\tau} \|u(\tau)\|_{q_0; \nu}^{q_0} = - \left( \frac{2}{q_0 + m - 1} \right)^2 m q_0 (q_0 - 1) \left\| \nabla \left( u^{\frac{q_0+m-1}{2}} \right) (\tau) \right\|_{2; \mu}^2. \quad (1.3.91)$$

Since  $u(\tau)$  has zero mean, we can apply Lemma 1.3.26 with the choice  $\Phi(y) = y^{(q_0+m-1)/2}$ , which provides us with a positive constant  $M_{(q_0+m-1)/2}$  such that

$$\|u(\tau)\|_{q_0+m-1; \nu}^{q_0+m-1} \leq M_{(q_0+m-1)/2}^2 \left\| \nabla \left( u^{\frac{q_0+m-1}{2}} \right) (\tau) \right\|_{2; \mu}^2. \quad (1.3.92)$$

In view of (1.3.91) and (1.3.92), we can proceed as in the proof of Theorem 1.2.5 to get the following bound:

$$\|u(t)\|_{q_0;\nu} \leq \frac{1}{\left(Dt + \|u_0\|_{q_0;\nu}^{1-m}\right)^{\frac{1}{m-1}}} \quad \forall t > 0, \quad (1.3.93)$$

where  $D > 0$  is a suitable constant depending on  $q_0$ ,  $m$ ,  $M_{(q_0+m-1)/2}$  and  $\nu(\Omega)$ . For any fixed  $\varrho \in (q_0, \infty)$ , using the smoothing estimate (1.3.7) between  $t$  and  $t/2$  and (1.3.93) evaluated at  $t/2$ , we obtain:

$$\|u(t)\|_{\varrho;\nu} \leq K \left[ t^{-\frac{\varrho-q_0}{\varrho(m-1)}} \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{q_0}{\varrho(m-1)}} + \left(t + \|u_0\|_{q_0;\nu}^{1-m}\right)^{-\frac{1}{m-1}} \right] \quad \forall t > 0, \quad (1.3.94)$$

where  $K > 0$  is a suitable constant as in the statement, which we do not relabel in the sequel. It is immediate to see that (1.3.94) implies (1.3.15). Hence, it remains to prove the validity of the latter down to  $q_0 = 1$ . To this end, given  $\varrho \in (1, \infty)$ , a trivial interpolation plus estimate (1.3.15) with  $q_0 = (\varrho + 1)/2$  yield

$$\|u(t)\|_{\frac{\varrho+1}{2};\nu} \leq \|u(t)\|_{\varrho;\nu}^{\frac{\varrho}{\varrho+1}} \|u_0\|_{1;\nu}^{\frac{1}{\varrho+1}} \leq K (t/2)^{-\frac{\varrho-1}{2(\varrho+1)(m-1)}} \|u(t/2)\|_{\frac{\varrho+1}{2};\nu}^{\frac{1}{2}} \|u_0\|_{1;\nu}^{\frac{1}{\varrho+1}} \quad \forall t > 0. \quad (1.3.95)$$

A standard iteration allows us to infer from (1.3.95) the validity of

$$\|u(t)\|_{\frac{\varrho+1}{2};\nu} \leq K t^{-\frac{\varrho-1}{(\varrho+1)(m-1)}} \|u_0\|_{1;\nu}^{\frac{2}{\varrho+1}} \quad \forall t > 0. \quad (1.3.96)$$

Thanks to (1.3.94) (with  $q_0 = (\varrho + 1)/2$ ) and (1.3.96), in order to obtain estimate (1.3.15) for  $q_0 = 1$  one argues exactly as in the end of proof of Theorem 1.2.5.  $\square$

In order to handle solutions with non-zero mean, we partially rely on a result which, at least in the case  $\rho_\nu = \rho_\mu$  and for nonnegative solutions, had already been proved in [82] (see Theorem 4.5 there). For the sake of completeness, we repeat the main lines of the proof in our framework.

**Lemma 1.3.27.** *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^1(\Omega; \nu)$ , with  $\bar{u}_0 \neq 0$ . Suppose that the Poincaré-type inequality (1.3.14) holds true. Then*

$$\|u(t) - \bar{u}\|_{2;\nu} \leq K t^{-\frac{1}{m-1}} \quad \forall t > 0, \quad (1.3.97)$$

where  $K > 0$  is a constant that depends only on  $m$ ,  $M_P$  and  $\nu(\Omega)$ .

*Proof.* For any  $\alpha \in (0, 1]$ , Proposition 2.2 of [82] (apply it to  $f_+$  and  $f_-$ ) provides us with the following inequality:

$$\|v^\alpha - \bar{v}^\alpha\|_{2;\nu} \leq M_{P,\alpha} \|\nabla v\|_{2;\mu}^\alpha \quad \forall v \in H^1(\Omega; \nu, \mu), \quad (1.3.98)$$

where one can choose e.g.

$$M_{P,\alpha} = 2^{1-\frac{\alpha}{2}} \nu(\Omega)^{\frac{1-\alpha}{2}} M_P^\alpha.$$

Now notice that, in view of the conservation of mass,

$$\frac{d}{d\tau} \|u(\tau) - \bar{u}\|_{2;\nu}^2 = \frac{d}{d\tau} \|u(\tau)\|_{2;\nu}^2,$$

so that

$$\frac{d}{d\tau} \|u(\tau) - \bar{u}\|_{2;\nu}^2 = -\frac{8m}{(m+1)^2} \left\| \nabla \left( u^{\frac{m+1}{2}} \right) (\tau) \right\|_{2;\mu}^2 \leq -\frac{8m}{M_{P,2/(m+1)}^{m+1} (m+1)^2} \|u(\tau) - \bar{u}\|_{2;\nu}^{m+1},$$

where we used (1.3.98) with  $\alpha = 2/(m+1)$  and  $v = u^{(m+1)/2}(\tau)$ . Solving the above differential inequality in the variable  $y(\tau) := \|u(\tau) - \bar{u}\|_{2;\nu}^2$  we end up with the same bound as (1.3.93), with  $q_0 = 2$  and  $u$  replaced by  $u - \bar{u}$ , which entails (1.3.97).  $\square$

We are now in position to prove Theorem 1.3.18.

*Proof of Theorem 1.3.18.* Estimate (1.3.23) for  $\varrho \in [1, 2]$  is just a consequence of Lemma (1.3.27) and the finiteness of the measure. In order to get (1.3.24) for  $\varrho \in (2, \infty)$ , consider first the following interpolation inequality:

$$\|u(t) - \bar{u}\|_{\varrho;\nu} \leq \|u(t) - \bar{u}\|_{2;\nu}^{\frac{2}{\varrho} - \varepsilon(m-1)} \|u(t) - \bar{u}\|_{2 + \frac{2(\varrho-2)}{\varepsilon\varrho(m-1)};\nu}^{1 - \frac{2}{\varrho} + \varepsilon(m-1)}.$$

Thanks to (1.3.97), the first factor in the right-hand side gives the same time rate as in (1.3.24). Hence, we only need to bound the second factor. Applying the smoothing estimate (1.3.7) with  $q_0 = 1$  and  $\varrho$  replaced by

$$p = p(\varepsilon, \varrho, m) := 2 + \frac{2(\varrho - 2)}{\varepsilon\varrho(m - 1)},$$

we obtain:

$$\|u(t) - \bar{u}\|_{p;\nu} \leq 2 \|u(t)\|_{p;\nu} \leq 2K \left( \|u_0\|_{1;\nu}^{1/p} + \|u_0\|_{1;\nu} \right) \quad \forall t \geq 1,$$

which finally entails (1.3.24).

Consider now an initial datum  $u_0 \in L^\infty(\Omega)$  with  $\bar{u}_0 \neq 0$ , so that  $\|u(t)\|_\infty \leq \|u_0\|_\infty$  for all  $t > 0$ . Let  $w := u/\bar{u} - 1$ . Given  $\varrho \in (1, \infty)$ , by plugging the test function  $w^{\varrho-1}$  in the weak formulation (1.3.1) (we can justify this rigorously), we find:

$$\frac{d}{d\tau} \|w(\tau)\|_{\varrho;\nu}^\varrho = -\varrho(\varrho - 1)m|\bar{u}|^{m-1} \int_\Omega |w(x, \tau) + 1|^{m-1} |w(x, \tau)|^{\varrho-2} |\nabla w(x, \tau)|^2 d\mu. \quad (1.3.99)$$

If we let

$$\Phi(y) = \int_0^y |z|^{\frac{\varrho}{2}-1} |z + 1|^{\frac{m-1}{2}} dz,$$

then (1.3.99) reads

$$\frac{d}{d\tau} \|w(\tau)\|_{\varrho;\nu}^\varrho = -\varrho(\varrho - 1)m|\bar{u}|^{m-1} \int_\Omega |\nabla(\Phi(w))(x, \tau)|^2 d\mu. \quad (1.3.100)$$

At this point we aim to exploit Lemma 1.3.26 to  $\Phi$ . Such a function is certainly continuous and increasing, and by means of de l'Hôpital's Theorem it is straightforward to check that

$$\lim_{y \rightarrow 0} \frac{\Phi(y)}{y^{\frac{\varrho}{2}}} = \frac{2}{\varrho}, \quad \lim_{y \rightarrow \pm\infty} \frac{\Phi(y)}{y^{\frac{\varrho+m-1}{2}}} = \frac{2}{\varrho + m - 1}.$$

Actually, since  $\varrho/2 \neq (\varrho + m - 1)/2$ , we cannot apply Lemma 1.3.26 to the function  $\Phi$  as it is. However, we can circumvent this difficulty by taking advantage of the fact that  $u_0 \in L^\infty(\Omega)$  and the quantity  $\|w(t)\|_\infty$  does not increase in time (trivial consequence of (1.3.99)), so that

$$\|w(t)\|_\infty \leq \|w(0)\|_\infty = \frac{\|u_0 - \bar{u}\|_\infty}{|\bar{u}|} =: R \quad \forall t \geq 0.$$

In particular, the behaviour of  $\Phi(y)$  for  $|y|$  large does not affect  $\Phi(w(t))$ . Hence, we are allowed to modify  $\Phi$  *e.g.* as follows:

$$\Phi_R(y) := \begin{cases} \Phi(y) & \text{if } y \in [-R-2, R], \\ \Phi(R) + \int_R^y z^{\frac{\varrho}{2}-1} (R+1)^{\frac{m-1}{2}} dz & \text{if } y > R, \\ \Phi(-R-2) - \int_y^{-R-2} |z|^{\frac{\varrho}{2}-1} (R+1)^{\frac{m-1}{2}} dz & \text{if } y < -R-2. \end{cases} \quad (1.3.101)$$

It is plain that  $\Phi_R$  fulfils all the hypotheses of Lemma 1.3.26 since

$$\lim_{y \rightarrow +\infty} \frac{\Phi_R(y)}{y^{\frac{\varrho}{2}}} = \lim_{y \rightarrow -\infty} \frac{\Phi_R(y)}{y^{\frac{\varrho}{2}}} = \frac{2(R+1)^{\frac{m-1}{2}}}{\varrho}.$$

Recalling that  $\bar{w} = 0$ , we can therefore assert that there exists a positive constant  $M_{\Phi_R}$  (depending on  $\varrho, m, \Omega, \nu, \mu$  and  $R$ ) such that

$$\|\Phi_R(w)\|_{2;\nu} \leq M_{\Phi_R} \|\nabla \Phi_R(w)\|_{2;\mu}. \quad (1.3.102)$$

Moreover,  $\Phi(w) = \Phi_R(w)$  by definition of  $\Phi_R$ , so that (1.3.100) and (1.3.102) entail

$$\frac{d}{d\tau} \|w(\tau)\|_{\varrho;\nu}^{\varrho} \leq -\frac{\varrho(\varrho-1)m|\bar{u}|^{m-1}}{M_{\Phi_R}^2} \|\Phi_R(w(\tau))\|_{2;\mu}^2.$$

In view of (1.3.101), it is apparent that there exists a suitable constant  $D = D(\varrho, m) > 0$  such that

$$D^{-1}|y|^{\frac{\varrho}{2}} \leq |\Phi_0(y)| \leq |\Phi_R(y)| \quad \forall y \in \mathbb{R},$$

whence

$$\frac{d}{d\tau} \|w(\tau)\|_{\varrho;\nu}^{\varrho} \leq -\frac{\varrho(\varrho-1)m|\bar{u}|^{m-1}}{M_{\Phi_R}^2 D^2} \|w^{\varrho/2}(\tau)\|_{2;\nu}^2 = -C_{\varrho} |\bar{u}|^{m-1} \|w(\tau)\|_{\varrho;\nu}^{\varrho},$$

where  $C$  is a positive constant as in the statement. Solving the above differential inequality in the variable  $y(\tau) := \|w(\tau)\|_{\varrho;\nu}^{\varrho}$  and going back to  $u - \bar{u}$  we deduce (1.3.25).

In the case where  $\rho_{\nu}$  has better local regularity properties ( $\nabla \rho_{\nu} \in L_{\text{loc}}^{\infty}(\Omega)$  will do), from the results of [72] and [151] we infer that the solution  $u(t)$  is Hölder continuous in any compact set  $\Omega_K \Subset \Omega$ . More precisely, we can apply Theorem 1.1 of [151] to  $\hat{u} := \rho_{\nu} u$ , which (in particular) ensures that there exists a constant  $\alpha = \alpha(m, \Omega, \Omega_K, \rho_{\nu}, \rho_{\mu}, \|u_0\|_{\infty}) \in (0, 1)$  such that

$$\sup_{t \geq 1} |\hat{u}(t)|_{C^{\alpha}(\Omega_K)} =: B(m, \Omega, \Omega_K, \rho_{\nu}, \rho_{\mu}, \|u_0\|_{\infty}) < \infty. \quad (1.3.103)$$

Since  $\rho_{\nu}$  is locally Lipschitz and bounded away from zero, it is plain that (1.3.103) also holds for  $u(t)$ . We point out that in case  $\rho_{\nu}$  is constant one may apply [72, Theorem 1.2 and Remark 4.1] as well. Now let  $\Omega'_K$  be a suitable subdomain of  $\Omega$  with  $\Omega_K \Subset \Omega'_K \Subset \Omega$  (for instance,  $\Omega'_K = \Omega_K + B_{\epsilon}$ , where  $0 < \epsilon < \text{dist}(\Omega_K, \partial\Omega)$ ). Take a regular cut-off function  $\eta$  such that  $\eta \equiv 1$  in  $\Omega_K$  and  $\text{supp } \eta \Subset \Omega'_K$ . Reasoning likewise, it is not difficult to deduce that

$$\sup_{t \geq 1} |\eta(u(t) - \bar{u})|_{C^{\alpha}(\mathbb{R}^d)} < \infty \quad (1.3.104)$$

for another constant  $\alpha$  as above. A classical interpolation result due to Gagliardo and Nirenberg (see *e.g.* [142, p. 126], [30, Appendix A.3] or [100, 101]) entails the validity of the following inequality:

$$\|\eta(u(t) - \bar{u})\|_{L^{\infty}(\mathbb{R}^d)} \leq G \|\eta(u(t) - \bar{u})\|_{C^{\alpha}(\mathbb{R}^d)}^{\frac{d}{d+\alpha\varrho}} \|\eta(u(t) - \bar{u})\|_{L^{\varrho}(\mathbb{R}^d)}^{\frac{\alpha\varrho}{d+\alpha\varrho}}, \quad (1.3.105)$$

for a suitable  $G > 0$  that depends on  $\alpha, \varrho$  and  $d$ . Hence, estimate (1.3.26) follows from (1.3.25), (1.3.104) and (1.3.105).  $\square$

Note that, in the above proof of (1.3.25) (and (1.3.26)), the assumption  $u_0 \in L^\infty(\Omega)$  is crucial. In fact the way the constant  $M_{\Phi_R}$  in (1.3.102) depends on  $R$ , and so on  $\|u_0\|_\infty$ , is completely unknown: Lemma 1.3.26 takes advantage of an argument by contradiction. For the same reason, we can give no explicit bound over  $C$  (and  $C_K$ ).

### 1.3.4.2 Counterexamples to uniform convergence to the mean value

In Paragraph 1.3.3.2, by means of an explicit counterexample, we have seen that in general validity of the Poincaré-type inequality (1.3.6) yields a smoothing effect for solutions to (1.1.2) which works up to  $L^\varrho(\Omega; \nu)$ , with  $\varrho$  *strictly* smaller than infinity. However, in Theorem 1.3.18 we proved that, for bounded initial data with nonzero mean, as a consequence of the (zero-mean) Poincaré-type inequality (1.3.14) solutions converge (with exponential rate) locally uniformly to their mean value. It seems therefore natural to ask whether, at least for bounded initial data, such convergence could be *globally* uniform. By means of another counterexample, we shall now see that the answer is again negative (in general).

Let us consider problem (1.1.2) with the following choices:

$$\Omega = (0, 1), \quad \rho_\nu(x) = x^{\beta-2}, \quad \rho_\mu(x) = x^\beta, \quad \beta > 1, \quad m \geq 2.$$

In this context, the (zero-mean) Poincaré-type inequality (1.3.14) holds true (see Appendix 1.A, Paragraph 1.A.2.1). We are looking for a function  $r : \mathbb{R}^+ \mapsto (0, 1)$  which is regular, decreasing, with  $\lim_{t \rightarrow +\infty} r(t) = 0$  and such that

$$\hat{u}(x, t) := \begin{cases} 0 & \text{if } x \in [0, r(t)/2], \\ \frac{2x}{r(t)} - 1 & \text{if } x \in (r(t)/2, r(t)], \\ 1 & \text{if } x \in (r(t), 1] \end{cases} \quad (1.3.106)$$

is a supersolution to (1.1.2). Because  $\hat{u}_x(x, t)$  vanishes in neighbourhoods of  $x = 0$  and  $x = 1$  for all  $t > 0$ , it is enough to check that  $\hat{u}$  is a supersolution in the distributional sense. This amounts to requiring that

$$\rho_\nu(x) \hat{u}_t(x, t) \geq (\rho_\mu)_x(x) (\hat{u}^m)_x(x, t) + \rho_\mu(x) (\hat{u}^m)_{xx}(x, t) \quad \text{in } \mathcal{D}'((0, 1) \times \mathbb{R}^+). \quad (1.3.107)$$

After some straightforward computations, we get:

$$\begin{aligned} \hat{u}_t(x, t) &= -\frac{2r'(t)x}{r^2(t)} \chi_{(r(t)/2, r(t)]}(x), \\ (\hat{u}^m)_x(x, t) &= \frac{2m}{r(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-1} \chi_{(r(t)/2, r(t)]}(x), \\ (\hat{u}^m)_{xx}(x, t) &= v(x, t) - \frac{2m}{r(t)} \delta_{x=r(t)}(x, t), \end{aligned} \quad (1.3.108)$$

where we set

$$v(x, t) := \frac{4m(m-1)}{r^2(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-2} \chi_{(r(t)/2, r(t)]}(x).$$

Since the contribution of the Dirac mass in (1.3.108) is negative, we can neglect it, so that the validity of (1.3.107) is ensured provided

$$\begin{aligned} -x^{\beta-2} \frac{2r'(t)x}{r^2(t)} &\geq \frac{2m\beta x^{\beta-1}}{r(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-1} + \frac{4m(m-1)x^\beta}{r^2(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-2} \\ &\forall t > 0, \quad \forall x \in (r(t)/2, r(t)). \end{aligned} \quad (1.3.109)$$



Dividing (1.3.109) by  $x^{\beta-1}$ , we obtain:

$$-\frac{2r'(t)}{r^2(t)} \geq \frac{2m\beta}{r(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-1} + \frac{4m(m-1)x}{r^2(t)} \left( \frac{2x}{r(t)} - 1 \right)^{m-2}. \quad (1.3.110)$$

It is then apparent that, for any given  $t > 0$ , (1.3.110) holds true for all  $x \in (r(t)/2, r(t))$  if and only if it holds true at  $x = r(t)$  (recall that  $m \geq 2$ ). Hence, letting  $x = r(t)$  in the r.h.s. of (1.3.110), we get:

$$\frac{r'(t)}{r(t)} \leq -m(\beta + 2(m-1)) =: -c(m, \beta) < 0.$$

Integrating between 0 and  $t$  we end up with

$$r(t) \leq r(0)e^{-c(m, \beta)t}. \quad (1.3.111)$$

If we pick  $r(0)$  small enough, a function  $r(t)$  as in the r.h.s. of (1.3.111) fulfils all the requirements we listed in the beginning of the construction. So we have proved that there exists a supersolution to (1.1.2) of the type of (1.3.106). In particular, the *solution* to the same problem associated to any nonnegative (and nontrivial) initial datum  $u_0(x) \leq \hat{u}(x, 0)$  will be less than or equal to  $\hat{u}(x, t)$  for all  $t > 0$ : as a result, it cannot converge uniformly to its mean value  $\bar{u} > 0$  since it is forced to be zero in the interval  $(0, r(t)/2)$  for all  $t > 0$ . Thanks to a routine time-scaling argument it is easy to show that the same holds true for solutions corresponding to any uniformly bounded initial datum which is less than or equal to zero in a neighbourhood of  $x = 0$  and has positive mean.

As the reader may guess, the assumption  $m \geq 2$  in the above example cannot be relaxed, since for  $m \in (1, 2)$  the right-hand side of (1.3.109) blows up as  $x \rightarrow r(t)/2$ . However, by performing some modifications, we are still able to construct a suitable supersolution. Indeed, upon setting

$$\tilde{u}(x, t) := \begin{cases} \frac{x}{r(t)} & \text{if } x \in [0, r(t)], \\ 1 & \text{if } x \in (r(t), 1] \end{cases}$$

and proceeding as above, we get the condition

$$-\frac{r'(t)}{r^2(t)} \geq \frac{m\beta x^{m-1}}{r^m(t)} + \frac{m(m-1)x^{m-1}}{r^m(t)}, \quad (1.3.112)$$

which must be valid for all  $x \in (0, r(t))$ . The maximum of the r.h.s. of (1.3.112) is clearly attained at  $x = r(t)$ , so that by substituting such value in the latter and solving the corresponding differential inequality we end up again with (1.3.111) (up to a different positive constant  $c(m, \beta)$ ). Unlike  $\hat{u}$ , the supersolution  $\tilde{u}$  *does not* have a zero derivative at  $x = 0$ . Luckily, it turns out that this does not matter. In fact, the space of absolutely continuous functions in  $[0, 1]$  which vanish in a neighbourhood of  $x = 0$  is dense in  $H^1((0, 1); x^\alpha, x^\beta)$ , provided  $\alpha \in \mathbb{R}$  and  $\beta \geq 1$  (see *e.g.* the proof of [95, Theorem 2.11]). This means that in order to prove that  $\tilde{u}$  is a supersolution to (1.1.2) it is enough to test it against functions which vanish in a neighbourhood of  $x = 0$ . The fact that  $\tilde{u}$  is zero at  $x = 0$  for all  $t > 0$  makes sure that uniform convergence to the mean value  $\bar{u} > 0$  does not occur for a class of bounded data as above.

We point out that in the case  $d \geq 3$  (let  $d \in \mathbb{N}$ ) and  $\beta = d - 1$ , our analysis amounts to considering radial solutions to the Neumann problem (1.1.2) in  $B_1 \subset \mathbb{R}^d$ , with the singular weight  $\rho_\nu(x) = |x|^{-2}$  and  $\rho_\mu \equiv 1$ . In [111] the authors studied the same problem in the whole Euclidean space: our counterexamples, at least in a neighbourhood of the origin, are in agreement with [111, Theorem 1.1].

## 1.3.4.3 Sobolev- and Gagliardo-Nirenberg-type inequalities

If Sobolev- or Gagliardo-Nirenberg-type inequalities of the type of (1.3.16), (1.3.19) or (1.3.21) hold true, then the results we proved in Paragraph 1.3.4.1 leave room for improvement, as we shall now see. We begin with the case of zero-mean solutions.

*Proof of Theorems 1.3.15, 1.3.16 and 1.3.17.* Let  $q_0 \in (1, \infty)$  and, with no loss of generality,  $u_0 \in L^\infty(\Omega)$ , with  $\bar{u}_0 = 0$ . First of all, notice that estimate (1.3.93) holds true. Indeed, any of the inequalities (1.3.16), (1.3.19) or (1.3.21) implies the validity of the (zero-mean) Poincaré-type inequality (1.3.14) (Remark 1.3.21), which is the only functional tool one needs to prove (1.3.93) (recall the proof of Theorem 1.3.14). Hence, taking advantage of the latter together with the smoothing estimates (1.3.9), (1.3.11) or (1.3.13) evaluated between  $t$  and  $t/2$ , we obtain ( $K$  denotes a positive constant as in the statements):

$$\|u(t)\|_\infty \leq K \left[ t^{-\frac{\sigma}{(\sigma-1)q_0 + \sigma(m-1)}} \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{(\sigma-1)q_0}{(m-1)[(\sigma-1)q_0 + \sigma(m-1)]}} + \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{1}{m-1}} \right], \quad (1.3.113)$$

$$\|u(t)\|_\infty \leq K \left[ t^{-\frac{\lambda}{(\lambda+2)q_0 + \lambda(m-1)}} \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{(\lambda+2)q_0}{(m-1)[(\lambda+2)q_0 + \lambda(m-1)]}} + \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{1}{m-1}} \right] \quad (1.3.114)$$

or

$$\|u(t)\|_\infty \leq K \left[ t^{-\frac{1}{q_0 + m - 1}} \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{q_0}{(m-1)(q_0 + m - 1)}} + \left( t + \|u_0\|_{q_0; \nu}^{1-m} \right)^{-\frac{1}{m-1}} \right], \quad (1.3.115)$$

respectively, for all  $t > 0$ , from which the corresponding estimates (1.3.17), (1.3.20) and (1.3.22) follow. In order to prove the validity of such estimates down to  $q_0 = 1$ , we can reason exactly as in the proof of Theorem 1.3.14. That is, by interpolation and iteration arguments we infer the bounds

$$\|u(t)\|_{2; \nu} \leq K t^{-\frac{\sigma}{2(\sigma m - 1)}} \|u_0\|_{1; \nu}^{\frac{\sigma(m+1)-2}{2(\sigma m - 1)}} \quad \forall t > 0,$$

$$\|u(t)\|_{2; \nu} \leq K t^{-\frac{\lambda}{2\lambda m + 4}} \|u_0\|_{1; \nu}^{\frac{\lambda(m+1)+4}{2\lambda m + 4}} \quad \forall t > 0$$

or

$$\|u(t)\|_{2; \nu} \leq K t^{-\frac{1}{2m}} \|u_0\|_{1; \nu}^{\frac{m+1}{2m}} \quad \forall t > 0;$$

then we plug the latter in estimates (1.3.113)–(1.3.115), with  $q_0 = 2$  and the time origin shifted to  $t/2$ .  $\square$

As concerns solutions with nonzero mean, before proving Theorem 1.3.19 we need some preliminary steps, one of which is a delicate refinement of Lemma 1.3.26.

**Lemma 1.3.28.** *Given  $r \geq 1/2$ ,  $m > 1$  and a fixed number  $R > 1$ , let*

$$\Phi_{r,m}(y) := \int_0^y |z|^{r-1} |z+1|^{\frac{m-1}{2}} dz \quad \forall y \in [-R, R].$$

*Then there exist two positive constants  $C_1 = C_1(m)$  and  $C_2 = C_2(m, R)$  such that*

$$\frac{C_1}{r^{1+1\lfloor \frac{m-1}{2} \rfloor}} |y|^r \leq |\Phi_{r,m}(y)| \leq \frac{C_2}{r} |y|^r \quad \forall y \in [-R, R]. \quad (1.3.116)$$

*Proof.* One has to study the ratio  $|\Phi_{r,m}(y)|/|y|^r$ . The bound from above is easily achievable, since

$$|\Phi_{r,m}(y)| \leq (R+1)^{\frac{m-1}{2}} \left| \int_0^y |z|^{r-1} dz \right| = \frac{(R+1)^{\frac{m-1}{2}}}{r} |y|^r \quad \forall y \in [-R, R].$$

In order to get a lower bound, we begin with the case  $m = 3$ . Recall that, for  $y \neq 0$ , we use the convention  $y^r := |y|^{r-1}y$ . First of all notice that

$$\Phi_{r,m}(y)/y^r \geq \frac{\int_0^y z^{r-1} dz}{y^r} = \frac{1}{r} \quad \forall y \in (0, R]; \quad (1.3.117)$$

moreover,

$$\Phi_{r,3}(y)/y^r = \frac{\int_0^{|y|} z^{r-1} (1-z) dz}{|y|^r} = \frac{1}{r} - \frac{|y|}{r+1} \geq \frac{1}{r(r+1)} \quad \forall y \in [-1, 0). \quad (1.3.118)$$

So we are left with studying the minimum of  $\Phi_{r,3}(y)/y^r$  as  $y$  varies in  $[-R, -1)$ . We have:

$$\begin{aligned} f_r(y) := \Phi_{r,3}(y)/y^r &= \frac{\int_0^{|y|} z^{r-1} |z-1| dz}{|y|^r} = \frac{\int_0^1 z^{r-1} (1-z) dz + \int_1^{|y|} z^{r-1} (z-1) dz}{|y|^r} \\ &= \left( \frac{2}{r(r+1)} + \frac{|y|^{r+1}}{r+1} - \frac{|y|^r}{r} \right) / |y|^r \\ &= \frac{2}{r(r+1)} |y|^{-r} + \frac{|y|}{r+1} - \frac{1}{r} \quad \forall y \in [-R, -1). \end{aligned} \quad (1.3.119)$$

Since

$$\frac{d}{ds} \left( \frac{2}{r(r+1)} s^{-r} + \frac{s}{r+1} - \frac{1}{r} \right) = -\frac{2}{r+1} s^{-r-1} + \frac{1}{r+1},$$

whose zero is attained at  $s_0 := 2^{1/(r+1)}$ , there holds

$$f_r(y) \geq \frac{1}{r} \left( 2^{\frac{1}{r+1}} - 1 \right) \quad \forall y \in [-R, -1). \quad (1.3.120)$$

Finally, it is apparent that

$$\frac{1}{r} \left( 2^{\frac{1}{r+1}} - 1 \right) \geq \frac{\log 2}{r(r+1)}. \quad (1.3.121)$$

By collecting (1.3.117)–(1.3.121) we then get the lower bound in (1.3.116) for  $m = 3$ .

Now let us consider the case  $m > 3$ . Since the function  $s \mapsto |s|^{(m-1)/2}$  is convex, in view of Jensen's inequality we have:

$$\frac{|\Phi_{r,m}(y)|}{|y|^r} = \frac{1}{r} \frac{\int_0^{|y|} z^{r-1} |z-1|^{\frac{m-1}{2}} dz}{\int_0^{|y|} z^{r-1} dz} \geq r^{\frac{m-3}{2}} \left( \frac{\int_0^{|y|} z^{r-1} |z-1| dz}{|y|^r} \right)^{\frac{m-1}{2}} \quad \forall y \in [-R, 0),$$

and we can bound the right-hand side from below just by applying to it the above estimates for  $m = 3$ , so as to obtain

$$\frac{|\Phi_{r,m}(y)|}{|y|^r} \geq \frac{C_1}{r^{1+\frac{m-1}{2}}} \quad \forall y \in [-R, 0) \quad (1.3.122)$$

for some positive constant  $C_1 = C_1(m)$ . Hence, (1.3.117) and (1.3.122) provide us with the claimed lower bound in (1.3.116).

We still have to deal with the case  $m \in (1, 3)$ , that is when the function  $s \mapsto |s|^{(m-1)/2}$  is no more convex. Again, since (1.3.117) holds for all  $m > 1$ , we can limit ourselves to the analysis of the ratio  $|\Phi_{r,m}(y)|/|y|^r$  for  $y \in [-R, 0)$ . Since  $(m-1)/2 =: \alpha \in (0, 1)$ , straightforward computations show that

$$|z-1|^\alpha \geq |z^\alpha - 1| \quad \forall z \in [0, R],$$

whence

$$\frac{|\Phi_{r,m}(y)|}{|y|^r} = \frac{\int_0^{|y|} z^{r-1} |z-1|^{\frac{m-1}{2}} dz}{|y|^r} \geq \frac{\int_0^{|y|} z^{r-1} |z^{\frac{m-1}{2}} - 1| dz}{|y|^r} \quad \forall y \in [-R, 0). \quad (1.3.123)$$

Thanks to (1.3.123), there is no difficulty in proceeding as in the case  $m = 3$  in order to prove that also the r.h.s. of (1.3.123) is bounded from below by a constant times  $1/r^2$ .  $\square$

**Lemma 1.3.29.** *Suppose that there exists a constant  $M_{PS} > 0$  such that the Poincaré-Sobolev-type inequality*

$$\|v - \bar{v}\|_{2\sigma; \nu} \leq M_{PS} \|\nabla v\|_{2; \mu} \quad \forall v \in H^1(\Omega; \nu, \mu) \quad (1.3.124)$$

holds true for some  $\sigma \geq 1$ . Then for any fixed  $m > 1$  and  $R > 1$  there exists another constant  $M_* > 0$ , which depends on  $m, \Omega, \nu, \mu$  and  $R$ , such that the inequality

$$\|\Phi_{r,m}(\xi)\|_{2\sigma; \nu} \leq M_* \|\nabla \Phi_{r,m}(\xi)\|_{2; \mu}$$

holds true for all  $r \geq 1/2$  and every function  $\xi \in L^\infty(\Omega)$  with  $\bar{\xi} = 0$ ,  $\|\xi\|_\infty \leq R$  and  $\Phi_{r,m}(\xi) \in H^1(\Omega; \nu, \mu)$ .

*Proof.* As in the proof of Lemma 1.3.26, we argue by contradiction. If the assertion were false, then there would exist a sequence of numbers  $r_n \geq 1/2$  and a corresponding sequence of nontrivial functions  $\{\xi_n\}$  such that  $r_n \rightarrow \infty$ ,  $\|\xi_n\|_\infty \leq R$ ,  $\bar{\xi}_n = 0$ ,  $\Phi_{r_n,m}(\xi_n) \in H^1(\Omega; \nu, \mu)$  and

$$\|\nabla \Phi_{r_n,m}(\xi_n)\|_{2; \mu} \leq \frac{1}{r_n} \|\Phi_{r_n,m}(\xi_n)\|_{2\sigma; \nu} \quad \forall n \in \mathbb{N}. \quad (1.3.125)$$

The fact that  $\{r_n\}$  cannot accumulate at some finite  $r$  and is therefore forced to go to infinity is a direct consequence of the method of proof of Lemma 1.3.26. Let

$$a_n := \|\Phi_{r_n,m}(\xi_n)\|_{2\sigma; \nu}$$

and

$$\Psi_n := \frac{\Phi_{r_n,m}(\xi_n)}{a_n}.$$

From (1.3.125) we deduce that

$$\|\Psi_n\|_{2\sigma; \nu} = 1, \quad \|\nabla \Psi_n\|_{2; \mu} \leq \frac{1}{r_n} \quad \forall n \in \mathbb{N}.$$

Applying the Poincaré-Sobolev-type inequality (1.3.124) to  $\Psi_n$  we find

$$\|\Psi_n - \bar{\Psi}_n\|_{2\sigma; \nu} \leq \frac{M_{PS}}{r_n} \quad \forall n \in \mathbb{N}. \quad (1.3.126)$$

In particular, since  $\|\Psi_n\|_{2\sigma; \nu} = 1$ , the sequence of real numbers  $\{\bar{\Psi}_n\}$  is bounded and thus converges to some finite number  $c_0$  (up to a subsequence which we do not relabel). In view of (1.3.126),  $\{\Psi_n\}$  also converges in  $L^{2\sigma}(\Omega; \nu)$  to such constant, so that  $c_0 \neq 0$  since  $\|\Psi_n\|_{2\sigma; \nu} = 1$ .

Consider now the sequence

$$\mathcal{Z}_n(x) := \frac{\xi_n(x)}{a_n^{1/r_n}}.$$

First of all, we aim to prove that  $\{\mathcal{Z}_n\}$  converges pointwise to a nonzero constant. Indeed, still up to subsequences, we know that  $\{\Psi_n\}$  converges pointwise to the nonzero constant  $c_0$ . Let us rewrite  $\mathcal{Z}_n$  in the following way:

$$\mathcal{Z}_n(x) = \left[ \frac{|\xi_n(x)|^{r_n}}{|\Phi_{r_n,m}(\xi_n(x))|} \Psi_n(x) \right]^{\frac{1}{r_n}}.$$

Thanks to Lemma 1.3.28 and to the fact that  $|\xi_n(x)| \leq R$ , we have:

$$\left[ \frac{r_n}{C_2} \right]^{\frac{1}{r_n}} |\Psi_n(x)|^{\frac{1}{r_n}} \leq |\mathcal{Z}_n(x)| \leq \left[ \frac{r_n^{1+1\lfloor \frac{m-1}{2} \rfloor}}{C_1} \right]^{\frac{1}{r_n}} |\Psi_n(x)|^{\frac{1}{r_n}}.$$

Letting  $n \rightarrow \infty$  we then infer that  $\{|\mathcal{Z}_n|\}$  converges pointwise to 1, so that  $\{\mathcal{Z}_n\}$  converges pointwise to 1 if  $c_0 > 0$  or to  $-1$  if  $c_0 < 0$ , in any case to a nonzero constant. We can finally prove that  $\{\mathcal{Z}_n\}$  also converges in  $L^1(\Omega; \nu)$  to such a nonzero constant. To this end, thanks to Egoroff's Theorem, it is enough to show that  $\int_E |\mathcal{Z}_n(x)| d\nu$  vanishes uniformly as  $n \rightarrow \infty$  and  $\nu(E) \rightarrow 0$ . We have:

$$\begin{aligned} \int_E |\mathcal{Z}_n(x)| d\nu &= \int_E \frac{|\xi_n(x)|}{a_n^{1/r_n}} d\nu \leq \left[ \frac{r_n^{1+1\lfloor \frac{m-1}{2} \rfloor}}{C_1} \right]^{\frac{1}{r_n}} \int_E |\Psi_n(x)|^{\frac{1}{r_n}} d\nu \\ &\leq \nu(E)^{1-\frac{1}{r_n}} \left[ \frac{r_n^{1+1\lfloor \frac{m-1}{2} \rfloor}}{C_1} \right]^{\frac{1}{r_n}} \|\Psi_n\|_{1;\nu}^{\frac{1}{r_n}}. \end{aligned}$$

The assertion follows since  $\{\Psi_n\}$  converges in  $L^{2\sigma}(\Omega; \nu)$  and in particular is bounded in  $L^1(\Omega; \nu)$ . We have therefore proved that the sequence of zero-mean functions  $\{\mathcal{Z}_n\}$  converges in  $L^1(\Omega; \nu)$  to a nonzero constant, a contradiction.  $\square$

With respect to the family of functions  $\Phi_{r,m}$ , Lemma 1.3.29 improves Lemma 1.3.26 in the sense that, basically, it ensures that the constant  $M_{\Phi_{r,m}}$  appearing in (1.3.87) does not depend on  $r$ : this turns out to be a crucial property that allows us to prove the next result, namely *uniform* convergence of solutions with nonzero mean to their mean value.

**Proposition 1.3.30.** *Let  $\nu(\Omega) < \infty$  and  $u_0 \in L^\infty(\Omega)$ , with  $\bar{u}_0 \neq 0$ . Suppose that one between the Sobolev-type inequality (1.3.16), the one-dimensional Gagliardo-Nirenberg-type inequality (1.3.19) and the two-dimensional Gagliardo-Nirenberg-type inequality (1.3.21) holds true. Then the solution  $u$  to (1.1.2) corresponding to the initial datum  $u_0$  converges uniformly to its mean value  $\bar{u}$ .*

*Proof.* We proceed by means of a Moser iteration. Indeed, for a given  $\varrho > q_0 > 1$ , let us plug the test function  $w^{\varrho-1}$  in the weak formulation (1.3.1), where we set  $w := u/\bar{u} - 1$ . We obtain:

$$\|w(T)\|_{\varrho;\nu}^\varrho + \varrho(\varrho-1)m|\bar{u}|^{m-1} \int_s^T \left\| \nabla \Phi_{\varrho/2,m}(w(\tau)) \right\|_{2;\mu}^2 d\tau = \|w(s)\|_{\varrho;\nu}^\varrho \quad \forall T > s \geq 0. \quad (1.3.127)$$

Since  $\|w(\tau)\|_\infty \leq \|w(0)\|_\infty + 1 =: R$ , we can apply to the l.h.s. of (1.3.127) Lemma 1.3.29 (with  $r = \varrho/2$ ), which entails

$$\|w(T)\|_{\varrho;\nu}^\varrho + \frac{\varrho(\varrho-1)m|\bar{u}|^{m-1}}{M_*^2} \int_s^T \left\| \Phi_{\varrho/2,m}(w(\tau)) \right\|_{2\sigma;\nu}^2 d\tau \leq \|w(s)\|_{\varrho;\nu}^\varrho \quad \forall T > s \geq 0. \quad (1.3.128)$$

Notice that either of the Gagliardo-Nirenberg-type inequalities (1.3.19) or (1.3.21) implies the validity of a Sobolev-type inequality, so we assumed with no loss of generality that the latter holds true for some  $\sigma > 1$ . Thanks to estimate (1.3.116) (again with  $r = \varrho/2$ ), (1.3.128) yields

$$\|w(T)\|_{\varrho;\nu}^\varrho + \frac{\varrho(\varrho-1)m|\bar{u}|^{m-1}}{M_*^2} \int_s^T \frac{C_1^2}{\left(\frac{\varrho}{2}\right)^{2+2\nu(m-1)}} \|w^{\varrho/2}(\tau)\|_{2\sigma;\nu}^2 d\tau \leq \|w(s)\|_{\varrho;\nu}^\varrho \quad \forall T > s \geq 0,$$

that is

$$\|w(T)\|_{\varrho;\nu}^\varrho + \frac{K}{\varrho^{2\nu(m-1)}} \int_s^T \|w(\tau)\|_{\sigma\varrho;\nu}^\varrho d\tau \leq \|w(s)\|_{\varrho;\nu}^\varrho \quad \forall T > s \geq 0, \quad (1.3.129)$$

where  $K > 0$  is a suitable constant that depends on  $q_0, m, M_*, \sigma$  and  $|\bar{u}|$  (but is independent of  $\varrho$ ), which we do not relabel below. Given any  $t > 0$ , let us define the sequences  $t_k := (1-2^{-k})t$  and  $p_{k+1} = \sigma p_k$ , where  $p_0 := q_0$ . Using (1.3.129) with  $s = t_k, T = t_{k+1}, \varrho = p_k$  and taking advantage of the fact that  $\|w(\tau)\|_{p;\nu}$  is non-increasing in  $\tau$  for all  $p \in [1, \infty]$ , we obtain:

$$\|w(t_{k+1})\|_{p_{k+1};\nu} \leq \left( \frac{p_k^{2\nu(m-1)} 2^{k+1}}{K} \right)^{\frac{1}{p_k}} t^{-\frac{1}{p_k}} \|w(t_k)\|_{p_k;\nu},$$

or equivalently

$$\|w(t_{k+1})\|_{p_{k+1};\nu} \leq K^{\frac{k+1}{\sigma^k q_0}} t^{-\frac{1}{\sigma^k q_0}} \|w(t_k)\|_{p_k;\nu}. \quad (1.3.130)$$

Performing an iteration of (1.3.130), we end up with

$$\|w(t_{k+1})\|_{p_{k+1};\nu} \leq K^{\sum_{h=0}^k \frac{h+1}{\sigma^h q_0}} t^{-\sum_{h=0}^k \frac{1}{\sigma^h q_0}} \|w(0)\|_{q_0;\nu}.$$

Hence,

$$\|w(t)\|_\infty = \lim_{k \rightarrow \infty} \|w(t)\|_{p_{k+1};\nu} \leq \liminf_{k \rightarrow \infty} \|w(t_{k+1})\|_{p_{k+1};\nu} \leq K t^{-\frac{\sigma}{(\sigma-1)q_0}} \|w(0)\|_{q_0;\nu}, \quad (1.3.131)$$

and the assertion follows by letting  $t \rightarrow \infty$ .  $\square$

We are now ready to prove the main asymptotic result of this subsection: as a consequence of the validity of a Sobolev-type or Gagliardo-Nirenberg-type inequality, solutions to (1.1.2) with nonzero mean converge uniformly to their mean value with a precise exponential rate.

*Proof of Theorem 1.3.19.* We first assume  $u_0 \in L^\infty(\Omega)$ . In view of Proposition 1.3.30, in particular there exists a suitable  $\tau_0 = \tau_0(m, M_*, \sigma, |\bar{u}|, \|u_0\|_\infty) > 0$  such that

$$\operatorname{ess\,inf}_{x \in \Omega} |u(x, t)| \geq \frac{|\bar{u}|}{2} \quad \forall t \geq \tau_0. \quad (1.3.132)$$

By combining (1.3.127), (1.3.132) and recalling the definition of  $\Phi_{r,m}$ , it is straightforward to deduce the following inequality (let  $\varrho > 1$ ):

$$\frac{d}{d\tau} \|u(\tau) - \bar{u}\|_{\varrho;\nu}^\varrho \leq -\frac{4(\varrho-1)m}{\varrho} \left( \frac{|\bar{u}|}{2} \right)^{m-1} \left\| \nabla \left[ (u(\tau) - \bar{u})^{\frac{\varrho}{2}} \right] \right\|_{2;\mu}^2 \quad \forall \tau \geq \tau_0. \quad (1.3.133)$$

The choice  $\varrho = 2$  in (1.3.133) immediately leads to

$$\|u(t) - \bar{u}\|_{2;\nu} \leq e^{-\frac{m}{M_P^2} \left(\frac{|\bar{u}|}{2}\right)^{m-1} (t-\tau_0)} \|u(\tau_0) - \bar{u}\|_{2;\nu} \quad \forall t \geq \tau_0, \quad (1.3.134)$$

where in the r.h.s. we exploited the Poincaré-type inequality (1.3.14), which trivially holds true as a consequence of (1.3.16), (1.3.19) or (1.3.21). Replacing  $t$  with  $(t + \tau_0)/2$  in (1.3.134) and using estimate (1.3.131) with  $q_0 = 2$  and the time origin shifted to  $(t + \tau_0)/2$ , we end up with the bound

$$\|u(t) - \bar{u}\|_\infty \leq G e^{-Ct} \|u_0 - \bar{u}\|_{2;\nu} \quad \forall t \geq \tau_0 + 1, \quad (1.3.135)$$

where  $G > 0$  is a suitable constant that depends on  $m, \Omega, \nu, \mu, |\bar{u}|, \|u_0\|_\infty$  (which may change from line to line and we do not relabel) and

$$C := \frac{m|\bar{u}|^{m-1}}{M_P^2 2^m}.$$

Now let us pick  $\tau_1 \geq \tau_0 + 1$  so large that

$$\frac{1}{2}|\bar{u}| \leq |\bar{u}| - G e^{-Ct} \|u_0 - \bar{u}\|_{2;\nu} \quad \forall t \geq \tau_1. \quad (1.3.136)$$

Since

$$|u(x, \tau)| \geq |\bar{u}| - |u(x, \tau) - \bar{u}| \geq |\bar{u}| - G e^{-C\tau} \|u_0 - \bar{u}\|_{2;\nu} \quad \forall \tau \geq \tau_1, \quad \text{for a.e. } x \in \Omega,$$

we have:

$$\begin{aligned} \frac{d}{d\tau} \|u(\tau) - \bar{u}\|_{2;\nu}^2 &= -2m \int_\Omega |u(x, \tau)|^{m-1} |\nabla u(x, \tau)|^2 d\mu \\ &\leq -2m \left( |\bar{u}| - G e^{-C\tau} \|u_0 - \bar{u}\|_{2;\nu} \right)^{m-1} \|\nabla u(\tau)\|_{2;\mu}^2 \\ &\leq -\frac{2m}{M_P^2} \left( |\bar{u}| - G e^{-C\tau} \|u_0 - \bar{u}\|_{2;\nu} \right)^{m-1} \|u(\tau) - \bar{u}\|_{2;\nu}^2 \quad \forall \tau \geq \tau_1. \end{aligned} \quad (1.3.137)$$

Integrating (1.3.137) between  $\tau_1$  and  $t$ , we get (notice that here and above we exploit the fact that  $\|u(\tau) - \bar{u}\|_{2;\nu}$  is non-increasing in  $\tau$ ):

$$\|u(t) - \bar{u}\|_{2;\nu}^2 \leq \|u_0 - \bar{u}\|_{2;\nu}^2 e^{-\frac{2m}{M_P^2} \int_{\tau_1}^t (|\bar{u}| - G e^{-C\tau} \|u_0 - \bar{u}\|_{2;\nu})^{m-1} d\tau}. \quad (1.3.138)$$

Letting  $\epsilon := G e^{-C\tau} \|u_0 - \bar{u}\|_{2;\nu}$ , in view of the elementary inequalities (recall (1.3.136))

$$(|\bar{u}| - \epsilon)^{m-1} \geq |\bar{u}|^{m-1} - (m-1)|\bar{u}|^{m-2}\epsilon \quad \forall m \geq 2,$$

$$(|\bar{u}| - \epsilon)^{m-1} \geq |\bar{u}|^{m-1} - (m-1)(|\bar{u}| - \epsilon)^{m-2}\epsilon \geq |\bar{u}|^{m-1} - 2^{2-m}(m-1)|\bar{u}|^{m-2}\epsilon \quad \forall m \in (1, 2)$$

and of a straightforward computation of the integral appearing at the exponent, estimate (1.3.138) yields

$$\|u(t) - \bar{u}\|_{2;\nu} \leq G e^{-\frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall t \geq \tau_1. \quad (1.3.139)$$

Taking advantage of (1.3.135) with the time origin shifted to  $t - \tau_0 - 1$  and (1.3.139) evaluated at time  $t - \tau_0 - 1$  (in place of  $t$ ) we finally obtain

$$\|u(t) - \bar{u}\|_\infty \leq G e^{-\frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall t \geq \tau_2, \quad (1.3.140)$$

where we set  $\tau_2 := \tau_0 + \tau_1 + 1$ . The fact that in (1.3.27) we can pick *e.g.*  $\tau_2 = 1$  and any  $u_0 \in L^1(\Omega; \nu)$  with  $\bar{u}_0 \neq 0$  (we also replace the dependence of  $G$  on  $\|u_0\|_\infty$  with  $\|u_0\|_{1;\nu}$  accordingly) is just a consequence of (1.3.140) and the smoothing effects (1.3.9), (1.3.11) or (1.3.13).  $\square$

As the reader might have noticed, in the proofs of Theorems 1.3.14–1.3.19 at some point we computed time derivatives of  $L^p(\Omega; \nu)$  norms of  $u$  (or  $u - \bar{u}$ ). This can be justified rigorously, for instance, since solutions are *strong*, as we recalled in Paragraph 1.2.3.1 above (similar remarks apply to the Neumann problem as well). Alternatively, one can prove all the key energy estimates (like *e.g.* (1.3.127)) on the regular solutions to the approximate problems we constructed in the the proof of Theorem 1.3.4 and then pass to the limit.

#### 1.3.4.4 Improvements on previous results and comments on sharpness

As concerns zero-mean solutions, in [27, Corollary 1.3] the authors established the following estimate (in the framework of compact Riemannian manifolds without boundary where the Sobolev inequality holds true, see Paragraph 1.3.3.5):

$$\|u(t)\|_\infty \leq \frac{C}{\left(B(t-1) + \|u_0\|_{q_0}^{1-m}\right)^{\frac{1-\theta}{m-1}}} \quad \forall t > 1 \quad (1.3.141)$$

for suitable positive constants  $B$  and  $C$ , where  $\theta \in (0, 1)$  is defined as in (1.3.84). It is apparent that (1.3.141) is weaker than (1.3.17) since it provides a slower decay rate for  $\|u(t)\|_\infty$  as  $t \rightarrow \infty$  ( $t^{-(1-\theta)/(m-1)}$  versus  $t^{-1/(m-1)}$ ). Actually, at least in the framework of regular bounded Euclidean domains, a decay rate of order  $t^{-1/(m-1)}$  was already known to hold for bounded initial data due to the results of the pioneering work [3] (see in particular Theorems 3.1 and 4.1), even though the estimate given there still depended in a nontrivial way on the  $L^\infty$  norm of the initial datum and in particular could not be extended to general  $L^1$  data. We point out that such rate is in fact sharp: one can consider the separable solutions constructed in [3, Section 2], whose  $L^\infty$  decay rate is precisely  $t^{-1/(m-1)}$ .

As we have seen throughout Subsection 1.3.4, solutions with nonzero mean converge *uniformly* to their mean value. In this regard, an immediate consequence of Proposition 1.3.30 is the following.

**Corollary 1.3.31.** *Let  $\nu(\Omega) < \infty$  and let one between the (zero-mean) Sobolev-type inequality (1.3.16) and the Gagliardo-Nirenberg-type inequalities (1.3.19) or (1.3.21) hold true. Then the support of any solution  $u(t)$  of (1.1.2) corresponding to a compactly supported initial datum  $u_0$  with nonzero mean becomes the whole  $\Omega$  for  $t$  large enough.*

The analysis of the properties of the support of solutions is a widely investigated topic in the literature. Without any claim of completeness, we quote *e.g.* the papers [107, 117, 118, 147, 155, 157, 168]. The interesting point in Corollary 1.3.31 lies in the fact that, at least for a Neumann-type problem like (1.1.2), solutions starting from nonzero-mean data with compact support lose this property in finite time *as a sole consequence of the validity of a suitable functional inequality*. A result of this type seems to have no analogue in the literature. In [27] uniform convergence to the mean value is also established, but in order to do so the authors take advantage of the uniform spatial Hölder continuity of solutions, which may not hold in the general framework we deal with.

Let us now compare Theorem 1.3.19 to some previous results. In [116] a convergence analysis is carried out in one-dimensional Euclidean space: actually, more general nonlinearities and more general weights are considered there, but only local uniform convergence to the mean value (without rates) for nonnegative solutions is proved. We have to remark, however, that for such weights Sobolev-type or Gagliardo-Nirenberg-type inequalities may or may not hold (depending on the behaviour of the weight at infinity), and therefore one cannot expect uniform convergence in general. On the other hand, in [27] the authors provided a



convergence rate of order

$$e^{-\frac{(1-\theta)m}{M_P^2 2^{m-1}} |\bar{u}|^{m-1} t}$$

(as recalled above, for nonzero-mean solutions on compact Riemannian manifolds without boundary), which is clearly weaker than the one we obtained here. It is worth mentioning that in fact an exponential rate of the type of

$$e^{-\varepsilon \frac{m}{M_P^2} |\bar{u}|^{m-1} t} \quad \forall \varepsilon \in (0, 1)$$

could have also been proved by exploiting the uniform convergence to the mean value together with Theorem 3.3 of [3]. Theorem 1.3.19 ensures that one can actually choose  $\varepsilon = 1$ .

Let  $\rho_\nu = \rho_\mu \equiv 1$  and let  $\Omega$  be a regular bounded domain of  $\mathbb{R}^d$ , for simplicity. By writing  $u(x, t) = \bar{u} + f(x, t)$  and linearizing (1.1.2) about the constant  $\bar{u}$  one formally gets the equation  $f_t = m|\bar{u}|^{m-1} \Delta f$ , so that the rate given in Theorem 1.3.19 coincides with *the one predicted by linearization*. This is not by chance: in the following, we shall prove that estimate (1.3.27) is sharp by providing suitable data for which the latter is matched by a lower bound.

*Proof of Proposition 1.3.20.* We consider first the solution to (1.1.2) corresponding to a regular initial datum  $u_0$  such that  $\bar{u}_0 = 1$  and  $\|u_0 - 1\|_\infty \leq \delta < 1$ , where  $\delta > 0$  will be chosen afterwards. Since  $\|u(t) - 1\|_\infty$  is non-increasing in  $t$ , we have that  $|u(x, t)| \geq 1 - \delta$  for all  $x \in \Omega$  and  $t > 0$ . Hence, by exploiting such bound in the first line of (1.3.137) and solving similarly the associated differential inequality, we end up with the following estimate:

$$\|f(t)\|_2 \leq \|f_0\|_2 e^{-\lambda_1 m(1-\delta)^{m-1} t} \quad \forall t > 0, \quad (1.3.142)$$

where we set  $f(x, t) := u(x, t) - 1$  and  $f_0(x) := u_0(x) - 1$ . Because the equation is in fact non-degenerate, from standard quasilinear theory (see the monograph [120]) we have that  $f \in C^\infty(\bar{\Omega} \times (0, \infty))$ . Moreover, straightforward computations show that

$$f_t = m\Delta f + F, \quad (1.3.143)$$

with

$$F := m(m-1)(1+f)^{m-2} |\nabla f|^2 + m[(1+f)^{m-1} - 1] \Delta f.$$

Upon choosing  $f_0 = c_1 \psi_1$  as initial datum, under the condition

$$|c_1| \leq \frac{\delta}{\|\psi_1\|_\infty}, \quad (1.3.144)$$

(1.3.142) reads

$$\|f(t)\|_2 \leq |c_1| e^{-\lambda_1 m(1-\delta)^{m-1} t} \quad \forall t > 0. \quad (1.3.145)$$

Let us define

$$|f|_{C^0(\bar{\Omega})} := \|f\|_\infty$$

and, for any multi-index  $\eta := (\eta_1, \dots, \eta_d)$  (let  $|\eta| := \eta_1 + \dots + \eta_d$ ), the seminorms

$$|f|_{C^k(\bar{\Omega})} := \max_{|\eta|=k} \|\partial^\eta f\|_\infty, \quad k \in \mathbb{N}.$$

Again, since the equation at hand is uniformly parabolic, we infer the existence of a constant  $C = C(\delta, k, m, \Omega) > 0$  such that

$$|f(t)|_{C^k(\bar{\Omega})} \leq C \quad \forall t > 0, \quad \forall k \in \mathbb{N}. \quad (1.3.146)$$

We now recall the generalized interpolation inequalities

$$|g|_{C^j(\bar{\Omega})} \leq Q \left( |g|_{C^k(\bar{\Omega})}^{\frac{d+jp}{d+kp}} \|g\|_p^{\frac{p(k-j)}{d+kp}} + \|g\|_p \right), \quad (1.3.147)$$

valid for all integers  $k > j \geq 0$ , real numbers  $p \geq 1$  and a suitable positive constant  $Q = Q(j, k, p, d, \Omega)$  (we refer the reader to [142, Theorem at p. 125 and comment n. 5 at p. 126] or, for a short review on the same topics, [30, Appendix A.3]). Combining (1.3.145), (1.3.146) and (1.3.147) (with  $g = f(t)$ ) we deduce that

$$|f(t)|_{C^\ell(\bar{\Omega})} \leq C' \left( |c_1|^{1-\epsilon} + |c_1| \right) e^{-\lambda_1 m(1-\delta)^{m-1}(1-\epsilon)t} \quad \text{for } \ell = 0, 1, 2, \quad \forall t > 0, \quad \forall \epsilon \in (0, 1),$$

where  $C' = C'(\epsilon, \delta, m, d, \Omega)$  is another positive constant independent of  $c_1$  subject to (1.3.144). From the very definition of  $F$  and from the fact that the  $L^\infty$  norm of  $f(t)$  is non-increasing in  $t$  we infer that, for suitable positive constants  $C'' = C''(\delta, m, d)$  and  $C''' = C'''(\epsilon, \delta, m, d, \Omega)$ ,

$$\begin{aligned} \|F(t)\|_\infty &\leq C'' \left( |f(t)|_{C^1(\bar{\Omega})}^2 + \|f(t)\|_\infty |f(t)|_{C^2(\bar{\Omega})} \right) \\ &\leq C''' \left( |c_1|^{1-\epsilon} + |c_1| \right)^2 e^{-2\lambda_1 m(1-\delta)^{m-1}(1-\epsilon)t} \quad \forall t > 0, \quad \forall \epsilon \in (0, 1). \end{aligned} \quad (1.3.148)$$

We aim to study the asymptotic behaviour of  $\alpha_1(t) := \langle f(t), \psi_1 \rangle_{L^2(\Omega)}$ , that is the first Fourier coefficient of  $f$  along the evolution. In order to do that, let us multiply equation (1.3.143) by  $\psi_1$  and integrate in  $\Omega$ , so as to obtain the following differential equation for  $\alpha_1(t)$ :

$$\dot{\alpha}_1(t) = -\lambda_1 m \alpha_1(t) + \langle F(t), \psi_1 \rangle_{L^2(\Omega)}.$$

Duhamel principle entails that  $\alpha_1$  must satisfy the nonlinear integral equation (recall that by construction  $\alpha_1(0) = c_1$ )

$$\alpha_1(t) = e^{-\lambda_1 m t} \left[ c_1 + \int_0^t e^{\lambda_1 m s} \langle F(s), \psi_1 \rangle_{L^2(\Omega)} ds \right]. \quad (1.3.149)$$

If we choose  $\delta$  and  $\epsilon$  sufficiently small, namely such that  $2(1-\delta)^{m-1}(1-\epsilon) > 1$ , thanks to (1.3.148) we easily deduce that the integral in (1.3.149) can be bounded as follows:

$$\left| \int_0^t e^{\lambda_1 m s} \langle F(s), \psi_1 \rangle_{L^2(\Omega)} ds \right| \leq D \left( |c_1|^{1-\epsilon} + |c_1| \right)^2,$$

where  $D$  is another suitable positive constant independent of  $c_1$  subject to (1.3.144) and of  $t \geq 0$ . Now pick  $|c_1|$  so small that

$$|c_1| > D \left( |c_1|^{1-\epsilon} + |c_1| \right)^2,$$

this being possible since  $2(1-\epsilon) > 1$ . Under such a constraint on  $|c_1|$ , (1.3.149) entails that the *exact* decay rate of  $\alpha_1(t)$  is  $e^{-\lambda_1 m t}$ , in the sense that  $|\alpha_1(t)| \sim e^{-\lambda_1 m t}$  as  $t \rightarrow \infty$ . Hence, (1.3.28) holds true since there exists a suitable constant  $\underline{G} > 0$  such that

$$\|u(t) - 1\|_\infty = \|f(t)\|_\infty \geq |\Omega|^{-\frac{1}{2}} \|f(t)\|_2 \geq |\Omega|^{-\frac{1}{2}} |\alpha_1(t)| \geq \underline{G} e^{-\lambda_1 m t} \quad \forall t > 0.$$

The case of initial data with nonzero mean  $\bar{u}_0 \neq 1$  can be brought back to the case  $\bar{u}_0 = 1$  by means of a standard time-scaling argument.  $\square$

## 1.4 A nonhomogeneous Dirichlet problem in Euclidean space

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After having analysed the homogeneous problems (1.1.1) and (1.1.2), in this section we deal with existence and uniqueness issues for the special problem (1.1.21) in Euclidean space, where nonhomogeneous Dirichlet conditions of the type of (1.1.25) are prescribed at infinity.

### 1.4.1 Statements of the main results

We first provide a suitable notion of solution to (1.1.21). For the moment, let us just assume that  $\rho$  and  $G$  are continuous functions and that  $u_0$  is a bounded initial datum.

**Definition 1.4.1.** *Given  $T > 0$ , a very weak solution to problem (1.1.21) is a function  $u \in C(S_T) \cap L^\infty(S_T)$  such that*

$$\begin{aligned} & \int_0^t \int_\Omega [\rho(x)u(x, \tau)\varphi_t(x, \tau) + G(u(x, \tau))\Delta\varphi(x, \tau)] dx d\tau \\ &= \int_\Omega \rho(x) [u(x, t)\varphi(x, t) - u_0(x)\varphi(x, 0)] dx + \int_0^t \int_{\partial\Omega} G(u(\sigma, \tau)) \frac{\partial\varphi}{\partial\mathbf{n}}(\sigma, \tau) d\sigma d\tau \end{aligned} \quad (1.4.1)$$

for all smooth bounded domain  $\Omega \subset \mathbb{R}^d$ ,  $t \in (0, T]$  and any  $\varphi \in C^{2,1}(\bar{\Omega} \times [0, t])$  with  $\varphi \equiv 0$  on  $\partial\Omega \times [0, t]$ .

In agreement with the above definition, a (very weak) supersolution [subsolution] to (1.1.21) is a function  $u \in C(S_T) \cap L^\infty(S_T)$  which satisfies (1.4.1) with “=” replaced by “ $\leq$ ” [“ $\geq$ ”] and in addition  $\varphi \geq 0$ .

As already mentioned in the Introduction, in order to establish well-posedness results for (1.1.21) we shall mostly deal with densities  $\rho$ , nonlinearities  $G$  and initial data  $u_0$  which comply with the following assumptions:

$$\begin{cases} (i) & \rho \in C(\mathbb{R}^d), \text{ with } \rho > 0; \\ (ii) & G \in C^1(\mathbb{R}), \text{ with } G(0) = 0 \text{ and } G' > 0 \text{ in } \mathbb{R} \setminus \{0\}; \\ (iii) & \text{if } G'(0) = 0 \text{ then } G' \text{ is decreasing in } (-\delta, 0) \text{ and increasing in } (0, \delta) \text{ for some } \delta > 0; \\ (iv) & u_0 \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{cases} \quad (H_0)$$

As concerns existence, we begin with the case of *non-degenerate* nonlinearities.

**Theorem 1.4.2.** *Let  $d \geq 3$ . Suppose that  $\rho \in C(\mathbb{R}^d)$  with  $\rho > 0$ ,  $G \in C^1(\mathbb{R})$  with  $G(0) = 0$ ,  $G' \geq \alpha_0$  for some  $\alpha_0 > 0$  and  $u_0 \in C(\mathbb{R}^d)$  with  $\lim_{|x| \rightarrow \infty} u_0(x)$  existing and being finite. Assume in addition that there exist  $R_0 > 0$  and a nonnegative  $\rho_0 \in C([R_0, \infty))$  with  $\int_{R_0}^\infty r\rho_0(r) dr < \infty$  such that*

$$\rho(x) \leq \rho_0(|x|) \quad \forall x \in B_{R_0}^c. \quad (1.4.2)$$

Finally, take  $T > 0$  and any  $a \in C([0, T])$  satisfying

$$a(0) = \lim_{|x| \rightarrow \infty} u_0(x). \quad (1.4.3)$$

Then there exists a solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{uniformly as } t \in [0, T]. \quad (1.4.4)$$

For possibly *degenerate* nonlinearities of porous medium type, we shall prove the following results.

**Theorem 1.4.3.** *Let  $d \geq 3$ . Let assumption  $(H_0)$  be satisfied. Suppose in addition that*

$$\lim_{|x| \rightarrow \infty} u_0(x) = a_0 \quad (1.4.5)$$

for some  $a_0 \in \mathbb{R}$ . Then there exists a solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that

$$\lim_{|x| \rightarrow \infty} u(x, t) = a_0 \quad \text{uniformly as } t \in [0, T],$$

for all  $T > 0$ .

**Theorem 1.4.4.** *Let  $d \geq 3$ . Let assumption  $(H_0)$  be satisfied, with  $\lim_{|x| \rightarrow \infty} u_0(x)$  existing and being finite. Suppose in addition that there exist  $R_0 > 0$  and a nonnegative  $\rho_0 \in C([R_0, \infty))$  with  $\int_{R_0}^{\infty} r \rho_0(r) dr < \infty$  such that (1.4.2) holds true. Take  $T > 0$  and any  $a \in C([0, T])$  complying with (1.4.3) and with  $\min_{t \in [0, T]} a(t) > 0$ . Then there exists a solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, which satisfies (1.4.4).*

**Remark 1.4.5.** Let us assume the same hypotheses of Theorem 1.4.3. Suppose moreover that  $\rho$  is a density which does not decay too fast at infinity, in the sense that there exist  $R_1 > 0$  and a nonnegative  $\rho_1 \in C([R_1, \infty))$  with  $\int_{R_1}^{\infty} r \rho_1(r) dr = \infty$  such that

$$\rho(x) \geq \rho_1(|x|) > 0 \quad \forall x \in B_{R_1}^c.$$

Then, in view of the uniqueness result recalled in the Introduction and as a consequence of Theorem 1.4.3, we can assert that the *unique* solution to problem (1.1.21) (without prescribed conditions at infinity) necessarily satisfies

$$\lim_{|x| \rightarrow \infty} u(x, t) = \lim_{|x| \rightarrow \infty} u_0(x) \quad \text{uniformly as } t \in [0, T],$$

for all  $T > 0$ .

**Remark 1.4.6.** As we shall explain in the end of Subsection 1.4.2, with minor changes in the corresponding proofs the above results admit some generalization. More precisely:

- (i) In Theorem 1.4.2, if we drop assumption (1.4.3), the conclusion remains true upon replacing (1.4.4) with the following property:

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{uniformly as } t \in [t_0, T] \tag{1.4.6}$$

for all  $t_0 \in (0, T)$ ;

- (ii) In Theorem 1.4.4, if we drop assumption (1.4.3), the conclusion remains true upon replacing (1.4.4) with (1.4.6) and requiring in addition that there exist  $R^*, \epsilon > 0$  such that

$$2G(I) > G(S), \quad I := \inf_{x \in B_{R^*}^c} u_0(x), \quad S := \sup_{t \in (0, \epsilon)} a(t). \tag{1.4.7}$$

Clearly (1.4.7) is a technical assumption, but it is essential in order to make our proof work: we do not know whether it can be removed. Notice that the latter certainly holds if  $I$  is large enough compared to  $S$ , so that possible problems occur, to some extent, only if the initial datum  $u_0$  is too small at infinity with respect to the “boundary” datum  $a$ .

**Remark 1.4.7.** Note that the hypotheses of Theorem 1.4.4 allow us to take as initial data functions  $u_0$  which may be nonpositive in some compact set  $K \subset \mathbb{R}^d$ : we only need them to be strictly positive outside a ball. On the other hand, it is apparent that an analogous result holds true for  $a < 0$  and initial data which are strictly negative outside a ball (just exploit Theorem 1.4.4 with  $u_0$  replaced by  $-u_0$ ,  $a$  replaced by  $-a$  and  $G(u)$  replaced by  $-G(-u)$ ); the same comment applies to Remark 1.4.6-(ii) as well.

Our uniqueness result reads as follows.

**Theorem 1.4.8.** *Let  $d \geq 3$ . Let assumption  $(H_0)$  be satisfied and suppose in addition that  $\rho \in L^\infty(\mathbb{R}^d)$  and  $a \in L^\infty((0, T))$  for some  $T > 0$ . Then there exists at most one solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that*

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{for a.e. } t \in (0, T).$$

Theorems 1.4.2–1.4.4 and 1.4.8 entail immediate corollaries.

**Corollary 1.4.9.** *Let the assumptions of Theorem 1.4.2 be satisfied and suppose in addition that  $\rho \in L^\infty(\mathbb{R}^d)$ . Then there exists a unique solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that*

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{uniformly as } t \in [0, T]. \quad (1.4.8)$$

**Corollary 1.4.10.** *Let the assumptions of Theorem 1.4.3 be satisfied and suppose in addition that  $\rho \in L^\infty(\mathbb{R}^d)$ . Then there exists a unique solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that*

$$\lim_{|x| \rightarrow \infty} u(x, t) = a_0 \quad \text{uniformly as } t \in [0, T],$$

for all  $T > 0$ .

**Corollary 1.4.11.** *Let the assumptions of Theorem 1.4.4 be satisfied and suppose in addition that  $\rho \in L^\infty(\mathbb{R}^d)$ . Then there exists a unique solution  $u$  to problem (1.1.21), in the sense of Definition 1.4.1, such that (1.4.8) holds true.*

**Remark 1.4.12.** Notice that, when  $(H_0)$  is satisfied,  $\rho$  belongs to  $L^\infty(\mathbb{R}^d)$  and does not decay too fast at infinity, the discussion in Remark 1.4.5 is in agreement with Corollary 1.4.10.

Theorems 1.4.2–1.4.4 will be proved in Subsection 1.4.2, whereas Theorem 1.4.8 will be proved in Subsection 1.4.3.

## 1.4.2 Existence

In view of the assumptions on the density  $\rho$  made in the hypotheses of Theorems 1.4.2 or 1.4.4, there exists a *radial* function  $V \in C^2(B_{R_0}^c)$  such that

$$\Delta V \leq -\rho \quad \text{in } B_{R_0}^c, \quad (1.4.9)$$

$$V(x) > 0 \quad \forall x \in B_{R_0}^c,$$

$$|x| \mapsto V(x) \text{ is nonincreasing,}$$

$$\lim_{|x| \rightarrow \infty} V(x) = 0, \quad (1.4.10)$$

where  $R_0 > 0$  can be assumed to be equal to the parameter that appears in the hypotheses of the quoted theorems. Such a function, which can be constructed through standard Riesz potential techniques (see *e.g.* Chapter 2, Subsection 2.3.2 for similar issues in the fractional framework), will play a central role in the sequel.

Before proving our existence results, let us mention that below we shall often make use of the function  $G^{-1}$ , whose domain need not coincide with  $\mathbb{R}$ . Nevertheless, as we deal with bounded initial data  $u_0$  (and, by comparison principles, with bounded solutions), this is not a problem since one can modify the definition of  $G(u)$  for  $|u|$  large so that it becomes a bijection from  $\mathbb{R}$  to itself and it still complies with the corresponding assumptions (like *e.g.*  $(H_0)$ ), without changing the evolution of  $u_0$ .

Hereafter we shall denote as  $\{\zeta_j\}$  a given sequence of cut-off functions having the following properties:

$$\zeta_j \in C_c^\infty(B_j), \quad 0 \leq \zeta_j \leq 1 \quad \text{in } \mathbb{R}^d, \quad \zeta_j \equiv 1 \quad \text{in } B_{j/2}.$$

*Proof of Theorem 1.4.2.* Since  $a \in C([0, T])$  and  $G \in C^1(\mathbb{R})$  is increasing, for any  $t_0 \in [0, T]$  and  $\sigma > 0$  there exists  $\delta = \delta(\sigma) > 0$  such that

$$G^{-1}(G(a(t_0)) - \sigma) \leq a(t) \leq G^{-1}(G(a(t_0)) + \sigma) \quad \forall t \in [t_\delta, \bar{t}_\delta], \quad (1.4.11)$$

where we set  $t_\delta := \max\{t_0 - \delta, 0\}$  and  $\bar{t}_\delta := \min\{t_0 + \delta, T\}$ . Moreover, in view of the assumptions on  $u_0$ , for any  $\sigma > 0$  there exists  $R = R(\sigma) > R_0$  such that

$$G^{-1}(G(a(0)) - \sigma) \leq u_0(x) \leq G^{-1}(G(a(0)) + \sigma) \quad \forall x \in B_R^c. \quad (1.4.12)$$

For any  $j \in \mathbb{N}$ , let  $u_j \in C(\bar{B}_j \times [0, T])$  be the unique solution to the quasilinear problem (see e.g. [120])

$$\begin{cases} \rho(x)u_t = \Delta[G(u)] & \text{in } B_j \times (0, T), \\ u = a(t) & \text{on } \partial B_j \times (0, T), \\ u = u_{0j} & \text{on } \bar{B}_j \times \{0\}, \end{cases} \quad (1.4.13)$$

where

$$u_{0j} := \zeta_j u_0 + (1 - \zeta_j)a(0) \quad \text{in } \bar{B}_j.$$

By standard comparison principles,

$$|u_j| \leq K := \max\{\|u_0\|_\infty, \|a\|_\infty\} \quad \text{in } B_j \times (0, T). \quad (1.4.14)$$

It is then a matter of usual compactness arguments to show that there exists a subsequence  $\{u_{j_k}\} \subseteq \{u_j\}$  which converges, as  $k \rightarrow \infty$ , *locally uniformly* in  $\mathbb{R}^d \times (0, T)$  to a solution  $u$  to problem (1.1.21). We are therefore left with proving that

$$\lim_{|x| \rightarrow \infty} u(x, t) = a(t) \quad \text{uniformly as } t \in [0, T].$$

To this end, let  $t_0 \in [0, T]$  and

$$\underline{w}(x, t) := G^{-1}\left(-\underline{M}V(x) - \sigma + G(a(t_0)) - \underline{\lambda}(t - t_0)^2\right) \quad \forall (x, t) \in B_{R_0}^c \times (t_\delta, \bar{t}_\delta),$$

where  $\underline{M} > 0$  and  $\underline{\lambda} > 0$  are suitable constants to be chosen later. Thanks to the assumptions and to (1.4.9), there holds

$$\rho(x)\underline{w}_t - \Delta[G(\underline{w})] = -\rho(x)\frac{2\underline{\lambda}(t - t_0)}{G'(\underline{w})} + \underline{M}\Delta V \leq \rho(x)\left(\frac{2\underline{\lambda}\delta}{\alpha_0} - \underline{M}\right) \leq 0 \quad \text{in } B_{R_0}^c \times (t_\delta, \bar{t}_\delta) \quad (1.4.15)$$

providing that

$$\underline{M} \geq \frac{2\underline{\lambda}\delta}{\alpha_0}. \quad (1.4.16)$$

For any  $j \in \mathbb{N}$ , with  $j > R$ , let us set

$$N_{R,j} := B_j \setminus \bar{B}_R, \quad (1.4.17)$$

where  $R$  is as in (1.4.12); we have

$$\underline{w}(x, t) \leq -K \quad \forall (x, t) \in \partial B_R \times (t_\delta, \bar{t}_\delta) \quad (1.4.18)$$

as long as

$$\underline{M} \geq \frac{G(\|a\|_\infty) - G(-K)}{V(R)}. \quad (1.4.19)$$

Furthermore,

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \sigma) \quad \forall (x, t) \in \partial B_j \times (t_\delta, \bar{t}_\delta). \quad (1.4.20)$$

If  $t_\delta = 0$  there holds

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \sigma) \quad \forall (x, t) \in \bar{N}_{R,j} \times \{0\}, \quad (1.4.21)$$

otherwise

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \lambda \delta^2) \leq -K \quad \forall (x, t) \in \bar{N}_{R,j} \times \{t_\delta\} \quad (1.4.22)$$

provided

$$\lambda \geq \frac{G(\|a\|_\infty) - G(-K)}{\delta^2}. \quad (1.4.23)$$

Suppose now that conditions (1.4.16), (1.4.19) and (1.4.23) are fulfilled. From (1.4.15), (1.4.18) and (1.4.20)–(1.4.22) we infer that  $\underline{w}$  is a subsolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } N_{R,j} \times (t_\delta, \bar{t}_\delta), \\ u = -K & \text{on } \partial B_R \times (t_\delta, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \partial B_j \times (t_\delta, \bar{t}_\delta), \\ u = -K & \text{on } \bar{N}_{R,j} \times \{t_\delta\} \end{cases} \quad (1.4.24)$$

in case  $t_\delta > 0$ , whereas it is a subsolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } N_{R,j} \times (0, \bar{t}_\delta), \\ u = -K & \text{on } \partial B_R \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \partial B_j \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \bar{N}_{R,j} \times \{0\} \end{cases} \quad (1.4.25)$$

in case  $t_\delta = 0$ . On the other hand, (1.4.11), (1.4.12) and (1.4.14) show that the data for the solutions to (1.4.13), (1.4.24) and (1.4.25) are ordered on each side of the parabolic boundary of  $N_{R,j} \times (t_\delta, \bar{t}_\delta)$ . In particular, we deduce that  $u_j$  is a supersolution to problem (1.4.24) or to problem (1.4.25) depending on whether  $t_\delta > 0$  or  $t_\delta = 0$ . Hence, by comparison principles,

$$\underline{w} \leq u_j \quad \text{in } N_{R,j} \times (t_\delta, \bar{t}_\delta). \quad (1.4.26)$$

We also aim to obtain an estimate from above for  $u_j$ . To this end, let

$$\bar{w}(x, t) := G^{-1}(\bar{M}V(x) + \sigma + G(a(t_0)) + \bar{\lambda}(t - t_0)^2) \quad \forall (x, t) \in B_{R_0}^c \times (t_\delta, \bar{t}_\delta),$$

with

$$\bar{M} \geq \max \left\{ \frac{2\bar{\lambda}\delta}{\alpha_0}, \frac{G(K) - G(-\|a\|_\infty)}{V(R)} \right\}$$

and

$$\bar{\lambda} \geq \frac{G(K) - G(-\|a\|_\infty)}{\delta^2}.$$

Reasoning as above, we find that  $\bar{w}$  is a supersolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } N_{R,j} \times (t_\delta, \bar{t}_\delta), \\ u = K & \text{on } \partial B_R \times (t_\delta, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) + \sigma) & \text{on } \partial B_j \times (t_\delta, \bar{t}_\delta), \\ u = K & \text{on } \bar{N}_{R,j} \times \{t_\delta\} \end{cases} \quad (1.4.27)$$

in case  $\underline{t}_\delta > 0$ , whereas it is a supersolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } N_{R,j} \times (0, \bar{t}_\delta), \\ u = K & \text{on } \partial B_R \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) + \sigma) & \text{on } \partial B_j \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) + \sigma) & \text{on } \bar{N}_{R,j} \times \{0\} \end{cases} \quad (1.4.28)$$

in case  $\underline{t}_\delta = 0$ . Collecting (1.4.11), (1.4.12) and (1.4.14) we then deduce that  $u_j$  is a subsolution to problem (1.4.27) or to problem (1.4.28) depending on whether  $\underline{t}_\delta > 0$  or  $\underline{t}_\delta = 0$ , so that comparison principles entail

$$u_j \leq \bar{w} \quad \text{in } N_{R,j} \times (\underline{t}_\delta, \bar{t}_\delta). \quad (1.4.29)$$

Thanks to (1.4.26) and (1.4.29) with  $j = j_k$ , letting  $k \rightarrow \infty$  we finally get

$$\underline{w} \leq u \leq \bar{w} \quad \text{in } B_R^c \times (\underline{t}_\delta, \bar{t}_\delta). \quad (1.4.30)$$

Hence, in view of (1.4.10) and (1.4.30), we infer that for  $|x|$  large enough, *independently of*  $t_0 \in [0, T]$ , there holds

$$G^{-1}(G(a(t_0)) - 2\sigma) \leq u(x, t_0) \leq G^{-1}(G(a(t_0)) + 2\sigma).$$

In order to complete the proof one just lets  $\sigma \downarrow 0$ . □

*Proof of Theorem 1.4.3.* As above, thanks to (1.4.5) we can assert that for any  $\sigma > 0$  there exists  $R = R(\sigma) > 0$  such that

$$G^{-1}(G(a_0) - \sigma) \leq u_0(x) \leq G^{-1}(G(a_0) + \sigma) \quad \forall x \in B_R^c. \quad (1.4.31)$$

In view of assumption  $(H_0)$ , standard results (see *e.g.* [8]) ensure existence and uniqueness, for all  $j \in \mathbb{N}$ , of the solution  $u_j$  to the problem

$$\begin{cases} \rho(x)u_t = \Delta [G(u)] & \text{in } B_j \times (0, T), \\ u = a_0 & \text{on } \partial B_j \times (0, T), \\ u = u_{0j} & \text{on } \bar{B}_j \times \{0\}, \end{cases}$$

where

$$u_{0j} := \zeta_j u_0 + (1 - \zeta_j) a_0 \quad \text{in } \bar{B}_j.$$

Note that, as a consequence of the regularity results in [71] (the presence of the density  $\rho$  here is not an issue),  $u_j \in C(\bar{B}_j \times [0, T])$ . Moreover, by comparison principles,

$$|u_j| \leq K := \max\{\|u_0\|_\infty, |a_0|\} \quad \text{in } B_j \times (0, T).$$

Routine compactness arguments (one can use *e.g.* [70, Lemma 5.2] plus a diagonal procedure) entail the existence of a subsequence  $\{u_{j_k}\} \subset \{u_j\}$  which converges, as  $k \rightarrow \infty$ , locally uniformly in  $\mathbb{R}^d \times (0, T)$  to a solution  $u$  of problem (1.1.21). Now let

$$\Gamma(x) := |x|^{2-d} \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

It is apparent that

$$\begin{aligned} \Delta \Gamma &= 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}, \\ \Gamma &> 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}, \end{aligned}$$



and

$$\lim_{|x| \rightarrow \infty} \Gamma(x) = 0. \quad (1.4.32)$$

Let us define the function

$$\bar{W}(x) := G^{-1}(\bar{M}\Gamma(x) + \sigma + G(a_0)) \quad \forall x \in \mathbb{R}^d \setminus \{0\}$$

under the assumption

$$\bar{M} \geq \frac{G(K) - G(a_0)}{\Gamma(R)},$$

so that, thanks to the properties of  $G$  and  $\Gamma$ , there hold

$$\Delta[G(\bar{W})] = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}, \quad (1.4.33)$$

$$\bar{W}(x) \geq K \quad \forall x \in \partial B_R, \quad (1.4.34)$$

$$\bar{W}(x) \geq a_0 \quad \forall x \in \partial B_j \quad (1.4.35)$$

and

$$\bar{W}(x) \geq G^{-1}(G(a_0) + \sigma) \quad \forall x \in \bar{N}_{R,j}, \quad (1.4.36)$$

where  $N_{R,j}$  is as in (1.4.17). From (1.4.33)–(1.4.36) we then deduce that  $\bar{W}$  is a supersolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta[G(u)] & \text{in } N_{R,j} \times (0, T), \\ u = K & \text{on } \partial B_R \times (0, T), \\ u = a_0 & \text{on } \partial B_j \times (0, T), \\ u = G^{-1}(G(a_0) + \sigma) & \text{on } \bar{N}_{R,j} \times \{0\}. \end{cases} \quad (1.4.37)$$

On the other hand,  $u_j$  is a subsolution to (1.4.37) (recall also (1.4.31)); hence, by comparison,

$$u_j \leq \bar{W} \quad \text{in } N_{R,j} \times (0, T). \quad (1.4.38)$$

In order to get a similar estimate from below, let

$$\underline{W}(x) := G^{-1}(-\underline{M}\Gamma(x) - \sigma + G(a_0)) \quad \forall x \in \mathbb{R}^d \setminus \{0\},$$

where

$$\underline{M} \geq \frac{G(a_0) - G(-K)}{\Gamma(R)}.$$

By means of arguments similar to the ones we used above, we can infer that  $\underline{W}$  and  $u_j$  are a subsolution and a supersolution, respectively, to the problem

$$\begin{cases} \rho(x)u_t = \Delta[G(u)] & \text{in } N_{R,j} \times (0, T), \\ u = -K & \text{on } \partial B_R \times (0, T), \\ u = a_0 & \text{on } \partial B_j \times (0, T), \\ u = G^{-1}(G(a_0) - \sigma) & \text{on } \bar{N}_{R,j} \times \{0\}, \end{cases}$$

so that by comparison principles

$$\underline{W} \leq u_j \quad \text{in } N_{R,j} \times (0, T). \quad (1.4.39)$$

By exploiting (1.4.38) and (1.4.39) with  $j = j_k$  and letting  $k \rightarrow \infty$ , we finally obtain

$$\underline{W} \leq u \leq \bar{W} \quad \text{in } B_R^c \times (0, T). \quad (1.4.40)$$

Taking the limit of (1.4.40) as  $|x| \rightarrow \infty$ , thanks to (1.4.32) we deduce that for  $|x|$  large enough, independently of  $t \in [0, T]$ , there holds

$$G^{-1}(G(a_0) - 2\sigma) \leq u(x, t) \leq G^{-1}(G(a_0) + 2\sigma).$$

The proof is then complete upon letting  $\sigma \downarrow 0$ . □

Theorem 1.4.4 combines the cases of a possibly degenerate nonlinearity  $G$  with a condition at infinity  $a(t)$  which may depend on time. Before proving it, we need some intermediate results.

**Lemma 1.4.13.** *Let  $d \geq 3$ . For any  $\alpha, R, M > 0$  there exists a subsolution  $\underline{u}_0$  to the equation*

$$-\Delta[G(u)] = 0 \quad \text{in } \mathbb{R}^d$$

*which is bounded, continuous, radial, nondecreasing as a function of  $|x|$ , satisfies*

$$\lim_{|x| \rightarrow \infty} \underline{u}_0(x) = \alpha$$

*and is equal to  $-M$  in  $B_R$ .*

*Proof.* First of all, let us define the function

$$\tilde{U}_0(x) := G(\alpha) - \frac{\beta}{|x|} \quad \forall x \in B_\varepsilon^c,$$

where

$$0 < \varepsilon < \gamma := \frac{\beta}{G(\alpha) - G(-M)}.$$

It is immediate to check that

$$-\Delta \tilde{U}_0 \leq 0 \quad \text{in } B_\varepsilon^c,$$

so that  $\tilde{u}_0 := G^{-1}(\tilde{U}_0)$  is a subsolution to  $-\Delta[G(u)] = 0$  in  $B_\varepsilon^c$ . Now consider the function

$$\hat{u}_0(x) := \max\{\tilde{u}_0(x), -M\} \quad \forall x \in B_\varepsilon^c.$$

Thanks to Kato's inequality, we deduce that  $\hat{u}_0$  is also a subsolution to  $-\Delta[G(u)] = 0$  in  $\mathbb{R}^d \setminus \overline{B}_\varepsilon$ . Upon noticing that  $\hat{u}_0 = -M$  in  $B_\gamma \setminus B_\varepsilon$ , it is readily seen that the function

$$\underline{u}_0 := \begin{cases} \hat{u}_0 & \text{in } B_\varepsilon^c, \\ -M & \text{in } B_\varepsilon \end{cases}$$

is a subsolution to  $-\Delta[G(u)] = 0$  in the whole of  $\mathbb{R}^d$ . The fact that  $\underline{u}_0$  is bounded, continuous, radial, nondecreasing as a function of  $|x|$  and satisfies the claimed limit property at infinity is clear by construction. The constant condition in  $B_R$  is then achieved by picking  $\beta = R(G(\alpha) - G(-M))$ . □

**Lemma 1.4.14.** *Suppose that, besides the assumptions of Theorem 1.4.4, there exists a function  $\underline{u}_0$  having all the properties stated in Lemma 1.4.13 and such that, for a suitable  $\varepsilon > 0$ , there hold*

$$\begin{aligned} u_0(x) &\geq \underline{u}_0(x) \quad \forall x \in \mathbb{R}^d, \\ \lim_{|x| \rightarrow \infty} \underline{u}_0(x) &= \min_{t \in [0, T]} a(t) - \varepsilon > 0 \end{aligned} \tag{1.4.41}$$

and

$$2G\left(\min_{t \in [0, T]} a(t) - \varepsilon\right) > G(\|a\|_\infty). \tag{1.4.42}$$

*Then there exists a solution to problem (1.1.21), in the sense of Definition 1.4.1, which satisfies (1.4.4).*

*Proof.* First we repeat the proof of Theorem 1.4.2 up to the construction of the sequence  $\{u_j\}$ , keeping the same notations. Notice that, as in the proof of Theorem 1.4.3, when we allow for a degenerate nonlinearity  $G$ , in view of the hypothesis  $(H_0)$  the well-posedness of problem (1.4.13) is known (see *e.g.* [8]). Again, from the regularity results of [71],  $u_j \in C(\bar{B}_j \times [0, T])$ .

Thanks to the properties of  $\underline{u}_0$ , (1.4.41), (1.4.42) and  $(H_0)$ , we can pick  $R_1 > R_0$  and  $\beta > 0$  such that

$$\beta < \bar{u}_0(R)$$

and

$$2G(\bar{u}_0(R)) - G(\beta) - G(\|a\|_\infty) > 0 \quad (1.4.43)$$

for all  $R \geq R_1$ . Moreover, by definition of  $\underline{u}_0$ , it is plain that the latter is a subsolution to problem (1.4.13). Hence, by comparison,

$$\underline{u}_0(x) \leq u_j(x, t) \leq K \quad \forall (x, t) \in B_j \times (0, T),$$

where  $K$  is as in (1.4.14). Recalling that  $\underline{u}_0(x)$  is nonincreasing as a function of  $|x|$ , we then infer

$$\underline{u}_0(R) \leq u_j(x, t) \leq K \quad \forall (x, t) \in N_{R,j} \times (0, T). \quad (1.4.44)$$

Now let

$$\gamma := \min_{u \in [\beta, K]} G'(u). \quad (1.4.45)$$

Given  $\sigma > 0$ , in view of (1.4.10) we can fix  $R = R(\sigma) > R_1$  in (1.4.12) so large that in (1.4.11) we are allowed to choose

$$\delta = \frac{2}{\gamma} V(R). \quad (1.4.46)$$

Note that  $\beta$  and  $\gamma$  are independent of  $R$  and  $\delta$ . Let  $t_0 \in [0, T]$  and set

$$\underline{\lambda} := \frac{G(a(t_0)) - G(\underline{u}_0(R))}{\delta^2}, \quad \underline{M} := \frac{2\lambda\delta}{\gamma}. \quad (1.4.47)$$

By combining (1.4.43), (1.4.46) and (1.4.47) there follow

$$\underline{M} = \frac{G(a(t_0)) - G(\underline{u}_0(R))}{V(R)} \quad (1.4.48)$$

and

$$-\underline{M}V(R) - \sigma + G(a(t_0)) - \lambda\delta^2 > G(\beta) \quad (1.4.49)$$

provided  $\sigma > 0$  is small enough. Let us define the function

$$\underline{w}(x, t) := G^{-1} \left( -\underline{M}V(x) - \sigma + G(a(t_0)) - \underline{\lambda}(t - t_0)^2 \right) \quad \forall (x, t) \in B_{R_0}^c \times (\underline{t}_\delta, \bar{t}_\delta).$$

Recalling that  $|x| \mapsto V(x)$  is nonincreasing, from (1.4.49) we deduce

$$\underline{w}(x, t) \geq \beta \quad \forall (x, t) \in N_{R,j} \times (\underline{t}_\delta, \bar{t}_\delta). \quad (1.4.50)$$

Moreover,  $(H_0)$ -(ii), (1.4.9), (1.4.45), (1.4.47) and (1.4.50) entail

$$\rho(x)\underline{w}_t - \Delta[G(\underline{w})] = -\rho(x) \frac{2\lambda(t - t_0)}{G'(\underline{w})} + \underline{M} \Delta V \leq \rho(x) \left( \frac{2\lambda\delta}{\gamma} - \underline{M} \right) = 0 \quad \text{in } B_{R_0}^c \times (\underline{t}_\delta, \bar{t}_\delta). \quad (1.4.51)$$

In view of (1.4.48), we have that

$$\underline{w}(x, t) \leq \underline{u}_0(R) \quad \forall (x, t) \in \partial B_R \times (\underline{t}_\delta, \bar{t}_\delta); \quad (1.4.52)$$

in addition,

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \sigma) \quad \forall (x, t) \in \partial B_j \times (t_\delta, \bar{t}_\delta). \quad (1.4.53)$$

If  $t_\delta = 0$  there holds

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \sigma) \quad \forall (x, t) \in \bar{N}_{R,j} \times \{0\}, \quad (1.4.54)$$

otherwise

$$\underline{w}(x, t) \leq G^{-1}(G(a(t_0)) - \lambda\delta^2) = \underline{u}_0(R) \quad \forall (x, t) \in \bar{N}_{R,j} \times \{t_\delta\}. \quad (1.4.55)$$

Notice that in (1.4.55) we took advantage of (1.4.47). Thanks to (1.4.51)–(1.4.55) we then infer that  $\underline{w}$  is a subsolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta[G(u)] & \text{in } N_{R,j} \times (t_\delta, \bar{t}_\delta), \\ u = \underline{u}_0(R) & \text{on } \partial B_R \times (t_\delta, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \partial B_j \times (t_\delta, \bar{t}_\delta), \\ u = \underline{u}_0(R) & \text{on } \bar{N}_{R,j} \times \{t_\delta\} \end{cases} \quad (1.4.56)$$

in case  $t_\delta > 0$ , whereas it is a subsolution to the problem

$$\begin{cases} \rho(x)u_t = \Delta[G(u)] & \text{in } N_{R,j} \times (0, \bar{t}_\delta), \\ u = \underline{u}_0(R) & \text{on } \partial B_R \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \partial B_j \times (0, \bar{t}_\delta), \\ u = G^{-1}(G(a(t_0)) - \sigma) & \text{on } \bar{N}_{R,j} \times \{0\} \end{cases} \quad (1.4.57)$$

in case  $t_\delta = 0$ . On the other hand, from (1.4.11), (1.4.12) (which, recall, holds true as a consequence of (1.4.3)) and (1.4.44) we easily deduce that  $u_j$  is a supersolution to problem (1.4.56) or to problem (1.4.57) depending on whether  $t_\delta > 0$  or  $t_\delta = 0$ . Hence, by comparison,

$$\underline{w} \leq u_j \quad \text{in } N_{R,j} \times (t_\delta, \bar{t}_\delta).$$

In order to get an analogous estimate from above, let

$$\bar{w}(x, t) := G^{-1}(\bar{M}V(x) + \sigma + G(a(t_0)) + \bar{\lambda}(t - t_0)^2) \quad \forall (x, t) \in B_{R_0}^c \times (t_\delta, \bar{t}_\delta).$$

By construction,

$$\bar{w} \geq \min_{t \in [0, T]} a(t) \quad \text{in } B_{R_0}^c \times (t_\delta, \bar{t}_\delta).$$

Choose any

$$\bar{M} \geq \max \left\{ \frac{2\bar{\lambda}\delta}{\min_{t \in [0, T]} G'(a(t))}, \frac{G(K) - G(-\|a\|_\infty)}{V(R)} \right\}$$

and

$$\bar{\lambda} \geq \frac{G(K) - G(-\|a\|_\infty)}{\delta^2}.$$

Reasoning in a similar way as above, there is no difficulty in showing that  $\bar{w}$  is a supersolution to problem (1.4.27) or to problem (1.4.28) depending on whether  $t_\delta > 0$  or  $t_\delta = 0$ . On the other hand, by exploiting (1.4.11), (1.4.12) and (1.4.14) we easily deduce that  $u_j$  is a subsolution to problem (1.4.27) or to problem (1.4.28) depending on whether  $t_\delta > 0$  or  $t_\delta = 0$ . Hence, as in the proof of Theorem 1.4.3, by means of a compactness argument which makes use *e.g.* of [70, Lemma 5.2] (again, the presence of the density  $\rho$  is not an issue) and a diagonal procedure, we deduce that there exists a subsequence  $\{u_{j_k}\} \subset \{u_j\}$  which converges, as  $k \rightarrow \infty$ , locally uniformly in  $\mathbb{R}^d \times (0, T)$  to a solution  $u$  to problem (1.1.21). We then conclude by arguing as in the end of proof of Theorem 1.4.2.  $\square$

*Proof of Theorem 1.4.4.* Consider first a datum  $a(t)$  such that, for some  $\varepsilon > 0$ ,

$$\min_{t \in [0, T]} a(t) - \varepsilon > 0$$

and (1.4.42) holds true. Then pick a function  $\underline{u}_0$  as in Lemma 1.4.13 with the choices

$$\alpha = \min_{t \in [0, T]} a(t) - \varepsilon,$$

$R$  so large that

$$u_0(x) \geq \min_{t \in [0, T]} a(t) - \varepsilon \quad \forall x \in B_R^c$$

and

$$M = \max \left\{ 0, - \inf_{x \in \mathbb{R}^d} u_0(x) \right\}.$$

Under these assumptions, it is plain that  $u_0(x) \geq \underline{u}_0(x)$  for all  $x \in \mathbb{R}^d$ . Hence, in this case, we are in position to apply Lemma 1.4.14 and the assertion is proved.

If there exists no  $\varepsilon > 0$  such that  $a(t)$  satisfies (1.4.42) in the time interval  $[0, T]$ , we can always find  $\varepsilon, \tau > 0$  so small that

$$2G \left( \min_{s \in [t, (t+\tau) \wedge T]} a(s) - \varepsilon \right) > G \left( \max_{s \in [t, (t+\tau) \wedge T]} a(s) \right) \quad \forall t \in [0, T].$$

This is a straightforward consequence of the uniform continuity of  $G(a(t))$  and of its strict positivity in  $[0, T]$ . By exploiting again Lemma 1.4.14 we get existence in the time interval  $[0, \tau]$ . Repeating the same procedure starting from  $t = \tau$  we get existence in  $[\tau, 2\tau \wedge T]$  with initial datum  $u(\tau)$  and so, by Definition 1.4.1, existence in the whole of  $[0, 2\tau \wedge T]$ . A finite number of iterations then yields the claim.  $\square$

As concerns Remark 1.4.6, note that point (i) follows by means of the same proof as Theorem 1.4.2, provided one chooses  $\delta \in (0, t_0)$  in (1.4.11). As for point (ii), we just stress that condition (1.4.7) permits us to repeat the proof of Theorem 1.4.4, upon picking  $R > R^*$  and  $\tau \leq \epsilon$ . In both cases, the assumption that  $\lim_{|x| \rightarrow \infty} u_0(x)$  exists finite can be replaced by  $u_0 \in L^\infty(\mathbb{R}^d)$ .

### 1.4.3 Uniqueness

Let  $u_1$  and  $u_2$  be any two solutions of problem (1.1.21). Define the function

$$q(x, t) := \begin{cases} \frac{G(u_1(x, t)) - G(u_2(x, t))}{u_1(x, t) - u_2(x, t)} & \text{if } u_1(x, t) \neq u_2(x, t), \\ 0 & \text{if } u_1(x, t) = u_2(x, t), \end{cases}$$

for all  $(x, t) \in S_T$ . Notice that, in view of assumption  $(H_0)$ -*(ii)*, we have  $q \geq 0$  in  $S_T$  and  $q \in L^\infty(S_T)$ . Given  $\tau \in (0, T)$ , one can show that there exists a sequence  $\{q_n\} \subset C^\infty(S_T)$  such that, for every  $n \in \mathbb{N}$ , there hold

$$\frac{1}{n^{d+1}} \leq q_n \leq \|q\|_{L^\infty(S_T)} + \frac{1}{n^{d+1}} \quad \text{in } Q_{n, \tau} := B_n \times (0, \tau)$$

and

$$\lim_{n \rightarrow \infty} \left\| \frac{(q_n - q)}{\sqrt{q_n}} \right\|_{L^2(Q_{n, \tau})} = 0. \tag{1.4.58}$$

In fact, it is enough to choose (for instance) a regular approximation of

$$q \vee \left( \frac{1}{n^{d+1}} \right).$$

Hereafter we shall assume, in addition, that the density  $\rho$  is regular. We shall then explain in the end of the proof of Theorem 1.4.8 how it is possible to consider general  $\rho \in C(\mathbb{R}^d)$ .

For any  $n \in \mathbb{N}$ , let  $\psi_n$  be the unique regular solution to the backward parabolic (dual) problem

$$\begin{cases} \rho(x)(\psi_n)_t + q_n(x, t) \Delta \psi_n = 0 & \text{in } Q_{n, \tau}, \\ \psi_n = 0 & \text{on } \partial B_n \times (0, \tau), \\ \psi_n = \chi & \text{on } \overline{B}_n \times \{\tau\}, \end{cases} \quad (1.4.59)$$

where  $\chi \in C^\infty(\mathbb{R}^d)$  is any function such that  $0 \leq \chi \leq 1$  and  $\text{supp } \chi \subset B_{n_0}$  for a fixed  $n_0 \in \mathbb{N}$ . Such solutions enjoy some crucial properties.

**Lemma 1.4.15.** *For all  $n > n_0$ , there hold*

$$0 \leq \psi_n \leq 1 \quad \text{in } Q_{n, \tau} \quad (1.4.60)$$

and

$$-\frac{C}{n^{d-1}} \leq \langle \nabla \psi_n, \mathbf{n}_n \rangle \leq 0 \quad \text{on } \partial B_n \times (0, \tau), \quad (1.4.61)$$

where  $C$  is a positive constant independent of  $n$  and  $\mathbf{n}_n = \mathbf{n}_n(\sigma)$  is the outer normal at  $\sigma \in \partial B_n$ .

*Proof.* The bounds (1.4.60) just follow by standard comparison principles (see *e.g.* [98]) because  $\underline{\psi} \equiv 0$  and  $\overline{\psi} \equiv 1$  are a subsolution and a supersolution, respectively, to problem (1.4.59). As for (1.4.61), first notice that (1.4.60) entails

$$\langle \nabla \psi_n, \mathbf{n}_n \rangle \leq 0 \quad \text{on } \partial B_n \times (0, \tau), \quad (1.4.62)$$

since

$$\psi_n = 0 \quad \text{on } \partial B_n \times (0, \tau).$$

Now let

$$E_n := B_n \setminus \overline{B}_{n_0}.$$

From (1.4.60) and the fact that  $\text{supp } \chi \subset B_{n_0}$  we infer that  $\psi_n$  is a *subsolution* to the problem

$$\begin{cases} \rho(x)\psi_t + q_n(x, t) \Delta \psi = 0 & \text{in } E_n \times (0, \tau), \\ \psi = 1 & \text{on } \partial B_{n_0} \times (0, \tau), \\ \psi = 0 & \text{on } \partial B_n \times (0, \tau), \\ \psi = 0 & \text{on } \overline{E}_n \times \{\tau\}. \end{cases} \quad (1.4.63)$$

Consider the function

$$z(x) := C_0 \frac{|x|^{2-d} - n^{2-d}}{1 - n^{2-d}} \quad \forall x \in E_n,$$

where  $C_0$  is a suitable positive constant. It is readily seen that, for  $C_0 = C_0(n_0)$  large enough,  $z$  is a *supersolution* to problem (1.4.63) (recall that  $d \geq 3$ ). Furthermore,

$$\psi_n = z = 0 \quad \text{on } \partial B_n \times (0, \tau),$$

whence

$$\langle \nabla \psi_n, \mathbf{n}_n \rangle \geq \langle \nabla z, \mathbf{n}_n \rangle = -\frac{(d-2)C_0 n^{1-d}}{1-n^{2-d}} \quad \text{on } \partial B_n \times (0, \tau). \quad (1.4.64)$$

Thanks to (1.4.62) and (1.4.64) there follows (1.4.61) with  $C := (d-2)C_0/(1-n_0^{2-d})$ , and the proof is complete.  $\square$

We are now in position to prove our uniqueness result.

*Proof of Theorem 1.4.8.* Let  $u_1$  and  $u_2$  be two bounded solutions to problem (1.1.21) (with the same  $u_0$ ), in the sense of Definition 1.4.1, satisfying

$$\lim_{|x| \rightarrow \infty} u_1(x, t) = \lim_{|x| \rightarrow \infty} u_2(x, t) = a(t) \quad \text{for a.e. } t \in (0, T).$$

By dominated convergence, this implies that for any  $\tau \in (0, T)$

$$\lim_{R \rightarrow \infty} \frac{1}{R^{d-1}} \int_0^\tau \int_{\partial B_R} |G(u_1(\sigma, t)) - G(u_2(\sigma, t))| \, d\sigma dt = 0. \quad (1.4.65)$$

It is apparent that  $w := u_1 - u_2$  solves

$$\begin{aligned} & \int_\Omega \rho(x) w(x, \tau) \varphi(x, \tau) \, dx \\ &= \int_0^\tau \int_\Omega \{ \rho(x) w(x, t) \varphi_t(x, t) + [G(u_1(x, t)) - G(u_2(x, t))] \Delta \varphi(x, t) \} \, dx dt \\ & \quad - \int_0^\tau \int_{\partial \Omega} [G(u_1(\sigma, t)) - G(u_2(\sigma, t))] \frac{\partial \varphi}{\partial \mathbf{n}}(\sigma, t) \, d\sigma dt \end{aligned} \quad (1.4.66)$$

for any  $\Omega$  and  $\varphi$  as in Definition 1.4.1 (just swap  $t$  and  $\tau$  there). Now note that, by multiplying the differential equation in (1.4.59) by  $\Delta \psi_n / \rho$  and integrating by parts we obtain

$$\int_0^\tau \int_{B_n} q_n(x, t) [\Delta \psi_n(x, t)]^2 \, dx dt \leq \tilde{C} \quad (1.4.67)$$

for some constant  $\tilde{C} > 0$  independent of  $n$  (recall that  $\rho \in L^\infty(\mathbb{R}^d)$ ). Choosing  $\Omega = B_n$  and  $\varphi = \psi_n$  in (1.4.66) we get

$$\begin{aligned} \int_{B_n} \rho(x) w(x, \tau) \chi(x) \, dx &= \int_0^\tau \int_{B_n} [q(x, t) - q_n(x, t)] w(x, t) \Delta \psi_n(x, t) \, dx dt \\ & \quad - \int_0^\tau \int_{\partial B_n} q(\sigma, t) w(\sigma, t) \langle \nabla \psi_n, \mathbf{n}_n \rangle(\sigma, t) \, d\sigma dt. \end{aligned} \quad (1.4.68)$$

We shall prove that both integrals in the right-hand side of (1.4.68) vanish as  $n \rightarrow \infty$ . As for the first one, (1.4.58) and (1.4.67) yield

$$\begin{aligned} & \left( \int_0^\tau \int_{B_n} [q(x, t) - q_n(x, t)] w(x, t) \Delta \psi_n(x, t) \, dx dt \right)^2 \\ & \leq B \left\| \frac{(q_n - q)}{\sqrt{q_n}} \right\|_{L^2(Q_{n, \tau})}^2 \int_0^\tau \int_{B_n} q_n(x, t) [\Delta \psi_n(x, t)]^2 \, dx dt \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (1.4.69)$$

where  $B := (\|u_1\|_\infty + \|u_2\|_\infty)^2$ . On the other hand, from (1.4.61) and (1.4.65) we deduce

that

$$\begin{aligned}
 & \left| \int_0^\tau \int_{\partial B_n} q(\sigma, t) w(\sigma, t) \langle \nabla \psi_n, \mathbf{n}_n \rangle(\sigma, t) \, d\sigma dt \right| \\
 &= \left| \int_0^\tau \int_{\partial B_n} [G(u_1(\sigma, t)) - G(u_2(\sigma, t))] \langle \nabla \psi_n, \mathbf{n}_n \rangle(\sigma, t) \, d\sigma dt \right| \\
 &\leq \sup_{\partial B_n \times (0, \tau)} |\langle \nabla \psi_n, \mathbf{n}_n \rangle| \int_0^\tau \int_{\partial B_n} |G(u_1(\sigma, t)) - G(u_2(\sigma, t))| \, d\sigma dt \\
 &\leq \frac{C}{n^{d-1}} \int_0^\tau \int_{\partial B_n} |G(u_1(\sigma, t)) - G(u_2(\sigma, t))| \, d\sigma dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{1.4.70}$$

Letting  $n \rightarrow \infty$  in (1.4.68), in view of (1.4.69) and (1.4.70) there follows

$$\int_{\mathbb{R}^d} \rho(x) \chi(x) w(x, \tau) \, dx = 0, \tag{1.4.71}$$

which in fact holds true for any  $\tau \in (0, T)$  and any  $\chi \in C_c^\infty(\mathbb{R}^d)$  with  $0 \leq \chi \leq 1$ . Now fix any compact subset  $K \Subset \mathbb{R}^d$  and let

$$\zeta(x, \tau) := \begin{cases} 1 & \text{if } x \in K \text{ and } w(x, \tau) > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Pick a sequence  $\{\chi_n\} \subset C_c^\infty(\mathbb{R}^d)$ , with  $0 \leq \chi_n \leq 1$ , which converges pointwise to  $\zeta$ . Identity (1.4.71) gives

$$\int_{\mathbb{R}^d} \rho(x) \chi_n(x) w(x, \tau) \, dx = 0,$$

so that by taking the limit as  $n \rightarrow \infty$  we get (*e.g.* by dominated convergence)

$$\int_{K \cap \{w(\tau) > 0\}} \rho(x) w(x, \tau) \, dx = 0.$$

As a result,  $w(x, \tau) \leq 0$  for all  $x \in K$ . Since the compact subset  $K \Subset \mathbb{R}^d$  and  $\tau \in (0, T)$  are arbitrarily chosen, we then obtain

$$w \leq 0 \quad \text{in } \mathbb{R}^d \times (0, T),$$

that is

$$u_1 \leq u_2 \quad \text{in } \mathbb{R}^d \times (0, T).$$

Swapping the role of  $u_1$  and  $u_2$  we also find the opposite inequality, which completes the proof.

Finally, we still have to get rid of the hypothesis that  $\rho$  is regular. The latter property has been used in order to justify rigorously (1.4.61), (1.4.67) and (1.4.68), where we need *e.g.*  $\psi_n \in C^2(\overline{B_n} \times [0, \tau])$ . The idea is just to replace, for any given  $n > n_0$ , the density  $\rho$  with its mollification  $\rho_\varepsilon$  in problem (1.4.59), whose solution we denote as  $\psi_{n,\varepsilon}$ . It is easy to check that estimates (1.4.61) and (1.4.67) are independent of  $\varepsilon > 0$ . The only difference is in the r.h.s. of (1.4.68), where the term

$$\int_0^\tau \int_{B_n} [\rho(x) - \rho_\varepsilon(x)] w(x, t) (\psi_{n,\varepsilon})_t(x, t) \, dx dt \tag{1.4.72}$$

appears. However, it is plain that (1.4.72) vanishes as  $\varepsilon \rightarrow 0$ . Hence, by letting first  $\varepsilon \rightarrow 0$  and then  $n \rightarrow \infty$  we recover (1.4.71).  $\square$



## 1.A Some examples of weighted inequalities

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In this section we shall provide pairs of weights  $(\rho_\nu, \rho_\mu)$  for which Poincaré-, Sobolev- and Gagliardo-Nirenberg-type inequalities hold true, so as to show some explicit frameworks where the results of Sections 1.2 and 1.3 apply.

### 1.A.1 The Dirichlet case: functions in $H_0^1(\Omega; \nu, \mu)$

Here we shall consider inequalities of the type of (1.2.5), (1.2.7), (1.2.12) and (1.2.15). For the weights involved, the smoothing and decay estimates of Section 1.2 concerning solutions to the Dirichlet problem (1.1.1) hold true.

#### 1.A.1.1 Poincaré-type inequalities

We begin with the one-dimensional case, where necessary and sufficient conditions are available. Afterwards we shall consider domains of  $\mathbb{R}^d$  with  $d > 1$ .

Let  $\Omega = (a, b)$ , the cases  $a = -\infty$  and  $b = +\infty$  being allowed. We look for weights  $\rho_\nu$  and  $\rho_\mu$  such that the Poincaré-type inequality (also referred to as *Hardy-type inequality*)

$$\|\phi\|_{2;\nu} \leq C_P \|\phi'\|_{2;\mu} \quad (1.A.1)$$

holds true for some  $C_P > 0$  and all  $\phi$  belonging to a suitable functional space. We shall mainly refer to the monograph [145]. According to the notations introduced therein, we denote as  $AC_{\mathcal{L}}(a, b)$  the space of all functions  $\phi : (a, b) \mapsto \mathbb{R}$  which are locally absolutely continuous and such that  $\lim_{x \rightarrow a^+} \phi(x) = 0$ . The space  $AC_{\mathcal{R}}(a, b)$  is meant likewise, upon replacing  $a^+$  with  $b^-$ , while  $AC_{\mathcal{LR}} := AC_{\mathcal{L}} \cap AC_{\mathcal{R}}$ . Since  $C_c^\infty(a, b)$  is included in  $AC_{\mathcal{LR}}(a, b)$  and it is dense in  $H_0^1((a, b); \nu, \mu)$  by definition, the validity of (1.A.1) in  $AC_{\mathcal{LR}}(a, b)$  implies in turn the validity of the latter in  $H_0^1((a, b); \nu, \mu)$ . Theorem 1.14 of [145] ensures that (1.A.1) holds true in  $AC_{\mathcal{L}}(a, b)$  if and only if the weights satisfy the following condition:

$$\mathcal{B}_{\mathcal{L}}(a, b, \nu, \mu) := \sup_{x \in (a, b)} \left( \int_x^b \rho_\nu(y) \, dy \right) \left( \int_a^x \rho_\mu(y)^{-1} \, dy \right) < \infty.$$

On the other hand, (1.A.1) holds true in  $AC_{\mathcal{R}}(a, b)$  if and only if

$$\mathcal{B}_{\mathcal{R}}(a, b, \nu, \mu) := \sup_{x \in (a, b)} \left( \int_a^x \rho_\nu(y) \, dy \right) \left( \int_x^b \rho_\mu(y)^{-1} \, dy \right) < \infty.$$

It is then possible to show [145, Theorem 8.8] that the existence of a constant  $c \in [a, b]$  such that

$$\mathcal{B}_{\mathcal{L}}(a, c, \nu, \mu) + \mathcal{B}_{\mathcal{R}}(c, b, \nu, \mu) < \infty, \quad (1.A.2)$$

where we conventionally set  $\mathcal{B}_{\mathcal{L}}(a, a, \cdot, \cdot) = \mathcal{B}_{\mathcal{R}}(b, b, \cdot, \cdot) = 0$ , is necessary and sufficient for the validity of (1.A.1) in  $AC_{\mathcal{LR}}(a, b)$ . Actually, one can show that the same result is true if we replace  $AC_{\mathcal{LR}}(a, b)$  with the space of absolutely continuous functions in  $(a, b)$  with compact support. As a consequence, at least for weights that are locally bounded and bounded away from zero, we can assert that condition (1.A.2) is necessary and sufficient for the validity of (1.A.1) in  $H_0^1((a, b); \nu, \mu)$ . For a more general discussion about inequality (1.A.1), possibly involving weights which are not absolutely continuous with respect to the Lebesgue measure, see *e.g.* [131, Section 1.3] or also the pioneering works [138, 139].

In dimension  $d \geq 1$ , it is well known (see *e.g.* [145, Chapter 15]) that the so-called *Muckenhoupt classes* of weights originally introduced in [139] have an important role in weighted

functional inequalities. In fact, Muckenhoupt classes are defined by a sort of generalization of condition (1.A.2) for  $d$ -dimensional domains. We shall not investigate further such theory: we prefer to give some explicit examples instead.

Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. We denote as  $\delta(x) := \text{dist}(x, \partial\Omega)$  the distance of  $x \in \Omega$  with respect to the boundary. Consider a parameter  $\beta < 1$ . From [145, Theorem 21.5] we have that the Poincaré-type inequality

$$\|v\|_{2;\nu} \leq C_P \|\nabla v\|_{2;\mu} \quad (1.A.3)$$

holds true for all  $v \in H_0^1(\Omega; \delta^\alpha, \delta^\beta)$  if and only if  $\alpha \geq \beta - 2$ . Moreover, Sobolev-type inequalities hold if in addition either  $\beta \leq 2 - d$  or  $\beta > 2 - d$  and  $\alpha > \beta - 2$  (see the next Paragraph 1.A.1.2). Finally, if  $\beta \geq 1$  it is possible to prove that  $H_0^1(\Omega; \delta^\beta, \delta^\beta) = H^1(\Omega; \delta^\beta, \delta^\beta)$  [95, Theorem 2.11]: this is enough in order to assert that for any  $\alpha \in \mathbb{R}$  and  $\beta \geq 1$  inequality (1.A.3) cannot be valid in  $H_0^1(\Omega; \delta^\alpha, \delta^\beta)$ .

Consider now an exterior domain  $\Omega \subset \mathbb{R}^d$  (namely the complement of any compact set) such that  $\inf_{x \in \Omega} |x| > 0$ . Then the Poincaré-type inequality (1.A.3) holds true in

$$H_0^1(\Omega; |x|^\alpha, |x|^\beta)$$

if and only if  $\beta \neq 2 - d$  and  $\alpha \leq \beta - 2$  [145, Example 21.10]. Under such assumptions, Sobolev-type inequalities are valid either if  $\beta > 2 - d$  or if  $\beta < 2 - d$  and  $\alpha < \beta - 2$  (see again Paragraph 1.A.1.2). The same results apply to the whole Euclidean space  $\mathbb{R}^d$  upon replacing the weights  $|x|^\alpha$  and  $|x|^\beta$  with  $(|x| + 1)^\alpha$  and  $(|x| + 1)^\beta$ , respectively, provided  $\beta > 2 - d$  (see *e.g.* [22, 24]). On the other hand, if  $\beta < 2 - d$  then the Poincaré-type inequality (1.A.3) does not hold since, for instance, one can prove that constants belong to

$$H_0^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta)$$

provided  $\alpha \leq \beta - 2 < -d$ . As concerns the limit case  $\beta = 2 - d$  and peculiar related functional inequalities, we refer the reader to [24, 30] for a thorough analysis. Still in the framework of exterior domains, in [145, Example 21.12] exponential weights are also dealt with. In particular, one can prove that the Poincaré-type inequality (1.A.3) holds true in

$$H_0^1(\Omega; e^{\alpha|x|}, e^{\beta|x|}), \quad \beta \neq 0,$$

provided  $\alpha \leq \beta$ . Sobolev-type inequalities also hold if in addition either  $\beta > 0$  or  $\beta < 0$  and  $\alpha < \beta$ .

In the special case  $\Omega = \mathbb{R}^d \setminus \{0\}$  (see again [145, Example 21.10]), the (Hardy) Poincaré-type inequality (1.A.3) is also satisfied with respect to the weights  $\rho_\nu(x) = |x|^{\beta-2}$  and  $\rho_\mu(x) = |x|^\beta$ , under the only assumption  $\beta \neq 2 - d$  (and no Sobolev-type inequality holds).

In view of the above discussion, we can now list a collection of some explicit domains  $\Omega$  and pairs of weights  $(\rho_\nu, \rho_\mu)$  for which the Poincaré-type inequality (1.A.3) holds true in  $H_0^1(\Omega; \nu, \mu)$  but Sobolev-type embeddings fail. In the one-dimensional setting we exploit condition (1.A.2) and condition (1.A.5) below.

• *Intervals:*

- $(x^{\beta-2}, x^\beta)$  for  $\beta > 1$  in  $\Omega = (0, b)$ , with  $b > 0$ ;
- $(x^{\beta-2}, x^\beta)$  for  $\beta < 1$  in  $\Omega = (a, +\infty)$ , with  $a > 0$ ;
- $(x^{\beta-2}, x^\beta)$  for  $\beta \neq 1$  in  $\Omega = (0, +\infty)$ ;
- $(x^{-1}|\log x|^{\beta-2}, x|\log x|^\beta)$  for  $\beta \neq 1$  in  $\Omega = (0, 1)$ ;
- $(e^{\alpha x}, e^{\alpha x})$  for  $\alpha \neq 0$  in  $\Omega = \mathbb{R}$ ;

- *Bounded Lipschitz domains* ( $d \geq 2$ ):
  - $(\delta^{\beta-2}, \delta^\beta)$  for  $2 - d < \beta < 1$ ;
- *Exterior domains*:
  - $(|x|^{\beta-2}, |x|^\beta)$  for  $\beta < 2 - d$ ;
  - $(e^{-\alpha|x|}, e^{-\alpha|x|})$  for  $\alpha > 0$ ;
- *Euclidean space*:
  - $(|x|^{\beta-2}, |x|^\beta)$  for  $\beta \neq 2 - d$  in  $\mathbb{R}^d \setminus \{0\}$ .

### 1.A.1.2 Sobolev-type inequalities

As in the Poincaré case, necessary and sufficient conditions are also known for the validity of Sobolev-type inequalities in the one-dimensional context. More precisely, given  $a \in [-\infty, +\infty)$  and  $b \in (-\infty, +\infty]$ , with  $a < b$ , the Sobolev-type inequality

$$\|\phi\|_{2\sigma;\nu} \leq C_S \|\phi'\|_{2;\mu} \quad (1.A.4)$$

holds true for some  $\sigma > 1$ ,  $C_S > 0$  and all  $\phi \in AC_{\mathcal{L}}(a, b)$  if and only if

$$\mathcal{C}_{\mathcal{L}}(a, b, \nu, \mu, \sigma) := \sup_{x \in (a, b)} \left( \int_x^b \rho_\nu(y) \, dy \right)^{\frac{1}{\sigma}} \left( \int_a^x \rho_\mu(y)^{-1} \, dy \right) < \infty.$$

On the other hand, (1.A.4) holds true in  $AC_{\mathcal{R}}(a, b)$  if and only if

$$\mathcal{C}_{\mathcal{R}}(a, b, \nu, \mu, \sigma) := \sup_{x \in (a, b)} \left( \int_a^x \rho_\nu(y) \, dy \right)^{\frac{1}{\sigma}} \left( \int_x^b \rho_\mu(y)^{-1} \, dy \right) < \infty.$$

These results are still proved in [145, Theorem 1.14], whereas [145, Theorem 8.8] ensures that the existence of a constant  $c \in [a, b]$  such that

$$\mathcal{C}_{\mathcal{L}}(a, c, \nu, \mu, \sigma) + \mathcal{C}_{\mathcal{R}}(c, b, \nu, \mu, \sigma) < \infty \quad (1.A.5)$$

is necessary and sufficient for the validity of (1.A.4) in  $AC_{\mathcal{LR}}(a, b)$ .

Hereafter, for simplicity, we shall mainly focus on domains and weights for which the Poincaré-type inequality (1.A.1) also holds true: we point out that such property (*i.e.* the validity of both Poincaré-type and Sobolev-type inequalities) need not be a feature of only domains with finite measure. Moreover, in order to make sure that no Gagliardo-Nirenberg-type inequality of the type of (1.2.12) or (1.2.15) holds, we shall only consider cases where  $\sigma$  has a finite upper bound.

As concerns bounded Lipschitz domains  $\Omega \subset \mathbb{R}^d$  (we use the same notations as in Paragraph 1.A.1.1), from [145, Theorem 21.5] we have that the Sobolev-type inequality

$$\|v\|_{2\sigma;\nu} \leq C_S \|\nabla v\|_{2;\mu} \quad (1.A.6)$$

holds true in  $H_0^1(\Omega; \delta^\alpha, \delta^\beta)$  provided  $d \geq 3$ ,  $\beta < 1$ ,  $\beta \leq 2 - d$ ,  $\alpha \geq \beta - 2$  and  $\sigma \in [1, 1^*]$  or  $d \geq 2$ ,  $2 - d < \beta < 1$ ,  $\alpha > \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$ , where we set

$$1^* := d/(d - 2) \text{ if } d \geq 3, \quad 1^* := +\infty \text{ otherwise} \quad (1.A.7)$$

and

$$\kappa := \frac{\alpha + d}{\beta - 2 + d}. \quad (1.A.8)$$

Given an exterior domain  $\Omega \subset \mathbb{R}^d$  with  $\inf_{x \in \Omega} |x| > 0$ , Example 21.10 of [145] entails that the Sobolev-type inequality (1.A.6) holds true in  $H_0^1(\Omega; |x|^\alpha, |x|^\beta)$  provided  $\beta < 2 - d$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$  or  $d \geq 3$ ,  $\beta > 2 - d$ ,  $\alpha \leq \beta - 2$  and  $\sigma \in [1, 1^*]$ . For the latter choice of the parameters, as mentioned above, the same is true in

$$H_0^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta).$$

In the case of exponential weights, from [145, Example 21.12] we deduce that (1.A.6) holds true in

$$H_0^1(\Omega; e^{\alpha|x|}, e^{\beta|x|})$$

provided  $\beta < 0$ ,  $\alpha < \beta$  and  $\sigma \in [1, 1^* \wedge (\alpha/\beta)]$  or  $d \geq 3$ ,  $\beta > 0$ ,  $\alpha \leq \beta$  and  $\sigma \in [1, 1^*]$ . For the latter choice of the parameters it is not difficult to show that the same result is valid with  $\Omega = \mathbb{R}^d$ .

Finally, let us consider the special case  $\Omega = \mathbb{R}^d \setminus \{0\}$ . Still from [145, Example 21.10], we have that the Sobolev-type inequality (1.A.6) holds true in  $H_0^1(\Omega; |x|^\alpha, |x|^\beta)$  provided  $\beta > 2 - d$ ,  $\alpha > \beta - 2$  (with in addition  $\alpha \leq \beta 1^*$  if  $d \geq 3$ ) and  $\sigma = \kappa$  or  $\beta < 2 - d$ ,  $\alpha < \beta - 2$  (with in addition  $\alpha \geq \beta 1^*$  if  $d \geq 3$ ) and  $\sigma = \kappa$ .

We can now summarize the above examples by listing a collection of explicit domains  $\Omega$  and pairs of weights  $(\rho_\nu, \rho_\mu)$  for which the Sobolev-type inequality (1.A.6) holds true in  $H_0^1(\Omega; \nu, \mu)$  (but one- and two-dimensional Gagliardo-Nirenberg-type inequalities fail). In the one-dimensional setting we exploit condition (1.A.5).

- *Intervals:*

- $(x^\alpha, x^\beta)$  in  $\Omega = (0, b)$  (with  $b > 0$ ) for  $\beta > 1$ ,  $\alpha > \beta - 2$  and  $\sigma \in [1, (\alpha + 1)/(\beta - 1)]$ ;
- $(x^\alpha, x^\beta)$  in  $\Omega = (a, +\infty)$  (with  $a > 0$ ) for  $\beta < 1$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, (\alpha + 1)/(\beta - 1)]$ ;
- $(x^\alpha, x^\beta)$  in  $\Omega = (0, +\infty)$  for  $\beta > 1$ ,  $\alpha > \beta - 2$  and  $\sigma = (\alpha + 1)/(\beta - 1)$  or for  $\beta < 1$ ,  $\alpha < \beta - 2$  and  $\sigma = (\alpha + 1)/(\beta - 1)$ ;
- $(x^{-1}|\log x|^\alpha, |x| \log x|^\beta)$  in  $\Omega = (0, 1)$  for  $\beta > 1$ ,  $\alpha > \beta - 2$  and  $\sigma = (\alpha + 1)/(\beta - 1)$  or for  $\beta < 1$ ,  $\alpha < \beta - 2$  and  $\sigma = (\alpha + 1)/(\beta - 1)$ ;
- $(e^{\alpha x}, e^{\beta x})$  in  $\Omega = \mathbb{R}$  for  $\alpha > \beta > 0$  and  $\sigma = \alpha/\beta$ ;

- *Bounded Lipschitz domains:*

- $(\delta^\alpha, \delta^\beta)$  for  $d \geq 3$ ,  $\beta < 1$ ,  $\beta \leq 2 - d$ ,  $\alpha \geq \beta - 2$  and  $\sigma \in [1, 1^*]$  or for  $d \geq 2$ ,  $2 - d < \beta < 1$ ,  $\alpha > \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$ , where  $1^*$  and  $\kappa$  are as in (1.A.7) and (1.A.8), respectively;

- *Exterior domains:*

- $(|x|^\alpha, |x|^\beta)$  for  $\beta < 2 - d$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$  or for  $d \geq 3$ ,  $\beta > 2 - d$ ,  $\alpha \leq \beta - 2$  and  $\sigma \in [1, 1^*]$ ;
- $(e^{\alpha|x|}, e^{\beta|x|})$  for  $\beta < 0$ ,  $\alpha < \beta$  and  $\sigma \in [1, 1^* \wedge (\alpha/\beta)]$  or for  $d \geq 3$ ,  $\beta > 0$ ,  $\alpha \leq \beta$  and  $\sigma \in [1, 1^*]$ ;

- *Euclidean space:*

- $(|x|^\alpha, |x|^\beta)$  in  $\mathbb{R}^d$  for  $d \geq 3$ ,  $\beta > 2 - d$ ,  $\alpha \leq \beta - 2$  and  $\sigma \in [1, 1^*]$ ;
- $(e^{\alpha|x|}, e^{\beta|x|})$  in  $\mathbb{R}^d$  for  $d \geq 3$ ,  $\beta > 0$ ,  $\alpha \leq \beta$  and  $\sigma \in [1, 1^*]$ ;
- $(|x|^\alpha, |x|^\beta)$  in  $\mathbb{R}^d \setminus \{0\}$  for  $\beta > 2 - d$ ,  $\alpha > \beta - 2$  (with  $\alpha \leq \beta 1^*$  if  $d \geq 3$ ) and  $\sigma = \kappa$  or for  $\beta < 2 - d$ ,  $\alpha < \beta - 2$  (with  $\alpha \geq \beta 1^*$  if  $d \geq 3$ ) and  $\sigma = \kappa$ .

### 1.A.1.3 Gagliardo-Nirenberg-type inequalities

Unlike Poincaré- and Sobolev-type inequalities, as far as we know no necessary and sufficient conditions on the weights (in dimension one) are available for one- or two-dimensional

Gagliardo-Nirenberg-type inequalities. Nevertheless, we can still construct explicit examples in our setting relying on the well-known Caffarelli-Kohn-Nirenberg inequalities (see the original reference [41]). In particular, given  $\beta \in (-1, 1)$  and  $\alpha \geq -\beta$ , the inequalities

$$\left( \int_{\mathbb{R}} |\phi(x)|^r |x|^\alpha dx \right)^{\frac{1}{r}} \leq C_{G_1} \left( \int_{\mathbb{R}} |\phi'(x)|^2 |x|^\beta dx \right)^{\frac{\lambda(r-s)}{2r(s+\lambda)}} \left( \int_{\mathbb{R}} |\phi(x)|^s |x|^\alpha dx \right)^{\frac{r+\lambda}{r(s+\lambda)}} \quad \forall \phi \in C_c^\infty(\mathbb{R}) \quad (1.A.9)$$

hold true for all  $r > s > 0$  and some  $C_{G_1} = C_{G_1}(r, s, \alpha, \beta) > 0$ , where

$$\lambda := 2 \frac{\alpha + 1}{1 - \beta}. \quad (1.A.10)$$

This is in fact a special case of the main theorem in [41], at least for  $s \geq 1$ . In view of [13, Theorem 3.2], we then infer that (1.A.9) is equivalent to

$$\|\phi\|_\infty \leq C_{G_1} \left( \int_{\mathbb{R}} |\phi'(x)|^2 |x|^\beta dx \right)^{\frac{\lambda}{2(s+\lambda)}} \left( \int_{\mathbb{R}} |\phi(x)|^s |x|^\alpha dx \right)^{\frac{1}{s+\lambda}} \quad \forall \phi \in C_c^\infty(\mathbb{R}), \quad (1.A.11)$$

for another positive constant  $C_{G_1}$  independent of  $s > 0$ . We point out that the weights  $|x|^\alpha$  and  $|x|^\beta$  are not locally bounded and locally bounded away from zero in  $\Omega = \mathbb{R}$ , a property that we required in the theory above. However, notice that (1.A.11) trivially implies the validity of the same inequality *e.g.* in  $C_c^\infty((0, +\infty))$ , and therefore in the whole of

$$H_0^1((0, +\infty); |x|^\alpha, |x|^\beta) \cap L^s((0, +\infty); |x|^\alpha)$$

by density.

As concerns two-dimensional Gagliardo-Nirenberg-type inequalities, for simplicity we still provide examples in the real line. In order to avoid confusion, let us remark that the terminology “two dimensional” only refers to the type of inequality and not to the setting. Given  $\alpha > -1$ , the inequalities

$$\left( \int_{\mathbb{R}} |\phi(x)|^r |x|^\alpha dx \right)^{\frac{1}{r}} \leq C_{G_2} \left( \int_{\mathbb{R}} |\phi'(x)|^2 |x| dx \right)^{\frac{r-s}{2r}} \left( \int_{\mathbb{R}} |\phi(x)|^s |x|^\alpha dx \right)^{\frac{1}{r}} \quad \forall \phi \in C_c^\infty(\mathbb{R}) \quad (1.A.12)$$

hold true for all  $r > s > 0$  and some  $C_{G_2} = C_{G_2}(r, s, \alpha) > 0$ , still as a consequence of the results in [41] and of [13, Theorem 3.3]. Reasoning as above, we can then assert that (1.A.12) is also valid, for instance, in the whole of

$$H_0^1((0, +\infty); |x|^\alpha, |x|) \cap L^s((0, +\infty); |x|^\alpha),$$

for another  $C_{G_2} > 0$  that depends on  $r, s$  only through a finite upper bound on  $r$ . Moreover, in this case, a standard scaling argument (replace  $\phi(x)$  with  $\phi(x/\varepsilon)$  and let  $\varepsilon \downarrow 0$ ) prevents the possibility that a one-dimensional inequality of the type of (1.A.11) holds.

## 1.A.2 The Neumann case: functions in $H^1(\Omega; \nu, \mu)$ with zero mean

We now consider inequalities of the type of (1.3.14), (1.3.16), (1.3.19) and (1.3.21). Accordingly, the smoothing and long-time estimates of Section 1.3 concerning solutions to the Neumann problem (1.1.2) hold true for the weights involved.

### 1.A.2.1 Poincaré-type inequalities

As concerns Poincaré-type inequalities for zero-mean functions, in the one-dimensional context necessary and sufficient (and easy-to-check) conditions are still available. More precisely, let  $\Omega = (a, b)$  (the cases  $a = -\infty$  and  $b = +\infty$  being admissible) and take a pair of

weights  $(\rho_\nu, \rho_\mu)$  defined in  $\Omega$ , with  $\nu(\Omega) < \infty$ . Consider the quantities

$$\begin{aligned}\mathcal{K}_{\mathcal{L}}(a, b, \nu, \mu) &:= \sup_{x \in (a, b)} \left( \int_x^b \rho_\nu(y) \, dy \right) \left( \int_a^x \left( \int_a^y \rho_\nu(t) \, dt \right)^2 \rho_\mu(y)^{-1} \, dy \right), \\ \mathcal{K}_{\mathcal{R}}(a, b, \nu, \mu) &:= \sup_{x \in (a, b)} \left( \int_a^x \rho_\nu(y) \, dy \right) \left( \int_x^b \left( \int_y^b \rho_\nu(t) \, dt \right)^2 \rho_\mu(y)^{-1} \, dy \right).\end{aligned}$$

Theorem 1.4 of [54] states that the Poincaré-type inequality

$$\|v - \bar{v}\|_{2; \nu} \leq M_P \|v'\|_{2; \mu}, \quad \bar{v} := \frac{\int_a^b v(x) \rho_\nu(x) \, dx}{\int_a^b \rho_\nu(x) \, dx}$$

holds true for some  $M_P > 0$  and all  $v \in H^1((a, b); \nu, \mu)$  if and only if

$$\mathcal{K}_{\mathcal{L}}(a, b, \nu, \mu) + \mathcal{K}_{\mathcal{R}}(a, b, \nu, \mu) < \infty. \quad (1.A.13)$$

We refer the reader to the end of this paragraph for some explicit examples of one-dimensional weights complying with (1.A.13).

Let  $\Omega \subset \mathbb{R}^d$  be a bounded star-shaped domain and let  $w : (0, +\infty) \mapsto (0, +\infty)$  be any increasing function such that  $w(sr) \geq sw(r)$  for all  $r > 0$  and  $s \in (0, 1)$ . Then, for any  $k \in \mathbb{N}$ , there exists a constant  $M_P > 0$  such that (recall that  $\delta$  here denotes the distance function w.r.t. the boundary)

$$\begin{aligned}\int_{\Omega} |\phi(x) - \bar{\phi}|^2 w(\delta(x))^k \, dx &\leq M_P^2 \int_{\Omega} |\nabla \phi(x)|^2 w(\delta(x))^k \, dx \\ \forall \phi &\in C^1(\Omega) \cap H^1(\Omega; (w \circ \delta)^k, (w \circ \delta)^k).\end{aligned} \quad (1.A.14)$$

This is the content of [39, Theorem 1]. In view of Proposition 1.1.3 the validity of (1.A.14) is readily extended to the whole of  $H^1(\Omega; (w \circ \delta)^k, (w \circ \delta)^k)$ . In the case where  $\Omega$  is a ball, the result remains true under the sole assumption that  $w$  is increasing, see *e.g.* [88] and references therein. Nevertheless, here we are mainly interested in weights for which the Poincaré-type inequality

$$\|v - \bar{v}\|_{2; \nu} \leq M_P \|\nabla v\|_{2; \mu}, \quad \bar{v} := \frac{\int_{\Omega} v(x) \rho_\nu(x) \, dx}{\int_{\Omega} \rho_\nu(x) \, dx} \quad (1.A.15)$$

holds true and no Sobolev-type inequality does. For instance, from (1.A.14) we deduce that (1.A.15) is satisfied in  $H^1(\Omega; \delta^\beta, \delta^\beta)$  for all  $\beta > 0$  (see also [110] for similar results in less regular domains), but in the latter case also Sobolev-type inequalities are valid (we refer to Paragraph 1.A.2.2).

If  $\Omega \subset \mathbb{R}^d$  is a bounded convex domain, then the Poincaré-type inequality (1.A.15) holds true in  $H^1(\Omega; \delta^{\beta-2}, \delta^\beta)$  as a consequence of [55, Theorem 1.1], provided  $\beta \geq 2$ . Moreover, in this case there is no Sobolev-type embedding (we refer again the reader to the next Paragraph 1.A.2.2). Actually, by exploiting the results of [87] (see Theorem 3.3 there), where the authors establish some Muckenhoupt-type condition for rather general domains, we can assert that the same holds true for  $\beta \in (1, 2)$  as well. Notice that, by means of an elementary change of variable, we can relax the hypothesis that  $\Omega$  is convex by requiring, for instance, that there exists a bi-Lipschitz function which maps it to a convex domain.

We now consider the whole Euclidean space  $\mathbb{R}^d$  with power-law weights. In view of the results of [22, 24], we have that for  $d \geq 3$  the Poincaré-type inequality (1.A.15) holds true in

$$H^1(\mathbb{R}^d; (|x| + 1)^{\beta-2}, (|x| + 1)^\beta)$$

provided  $\beta < 2 - d$ . Furthermore, in this case there is no Sobolev-type embedding (see *e.g.* [145, Example 20.6]).

Finally, as for exponential weights, we limit ourselves to mentioning the well-know Poincaré inequality with respect to the Gaussian measure, namely the fact that (1.A.15) holds true in

$$H^1(\mathbb{R}^d; e^{-\alpha|x|^2}, e^{-\alpha|x|^2})$$

for all  $\alpha > 0$  (and Sobolev-type embeddings fail). Several references are available in the literature on this classical topic: we refer the reader *e.g.* to [106], [59, Chapter 2 and Section 4.3] and [92]. In this regard, see also the paper [12] and references therein for similar results concerning log-concave (and more general) measures.

For the reader's convenience, we list the most significant examples we discussed above where the Poincaré-type inequality (1.A.15) holds true in  $H^1(\Omega; \nu, \mu)$  but Sobolev-type ones do not. In the one-dimensional case we exploit condition (1.A.13) and condition (1.A.17) below.

- *Intervals:*
  - $(x^{\beta-2}, x^\beta)$  for  $\beta > 1$  in  $\Omega = (0, b)$ , with  $b > 0$ ;
  - $(x^{\beta-2}, x^\beta)$  for  $\beta < 1$  in  $\Omega = (a, +\infty)$ , with  $a > 0$ ;
  - $(x^{-1}|\log x|^{\beta-2}, x|\log x|^\beta)$  for  $\beta < 1$  in  $\Omega = (0, c)$ , with  $c \in (0, 1)$ ;
  - $(e^{-\alpha|x|}, e^{-\alpha|x|})$  for  $\alpha > 0$  in  $\Omega = \mathbb{R}$ ;
- *Bounded convex domains:*
  - $(\delta^{\beta-2}, \delta^\beta)$  for  $\beta > 1$ ;
- *Euclidean space:*
  - $[(|x| + 1)^{\beta-2}, (|x| + 1)^\beta]$  for  $d \geq 3$  and  $\beta < 2 - d$  in  $\mathbb{R}^d$ ;
  - $(e^{-\alpha|x|^2}, e^{-\alpha|x|^2})$  for  $\alpha > 0$  in  $\mathbb{R}^d$ .

### 1.A.2.2 Sobolev-type inequalities

In the one-dimensional context, Theorem 1.4 of [54] also deals with the Sobolev-type inequality

$$\|v - \bar{v}\|_{2\sigma; \nu} \leq M_S \|v'\|_{2; \mu}. \quad (1.A.16)$$

That is, given  $a \in [-\infty, +\infty)$  and  $b \in (-\infty, +\infty]$ , with  $a < b$ , (1.A.16) holds true for some  $\sigma > 1$ ,  $M_S > 0$  and all  $v \in H^1((a, b); \nu, \mu)$  if and only if

$$\mathcal{Q}_{\mathcal{L}}(a, b, \nu, \mu, \sigma) + \mathcal{Q}_{\mathcal{R}}(a, b, \nu, \mu, \sigma) < \infty, \quad (1.A.17)$$

where

$$\mathcal{Q}_{\mathcal{L}}(a, b, \nu, \mu, \sigma) := \sup_{x \in (a, b)} \left( \int_x^b \rho_\nu(y) dy \right)^{\frac{1}{\sigma}} \left( \int_a^x \left( \int_a^y \rho_\nu(t) dt \right)^2 \rho_\mu(y)^{-1} dy \right)$$

and

$$\mathcal{Q}_{\mathcal{R}}(a, b, \nu, \mu, \sigma) := \sup_{x \in (a, b)} \left( \int_a^x \rho_\nu(y) dy \right)^{\frac{1}{\sigma}} \left( \int_x^b \left( \int_y^b \rho_\nu(t) dt \right)^2 \rho_\mu(y)^{-1} dy \right).$$

Again, in the end of the paragraph we shall list some explicit examples of one-dimensional weights which comply with (1.A.17).

In general dimensions, once the embedding of  $H^1(\Omega; \nu, \mu)$  in  $L^{2\sigma}(\Omega; \nu)$  is continuous and the one in  $L^2(\Omega; \nu)$  is *compact*, one can readily prove the validity of the Sobolev-type inequality

$$\|v - \bar{v}\|_{2\sigma; \nu} \leq M_S \|\nabla v\|_{2; \mu} \quad (1.A.18)$$

by a means of a standard argument by contradiction (of course, provided  $\nu(\Omega) < \infty$ ). As a consequence, below we shall only mention compactness properties, taking for granted that the latter imply (1.A.18). We point out that this approach is in principle possible for the Poincaré-type inequality (1.A.15) as well, except that in this case lack of Sobolev-type embeddings is usually associated with lack of compactness.

Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain, with  $d \geq 2$ . If  $\beta > 1$  and  $\alpha > \beta - 2$ , then Theorems 19.9 and 19.11 of [145] establish that the injection of  $H^1(\Omega; \delta^\beta, \delta^\beta)$  in  $L^{2\sigma}(\Omega; \delta^\alpha)$  is continuous for all  $\sigma \in [1, 1^* \wedge \kappa]$  ( $1^*$  and  $\kappa$  are as in (1.A.7) and (1.A.8), respectively) and compact for all  $\sigma \in [1, 1^* \wedge \kappa)$ . Using compactness it is easy to show that in fact, in this case,  $H^1(\Omega; \delta^\alpha, \delta^\beta)$  is continuously embedded in  $H^1(\Omega; \delta^\beta, \delta^\beta)$ , so that the just mentioned results also hold upon replacing the latter with the former. We point out that for  $\alpha = \beta - 2$  no Sobolev-type embedding takes place (see *e.g.* [145, Example 18.15]). On the other hand, if  $\beta \leq 1$  and  $\alpha > -1$  (recall that we need  $\delta^\alpha(x)dx$  to be a finite measure), still Theorems 19.9 and 19.11 of [145] assert that the embedding of  $H^1(\Omega; \delta^\beta, \delta^\beta)$  in  $L^{2\sigma}(\Omega; \delta^\alpha)$  is continuous for all  $\sigma \in [1, 1^*] \cap [1, (\alpha + d)/(d - 1))$  and compact for all  $\sigma \in [1, 1^*) \cap [1, (\alpha + d)/(d - 1))$ . Actually, one can prove that the same holds in  $H^1(\Omega; \delta^\alpha, \delta^\beta)$ . Indeed, if  $\alpha > -1$  then for every  $\beta_0 > 1$  small enough we have  $\alpha > \beta_0 - 2$ . Hence, in view of first part of the discussion,  $H^1(\Omega; \delta^\alpha, \delta^{\beta_0})$  is continuously embedded in  $L^{2\sigma}(\Omega; \delta^\alpha)$  for all  $\sigma \in [1, 1^* \wedge (\alpha + d)/(\beta_0 - 2 + d)]$ , with compactness provided  $\sigma \in [1, 1^* \wedge (\alpha + d)/(\beta_0 - 2 + d))$ . Since  $\beta_0 > \beta$ , so that  $H^1(\Omega; \delta^\alpha, \delta^\beta)$  is in turn continuously embedded in  $H^1(\Omega; \delta^\alpha, \delta^{\beta_0})$ , the result follows by letting  $\beta_0 \downarrow 1$ .

We finally consider the whole Euclidean space  $\mathbb{R}^d$ , with power-law and exponential weights. If  $\beta < 2 - d$  and  $\alpha < \beta - 2$ , by means of elementary cut-off arguments it is not difficult to show that

$$H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta) = H_0^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta);$$

In particular, we are allowed to take advantage of the results of [145, Examples 20.6 and 21.10], which ensure that the embedding

$$H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta) \hookrightarrow L^{2\sigma}(\mathbb{R}^d; (|x| + 1)^\alpha)$$

is continuous for all  $\sigma \in [1, 1^* \wedge \kappa]$  and compact for all  $\sigma \in [1, 1^* \wedge \kappa)$ . On the other hand, if  $\beta \geq 2 - d$  and  $\alpha < -d$  we can reason as follows (let  $d \geq 3$ ). For every  $\beta_0 < 2 - d$  such that  $\alpha < \beta_0 - 2$ , in view of the just recalled results we know that the embedding

$$H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^{\beta_0}) \hookrightarrow L^{2\sigma}(\mathbb{R}^d; (|x| + 1)^\alpha)$$

is continuous for all  $\sigma \in [1, 1^* \wedge (\alpha + d)/(\beta_0 - 2 + d)]$  and compact for all  $\sigma \in [1, 1^* \wedge (\alpha + d)/(\beta_0 - 2 + d))$ . Since the embedding

$$H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta) \hookrightarrow H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^{\beta_0})$$

is trivially continuous, by letting  $\beta_0 \uparrow 2 - d$  we then deduce that in this case the embedding

$$H^1(\mathbb{R}^d; (|x| + 1)^\alpha, (|x| + 1)^\beta) \hookrightarrow L^{2\sigma}(\mathbb{R}^d; (|x| + 1)^\alpha)$$

is continuous for all  $\sigma \in [1, 1^*]$  and compact for all  $\sigma \in [1, 1^*)$ . When dealing with exponential weights, one can also prove that

$$H^1(\mathbb{R}^d; e^{\alpha|x|}, e^{\beta|x|}) = H_0^1(\mathbb{R}^d; e^{\alpha|x|}, e^{\beta|x|})$$

provided (for instance)  $\alpha < \beta < 0$ . Hence, by exploiting [145, Examples 20.8 and 21.12], we deduce that the embedding

$$H^1(\mathbb{R}^d; e^{\alpha|x|}, e^{\beta|x|}) \hookrightarrow L^{2\sigma}(\mathbb{R}^d; e^{\alpha|x|})$$



is continuous for all  $\sigma \in [1, 1^* \wedge (\alpha/\beta)]$  and compact for all  $\sigma \in [1, 1^* \wedge (\alpha/\beta))$ . If  $\alpha < 0$  and  $\beta > 0$ , arguing as above we can then assert that the same embedding is continuous for all  $\sigma \in [1, 1^*]$  and compact for all  $\sigma \in [1, 1^*)$  (let  $d \geq 3$ ).

Let us now summarize the above examples by listing some explicit domains  $\Omega$  and pairs of weights  $(\rho_\nu, \rho_\mu)$  for which the Sobolev-type inequality (1.A.18) holds true in  $H^1(\Omega; \nu, \mu)$ . In the one-dimensional setting we exploit condition (1.A.17).

- *Intervals:*
  - $(x^\alpha, x^\beta)$  in  $\Omega = (0, b)$  (with  $b > 0$ ) for  $\beta > 1$ ,  $\alpha > \beta - 2$  and  $\sigma \in [1, (\alpha + 1)/(\beta - 1)]$ ;
  - $(x^\alpha, x^\beta)$  in  $\Omega = (a, +\infty)$  (with  $a > 0$ ) for  $\beta < 1$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, (\alpha + 1)/(\beta - 1)]$ ;
  - $(x^{-1}|\log x|^\alpha, x|\log x|^\beta)$  in  $\Omega = (0, c)$  (with  $c \in (0, 1)$ ) for  $\beta < 1$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, (\alpha + 1)/(\beta - 1)]$ ;
  - $(e^{-\alpha|x|}, e^{-\beta|x|})$  in  $\Omega = \mathbb{R}$  for  $\alpha > \beta > 0$  and  $\sigma \in [1, \alpha/\beta]$ ;
- *Bounded Lipschitz domains* ( $d \geq 2$ ):
  - $(\delta^\alpha, \delta^\beta)$  for  $\beta \leq 1$ ,  $\alpha > -1$  and  $\sigma \in [1, 1^*] \cap [1, (\alpha + d)/(d - 1)]$  or for  $\beta > 1$ ,  $\alpha > \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$ , where  $1^*$  and  $\kappa$  are as in (1.A.7) and (1.A.8), respectively;
- *Euclidean space:*
  - $(|x| + 1)^\alpha, (|x| + 1)^\beta$  in  $\mathbb{R}^d$  for  $d \geq 3$ ,  $\beta \geq 2 - d$ ,  $\alpha < -d$  and  $\sigma \in [1, 1^*]$  or for  $\beta < 2 - d$ ,  $\alpha < \beta - 2$  and  $\sigma \in [1, 1^* \wedge \kappa]$ ;
  - $(e^{\alpha|x|}, e^{\beta|x|})$  in  $\mathbb{R}^d$  for  $\beta < 0$ ,  $\alpha < \beta$  and  $\sigma \in [1, 1^* \wedge (\alpha/\beta)]$  or for  $d \geq 3$ ,  $\beta > 0$ ,  $\alpha < 0$  and  $\sigma \in [1, 1^*]$ .

### 1.A.2.3 Gagliardo-Nirenberg-type inequalities

Recalling the one-dimensional example of Paragraph 1.A.1.3, for  $\beta \in (-1, 1)$ ,  $\alpha \geq -\beta$  and  $\lambda \geq 2$  as in (1.A.10) it is easy to show that the family of inequalities (1.A.11) also holds upon replacing  $C_c^\infty(\mathbb{R})$  with the space  $X$  of functions belonging to  $C^\infty([0, 2])$  which vanish at  $x = 2$ . By means of classical cut-off and mollification arguments, one can prove that  $X$  is dense in the Hilbert space  $Y$  of functions belonging to  $H^1((0, 2); |x|^\alpha, |x|^\beta)$  which vanish at  $x = 2$ , so that (1.A.11) holds in  $Y$  as well. It is then apparent that there exists an extension operator  $\mathcal{E} : H^1((0, 1); |x|^\alpha, |x|^\beta) \mapsto Y$  which is linear and continuous. Picking for instance  $s = 2$  in (1.A.11) we infer that

$$\|\mathcal{E}(v)\|_\infty \leq W_{G_1} \left( \int_{\mathbb{R}} |\mathcal{E}(v)'(x)|^2 |x|^\beta dx \right)^{\frac{\lambda}{2(2+\lambda)}} \left( \int_{\mathbb{R}} |\mathcal{E}(v)(x)|^2 |x|^\alpha dx \right)^{\frac{1}{2+\lambda}} \\ \forall v \in H^1((0, 1); |x|^\alpha, |x|^\beta),$$

whence

$$\|v\|_\infty \leq W_{G_1} \left( \|v'\|_{2;|x|^\beta} + \|v\|_{2;|x|^\alpha} \right)^{\frac{\lambda}{2+\lambda}} \|v\|_{2;|x|^\alpha}^{\frac{2}{2+\lambda}} \quad \forall v \in H^1((0, 1); |x|^\alpha, |x|^\beta) \quad (1.A.19)$$

for a suitable  $W_{G_1} > 0$  that may change from line to line (we do not relabel it). It is possible to prove, *e.g.* using compactness (see [145, Section 7.10, ii]) or the necessary and sufficient conditions provided by [54, Theorem 1.4], that the Poincaré-type inequality

$$\|v - \bar{v}\|_{2;|x|^\alpha} \leq M_P \|v'\|_{2;|x|^\beta} \quad \forall v \in H^1((0, 1); |x|^\alpha, |x|^\beta) \quad (1.A.20)$$

holds true for some  $M_P > 0$ . Hence, taking advantage of (1.A.19), (1.A.20) and reasoning as in Remark 1.3.21 we end up with the validity of the one-dimensional family

$$\|v - \bar{v}\|_\infty \leq W_{G_1} \|v'\|_{2;|x|^\beta}^{\frac{\lambda}{s+\lambda}} \|v - \bar{v}\|_{s;|x|^\alpha}^{\frac{s}{s+\lambda}} \quad \forall s > 0, \quad \forall v \in H^1((0, 1); |x|^\alpha, |x|^\beta). \quad (1.A.21)$$

As for two-dimensional inequalities we only mention that, given  $\alpha > -1$ , starting from (1.A.12) and arguing as above it is not difficult to deduce the validity of the family

$$\|v\|_{r;|x|^\alpha} \leq W_{G_2} \left( \|v'\|_{2;|x|} + \|v\|_{2;|x|^\alpha} \right)^{1-\frac{2}{r}} \|v\|_{2;|x|^\alpha}^{\frac{2}{r}} \quad \forall r > 2, \quad \forall v \in H^1((0, 1); |x|^\alpha, |x|).$$

for a suitable  $W_{G_2} > 0$ . In fact, even though functions belonging to  $H^1((0, 1); |x|^\alpha, |x|)$  are not necessarily bounded, standard truncation and mollification techniques entail the density of restrictions of functions in  $C^\infty([0, 2])$  in  $H^1((0, 1); |x|^\alpha, |x|)$ . Also the Poincaré inequality continues to hold thanks to [54, Theorem 1.4]. We can therefore claim the validity of the two-dimensional family

$$\|v - \bar{v}\|_{r;|x|^\alpha} \leq M_{G_2} \|v'\|_{2;|x|}^{1-\frac{s}{r}} \|v - \bar{v}\|_{s;|x|^\alpha}^{\frac{s}{r}} \quad \forall r > s > 0, \quad \forall v \in H^1((0, 1); |x|^\alpha, |x|)$$

for a constant  $M_{G_2} > 0$  that depends on  $r, s$  only through a finite upper bound on  $r$ . Finally, the same scaling argument as in Paragraph 1.A.1.3 ensures that no one-dimensional inequality of the type of (1.A.21) can hold. Alternatively, one can exploit the already mentioned fact that functions in  $H^1((0, 1); |x|^\alpha, |x|)$  are not necessarily bounded (take for instance  $x \mapsto \log |\log x|$ ).



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## Fractional porous medium equations with power-type densities

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### 2.1 Introduction

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The main topic of this chapter is the analysis of the asymptotic behaviour of nonnegative solutions to the following nonlocal, nonlinear and degenerate parabolic problem:

$$\begin{cases} \rho(x)u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.1.1)$$

where  $m > 1$ ,  $s \in (0, 1)$ ,  $d > 2s$  and the symbol  $(-\Delta)^s$  denotes the *fractional Laplacian* operator on  $\mathbb{R}^d$ , that is

$$(-\Delta)^s(\phi)(x) := p.v. \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x - y|^{d+2s}} dy \quad \forall x \in \mathbb{R}^d, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d), \quad (2.1.2)$$

$C_{d,s}$  being a suitable positive constant depending only on  $d$  and  $s$  (if  $\phi$  is less regular, its fractional Laplacian is meant in the distributional sense), see *e.g.* [69, Sections 3, 4]. The initial datum  $u_0$  in (2.1.1) is assumed to be nonnegative and to belong to the weighted  $L^1$  space

$$L_\rho^1(\mathbb{R}^d) := \left\{ u \text{ measurable} : \|u\|_{1,\rho} := \int_{\mathbb{R}^d} |u(x)| \rho(x) dx < \infty \right\}.$$

The densities  $\rho$  we consider are nonnegative Lebesgue measurable functions, locally essentially bounded away from zero (namely  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ ) and satisfying suitable power-law decay conditions at infinity. More precisely, we say that  $\rho$  is a *slowly decaying density* if there exist two positive constants  $c < C$  such that

$$c|x|^{-\gamma_0} \leq \rho(x) \leq C|x|^{-\gamma_0} \quad \text{a.e. in } B_1 \quad \text{and} \quad c|x|^{-\gamma} \leq \rho(x) \leq C|x|^{-\gamma} \quad \text{a.e. in } B_1^c \quad (2.1.3)$$

for some  $\gamma \in (0, 2s)$  and  $\gamma_0 \in [0, \gamma]$ , where  $B_r := B_r(0)$  and  $B_r^c$  denotes its complement. Note that possible singularities are allowed at  $x = 0$ . For this kind of densities, we shall also study

the well-posedness of the more general problem

$$\begin{cases} \rho(x)u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \rho(x)u = \mu & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.1.4)$$

where  $\mu$  is a positive finite Radon measure on  $\mathbb{R}^d$  (here we let  $\gamma \in [0, 2s)$ , since our results are new even in the non-weighted case). In fact, as we shall explain below, the solution to (2.1.4) corresponding to the choices  $\rho(x) = |x|^{-\gamma}$  and  $\mu = \delta$  will be key to the asymptotics of general solutions. Conversely, we say that  $\rho$  is a *rapidly decaying density* if  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  and there exists a positive constant  $C$  such that

$$\rho(x) \leq C(1 + |x|)^{-\gamma} \quad \text{a.e. in } \mathbb{R}^d \quad (2.1.5)$$

for some  $\gamma > 2s$ . In contrast with the slowly decaying case, for such densities an essential role is played by a separable solution of the form  $t^{-1/(m-1)}w^{1/m}(x)$ , where  $w$  is the solution of a suitable *fractional sublinear elliptic equation*.

The local version of (2.1.1), that is

$$\begin{cases} \rho(x)u_t - \Delta(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.1.6)$$

known as *inhomogeneous porous medium equation*, has been largely studied in the literature. As concerns well-posedness, we quote the papers [89, 90, 112, 116, 153, 163]. In particular, for  $d \geq 3$  it is shown that (2.1.6) admits a unique *very weak* solution if  $\rho(x)$  decays slowly as  $|x| \rightarrow \infty$  (e.g. like  $|x|^{-\gamma}$  with  $\gamma \in (0, 2)$ ), while nonuniqueness prevails when  $\rho(x)$  decays rapidly as  $|x| \rightarrow \infty$  (e.g. like  $|x|^{-\gamma}$  with  $\gamma > 2$ ). In the latter case, uniqueness can be restored by imposing proper extra conditions at infinity (see Chapter 1, Section 1.4) or by looking for solutions in suitable energy spaces (*weak energy solutions*). In this regard, we refer the reader to the results of Chapter 1, Subsection 1.2.2, which hold independently of the behaviour of the weights at the boundary (*i.e.* the behaviour of  $\rho$  at infinity).

The long time behaviour of solutions to (2.1.6) has been analysed in detail in [114, 162, 164]. In [164] the authors prove that, if  $\rho > 0$ ,  $\rho(x) \approx |x|^{-\gamma}$  as  $|x| \rightarrow \infty$  for some  $\gamma \in (0, 2)$  and  $\|u_0\|_{1,\rho} =: M > 0$ , then

$$\lim_{t \rightarrow \infty} \|u(t) - U_M(t)\|_{1,\rho} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} t^\alpha \|u(t) - U_M(t)\|_\infty = 0.$$

The function  $U_M$  above is the *Barenblatt solution* with mass  $\|U_M(t)\|_{1,\rho} = M$  (it is preserved in time), which has an explicit self-similar profile:

$$U_M(x, t) := t^{-\alpha} F(t^{-\kappa}|x|) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where

$$F(\xi) := (C - b\xi^{2-\gamma})_+^{\frac{1}{m-1}} \quad \forall \xi \geq 0$$

for suitable positive constants  $C$  and  $b$  depending on  $M, m, d$  and  $\gamma$ , while  $\alpha$  and  $\kappa$  are defined by the formulas

$$\alpha := (d - \gamma)\kappa, \quad \kappa := \frac{1}{d(m-1) + 2 - m\gamma}.$$

Analogous results are proved in [162], buy only in the case  $d = 2$ . We stress that  $U_M$  solves the singular problem

$$\begin{cases} |x|^{-\gamma}u_t - \Delta(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ |x|^{-\gamma}u = M\delta & \text{on } \mathbb{R}^d \times \{0\}. \end{cases}$$

Let us point out that these convergence results are similar to the ones previously obtained in the non-weighted case (see *e.g.* [99] and [170]), namely for  $\rho \equiv 1$  and so  $\gamma = 0$ . For rapidly decaying densities, the situation is quite different. In fact, in [114] the authors prove that if  $\rho > 0$  and  $\rho(x) \sim |x|^{-\gamma}$  as  $|x| \rightarrow \infty$  for some  $\gamma > 2$ , then the *minimal* solution to problem (2.1.6) (namely the one obtained as monotone limit of solutions to homogeneous Dirichlet problems in balls of increasing radii), which is unique in the class of solutions satisfying

$$\frac{1}{R^{d-1}} \int_{\partial B_R} \int_0^t u^m(\sigma, \tau) \, d\tau d\sigma \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

for all  $t > 0$ , converges to a *separable solution*:

$$t^{\frac{1}{m-1}} u(x, t) \rightarrow (m-1)^{-\frac{1}{m-1}} W^{\frac{1}{m}}(x) \quad \text{as } t \rightarrow \infty,$$

uniformly w.r.t.  $x \in \mathbb{R}^d$ . Here  $W$  is the minimal positive solution to the sublinear elliptic equation

$$-\Delta W = \rho W^{\frac{1}{m}} \quad \text{in } \mathbb{R}^d,$$

and it satisfies

$$\lim_{|x| \rightarrow \infty} W(x) = 0.$$

Problem (2.1.1) with  $\rho \equiv 1$ , known as *fractional porous medium equation*, which reads

$$\begin{cases} u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.1.7)$$

has recently been addressed in the breakthrough papers [60, 61] by J. L. Vázquez et al.: existence, uniqueness and qualitative properties of weak energy solutions (with  $L^1$  data) are analysed there, also for  $m < 1$ . The asymptotic behaviour of such solutions as  $t \rightarrow \infty$  is then investigated in [174]. As one may guess from the local case ( $s = 1$ ), a key preliminary result consists in proving that, for any  $M > 0$ , there exists a unique solution  $\widehat{U}_M$  to the problem

$$\begin{cases} u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = M\delta & \text{on } \mathbb{R}^d \times \{0\}. \end{cases}$$

Such solution still has a self-similar profile:

$$\widehat{U}_M(x, t) := t^{-\alpha} f(t^{-\kappa}|x|) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where now

$$\alpha = d\kappa, \quad \kappa = \frac{1}{d(m-1) + 2s}.$$

The function  $f : [0, \infty) \mapsto (0, \infty)$  is bounded, Hölder continuous, decreasing and satisfies  $\lim_{\xi \rightarrow \infty} f(\xi) = 0$ . Thanks to these properties,  $\widehat{U}_M$  is called again a *Barenblatt solution*. However, in contrast with the local case,  $f$ , and so  $\widehat{U}_M$ , is no more explicit. It is then proved that  $\widehat{U}_M$  actually plays the role of an attractor for any nonnegative solution to (2.1.7) with mass  $\|u_0\|_1 = M$ :

$$\lim_{t \rightarrow \infty} \|u(t) - \widehat{U}_M(t)\|_1 = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} t^\alpha \|u(t) - \widehat{U}_M(t)\|_\infty = 0. \quad (2.1.8)$$

Inspired by the above results, here we carry out the asymptotic analysis for nonnegative solutions to (2.1.1), which is both a weighted and a fractional problem.

In the case of *slowly decaying densities* (Section 2.2) we first perform a full well-posedness analysis of problem (2.1.4), where any positive finite Radon measure  $\mu$  is allowed as initial datum. More precisely, under the additional assumption  $\gamma \leq d - 2s$ , we prove existence and uniqueness of weak solutions for any  $\rho$  complying with (2.1.3). We stress that our techniques crucially exploit the validity of the fractional Hardy-Sobolev inequality

$$\left( \int_{\mathbb{R}^d} |v(x)|^{2_\gamma^*} |x|^{-\gamma} dx \right)^{\frac{1}{2_\gamma^*}} \leq C_\gamma \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2, \quad 2_\gamma^* := 2 \frac{d - \gamma}{d - 2s}, \quad C_\gamma > 0$$

and the Stroock-Varopoulos inequality (let  $q > 1$  and  $v \geq 0$ )

$$\int_{\mathbb{R}^d} v^{q-1}(x) (-\Delta)^s(v)(x) dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} \left( v^{\frac{q}{2}} \right) (x) \right|^2 dx.$$

As it might be guessed, the hardest part to tackle is uniqueness. To this end, the first issue we have to deal with is the essential self-adjointness, in suitably weighted  $L^2$  spaces, of the operator formally defined as  $\rho^{-1}(-\Delta)^s$  on test functions, and the validity of the Markov property for the associated linear evolution (Appendix 2.A). In fact, in order to prove uniqueness, we need to adapt a delicate argument originally developed by M. Pierre in [148] for the same problem, but in the local case and without weight (porous medium equation). Such an argument crucially exploits the properties of solutions to a dual problem, which is a linear modification of the original one; moreover, it strongly relies on *Riesz potential* techniques, which means that one considers the equation solved by the Riesz potential of  $\rho u(t)$  and works with the latter, rather than with the original one. Though this kind of techniques is proving to be extremely powerful in the fractional framework (see *e.g.* [32, 33]), the drawback in our setting is that, in order to justify rigorously the equation solved by the Riesz potential, we have to require the additional assumption  $\gamma \leq d - 2s$ .

Existence and uniqueness of nonnegative bounded solutions to problem (2.1.1) for nonnegative initial data  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and bounded, strictly positive densities  $\rho$  have been investigated in [159, 160], where it is proved that if there exists  $C > 0$  such that

$$\rho(x) \geq C|x|^{-\gamma} \quad \text{a.e. in } B_1^c$$

for some  $\gamma \in [0, 2s)$ , then (2.1.1) admits a unique bounded solution. Here we shall exploit such results in order to prove existence of solutions to (2.1.4), by means of an approximation procedure.

The study of porous medium and fast diffusion equations with measure data can be tracked back to the pioneering papers [9, 37, 56, 148] (see also [173, Chapter 13] for details and additional references). The fast diffusion case, which will not be dealt with here, is investigated in [48, 49]: notice that for such evolutions a Dirac delta may not be smoothed into a regular solution, so that different techniques must be used; a general approach can be found in [150]. Semilinear heat equations with measure data have a long history as well and have recently been studied also in the fractional framework (we refer the reader *e.g.* to [50, 130] and references quoted). Let us point out that the terminology “measure data” sometimes refers to different contexts in which a measure appears as a source term in certain evolution equations, see for instance [133].

Once existence and uniqueness of solutions to (2.1.4) is established, one can start the asymptotic analysis. To this end, in agreement with the local case, it is crucial to assume that, in addition to (2.1.3),  $\rho$  satisfies

$$\lim_{|x| \rightarrow \infty} \rho(x)|x|^\gamma =: c_\infty > 0, \tag{2.1.9}$$

namely  $\rho(x) \approx |x|^{-\gamma}$  as  $|x| \rightarrow \infty$  (up to a multiplicative constant). We are then able to prove that the asymptotics is ruled by the solution  $u_M$  to the following singular problem:

$$\begin{cases} c_\infty |x|^{-\gamma} u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ c_\infty |x|^{-\gamma} u = M\delta & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.1.10)$$

where  $\|u_0\|_{1,\rho} = M > 0$  is the (conserved) mass and  $c_\infty$  is as in (2.1.9). More precisely, there holds

$$\lim_{t \rightarrow \infty} \|u(t) - u_M(t)\|_{1,|x|^{-\gamma}} = \lim_{t \rightarrow \infty} \|t^\alpha u(t^\kappa \cdot, t) - u_M(\cdot, 1)\|_{1,|x|^{-\gamma}} = 0, \quad (2.1.11)$$

where

$$\alpha = (d - \gamma)\kappa, \quad \kappa = \frac{1}{d(m - 1) + 2s - m\gamma}.$$

In the proof of (2.1.11) we partially follow the general strategy used in the literature to establish similar convergence results, see [99, 163, 170, 173, 174]. However, in the present context, some technical difficulties arise, due to the simultaneous presence of the weight  $\rho(x)$  and of the nonlocal operator  $(-\Delta)^s$ . An essential tool is the well-posedness of problem (2.1.10), which falls in the framework of the more general problems (2.1.4) discussed above. In particular, uniqueness ensures that the solution to (2.1.10) is self-similar (though not explicit), a crucial fact in order to prove the asymptotic result. We remark, however, that the lack of known Hölder regularity properties for the solutions to the equation at hand, which on the contrary hold true in the non-weighted case in view of the theory developed in [10] (see also [175] for improved regularity), makes the final convergence step more difficult with respect to [174] and prevents us from proving an  $L^\infty$  asymptotic result of the type of (2.1.8).

In the case of *rapidly decaying densities* (Section 2.3), the well-posedness of problem (2.1.1) was first studied in [159]. Here we shall improve on such results, establishing existence of a *minimal* solution within a more general class of solutions, which satisfies a suitable decay condition at infinity. Moreover, under the stronger assumptions  $d > 4s$  and  $\gamma \in (2s, d - 2s) \cap (4s, \infty)$ , we shall prove new uniqueness theorems dealing with integrability properties of solutions. As in the local framework, let us notice that for generic densities  $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^d)$  such that  $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$  one can also prove uniqueness of *weak energy solutions* in the same spirit as Chapter 1, Subsection 1.2.2, *i.e.* regardless of the behaviour of  $\rho$  at infinity. Again, it is just a matter of energy spaces in which one looks for the solutions: the more such spaces include themselves some “condition at infinity” (like  $\dot{H}^s(\mathbb{R}^d)$ , for instance), the less one expects to have to ask extra conditions to restore uniqueness. Also existence of a minimal solution is ensured, provided a local comparison principle holds true for the classes of solutions involved, independently of the behaviour of  $\rho$  at infinity: the latter only plays a role in restricting or enlarging the class of functions where the minimal one is the *unique* solution.

The long time behaviour of the minimal solution to (2.1.1) in the rapidly decaying case is deeply linked with the minimal positive solution  $w$  to the following fractional sublinear elliptic equation:

$$(-\Delta)^s w = \rho w^\ell \quad \text{in } \mathbb{R}^d, \quad (2.1.12)$$

where  $\ell := 1/m \in (0, 1)$ . The local problem ( $s = 1$ ) has been thoroughly studied in [38], while the fractional problem with  $\rho \equiv 1$  has been recently addressed in [36] (on bounded domains). Equation (2.1.12) is partially analysed in [158], where the authors deal with *weak energy solutions* under the hypotheses that (2.1.5) holds true for  $\gamma > d$  (with  $d > 4s$ ) and



$\rho \geq 0$  (with  $\rho \not\equiv 0$ ). Here we prove existence of nontrivial *very weak solutions* provided (2.1.5) is valid for  $\gamma > 2s$  (with  $d > 2s$ ) and  $\rho$  is locally bounded away from zero. To our purposes, a central role will be played by the solution to the linear equation

$$(-\Delta)^s V = \rho \quad \text{in } \mathbb{R}^d.$$

Assuming that (2.1.5) holds true for  $d, \gamma > 4s$ , we shall also prove uniqueness of very weak solutions, which are required to be, in addition, *local weak solutions* and to comply with proper integrability conditions.

Once the analysis of (2.1.12) has been performed, we are able to prove that, if  $u$  is the minimal solution to (2.1.1) and  $w$  is the minimal solution to (2.1.12) for  $\ell = 1/m$ , then

$$\lim_{t \rightarrow \infty} t^{\frac{1}{m-1}} u(x, t) = (m-1)^{-\frac{1}{m-1}} w^{\frac{1}{m}}(x) \quad (2.1.13)$$

*e.g.* in  $L^p_{\text{loc}}(\mathbb{R}^d)$  for all  $p \in [1, \infty)$ . The just mentioned uniqueness results allow us to extend the validity of (2.1.13) to a larger class of solutions. Let us remark that (2.1.13) is in agreement with the content of the very recent paper [32], where the authors study problem (2.1.1) with  $\rho \equiv 1$  and on bounded domains, obtaining (in particular) sharp convergence results with rates.

In [114] it is proved that actually, for the weighted local problem (2.1.6), there is an exponent  $\gamma_2 = \gamma_2(m, d) \in (2, d)$  such that, for  $\gamma \in (2, \gamma_2)$ , a family of Barenblatt solutions continues to exist (though they become unbounded at the origin) and still plays a role for the asymptotics in “outer sets” of the form  $\{|x| \geq Kt^\kappa\}$  (let  $K > 0$ ). In fact, in such sets, Barenblatt solutions provide more precise information with respect to the separable solution. Here we do not address this issue: indeed, unlike the local problem, no explicit Barenblatt profile is available, and the analysis of (2.1.10) with the techniques we used for  $\gamma < 2s$  completely fails when  $\gamma > 2s$ .

Finally, let us mention the critical case  $\gamma = 2s$ . In the local framework it corresponds to  $\gamma = 2$  and has been investigated only recently, mainly in the papers [141] and [111]. Moreover, the analysis of the spectral properties of operators which are modelled on the critical operator formally given by  $|x|^2(-\Delta)$  has been performed in [67]. The main results of [141], for problem (2.1.6) with a regular density  $\rho(x)$  that behaves like  $|x|^{-2}$  at infinity, concern asymptotic estimates which are, to some extent, similar to the ones obtained in the subcritical case  $\gamma < 2$ . They rely on existence of explicit Barenblatt profiles, and the fact that the latter have a logarithmic singularity at the origin complicates the analysis (see also Chapter 1, Paragraph 1.2.3.2). In [111] the authors study problem (2.1.6) with  $\rho(x) = |x|^{-2}$ . They mainly focus on radial solutions and on their different behaviour near the origin, depending on whether the initial datum vanishes or not at the origin. Here an analysis similar to the one developed in [141] is in principle possible. However, the study of problem (2.1.10) when  $\gamma = 2s$  is much more delicate than in the subcritical case (in particular, we expect solutions to be unbounded).

As for applied context in which the fractional porous medium equation is used as a model, we refer the reader to Appendix B of [35], where a brief overview of the topic is given, along with several references.

### 2.1.1 Preliminary tools

In this subsection we recall some basic notions and properties concerning weighted Lebesgue spaces, positive finite Radon measures, fractional Laplacians, fractional Sobolev spaces and Riesz potentials, which will be extremely useful throughout.

### 2.1.1.1 Weighted Lebesgue spaces

For a given measurable function  $\rho : \mathbb{R}^d \mapsto \mathbb{R}^+$  (that is, a weight or density) and  $p \in [1, \infty)$ , we denote as  $L^p_\rho(\mathbb{R}^d)$  the Banach space constituted by all (equivalence classes of) measurable functions  $f : \mathbb{R}^d \mapsto \mathbb{R}$  such that

$$\|f\|_{p,\rho} := \left( \int_{\mathbb{R}^d} |f(x)|^p \rho(x) dx \right)^{1/p} < \infty.$$

Occasionally, we shall also let  $p \in (0, 1)$ : the definition is the same, but of course  $\|\cdot\|_{p,\rho}$  is no longer a norm. In the special case  $\rho(x) = |x|^\alpha$  (let  $\alpha \in \mathbb{R}$ ) we simplify notation, replacing  $L^p_\rho(\mathbb{R}^d)$  with  $L^p_\alpha(\mathbb{R}^d)$  and  $\|f\|_{p,\rho}$  with  $\|f\|_{p,\alpha}$ . If  $\alpha = 0$  we keep the usual symbol  $L^p(\mathbb{R}^d)$ , denoting the corresponding norm as  $\|f\|_p$  or  $\|f\|_{L^p(\mathbb{R}^d)}$ .

At some point we shall also deal with weighted  $L^p$  spaces in domains  $\Omega \subset \mathbb{R}^d$ . In such cases, we shall use the more explicit notations  $L^p_\rho(\Omega)$  and  $\|f\|_{L^p_\rho(\Omega)}$ .

### 2.1.1.2 Positive finite Radon measures on $\mathbb{R}^d$

With a slight abuse of notation, we denote as  $\mathcal{M}(\mathbb{R}^d)$  the cone of positive finite Radon measures on  $\mathbb{R}^d$  (this is actually the usual symbol for the space of *signed* measures). To begin with, consider a sequence  $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$ . Following the same notation as [148], we say that  $\{\mu_n\}$  converges to  $\mu \in \mathcal{M}(\mathbb{R}^d)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  if there holds

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \phi d\mu_n = \int_{\mathbb{R}^d} \phi d\mu \quad \forall \phi \in C_c(\mathbb{R}^d), \quad (2.1.14)$$

where  $C_c(\mathbb{R}^d)$  is the space of continuous, compactly supported functions in  $\mathbb{R}^d$ . This is usually referred to as *local weak\* convergence* (see [4, Definition 1.58]). A classical theorem in measure theory asserts that if

$$\sup_n \mu_n(\mathbb{R}^d) < \infty \quad (2.1.15)$$

then there exists  $\mu \in \mathcal{M}(\mathbb{R}^d)$  such that  $\{\mu_n\}$  converges to  $\mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  up to subsequences [4, Theorem 1.59]. The same holds true if we replace  $C_c(\mathbb{R}^d)$  with  $C_0(\mathbb{R}^d)$ , the latter being the closure of the former w.r.t. the  $L^\infty$  norm. A stronger notion of convergence is the following. A sequence  $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$  is said to converge to  $\mu \in \mathcal{M}(\mathbb{R}^d)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \phi d\mu_n = \int_{\mathbb{R}^d} \phi d\mu \quad \forall \phi \in C_b(\mathbb{R}^d), \quad (2.1.16)$$

where  $C_b(\mathbb{R}^d)$  is the space of continuous, bounded functions in  $\mathbb{R}^d$ . Clearly, (2.1.16) implies (2.1.14). The opposite is true under a further hypothesis: that is, if  $\{\mu_n\}$  converges to  $\mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  and

$$\lim_{n \rightarrow \infty} \mu_n(\mathbb{R}^d) = \mu(\mathbb{R}^d),$$

then  $\{\mu_n\}$  converges to  $\mu$  also in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  [4, Proposition 1.80]. Notice that if  $\{\mu_n\}$  converges to  $\mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  and (2.1.15) holds, *a priori* we only have a weak\* lower semi-continuity property:

$$\mu(\mathbb{R}^d) \leq \liminf_{n \rightarrow \infty} \mu_n(\mathbb{R}^d)$$

(see again [4, Theorem 1.59]).

### 2.1.1.3 Fractional Laplacians and fractional Sobolev spaces

For all  $s \in (0, 1)$  and  $\phi \in \mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$  the fractional Laplacian  $(-\Delta)^s(\phi)$  is defined as in (2.1.2). However, in order to reformulate problem (2.1.1) in a suitable weak sense (see Definition 2.2.1 below), it is necessary to introduce some fractional Sobolev spaces. Here we shall mainly deal with  $\dot{H}^s(\mathbb{R}^d)$ , that is the closure of  $\mathcal{D}(\mathbb{R}^d)$  w.r.t. the norm

$$\|\phi\|_{\dot{H}^s}^2 := \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{d+2s}} dx dy \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

Notice that the space usually denoted as  $H^s(\mathbb{R}^d)$  is just  $L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ . For definitions and properties of the general fractional Sobolev spaces  $W^{r,p}(\mathbb{R}^d)$  we refer the reader to the survey paper [69].

The link between the fractional Laplacian and the space  $\dot{H}^s(\mathbb{R}^d)$  is made clear by the identity

$$\begin{aligned} \langle \phi, \psi \rangle_{\dot{H}^s} &= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x - y|^{d+2s}} dx dy = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(\phi)(x)(-\Delta)^{\frac{s}{2}}(\psi)(x) dx \\ &= \int_{\mathbb{R}^d} \phi(x)(-\Delta)^s(\psi)(x) dx, \end{aligned} \quad (2.1.17)$$

valid for all  $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$  (see [69, Section 3]). In particular,

$$\|\phi\|_{\dot{H}^s}^2 = \|(-\Delta)^{\frac{s}{2}}(\phi)\|_2^2 \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \quad (2.1.18)$$

Notice that (2.1.17) can be shown to hold, by approximation, also when  $\phi \in \mathcal{D}(\mathbb{R}^d)$  is replaced by any  $v \in \dot{H}^s(\mathbb{R}^d)$ , where now  $(-\Delta)^{\frac{s}{2}}(v)$  is meant in the sense of distributions. A further approximation entails

$$\frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2s}} dx dy = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(v)(x)(-\Delta)^{\frac{s}{2}}(w)(x) dx \quad (2.1.19)$$

for all  $v, w \in \dot{H}^s(\mathbb{R}^d)$ . Hence, if we set  $v = w$  in (2.1.19) we deduce that (2.1.18) holds true in  $\dot{H}^s(\mathbb{R}^d)$  as well.

### 2.1.1.4 Riesz potentials

Another mathematical object deeply linked with the fractional Laplacian is its Riesz kernel, namely the function

$$I_{2s}(x) := \frac{k_{d,s}}{|x|^{d-2s}},$$

where  $k_{d,s}$  is again a positive constant depending only on  $d$  and  $s$ . For a given (possibly signed) finite Radon measure  $\nu$ , one can show that the convolution

$$U^\nu := I_{2s} * \nu$$

yields an  $L^1_{\text{loc}}(\mathbb{R}^d)$  function referred to as the *Riesz potential* of  $\nu$ , which formally satisfies

$$(-\Delta)^s(U^\nu) = \nu;$$

that is, still at a formal level, the convolution against  $I_{2s}$  coincides with the operator  $(-\Delta)^{-s}$ . One of the most important and classical references for Riesz potentials is the monograph [121]. Throughout Subsections 2.2.2 and 2.2.3 we shall exploit some crucial properties of Riesz potentials collected in [121], along with their connections with the fractional Laplacian.

### 2.1.2 Outline of the chapter

The chapter is organized as follows.

Section 2.2 deals with the case of slowly decaying densities. In Subsection 2.2.1 we state our main results. In Subsection 2.2.2 we prove existence of weak solutions with measure data. Paragraphs 2.2.2.1 and 2.2.2.2 give hints of the proof of two standard properties of such solutions. Subsection 2.2.3 is devoted to uniqueness: since the argument is long and technical, we split it into Paragraphs 2.2.3.1–2.2.3.3. In Subsection 2.2.4 we finally study the asymptotics of solutions.

Section 2.3 deals with the case of rapidly decaying densities. In Subsection 2.3.1 we provide the statements of the main theorems. In Subsection 2.3.2 we analyse a fractional sublinear elliptic equation deeply connected with the asymptotics: existence and uniqueness results are proved in Paragraphs 2.3.2.1 and 2.3.2.2, respectively. Subsection 2.3.3 is devoted to the asymptotic analysis. Finally, in Subsection 2.3.4 we collect some well-posedness results for various kinds of solutions (Paragraphs 2.3.4.1–2.3.4.3) to the parabolic problem.

In Appendix 2.A we study the properties of a suitable weighted fractional Laplacian operator, which are crucial for uniqueness in the slowly decaying case. Subsection 2.A.1 recalls some basic scaling properties of fractional Laplacians. In Subsection 2.A.2 we prove that the standard mollification of a function converges to the latter not only in the usual  $L^p$  spaces but also in some power-weighted  $L^p$  spaces. This is an important tool for the analysis in Subsection 2.A.3, which is devoted to the proof of the self-adjointness property and integration by parts formulas for the weighted fractional Laplacian operator mentioned above.

## 2.2 Slowly decaying densities

In this section we mainly deal with existence and uniqueness of weak solutions to problem (2.1.4) in the case of a slowly decaying density  $\rho$ , and related consequences for the asymptotics.

### 2.2.1 Statements of the main results

Here we provide a suitable notion of weak solution to (2.1.4), in the spirit of [61] (and [160]).

**Definition 2.2.1.** *Given a positive finite Radon measure  $\mu$ , by a weak solution to problem (2.1.4) we mean a nonnegative function  $u$  such that*

$$u \in L^\infty((0, \infty); L^1_\rho(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty)) \quad \forall \tau > 0, \quad (2.2.1)$$

$$u^m \in L^2_{\text{loc}}((0, \infty); \dot{H}^s(\mathbb{R}^d)), \quad (2.2.2)$$

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt = 0 \quad (2.2.3)$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$$

and

$$\lim_{t \rightarrow 0} \rho u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d)). \quad (2.2.4)$$

Our first main result concerns existence of such weak solutions.

**Theorem 2.2.2.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $\mu$  be a positive finite Radon measure. Then there exists a weak solution  $u$  to (2.1.4), according to Definition 2.2.1, which conserves the mass in the sense that  $\mu(\mathbb{R}^d) = \int_{\mathbb{R}^d} u(x, t) \rho(x) dx$  for all  $t > 0$  and satisfies the smoothing estimate*

$$\|u(t)\|_{\infty} \leq K t^{-\alpha} \mu(\mathbb{R}^d)^{\beta} \quad \forall t > 0, \quad (2.2.5)$$

where  $K$  is a suitable positive constant depending only on  $m, C, \gamma, d, s$  and

$$\alpha := \frac{d - \gamma}{(m - 1)(d - \gamma) + 2s - \gamma}, \quad \beta := \frac{2s - \gamma}{(m - 1)(d - \gamma) + 2s - \gamma}. \quad (2.2.6)$$

For given solutions to the differential equation in (2.1.4) (namely without a prescribed initial datum), the following result on existence and uniqueness of their initial trace, in the spirit of [35, Section 7] and [15], will also be proved.

**Theorem 2.2.3.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d - 2s]$  and  $\gamma_0 \in [0, \gamma]$ . Consider a weak solution  $u$  to  $\rho u_t + (-\Delta)^s(u^m) = 0$  in the sense that  $u$  complies with (2.2.1), (2.2.2) and (2.2.3). Then there exists a unique positive finite Radon measure  $\mu$  which is the initial trace of  $u$  in the sense of (2.2.4). In particular,  $\mu(\mathbb{R}^d) = \int_{\mathbb{R}^d} u(x, t) \rho(x) dx$  for all  $t > 0$ . The same result holds true if the condition  $u \in L^{\infty}(\mathbb{R}^d \times (\tau, \infty))$  in (2.2.1) is replaced by the weaker condition  $\int_{t_1}^{t_2} u^m(\tau) d\tau \in L^1_{\rho}(\mathbb{R}^d)$  for all  $t_2 > t_1 > 0$ .*

As for uniqueness of weak solutions, we shall prove the next result.

**Theorem 2.2.4.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $u_1$  and  $u_2$  be two weak solutions to (2.1.4), according to Definition 2.2.1. Suppose that their initial datum, in the sense of (2.2.4), is the same positive finite Radon measure  $\mu$ . Then  $u_1 = u_2$ .*

We point out that, in order to prove Theorem 2.2.4, we shall crucially exploit the properties of the operator  $A := \rho^{-1}(-\Delta)^s$  provided by Theorems 2.A.12 and 2.A.13 in Appendix 2.A below.

In the analysis of the long time behaviour of solutions, we shall assume in addition that  $\rho$  satisfies (2.1.9). In agreement with the local case ( $s = 1$ ), a major role is played by the solution to the same problem in the particular case  $\rho(x) = c_{\infty}|x|^{-\gamma}$  and  $\mu = M\delta$  (namely the solution to (2.1.10)), where  $M$  is the mass of the initial datum. Our main result reads as follows.

**Theorem 2.2.5.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Suppose in addition that*

$$\lim_{|x| \rightarrow \infty} \rho(x)|x|^{\gamma} =: c_{\infty} > 0. \quad (2.2.7)$$

Let  $u$  be the unique weak solution to problem (2.1.4), according to Definition 2.2.1, corresponding to  $\mu \in \mathcal{M}(\mathbb{R}^d)$ , with  $\mu(\mathbb{R}^d) =: M > 0$ . Then,

$$\lim_{t \rightarrow \infty} \|u(t) - u_M(t)\|_{1, -\gamma} = 0, \quad (2.2.8)$$

or equivalently

$$\lim_{t \rightarrow \infty} \|t^{\alpha} u(t^{\kappa} \cdot, t) - u_M(\cdot, 1)\|_{1, -\gamma}, \quad (2.2.9)$$

where  $u_M$  is the Barenblatt solution defined as the unique solution to problem (2.1.10),

$$\kappa := \frac{1}{(m-1)(d-\gamma) + 2s - \gamma} \quad (2.2.10)$$

and  $\alpha$  is as in (2.2.6).

**Remark 2.2.6.** Notice that, if  $d \geq 4s$ , then the above assumptions on  $\gamma$  amount to  $\gamma \in [0, 2s)$ .

Theorems 2.2.2 and 2.2.3 will be proved in Subsection 2.2.2, Theorem 2.2.4 will be proved in Subsection 2.2.3 and Theorem 2.2.5 will be proved in Subsection 2.2.4.

### 2.2.2 Existence of weak solutions

Before proceeding with the proof of Theorem 2.2.2 (and related lemmas), we shall show a first, direct consequence of Definition 2.2.1, namely the conservation in time of the mass  $\int_{\mathbb{R}^d} u(x, t) \rho(x) dx$ , that is the  $L^1_\rho(\mathbb{R}^d)$  norm of  $u(t)$  since we are considering nonnegative solutions.

**Proposition 2.2.7.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s)$  and  $\gamma_0 \in [0, \gamma]$ . Let  $u$  be a weak solution to (2.1.4) according to Definition 2.2.1. Then*

$$\|u(t)\|_{1,\rho} = \int_{\mathbb{R}^d} u(x, t) \rho(x) dx = \mu(\mathbb{R}^d) \quad \text{for a.e. } t > 0, \quad (2.2.11)$$

namely we have conservation of mass.

*Proof.* In order to prove (2.2.11) we first plug in (2.2.3) the following test function:

$$\varphi_R(x, t) := \vartheta(t) \xi_R(x),$$

where  $\xi_R$  is a standard family of cut-off functions (see Lemma 2.A.3 below) and  $\vartheta$  is a suitable family of positive, regular and compactly supported approximations of  $\chi_{[t_1, t_2]}$  (let  $t_2 > t_1 > 0$ ). In view of (2.1.17), there holds

$$\int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi_R)(x, t) dx dt = \int_0^\infty \int_{\mathbb{R}^d} u^m(x, t) (-\Delta)^s(\varphi_R)(x, t) dx dt.$$

Recalling Lemma 2.A.4 and the assumptions on  $\rho$ , we have:

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}^d} u^m(x, t) (-\Delta)^s(\varphi_R)(x, t) dx dt \right| \\ & \leq c^{-1} \|\vartheta\|_\infty \left( \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} + \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \right) \left| \int_{t_*}^{t^*} \int_{\mathbb{R}^d} u^m(x, t) \rho(x) dx dt \right|, \end{aligned} \quad (2.2.12)$$

where  $t^* > t_* > 0$  are chosen so that  $\text{supp } \vartheta \subset [t_*, t^*]$  independently of  $\theta$ . The integral in the r.h.s. of (2.2.12) is finite thanks to (2.2.1). Hence, letting  $R \rightarrow \infty$  in (2.2.3), we deduce that

$$\int_0^\infty \int_{\mathbb{R}^d} u(x, t) \vartheta'(t) \rho(x) dx dt = 0. \quad (2.2.13)$$

If we then let  $\vartheta \rightarrow \chi_{[t_1, t_2]}$  in (2.2.13) and use Lebesgue's differentiation Theorem we find that  $\|u(t_2)\|_{1,\rho} = \|u(t_1)\|_{1,\rho}$  for a.e.  $t_2 > t_1 > 0$ . This property and (2.2.4) finally yield (2.2.11).  $\square$

The proof of existence of weak solutions to (2.1.4) is based on an approximation procedure. That is, one approximates the measure  $\mu$  with data  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and shows that the corresponding solutions suitably converge to a solution of (2.1.4). This calls, first of all, for an existence result for the following problem:

$$\begin{cases} \rho(x)u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \quad (2.2.14)$$

To this aim, we need a preliminary lemma.

**Lemma 2.2.8.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, d + 2s]$  and  $\gamma_0 \geq 0$ . Consider a function  $v \in L^2_{\text{loc}}((0, \infty); \dot{H}^s(\mathbb{R}^d))$  such that, for all  $t_2 > t_1 > 0$ ,*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v(x, t)|^2 \rho(x) dx dt \leq K, \quad (2.2.15)$$

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(v)(x, t)|^2 dx dt \leq K \quad (2.2.16)$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v_t(x, t)|^2 \rho(x) dx dt \leq K, \quad (2.2.17)$$

where  $K$  is a positive constant depending only on  $t_1$  and  $t_2$ . Take any cut-off functions  $\xi_1 \in C_c^\infty(\mathbb{R}^d)$ ,  $\xi_2 \in C_c^\infty((0, \infty))$  and define  $v_c : \mathbb{R}^d \mapsto \mathbb{R}$  as follows:

$$v_c(x, t) := \xi_1(x)\xi_2(t)v(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R},$$

where we implicitly assume  $\xi_2$  and  $v$  to be zero for  $t < 0$ . Then

$$\|v_c\|_{\dot{H}^s(\mathbb{R}^{d+1})}^2 = \|v_c\|_{L^2(\mathbb{R}^{d+1})}^2 + \|v_c\|_{\dot{H}^s(\mathbb{R}^{d+1})}^2 \leq C' \quad (2.2.18)$$

for a positive constant  $C'$  that depends only on  $\xi_1$ ,  $\xi_2$ ,  $\gamma$ ,  $c$ ,  $K$ ,  $d$  and  $s$ .

*Proof.* The validity of

$$\|v_c\|_{L^2(\mathbb{R}^{d+1})}^2 \leq C' \quad (2.2.19)$$

is an immediate consequence of (2.2.15) and of the fact that  $\rho$  is bounded away from zero on compact sets (from now on  $C'$  will be a constant as in the statement that we shall not relabel). Moreover, since

$$(v_c)_t = \xi_1 \xi_2' v + \xi_1 \xi_2 v_t,$$

by (2.2.15), (2.2.17) and again the fact that  $\rho$  is bounded away from zero on compact sets we deduce that

$$\|(v_c)_t\|_{L^2(\mathbb{R}^{d+1})}^2 \leq C'. \quad (2.2.20)$$

Now we have to deal with the spatial regularity of  $v_c$ . Straightforward computations show that

$$\begin{aligned} \|v_c(t)\|_{\dot{H}^s(\mathbb{R}^d)}^2 &= \frac{C_{d,s}}{2} \xi_2^2(t) \int_{\mathbb{R}^d} \xi_1^2(x) \left( \int_{\mathbb{R}^d} \frac{(v(x, t) - v(y, t))^2}{|x - y|^{d+2s}} dy \right) dx \\ &\quad + \frac{C_{d,s}}{2} \xi_2^2(t) \int_{\mathbb{R}^d} |v(y, t)|^2 \left( \int_{\mathbb{R}^d} \frac{(\xi_1(x) - \xi_1(y))^2}{|x - y|^{d+2s}} dx \right) dy \\ &\quad + C_{d,s} \xi_2^2(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_1(x)v(y, t) \frac{(v(x, t) - v(y, t))(\xi_1(x) - \xi_1(y))}{|x - y|^{d+2s}} dx dy. \end{aligned} \quad (2.2.21)$$

The Cauchy-Schwarz inequality allows us to bound the third integral in the r.h.s. of (2.2.21) with the first two integrals. As concerns the first one, we have:

$$\frac{C_{d,s}}{2} \xi_2^2(t) \int_{\mathbb{R}^d} \xi_1^2(x) \left( \int_{\mathbb{R}^d} \frac{(v(x,t) - v(y,t))^2}{|x-y|^{d+2s}} dy \right) dx \leq \chi_{\text{supp } \xi_2}(t) \|\xi_2\|_\infty^2 \|\xi_1\|_\infty^2 \|v(t)\|_{H^s(\mathbb{R}^d)}^2. \quad (2.2.22)$$

In order to bound the second integral, it is important to recall that the function  $l_s(\xi_1)(y)$  is regular and decays at least like  $|y|^{-d-2s}$  as  $|y| \rightarrow \infty$  (for the definition and properties of  $l_s$  see Lemmas 2.A.2–2.A.4 in Appendix 2.A). Hence, thanks to the assumptions on  $\rho$  and  $\gamma$ , we infer that

$$\xi_2^2(t) \int_{\mathbb{R}^d} |v(y,t)|^2 \left( \int_{\mathbb{R}^d} \frac{(\xi_1(x) - \xi_1(y))^2}{|x-y|^{d+2s}} dx \right) dy \leq C' \chi_{\text{supp } \xi_2}(t) \|\xi_2\|_\infty^2 \int_{\mathbb{R}^d} |v(y,t)|^2 \rho(y) dy. \quad (2.2.23)$$

Integrating in time (2.2.21), using (2.2.22), (2.2.23), (2.2.15), (2.2.16) and identity (2.1.18) evaluated at  $\phi = v_c$ , we then get

$$\|(-\Delta)^{\frac{s}{2}}(v_c)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq C'. \quad (2.2.24)$$

By combining (2.2.19), (2.2.20) and (2.2.24) one can deduce (2.2.18) *e.g.* using Fourier transform methods. In fact, upon denoting as  $\mathcal{F}(f)(x',t')$  the Fourier transform in  $\mathbb{R}^{d+1}$  of a function  $f(x,t)$ , estimate (2.2.20) entails

$$\int_{\mathbb{R}^{d+1}} |t'|^2 |\mathcal{F}(v_c)(x',t')|^2 dx' dt' = \int_{\mathbb{R}^{d+1}} (v_c)_t^2(x,t) dx dt \leq C', \quad (2.2.25)$$

whereas (2.2.24) gives

$$\int_{\mathbb{R}^{d+1}} |x'|^{2s} |\mathcal{F}(v_c)(x',t')|^2 dx' dt' = \int_{\mathbb{R}^{d+1}} |(-\Delta)^{\frac{s}{2}}(v_c)(x,t)|^2 dx dt \leq C'. \quad (2.2.26)$$

Thus, in view of (2.2.19), (2.2.25) and (2.2.26) we finally get the estimate

$$\int_{\mathbb{R}^{d+1}} (1 + |x'|^2 + |t'|^2)^s |\mathcal{F}(v_c)(x',t')|^2 dx' dt' \leq C',$$

which is equivalent to (2.2.18) (see *e.g.* [69, Section 3]).  $\square$

We are now able to prove existence of weak solutions to (2.2.14). Such solutions are understood in the sense of Definition 2.2.1, just upon replacing  $\mu$  with  $\rho u_0$ . The idea is to regularize the density  $\rho(x)$  in a neighbourhood of  $x = 0$  (where it can be singular) and use the results of [159].

**Lemma 2.2.9.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s)$  and  $\gamma_0 \in [0, \gamma]$ . Let  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , with  $u_0 \geq 0$ . Then there exists a weak solution  $u$  to (2.2.14) which satisfies the following energy estimates:*

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(u^m)(x,t)|^2 dx dt + \frac{1}{m+1} \int_{\mathbb{R}^d} u^{m+1}(x,t_2) \rho(x) dx \\ &= \frac{1}{m+1} \int_{\mathbb{R}^d} u^{m+1}(x,t_1) \rho(x) dx \end{aligned} \quad (2.2.27)$$

for all  $t_2 > t_1 \geq 0$  and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |z_t(x,t)|^2 \rho(x) dx dt \leq K \quad \forall t_2 > t_1 > 0, \quad (2.2.28)$$

where  $z := u^{(m+1)/2}$  and  $K$  depends only on  $m, t_1, t_2$  and  $u_0$  *e.g.* through  $\|u(t_1/2)\|_{m+1,\rho}$ .



*Proof.* We introduce the following approximation of problem (2.2.14):

$$\begin{cases} \rho_\eta(x)(u_\eta)_t + (-\Delta)^s(u_\eta^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u_\eta = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \quad (2.2.29)$$

where  $\{\rho_\eta\} \subset C(\mathbb{R}^d)$  is a family of strictly positive densities which behave like  $|x|^{-\gamma}$  at infinity and approximate  $\rho(x)$  monotonically from below. In particular,  $u_0 \in L^1_{\rho_\eta}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . Existence (and uniqueness) of weak solutions to (2.2.29) for such densities and initial data have been established in [159, Theorem 3.1]. Actually the solutions constructed there also belong to  $C([0, \infty); L^1_{\rho_\eta}(\mathbb{R}^d))$  and satisfy the bound

$$\|u_\eta\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

Moreover, one can show that each  $u_\eta$  satisfies a weak formulation which is slightly stronger than the one of Definition 2.2.1:

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^d} u_\eta(x, t) \varphi_t(x, t) \rho_\eta(x) dx dt + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u_\eta^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt \\ & = \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) \rho_\eta(x) dx \end{aligned} \quad (2.2.30)$$

for all  $T > 0$  and  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T])$  such that  $\varphi(T) \equiv 0$ , where  $u_\eta^m \in L^2((0, \infty); \dot{H}^s(\mathbb{R}^d))$ . The latter property follows from the validity of the key energy identity

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(u_\eta^m)(x, t)|^2 dx dt + \frac{1}{m+1} \int_{\mathbb{R}^d} u_\eta^{m+1}(x, t_2) \rho_\eta(x) dx \\ & = \frac{1}{m+1} \int_{\mathbb{R}^d} u_\eta^{m+1}(x, t_1) \rho_\eta(x) dx \end{aligned} \quad (2.2.31)$$

for all  $t_2 > t_1 \geq 0$ . Formally, (2.2.31) can be proved by plugging the test function  $\varphi(x, t) = \vartheta(t)u_\eta^m(x, t)$  in the weak formulation (2.2.30) and letting  $\vartheta$  tend to  $\chi_{[t_1, t_2]}$  as in the proof of Proposition 2.2.7. The problem is that, *a priori*, such a  $\varphi$  is not admissible as a test function. In order to justify (2.2.31) rigorously one must then proceed as in [61, Section 8]. A crucial point concerns the fact that our solutions are *strong*, which follows by means of techniques analogous to the ones used in [61, Subsection 8.1]. We refer the reader to the Paragraphs 2.2.2.1 and 2.2.2.2 below for more details. Another fundamental estimate is

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(z_\eta)_t(x, t)|^2 \rho_\eta(x) dx dt \leq K \quad \forall t_2 > t_1 > 0, \quad (2.2.32)$$

where  $z_\eta := u_\eta^{(m+1)/2}$  and  $K$  is a suitable positive constant that depends only on  $t_1, t_2$  and  $u_0$  through  $\|u_\eta(t_1/2)\|_{m+1, \rho} \leq \|u_0\|_{m+1, \rho}$ . The proof of (2.2.32) follows exactly as in [61, Lemma 8.1]. Since

$$\left(u_\eta^m\right)_t = \frac{2m}{m+1} z_\eta^{\frac{m-1}{m+1}} (z_\eta)_t \quad \text{and} \quad \|z_\eta\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} = \|u_\eta\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)}^{\frac{m+1}{2}} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}^{\frac{m+1}{2}},$$

from (2.2.32) we deduce that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \left(u_\eta^m\right)_t(x, t) \right|^2 \rho_\eta(x) dx dt \leq \left(\frac{2m}{m+1}\right)^2 \|u_0\|_\infty^{m-1} K \quad \forall t_2 > t_1 > 0.$$

Furthermore, the validity of

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |u_\eta^m(x, t)|^2 \rho_\eta(x) dx dt \leq C' \quad \forall t_2 > t_1 \geq 0$$

for another suitable positive constant  $C'$  that depends only on  $t_1$ ,  $t_2$  and  $u_0$  is ensured by the conservation of mass (2.2.11) (with  $\rho = \rho_\eta$ ) and by the bound over  $\|u_\eta\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)}$ . Applying Lemma 2.2.8 to  $v = u_\eta^m$  and using the fact that  $H^s(\mathbb{R}^{d+1})$  is compactly embedded in  $L^2_{\text{loc}}(\mathbb{R}^{d+1})$  [69, Theorem 7.1], one can pass to the limit in (2.2.30) as  $\eta \rightarrow \infty$  and get that the weak limit  $u^m$  of  $\{u_\eta^m\}$  in  $L^2((0, T); \dot{H}^s(\mathbb{R}^d))$  (for any  $T > 0$ ) satisfies

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt \\ & = \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) \rho(x) dx \end{aligned} \quad (2.2.33)$$

for all  $T > 0$  and  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T])$  such that  $\varphi(T) \equiv 0$ . The validity of (2.2.4) follows by plugging in (2.2.33) the test function  $\varphi(x, t) = \vartheta(t) \xi_R(x)$ , where  $\xi_R$  is a cut-off function as in Lemma 2.A.3 and  $\vartheta$  is a regular approximation of  $\chi_{[0, t_2]}$ . One then lets  $t_2 \rightarrow 0$  and  $R \rightarrow \infty$ .

Finally, the energy estimates (2.2.27) and (2.2.28) can be obtained reasoning exactly as we did in the proof of (2.2.31) and (2.2.32) (we exploit again the fact that solutions are strong).  $\square$

Having at our disposal an existence result for problem (2.2.14), we can now let  $\rho u_0$  approximate  $\mu$ . In order to show that the corresponding solutions converge to a solution of (2.1.4), we first need some technical lemmas. We begin with a modification of the classical Stroock-Varopoulos inequality: it is proved here for a precise set of functions that we shall deal with later on. A simple proof of such inequality (with different assumptions on the functions involved), which exploits the extension in the upper plane, can be found in [61, Section 5]. See also [59, formula (2.2.7)] for a similar result involving general Dirichlet forms.

**Proposition 2.2.10.** *Let  $d > 2s$ . For all nonnegative  $v \in L^\infty(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$  such that  $(-\Delta)^s(v) \in L^1(\mathbb{R}^d)$  the function  $v^{q/2}$  belongs to  $\dot{H}^s(\mathbb{R}^d)$  and the inequality*

$$\int_{\mathbb{R}^d} v^{q-1}(x) (-\Delta)^s(v)(x) dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(v^{\frac{q}{2}})(x)|^2 dx \quad (2.2.34)$$

holds true for any  $q > 1$ .

*Proof.* We shall assume, with no loss of generality, that  $v$  is a regular function. Indeed, by standard mollification arguments, one can always pick a sequence  $\{v_n\} \subset C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$  such that  $\{v_n\}$  converges pointwise to  $v$ ,  $\|v_n\|_\infty \leq \|v\|_\infty$  and  $\{(-\Delta)^s(v_n)\}$  converges to  $(-\Delta)^s(v)$  in  $L^1(\mathbb{R}^d)$ . This is enough to pass to the limit as  $n \rightarrow \infty$  in the l.h.s. of (2.2.34), while in the r.h.s. one exploits the weak lower semi-continuity of the  $L^2$  norm.

Consider the following sequences of functions:

$$\begin{aligned} \psi_n(x) & := \int_0^{x \wedge \frac{1}{n}} y^{\frac{4s}{d-2s}} dy + (q-1) \int_{\frac{1}{n}}^{x \vee \frac{1}{n}} y^{q-2} dy \quad \forall x \in \mathbb{R}^+, \\ \Psi_n(x) & := \int_0^{x \wedge \frac{1}{n}} y^{\frac{2s}{d-2s}} dy + (q-1)^{\frac{1}{2}} \int_{\frac{1}{n}}^{x \vee \frac{1}{n}} y^{\frac{q}{2}-1} dy \quad \forall x \in \mathbb{R}^+. \end{aligned}$$

It is plain that  $\psi_n$  and  $\Psi_n$  are absolutely continuous, monotone increasing functions such that  $\psi'_n(x) = [\Psi'_n(x)]^2$  for all  $x \in \mathbb{R}^+$ . For any  $R > 0$ , take a cut-off function  $\xi_R$  as in Lemma 2.A.3. To the function  $\xi_R v$  one can apply Lemma 5.2 of [61] with the choices  $\psi = \psi_n$  and  $\Psi = \Psi_n$ , which yields

$$\int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) (-\Delta)^s(\xi_R v)(x) dx \geq \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(\Psi_n(\xi_R v))(x)|^2 dx. \quad (2.2.35)$$

Computing the fractional Laplacian of the product of two functions, we get that the l.h.s. of (2.2.35) equals

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) \xi_R(x) (-\Delta)^s(v)(x) dx + \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) (-\Delta)^s(\xi_R)(x) v(x) dx \\ & + 2 C_{d,s} \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))(v(x) - v(y))}{|x - y|^{d+2s}} dy dx. \end{aligned} \quad (2.2.36)$$

By dominated convergence,

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) \xi_R(x) (-\Delta)^s(v)(x) dx = \int_{\mathbb{R}^d} \psi_n(v)(x) (-\Delta)^s(v)(x) dx.$$

Our aim is to show that the other two integrals in (2.2.36) vanish as  $R \rightarrow \infty$ . We have:

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) (-\Delta)^s(\xi_R)(x) v(x) dx \right| \\ & \leq \|(-\Delta)^s(\xi_R)\|_\infty \left( \frac{d-2s}{d+2s} \int_{\{v \leq 1/n\}} v^{\frac{2d}{d-2s}}(x) dx + \psi_n(\|v\|_\infty) \|v\|_\infty \int_{\{v > 1/n\}} dx \right) \end{aligned} \quad (2.2.37)$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))(v(x) - v(y))}{|x - y|^{d+2s}} dy dx \right| \\ & \leq \|v\|_{\dot{H}^s} \left( \int_{\mathbb{R}^d} [\psi_n(\xi_R v)(x)]^2 \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dy dx \right)^{\frac{1}{2}} \\ & \leq \|v\|_{\dot{H}^s} \|l_s(\xi_R)\|_\infty^{\frac{1}{2}} \left( \left[ \frac{d-2s}{d+2s} \right]^2 \int_{\{v \leq 1/n\}} v^{\frac{2d+2s}{d-2s}}(x) dx + [\psi_n(\|v\|_\infty)]^2 \int_{\{v > 1/n\}} dx \right)^{\frac{1}{2}}, \end{aligned} \quad (2.2.38)$$

where  $l_s$  is defined in Lemma 2.A.2. In view of Lemma 2.A.4,  $\lim_{R \rightarrow \infty} \|(-\Delta)^s(\xi_R)\|_\infty = \lim_{R \rightarrow \infty} \|l_s(\xi_R)\|_\infty = 0$ . Moreover,

$$v \in L^{\frac{2d}{d-2s}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

(see [69, Section 6] or Lemma 2.2.11 below). In particular,  $v$  also belongs to

$$L^{\frac{2d+2s}{d-2s}}(\mathbb{R}^d).$$

Letting  $R \rightarrow \infty$  in (2.2.37) and (2.2.38), we therefore deduce that the last two integrals in (2.2.36) actually go to zero, so that we can pass to the limit in the l.h.s. of (2.2.35). In the r.h.s. we just use the fact that  $\{(-\Delta)^{\frac{s}{2}}(\Psi_n(\xi_R v))\}$  converges to  $(-\Delta)^{\frac{s}{2}}(\Psi_n(v))$  weakly in  $L^2(\mathbb{R}^d)$ , i.e.  $\{\Psi_n(\xi_R v)\}$  converges to  $\Psi_n(v)$  weakly in  $\dot{H}^s(\mathbb{R}^d)$ . This proves the validity of

$$\int_{\mathbb{R}^d} \psi_n(v)(x) (-\Delta)^s(v)(x) dx \geq \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}}(\Psi_n(v))(x) \right|^2 dx. \quad (2.2.39)$$

The final step is to let  $n \rightarrow \infty$  in (2.2.39). It is clear that the sequence  $\{\psi_n(x)\}$  converges locally uniformly to the function  $x^{q-1}$ , while  $\{\Psi_n(x)\}$  converges locally uniformly to the function  $2(q-1)^{\frac{1}{2}} x^{\frac{q}{2}}/q$ . Hence,  $\{\psi_n(v)\}$  and  $\{\Psi_n(v)\}$  converge in  $L^\infty(\mathbb{R}^d)$  to  $v^{q-1}$  and  $2(q-1)^{\frac{1}{2}} v^{\frac{q}{2}}/q$ , respectively, so that we can pass to the limit in (2.2.39) and obtain (2.2.34).  $\square$

**Lemma 2.2.11.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s)$  and  $\gamma_0 \in [0, \gamma]$ . There exists a positive constant  $C_{CKN} = C_{CKN}(C, \gamma, d, s)$  such that the family of fractional Caffarelli-Kohn-Nirenberg inequalities*

$$\|v\|_{q,\rho} \leq C_{CKN} \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2^{\frac{1}{a+1}} \|v\|_{p,\rho}^{\frac{a}{a+1}} \quad \forall v \in L_\rho^p(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d) \quad (2.2.40)$$

holds true for any  $a \geq 0$ ,  $p > 0$  and

$$q = \frac{2(d-\gamma)(a+1)}{2(d-\gamma)\frac{a}{p} + d - 2s}.$$

For  $a = 0$  we recover the fractional Sobolev inequality

$$\|v\|_{2\frac{d-\gamma}{d-2s},\rho} \leq C_{CKN} \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d). \quad (2.2.41)$$

*Proof.* The fractional Hardy-Sobolev inequality

$$\|v\|_{2\frac{d-\gamma}{d-2s},-\gamma} \leq C_\gamma \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d) \quad (2.2.42)$$

can be proved by interpolation between the fractional Sobolev inequality

$$\|v\|_{\frac{2d}{d-2s}} \leq C_S \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d)$$

(see *e.g.* [69, Theorem 6.5]) and the fractional Hardy inequality

$$\|v\|_{2,-2s} \leq C_H \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d)$$

(see *e.g.* [96] and references quoted therein). Hence, (2.2.41) follows from (2.2.42) in view of the hypotheses on  $\rho$ : here it is crucial that  $\gamma \in [0, 2s)$  and that  $\gamma_0 \leq \gamma$ . Once we have (2.2.41), another interpolation yields (2.2.40).

We point out that, for  $\rho(x) = |x|^{-\gamma}$ , inequality (2.2.40) is just a particular case of [57, Theorem 1.8].  $\square$

Proposition 2.2.10 and Lemma 2.2.11 provide us with some functional inequalities which are key to prove the following smoothing effect for solutions to (2.2.14).

**Proposition 2.2.12.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s)$  and  $\gamma_0 \in [0, \gamma]$ . There exists a constant  $K > 0$  depending only on  $m$ ,  $C$ ,  $\gamma$ ,  $d$  and  $s$  such that, for all nonnegative initial data  $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and the corresponding weak solution  $u$  to (2.2.14) constructed in Lemma 2.2.9, the following  $L_\rho^{p_0}$ - $L^\infty$  smoothing effect holds true for any  $p_0 \in [1, \infty)$ :*

$$\|u(t)\|_\infty \leq K t^{-\alpha_0} \|u_0\|_{p_0,\rho}^{\beta_0} \quad \forall t > 0, \quad (2.2.43)$$

where

$$\alpha_0 := \frac{d-\gamma}{(m-1)(d-\gamma) + (2s-\gamma)p_0}, \quad \beta_0 := \frac{(2s-\gamma)p_0}{(m-1)(d-\gamma) + (2s-\gamma)p_0}. \quad (2.2.44)$$

*Proof.* We proceed exactly as in the proof of [61, Theorem 8.2], *i.e.* through a standard parabolic Moser iteration, so we just stress the main steps for the convenience of the reader. In fact, the method of proof is the same as Theorem 1.2.11 in Chapter 1, Paragraph 1.2.3.4: here the additional difficulty due to the presence of the fractional Laplacian is overcome by means of the above Stroock-Varopoulos inequality.

First of all, let us fix  $t > 0$  and consider the time sequence  $t_k := (1 - 2^{-k})t$ . Denote as  $\{p_k\} \subset (1, \infty)$  another increasing numerical sequence to be chosen later. Multiplying the differential equation in (2.2.14) by  $u^{p_k-1}(x, t)$ , integrating over  $\mathbb{R}^d \times [t_k, t_{k+1}]$ , applying Proposition 2.2.10 to the function  $v = u^m$  (with the choice  $q = (p_k + m - 1)/m$ ) and exploiting the fact that the  $L_\rho^p$  norms do not increase along the evolution (see Paragraph 2.2.2.2 below), we get:

$$\|u(t_k)\|_{p_k, \rho}^{p_k} \geq \frac{c_k}{\|u(t_k)\|_{p_k, \rho}^{p_k}} \int_{t_k}^{t_{k+1}} \left\| (-\Delta)^{\frac{s}{2}} \left( u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k, \rho}^{p_k} d\tau, \quad (2.2.45)$$

where  $c_k := 4mp_k(p_k - 1)/(p_k + m - 1)^2$ . The above computations are justified rigorously since, as we recall in Paragraph 2.2.2.1, our solutions are strong. In particular, both sides of the differential equation in (2.2.14) belong to  $L^1(\mathbb{R}^d)$ . Exploiting (2.2.40) with the choices  $p = 2p_k/(p_k + m - 1)$  and  $a = p_k/(p_k + m - 1)$ , we obtain:

$$\left\| (-\Delta)^{\frac{s}{2}} \left( u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k, \rho}^{p_k} \geq C_{CKN}^{-2\frac{2p_k+m-1}{p_k+m-1}} \|u(\tau)\|_{\frac{2p_k+m-1}{(d-\gamma)(2p_k+m-1)}, \rho}^{\frac{2p_k+m-1}{2d-\gamma-2s}}. \quad (2.2.46)$$

Thanks to (2.2.46) and again to the fact that the  $L_\rho^p$  norms of  $u(\tau)$  do not grow in time, we deduce:

$$\int_{t_k}^{t_{k+1}} \left\| (-\Delta)^{\frac{s}{2}} \left( u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k, \rho}^{p_k} d\tau \geq C_{CKN}^{-2\frac{2p_k+m-1}{p_k+m-1}} 2^{-(k+1)t} \|u(t_{k+1})\|_{\frac{2p_k+m-1}{(d-\gamma)(2p_k+m-1)}, \rho}^{\frac{2p_k+m-1}{2d-\gamma-2s}}. \quad (2.2.47)$$

Gathering (2.2.45) and (2.2.47) we find the recursive inequality

$$\|u(t_{k+1})\|_{p_{k+1}, \rho} \leq \left( \frac{2^{k+1} C_{CKN}^{2\frac{2p_k+m-1}{p_k+m-1}}}{c_k t} \right)^{\frac{\sigma}{2p_{k+1}}} \|u(t_k)\|_{p_k, \rho}^{\frac{\sigma p_k}{p_{k+1}}},$$

where

$$p_{k+1} = \frac{\sigma}{2} (2p_k + m - 1), \quad \sigma := \frac{2(d - \gamma)}{2d - \gamma - 2s}.$$

Note that, since  $\gamma \in [0, 2s)$ ,  $\sigma > 1$ . Furthermore, upon taking  $p_0 > 1$ , it is easy to check that

$$p_k = A(\sigma^k - 1) + p_0, \quad A := p_0 + \frac{(d - \gamma)(m - 1)}{2s - \gamma} > 0,$$

whence  $p_{k+1} > p_k$  and  $\lim_{k \rightarrow \infty} p_k = \infty$ . Thus, letting  $U_k := \|u(t_k)\|_{p_k, \rho}$ , we can find a constant  $c_0 = c_0(p_0, m, C, \gamma, d, s) > 0$  (in particular, independent of  $k$ ) such that

$$U_{k+1} \leq c_0^{\frac{k}{p_{k+1}}} t^{-\frac{\sigma}{2p_{k+1}}} U_k^{\frac{\sigma p_k}{p_{k+1}}}; \quad (2.2.48)$$

applying (2.2.48) iteratively we get

$$U_k \leq c_0^{\frac{1}{p_k} \sum_{j=1}^{k-1} (k-j)\sigma^j} t^{-\frac{1}{2p_k} \sum_{j=1}^k \sigma^j} U_0^{\frac{\sigma^k p_0}{p_k}}. \quad (2.2.49)$$

Passing to the limit in (2.2.49) as  $k \rightarrow \infty$  we end up with the estimate

$$\|u(t)\|_\infty \leq C' t^{-\frac{d-\gamma}{(m-1)(d-\gamma)+(2s-\gamma)p_0}} \|u_0\|_{\frac{(2s-\gamma)p_0}{(m-1)(d-\gamma)+(2s-\gamma)p_0}, \rho}^{\frac{(2s-\gamma)p_0}{(m-1)(d-\gamma)+(2s-\gamma)p_0}} \quad (2.2.50)$$

for some positive constant  $C' = C'(p_0, m, C, \gamma, d, s) > 0$ , namely (2.2.43). However, we recall that (2.2.50) has been proved under the assumption  $p_0 > 1$ . In order to establish its validity down to  $p_0 = 1$  one can proceed as in the proof of [61, Corollary 8.1] (see also the proof of Theorem 1.2.9 in Chapter 1, Paragraph 1.2.3.4).  $\square$

Before proving Theorem 2.2.2, we still need a preliminary result concerning Riesz potentials.

**Lemma 2.2.13.** *Let  $d > 2s$  and  $\phi : \mathbb{R}^d \mapsto \mathbb{R}$  be a continuous function which belongs to  $L^1(\mathbb{R}^d)$  and decays at least like  $|x|^{-d}$  as  $|x| \rightarrow \infty$ . Then the convolution  $I_{2s} * \phi$  (namely the Riesz potential of  $\phi$ ) is also a continuous function, decaying at least like  $|x|^{-d+2s}$  as  $|x| \rightarrow \infty$ .*

*Proof.* The fact that  $I_{2s} * \phi$  is continuous easily follows from continuity and integrability properties of both  $I_{2s}$  and  $\phi$ . In order to prove the claimed decay behaviour as  $|x| \rightarrow \infty$  we have to work a bit more. To begin with, let us split the convolution in this way:

$$(I_{2s} * \phi)(x) = \int_{\mathbb{R}^d} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy = \underbrace{\int_{B_{2|x|}} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy}_{F_1(x)} + \underbrace{\int_{B_{2|x|}^c} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy}_{F_2(x)}.$$

As concerns  $F_2$ , we have:

$$|F_2(x)| = \left| \int_{B_{2|x|}^c} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right| \leq 2^{d-2s} C k_{d,s} \int_{B_{2|x|}^c} |y|^{-2d+2s} dy \leq \frac{C k_{d,s} d |B_1|}{(d-2s)|x|^{d-2s}}, \quad (2.2.51)$$

where we used the inequalities

$$|\phi(y)| \leq \frac{C}{|y|^d} \quad \forall y \in \mathbb{R}^d, \quad |x-y| \geq \frac{|y|}{2} \quad \forall y \in B_{2|x|}^c, \quad (2.2.52)$$

valid for some  $C > 0$ . On the other hand,  $F_1$  can be estimated as follows:

$$|F_1(x)| = \left| \int_{B_{2|x|}} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right| \leq \left| \int_{B_{|x|/2}(x)} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right| + \left| \int_{B_{2|x|} \cap B_{|x|/2}(x)^c} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right|. \quad (2.2.53)$$

Since

$$\left| \int_{B_{|x|/2}(x)} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right| \leq \frac{2^d C k_{d,s}}{|x|^d} \left| \int_{B_{|x|/2}(x)} |x-y|^{-d+2s} dy \right| \leq \frac{2^{d-2s-1} C k_{d,s} d |B_1|}{s|x|^{d-2s}}$$

and

$$\left| \int_{B_{2|x|} \cap B_{|x|/2}(x)^c} \frac{k_{d,s} \phi(y)}{|x-y|^{d-2s}} dy \right| \leq \frac{2^{d-2s} k_{d,s}}{|x|^{d-2s}} \|\phi\|_1, \quad (2.2.54)$$

gathering (2.2.51)–(2.2.54) we finally deduce the claimed decay property of  $I_{2s} * \phi$ .  $\square$

*Proof of Theorem 2.2.2.* Suppose first that  $\mu$  is a compactly supported measure. Consider the family  $\{u_\varepsilon\}$  of weak solutions to (2.1.4) that take the regular initial data  $\mu_\varepsilon := \psi_\varepsilon * \mu$  (let  $\varepsilon > 0$ ), where  $\psi_\varepsilon(x) := \varepsilon^{-d} \psi(x/\varepsilon)$  with  $\psi \in \mathcal{D}(\mathbb{R}^d)$ ,  $\psi \geq 0$  and  $\|\psi\|_1 = 1$ . The existence of such a family is ensured by Lemma 2.2.9, upon letting  $u_0 = \rho^{-1} \mu_\varepsilon \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . We shall prove that  $\{u_\varepsilon\}$  converges (up to subsequences), as  $\varepsilon \rightarrow 0$ , to a function  $u$  which satisfies (2.2.1), (2.2.2) and (2.2.3). Afterwards we shall deal with (2.2.4).

Combining the smoothing effect (2.2.43) (for  $p_0 = 1$ ) with the fact that  $\|\mu_\varepsilon\|_1 = \mu(\mathbb{R}^d)$  and with the conservation of mass (2.2.11), we obtain:

$$\int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x, t) \rho(x) dx \leq \|u_\varepsilon(t)\|_\infty^m \|\mu_\varepsilon\|_1 \leq K^m t^{-\alpha m} \mu(\mathbb{R}^d)^{1+\beta m} \quad \forall t > 0. \quad (2.2.55)$$

Hence, using (2.2.27), (2.2.28) and (2.2.55), we get:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} (u_\varepsilon^m)(x, t) \right|^2 dx dt + \frac{1}{m+1} \int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x, t_2) \rho(x) dx \leq \frac{K^m}{m+1} t_1^{-\alpha m} \mu(\mathbb{R}^d)^{1+\beta m}, \quad (2.2.56)$$

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(z_\varepsilon)_t(x, t)|^2 \rho(x) dx dt \leq K' \quad (2.2.57)$$

for all  $t_2 > t_1 > 0$ , where  $z_\varepsilon := u_\varepsilon^{(m+1)/2}$  and  $K'$  is a positive constant that depends on  $t_1$ ,  $t_2$  and  $\mu(\mathbb{R}^d)$  (via (2.2.55) evaluated *e.g.* at  $t_1/2$ ) but is independent of  $\varepsilon$ . Thanks to (2.2.56), (2.2.57), the conservation of mass and the smoothing effect (which in particular permits us to bound  $\{u_\varepsilon\}$  in  $L^\infty(\mathbb{R}^d \times (\tau, \infty))$  for all  $\tau > 0$ , independently of  $\varepsilon$ ), we can proceed exactly as in the proof of Lemma 2.2.9. That is, we find that the pointwise limit  $u$  of  $\{u_\varepsilon\}$ , up to subsequences, satisfies (2.2.1), (2.2.2) and (2.2.3).

In order to prove that the initial datum assumed by  $u$  (in the sense of (2.2.4)) is actually  $\mu$ , we exploit some results in potential theory, following [148] or [174]. To begin with, let us introduce the Riesz potential  $U_\varepsilon(t)$  of  $\rho u_\varepsilon(t)$ , namely

$$U_\varepsilon(x, t) := [I_{2s} * \rho u_\varepsilon(t)](x) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

The equation solved by  $u_\varepsilon$  is

$$\rho(x)(u_\varepsilon)_t(x, t) = -(-\Delta)^s (u_\varepsilon^m)(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+. \quad (2.2.58)$$

Applying to both sides of (2.2.58) the operator  $(-\Delta)^{-s}$ , *i.e.* the convolution against the Riesz kernel  $I_{2s}$  (see Paragraph 2.1.1.4), we formally obtain

$$(U_\varepsilon)_t(x, t) = -u_\varepsilon^m(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+. \quad (2.2.59)$$

We aim at proving (2.2.59) rigorously. To this end, let us plug in (2.2.3) (with  $u = u_\varepsilon$ ) the test function  $\varphi(y, t) = \vartheta(t)\phi(y)$ , where  $\vartheta$  is a smooth and compactly supported approximation of  $\chi_{[t_1, t_2]}$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ . Integrating by parts (in space), letting  $\vartheta$  tend to  $\chi_{[t_1, t_2]}$  and replacing  $\phi(y)$  with  $\phi(y+x)$  (for any fixed  $x$ ), we get:

$$\begin{aligned} & \int_{\mathbb{R}^d} u_\varepsilon(y, t_2) \phi(y+x) \rho(y) dy - \int_{\mathbb{R}^d} u_\varepsilon(y, t_1) \phi(y+x) \rho(y) dy \\ &= - \int_{\mathbb{R}^d} \left( \int_{t_1}^{t_2} u_\varepsilon^m(y, t) dt \right) (-\Delta)^s(\phi)(y+x) dy. \end{aligned} \quad (2.2.60)$$

Integrating (2.2.60) against the Riesz kernel  $I_{2s}(x)$  and exploiting Fubini's Theorem we end up with the identity (let  $z = y+x$ )

$$\begin{aligned} & \int_{\mathbb{R}^d} U_\varepsilon(z, t_2) \phi(z) dz - \int_{\mathbb{R}^d} U_\varepsilon(z, t_1) \phi(z) dz \\ &= - \int_{\mathbb{R}^d} \left( \int_{t_1}^{t_2} u_\varepsilon^m(y, t) dt \right) \left( \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(y+x) I_{2s}(x) dx \right) dy \\ &= - \int_{\mathbb{R}^d} \left( \int_{t_1}^{t_2} u_\varepsilon^m(y, t) dt \right) \phi(y) dy. \end{aligned} \quad (2.2.61)$$

The application of Fubini's Theorem here is justified by means of Lemma 2.2.13, Lemma 2.A.1, (2.2.1) and the assumptions on  $\rho$ . In fact, both the functions

$$y \longrightarrow \int_{\mathbb{R}^d} |\phi(y+x)| I_{2s}(x) dx \quad (2.2.62)$$

and

$$y \longrightarrow \int_{\mathbb{R}^d} |(-\Delta)^s(\phi)(y+x)| I_{2s}(x) dx \quad (2.2.63)$$

are continuous and decay at least like  $|y|^{-d+2s}$  as  $|y| \rightarrow \infty$ . This implies that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\rho(y)u_\varepsilon(y,t)\phi(y+x)I_{2s}(x)| dydx < \infty \quad \text{for } t \in \{t_1, t_2\}$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \left( \int_{t_1}^{t_2} u_\varepsilon^m(y,t) dt \right) (-\Delta)^s(\phi)(y+x)I_{2s}(x) \right| dydx < \infty$$

since (2.2.62) is bounded and (2.2.63) decays at least like  $\rho(y)$  at infinity (here the assumption  $d - 2s \geq \gamma$  is crucial). By Lemma 2.2.9 and Definition 2.2.1, we know that  $\rho u_\varepsilon(t)$  converges to  $\mu_\varepsilon$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  as  $t \rightarrow 0$ . Hence, letting  $t_1 \rightarrow 0$  in (2.2.61), we find that

$$\int_{\mathbb{R}^d} U_\varepsilon(x, t_2)\phi(x) dx - \int_{\mathbb{R}^d} U^{\mu_\varepsilon}(x)\phi(x) dx = - \int_{\mathbb{R}^d} \left( \int_0^{t_2} u_\varepsilon^m(x,t) dt \right) \phi(x) dx \quad (2.2.64)$$

for all  $t_2 > 0$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ . In fact,

$$\begin{aligned} \int_{\mathbb{R}^d} U_\varepsilon(x, t_1)\phi(x) dx &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_{2s}(x-y) \rho(y)u_\varepsilon(y, t_1) dy \right) \phi(x) dx \\ &= \int_{\mathbb{R}^d} \underbrace{\left( \int_{\mathbb{R}^d} I_{2s}(y-x) \phi(x) dx \right)}_{U^\phi(y)} \rho(y)u_\varepsilon(y, t_1) dy, \end{aligned}$$

and in view of Lemma 2.2.13 we know that, in particular,  $U^\phi \in C_0(\mathbb{R}^d)$ , which allows us to pass to the limit in the integral as  $t_1 \rightarrow 0$ . Thanks to the smoothing effect, the conservation of mass and the hypotheses on  $\rho$ , we can provide the following bound for (2.2.64):

$$\left| \int_{\mathbb{R}^d} U_\varepsilon(x, t_2)\phi(x) dx - \int_{\mathbb{R}^d} U^{\mu_\varepsilon}(x)\phi(x) dx \right| \leq \|\rho^{-1}\phi\|_\infty K^{m-1} \mu(\mathbb{R}^d)^{1+\beta(m-1)} \int_0^{t_2} t^{-\alpha(m-1)} dt. \quad (2.2.65)$$

Note that the time integral in the r.h.s. is finite since  $\alpha(m-1) < 1$  (recall (2.2.44) for  $p_0 = 1$ ). We proved above that  $\{u_\varepsilon\}$  converges pointwise a.e. (up to subsequences) to a function  $u$  which satisfies (2.2.1), (2.2.2) and (2.2.3). If we exploit once again the smoothing effect and the conservation of mass, we easily infer that such convergence also takes place in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_0(\mathbb{R}^d))$ :

$$\lim_{\varepsilon \rightarrow 0} \rho u_\varepsilon(t) = \rho u(t) \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_0(\mathbb{R}^d)), \text{ for a.e. } t > 0. \quad (2.2.66)$$

Using (2.2.66), the fact that  $\mu_\varepsilon \rightarrow \mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  and proceeding exactly as we did in the proof of (2.2.64), we can let  $\varepsilon \rightarrow 0$  in (2.2.65) to get

$$\left| \int_{\mathbb{R}^d} U(x, t_2)\phi(x) dx - \int_{\mathbb{R}^d} U^\mu(x)\phi(x) dx \right| \leq \|\rho^{-1}\phi\|_\infty K^{m-1} \mu(\mathbb{R}^d)^{1+\beta(m-1)} \int_0^{t_2} t^{-\alpha(m-1)} dt \quad (2.2.67)$$

for a.e.  $t_2 > 0$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , where we denote as  $U$  the potential of  $\rho u$ . Note that, passing to the limit in (2.2.64) for any nonnegative  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , we deduce in particular that  $U(x, t)$  is nonincreasing in  $t$ . Moreover, (2.2.67) implies that  $U(t)$  converges to  $U^\mu$  in  $L^1_{\text{loc}}(\mathbb{R}^d)$ , whence

$$\lim_{t \rightarrow 0} U(x, t) = U^\mu(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.2.68)$$



Letting  $\varepsilon \rightarrow 0$  in the conservation of mass (2.2.11) (applied to  $u = u_\varepsilon$  and  $\mu = \mu_\varepsilon$ ), by means *e.g.* of Fatou's Lemma we obtain

$$\|u(t)\|_{1,\rho} \leq \mu(\mathbb{R}^d) \quad \text{for a.e. } t > 0. \quad (2.2.69)$$

Due to the compactness results recalled in Paragraph 2.1.1.2, from (2.2.69) we infer that (almost) every sequence  $t_n \rightarrow 0$  admits a subsequence  $\{t_{n_k}\}$  such that  $\{\rho u(t_{n_k})\}$  converges to a certain positive finite Radon measure  $\nu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ . Thanks to (2.2.68) and [121, Theorem 3.8] we have that  $U^\nu(x) = U^\mu(x)$  almost everywhere. Alternatively, such identity can be proved by passing to the limit in  $\int_{\mathbb{R}^d} U(x, t_{n_k}) \phi(x) dx$ , recalling that  $U(t_{n_k}) \rightarrow U^\mu$  in  $L^1_{\text{loc}}(\mathbb{R}^d)$  as  $k \rightarrow \infty$ . Theorem 1.12 of [121] then ensures that two positive finite Radon measures whose potentials are equal almost everywhere must coincide. Hence,  $\nu = \mu$  and the limit measure does not depend on the particular subsequence, so that

$$\lim_{t \rightarrow 0} \rho u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d)).$$

In order to show that convergence also takes place in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ , it is enough to establish that

$$\lim_{t \rightarrow 0} \|u(t)\|_{1,\rho} = \mu(\mathbb{R}^d). \quad (2.2.70)$$

Since  $\rho u(t)$  converges to  $\mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  as  $t \rightarrow 0$ , we know that

$$\mu(\mathbb{R}^d) \leq \liminf_{t \rightarrow 0} \|u(t)\|_{1,\rho}, \quad (2.2.71)$$

see again Paragraph 2.1.1.2. But (2.2.71) and (2.2.69) entail (2.2.70).

Finally, the validity of the smoothing estimate (2.2.5) just follows by passing to the limit in (2.2.43) (applied to  $u_\varepsilon$  and  $p_0 = 1$ ) as  $\varepsilon \rightarrow 0$ , recalling that  $\{u_\varepsilon\}$  converges to  $u$  almost everywhere.

At the beginning of the proof we assumed  $\mu$  to be compactly supported. Otherwise, take a sequence of compactly supported measures  $\{\mu_n\}$  converging to  $\mu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  and consider the corresponding sequence of solutions  $\{u_n\}$  to (2.1.4). The energy estimates (2.2.56) and (2.2.57) (as well as the conservation of mass and the smoothing effect) are clearly stable as  $\varepsilon \rightarrow 0$ , so that they hold true with  $u_\varepsilon$  replaced by  $u_n$  and  $\mu_\varepsilon$  replaced by  $\mu_n$ . Hence, using the same techniques as above, one can prove that  $\{u_n\}$  converges to a solution  $u$  of (2.1.4) starting from  $\mu$ .  $\square$

**Lemma 2.2.14.** *Let  $\nu$  be a signed finite measure, namely the difference between two positive finite Radon measures, such that  $U^\nu \geq 0$  almost everywhere. Then  $\nu(\mathbb{R}^d) \geq 0$ .*

*Proof.* From the assumptions on  $U^\nu$  and thanks to Fubini's Theorem, there holds

$$\int_{\mathbb{R}^d} \chi_{B_n}(y) U^\nu(y) dy = \int_{\mathbb{R}^d} (I_{2s} * \chi_{B_n})(x) d\nu = k_{d,s} \int_{\mathbb{R}^d} \left( \int_{B_n} |x-y|^{-d+2s} dy \right) d\nu \geq 0 \quad (2.2.72)$$

for all  $n \in \mathbb{N}$ . Performing the change of variable  $z = y/n$ , the last inequality in (2.2.72) reads

$$\int_{\mathbb{R}^d} \left( \int_{B_1} |x/n - z|^{-d+2s} dz \right) d\nu \geq 0 \quad \forall n \in \mathbb{N}. \quad (2.2.73)$$

It is plain that for every  $x \in \mathbb{R}^d$  the sequence  $\{\int_{B_1} |x/n - z|^{-d+2s} dz\}$  converges to the positive constant  $\int_{B_1} |z|^{-d+2s} dz$  and it is dominated by the latter. Passing to the limit as  $n \rightarrow \infty$  in (2.2.73), we get the assertion by dominated convergence (recall that  $\nu$  is a finite measure).  $\square$

*Proof of Theorem 2.2.3.* Consider a function  $u$  satisfying (2.2.1), (2.2.2) and (2.2.3). Monotonicity in time of the associated potential is a consequence of the analogue of (2.2.61), which here can be proved exactly as above: for such an argument to work, the running assumptions on  $\gamma$  are necessary. Actually the same proof holds if, instead of  $u \in L^\infty(\mathbb{R}^d \times (\tau, \infty))$ ,  $u$  is only supposed to satisfy  $\int_{t_1}^{t_2} u^m(\tau) d\tau \in L_\rho^1(\mathbb{R}^d)$  for all  $t_2 > t_1 > 0$ . Existence of an initial trace  $\mu$ , meant as convergence in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  along subsequences of a given sequence of times going to zero, follows by compactness, since we are assuming that  $u \in L^\infty((0, \infty); L_\rho^1(\mathbb{R}^d))$ . Uniqueness of such a trace is established as we did after (2.2.68), *i.e.* using monotonicity of potentials and [121, Theorems 1.12 and 3.8].

We are left with proving that convergence to  $\mu$  takes places also in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ , namely that  $\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} u(x, t) \rho(x) dx = \mu(\mathbb{R}^d)$ . By weak\* lower semi-continuity, it is enough to show that  $\limsup_{t \rightarrow 0} \int_{\mathbb{R}^d} u(x, t) \rho(x) dx \leq \mu(\mathbb{R}^d)$ . Let  $U(t)$  be the potential of  $\rho u(t)$ . Again, the monotonicity in time of  $U(t)$  and the first part of the proof ensure that  $U^\mu - U(t) \geq 0$  almost everywhere. Therefore, Lemma 2.2.14 applied to the signed finite measure  $d\nu = d\mu - u(x, t)\rho(x)dx$  entails  $\mu(\mathbb{R}^d) \geq \int_{\mathbb{R}^d} u(x, t) \rho(x) dx$ . Letting  $t \rightarrow 0$  we conclude the proof.  $\square$

### 2.2.2.1 Strong solutions

In order to justify rigorously some of the above computations, it is essential to show that the weak solutions constructed in Lemma 2.2.9 are strong. By a “strong solution”, following [61, Subsection 6.2], we mean a weak solution  $u$  (in the sense of Definition 2.2.1) having the additional property

$$u_t \in L^\infty((\tau, \infty), L_\rho^1(\mathbb{R}^d)) \quad \forall \tau > 0. \quad (2.2.74)$$

Here we shall only give a sketch of how it is possible to prove that our solutions are indeed strong, as techniques are completely analogous to the ones used in [61, Subsection 8.1]. The first step consists in establishing that  $\rho u_t(t)$  is a bounded Radon measure satisfying the estimate

$$\|\rho u_t(t)\|_{\mathcal{M}(\mathbb{R}^d)} \leq \frac{2 \|u_0\|_{1,\rho}}{(m-1)t} \quad \forall t > 0, \quad (2.2.75)$$

where now, by  $\mathcal{M}(\mathbb{R}^d)$ , we mean the Banach space of *signed* measures on  $\mathbb{R}^d$ , equipped with the usual norm of the total variation. This can be proved proceeding exactly as in [173, Lemma 8.5], that is by exploiting the validity of the  $L_\rho^1(\mathbb{R}^d)$  contraction principle

$$\int_{\mathbb{R}^d} [u(x, t) - \tilde{u}(x, t)]_+ \rho(x) dx \leq \int_{\mathbb{R}^d} [u_0(x) - \tilde{u}_0(x)]_+ \rho(x) dx \quad \forall t > 0, \quad (2.2.76)$$

where  $u$  and  $\tilde{u}$  are the solutions to (2.2.14) *constructed in Lemma 2.2.9* corresponding to the initial data  $u_0$  and  $\tilde{u}_0$ , respectively. Such principle does hold for the approximate solutions  $u_\eta$  and  $\tilde{u}_\eta$  used in the proof of Lemma 2.2.9:

$$\int_{\mathbb{R}^d} [u_\eta(x, t) - \tilde{u}_\eta(x, t)]_+ \rho_\eta(x) dx \leq \int_{\mathbb{R}^d} [u_0(x) - \tilde{u}_0(x)]_+ \rho_\eta(x) dx \quad \forall t > 0. \quad (2.2.77)$$

This is proved in [159, Proposition 3.3]. Hence, (2.2.76) is just a consequence of letting  $\eta \rightarrow \infty$  in (2.2.77).

Afterwards one proves that  $z := u^{(m+1)/2}$  is a function satisfying estimate (2.2.28), with a constant  $K$  that depends on  $t_1$ ,  $t_2$  and *e.g.* on

$$\int_{t_1/2}^{2t_2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}}(u^m)(x, t) \right|^2 dx dt.$$

In order to do that, one just repeats the proof of [61, Lemma 8.1] (the idea is to use Steklov averages). In particular,

$$z_t \in L^2_{\text{loc}}((0, \infty); L^2_\rho(\mathbb{R}^d)). \quad (2.2.78)$$

The dependence of  $K$  on the initial datum through  $\|u(t_1/2)\|_{m+1, \rho}$  is then a consequence of the energy identity (2.2.27) (the proof of which requires however that solutions are strong, see the next Paragraph 2.2.2.2). Having at our disposal (2.2.75) and (2.2.78), we can apply the general result [17, Theorem 1.1], which ensures (in particular) that  $u_t$  is actually a function satisfying

$$u_t \in L^1_{\text{loc}}((0, \infty); L^1_\rho(\mathbb{R}^d)). \quad (2.2.79)$$

In view of (2.2.75) and (2.2.79) we then get the estimate

$$\|u_t(t)\|_{1, \rho} \leq \frac{2 \|u_0\|_{1, \rho}}{(m-1)t}. \quad (2.2.80)$$

In particular, (2.2.74) holds true and solutions are strong.

**Remark 2.2.15.** We have shown that the weak solutions to (2.2.14) constructed in Lemma 2.2.9 are strong. Since, for any  $\tau > 0$ , every weak solution  $u$  to (2.1.4) is a weak solution to (2.2.14) corresponding to the initial datum  $u(\tau) \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , one may claim that also such  $u$  is a strong solution. This is actually true: however, in order to prove it rigorously, we need the uniqueness Theorem 2.2.4.

Knowing that also the weak solutions provided by Theorem 2.2.2 are strong allows us, *a posteriori*, to state some properties of such solutions *for all*  $t > 0$  rather than only *for a.e.*  $t > 0$ .

### 2.2.2.2 Decrease of the norms

An important consequence of the fact that the solutions constructed in Lemma 2.2.9 are strong is the decrease of their  $L^p_\rho$  norms for any  $p \in [1, \infty]$ . Indeed, thanks to (2.2.74), we are allowed to multiply the differential equation in (2.2.14) by  $u^{p-1}$  and integrate in  $\mathbb{R}^d \times [t_1, t_2]$ . Exploiting the Stroock-Varopoulos inequality (2.2.34) (let  $v = u^m$  and  $q = (p+m-1)/m$  there), we obtain

$$\int_{\mathbb{R}^d} u^p(x, t_2) \rho(x) dx - \int_{\mathbb{R}^d} u^p(x, t_1) \rho(x) dx = -p \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{p-1}(x, t) (-\Delta)^s(u^m)(x, t) dx dt \leq 0 \quad (2.2.81)$$

for all  $t_2 > t_1 > 0$ . The validity of (2.2.81) down to  $t_1 = 0$  cannot be proved just by letting  $t_1 \rightarrow 0$ , since *a priori* we have no information over the continuity of  $\|u(t)\|_{p, \rho}$  at  $t = 0$ . However, reasoning as above, we have that also the approximate solutions  $u_\eta$  of Lemma 2.2.9 are strong and therefore satisfy

$$\int_{\mathbb{R}^d} u_\eta^p(x, t_2) \rho_\eta(x) dx \leq \int_{\mathbb{R}^d} u_\eta^p(x, t_1) \rho_\eta(x) dx. \quad (2.2.82)$$

Moreover, here we are indeed allowed to let  $t_1 \rightarrow 0$  in (2.2.82), which yields

$$\int_{\mathbb{R}^d} u_\eta^p(x, t) \rho_\eta(x) dx \leq \int_{\mathbb{R}^d} u_0^p(x) \rho_\eta(x) dx \quad \forall t > 0; \quad (2.2.83)$$

in fact,  $u_\eta \in C([0, \infty); L^1_{\rho_\eta}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0, \infty))$  (see [159, Definition 2.2 and Theorem 3.1] and also Subsection 2.3.4). Hence, letting  $\eta \rightarrow \infty$  and using *e.g.* Fatou's Lemma in the l.h.s. of (2.2.83), we find that

$$\int_{\mathbb{R}^d} u^p(x, t_2) \rho(x) dx \leq \int_{\mathbb{R}^d} u^p(x, t_1) \rho(x) dx$$

for all  $t_2 > t_1 \geq 0$ .

We point out that the above computations are rigorous provided  $p \in (1, \infty)$ . However, we know that  $\|u(t)\|_{1,\rho}$  is preserved, while the case  $p = \infty$  can be dealt with by taking limits. Notice that, for  $p = m + 1$ , (2.2.81) becomes exactly the energy identity (2.2.27).

### 2.2.3 Uniqueness of weak solutions

In this subsection we borrow some ideas from the pioneering paper [148], which need to be carefully modified in order to deal with our fractional, weighted problem. In particular, the Markov property for the linear semigroup associated to the operator  $A := \rho^{-1}(-\Delta)^s$ , which is addressed in Appendix 2.A, will play a crucial role.

Prior to the proof of Theorem 2.2.4, below we prove some basic lemmas and introduce our strategy.

**Lemma 2.2.16.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \geq 0$  and  $\gamma_0 \in [0, 2s)$ . Let  $v \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and  $U_\rho^v$  be the Riesz potential of  $\rho v$ . Then,  $U_\rho^v$  belongs to  $C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  for all  $p$  such that*

$$p \in \left( \frac{d}{d-2s}, \infty \right]. \quad (2.2.84)$$

*Proof.* In order to prove that  $U_\rho^v$  belongs to  $C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.84), we first split the convolution as follows:

$$U_\rho^v(x) = \underbrace{\int_{B_1} \rho(y) v(y) I_{2s}(x-y) dy}_{U_{\rho,1}^v(x)} + \underbrace{\int_{\mathbb{R}^d} \chi_{B_1^c}(y) \rho(y) v(y) I_{2s}(x-y) dy}_{U_{\rho,2}^v(x)}.$$

Exploiting the fact that  $v \in L^\infty(\mathbb{R}^d)$  and  $\gamma_0 < 2s$  (so that  $|y|^{-d+2s}\rho(y) \in L^1_{\text{loc}}(\mathbb{R}^d)$ ), it is easily seen that  $U_{\rho,1}^v(x)$  is a continuous function which decays at least like  $|x|^{-d+2s}$  as  $|x| \rightarrow \infty$ . In particular, it belongs to  $L^p(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.84). As concerns  $U_{\rho,2}^v(x)$ , notice that, since  $v \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , the function  $\chi_{B_1^c}\rho v$  belongs to  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . Hence  $U_{\rho,2}^v(x)$  is continuous as well. To prove that it belongs to  $L^p(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.84), we write:

$$U_{\rho,2}^v = (\chi_{B_1} I_{2s}) * (\chi_{B_1^c} \rho v) + (\chi_{B_1^c} I_{2s}) * (\chi_{B_1^c} \rho v). \quad (2.2.85)$$

Because  $\chi_{B_1} I_{2s} \in L^1(\mathbb{R}^d)$  and  $\chi_{B_1^c} \rho v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , the first convolution in (2.2.85) belongs to  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . Using the fact that  $\chi_{B_1^c} I_{2s} \in L^p(\mathbb{R}^d)$  for all  $p$  as in (2.2.84) and  $\chi_{B_1^c} \rho v \in L^1(\mathbb{R}^d)$ , we infer that the second convolution in (2.2.85) belongs to  $L^p(\mathbb{R}^d)$  for all such  $p$ . The latter property is then inherited by  $U_{\rho,2}^v$ .  $\square$

In the following, with the term “weak solution”, we shall always mean a solution to (2.1.4) in the sense of Definition 2.2.1.

**Lemma 2.2.17.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, d-2s]$  and  $\gamma_0 \geq 0$ . Let  $u$  be a weak solution to (2.1.4), starting from the positive finite Radon measure  $\mu$  whose potential is  $U^\mu$ , and denote as  $U$  the potential of  $\rho u$ . Then,*

$$\lim_{t \rightarrow 0} U(x, t) = U^\mu(x) \quad \forall x \in \mathbb{R}^d. \quad (2.2.86)$$

*Proof.* It is actually a direct application of [121, Theorem 3.9], but for the reader’s convenience we give some details.

Thanks to Theorem 3.8 of [121] and to the monotonicity in time of  $U(x, t)$  (see the proof of Theorem 2.2.2 above, one only needs  $\gamma \leq d - 2s$  to justify (2.2.61)), we have that the limit in (2.2.86) is taken at least for a.e.  $x \in \mathbb{R}^d$ . Nevertheless, Lemma 1.12 of [121] shows that, as a consequence of monotonicity, there exists a positive measure  $\nu$ , whose potential is denoted by  $U^\nu$ , and a constant  $A \geq 0$  such that

$$\lim_{t \rightarrow 0} U(x, t) = U^\nu(x) + A \quad \forall x \in \mathbb{R}^d.$$

Since we have just proved that (2.2.86) holds almost everywhere, in particular

$$U^\mu(x) = U^\nu(x) + A \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.2.87)$$

However, in view of the corollary at page 129 of [121], from (2.2.87) we deduce that  $A = 0$ . Hence,  $U^\nu = U^\mu$  almost everywhere, and by Theorem 1.12 of [121] we know that two potentials coinciding a.e. in fact coincide everywhere, whence (2.2.86).  $\square$

Now let  $u_1$  and  $u_2$  be two weak solutions to (2.1.4) which take the same positive finite Radon measure  $\mu$  as initial datum. We denote as  $U_1$  and  $U_2$  the potentials of  $\rho u_1$  and  $\rho u_2$ , respectively. Fix once for all the parameters  $h, T > 0$  and consider the function

$$g(x, t) := U_2(x, t + h) - U_1(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times (0, T]. \quad (2.2.88)$$

Thanks to the hypothesis  $\gamma \leq d - 2s$ , we can proceed again as in the proof of (2.2.61) to find that  $g(t)$  is an absolutely continuous curve in  $L^1_{\text{loc}}(\mathbb{R}^d)$  which satisfies

$$\rho(x)g_t(x, t) = \rho(x)(u_1^m(x, t) - u_2^m(x, t + h)) = -a(x, t)(-\Delta)^s(g)(x, t) \quad (2.2.89)$$

for a.e.  $(x, t) \in \mathbb{R}^d \times (0, T)$ , where

$$(-\Delta)^s(g)(x, t) = \rho(x)u_2(x, t + h) - \rho(x)u_1(x, t)$$

and we define the function  $a$  as

$$a(x, t) := \begin{cases} \frac{u_1^m(x, t) - u_2^m(x, t + h)}{u_1(x, t) - u_2(x, t + h)} & \text{if } u_1(x, t) \neq u_2(x, t + h), \\ 0 & \text{if } u_1(x, t) = u_2(x, t + h). \end{cases} \quad (2.2.90)$$

Notice that, since  $m > 1$  and  $u_1, u_2 \in L^\infty(\mathbb{R}^d \times (\tau, \infty))$  for all  $\tau > 0$ ,  $a$  is a nonnegative function belonging to  $L^\infty(\mathbb{R}^d \times (\tau, \infty))$  for all  $\tau > 0$ .

In the sequel, a key role will be played by a suitable family of positive finite Radon measures  $\{\nu(t)\}$ , which is somehow related to equation (2.2.89). More precisely,  $\nu(t)$  is the limit in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ , as  $\varepsilon \rightarrow 0$ , of  $\{\rho\psi_\varepsilon(t)\}$ , while  $\psi_\varepsilon$  is in turn the weak limit in  $L^2_\rho(\mathbb{R}^d \times (\tau, T))$  (for all  $\tau \in (0, T)$ ), as  $n \rightarrow \infty$ , of a suitable sequence  $\{\psi_{n,\varepsilon}\}$ . The latter is defined, for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , to be the solution (in a sense that will be clarified later) of

$$\begin{cases} \rho(x)(\psi_{n,\varepsilon})_t = (-\Delta)^s[(a_n + \varepsilon)\psi_{n,\varepsilon}] & \text{in } \mathbb{R}^d \times (0, T), \\ \psi_{n,\varepsilon} = \psi & \text{on } \mathbb{R}^d \times \{T\}, \end{cases} \quad (2.2.91)$$

where  $\psi$  is an arbitrary nonnegative function of  $\mathcal{D}(\mathbb{R}^d)$ . The sequence  $\{a_n\}$  is a suitable approximation of the function  $a$  above. In particular we suppose that, for every  $n \in \mathbb{N}$ ,  $a_n(x, t)$  is a piecewise constant function of  $t$  (regular in  $x$ ) in the time intervals  $(T - (k + 1)T/n, T - kT/n]$ , for any  $k \in \{0, \dots, n - 1\}$ . Thanks to Theorem 2.A.13, we are then able to treat problem (2.2.91) by means of standard semigroup theory; in [148, Theorem 1] this is not needed since parabolic regularity holds true for the corresponding equation.

2.2.3.1 Construction and properties of the family  $\{\psi_{n,\varepsilon}\}$ 

We begin the proof of Theorem 2.2.4 by introducing more precisely the functions  $\psi_{n,\varepsilon}$  that solve (2.2.91).

**Lemma 2.2.18.** *Assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s)$  and  $\gamma_0 \in [0, d)$ . Let  $\{a_n\}$  be a sequence of nonnegative functions such that:*

- for any  $n \in \mathbb{N}$  and  $t > 0$ ,  $a_n(x, t)$  is regular in  $x$ ;
- for any  $n \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ ,  $a_n(x, t)$  is piecewise constant in  $t$  on the time intervals  $(T - (k+1)T/n, T - kT/n]$ , for any  $k \in \{0, \dots, n-1\}$ ;
- $\{\|a_n\|_{L^\infty(\mathbb{R}^d \times (\tau, \infty))}\}$  is uniformly bounded in  $n$  for any  $\tau > 0$ .

Then, for any  $\varepsilon > 0$  and any  $\psi \in \mathcal{D}(\mathbb{R}^d)$  with  $\psi \geq 0$ , there exists a nonnegative solution  $\psi_{n,\varepsilon}$  to problem (2.2.91), in the sense that  $\psi_{n,\varepsilon}(t)$  is a continuous curve in  $L^p_\rho(\mathbb{R}^d)$  (for all  $p \in (1, \infty)$ ) satisfying  $\psi_{n,\varepsilon}(T) = \psi$  and it is absolutely continuous on  $(T - (k+1)T/n, T - kT/n)$  for all  $k \in \{0, \dots, n-1\}$ , so that the identity

$$\begin{aligned} \psi_{n,\varepsilon}(t_2) - \psi_{n,\varepsilon}(t_1) &= \int_{t_1}^{t_2} \rho^{-1}(-\Delta)^s [(a_n + \varepsilon)\psi_{n,\varepsilon}](\tau) \, d\tau \\ \forall t_1, t_2 \in \left(T - \frac{(k+1)T}{n}, T - \frac{kT}{n}\right), \quad \forall k \in \{0, \dots, n-1\} \end{aligned} \quad (2.2.92)$$

holds true in  $L^p_\rho(\mathbb{R}^d)$  for all  $p \in (1, \infty)$ . Moreover,

$$\psi_{n,\varepsilon} \in L^\infty((0, T); L^p_\rho(\mathbb{R}^d)) \quad \forall p \in [1, \infty] \quad \text{and} \quad \|\psi_{n,\varepsilon}(t)\|_{1,\rho} \leq \|\psi\|_{1,\rho} \quad \forall t \in [0, T]. \quad (2.2.93)$$

*Proof.* Let us first define  $\zeta_1$  as the solution of

$$\begin{cases} \rho(\zeta_1)_t = (-\Delta)^s [(a_n(T) + \varepsilon)\zeta_1] & \text{in } \mathbb{R}^d \times \left(T - \frac{T}{n}, T\right), \\ \zeta_1 = \psi & \text{on } \mathbb{R}^d \times \{T\}. \end{cases} \quad (2.2.94)$$

To obtain such a  $\zeta_1$ , one can for instance exploit the change of variable

$$\phi_1(x, t) := (a_n(x, T) + \varepsilon)\zeta_1(x, t) \quad (2.2.95)$$

and see  $\phi_1$  as the solution of

$$\begin{cases} (\phi_1)_t = (a_n(T) + \varepsilon)\rho^{-1}(-\Delta)^s(\phi_1) & \text{in } \mathbb{R}^d \times \left(T - \frac{T}{n}, T\right), \\ \phi_1 = (a_n(T) + \varepsilon)\psi & \text{on } \mathbb{R}^d \times \{T\}. \end{cases} \quad (2.2.96)$$

Problem (2.2.96) is indeed solvable by means of standard semigroup theory. In fact, letting

$$\rho_1(x) := (a_n(x, T) + \varepsilon)^{-1}\rho(x),$$

we know that the operator  $\rho_1^{-1}(-\Delta)^s$  with domain  $X_{s,\rho_1} = X_{s,\rho}$  (see Definition 2.A.8 below) is nonnegative, self-adjoint and generates a Markov semigroup on  $L^2_{\rho_1}(\mathbb{R}^d)$  (Theorem 2.A.13). Our final datum  $(a_n(T) + \varepsilon)\psi$  is clearly nonnegative and belongs to  $L^p_{\rho_1}(\mathbb{R}^d)$  for all  $p \in [1, \infty]$ , which is enough in order to have a nonnegative solution to (2.2.96) that is continuous up to  $t = T$  and absolutely continuous on  $(T - T/n, T)$  in  $L^p_{\rho_1}(\mathbb{R}^d)$  for all  $p \in (1, \infty)$ . This is a consequence of the fact that the semigroup is actually analytic in  $L^p_{\rho_1}(\mathbb{R}^d)$  for  $p \in (1, \infty)$  (we refer again the reader to Theorem 2.A.13 below): by classical results, see for instance [146, Theorem 5.2 at p. 61], such property ensures in particular that problem (2.2.96) is solved by a differentiable curve  $\phi_1(t)$ .

Going back to the original variable  $\zeta_1$  through (2.2.95), we deduce that the latter solves (2.2.94) in the same sense in which  $\phi_1$  solves (2.2.96). Having at our disposal such a  $\zeta_1$ , we can then solve the problem

$$\begin{cases} \rho(\zeta_2)_t = (-\Delta)^s \left[ \left( a_n \left( T - \frac{T}{n} \right) + \varepsilon \right) \zeta_2 \right] & \text{in } \mathbb{R}^d \times \left( T - \frac{2T}{n}, T - \frac{T}{n} \right), \\ \zeta_2 = (a_n(x, T) + \varepsilon)^{-1} \phi_1 & \text{on } \mathbb{R}^d \times \left\{ T - \frac{T}{n} \right\}, \end{cases}$$

just by proceeding as above. That is, we perform the change of variable

$$\phi_2(x, t) := (a_n(x, T - T/n) + \varepsilon) \zeta_2(x, t)$$

and consider  $\phi_2$  as the solution of

$$\begin{cases} (\phi_2)_t = \left( a_n \left( T - \frac{T}{n} \right) + \varepsilon \right) \rho^{-1} (-\Delta)^s (\phi_2) & \text{in } \mathbb{R}^d \times \left( T - \frac{2T}{n}, T - \frac{T}{n} \right), \\ \phi_2 = \left( a_n \left( T - \frac{T}{n} \right) + \varepsilon \right) \zeta_1 = \frac{a_n(T - T/n) + \varepsilon}{a_n(T) + \varepsilon} \phi_1 & \text{on } \mathbb{R}^d \times \left\{ T - \frac{T}{n} \right\}. \end{cases}$$

It is apparent how the procedure goes on and permits us to obtain a solution  $\psi_{n,\varepsilon}$  to (2.2.91) in the sense of the statement, just by defining it as

$$\psi_{n,\varepsilon}(t) := \zeta_{k+1}(t) \quad \forall t \in \left( T - \frac{(k+1)T}{n}, T - \frac{kT}{n} \right], \quad \forall k \in \{0, \dots, n-1\}.$$

Finally, since  $\rho_{k+1}^{-1}(-\Delta)^s$  generates a contraction semigroup on  $L^p_{\rho_{k+1}}(\mathbb{R}^d)$  for all  $p \in [1, \infty]$ , where

$$\rho_{k+1}(x) := (a_n(x, T - kT/n) + \varepsilon)^{-1} \rho(x), \quad (2.2.97)$$

the inequalities

$$\begin{aligned} \|\phi_{k+1}(t)\|_{p, \rho_{k+1}} &\leq \left\| \frac{a_n(T - kT/n) + \varepsilon}{a_n(T - (k-1)T/n) + \varepsilon} \phi_k(T - kT/n) \right\|_{p, \rho_{k+1}} \\ &\forall t \in \left( T - \frac{(k+1)T}{n}, T - \frac{kT}{n} \right], \quad \forall p \in [1, \infty] \end{aligned} \quad (2.2.98)$$

hold true for any  $k \in \{0, \dots, n-1\}$  (in the r.h.s. of (2.2.98) for  $k=0$  we conventionally set  $\phi_0 = \psi$  and  $a_n(T + T/n) + \varepsilon = 1$ ). Going back to the variables  $\zeta_{k+1}$ , from (2.2.97) and (2.2.98) one deduces (2.2.93): in fact, for  $p=1$  it is easy to check that the factors containing  $a_n$  cancel out and give the corresponding inequality, while for  $p>1$  such terms remain and one obtains an inequality of the type  $\|\psi_{n,\varepsilon}(t)\|_{p,\rho} \leq C(n,\varepsilon) \|\psi\|_{p,\rho}$ , where  $C(n,\varepsilon)$  is a positive constant depending on  $n$  and  $\varepsilon$ .  $\square$

**Lemma 2.2.19.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $g$  be as in (2.2.88),  $a$  as in (2.2.90) and  $a_n, \psi_{n,\varepsilon}, \psi$  as in Lemma 2.2.18. Then the identity*

$$\begin{aligned} &\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx - \int_{\mathbb{R}^d} g(x, t) \psi_{n,\varepsilon}(x, t) \rho(x) dx \\ &= \int_t^T \int_{\mathbb{R}^d} (a_n(x, \tau) + \varepsilon - a(x, \tau)) (-\Delta)^s(g)(x, \tau) \psi_{n,\varepsilon}(x, \tau) dx d\tau \end{aligned} \quad (2.2.99)$$

holds true for all  $t \in (0, T]$ .

*Proof.* To begin with, set

$$t_k := T - \frac{kT}{n} \quad \forall k \in \{0, \dots, n\}.$$

Recall that, by Lemma 2.2.18,  $\psi_{n,\varepsilon}(t)$  is a continuous curve in  $L^p_\rho(\mathbb{R}^d)$  on  $(0, T]$ , absolutely continuous on any interval  $(t_{k+1}, t_k)$  for  $k \in \{0, \dots, n-1\}$  and satisfying the differential equation in (2.2.91) on such intervals, for all  $p \in (1, \infty)$ . Moreover,  $g(t)$  is an absolutely continuous curve in  $L^p_\rho(\mathbb{R}^d)$  on  $(0, T]$  for all  $p$  such that

$$p \in \left( \frac{d-\gamma}{d-2s}, \infty \right). \quad (2.2.100)$$

Since  $g(x, t)$  is a continuous function of  $x$  (recall Lemma 2.2.16) and the weight  $\rho(x)$  is locally integrable, in order to prove that  $g(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p$  as in (2.2.100) it suffices to show that  $g(t) \in L^p_\rho(B_1^c)$ . To this end, still Lemma 2.2.16 ensures in particular that  $g(t) \in L^p(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.84): the latter property and Hölder's inequality readily imply that  $g(t) \in L^p_\rho(B_1^c)$  for all  $p$  as in (2.2.100). The fact that  $g(t)$  is also absolutely continuous in  $L^p_\rho(\mathbb{R}^d)$  on the time interval  $(0, T]$  is then a consequence of (2.2.89) and of the integrability properties of  $u_1$  and  $u_2$ . Hence, due to Lemma 2.2.18, we get that

$$t \mapsto \int_{\mathbb{R}^d} g(x, t) \psi_{n,\varepsilon}(x, t) \rho(x) dx \quad (2.2.101)$$

is a continuous function on  $(0, T]$ , absolutely continuous on each interval  $(t_{k+1}, t_k)$ , which satisfies

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} g(x, t) \psi_{n,\varepsilon}(x, t) \rho(x) dx \\ &= \int_{\mathbb{R}^d} \{-a(x, t)(-\Delta)^s(g)(x, t) \psi_{n,\varepsilon}(x, t) + g(x, t)(-\Delta)^s[(a_n + \varepsilon) \psi_{n,\varepsilon}](x, t)\} dx \end{aligned} \quad (2.2.102)$$

there. We have just shown that  $g(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.100); moreover,  $\rho^{-1}(-\Delta)^s(g)(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p \in [1, \infty]$ . As a consequence of Lemma 2.2.18, we have that  $(a_n(t) + \varepsilon) \psi_{n,\varepsilon}(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p \in [1, \infty]$  and  $\rho^{-1}(-\Delta)^s[(a_n(t) + \varepsilon) \psi_{n,\varepsilon}(t)] \in L^p_\rho(\mathbb{R}^d)$  for all  $p \in (1, \infty)$ . We are therefore in position to apply Theorem 2.A.12 to the r.h.s. of (2.2.102) (note that the interval  $((d-\gamma)/(d-2s), \infty) \cap [2, 2(d-\gamma)/(d-2s))$  is not empty), which yields

$$\frac{d}{dt} \int_{\mathbb{R}^d} g(x, t) \psi_{n,\varepsilon}(x, t) \rho(x) dx = \int_{\mathbb{R}^d} (a_n(x, t) + \varepsilon - a(x, t)) (-\Delta)^s(g)(x, t) \psi_{n,\varepsilon}(x, t) dx. \quad (2.2.103)$$

As the r.h.s. of (2.2.103) is in  $L^1((\tau, T))$  for any  $\tau \in (0, T)$ , we deduce that (2.2.101) is absolutely continuous on the whole of  $(0, T]$  and not only on  $(t_{k+1}, t_k)$ . Hence, integrating (2.2.103) between  $t$  and  $T$ , we end up with (2.2.99).  $\square$

Next we prove that  $\psi_{n,\varepsilon}$  preserves the mass.

**Lemma 2.2.20.** *Under the same assumptions as in Lemma 2.2.18, the  $L^1_\rho$  norm of  $\psi_{n,\varepsilon}(t)$  is preserved, that is*

$$\int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t) \rho(x) dx = \int_{\mathbb{R}^d} \psi(x) \rho(x) dx \quad \forall t \in [0, T]. \quad (2.2.104)$$



*Proof.* Multiplying (2.2.92) by  $\rho\phi$  for any  $\phi \in \mathcal{D}(\mathbb{R}^d)$  and integrating in  $\mathbb{R}^d$ , we obtain the identity

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t^*)\phi(x) \rho(x) dx - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t_*)\phi(x) \rho(x) dx \\ &= \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x) \left( \int_{t_*}^{t^*} (a_n(x, \tau) + \varepsilon) \psi_{n,\varepsilon}(x, \tau) d\tau \right) dx \end{aligned} \quad (2.2.105)$$

for all  $t_*, t^* \in (t_{k+1}, t_k)$ . Since the  $L^1_\rho$  norm of  $\psi_{n,\varepsilon}(t)$  is bounded by the  $L^1_\rho$  norm of the final datum  $\psi$  (recall (2.2.93)), from (2.2.105) we get:

$$\left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t^*)\phi(x) \rho(x) dx - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t_*)\phi(x) \rho(x) dx \right| \leq K |t^* - t_*| \|\psi\|_{1,\rho} \|\rho^{-1}(-\Delta)^s(\phi)\|_\infty, \quad (2.2.106)$$

where  $K := \|a_n + \varepsilon\|_{L^\infty(\mathbb{R}^d \times (t_* \wedge t^*, T))}$  is a positive constant independent of  $n$  and  $\varepsilon$ . Replacing  $\phi$  with a usual cut-off function  $\xi_R$  (see Lemma 2.A.3 below) and estimating the r.h.s. of (2.2.106) as in the proof of Proposition 2.2.7, we end up with

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t^*)\xi_R(x)\rho(x) dx - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t_*)\xi_R(x)\rho(x) dx \right| \\ & \leq K |t^* - t_*| \|\psi\|_{1,\rho} c^{-1} \left( \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} + \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \right) \end{aligned} \quad (2.2.107)$$

for all  $R > 0$  and  $t_*, t^* \in (t_{k+1}, t_k)$ , the positive constant  $c$  being as in (2.1.3). Recalling that  $\psi_{n,\varepsilon}(t)$  is a continuous curve (for instance in  $L^1_{\rho, \text{loc}}(\mathbb{R}^d)$ ) on  $[0, T]$ , we can extend the validity of (2.2.107) (and (2.2.106)) to any  $t_*, t^* \in [0, T]$ . Picking  $t^* = T$  and letting  $R \rightarrow \infty$  in (2.2.107), we finally get (2.2.104).  $\square$

In the following lemma we introduce the Riesz potential of  $\rho\psi_{n,\varepsilon}$ , which will play a crucial role below.

**Lemma 2.2.21.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s) \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $a_n, \psi_{n,\varepsilon}, \psi$  be as in Lemma 2.2.18, and denote as  $H_{n,\varepsilon}(t)$  the potential of  $\rho\psi_{n,\varepsilon}(t)$ , that is*

$$H_{n,\varepsilon}(x, t) := [I_{2s} * \rho\psi_{n,\varepsilon}(t)](x) \quad \forall (x, t) \in \mathbb{R}^d \times [0, T].$$

Then  $H_{n,\varepsilon}(t) \in \dot{H}^s(\mathbb{R}^d)$  and the identity

$$\|I_{2s} * \rho\psi\|_{\dot{H}^s}^2 = \|H_{n,\varepsilon}(t)\|_{\dot{H}^s}^2 + 2 \int_t^T \int_{\mathbb{R}^d} (a_n(x, \tau) + \varepsilon) \psi_{n,\varepsilon}^2(x, \tau) \rho(x) dx d\tau \quad (2.2.108)$$

holds true for all  $t \in [0, T]$ .

*Proof.* First notice that  $\rho^{-1}(-\Delta)^s(H_{n,\varepsilon})(t) = \psi_{n,\varepsilon}(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p \in [1, \infty]$  (recall (2.2.93)) and  $H_{n,\varepsilon}(t) \in L^p_\rho(\mathbb{R}^d)$  for all  $p$  satisfying (2.2.100) (this can be proved by using Lemma 2.2.16 exactly as in the proof of Lemma 2.2.19). Again, since the interval  $((d - \gamma)/(d - 2s), \infty) \cap [2, 2(d - \gamma)/(d - 2s)]$  is never empty, applying Lemma 2.A.11 and Theorem 2.A.12 we get that  $H_{n,\varepsilon}(t) \in \dot{H}^s(\mathbb{R}^d)$  and the identity

$$\|H_{n,\varepsilon}(t)\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x, t) (-\Delta)^s(H_{n,\varepsilon})(x, t) dx = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x, t) \psi_{n,\varepsilon}(x, t) \rho(x) dx \quad (2.2.109)$$

holds true. Thanks to the validity of the differential equation

$$(H_{n,\varepsilon})_t(x, t) = (a_n(x, t) + \varepsilon)\psi_{n,\varepsilon}(x, t) \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times (0, T), \quad (2.2.110)$$

which can be justified as we did for (2.2.89), taking the time derivative of (2.2.109) in the intervals  $(t_{k+1}, t_k)$ , using (2.2.110), (2.2.91) and again Theorem 2.A.12, we obtain:

$$\frac{d}{dt} \|H_{n,\varepsilon}(t)\|_{\dot{H}^s}^2 = 2 \int_{\mathbb{R}^d} (a_n(x, t) + \varepsilon) \psi_{n,\varepsilon}^2(x, t) \rho(x) dx. \quad (2.2.111)$$

*A priori*, in view of (2.2.109), we have that  $\|H_{n,\varepsilon}(t)\|_{\dot{H}^s}^2$  is continuous on  $[0, T]$  and absolutely continuous only on  $(t_{k+1}, t_k)$ . Nevertheless, the r.h.s. of (2.2.111) is in  $L^1((0, T))$ ; hence, (2.2.108) just follows by integrating (2.2.111) from  $t$  to  $T$ .  $\square$

### 2.2.3.2 Passing to the limit as $n \rightarrow \infty$

The goal of the next lemma is to show that, if  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , then sequence  $\{\psi_{n,\varepsilon}\}$  suitably converges (up to subsequences) to a limit function  $\psi_\varepsilon$  having some crucial properties.

**Lemma 2.2.22.** *Let  $d > 2s$  and assume that  $\rho$  satisfies (2.1.3) for some  $\gamma \in [0, 2s] \cap [0, d-2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $u_1$  and  $u_2$  be two weak solutions to problem (2.1.4), taking the same positive finite Radon measure  $\mu$  as initial datum. Let  $g$  be as in (2.2.88),  $a$  as in (2.2.90) and  $\psi_{n,\varepsilon}, \psi$  as in Lemma 2.2.18, where we suppose in addition that  $a_n \rightarrow a$  almost everywhere. Then, up to subsequences,  $\{\psi_{n,\varepsilon}\}$  converges weakly in  $L_\rho^2(\mathbb{R}^d \times (\tau, T))$  (for all  $\tau \in (0, T)$ ) to a suitable nonnegative function  $\psi_\varepsilon$  and  $\{\rho\psi_{n,\varepsilon}(t)\}$  converges to  $\rho\psi_\varepsilon(t)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  for a.e.  $t \in (0, T)$ . Moreover,  $\psi_\varepsilon$  enjoys the following properties:*

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x, t) \rho(x) dx = \int_{\mathbb{R}^d} \psi(x) \rho(x) dx, \quad (2.2.112)$$

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi(x) \phi(x) \rho(x) dx - \int_{\mathbb{R}^d} \psi_\varepsilon(x, t) \phi(x) \rho(x) dx \\ &= \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x) \left( \int_t^T (a(x, \tau) + \varepsilon) \psi_\varepsilon(x, \tau) d\tau \right) dx, \end{aligned} \quad (2.2.113)$$

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx - \int_{\mathbb{R}^d} g(x, t) \psi_\varepsilon(x, t) \rho(x) dx \right| \\ & \leq \varepsilon (T - t) \|\psi\|_{1,\rho} \|u_2(\cdot + h) - u_1\|_{L^\infty(\mathbb{R}^d \times (t, T))} \end{aligned} \quad (2.2.114)$$

for a.e.  $t \in (0, T)$  and any  $\phi \in \mathcal{D}(\mathbb{R}^d)$ .

*Proof.* From (2.2.108) we deduce that, up to subsequences,  $\{\psi_{n,\varepsilon}\}$  converges weakly in  $L_\rho^2(\mathbb{R}^d \times (\tau, T))$ , for all  $\tau \in (0, T)$ , to a suitable function  $\psi_\varepsilon$ . Moreover, thanks to the uniform boundedness of  $\{\rho\psi_{n,\varepsilon}(t)\}$  in  $L^1(\mathbb{R}^d)$  ensured by (2.2.93), for every  $t \in (0, T)$  there exists a subsequence (*a priori* depending on  $t$ ) along which  $\{\rho\psi_{n,\varepsilon}(t)\}$  converges in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  to some positive finite Radon measure  $\nu(t)$  (recall Paragraph 2.1.1.2). We aim at identifying, at least for a.e.  $t \in (0, T)$ ,  $\nu(t)$  with  $\rho\psi_\varepsilon(t)$ , so that *a posteriori* the subsequence does not depend on  $t$ . In order to do that, let  $t \in (0, T)$  be a Lebesgue point of  $\psi_\varepsilon(t)$  (e.g. as a curve in  $L^1((\tau, T); L_\rho^2(\mathbb{R}^d))$ ). Given any  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , using (2.2.106) we obtain:

$$\begin{aligned} & \left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, \tau) \phi(x) \rho(x) dx d\tau - \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t) \phi(x) \rho(x) dx d\tau \right| \\ & \leq \int_t^{t+\delta} \left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, \tau) \phi(x) \rho(x) dx - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x, t) \phi(x) \rho(x) dx \right| d\tau \\ & \leq \int_t^{t+\delta} K(\tau - t) \|\psi\|_{1,\rho} \|\rho^{-1}(-\Delta)^s(\phi)\|_\infty d\tau = \frac{\delta^2}{2} K \|\psi\|_{1,\rho} \|\rho^{-1}(-\Delta)^s(\phi)\|_\infty \end{aligned} \quad (2.2.115)$$

for all  $\delta > 0$  sufficiently small. The passage to the limit in (2.2.115) as  $n \rightarrow \infty$  (up to subsequences) yields

$$\left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_\varepsilon(x, \tau) \phi(x) \rho(x) dx d\tau - \delta \int_{\mathbb{R}^d} \phi(x) d\nu(t) \right| \leq \frac{\delta^2}{2} K \|\psi\|_{1,\rho} \|\rho^{-1}(-\Delta)^s(\phi)\|_\infty. \quad (2.2.116)$$

Dividing (2.2.116) by  $\delta$  and letting  $\delta \rightarrow 0$  we find the identity (recall that  $t$  is a Lebesgue point for  $\psi_\varepsilon(t)$ )

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x, t) \phi(x) \rho(x) dx = \int_{\mathbb{R}^d} \phi(x) d\nu(t),$$

which is valid for any  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , whence  $\psi_\varepsilon(x, t) \rho(x) dx = d\nu(t)$ .

We finally prove the claimed properties of  $\psi_\varepsilon$ . Letting  $n \rightarrow \infty$  in (2.2.107) (with  $t^* = T$  and  $t_* = t$ ) and using the just proved convergence of  $\{\rho\psi_{n,\varepsilon}(t)\}$  to  $\rho\psi_\varepsilon(t)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ , we get:

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi(x) \xi_R(x) \rho(x) dx - \int_{\mathbb{R}^d} \psi_\varepsilon(x, t) \xi_R(x) \rho(x) dx \right| \\ & \leq K(T-t) \|\psi\|_{1,\rho} c^{-1} \left( \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} + \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \right) \end{aligned} \quad (2.2.117)$$

for a.e.  $t \in (0, T)$ . Letting  $R \rightarrow \infty$  in (2.2.117) we then deduce (2.2.112). Thanks to (2.2.104) and (2.2.112), in particular

$$\lim_{n \rightarrow \infty} \|\psi_{n,\varepsilon}(t)\|_{1,\rho} = \|\psi_\varepsilon(t)\|_{1,\rho},$$

so that the convergence of  $\{\rho\psi_{n,\varepsilon}(t)\}$  to  $\rho\psi_\varepsilon(t)$  takes place in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  as well. Recalling that  $g(t)$  belongs to  $C_b(\mathbb{R}^d)$  (Lemma 2.2.16), we can let  $n \rightarrow \infty$  in (2.2.99) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx - \int_{\mathbb{R}^d} g(x, t) \psi_\varepsilon(x, t) \rho(x) dx \\ & = \lim_{n \rightarrow \infty} \int_t^T \int_{\mathbb{R}^d} (a_n(x, \tau) + \varepsilon - a(x, \tau)) (-\Delta)^s(g)(x, \tau) \psi_{n,\varepsilon}(x, \tau) dx d\tau \\ & = \lim_{n \rightarrow \infty} \int_t^T \int_{\mathbb{R}^d} (a_n(x, \tau) + \varepsilon - a(x, \tau)) (u_2(x, \tau + h) - u_1(x, \tau)) \psi_{n,\varepsilon}(x, \tau) \rho(x) dx d\tau \\ & = \varepsilon \int_t^T \int_{\mathbb{R}^d} (u_2(x, \tau + h) - u_1(x, \tau)) \psi_\varepsilon(x, \tau) \rho(x) dx d\tau \quad \text{for a.e. } t \in (0, T), \end{aligned} \quad (2.2.118)$$

where in the last integral we are allowed to pass to the limit since  $\{\psi_{n,\varepsilon}\}$  converges to  $\psi_\varepsilon$  weakly in  $L^2_\rho(\mathbb{R}^d \times (t, T))$ ,  $\{a_n\}$  converges to  $a$  pointwise and it is uniformly bounded in  $L^\infty(\mathbb{R}^d \times (t, T))$  and  $u_1, u_2$  belong to  $L^p_\rho(\mathbb{R}^d \times (t, T+h))$  for all  $p \in [1, \infty]$ . In particular, (2.2.118) and (2.2.112) entail (2.2.114). In a similarly way, we can pass to the limit in (2.2.105) (which actually holds for any  $t_*, t^* \in (0, T)$ ) to get (2.2.113).  $\square$

### 2.2.3.3 Passing to the limit as $\varepsilon \rightarrow 0$ and proof of Theorem 2.2.4

We are now in position to prove Theorem 2.2.4, using the strategy of [148]: we give some detail for the reader's convenience.

*Proof of Theorem 2.2.4.* To begin with, we introduce the Riesz potential  $H_\varepsilon$  of  $\rho\psi_\varepsilon$ , that is  $H_\varepsilon(x, t) := [I_{2s} * \rho\psi_\varepsilon(t)](x)$ . Since we only know that  $\rho\psi_\varepsilon(t) \in L^1(\mathbb{R}^d)$ , we have no information over the integrability of  $H_\varepsilon(t)$  other than  $L^1_{\text{loc}}(\mathbb{R}^d)$  (by classical results, see e.g. [121, p. 61]).

Nevertheless, exploiting (2.2.113) and proceeding once again as in the proof of (2.2.61), we obtain the identity

$$I_{2s} * \rho\psi - H_\varepsilon(t) = \int_t^T (a(\tau) + \varepsilon) \psi_\varepsilon(\tau) d\tau \geq 0 \quad \text{for a.e. } t \in (0, T),$$

whence, in particular,

$$0 \leq H_\varepsilon(x, t_1) \leq H_\varepsilon(x, t_2) \leq H_\varepsilon(x, T) = (I_{2s} * \rho\psi)(x) \quad (2.2.119)$$

for a.e.  $0 < t_1 \leq t_2 \leq T$  and a.e.  $x \in \mathbb{R}^d$ . In view of the above inequality we deduce that  $H_\varepsilon(t)$  belongs to  $L^p(\mathbb{R}^d)$  at least for the same  $p$  for which  $H_\varepsilon(T)$  does, namely for any  $p \in (d/(d-2), \infty]$ .

Our next goal is to let  $\varepsilon \rightarrow 0$  (along a fixed sequence whose index for the moment we omit, in order to simplify readability). Thanks to the boundedness of  $\{\rho\psi_\varepsilon(t)\}$  in  $L^1(\mathbb{R}^d)$  (trivial consequence of (2.2.112)), for a.e.  $t \in (0, T)$  there exists a subsequence  $\{\varepsilon_n\}$ , *a priori* depending on  $t$ , such that  $\{\rho\psi_{\varepsilon_n}(t)\}$  converges to a positive finite Radon measure  $\nu(t)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ . In order to overcome the possible dependence of  $\{\varepsilon_n\}$  on  $t$ , it is convenient to exploit the properties of  $\{H_\varepsilon\}$ . First of all, notice that (2.2.119) ensures the uniform boundedness of  $\{H_\varepsilon\}$  in  $L^p(\mathbb{R}^d \times (0, T))$  for any  $p \in (d/(d-2), \infty]$ . This entails the existence of a decreasing subsequence  $\{\varepsilon_m\}$  such that  $\{H_{\varepsilon_m}\}$  converges weakly in  $L^p(\mathbb{R}^d \times (0, T))$  to a suitable limit function  $H$ . By Mazur's Lemma, there exists a sequence  $\{H_k\}$  of convex combinations of  $\{H_{\varepsilon_m}\}$  that converges *strongly* to  $H$  in  $L^p(\mathbb{R}^d \times (0, T))$ . By definition,  $\{H_k\}$  is of the form

$$H_k = \sum_{m=1}^{M_k} \lambda_{m,k} H_{\varepsilon_m}, \quad \sum_{m=1}^{M_k} \lambda_{m,k} = 1$$

for some sequence  $\{M_k\} \subset \mathbb{N}$  and a suitable choice of the coefficients  $\lambda_{m,k} \in [0, 1]$ . With no loss of generality we shall assume that

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{M_k} \varepsilon_m \lambda_{m,k} = 0.$$

This can be readily justified by applying iteratively Mazur's Lemma on suitable subsequences of  $\{H_{\varepsilon_m}\}$ , exploiting the fact that also the sequence  $\{H_{\varepsilon_{m+\ell}}\}$  converges to  $H$  weakly in  $L^p(\mathbb{R}^d \times (0, T))$  for all  $\ell \in \mathbb{N}$ . Now notice that the function whose Riesz potential is  $H_k$  is just

$$f_k(x, t) := \sum_{m=1}^{M_k} \lambda_{m,k} \rho(x) \psi_{\varepsilon_m}(x, t).$$

Multiplying (2.2.114) (with  $\varepsilon = \varepsilon_m$ ) by  $\lambda_{m,k}$  and summing over  $k$ , one finds that  $f_k$  satisfies the estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx - \int_{\mathbb{R}^d} g(x, t) f_k(x, t) dx \right| \\ & \leq \left( \sum_{m=1}^{M_k} \varepsilon_m \lambda_{m,k} \right) (T - t) \|\psi\|_{1,\rho} \|u_2(\cdot + h) - u_1\|_{L^\infty(\mathbb{R}^d \times (t, T))} \end{aligned} \quad (2.2.120)$$

for a.e.  $t \in (0, T)$ , whereas from (2.2.112) and (2.2.117) we infer that

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi(x) \xi_R(x) \rho(x) dx - \int_{\mathbb{R}^d} f_k(x, t) \xi_R(x) dx \right| \\ & \leq K(T - t) \|\psi\|_{1,\rho} c^{-1} \left( \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} + \frac{\||x|^\gamma(-\Delta)^s(\xi)\|_\infty}{R^{2s-\gamma}} \right) \end{aligned} \quad (2.2.121)$$

and

$$\int_{\mathbb{R}^d} \psi(x) \rho(x) dx = \int_{\mathbb{R}^d} f_k(x, t) dx \quad (2.2.122)$$

for a.e.  $t \in (0, T)$ . Letting  $k \rightarrow \infty$  we find that, for almost every  $t \in (0, T)$ , there exists a subsequence of  $\{f_k(t)\}$  (*a priori* depending on  $t$ ) that converges in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  to a positive finite Radon measure  $\nu(t)$ . Nonetheless, the fact that  $\{H_k\}$  converges strongly in  $L^p(\mathbb{R}^d \times (0, T))$  to  $H$  forces the potential of  $\nu(t)$  to coincide a.e. with  $H(t)$ . This is a consequence *e.g.* of [121, Theorem 3.8]. By [121, Theorem 1.12] we therefore deduce that the limit  $\nu(t)$  is uniquely determined by its potential  $H(t)$ . This identification allows us to assert that the *whole* sequence  $\{f_k(t)\}$  converges to  $\nu(t)$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ , independently of  $t$ . Passing to the limit in (2.2.119) (after having set  $\varepsilon = \varepsilon_m$ , multiplied by  $\lambda_{m,k}$  and summed over  $k$ ) we deduce that also the potentials  $H(t)$  of  $\nu(t)$  are ordered and bounded above by  $I_{2s} * \rho\psi$ :

$$0 \leq H(x, t_1) \leq H(x, t_2) \leq (I_{2s} * \rho\psi)(x) \quad \text{for a.e. } 0 < t_1 \leq t_2 \leq T, \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.2.123)$$

Letting  $k \rightarrow \infty$  in (2.2.121) we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \psi(x) \xi_R(x) \rho(x) dx - \int_{\mathbb{R}^d} \xi_R(x) d\nu(t) \right| \\ & \leq K(T-t) \|\psi\|_{1,\rho} c^{-1} \left( \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} + \frac{\||x|^\gamma(-\Delta)^s(\xi)\|_\infty}{R^{2s-\gamma}} \right) \end{aligned} \quad (2.2.124)$$

for a.e.  $t \in (0, T)$ , whence, letting  $R \rightarrow \infty$ ,

$$\int_{\mathbb{R}^d} \psi(x) \rho(x) dx = \int_{\mathbb{R}^d} d\nu(t) \quad \text{for a.e. } t \in (0, T). \quad (2.2.125)$$

Gathering (2.2.122) and (2.2.125) we infer that  $\{f_k(t)\}$  converges to  $\nu(t)$  also in the stronger sense  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ : this permits us to pass to the limit in (2.2.120) to get the identity

$$\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx = \int_{\mathbb{R}^d} g(x, t) d\nu(t) \quad \text{for a.e. } t \in (0, T). \quad (2.2.126)$$

As a consequence of the monotonicity in time of  $H(x, t)$  and thanks to (2.2.124)–(2.2.126), the curve  $\nu(t)$  can actually be extended to *every*  $t \in (0, T]$  so that it still complies with (2.2.123)–(2.2.126) (one uses again *e.g.* [121, Theorem 3.8]). Recalling that  $g(x, t) = U_2(x, t+h) - U_1(x, t)$  and that  $U_1, U_2$  are continuous in space and do not increase in time, we have that  $g(x, t) \leq U_2(x, h) - U_1(x, t_0)$  for every  $x \in \mathbb{R}^d$  and  $t_0 > t$ . Because  $\nu(t)$  is a positive finite measure, this fact and (2.2.126) imply that

$$\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx \leq \int_{\mathbb{R}^d} (U_2(x, h) - U_1(x, t_0)) d\nu(t) \quad \forall t_0 > t. \quad (2.2.127)$$

The next step is to let  $t$  tend to zero in (2.2.127). Since the mass of  $\nu(t)$  is constant (formula (2.2.125)), up to subsequences  $\nu(t)$  converges to a suitable positive finite Radon measure  $\nu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ . Moreover, by (2.2.123), we know that the potentials  $H(t)$  of  $\nu(t)$  are nondecreasing in  $t$  (for a.e.  $x$ ): in particular,  $H(t)$  admits a pointwise limit almost everywhere  $H_0$  as  $t \downarrow 0$ . Theorem 3.8 of [121] ensures that  $H_0$  coincides almost everywhere with the potential of the limit measure  $\nu$  (which therefore does not depend on the particular sequence  $t_k \downarrow 0$ ). We can then pass to the limit in the integral

$$\int_{\mathbb{R}^d} U_1(x, t_0) d\nu(t). \quad (2.2.128)$$

Indeed, by Fubini's Theorem, (2.2.128) is equal to

$$\int_{\mathbb{R}^d} u_1(x, t_0) H(x, t) \rho(x) dx. \quad (2.2.129)$$

Letting  $t \downarrow 0$  in (2.2.129) we get that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^d} u_1(x, t_0) H(x, t) \rho(x) dx = \int_{\mathbb{R}^d} u_1(x, t_0) H_0(x) \rho(x) dx \quad (2.2.130)$$

*e.g.* by dominated convergence, since  $H(t)$  is nonincreasing as  $t \downarrow 0$  and converges a.e. to  $H_0$ . Recalling that  $H_0$  is the potential of  $\nu$ , and using again Fubini's Theorem, (2.2.130) can be rewritten as

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^d} U_1(x, t_0) d\nu(t) = \int_{\mathbb{R}^d} U_1(x, t_0) d\nu.$$

One proceeds similarly for the integral

$$\int_{\mathbb{R}^d} U_2(x, h) d\nu(t).$$

Hence, passing to the limit as  $t \downarrow 0$  in (2.2.127), we find that

$$\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx \leq \int_{\mathbb{R}^d} (U_2(x, h) - U_1(x, t_0)) d\nu \quad \forall t_0 > 0. \quad (2.2.131)$$

Now we let  $t_0 \downarrow 0$  in (2.2.131). By monotone convergence (see Lemma 2.2.17) we obtain

$$\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx \leq \int_{\mathbb{R}^d} (U_2(x, h) - U^\mu(x)) d\nu. \quad (2.2.132)$$

In this step it is crucial that the limit of  $U_1(x, t_0)$  to  $U^\mu(x)$  is taken *for every*  $x \in \mathbb{R}^d$ , because we have no information over  $\nu$  besides the fact that it is a positive finite Radon measure. Still monotonicity implies that  $U_2(x, h) \leq U^\mu(x)$  for every  $x \in \mathbb{R}^d$ . Thus, from (2.2.132) there follows

$$\int_{\mathbb{R}^d} g(x, T) \psi(x) \rho(x) dx \leq 0. \quad (2.2.133)$$

Since (2.2.133) holds true for any  $h, T > 0$  and any nonnegative  $\psi \in \mathcal{D}(\mathbb{R}^d)$ , we infer that  $U_2 \leq U_1$ . Swapping  $u_1$  and  $u_2$  we get that  $U_1 \leq U_2$ , whence  $U_1 = U_2$  and  $u_1 = u_2$ .  $\square$

**Remark 2.2.23.** We point out that the proofs of Lemmas 2.2.19, 2.2.21, 2.2.22 and Theorem 2.2.4 actually work under the less restrictive assumption  $\gamma_0 \in [0, 2s)$ . However, since in the existence Theorem 2.2.2 we needed the stronger hypothesis  $\gamma_0 \in [0, \gamma]$ , we preferred to leave it unmodified in the statement of Theorem 2.2.4.

### 2.2.4 Asymptotic behaviour

Take a positive finite Radon measure  $\mu$ , with  $\mu(\mathbb{R}^d) =: M > 0$ , and denote as  $u_M$  the solution to the singular problem (2.1.10), where  $c_\infty$  is the positive constant appearing in (2.2.7). Upon defining the positive parameters  $\alpha, \kappa$  as in (2.2.6) and (2.2.10), it is immediate to check that, for any given  $\lambda > 0$ , the function

$$u_{M,\lambda}(x, t) := \lambda^\alpha u_M(\lambda^\kappa x, \lambda t)$$

is still a solution to (2.1.10). Hence, as a consequence of our uniqueness Theorem 2.2.4,  $u_{M,\lambda}$  and  $u_M$  must necessarily coincide, that is

$$u_M(x, t) = \lambda^\alpha u_M(\lambda^\kappa x, \lambda t) \quad \forall t, \lambda > 0, \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.2.134)$$

Note that the validity of (2.2.134) for every  $t, \lambda > 0$  is a consequence of the fact that solutions are strong, see Paragraph 2.2.2.1. As we shall see below, in view of this self-similarity identity the special solution  $u_M$  will be key in the study of the asymptotic behaviour of any solution to (2.1.4).

**Lemma 2.2.24.** *Let  $d > 2s$ . Assume that  $\rho$  satisfies (2.1.3) and (2.2.7) for some  $\gamma \in [0, 2s] \cap [0, d - 2s]$  and  $\gamma_0 \in [0, \gamma]$ . Let  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , with  $u_0 \geq 0$  and  $\|u_0\|_{1,\rho} =: M$ . Let  $u$  be the weak solution to (2.2.14) provided by Lemma 2.2.9. For any  $\lambda > 0$ , set*

$$u_\lambda(x, t) := \lambda^\alpha u(\lambda^\kappa x, \lambda t).$$

*Then, given any sequence  $\lambda_n \rightarrow \infty$ ,  $\{u_{\lambda_n}\}$  converges to  $u_M$  almost everywhere in  $\mathbb{R}^d \times (0, \infty)$  along subsequences.*

*Proof.* For notational simplicity, and with no loss of generality, we shall only consider the case  $c_\infty = 1$ . We shall not give a fully detailed proof, since the procedure follows closely the one performed in the proof of Theorem 2.2.2.

To begin with, let us notice that  $u_\lambda$  solves

$$\begin{cases} \rho_\lambda(x)u_t + (-\Delta)^s(u_\lambda^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ u_\lambda = u_{0\lambda} & \text{on } \mathbb{R}^d \times \{0\}, \end{cases}$$

where

$$\rho_\lambda(x) := \lambda^{\kappa\gamma} \rho(\lambda^\kappa x), \quad u_{0\lambda}(x) := \lambda^\alpha u_0(\lambda^\kappa x).$$

Since  $\alpha = (d - \gamma)\kappa$ , due to the conservation of mass (2.2.11) there holds

$$\|u_\lambda(t)\|_{1,\rho_\lambda} = \|u_{0\lambda}\|_{1,\rho_\lambda} = M \quad \forall t, \lambda > 0. \quad (2.2.135)$$

**Claim 1:** *There exists a subsequence  $\{u_{\lambda_m}\} \subset \{u_{\lambda_n}\}$  that converges pointwise a.e. in  $\mathbb{R}^d \times \mathbb{R}^+$  to some function  $u$ , which satisfies (2.2.1), (2.2.2) and (2.2.3) with  $\rho(x) = |x|^{-\gamma}$ .*

First of all observe that, in view of (2.1.3),

$$\frac{c}{1 + |x|^\gamma} \leq \rho_\lambda(x) \leq \frac{C}{|x|^\gamma} \quad \forall \lambda \geq 1. \quad (2.2.136)$$

Combining the smoothing effect (2.2.43) with (2.2.135), we obtain:

$$\|u_\lambda(t)\|_\infty \leq K t^{-\alpha} M^\beta \quad \forall t, \lambda > 0, \quad (2.2.137)$$

where  $K > 0$  is a constant depending only on  $m, C, \gamma, d$  and  $s$ . In particular,

$$\int_{\mathbb{R}^d} u_\lambda^{m+1}(x, t) \rho_\lambda(x) dx \leq K^m t^{-\alpha m} M^{\beta m+1} \quad \forall t, \lambda > 0. \quad (2.2.138)$$

By (2.2.27) and (2.2.138), we infer that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}(u_\lambda^m)(x, t)|^2 dx dt + \frac{1}{m+1} \int_{\mathbb{R}^d} u_\lambda^{m+1}(x, t_2) \rho_\lambda(x) dx \leq \frac{K^m}{m+1} t_1^{-\alpha m} M^{\beta m+1} \quad (2.2.139)$$

for all  $\lambda > 0$  and all  $t_2 > t_1 > 0$ . On the other hand, thanks to (2.2.28),

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(z_\lambda)_t(x, t)|^2 \rho_\lambda(x) dx dt \leq C' \quad \forall t_2 > t_1 > 0, \quad \forall \lambda > 0, \quad (2.2.140)$$

where  $z_\lambda := u_\lambda^{(m+1)/2}$  and  $C'$  is another positive constant depending on  $m, \gamma, d, s, t_1, t_2, K, M$  but independent of  $\lambda$ . In view of (2.2.135)–(2.2.140), by standard compactness arguments (see the proofs of Lemma 2.2.9 and Theorem 2.2.2) the sequence  $\{u_{\lambda_m}\}$  admits a subsequence  $\{u_{\lambda_m}\}$  converging a.e. in  $\mathbb{R}^d \times \mathbb{R}^+$  to some function  $u$  that complies with (2.2.1) and (2.2.2). Moreover, recalling the assumptions on  $\rho$ , we have that (2.2.136) holds true and

$$\lim_{\lambda \rightarrow \infty} \rho_\lambda(x) = |x|^{-\gamma} \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.2.141)$$

We are therefore allowed to pass to the limit in the weak formulation solved by  $u_{\lambda_m}$  to find that  $u$  also satisfies (2.2.3), and Claim 1 is proved. In order to deal with the initial trace of  $u$ , it is convenient to introduce the Riesz potential  $U_\lambda$  of  $\rho_\lambda u_\lambda$ , that is  $U_\lambda(x, t) := [I_{2s} * \rho_\lambda u_\lambda(t)](x)$ .

**Claim 2:** For any  $\lambda > 0$ , the function  $U_\lambda$  satisfies

$$\int_{\mathbb{R}^d} U_\lambda(x, t_2) \phi(x) dx - \int_{\mathbb{R}^d} U_\lambda(x, t_1) \phi(x) dx = - \int_{\mathbb{R}^d} \left( \int_{t_1}^{t_2} u_\lambda^m(x, t) dt \right) \phi(x) dx \quad (2.2.142)$$

for all  $t_2 > t_1 > 0$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ .

Note that the above identity is just (2.2.61), with  $u_\varepsilon$  replaced by  $u_\lambda$ .

**Claim 3:** For any  $\lambda > 0$ , let  $U_{0\lambda} := I_{2s} * \rho_\lambda u_{0\lambda}$ . Then,

$$\left| \int_{\mathbb{R}^d} U_\lambda(x, t_2) \phi(x) dx - \int_{\mathbb{R}^d} U_{0\lambda}(x) \phi(x) dx \right| \leq \left\| \rho_\lambda^{-1} \phi \right\|_\infty K^{m-1} M^{1+\beta(m-1)} \int_0^{t_2} t^{-\alpha(m-1)} dt \quad (2.2.143)$$

for all  $t_2 > 0$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ .

The validity of (2.2.143) is just a consequence of (2.2.135), (2.2.137), (2.2.142) and of the definition of weak solution (see the proof of (2.2.65)).

**Claim 4:** The potential  $U$  of  $|x|^{-\gamma} u$ , that is  $U(x, t) := [I_{2s} * |\cdot|^{-\gamma} u(\cdot, t)](x)$ , satisfies

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} U(x, t_2) \phi(x) dx - \int_{\mathbb{R}^d} M I_{2s}(x) \phi(x) dx \right| \\ & \leq c^{-1} \left\| (1 + |x|^\gamma) \phi \right\|_\infty K^{m-1} M^{1+\beta(m-1)} \int_0^{t_2} t^{-\alpha(m-1)} dt \end{aligned} \quad (2.2.144)$$

for a.e.  $t_2 > 0$  and  $\phi \in \mathcal{D}(\mathbb{R}^d)$ .

Our goal is to let  $\lambda \rightarrow \infty$  in (2.2.143). In the r.h.s. we just exploit (2.2.136). Thanks to Claim 1, (2.2.135), (2.2.136), (2.2.137) and (2.2.141) we infer that

$$\lim_{m \rightarrow \infty} \rho_{\lambda_m} u_{\lambda_m}(t) = |x|^{-\gamma} u(t) \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_0(\mathbb{R}^d))$$

for a.e.  $t > 0$ . This is enough in order to pass to the limit in the first integral in the l.h.s. of (2.2.143) (for details, we refer the reader to the proof of (2.2.67)). The same holds true for the second integral, provided we can prove that  $\{\rho_\lambda u_{0\lambda}\}$  tends to  $M\delta$  e.g. in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$  as  $\lambda \rightarrow \infty$ . This is indeed the case: in fact,  $\|\rho_\lambda u_{0\lambda}\|_1 = M$  and for any  $\phi \in C_c(\mathbb{R}^d)$  one has

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^d} \phi(x) \rho_\lambda(x) u_{0\lambda}(x) dx \\ & = \lim_{\lambda \rightarrow \infty} \lambda^{\alpha+\kappa\gamma} \int_{\mathbb{R}^d} \phi(x) \rho(\lambda^\kappa x) u_0(\lambda^\kappa x) dx = \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^d} \phi(y/\lambda^\kappa) u_0(y) \rho(y) dy = M\phi(0). \end{aligned}$$

**Claim 5:** There holds

$$\lim_{t \rightarrow 0} |x|^{-\gamma} u(t) = M\delta \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d)). \quad (2.2.145)$$



Passing to the limit in (2.2.135) as  $\lambda = \lambda_m \rightarrow \infty$  we get

$$\| |x|^{-\gamma} u(t) \|_1 \leq M \quad \text{for a.e. } t > 0. \quad (2.2.146)$$

Estimate (2.2.146) implies, in particular, that  $|x|^{-\gamma} u(t)$  converges, up to subsequences, to some positive finite Radon measure  $\nu$  in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$  as  $t \rightarrow 0$ . In view of (2.2.144) we know that  $U(t)$  converges to  $MI_{2s} = I_{2s} * M\delta$  e.g. in  $L^1_{\text{loc}}(\mathbb{R}^d)$  as  $t \rightarrow 0$ , which entails  $\nu = M\delta$  (see the end of proof of Theorem 2.2.2). We have therefore proved (2.2.145) at least in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ . In order to recover convergence in  $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ , it suffices to show that

$$\lim_{t \rightarrow 0} \| |x|^{-\gamma} u(t) \|_1 = M;$$

but this is a consequence of (2.2.146) and weak\* lower semi-continuity.

From Claims 1 and 5 we conclude that  $u$  solves (2.1.10) in the sense of Definition 2.2.1, and therefore coincides with  $u_M$  in view of the uniqueness Theorem 2.2.4.  $\square$

We are now in position to prove Theorem 2.2.5.

*Proof of Theorem 2.2.5.* With no loss of generality, we shall suppose that  $\mu = \rho u_0$  with  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ : in fact, thanks to the smoothing effect (2.2.5), we know that solutions become instantaneously  $L^1_\rho \cap L^\infty$  functions.

Take any sequence  $\lambda_n \rightarrow \infty$ . We first aim at proving that, along any of the subsequences  $\{\lambda_m\} \subset \{\lambda_n\}$  given by Lemma 2.2.24, there holds

$$\lim_{m \rightarrow \infty} \int_{B_R} |u_{\lambda_m}(x, t) - u_M(x, t)| |x|^{-\gamma} dx = 0 \quad \forall R > 0, \quad \forall t > 0. \quad (2.2.147)$$

Thanks to the smoothing estimates (2.2.5), (2.2.137) and to the fact that for almost every  $t > 0$  we know that  $\{u_{\lambda_m}(t)\}$  converges pointwise almost everywhere to  $u_M(t)$ , by dominated convergence

$$\lim_{m \rightarrow \infty} \int_{B_R} |u_{\lambda_m}(x, t) - u_M(x, t)| dx = 0 \quad \forall R > 0, \quad \text{for a.e. } t > 0. \quad (2.2.148)$$

Moreover, estimate (2.2.80) for  $u_\lambda$  reads

$$\|(u_\lambda)_t(t)\|_{1, \rho_\lambda} \leq \frac{2M}{(m-1)t} \quad \text{for a.e. } t > 0. \quad (2.2.149)$$

Gathering (2.2.149) and (2.2.136), we can assert that for every  $R, \tau > 0$  there exists a positive constant  $C(R, \tau)$  (independent of  $\lambda$ ) such that

$$\|(u_\lambda)_t(t)\|_{L^1(B_R)} \leq C(R, \tau) \quad \text{for a.e. } t \geq \tau. \quad (2.2.150)$$

Of course (2.2.150) also holds for  $u_M$ . It is then possible to infer that (2.2.148) is actually valid for *every*  $t > 0$ :

$$\lim_{m \rightarrow \infty} \int_{B_R} |u_{\lambda_m}(x, t) - u_M(x, t)| dx = 0 \quad \forall R > 0, \quad \forall t > 0. \quad (2.2.151)$$

In fact, for any fixed  $t_0, \varepsilon > 0$ , there exists  $t > t_0$  such that (2.2.148) holds true and  $|t - t_0| \leq \varepsilon$ . Exploiting (2.2.150), we get:

$$\int_{B_R} |u_{\lambda_m}(x, t_0) - u_M(x, t_0)| dx \leq 2C(R, t_0)\varepsilon + \int_{B_R} |u_{\lambda_m}(x, t) - u_M(x, t)| dx,$$

whence

$$\limsup_{m \rightarrow \infty} \int_{B_R} |u_{\lambda_m}(x, t_0) - u_M(x, t_0)| dx \leq 2C(R, t_0) \varepsilon. \quad (2.2.152)$$

Letting now  $\varepsilon \rightarrow 0$  in (2.2.152) we deduce that (2.2.148) holds for  $t = t_0$  as well. The validity of (2.2.147) is then just a consequence of (2.2.151), the local integrability of  $|x|^{-\gamma}$  and the uniform bound over  $\|u_{\lambda_m}(t) - u_M(t)\|_\infty$  ensured by the smoothing estimates (2.2.5) and (2.2.137).

The consequence of Lemma 2.2.24 and what we proved above is that *any* sequence  $\lambda_n \rightarrow \infty$  satisfies (2.2.147) along subsequences. Hence,

$$\lim_{\lambda \rightarrow \infty} \int_{B_R} |u_\lambda(x, t) - u_M(x, t)| |x|^{-\gamma} dx = 0 \quad \forall R > 0, \quad \forall t > 0. \quad (2.2.153)$$

Letting  $t = 1$ , relabelling  $\lambda$  as  $t$  and recalling the definition of  $u_\lambda$ , we have that (2.2.153) reads

$$\lim_{t \rightarrow \infty} \int_{B_R} |t^\alpha u(t^\kappa x, t) - u_M(x, 1)| |x|^{-\gamma} dx = 0 \quad \forall R > 0.$$

Performing the change of variable  $y = t^\kappa x$  and using the fact that  $\alpha + \kappa(\gamma - d) = 0$ , we obtain:

$$\lim_{t \rightarrow \infty} \int_{B_{Rt^\kappa}} |u(y, t) - t^{-\alpha} u_M(t^{-\kappa} y, 1)| |y|^{-\gamma} dy = \lim_{t \rightarrow \infty} \int_{B_{Rt^\kappa}} |u(y, t) - u_M(y, t)| |y|^{-\gamma} dy = 0 \quad (2.2.154)$$

for all  $R > 0$ , where we used (2.2.134) with  $\lambda = t^{-1}$ .

From now on we shall denote as  $\varepsilon_R$  any function of the spatial variable (possibly constant) which is independent of  $t$  and vanishes uniformly as  $R \rightarrow \infty$ . Going back to the original variable  $x = t^{-\kappa} y$ , we find that

$$\int_{B_{Rt^\kappa}^c} u_M(y, t) |y|^{-\gamma} dy = \int_{B_R^c} u_M(x, 1) |x|^{-\gamma} dx = \varepsilon_R \quad \forall R > 0. \quad (2.2.155)$$

Therefore, the conservation of mass for  $u_M$ , (2.2.154) and (2.2.155) imply that

$$\lim_{t \rightarrow \infty} \int_{B_{Rt^\kappa}^c} u(y, t) |y|^{-\gamma} dy = M c_\infty^{-1} - \varepsilon_R \quad \forall R > 0. \quad (2.2.156)$$

Next we show that in fact

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^d} u(y, t) |y|^{-\gamma} dy = M c_\infty^{-1}. \quad (2.2.157)$$

To this end first notice that, thanks to (2.2.7), there holds

$$|y|^{-\gamma} = \frac{\rho(y)}{c_\infty + \varepsilon_R(y)} \quad \text{for a.e. } y \in B_R^c,$$

whence

$$\int_{\mathbb{R}^d} u(y, t) |y|^{-\gamma} dy = \int_{B_R} u(y, t) |y|^{-\gamma} dy + \int_{B_R^c} u(y, t) \frac{\rho(y)}{c_\infty + \varepsilon_R(y)} dy. \quad (2.2.158)$$

Due to (2.2.158) and the conservation of mass (2.2.11) for  $u$ , we get:

$$\begin{aligned} \left| \int_{\mathbb{R}^d} u(y, t) |y|^{-\gamma} dy - M c_\infty^{-1} \right| &= \left| \int_{\mathbb{R}^d} u(y, t) |y|^{-\gamma} dy - \int_{\mathbb{R}^d} u(y, t) \frac{\rho(y)}{c_\infty} dy \right| \\ &\leq \int_{B_R} u(y, t) |y|^{-\gamma} dy \\ &\quad + \int_{B_R} u(y, t) \frac{\rho(y)}{c_\infty} dy + \frac{\|\varepsilon_R\|_\infty}{c_\infty(c_\infty - \|\varepsilon_R\|_\infty)} \int_{B_R^c} u(y, t) \rho(y) dy. \end{aligned}$$

Letting  $t \rightarrow \infty$ , using the smoothing effect (2.2.5) (as a decay estimate) and the fact that both  $\rho(y)$  and  $|y|^{-\gamma}$  are locally integrable, we end up with

$$\limsup_{t \rightarrow \infty} \left| \int_{\mathbb{R}^d} u(y, t) |y|^{-\gamma} dy - M c_\infty^{-1} \right| \leq \frac{M \|\varepsilon_R\|_\infty}{c_\infty (c_\infty - \|\varepsilon_R\|_\infty)},$$

which entails (2.2.157). It is apparent that

$$\begin{aligned} \int_{\mathbb{R}^d} |u(y, t) - u_M(y, t)| |y|^{-\gamma} dy &\leq \int_{B_{Rt^\kappa}} |u(y, t) - u_M(y, t)| |y|^{-\gamma} dy \\ &\quad + \int_{B_{Rt^\kappa}^c} u(y, t) |y|^{-\gamma} dy + \int_{B_{Rt^\kappa}^c} u_M(y, t) |y|^{-\gamma} dy. \end{aligned} \quad (2.2.159)$$

Moreover, (2.2.156) and (2.2.157) yield

$$\lim_{t \rightarrow \infty} \int_{B_{Rt^\kappa}^c} u(y, t) |y|^{-\gamma} dy = \varepsilon_R. \quad (2.2.160)$$

Collecting (2.2.154), (2.2.155), (2.2.159) and (2.2.160) we finally get

$$\limsup_{t \rightarrow \infty} \int_{\mathbb{R}^d} |u(y, t) - u_M(y, t)| |y|^{-\gamma} dy \leq 2\varepsilon_R,$$

whence (2.2.8). The validity of (2.2.9) is just a consequence of (2.2.8) and the change of variable  $y = t^\kappa x$  (one exploits again the self-similar property (2.2.134) of  $u_M$ ).  $\square$

## 2.3 Rapidly decaying densities

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This section is focussed on the asymptotic behaviour of nonnegative solutions to (2.1.1) when  $\rho(x)$  decays sufficiently fast as  $|x| \rightarrow \infty$ . A related fractional sublinear elliptic problem is also studied in detail.

### 2.3.1 Statements of the main results

We start by providing a suitable notion of weak solution to (2.1.1). In the following, we shall always assume  $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  with  $\rho \geq 0$  and  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ .

**Definition 2.3.1.** *A nonnegative function  $u$  is a weak solution to problem (2.1.1) corresponding to the nonnegative initial datum  $u_0 \in L_\rho^1(\mathbb{R}^d)$  if:*

- $u \in C([0, \infty); L_\rho^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty))$  for all  $\tau > 0$ ;
- $u^m \in L_{\text{loc}}^2((0, \infty); \dot{H}^s(\mathbb{R}^d))$ ;
- for any  $\varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$  there holds

$$\int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt = 0;$$

- $\lim_{t \rightarrow 0} u(t) = u_0$  in  $L_\rho^1(\mathbb{R}^d)$ .

In agreement with the discussion in Paragraph 2.2.2.1, we say that  $u$  is a *strong solution* if, in addition,  $u_t \in L^\infty((\tau, \infty); L_\rho^1(\mathbb{R}^d))$  for every  $\tau > 0$ .

It is plain that for  $\mu = \rho u_0 \in L^1(\mathbb{R}^d)$  a solution to (2.1.1) with respect to Definition 2.3.1 is also a solution to (2.1.4) with respect to Definition 2.2.1. However, by means of the latter we are able to handle more general initial data (positive finite Radon measures). In particular, we cannot ask  $u \in C([0, \infty), L_\rho^1(\mathbb{R}^d))$ . Nevertheless, in view of the fundamental uniqueness

Theorem 2.2.4 we proved above, when  $\mu = \rho u_0 \in L^1(\mathbb{R}^d)$  such solutions do coincide, provided  $\rho, \gamma, \gamma_0, d$  and  $s$  meet the corresponding assumptions.

Existence and uniqueness of weak solutions to (2.1.1), which can be established by means of relatively standard techniques (see *e.g.* [60, 61, 159] and Chapter 1, Subsection 1.2.2), are discussed in Subsection 2.3.4. In this regard, we have the following result (for a sketch of proof see in particular Paragraphs 2.3.4.1 and 2.3.4.2 below).

**Proposition 2.3.2.** *Let  $d > 2s$ . Let  $\rho \in L^\infty(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ . Take  $u_0 \in L^1_\rho(\mathbb{R}^d)$ , with  $u_0 \geq 0$ . Then there exists a unique weak solution  $u$  to problem (2.1.1), in the sense of Definition 2.3.1, which is also a strong solution.*

When  $\rho$  is a rapidly decaying density, we shall often deal with solutions to (2.1.1) which are meant in a more general sense with respect to Definition 2.3.1, namely what we call *local weak solutions* and *local strong solutions*. The corresponding definitions are a bit technical, and we leave them to Subsection 2.3.4 (Definition 2.3.21). The first result we present below concerns existence and uniqueness of local strong solutions.

**Theorem 2.3.3.** *Let  $d > 2s$ . Let  $\rho \in L^\infty(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ . Assume in addition that  $\rho(x) \leq C|x|^{-\gamma}$  a.e. in  $B_1^c$  for some  $\gamma > 2s$  and  $C > 0$ . Take  $u_0 \in L^1_\rho(\mathbb{R}^d)$ , with  $u_0 \geq 0$ . Then the weak solution to problem (2.1.1) provided by Proposition 2.3.2 is the minimal solution in the class of local strong solutions (according to Definition 2.3.21 below) and satisfies*

$$\mathcal{U}(x, t_0, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (2.3.1)$$

where for any fixed  $t_0 > 0$  and  $t > t_0$  we set

$$\mathcal{U}(x, t_0, t) := \int_{t_0}^t u^m(x, \tau) \, d\tau.$$

More precisely, there holds

$$\mathcal{U}(x, t_0, t) \leq K(I_{2s} * \rho)(x) \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times (t_0, \infty) \quad (2.3.2)$$

and some  $K > 0$ , whence (2.3.1) follows by Lemma 2.3.12 below.

Furthermore, under the more restrictive assumption that  $d > 4s$  and  $\gamma \in (2s, d - 2s] \cup (4s, \infty)$ , the solution is unique in the class of local strong solutions satisfying

$$u^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (t_0, T)) \quad \forall T > t_0 > 0.$$

If in addition  $u_0 \in L^\infty(\mathbb{R}^d)$ , then  $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$  and all the above results hold true with  $t_0 = 0$  as well.

**Remark 2.3.4.** Notice that, as concerns uniqueness, for  $d \geq 6s$  the assumptions on  $\gamma$  amount to  $\gamma > 2s$ .

For the proof of Theorem 2.3.3 we refer the reader to Paragraph 2.3.4.3 in Subsection 2.3.4. As for asymptotics, in Subsection 2.3.3 we shall prove the following result.

**Theorem 2.3.5.** *Let  $d > 2s$ . Let  $\rho \in C^\sigma_{\text{loc}}(\mathbb{R}^d)$  (for some  $\sigma \in (0, 1)$ ) be strictly positive and such that  $\rho(x) \leq C|x|^{-\gamma}$  in  $B_1^c$  for some  $\gamma > 2s$  and  $C > 0$ . Take  $u_0 \in L^1_\rho(\mathbb{R}^d)$ , with  $u_0 \geq 0$  and  $u_0 \not\equiv 0$ . Denote as  $u$  the minimal solution to problem (2.1.1) provided by Theorem 2.3.3 and as  $w$  the nontrivial solution to the fractional sublinear elliptic equation (2.1.12) (with  $\ell = 1/m$ ) provided by Theorem 2.3.10 below, which is also minimal in the class of solutions specified by the corresponding statement. Then,*

$$\lim_{t \rightarrow \infty} t^{\frac{1}{m-1}} u(x, t) = (m-1)^{-\frac{1}{m-1}} w^{\frac{1}{m}}(x) \quad (2.3.3)$$

monotonically and in  $L^p_{\text{loc}}(\mathbb{R}^d)$  for all  $p \in [1, \infty)$ .

**Remark 2.3.6.** Under the same assumptions as in Theorem 2.3.5, with in addition  $d > 4s$  and  $\gamma > 4s$ , thanks to the uniqueness results of Theorem 2.3.3 and Theorem 2.3.11 below, we can read the above asymptotic result as follows: any nontrivial local strong solution  $u$  to (2.1.1) satisfying

$$u^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (t_0, T)) \quad \forall T > t_0 > 0$$

converges, in the sense of (2.3.3), to the unique nontrivial local weak and very weak solution  $w$  to (2.1.12) (with  $\ell = 1/m$ ) satisfying

$$w \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d).$$

### 2.3.2 A fractional sublinear elliptic equation

Here we study the sublinear equation (2.1.12) when  $\rho(x)$  decays fast as  $|x| \rightarrow \infty$ . Such an equation naturally arises from the asymptotic analysis, as we shall see in Subsection 2.3.3.

If  $\varphi$  is a smooth and compactly supported function defined in  $\mathbb{R}^d$ , we can consider its  $s$ -harmonic extension  $E(\varphi)$  to the upper half-space  $\mathbb{R}_+^{d+1} := \{(x, y) : x \in \mathbb{R}^d, y > 0\}$ , namely the unique smooth and bounded solution to the problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla E(\varphi)) = 0 & \text{in } \mathbb{R}_+^{d+1}, \\ E(\varphi) = \varphi & \text{on } \partial\mathbb{R}_+^{d+1} := \mathbb{R}^d \times \{y = 0\}. \end{cases}$$

It is well-known (see *e.g.* [40, 42, 61]) that

$$-\mu_s \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial E(\varphi)}{\partial y}(x, y) = (-\Delta)^s(\varphi)(x) \quad \forall x \in \mathbb{R}^d,$$

where  $\mu_s := \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}$ . It is therefore convenient to introduce the operators

$$L_s := \operatorname{div}(y^{1-2s}\nabla), \quad \frac{\partial}{\partial y^{2s}} := -\mu_s \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial}{\partial y}.$$

Furthermore, we denote as  $X^s$  the completion of  $C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$  w.r.t. the norm

$$\|\psi\|_{X^s} := \left( \mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} |\nabla \psi(x, y)|^2 dx dy \right)^{\frac{1}{2}} \quad \forall \psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}),$$

and as  $X_{\text{loc}}^s$  the space of all functions  $v$  such that  $\psi v \in X^s$  for any  $\psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$ . It is possible to prove that there exists a well defined notion of trace on  $\partial\mathbb{R}_+^{d+1}$  for functions in  $X^s$  (see *e.g.* [36, Section 2], [40, Subsection 3.1] or [61, Subsection 3.2]), and therefore in  $X_{\text{loc}}^s$  as well. Moreover, for every  $v \in \dot{H}^s(\mathbb{R}^d)$  there exists a unique extension  $E(v) \in X^s$  such that

$$E(v)(x, 0) = v(x) \quad \text{for a.e. } x \in \mathbb{R}^d$$

and

$$\mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla E(v), \nabla \psi \rangle(x, y) dx dy = \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) (-\Delta)^s(\psi)(x, 0) dx$$

for any  $\psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$  (that is,  $E(v)$  is the *harmonic extension* of  $v$ ).

Having at our disposal the above tools, we can provide suitable weak formulations of problem (2.1.12) which deal with the harmonic extension. In fact, at a formal level, looking

for a solution  $w$  to (2.1.12) is the same as looking for a pair of functions  $(w, \tilde{w})$  that solve the problem

$$\begin{cases} L_s \tilde{w} = 0 & \text{in } \mathbb{R}_+^{d+1}, \\ \tilde{w} = w & \text{on } \partial\mathbb{R}_+^{d+1}, \\ \frac{\partial \tilde{w}}{\partial y^{2s}} = \rho w^\ell & \text{on } \partial\mathbb{R}_+^{d+1}, \end{cases} \quad (2.3.4)$$

where  $0 < \ell < 1$ .

**Definition 2.3.7.** A nonnegative function  $w \in \dot{H}^s(\mathbb{R}^d)$  is a weak solution to problem (2.1.12) if it satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} w^\ell(x) \psi(x, 0) \rho(x) dx &= \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(w)(x) (-\Delta)^{\frac{s}{2}}(\psi)(x, 0) dx \\ &= \mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla E(w), \nabla \psi \rangle(x, y) dx dy \end{aligned} \quad (2.3.5)$$

for any  $\psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$ .

**Definition 2.3.8.** A local weak solution to problem (2.3.4) is a bounded nonnegative function  $w$  such that, for some nonnegative  $\tilde{w} \in X_{\text{loc}}^s \cap L_{\text{loc}}^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$  (what we call a local extension for  $w$ ), there holds  $\tilde{w}|_{\partial\mathbb{R}_+^{d+1}} = w$  and

$$\int_{\mathbb{R}^d} w^\ell(x) \psi(x, 0) \rho(x) dx = \mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla \tilde{w}, \nabla \psi \rangle(x, y) dx dy$$

for any  $\psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$ .

**Definition 2.3.9.** A bounded, nonnegative function  $w$  is a very weak solution to problem (2.1.12) if it satisfies

$$\int_{\mathbb{R}^d} w^\ell(x) \phi(x) \rho(x) dx = \int_{\mathbb{R}^d} w(x) (-\Delta)^s(\phi)(x) dx$$

for any  $\phi \in C_c^\infty(\mathbb{R}^d)$ .

Note that a bounded weak solution is also a solution both in the sense of Definition 2.3.8 and Definition 2.3.9. This section aims at studying existence and uniqueness of solutions to (2.1.12) and (2.3.4), according to the above definitions. Our main results are the following.

**Theorem 2.3.10** (existence). *Let  $d > 2s$  and  $\ell \in (0, 1)$ . Let  $\rho \in C_{\text{loc}}^\sigma(\mathbb{R}^d)$  (for some  $\sigma \in (0, 1)$ ) be strictly positive and such that  $\rho(x) \leq C|x|^{-\gamma}$  in  $B_1^c$  for some  $\gamma > 2s$  and  $C > 0$ . Then there exists a local weak solution  $w$  to problem (2.3.4), which is minimal in the class of non-identically zero local weak solutions (in the sense of Definition 2.3.8). Moreover,  $w$  is a very weak solution to (2.1.12) (in the sense of Definition 2.3.9) and satisfies the estimate*

$$w(x) \leq K(I_{2s} * \rho)(x) \quad \forall x \in \mathbb{R}^d \quad (2.3.6)$$

for some  $K > 0$ .

Finally, if  $\gamma$  complies with the more restrictive condition

$$\gamma > 2s + \frac{d-2s}{\ell+2}, \quad (2.3.7)$$

then  $w$  is also a weak solution to (2.1.12) (in the sense of Definition 2.3.7).

**Theorem 2.3.11** (uniqueness). *Let  $d > 4s$  and  $\ell \in (0, 1)$ . Let  $\rho \in C_{\text{loc}}^\sigma(\mathbb{R}^d)$  (for some  $\sigma \in (0, 1)$ ) be strictly positive and such that  $\rho(x) \leq C|x|^{-\gamma}$  in  $B_1^c$  for some  $\gamma > 4s$  and  $C > 0$ . Then the minimal solution to problem (2.3.4) provided by Theorem 2.3.10 is unique in the class of non-identically zero local weak solutions (in the sense of Definition 2.3.8) which are also very weak solutions to (2.1.12) (according to Definition 2.3.9) and belong to  $L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d)$ .*

The next lemma, which provides us with elementary estimates from above for the Riesz potential of  $\rho$ , is key to our analysis.

**Lemma 2.3.12.** *Let  $d > 2s$  and  $\rho \geq 0$  be a measurable function. Assume in addition that  $\rho(x) \leq C(1 + |x|)^{-\gamma}$  for some  $\gamma > 2s$  and  $C > 0$ . Then,  $I_{2s} * \rho$  is a nonnegative continuous function and there exists a constant  $K > 0$  such that*

$$(I_{2s} * \rho)(x) \leq K(1 + |x|)^{-\kappa} \quad \forall x \in \mathbb{R}^d,$$

where:

- (a) if  $\gamma < d$ ,  $\kappa = \gamma - 2s$ ;
- (b) if  $\gamma = d$ ,  $\kappa = d - 2s - \varepsilon$  for all  $\varepsilon > 0$  (with  $K = K(\varepsilon)$ );
- (c) if  $\gamma > d$ ,  $\kappa = d - 2s$ .

*Proof.* The proof goes exactly as the one of Lemma 2.2.13, hence we skip it. Let us just mention that the critical term, namely the one that gives the above decay rates, is (2.2.54) (with  $\phi$  replaced by  $\rho$ ).  $\square$

### 2.3.2.1 Existence

Below we shall prove all the properties of  $w$  claimed in Theorem 2.3.10 *except* the fact that  $w$  is a very weak solution to problem (2.1.12) in the sense of Definition 2.3.9. The latter will actually be an interesting consequence of the asymptotic analysis in Subsection 2.3.3.

Consider first the following problem: find  $(w_R, \tilde{w}_R)$  such that

$$\begin{cases} L_s \tilde{w}_R = 0 & \text{in } \Omega_R, \\ \tilde{w}_R = 0 & \text{on } \Sigma_R, \\ \tilde{w}_R = w_R & \text{on } \Gamma_R, \\ \frac{\partial \tilde{w}_R}{\partial y^{2s}} = \rho w_R^\ell & \text{on } \Gamma_R, \end{cases} \quad (2.3.8)$$

where  $\Omega_R := \{(x, y) \in \mathbb{R}_+^{d+1} : |(x, y)| < R\}$ ,  $\Sigma_R := \partial\Omega_R \cap \{y > 0\}$  and  $\Gamma_R := \partial\Omega_R \cap \{y = 0\}$ . We denote by  $X_0^s(\Omega_R)$  the completion of  $C_c^\infty(\Omega_R \cup \Gamma_R)$  w.r.t. the norm

$$\|\psi\|_{X_0^s(\Omega_R)} := \left( \mu_s \int_{\Omega_R} y^{1-2s} |\nabla \psi(x, y)|^2 dx dy \right)^{\frac{1}{2}} \quad \forall \psi \in C_c^\infty(\Omega_R \cup \Gamma_R),$$

*i.e.* the closure of  $C_c^\infty(\Omega_R \cup \Gamma_R)$  in  $X^s|_{\Omega_R}$ .

**Definition 2.3.13.** *A weak solution to problem (2.3.8) is a pair of nonnegative functions  $(w_R, \tilde{w}_R)$  such that:*

- $w_R^\ell \in L^1(B_R)$ ,  $\tilde{w}_R \in X_0^s(\Omega_R)$ ;
- $\tilde{w}_R|_{\Gamma_R} = w_R$ ;
- for any  $\psi \in C_c^\infty(\Omega_R \cup \Gamma_R)$  there holds

$$\int_{B_R} w_R^\ell(x) \psi(x, 0) \rho(x) dx = \mu_s \int_{\Omega_R} y^{1-2s} \langle \nabla \tilde{w}_R, \nabla \psi \rangle(x, y) dx dy. \quad (2.3.9)$$

Existence results for (2.3.8) can be proved by standard variational methods, see *e.g.* [36].

**Proposition 2.3.14.** *Let  $\ell \in (0, 1)$ . Let  $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ . Then there exists a non-identically zero weak solution  $(w_R, \tilde{w}_R)$  to problem (2.3.8), in the sense of Definition 2.3.13.*

The following positivity result and comparison principle for problem (2.3.8) will be crucial in the proof of Theorem 2.3.10, specially as concerns minimality.

**Proposition 2.3.15.** *Let  $\ell \in (0, 1)$ . Let  $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ .*

- (i) *Take a subsolution  $(f_R, \tilde{f}_R)$  and a supersolution  $(g_R, \tilde{g}_R)$  to problem (2.3.8) (in a weak sense, in agreement with Definition 2.3.13). Assume that  $\tilde{f}_R, \tilde{g}_R \geq 0$  a.e. in  $\Omega_R$ ,  $f_R \geq 0$  a.e. in  $B_R$ ,  $g_R > 0$  a.e. in  $B_R$  and  $\tilde{f}_R|_{\Sigma_R} \leq \tilde{g}_R|_{\Sigma_R}$  a.e. in  $\Sigma_R$ . Then  $f_R \leq g_R$  a.e. in  $B_R$  and  $\tilde{f}_R \leq \tilde{g}_R$  a.e. in  $\Omega_R$ .*
- (ii) *Suppose in addition that  $\rho \in C_{\text{loc}}^\sigma(\mathbb{R}^d)$  for some  $\sigma \in (0, 1)$ . Let  $(w_R, \tilde{w}_R)$  be a weak solution to problem (2.3.8), in the sense of Definition 2.3.13, such that  $w_R \in L^\infty(B_R)$  and  $\tilde{w}_R \in L^\infty(\Omega_R)$ . Then (in particular)  $\tilde{w}_R \in C(\bar{\Omega}_r)$  for all  $0 < r < R$  and either  $(w_R, \tilde{w}_R) \equiv (0, 0)$  or  $w_R > 0$  in  $B_R$  and  $\tilde{w}_R > 0$  in  $\Omega_R$ .*

*Proof.* Statement (i) follows by performing minor modifications to the proof of [36, Lemma 5.3]. Actually the strategy of proof goes back to the pioneering paper [38]: let us mention that the strict positivity of the supersolution and the fact that the nonlinearity is sublinear are essential.

Statement (ii) is due to the regularity results in [40]. In fact, since  $(w_R, \tilde{w}_R)$  is bounded and  $\rho(x)$  and  $f(w) := w^\ell$  are Hölder functions, Lemma 4.5 of [40] ensures that  $\tilde{w}_R$  and  $\partial \tilde{w}_R / \partial y^{2s}$  are also Hölder continuous in  $\bar{\Omega}_r$  for all  $0 < r < R$ . Corollary 4.12 of [40] then entails the assertion (the same argument works upon replacing  $d(x)u(x)$  there with  $-\rho(x)u^\ell(x)$ ).  $\square$

We are now in position to prove the part of Theorem 2.3.10 which deals with existence of a minimal local weak solution to (2.3.4). The fact that such solution is also a very weak solution to (2.1.12) (according to Definition 2.3.9) will be deduced in the end of proof of Theorem 2.3.5 below.

*Proof of Theorem 2.3.10 (first part).* For any  $R > 0$ , in view of Proposition 2.3.14 we know that there exists a nontrivial solution  $(w_R, \tilde{w}_R)$  to problem (2.3.8). Let  $(F_R, \tilde{F}_R)$  be the unique regular solution to

$$\begin{cases} L_s \tilde{F}_R = 0 & \text{in } \mathcal{C}_R, \\ \tilde{F}_R = 0 & \text{on } \partial \mathcal{C}_R \cap \{y > 0\}, \\ \tilde{F}_R = F_R & \text{on } \Gamma_R, \\ \frac{\partial \tilde{F}_R}{\partial y^{2s}} = \rho & \text{on } \Gamma_R, \end{cases} \quad (2.3.10)$$

where  $\mathcal{C}_R := B_R \times \{y > 0\}$ . By standard results (see *e.g.* [52]),

$$\tilde{F}_R(x, y) = \int_{B_R} G_R((x, y), z) \rho(z) dz \quad \forall (x, y) \in \bar{\mathcal{C}}_R, \quad (2.3.11)$$

where  $G_R((x, y), z)$  (let  $(x, y) \in \bar{\mathcal{C}}_R$  and  $z \in B_R$ ) is the Green function associated to problem



(2.3.10), namely the solution of

$$\begin{cases} L_s G_R(\cdot, z) = 0 & \text{in } \mathcal{C}_R, \\ G_R(\cdot, z) = 0 & \text{on } \partial\mathcal{C}_R \cap \{y > 0\}, \\ \frac{\partial G_R(\cdot, z)}{\partial y^{2s}} = \delta_z & \text{on } \Gamma_R. \end{cases}$$

It is well known that Green functions are positive and ordered w.r.t.  $R$ , that is

$$G_{R_1} > 0 \text{ in } \mathcal{C}_{R_1} \cup (\{B_{R_1}\} \times \{0\}), \quad G_{R_1} \leq G_{R_2} \text{ in } \bar{\mathcal{C}}_{R_1} \quad (2.3.12)$$

for all  $z \in B_{R_1}$  and  $R_2 > R_1 > 0$ . Furthermore, they are all bounded from above by the Green function  $G_+$  of the half-space:

$$G_R((x, y), z) \leq G_+((x, y), z) \quad \forall (x, y) \in \bar{\mathcal{C}}_R, \quad \forall z \in B_R, \quad \forall R > 0, \quad (2.3.13)$$

where

$$G_+((x, y), z) := \frac{k_{s,d}}{|(x - z, y)|^{d-2s}} \quad \forall (x, y) \in \mathbb{R}_+^{d+1}, \quad \forall z \in \mathbb{R}^d$$

and  $k_{d,s}$  is the same constant as in Paragraph 2.1.1.4. In fact,  $G_+$  solves

$$\begin{cases} L_s G_+(\cdot, z) = 0 & \text{in } \mathbb{R}_+^{d+1}, \\ \frac{\partial G_+(\cdot, z)}{\partial y^{2s}} = \delta_z & \text{on } \partial\mathbb{R}_+^{d+1} \end{cases}$$

for all  $z \in \mathbb{R}^d$  (see again [52] or [68]). From (2.3.11), (2.3.13) and Lemma 2.3.12 we deduce that, for any  $R > 0$  and  $(x, y) \in \bar{\mathcal{C}}_R$ ,

$$\begin{aligned} \tilde{F}_R(x, y) &\leq \int_{\mathbb{R}^d} G_+((x, y), z) \rho(z) dz \leq \int_{\mathbb{R}^d} G_+((x, 0), z) \rho(z) dz = (I_{2s} * \rho)(x) \\ &\leq \|I_{2s} * \rho\|_\infty =: \hat{C}. \end{aligned} \quad (2.3.14)$$

Now note that, by definition of  $(F_R, \tilde{F}_R)$ , for any test function  $\psi \in C_c^\infty(\Omega_R \cup \Gamma_R)$  there holds

$$\mu_s \int_{\Omega_R} y^{1-2s} \langle \nabla \tilde{F}_R, \nabla \psi \rangle(x, y) dx dy = \mu_s \int_{\mathcal{C}_R} y^{1-2s} \langle \nabla \tilde{F}_R, \nabla \psi \rangle(x, y) dx dy = \int_{B_R} \psi(x, 0) \rho(x) dx. \quad (2.3.15)$$

Upon picking  $C_0 \geq \hat{C}^{\frac{\ell}{1-\ell}}$ , the function  $(C_0 F_R, C_0 \tilde{F}_R)$  is a supersolution to problem (2.3.8). In fact, thanks to (2.3.14) and (2.3.15),

$$\begin{aligned} \mu_s \int_{\Omega_R} y^{1-2s} \langle \nabla(C_0 \tilde{F}_R), \nabla \psi \rangle(x, y) dx dy &= \int_{B_R} C_0 \psi(x, 0) \rho(x) dx \\ &\geq \int_{B_R} [C_0 F_R(x)]^\ell \psi(x, 0) \rho(x) dx \end{aligned}$$

for all *nonnegative*  $\psi$  as above. In view of (2.3.11) and (2.3.12), we are then in position to apply the comparison principle provided by Proposition 2.3.15-(i) with the choices  $(f_R, \tilde{f}_R) = (w_R, \tilde{w}_R)$  and  $(g_R, \tilde{g}_R) = (C_0 F_R, C_0 \tilde{F}_R)$ , to get:

$$\tilde{w}_R \leq C_0 \tilde{F}_R \quad \text{a.e. in } \Omega_R \quad (2.3.16)$$

and

$$w_R \leq C_0 F_R \quad \text{a.e. in } B_R. \quad (2.3.17)$$

In particular, from (2.3.14), (2.3.16) and (2.3.17) we infer that  $w_R \in L^\infty(B_R)$  and  $\tilde{w}_R \in L^\infty(\Omega_R)$ . We can therefore exploit Proposition 2.3.15-(ii), which yields

$$\tilde{w}_R > 0 \quad \text{in } \Omega_R \tag{2.3.18}$$

and

$$w_R > 0 \quad \text{in } B_R. \tag{2.3.19}$$

Let  $R_2 > R_1 > 0$ . The strict positivity ensured by (2.3.18) and (2.3.19) permits us to apply again Proposition 2.3.15-(i), this time with the choices  $(f_R, \tilde{f}_R) = (w_{R_1}, \tilde{w}_{R_1})$  and  $(g_R, \tilde{g}_R) = (w_{R_2}, \tilde{w}_{R_2})$ . We obtain:

$$\tilde{w}_{R_1} \leq \tilde{w}_{R_2} \quad \text{in } \Omega_{R_1}, \quad w_{R_1} \leq w_{R_2} \quad \text{in } B_{R_1} \quad \forall R_2 > R_1 > 0. \tag{2.3.20}$$

We aim at passing to the limit on  $(w_R, \tilde{w}_R)$  as  $R \rightarrow \infty$ . Given any fixed  $\eta \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$ , for every  $R > 0$  large enough we can pick (after a standard approximation)  $\psi = \tilde{w}_R \eta^2$  as a test function in Definition 2.3.13. It is straightforward to check that

$$\begin{aligned} & \mu_s \int_{\Omega_R} y^{1-2s} |\nabla \tilde{w}_R(x, y)|^2 \eta^2 \, dx dy \\ & \leq 2 \|w_R\|_\infty^{\ell+1} \int_{B_R} \eta^2(x, 0) \rho(x) dx + 4\mu_s \|\tilde{w}_R\|_\infty^2 \int_{\Omega_R} y^{1-2s} |\nabla \eta(x, y)|^2 \, dx dy. \end{aligned} \tag{2.3.21}$$

From (2.3.14), (2.3.16), (2.3.17) and (2.3.21) we deduce that, for any  $\Omega_0 \Subset \mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}$ , there holds

$$\int_{\Omega_0} y^{1-2s} |\nabla \tilde{w}_R(x, y)|^2 \, dx dy \leq C' \tag{2.3.22}$$

for a suitable positive constant  $C'$  independent of  $R > 0$ . Moreover, (2.3.20) entails the existence of the following (nontrivial) pointwise limits:

$$\lim_{R \rightarrow \infty} \tilde{w}_R =: \tilde{w} \in L^\infty(\mathbb{R}_+^{d+1}), \quad \lim_{R \rightarrow \infty} w_R =: w \in L^\infty(\mathbb{R}^d). \tag{2.3.23}$$

Thanks to (2.3.22), by standard compactness arguments we can pass to the limit in the weak formulation (2.3.9) to find that  $w$  is a local weak solution to (2.3.4) in the sense of Definition 2.3.8 (with local extension  $\tilde{w}$ ).

We are left with proving minimality. In order to avoid confusion, hereafter we shall denote as  $\underline{w}$  the solution constructed above and as  $w$  any other non-identically zero local weak solution to (2.3.4) (according to Definition 2.3.8). Note that, for  $R$  large enough,  $(w|_{B_R}, \tilde{w}|_{\Omega_R})$  is a nontrivial solution to problem (2.3.8), in the sense of Definition 2.3.13, except that  $\tilde{w}|_{\Omega_R}$  is not necessarily zero on  $\Sigma_R$ . That is,  $\tilde{w}$  has finite energy (the  $L_{y^{1-2s}}^2$  norm of the gradient) in  $\Omega_R$  but does not belong to  $X_0^s(\Omega_R)$ . Nevertheless, the regularity results of [40] hold regardless of boundary conditions on  $\Sigma_R$ : namely, Proposition 2.3.15-(ii) is applicable in this case as well, ensuring in particular that  $w > 0$  in  $B_R$ . Because  $(w_R, \tilde{w}_R)$  is also a weak solution to (2.3.8) and, trivially,  $\tilde{w}_R|_{\Sigma_R} \leq \tilde{w}|_{\Sigma_R}$ , from Proposition 2.3.15-(i) (with the choices  $(f_R, \tilde{f}_R) = (w_R, \tilde{w}_R)$  and  $(g_R, \tilde{g}_R) = (w|_{B_R}, \tilde{w}|_{\Omega_R})$ ) we deduce that

$$w_R \leq w|_{B_R} \quad \text{in } B_R,$$

whence  $\underline{w} \leq w$  in  $\mathbb{R}^d$  by letting  $R \rightarrow \infty$ . We have therefore proved that  $\underline{w}$  is indeed minimal. The upper bound (2.3.6) is just a consequence of (2.3.14), (2.3.17) and (2.3.23).

From the above method of proof it is direct to check that, under the more restrictive assumption (2.3.7),  $\underline{w}$  is also a *weak solution* to (2.1.12) in the sense of Definition 2.3.7. In fact, in this case, inequalities (2.3.14), (2.3.17) and Lemma 2.3.12 ensure that the  $L_\rho^1$  norm of  $w_R^{\ell+1}$  is uniformly bounded with respect to  $R$ . As a consequence, estimate (2.3.22) actually holds with  $\Omega_0 = \mathbb{R}_+^{d+1}$  (up to setting  $\tilde{w}_R \equiv 0$  in  $\Omega_R^c$ ). Hence, passing to the limit as  $R \rightarrow \infty$ , we deduce that  $\underline{w} \in X^s$ ,  $\underline{w} \in \dot{H}^s(\mathbb{R}^d)$ ,  $\underline{w} = E(\underline{w})$  and that  $\underline{w}$  satisfies (2.3.5).  $\square$

### 2.3.2.2 Uniqueness

Our strategy of proof of Theorem 2.3.11 strongly relies on the uniqueness result for the parabolic problem provided by Theorem 2.3.3.

*Proof of Theorem 2.3.11.* Throughout, we shall denote as  $\underline{w}$  the minimal solution provided by Theorem 2.3.10 and as  $w$  any other solution complying with the assumptions of the statement.

To begin with, set  $m = 1/\ell$  and

$$C_m := (m - 1)^{-\frac{1}{m-1}}.$$

For any  $k \in \mathbb{N}$ , let  $\zeta_k \in C^\infty(\mathbb{R}^d)$  with  $\zeta_k \equiv 1$  in  $B_k$ ,  $\zeta_k \equiv 0$  in  $B_{2k}^c$  and  $0 \leq \zeta_k \leq 1$  in  $\mathbb{R}^d$ . Take  $R > 2k$  and denote as  $(v_{R,k}, \tilde{v}_{R,k})$  the unique strong solution to the following parabolic problem (see Paragraph 2.3.4.2 in Subsection 2.3.4 below):

$$\begin{cases} L_s(\tilde{v}_{R,k}^m) = 0 & \text{in } \Omega_R \times \mathbb{R}^+, \\ \tilde{v}_{R,k}^m = 0 & \text{on } \Sigma_R \times \mathbb{R}^+, \\ \tilde{v}_{R,k}^m = v_{R,k}^m & \text{on } \Gamma_R \times \mathbb{R}^+, \\ \frac{\partial(\tilde{v}_{R,k}^m)}{\partial y^{2s}} = \rho \frac{\partial v_{R,k}}{\partial t} & \text{on } \Gamma_R \times \mathbb{R}^+, \\ v_{R,k} = C_m \zeta_k w^{\frac{1}{m}} & \text{on } B_R \times \{t = 0\}. \end{cases} \quad (2.3.24)$$

Let  $(w_R, \tilde{w}_R)$  be as in the proof of Theorem 2.3.10. Since by hypothesis  $w \in L^\infty(\mathbb{R}^d)$ , thanks to (2.3.19) and to the fact that  $w_{R+1} \in C(\bar{B}_R)$  we can select a suitable  $\tau_R > 0$  so that

$$\frac{w_{R+1}^{\frac{1}{m}}}{\tau_R^{\frac{1}{m-1}}} \geq w^{\frac{1}{m}} \quad \text{in } B_R. \quad (2.3.25)$$

Let

$$\tilde{U}_R := \frac{C_m \tilde{w}_{R+1}^{\frac{1}{m}}}{(t + \tau_R)^{\frac{1}{m-1}}} \leq \frac{C_m \tilde{w}_{R+1}^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} =: \tilde{U}_{0R} \quad \text{in } \bar{\Omega}_R \times (0, \infty),$$

with  $U_R(\cdot, t) := \tilde{U}_R(\cdot, 0, t)$  and  $U_{0R}(\cdot, t) := \tilde{U}_{0R}(\cdot, 0, t)$  for all  $t > 0$ . By definition of  $(U_R, \tilde{U}_R)$  and in view of (2.3.25), it is apparent that  $(U_R, \tilde{U}_R)$  is a strong supersolution to (2.3.24). Hence, applying the comparison principle provided by Proposition 2.3.20 below, we deduce that

$$v_{R,k} \leq U_R \leq U_{0R} \quad \text{a.e. in } B_R \times (0, \infty). \quad (2.3.26)$$

In addition to the above bounds, note that for any  $k_2 > k_1$  and  $R > 2k_2$  there holds

$$v_{R,k_1} \leq v_{R,k_2} \leq \frac{C_m w^{\frac{1}{m}}}{(t + 1)^{\frac{1}{m-1}}} =: V \quad \text{a.e. in } B_R \times (0, \infty). \quad (2.3.27)$$

Such inequalities still follow by Proposition 2.3.20:  $(V, \tilde{V})$  is a strong supersolution to (2.3.24) for all  $R > 0$  and  $k \in \mathbb{N}$ , while  $(v_{R,k_2}, \tilde{v}_{R,k_2})$  is a strong supersolution to (2.3.24) for  $k = k_1$ .

Since  $C_m \zeta_k w^{\frac{1}{m}} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , by standard arguments (see *e.g.* the proof of [159, Theorem 3.1] or Paragraph 2.3.4.2) there exists the limit

$$v_{\infty,k} := \lim_{R \rightarrow \infty} v_{R,k} \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+,$$

which solves problem

$$\begin{cases} \rho(v_{\infty,k})_t + (-\Delta)^s(v_{\infty,k}^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ v_{\infty,k} = C_m \zeta_k w^{\frac{1}{m}} & \text{on } \mathbb{R}^d \times \{0\} \end{cases}$$

both in the sense of Definition 2.3.1 (weak solution) and in the sense of Definition 2.3.23 (very weak solution). Moreover, as a consequence of (2.3.27), such limit satisfies the bounds

$$v_{\infty,k_1} \leq v_{\infty,k_2} \leq V \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+ \quad (2.3.28)$$

for all  $k_2 > k_1$ . Thanks to (2.3.28), by monotonicity we get existence of the pointwise limit

$$v_{\infty} := \lim_{k \rightarrow \infty} v_{\infty,k} \leq V \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+. \quad (2.3.29)$$

Passing to the limit as  $k \rightarrow \infty$  in the very weak formulation solved by  $v_{\infty,k}$ , we then infer that  $v_{\infty}$  is a very weak solution, in the sense of Definition 2.3.23, to the problem

$$\begin{cases} \rho(v_{\infty})_t + (-\Delta)^s(v_{\infty}^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ v_{\infty} = C_m w^{\frac{1}{m}} & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \quad (2.3.30)$$

Now notice that  $V$  is also a very weak solution to (2.3.30), because so is  $w$  with respect to (2.1.12). By hypothesis we know that  $w \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d)$ , whence  $V^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$  for all  $T > 0$ ; but then (2.3.29) implies that  $v_{\infty}^m$  belongs to  $L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$  as well. We are therefore in position to apply the uniqueness result of Theorem 2.3.3 (together with Remark 2.3.24), which yields

$$v_{\infty} = V \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+.$$

Letting  $R \rightarrow \infty$ ,  $k \rightarrow \infty$  in (2.3.26) and using (2.3.23) (replace  $w$  with  $\underline{w}$  there), we infer that

$$v_{\infty} \leq \frac{C_m \underline{w}^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+.$$

Hence,

$$w^{\frac{1}{m}} \leq \frac{(t+1)^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \underline{w}^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+. \quad (2.3.31)$$

Passing to the limit in (2.3.31) as  $t \rightarrow \infty$  we end up with

$$w^{\frac{1}{m}} \leq \underline{w}^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d,$$

which implies  $w = \underline{w}$  since  $w$  is nontrivial and  $\underline{w}$  is minimal.

Finally, the fact that  $\underline{w} \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d)$  is just a consequence of estimate (2.3.6) and Lemma 2.3.12 (recall the assumptions  $d, \gamma > 4s$ ).  $\square$

### 2.3.3 Asymptotic behaviour

Before proving Theorem 2.3.5 we need an intermediate result, which provides us with a crucial bound from above for the minimal solution to problem (2.1.1).

**Lemma 2.3.16.** *Under the same assumptions and with the same notations as in Theorem 2.3.5, there holds*

$$u(x, t) \leq (m-1)^{-\frac{1}{m-1}} t^{-\frac{1}{m-1}} w^{\frac{1}{m}}(x) \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times (0, \infty). \quad (2.3.32)$$

*Proof.* Suppose first that  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . Let us define  $C_m$ ,  $(w_R, \tilde{w}_R)$  and  $(U_R, \tilde{U}_R)$  (for a suitable  $\tau_R > 0$  to be chosen later) as in the proofs of Theorems 2.3.10 and 2.3.11. For any  $R > 0$ , let  $(u_R, \tilde{u}_R)$  be the unique strong solution to the following parabolic problem (see Paragraph 2.3.4.2 below):

$$\begin{cases} L_s(\tilde{u}_R^m) = 0 & \text{in } \Omega_R \times \mathbb{R}^+, \\ \tilde{u}_R^m = 0 & \text{on } \Sigma_R \times \mathbb{R}^+, \\ \tilde{u}_R^m = u_R^m & \text{on } \Gamma_R \times \mathbb{R}^+, \\ \frac{\partial \tilde{u}_R^m}{\partial y^{2s}} = \rho \frac{\partial u_R}{\partial t} & \text{on } \Gamma_R \times \mathbb{R}^+, \\ u_R = u_0 & \text{on } B_R \times \{t = 0\}. \end{cases} \quad (2.3.33)$$

By standard arguments (we refer again the reader to the proof of [159, Theorem 3.1] or to Paragraph 2.3.4.2), we have that

$$\lim_{R \rightarrow \infty} u_R = u \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+, \quad \lim_{R \rightarrow \infty} \tilde{u}_R^m = \tilde{u}^m = E(u^m) \quad \text{a.e. in } \mathbb{R}_+^{d+1} \times \mathbb{R}^+. \quad (2.3.34)$$

In view of the positivity properties of the solutions  $w_R$  recalled above, for any  $R > 0$  there holds

$$\min_{B_R} w_{R+1} > 0.$$

Since  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , we can therefore pick  $\tau_R > 0$  so that

$$\frac{C_m w_{R+1}^{\frac{1}{m}}}{\tau_R^{\frac{1}{m-1}}} \geq u_0 \quad \text{a.e. in } B_R. \quad (2.3.35)$$

Thanks to (2.3.35),  $(U_R, \tilde{U}_R)$  is a strong supersolution to problem (2.3.33). By comparison (see Proposition 2.3.20 below),

$$u_R \leq U_R \quad \text{a.e. in } B_R \times \mathbb{R}^+. \quad (2.3.36)$$

It is then apparent that (2.3.36) entails the fundamental estimate

$$u_R \leq C_m t^{-\frac{1}{m-1}} w_{R+1}^{\frac{1}{m}} \quad \text{a.e. in } B_R \times (0, \infty). \quad (2.3.37)$$

Letting  $R \rightarrow \infty$  in (2.3.37) and exploiting (2.3.23), (2.3.34), we finally get (2.3.32).

Consider now a general datum  $u_0 \in L^1_\rho(\mathbb{R}^d)$ . We have:

$$u = \lim_{n \rightarrow \infty} u_n \quad \text{a.e. in } \mathbb{R}^d \times \mathbb{R}^+,$$

where for every  $n \in \mathbb{N}$  we denote as  $u_n$  the solution to problem (2.1.1) corresponding to the initial datum  $u_{0n} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , and the sequence  $\{u_{0n}\}$  is such that  $0 \leq u_{0n} \leq u_0$  in  $\mathbb{R}^d$  for all  $n \in \mathbb{N}$  and  $u_{0n} \rightarrow u_0$  in  $L^1_\rho(\mathbb{R}^d)$  as  $n \rightarrow \infty$  (see [159, Subsection 6.2] or Subsection 2.3.4, Paragraphs 2.3.4.1 and 2.3.4.2). In view of the first part of the proof, we know that for every  $n \in \mathbb{N}$  there holds

$$u_n \leq C_m t^{-\frac{1}{m-1}} w^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty). \quad (2.3.38)$$

The assertion then follows by passing to the limit as  $n \rightarrow \infty$  in (2.3.38).  $\square$

**Remark 2.3.17.** As a consequence of the method of proof of Lemma 2.3.16 we also get the estimate

$$E(u^m) \leq C_m^m t^{-\frac{m}{m-1}} \tilde{w} \quad \text{a.e. in } \mathbb{R}_+^{d+1} \times (0, \infty), \quad (2.3.39)$$

where  $E(u^m)$  is the extension of  $u^m$  and  $\tilde{w}$  is the local extension of  $w$  (in agreement with Definition 2.3.8) provided along the first part of the proof of Theorem 2.3.10. In fact it is enough to notice that, thanks to the standard comparison principle for sub- and supersolutions to the elliptic problem  $L_s = 0$  in  $\Omega_R$ , (2.3.37) implies

$$\tilde{u}_R^m \leq C_m^m t^{-\frac{m}{m-1}} \tilde{w}_{R+1} \quad \text{a.e. in } \Omega_R \times (0, \infty),$$

whence (2.3.39) upon letting  $R \rightarrow \infty$ .

*Proof of Theorem 2.3.5 and end of proof of Theorem 2.3.10.* Let us denote as  $v(x, \tau)$  the following rescaling of  $u(x, t)$ :

$$u(x, t) =: e^{-\beta\tau} v(x, \tau), \quad t =: e^\tau, \quad \beta := \frac{1}{m-1}.$$

It is immediate to check that  $v$  is a (weak, and in particular very weak) solution of

$$\rho v_\tau = -(-\Delta)^s(v^m) + \beta \rho v \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+,$$

in the sense that

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{R}^d} v(x, \tau) \varphi_\tau(x, \tau) \rho(x) dx d\tau + \int_0^\infty \int_{\mathbb{R}^d} v^m(x, \tau) (-\Delta)^s(\varphi)(x, \tau) dx d\tau \\ & = \beta \int_0^\infty \int_{\mathbb{R}^d} v(x, \tau) \varphi(x, \tau) \rho(x) dx d\tau + \int_{\mathbb{R}^d} u(x, 1) \varphi(x, 0) \rho(x) dx \end{aligned} \quad (2.3.40)$$

for all  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$ . Moreover,  $E(v^m) \in L^2((0, T); X^s)$  and

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^d} v(x, \tau) \psi_\tau(x, 0, \tau) \rho(x) dx d\tau + \mu_s \int_0^T \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla E(v^m), \nabla \psi \rangle(x, y, \tau) dx dy d\tau \\ & = \beta \int_0^T \int_{\mathbb{R}^d} v(x, \tau) \psi(x, 0, \tau) \rho(x) dx d\tau \end{aligned} \quad (2.3.41)$$

for all  $T > 0$  and  $\psi \in C_c^\infty((\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}) \times (0, T))$ . In view of Lemma 2.3.16 (and Remark 2.3.17), we have:

$$v(x, \tau) \leq C_m w^{\frac{1}{m}}(x) \leq C_m \|w\|_{\frac{1}{m}} \quad \text{for a.e. } (x, \tau) \in \mathbb{R}^d \times (0, \infty), \quad (2.3.42)$$

$$E(v^m)(x, y, \tau) \leq C_m^m \tilde{w}(x, y) \leq C_m^m \|\tilde{w}\|_\infty \quad \text{for a.e. } (x, y, \tau) \in \mathbb{R}_+^{d+1} \times (0, \infty). \quad (2.3.43)$$

Furthermore, the following monotonicity properties hold true:

$$v(x, \tau_2) \geq v(x, \tau_1) \quad \text{for a.e. } x \in \mathbb{R}^d, \quad E(v^m)(x, y, \tau_2) \geq E(v^m)(x, y, \tau_1) \quad \text{for a.e. } (x, y) \in \mathbb{R}_+^{d+1} \quad (2.3.44)$$

and all  $\tau_2 \geq \tau_1 > 0$ . In fact, by proceeding exactly as in [173, p. 182] (see also the original reference [16]), one can prove the fundamental Bénylan-Crandall inequality

$$\rho u_t \geq -\frac{\rho u}{(m-1)t} \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty),$$

which reads

$$v_\tau \geq 0 \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty). \quad (2.3.45)$$

Thanks to (2.3.45) we get the first inequality in (2.3.44), and therefore also the second one because the extension operator is order preserving. Hence, from (2.3.42)–(2.3.44) we infer that there exist finite the limits

$$h(x) := \lim_{k \rightarrow \infty} v(x, \tau_k) \text{ for a.e. } x \in \mathbb{R}^d, \quad H(x, y) := \lim_{k \rightarrow \infty} E(v^m)(x, y, \tau_k) \text{ for a.e. } (x, y) \in \mathbb{R}_+^{d+1}, \quad (2.3.46)$$

where  $\{\tau_k\}$  is any time sequence tending to infinity. Note that, since  $u_0 \not\equiv 0$ , (2.3.44) implies that  $h \not\equiv 0$  and  $H \not\equiv 0$ , while (2.3.42) and (2.3.43) ensure that  $h \in L^\infty(\mathbb{R}^d)$  and  $H \in L^\infty(\mathbb{R}_+^{d+1})$ . Let

$$g := C_m^{-m} h^m, \quad \tilde{g} := C_m^{-m} H. \quad (2.3.47)$$

First we aim at proving that  $g$  (with the corresponding local extension  $\tilde{g}$ ) is a solution to problem (2.3.4) (for  $\ell = 1/m$ ) in the sense of Definition 2.3.8. To this end, for any fixed  $0 < \tau_1 < \tau_2$  and  $0 < \epsilon < (\tau_2 + \tau_1)/2$ , let  $\zeta_\epsilon(\tau)$  be a smooth approximation of the function  $\chi_{[\tau_1, \tau_2]}(\tau)$  such that

$$0 \leq \zeta_\epsilon(\tau) \leq 1 \quad \forall \tau \geq 0, \quad \zeta_\epsilon(\tau) = 0 \quad \forall \tau \notin [\tau_1, \tau_2], \quad \zeta_\epsilon(\tau) = 1 \quad \forall \tau \in [\tau_1 + \epsilon, \tau_2 - \epsilon].$$

Moreover, we can and shall assume that

$$\zeta'_\epsilon(\tau) \rightarrow \delta_{\tau_1} - \delta_{\tau_2}$$

as  $\epsilon \rightarrow 0$ . Consider now a cut-off function  $\eta$  as in the first part of the proof of Theorem 2.3.10 and plug in the weak formulation (2.3.41) the test function  $\psi = \zeta_\epsilon \eta^2 E(v^m)$  (after a standard approximation). Letting  $\epsilon \rightarrow 0$ , we get:

$$\begin{aligned} & \frac{1}{m+1} \int_{\mathbb{R}^d} v^{m+1}(x, \tau_2) \eta^2(x, 0) \rho(x) dx \\ & + \mu_s \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla E(v^m), \nabla[\eta^2 E(v^m)] \rangle(x, y, \tau) dx dy d\tau \\ & = \frac{1}{m+1} \int_{\mathbb{R}^d} v^{m+1}(x, \tau_1) \eta^2(x, 0) \rho(x) dx + \beta \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} v^{m+1}(x, \tau) \eta^2(x, 0) \rho(x) dx d\tau. \end{aligned}$$

In view of (2.3.42) and (2.3.43), by setting  $\tau_1 = \tau_k$ ,  $\tau_2 = \tau_k + 1$  and proceeding as in the proof of (2.3.22), we obtain the estimate

$$\int_{\tau_k}^{\tau_k+1} \int_{\Omega_0} y^{1-2s} |\nabla E(v^m)(x, y, \tau)|^2 dx dy d\tau \leq C' \quad (2.3.48)$$

for all  $\Omega_0 \Subset \mathbb{R}_+^{d+1} \cup \partial \mathbb{R}_+^{d+1}$  and a suitable constant  $C' > 0$  independent of  $k$ . Given any  $\phi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial \mathbb{R}_+^{d+1})$ , by plugging in (2.3.41) the test function  $\psi(x, y, \tau) = \phi(x, y) \zeta_\epsilon(\tau)$  and letting  $\epsilon \rightarrow 0$  we infer that

$$\begin{aligned} & \int_{\mathbb{R}^d} [v(x, \tau_k + 1) - v(x, \tau_k)] \phi(x, 0) \rho(x) dx \\ & + \mu_s \int_{\tau_k}^{\tau_k+1} \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla E(v^m)(x, y, \tau), \nabla \phi(x, y) \rangle dx dy d\tau \\ & = \beta \int_{\tau_k}^{\tau_k+1} \int_{\mathbb{R}^d} v(x, \tau) \phi(x, 0) \rho(x) dx d\tau. \end{aligned} \quad (2.3.49)$$

We point out that, still as a consequence of the time monotonicity ensured by (2.3.44), in

addition to (2.3.46) we also have

$$\begin{aligned} h(x) &= \lim_{k \rightarrow \infty} v(x, \tau_k + \lambda) \quad \text{for a.e. } (x, \lambda) \in \mathbb{R}^d \times (0, 1), \\ h(x) &= \lim_{k \rightarrow \infty} v(x, \tau_k + 1) \quad \text{for a.e. } x \in \mathbb{R}^d, \\ H(x, y) &= \lim_{k \rightarrow \infty} E(v^m)(x, y, \tau_k + \lambda) \quad \text{for a.e. } ((x, y), \lambda) \in \mathbb{R}_+^{d+1} \times (0, 1). \end{aligned} \quad (2.3.50)$$

Gathering (2.3.42), (2.3.43), (2.3.46), (2.3.48) and (2.3.50), we can pass to the limit safely in (2.3.49) to find that  $h$  and  $H$  satisfy

$$\mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla H, \nabla \phi \rangle(x, y) \, dx dy = \beta \int_{\mathbb{R}^d} h(x) \phi(x, 0) \rho(x) dx,$$

with  $H(x, 0) = h^m(x)$ . That is, the function  $g$  (with  $\tilde{g}$  as a local extension) defined in (2.3.47) is a local weak solution to (2.3.4) (for  $\ell = 1/m$ ) in the sense of Definition 2.3.8. Furthermore,  $g$  is also a very weak solution to (2.1.12) in the sense of Definition 2.3.9. In order to prove the latter assertion, we can proceed as above: for any  $\phi \in C_c^\infty(\mathbb{R}^d)$  plug in the weak formulation (2.3.40) the test function  $\varphi(x, \tau) = \zeta_\epsilon(\tau)\phi(x)$  and let  $\epsilon \rightarrow 0$  to get

$$\begin{aligned} & \int_{\mathbb{R}^d} [v(x, \tau_k + 1) - v(x, \tau_k)] \phi(x) \rho(x) dx \\ &= \int_{\tau_k}^{\tau_k + 1} \int_{\mathbb{R}^d} [-v^m(x, \tau)(-\Delta)^s(\phi)(x) + \beta v(x, \tau)\phi(x)\rho(x)] \, dx d\tau. \end{aligned}$$

Passing to the limit as  $k \rightarrow \infty$  and using (2.3.42), (2.3.46), (2.3.50), we end up with

$$0 = - \int_{\mathbb{R}^d} g(x)(-\Delta)^s(\phi)(x) \, dx + \int_{\mathbb{R}^d} g^{\frac{1}{m}}(x)\phi(x) \rho(x) dx \quad (2.3.51)$$

and the estimate

$$g(x) \leq w(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (2.3.52)$$

Since  $g$  is a non-identically zero local weak solution to (2.3.4), the minimality of  $w$  and (2.3.52) necessarily imply that  $g = w$ . In particular, thanks to (2.3.51), we can conclude the proof of Theorem 2.3.10 by inferring that the minimal solution provided by it is also a very weak solution to (2.1.12) (for  $\ell = 1/m$ ) in the sense of Definition 2.3.9.

Finally, the convergence of  $\{v(\tau_k)\}$  to  $C_m w^{1/m}$  in  $L_{\text{loc}}^p(\mathbb{R}^d)$  for  $p \in [1, \infty)$  is just a consequence of (2.3.42) and (2.3.46). The above arguments being independent of the particular sequence  $\{\tau_k\}$ , the proof is complete.  $\square$

### 2.3.4 Well-posedness of the parabolic problem

This subsection is primarily devoted to the proofs of Proposition 2.3.2, Theorem 2.3.3 and connected additional results.

#### 2.3.4.1 First construction of weak solutions

If  $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  is nonnegative and such that  $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ , and  $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  with  $u_0 \geq 0$ , then one can argue as in the proof of [61, Theorem 7.3 (first construction)] to get existence of a weak solution to problem (2.1.1) in the sense of Definition 2.3.1, which is moreover bounded in the whole of  $\mathbb{R}^d \times \mathbb{R}^+$ . Let us just mention that such a construction proceeds by means of a time discretization of the problem, which makes use of the fundamental Crandall-Liggett Theorem. Furthermore, the following  $L_\rho^1$  comparison principle holds true:

$$\int_{\mathbb{R}^d} [u_1(x, t) - u_2(x, t)]_+ \rho(x) dx \leq \int_{\mathbb{R}^d} [u_{01} - u_{02}]_+ \rho(x) dx \quad \forall t > 0, \quad (2.3.53)$$



where  $u_1$  and  $u_2$  are the solutions to (2.1.1), constructed as above, corresponding to the initial data  $u_{01} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and  $u_{02} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , respectively.

As for uniqueness, a standard result is the following.

**Proposition 2.3.18.** *Let  $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ . Let  $u$  and  $v$  be two nonnegative weak solutions to (2.1.1) corresponding to the same initial datum  $u_0 \in L^1_\rho(\mathbb{R}^d)$ , in the sense that:*

$$u, v \in L^{m+1}_\rho(\mathbb{R}^d \times \mathbb{R}^+), \quad (2.3.54)$$

$$u^m, v^m \in L^2_{\text{loc}}([0, \infty); \dot{H}^s(\mathbb{R}^d)) \quad (2.3.55)$$

and

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt \\ &= - \int_0^\infty \int_{\mathbb{R}^d} v(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(v^m)(x, t) (-\Delta)^{\frac{s}{2}}(\varphi)(x, t) dx dt \quad (2.3.56) \\ &= \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) \rho(x) dx \end{aligned}$$

for any  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$ . Then,  $u = v$ .

*Proof.* In view of the hypotheses on  $u$  and  $v$ , using a standard approximation technique one can show that the so-called *Oleřnik's test function*

$$\varphi(x, t) = \int_t^T [u^m(x, \tau) - v^m(x, \tau)] d\tau \quad \text{in } \mathbb{R}^d \times (0, T], \quad \varphi \equiv 0 \quad \text{in } \mathbb{R}^d \times (T, \infty),$$

is in fact an admissible test function in the weak formulations (2.3.56) (for all  $T > 0$ ). The conclusion then follows by reasoning exactly as in [61, Theorem 6.1] (see also the subsequent remark). An analogous argument, for  $s = 1$  and more general weights, is performed in the proof of Proposition 1.2.3 in Chapter 1, Subsection 1.2.2.  $\square$

Arguing as in the proof of Lemma 2.2.9 and as in Paragraphs 2.2.2.1, 2.2.2.2, it is not difficult to deduce that the above solutions satisfy the energy estimates (2.2.27), (2.2.28), that they are strong and that their  $L^p_\rho$  norms do not increase in time (to this end, a key role is played by the comparison estimate (2.3.53)).

Now suppose that, in addition,  $d > 2s$  and  $\rho \in L^\infty(\mathbb{R}^d)$ . As a consequence, the fractional Sobolev inequality (2.2.41), and so the fractional Caffarelli-Kohn-Nirenberg inequalities (2.2.40), hold true for  $\gamma = 0$ . Hence, we can repeat the proof of Proposition 2.2.12 to obtain the following  $L^{p_0}_\rho$ - $L^\infty$  smoothing effect for any  $p_0 \in [1, \infty)$ :

$$\|u(t)\|_\infty \leq K t^{-\alpha_0} \|u_0\|_{p_0, \rho}^{\beta_0} \quad \forall t > 0, \quad (2.3.57)$$

where

$$\alpha_0 := \frac{d}{d(m-1) + 2sp_0}, \quad \beta_0 := \frac{2sp_0}{d(m-1) + 2sp_0}$$

and  $K = K(m, \|\rho\|_\infty, d, s) > 0$ . It is then possible to construct solutions to (2.1.1) corresponding to any nonnegative datum  $u_0 \in L^1_\rho(\mathbb{R}^d)$ . To this aim, one picks a sequence of nonnegative data  $u_{0n} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  such that  $u_{0n} \rightarrow u_0$  in  $L^1_\rho(\mathbb{R}^d)$  and passes to the limit in the weak formulation of Definition 2.3.1 as  $n \rightarrow \infty$  by exploiting (2.3.53), the energy estimate (2.2.27) and the smoothing estimate (2.3.57) for  $p_0 = 1$  (see also [159, Theorem

6.5 and Remark 6.11]). Such solutions are still strong *e.g.* because the fundamental  $L^1_\rho$  comparison principle (2.3.53) is preserved at the limit. We have therefore proved the existence result contained in Proposition 2.3.2. As concerns uniqueness, one can reason as follows. Proposition 2.3.18, in particular, ensures that if  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  then the solution to (2.1.1) that we constructed above is unique in the class of weak solutions satisfying (2.3.54), (2.3.55) and (2.3.56). Moreover, any weak solution  $u(x, t)$  to (2.1.1), in the sense of Definition 2.3.1, is such that  $u(x, t + \varepsilon)$  is a weak solution to (2.1.1), corresponding to the initial datum  $u_0(x, \varepsilon) \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , satisfying (2.3.54), (2.3.55) and (2.3.56), for any  $\varepsilon > 0$ . Thanks to these properties, one can then proceed exactly as in the proof of [159, Theorem 6.7]: that is, given two solutions  $u_1$  and  $u_2$  corresponding to the same  $u_0 \in L^1_\rho(\mathbb{R}^d)$ , in view of Proposition 2.3.18 and (2.3.53) they satisfy  $\|u_1(t) - u_2(t)\|_{1,\rho} \leq \|u_1(\varepsilon) - u_2(\varepsilon)\|_{1,\rho}$  for all  $t \geq \varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0$  we infer that  $u_1 = u_2$ .

### 2.3.4.2 Second construction of weak solutions

We describe here another method for constructing weak solutions to problem (2.1.1), which, under suitable hypotheses, provides us with a *minimal solution*. Given again a non-negative initial datum  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , consider the following problem (for notations, we refer to the discussion at the beginning of Subsection 2.3.2):

$$\begin{cases} L_s(\tilde{u}_R^m) = 0 & \text{in } \Omega_R \times \mathbb{R}^+, \\ \tilde{u}_R^m = 0 & \text{on } \Sigma_R \times \mathbb{R}^+, \\ \tilde{u}_R^m = u_R^m & \text{on } \Gamma_R \times \mathbb{R}^+, \\ \frac{\partial(\tilde{u}_R^m)}{\partial y^{2s}} = \rho \frac{\partial u_R}{\partial t} & \text{on } \Gamma_R \times \mathbb{R}^+, \\ u_R = u_0 & \text{on } B_R \times \{t = 0\}. \end{cases} \quad (2.3.58)$$

**Definition 2.3.19.** *A weak solution to problem (2.3.58) is a pair of nonnegative functions  $(u_R, \tilde{u}_R)$  such that:*

- $u_R \in C([0, \infty); L^1_\rho(B_R)) \cap L^\infty(B_R \times (\tau, \infty))$  for all  $\tau > 0$ ;
- $\tilde{u}_R^m \in L^2_{\text{loc}}((0, \infty); X_0^s(\Omega_R))$ ;
- $\tilde{u}_R^m|_{\Gamma_R \times (0, \infty)} = u_R^m$ ;
- for any  $\psi \in C_c^\infty((\Omega_R \cup \Gamma_R) \times (0, \infty))$  there holds

$$-\int_0^\infty \int_{B_R} u_R(x, t) \psi_t(x, 0, t) \rho(x) dx dt + \mu_s \int_0^\infty \int_{\Omega_R} y^{1-2s} \langle \nabla(\tilde{u}_R^m), \nabla \psi \rangle(x, y, t) dx dy dt = 0;$$

- $\lim_{t \rightarrow 0} u_R(t) = u_0|_{B_R}$  in  $L^1_\rho(B_R)$ .

Weak sub- and supersolutions to (2.3.58) are meant in agreement with Definition 2.3.19. We say in addition that  $(u_R, \tilde{u}_R)$  is a *strong solution* if  $(u_R)_t \in L^\infty((\tau, \infty); L^1_\rho(B_R))$  for every  $\tau > 0$ . By means of the same arguments used in the proof of [61, Theorem 6.2], it is direct to deduce the next comparison principle:

**Proposition 2.3.20.** *Let  $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^d)$  be nonnegative and such that  $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ . Let  $(f_R, \tilde{f}_R)$  and  $(g_R, \tilde{g}_R)$  be a strong subsolution and a strong supersolution, respectively, to problem (2.3.58). Suppose that  $f_R \leq g_R$  on  $B_R \times \{t = 0\}$  and  $\tilde{f}_R \leq \tilde{g}_R$  on  $\Sigma_R \times (0, \infty)$ . Then,  $f_R \leq g_R$  in  $B_R \times (0, \infty)$  and  $\tilde{f}_R \leq \tilde{g}_R$  in  $\Omega_R \times (0, \infty)$ .*

Making use of standard tools (see again [61, 159]), one can prove that for any  $R > 0$  and nonnegative  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  there exists a unique strong solution  $(u_R, \tilde{u}_R)$  to problem (2.3.58), in the sense of Definition 2.3.19. Moreover, the family  $\{u_R\}$  is monotone

in  $R$  thanks to Proposition 2.3.20 and the limit function  $u := \lim_{R \rightarrow \infty} u_R$  is bounded in  $\mathbb{R}^d \times \mathbb{R}^+$  and satisfies (2.3.54), (2.3.55), (2.3.56). Hence, in view of Proposition 2.3.18,  $u$  necessarily coincides with the solution constructed in Paragraph 2.3.4.1: this in particular ensures that  $u \in C([0, \infty), L_\rho^1(\mathbb{R}^d))$ . Again, for general data  $u_0 \in L_\rho^1(\mathbb{R}^d)$ , we can select a sequence  $\{u_{0n}\} \subset L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  such that  $0 \leq u_{0n} \leq u_0$  and  $u_{0n} \rightarrow u_0$  in  $L_\rho^1(\mathbb{R}^d)$  and pass to the limit as  $n \rightarrow \infty$  in the weak formulation of Definition 2.3.1 to get a weak solution to (2.1.1), which still coincides with the one obtained in Paragraph 2.3.4.1 thanks to the uniqueness result of Proposition 2.3.2.

Finally, we point out that in [61] and [159] the approximate problems are slightly different from (2.3.58). Namely, cylinders in the upper plane are used instead of half-balls. Nevertheless, this change does not affect the construction of the solution  $u$ . In fact the present idea of using problem (2.3.58) is inspired by [60, Section 3], where the case  $s = 1/2$  and  $\rho \equiv 1$  is studied.

### 2.3.4.3 Local weak solutions and very weak solutions

Now let us address the following problem, which is the analogue of (2.3.58) in the whole upper plane:

$$\begin{cases} L_s(\tilde{u}^m) = 0 & \text{in } \mathbb{R}_+^{d+1} \times \mathbb{R}^+, \\ \tilde{u}^m = u^m & \text{on } \partial\mathbb{R}_+^{d+1} \times \mathbb{R}^+, \\ \frac{\partial(\tilde{u}^m)}{\partial y^{2s}} = \rho \frac{\partial u}{\partial t} & \text{on } \partial\mathbb{R}_+^{d+1} \times \mathbb{R}^+, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{t = 0\}. \end{cases} \quad (2.3.59)$$

**Definition 2.3.21.** *A nonnegative function  $u$  is a local weak solution to problem (2.3.59) corresponding to the nonnegative initial datum  $u_0 \in L_\rho^1(\mathbb{R}^d)$  if, for some nonnegative function  $\tilde{u}$  such that*

$$\tilde{u}^m \in L_{\text{loc}}^2((0, \infty); X_{\text{loc}}^s) \cap L^\infty(\mathbb{R}_+^{d+1} \times (\tau, \infty)) \quad \forall \tau > 0,$$

there hold:

- $u \in C([0, \infty); L_\rho^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty))$  for all  $\tau > 0$ ;
- $\tilde{u}^m|_{\partial\mathbb{R}_+^{d+1} \times (0, \infty)} = u^m$ ;
- for any  $\psi \in C_c^\infty((\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}) \times (0, \infty))$ ,

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x, t) \psi_t(x, 0, t) \rho(x) dx dt + \mu_s \int_0^\infty \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla(\tilde{u}^m), \nabla \psi \rangle(x, y, t) dx dy dt = 0 \quad (2.3.60)$$

(in fact  $\tilde{u}^m$  is a local extension for  $u^m$ );

- for any  $\varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$ ,

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} u^m(x, t) (-\Delta)^s(\varphi)(x, t) dx dt = 0; \quad (2.3.61)$$

- $\lim_{t \rightarrow 0} u(t) = u_0$  in  $L_\rho^1(\mathbb{R}^d)$ .

Moreover, we say that  $u$  is a local strong solution if, in addition,  $u_t \in L^\infty((\tau, \infty); L_{\rho, \text{loc}}^1(\mathbb{R}^d))$  for every  $\tau > 0$ .

We stress that (2.3.61) is related to the so-called *very weak* formulation of problem (2.1.1), see Definition 2.3.23 below. For local weak solutions, in general,  $u^m \notin L_{\text{loc}}^2((0, \infty); \dot{H}^s(\mathbb{R}^d))$ . Hence, equivalence between (2.3.60) and (2.3.61) cannot be established.

It is apparent that the criterion of Proposition 2.3.18 is not applicable in order to prove uniqueness of local weak solutions. However, one can restore the latter by imposing suitable

extra integrability conditions, as stated in Theorem 2.3.3. Before proving it, we need some preliminaries. Given a nonnegative  $f \in C_c^\infty(\mathbb{R}^d)$ , let  $h := I_{2s} * f \geq 0$ , so that

$$(-\Delta)^s(h) = f \quad \text{in } \mathbb{R}^d.$$

It is straightforward to check that  $h \in C^\infty(\mathbb{R}^d)$  and

$$h(x) + |\nabla h(x)| \leq K(1 + |x|)^{-d+2s} \quad \forall x \in \mathbb{R}^d$$

for some  $K > 0$  (use *e.g.* Lemma 2.3.12). Now take a cut-off function  $\xi \in C_c^\infty(\mathbb{R}^d)$  such that  $0 \leq \xi \leq 1$  in  $\mathbb{R}^d$ ,  $\xi \equiv 1$  in  $B_{1/2}$  and  $\xi \equiv 0$  in  $B_1^c$ . For any  $R > 0$ , let

$$\xi_R(x) := \xi(x/R) \quad \forall x \in \mathbb{R}^d. \quad (2.3.62)$$

We have:

$$(-\Delta)^s(h\xi_R)(x) = h(x)(-\Delta)^s(\xi_R)(x) + (-\Delta)^s(h)(x)\xi_R(x) + \mathcal{B}(h, \xi_R)(x) \quad \forall x \in \mathbb{R}^d,$$

where

$$\mathcal{B}(\phi_1, \phi_2)(x) := 2C_{d,s} \int_{\mathbb{R}^d} \frac{(\phi_1(x) - \phi_1(y))(\phi_2(x) - \phi_2(y))}{|x - y|^{d+2s}} dy \quad \forall x \in \mathbb{R}^d$$

and  $C_{d,s}$  is the positive constant appearing in (2.1.2) (see Subsection 2.A.3 for analogous computations). Lemma 3.1 of [161] in our context reads as follows.

**Lemma 2.3.22.** *Let  $f \in C_c^\infty(\mathbb{R}^d)$ , with  $f \geq 0$ ,  $h := I_{2s} * f$  and  $\xi_R$  be as in (2.3.62). Then, for any  $T > 0$  and  $v \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$ , there holds*

$$\lim_{R \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} |v(x, t) h(x) (-\Delta)^s(\xi_R)(x)| dx dt + \int_0^T \int_{\mathbb{R}^d} |v(x, t) \mathcal{B}(h, \xi_R)(x)| dx dt = 0.$$

We are now in position to prove Theorem 2.3.3.

*Proof of Theorem 2.3.3.* Let  $\underline{u}$  be the weak solution to problem (2.1.1) provided by Proposition 2.3.2. Its minimality in the class of solutions described by Definition 2.3.21, namely local strong solutions, is a consequence of the construction outlined in Paragraph 2.3.4.2 and the comparison principle given in Proposition 2.3.20: the approximate solutions whose limit is  $\underline{u}$ , let them be  $u_R$  or  $u_n$ , are smaller than any local strong solution. As concerns estimate (2.3.2), we only mention that it can be established by means of the same arguments as in the proof of [159, Theorem 5.5], combined with the smoothing effect (2.3.57) (see also [159, Remark 6.11]). We point out that the approximate solutions  $u_R$  used there are those obtained by solving (2.3.58) in cylinders  $\mathcal{C}_R$  rather than in half-balls  $\Omega_R$ . Nevertheless, by comparison, the solution to the Dirichlet problem in  $\Omega_R$  is below the one in  $\mathcal{C}_R$ , and this is clearly enough to get the upper bound (2.3.2). If  $u_0 \in L^\infty(\mathbb{R}^d)$  no smoothing effect is needed, so that the validity of (2.3.2) down to  $t_0 = 0$  is ensured by the fact that  $\underline{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$  (see again [159, Theorem 5.5]).

In order to prove our uniqueness results, let us first assume that  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . In this case, we have just shown that

$$\int_0^T \underline{u}^m(x, \tau) d\tau \leq K(I_{2s} * \rho)(x)$$

for all  $T > 0$ . In view of Lemma 2.3.12, it is straightforward to check that  $I_{2s} * \rho \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d)$  provided  $d > 4s$  and  $\gamma > 4s$ , whence  $\underline{u}^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$  for all  $T >$

0 and such values of the parameters. Moreover, since  $\underline{u} \in L^1_\rho(\mathbb{R}^d \times (0, T)) \cap L^\infty(\mathbb{R}^d \times (0, T))$  and so  $\underline{u}^m \in L^1_\rho(\mathbb{R}^d \times (0, T))$ , if  $\gamma \in (2s, d - 2s]$  we have again that  $\underline{u}^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$ .

Now take another local strong solution  $u$  to (2.1.1) corresponding to the same  $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , which for the moment we assume to be bounded as well in the whole of  $\mathbb{R}^d \times \mathbb{R}^+$ . Because both  $u$  and  $\underline{u}$  belong to

$$C([0, \infty); L^1_\rho(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^+),$$

by exploiting (2.3.61) it is direct to see that for any  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$  there holds

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} u^m(x, t) (-\Delta)^s(\varphi)(x, t) dx dt \\ &= - \int_0^\infty \int_{\mathbb{R}^d} \underline{u}(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} \underline{u}^m(x, t) (-\Delta)^s(\varphi)(x, t) dx dt \quad (2.3.63) \\ &= \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) \rho(x) dx. \end{aligned}$$

Let  $\eta \in C^\infty([0, +\infty))$  be such that

$$\eta(0) = 1, \quad \eta \equiv 0 \quad \text{in } [1, +\infty), \quad 0 < \eta < 1 \quad \text{in } (0, 1), \quad \eta' \leq 0 \quad \text{in } [0, +\infty).$$

For any  $T > 0$ , set

$$\eta_T(t) := \eta(t/T) \quad \forall t \geq 0.$$

Take the test function

$$\varphi(x, t) = h(x) \xi_R(x) \eta_T(t) \quad \forall (x, t) \in \mathbb{R}^d \times [0, +\infty),$$

where  $h$  and  $\xi_R$  are defined above, and plug it in the weak formulation (2.3.63) solved by  $u - \underline{u}$ . We get:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} f(x) \xi_R(x) \eta_T(t) [u^m(x, t) - \underline{u}^m(x, t)] dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} h(x) \xi_R(x) \eta_T'(t) [u(x, t) - \underline{u}(x, t)] \rho(x) dx dt \quad (2.3.64) \\ & \quad - \int_0^T \int_{\mathbb{R}^d} [h(x) (-\Delta)^s(\xi_R)(x) + \mathcal{B}(h, \xi_R)(x)] \eta_T(t) [u^m(x, t) - \underline{u}^m(x, t)] dx dt. \end{aligned}$$

Since  $\underline{u} \leq u$  and  $\eta_T' \leq 0$ , from (2.3.64) we find that

$$\begin{aligned} 0 &\leq \int_0^T \int_{\mathbb{R}^d} f(x) \xi_R(x) \eta_T(t) [u^m(x, t) - \underline{u}^m(x, t)] dx dt \\ &\leq \int_0^T \int_{\mathbb{R}^d} |h(x) (-\Delta)^s(\xi_R)(x) + \mathcal{B}(h, \xi_R)(x)| [u^m(x, t) + \underline{u}^m(x, t)] dx dt. \end{aligned} \quad (2.3.65)$$

Letting  $R \rightarrow \infty$  in (2.3.65) and applying Lemma 2.3.22 with  $v = u^m + \underline{u}^m$ , we then infer that  $u = \underline{u}$  in the region  $\text{supp } f \times (0, T)$ . Thanks to the arbitrariness of  $f$  and  $T$ , this means that  $u = \underline{u}$  in the whole of  $\mathbb{R}^d \times \mathbb{R}^+$ .

We finally need to get rid of the assumption  $u_0 \in L^\infty(\mathbb{R}^d)$ . First notice that, for any  $t_0 > 0$ ,  $u$  and  $\underline{u}$ , restricted to  $\mathbb{R}^d \times (t_0, \infty)$ , are *bounded* local strong solutions with initial data  $u(t_0) \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and  $\underline{u}(t_0) \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , respectively. Moreover,  $u^m, \underline{u}^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (t_0, T))$  for all  $T > t_0$ . Hence, in view of the uniqueness result we proved above, they both coincide with the corresponding minimal local strong solutions having the same initial data. But minimal solutions are, in fact, the ones constructed in Paragraphs 2.3.4.1 and 2.3.4.2, for which, in particular, the  $L^1_\rho$  comparison principle (2.3.53) holds true. As a consequence,  $\|u(t) - \underline{u}(t)\|_{1,\rho} \leq \|u(t_0) - \underline{u}(t_0)\|_{1,\rho}$  for all  $t > t_0 > 0$ . The conclusion follows by letting  $t_0 \rightarrow 0$  and recalling that  $u, \underline{u} \in C([0, \infty); L^1_\rho(\mathbb{R}^d))$ .  $\square$

Consider the next definition of *very weak* solution to problem (2.1.1).

**Definition 2.3.23.** *A nonnegative function  $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$  is a very weak solution to problem (2.1.1) corresponding to the nonnegative initial datum  $u_0 \in L^\infty(\mathbb{R}^d)$  if, for any  $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$ , there holds*

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{R}^d} u(x, t) \varphi_t(x, t) \rho(x) dx dt + \int_0^\infty \int_{\mathbb{R}^d} u^m(x, t) (-\Delta)^s(\varphi)(x, t) dx dt \\ & = \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) \rho(x) dx. \end{aligned}$$

It is plain that any bounded weak solution to (2.1.1) (according to Definition 2.3.1) is also a very weak solution in the sense of Definition 2.3.23.

**Remark 2.3.24.** As a byproduct of the method of proof of the uniqueness result in Theorem 2.3.3, it turns out that if  $u_1$  and  $u_2$  are *ordered very weak* solutions to problem to (2.1.1) (i.e.  $u_1 \leq u_2$  or  $u_2 \leq u_1$  in  $\mathbb{R}^d \times \mathbb{R}^+$ ) such that  $u_1^m, u_2^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$  for all  $T > 0$ , then  $u_1 = u_2$ .

## 2.A Self-adjointness of a weighted fractional Laplacian operator

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This appendix is devoted to the analysis of the operator formally defined as  $\rho^{-1}(-\Delta)^s$  in  $\mathbb{R}^d$  ( $s \in (0, 1)$ ,  $d \geq 1$ ), where  $\rho(x)$  is a weight which behaves approximately as a power of  $|x|$ , both as  $x \rightarrow 0$  and as  $|x| \rightarrow \infty$ .

### 2.A.1 Elementary scaling properties of fractional Laplacians

We omit the proofs of the first two lemmas, since they follow by exploiting the same strategy as [35, Lemma 2.1].

**Lemma 2.A.1.** *The fractional Laplacian  $(-\Delta)^s(\phi)(x)$  of any  $\phi \in \mathcal{D}(\mathbb{R}^d)$  is a regular function which decays (together with its derivatives) at least like  $|x|^{-d-2s}$  as  $|x| \rightarrow \infty$ .*

**Lemma 2.A.2.** *For any  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , the function*

$$l_s(\phi)(x) := \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{d+2s}} dy \quad \forall x \in \mathbb{R}^d$$

*is regular and decays (together with its derivatives) at least like  $|x|^{-d-2s}$  as  $|x| \rightarrow \infty$ .*

**Lemma 2.A.3.** *For any  $R > 0$ , let  $\xi_R$  be the cut-off function*

$$\xi_R(x) := \xi(x/R) \quad \forall x \in \mathbb{R}^d,$$

*where  $\xi$  is a positive, regular function such that  $\|\xi\|_\infty = 1$ ,  $\xi \equiv 1$  in  $B_1$  and  $\xi \equiv 0$  in  $B_2^c$ . Then,  $(-\Delta)^s(\xi_R)$  and  $l_s(\xi_R)$  enjoy the following scaling property:*

$$(-\Delta)^s(\xi_R)(x) = R^{-2s} (-\Delta)^s(\xi)(x/R), \quad l_s(\xi_R)(x) = R^{-2s} l_s(\xi)(x/R) \quad \forall x \in \mathbb{R}^d. \quad (2.A.1)$$

*Proof.* We only prove the result for  $l_s(\xi_R)$ , since the proof for  $(-\Delta)^s(\xi_R)$  is analogous. Letting  $\tilde{y} = y/R$ , one has:

$$l_s(\xi_R)(x) = \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dy = R^{-2s} \int_{\mathbb{R}^d} \frac{(\xi(x/R) - \xi(\tilde{y}))^2}{|x/R - \tilde{y}|^{d+2s}} d\tilde{y} = R^{-2s} l_s(\xi)(x/R).$$

□

**Lemma 2.A.4.** *Let  $\xi$  and  $\xi_R$  be as in Lemma 2.A.3. Let  $q \in [1, \infty)$  and  $\gamma \in [0, d + 2q's)$ . Then the norms*

$$\| |x|^\gamma (-\Delta)^s(\xi) \|_{q, -\gamma}, \quad \| |x|^\gamma l_s(\xi) \|_{q, -\gamma} \quad (2.A.2)$$

are finite. If in addition  $\gamma \in [0, d + 2s]$ , then also the norms

$$\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty, \quad \| |x|^\gamma l_s(\xi) \|_\infty \quad (2.A.3)$$

are finite. Moreover,

$$\| |x|^\gamma (-\Delta)^s(\xi_R) \|_{q, -\gamma} = \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_{q, -\gamma}}{R^{2s-\gamma-\frac{d-\gamma}{q}}}, \quad (2.A.4)$$

$$\| |x|^\gamma (-\Delta)^s(\xi_R) \|_\infty = \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \quad (2.A.5)$$

and

$$\| |x|^\gamma l_s(\xi_R) \|_{q, -\gamma} = \frac{\| |x|^\gamma l_s(\xi) \|_{q, -\gamma}}{R^{2s-\gamma-\frac{d-\gamma}{q}}}, \quad (2.A.6)$$

$$\| |x|^\gamma l_s(\xi_R) \|_\infty = \frac{\| |x|^\gamma l_s(\xi) \|_\infty}{R^{2s-\gamma}}. \quad (2.A.7)$$

*Proof.* The finiteness of (2.A.2) and (2.A.3) is ensured by the decay properties of  $(-\Delta)^s(\xi)(x)$  and  $l_s(\xi)(x)$  recalled in Lemmas 2.A.1 and 2.A.2. Similarly, identities (2.A.4)–(2.A.7) follow from the scaling property (2.A.1).  $\square$

## 2.A.2 Density of mollifications in weighted Lebesgue spaces

Hereafter, we shall assume that  $\rho$  is a positive, measurable function such that

$$c|x|^\lambda \leq \rho(x) \leq C|x|^\lambda \quad \text{a.e. in } B_1 \quad \text{and} \quad c|x|^\Lambda \leq \rho(x) \leq C|x|^\Lambda \quad \text{a.e. in } B_1^c \quad (2.A.8)$$

for some positive constants  $c < C$  and exponents  $\lambda, \Lambda \in \mathbb{R}$ .

**Theorem 2.A.5.** *Let  $p \in (1, \infty)$ . Assume that (2.A.8) holds true for some  $\lambda \in (-d, (p-1)d)$  and  $\gamma \in \mathbb{R}$ . Let  $f \in L^p_\rho(\mathbb{R}^d)$  and consider the standard mollification*

$$f_\varepsilon(x) := \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y) dy \quad \forall x \in \mathbb{R}^d, \quad (2.A.9)$$

where

$$\eta_\varepsilon(x) := \varepsilon^{-d} \eta(x/\varepsilon) \quad \forall x \in \mathbb{R}^d, \quad \forall \varepsilon > 0,$$

$\eta$  being a regular nonnegative function, supported in  $B_1$  and such that  $\int_{\mathbb{R}^d} \eta(x) dx = 1$ . Then,  $f_\varepsilon \in C^\infty(\mathbb{R}^d) \cap L^p_\rho(\mathbb{R}^d)$  and

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - f\|_{p, \rho} = 0. \quad (2.A.10)$$

*Proof.* For simplicity, we shall give the full proof only in the simpler case  $\rho(x) = |x|^\lambda$ . Minor modifications are required to deal with a more general weight as in the statement. In order to give sense to (2.A.9) and to prove that  $f_\varepsilon \in C^\infty(\mathbb{R}^d)$ , we first need to establish that  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ . To this end, by means of Hölder's inequality, for any  $R > 0$  we get:

$$\int_{B_R} |f(y)| dy \leq \left( \int_{B_R} |y|^{-\frac{\lambda}{p-1}} dy \right)^{\frac{p-1}{p}} \left( \int_{B_R} |f(y)|^p |y|^\lambda dy \right)^{\frac{1}{p}} \leq \frac{|\mathbb{S}_{d-1}|^{\frac{p-1}{p}} R^{\frac{d(p-1)-\lambda}{p}}}{\left(d - \frac{\lambda}{p-1}\right)^{\frac{p-1}{p}}} \|f\|_{p, \lambda}, \quad (2.A.11)$$

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## 2.A. Self-adjointness of a weighted fractional Laplacian operator

where  $\mathbb{S}_{d-1}$  is the unitary  $(d-1)$ -dimensional sphere. The validity of (2.A.10) is in fact implied by the validity of

$$\|f_\varepsilon\|_{p,\lambda} \leq K \|f\|_{p,\lambda} \quad \forall f \in L_\lambda^p(\mathbb{R}^d) \quad (2.A.12)$$

for a suitable positive constant  $K$  independent of  $\varepsilon$ . Indeed, once we have (2.A.12), we can proceed as follows. First of all, we pick a sequence of functions  $\{f_n\}$  which are compactly supported in  $\mathbb{R}^d \setminus \{0\}$  and such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{p,\lambda} = 0. \quad (2.A.13)$$

This is always possible: for any given  $n \in \mathbb{N}$ , one considers the truncated functions

$$f_n(x) := \begin{cases} f(x) & \text{if } \frac{1}{n} \leq |x| \leq n, \\ 0 & \text{elsewhere.} \end{cases}$$

It is plain that each  $f_n$  is by definition compactly supported in  $\mathbb{R}^d \setminus \{0\}$  and belongs to  $L^p(\mathbb{R}^d)$ , and that the sequence  $\{f_n\}$  converges to  $f$  in  $L_\lambda^p(\mathbb{R}^d)$  as  $n \rightarrow \infty$ . By standard results (see *e.g.* [1, Chapters 2, 3] or [91, Appendix C.4]) we know that, for any  $n \in \mathbb{N}$ , the mollification  $(f_n)_\varepsilon$  converges to  $f_n$  in  $L^p(\mathbb{R}^d)$ . Since  $(f_n)_\varepsilon$  is eventually supported in  $B_{2n} \cap B_{1/2n}^c$  and the weight  $|x|^\lambda$  is equivalent to 1 in such region, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \|(f_n)_\varepsilon - f_n\|_{p,\lambda} = 0. \quad (2.A.14)$$

Using the triangular inequality, the linearity of the mollification operator and (2.A.12), we get:

$$\|f_\varepsilon - f\|_{p,\lambda} \leq (K+1) \|f_n - f\|_{p,\lambda} + \|(f_n)_\varepsilon - f_n\|_{p,\lambda}. \quad (2.A.15)$$

Thanks to (2.A.13), for any given  $\delta > 0$  we can pick  $n_\delta$  so large that  $\|f_{n_\delta} - f\|_{p,\lambda} \leq \delta$ . Letting  $\varepsilon \rightarrow 0$  in (2.A.15) evaluated at  $n = n_\delta$  and using (2.A.14), we obtain:

$$\limsup_{\varepsilon \rightarrow 0} \|f_\varepsilon - f\|_{p,\lambda} \leq (K+1) \delta.$$

The arbitrariness of  $\delta$  entails (2.A.10). We are then left with proving the validity of (2.A.12). To this aim, let us first observe that

$$\|f_\varepsilon\|_{p,\lambda}^p = \int_{\mathbb{R}^d} |f_\varepsilon(x)|^p |x|^\lambda dx = \int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda dx + \int_{\mathbb{R}^d \setminus B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda dx. \quad (2.A.16)$$

We shall estimate the two integrals in the r.h.s. of (2.A.16) separately. As for the first one, we have:

$$\int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda dx \leq |\mathbb{S}_{d-1}| \frac{2^{d+\lambda} \varepsilon^{d+\lambda}}{d+\lambda} \|f_\varepsilon\|_{L^\infty(B_{2\varepsilon})}^p, \quad (2.A.17)$$

where, in view of (2.A.9) and (2.A.11) (the latter evaluated at  $R = 3\varepsilon$ ),

$$\|f_\varepsilon\|_{L^\infty(B_{2\varepsilon})} \leq \frac{\|\eta\|_\infty}{\varepsilon^d} \int_{B_{3\varepsilon}} |f(y)| dy \leq \frac{3^{\frac{d(p-1)-\lambda}{p}} |\mathbb{S}_{d-1}|^{\frac{p-1}{p}} \|\eta\|_\infty}{\varepsilon^{\frac{d+\lambda}{p}} \left(d - \frac{\lambda}{p-1}\right)^{\frac{p-1}{p}}} \|f\|_{p,\lambda}. \quad (2.A.18)$$

Combining (2.A.17) and (2.A.18) we obtain

$$\int_{B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda dx \leq \frac{2^{d+\lambda} 3^{d(p-1)-\lambda} |\mathbb{S}_{d-1}|^p \|\eta\|_\infty^p}{(d+\lambda) \left(d - \frac{\lambda}{p-1}\right)^{p-1}} \|f\|_{p,\lambda}^p. \quad (2.A.19)$$



Now we need to estimate the second integral in the r.h.s. of (2.A.16). We have:

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{2\varepsilon}} |f_\varepsilon(x)|^p |x|^\lambda dx &= \int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \left| \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y) dy \right|^p |x|^\lambda dx \\ &\leq \int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \left( \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) |f(y)|^p dy \right) |x|^\lambda dx \\ &= \int_{\mathbb{R}^d} |f(y)|^p \left( \int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda dx \right) dy, \end{aligned} \quad (2.A.20)$$

where we exploited Hölder's inequality, at fixed  $x$ , w.r.t. the probability measure  $\eta_\varepsilon(x-y)dy$ . Thanks to (2.A.20), we are only left with showing that there exists a positive constant  $K'$ , independent of  $\varepsilon$ , such that

$$\int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda dx \leq K' |y|^\lambda \quad \forall y \in \mathbb{R}^d. \quad (2.A.21)$$

First of all observe that

$$\int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \eta_\varepsilon(x-y) |x|^\lambda dx \leq \|\eta\|_\infty \frac{\int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \chi_{\{|x-y| \leq \varepsilon\}} |x|^\lambda dx}{\varepsilon^d}. \quad (2.A.22)$$

It is apparent that for  $|y| < \varepsilon$  the integral in the r.h.s. of (2.A.22) is identically zero, while for  $|y| > 2\varepsilon$ , we have:

$$\begin{aligned} \frac{\int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \chi_{\{|x-y| \leq \varepsilon\}} |x|^\lambda dx}{\varepsilon^d} &\leq \frac{\int_{B_\varepsilon(y)} |x|^\lambda dx}{\varepsilon^d} \leq \frac{|\mathbb{S}_{d-1}|}{d} \max \left\{ (|y| + \varepsilon)^\lambda, (|y| - \varepsilon)^\lambda \right\} \\ &\leq \frac{|\mathbb{S}_{d-1}|}{d} \max \left\{ \frac{3^\lambda}{2^\lambda} |y|^\lambda, \frac{1}{2^\lambda} |y|^\lambda \right\} \\ &= \frac{|\mathbb{S}_{d-1}|}{d2^\lambda} \max \left\{ 3^\lambda, 1 \right\} |y|^\lambda. \end{aligned} \quad (2.A.23)$$

Hence, it remains to estimate the r.h.s. of (2.A.22) as  $y$  varies in the set  $\{\varepsilon \leq |y| \leq 2\varepsilon\}$ . To this end, notice that in such region the following inequality holds true:

$$\frac{\int_{\mathbb{R}^d \setminus B_{2\varepsilon}} \chi_{\{|x-y| \leq \varepsilon\}} |x|^\lambda dx}{\varepsilon^d} \leq \frac{\int_{B_{|y|+\varepsilon} \setminus B_{2\varepsilon}} |x|^\lambda dx}{\varepsilon^d} = \frac{|\mathbb{S}_{d-1}|}{d+\lambda} \frac{(|y| + \varepsilon)^{d+\lambda} - 2^{d+\lambda} \varepsilon^{d+\lambda}}{\varepsilon^d}. \quad (2.A.24)$$

We can finally show that there exists a positive constant  $M$ , independent of  $\varepsilon$ , such that

$$\frac{(|y| + \varepsilon)^{d+\lambda} - 2^{d+\lambda} \varepsilon^{d+\lambda}}{\varepsilon^d} \leq M |y|^\lambda \quad \forall y \in \overline{B_{2\varepsilon}} \cap B_\varepsilon^c. \quad (2.A.25)$$

In fact, performing the change of variable  $z = |y|/\varepsilon$ , we infer that the validity of (2.A.25) is equivalent to the validity of

$$\frac{(z+1)^{d+\lambda} - 2^{d+\lambda}}{z^\lambda} \leq M \quad \forall z \in [1, 2].$$

Therefore, upon setting

$$M := \max_{z \in [1, 2]} \frac{(z+1)^{d+\lambda} - 2^{d+\lambda}}{z^\lambda}$$

and gathering (2.A.22)–(2.A.25), we deduce that (2.A.21) actually holds true for

$$K' = |\mathbb{S}_{d-1}| \|\eta\|_\infty \max \left\{ \frac{\max \left\{ 3^\lambda, 1 \right\}}{d2^\lambda}, \frac{M}{d+\lambda} \right\},$$

whence, recalling (2.A.19) and (2.A.20), there follows (2.A.12) with the choice

$$C = \left[ \frac{2^{d+\lambda} 3^{d(p-1)-\lambda} |\mathbb{S}_{d-1}|^p \|\eta\|_\infty^p}{(d+\lambda) \left(d - \frac{\lambda}{p-1}\right)^{p-1}} + |\mathbb{S}_{d-1}| \|\eta\|_\infty \max \left\{ \frac{\max \{3^\lambda, 1\}}{d2^\lambda}, \frac{M}{d+\lambda} \right\} \right]^{\frac{1}{p}}.$$

If  $\rho$  is a general weight as in the statement, one can split  $f$  in the sum  $f = f_1 + f_2$ , with  $f_1 := f\chi_{B_1}$  and  $f_2 := f\chi_{B_1^c}$ . By linearity,  $f_\varepsilon = (f_1)_\varepsilon + (f_2)_\varepsilon$ ; it is therefore enough to show that (2.A.10) holds true for  $f_1$  and  $f_2$  separately. As concerns  $f_1$ , since the latter and its mollifications are (eventually) supported *e.g.* in  $B_{3/2}$ , one can modify  $\rho(x)$  so that it behaves like  $|x|^\lambda$  also in  $B_{3/2}^c$  and then apply the first part of the theorem. Similarly, because  $f_2$  and its mollifications are eventually supported in  $B_{1/2}^c$ , the validity of the analogue of (2.A.12) (and so of (2.A.10)) is now implied by the validity of (2.A.21) in the region  $\{|y| > 1/2\}$ , which holds for all  $\lambda = \Lambda \in \mathbb{R}$  in view of (2.A.23).  $\square$

**Remark 2.A.6.** Notice that the above assumption  $\lambda \in (-d, (p-1)d)$  is necessary. In fact, consider first the following function:

$$g(x) := \frac{\chi_{B_{1/2}}}{|x|^d \log |x|}.$$

It is apparent that  $g \notin L_{\text{loc}}^1(\mathbb{R}^d)$ . Moreover, its mollification  $g_\varepsilon$  is equal to  $-\infty$  in a set of positive measure, for all  $\varepsilon > 0$ . However,  $g$  belongs to  $L_\lambda^p(\mathbb{R}^d)$  for all  $\lambda \geq (p-1)d$ .

As concerns the bound from below over  $\lambda$ , consider the function

$$h(x) := \chi_{B_1} |x|^{-\frac{\lambda}{p}} \in L_\lambda^p(\mathbb{R}^d).$$

For all  $\varepsilon > 0$  small enough, the mollification  $h_\varepsilon$  is *strictly positive* in a neighbourhood of 0. In particular, if  $\lambda \leq -d$  then  $h_\varepsilon \notin L_\lambda^p(\mathbb{R}^d)$ , so that (2.A.10) (for  $f = h$ ) cannot hold.

### 2.A.3 Self-adjointness and integration by parts formulas

For a function  $f$  belonging to  $L_{\text{loc}}^1(\mathbb{R}^d) \cap L_{-d-2s}^1(B_1^c)$ , a property that any element of  $L_\rho^p(\mathbb{R}^d)$  enjoys if  $\rho$  satisfies (2.A.8) with

$$\lambda < (p-1)d \quad \text{and} \quad \Lambda > -d - 2ps,$$

the action

$$\phi \longrightarrow \int_{\mathbb{R}^d} f(x) (-\Delta)^s(\phi)(x) dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d) \tag{2.A.26}$$

is actually an element of  $\mathcal{D}'(\mathbb{R}^d)$ . This is an immediate consequence of the fact that the notion of convergence of a sequence  $\{\phi_n\} \subset \mathcal{D}(\mathbb{R}^d)$  to  $\phi$  in  $\mathcal{D}(\mathbb{R}^d)$  implies, in particular, the pointwise convergence of  $\{(-\Delta)^s(\phi_n)\}$  to  $(-\Delta)^s(\phi)$  and the validity of the bound

$$|(-\Delta)^s(\phi_n)(x)| \leq C(1 + |x|)^{-d-2s} \quad \forall x \in \mathbb{R}^d \tag{2.A.27}$$

for a suitable positive constant  $C$  independent of  $n$  (recall Lemma 2.A.1). The integrability properties of  $f$  and (2.A.27) then permit us to pass to the limit in (2.A.26) (with  $\phi = \phi_n$ ) as  $n \rightarrow \infty$ .

In the next lemma we show that, for regular functions having suitable integrability properties at infinity, the distributional fractional Laplacian and the classical one coincide.

**Lemma 2.A.7.** *Let  $v \in C^\infty(\mathbb{R}^d) \cap L_\Lambda^p(B_1^c)$ , with  $p \in (1, \infty)$  and  $\Lambda \geq -d - ps$ . Then the classical fractional Laplacian of  $v$ , defined as*

$$(-\Delta)^s(v)(x) := C_{d,s} p.v. \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x - y|^{d+2s}} dy \quad \forall x \in \mathbb{R}^d, \quad (2.A.28)$$

is a continuous function which coincides with its distributional fractional Laplacian, in the sense that

$$\int_{\mathbb{R}^d} v(x)(-\Delta)^s(\phi)(x) dx = \int_{\mathbb{R}^d} (-\Delta)^s(v)(x)\phi(x) dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \quad (2.A.29)$$

*Proof.* To begin with, we prove that formula (2.A.28) provides us with a locally bounded function of  $x$ . To this end, fix  $R > 0$  and let  $x$  vary in  $B_R$ . It is direct to see that the integral

$$p.v. \int_{B_{2R}} \frac{v(x) - v(y)}{|x - y|^{d+2s}} dy \quad \forall x \in B_R$$

is bounded in modulus by a constant (depending on  $R$ ) times  $\|\nabla^2 v\|_{L^\infty(B_{2R})}$ . Moreover, still for  $x$  varying in  $B_R$ , we have:

$$\begin{aligned} \int_{B_{2R}^c} \frac{|v(x) - v(y)|}{|x - y|^{d+2s}} dy &\leq 2^{d+2s} \|v\|_{L^\infty(B_R)} \int_{B_{2R}^c} |y|^{-d-2s} dy + 2^{d+2s} \int_{B_{2R}^c} |v(y)| |y|^{-d-2s} dy \\ &\leq 2^{d+2s} \|v\|_{L^\infty(B_R)} \int_{B_{2R}^c} |y|^{-d-2s} dy \\ &\quad + 2^{d+2s} \left( \int_{B_{2R}^c} |v(y)|^p |y|^\Lambda dy \right)^{\frac{1}{p}} \left( \int_{B_{2R}^c} |y|^{-\frac{p(d+2s)+\Lambda}{p-1}} dy \right)^{\frac{1}{p'}}. \end{aligned}$$

Note that the r.h.s. does not depend on  $x \in B_R$  and is finite since  $v \in C^\infty(\mathbb{R}^d) \cap L_\Lambda^p(B_1^c)$  with  $\Lambda \geq -d - ps > -d - 2ps$ . We have therefore proved that  $(-\Delta)^s(v)$  is locally bounded. Continuity follows by similar arguments, which we omit.

Now we aim at proving that  $(-\Delta)^s(v)$  is in fact the distributional fractional Laplacian of  $v$ , namely the validity of (2.A.29). Let us first consider the truncated function  $\xi_R v \in \mathcal{D}(\mathbb{R}^d)$  ( $\xi_R$  is as in Lemma 2.A.3) and observe that, for the latter, the identity

$$\int_{\mathbb{R}^d} \xi_R(x)v(x)(-\Delta)^s(\phi)(x) dx = \int_{\mathbb{R}^d} (-\Delta)^s(\xi_R v)(x)\phi(x) dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d) \quad (2.A.30)$$

holds true (see *e.g.* [69, Section 3]). Using the product formula

$$\begin{aligned} (-\Delta)^s(\xi_R v)(x) &= \xi_R(x)(-\Delta)^s(v)(x) + (-\Delta)^s(\xi_R)(x)v(x) \\ &\quad + 2C_{d,s} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dy \end{aligned} \quad (2.A.31)$$

and plugging it in (2.A.30), we get:

$$\begin{aligned} &\int_{\mathbb{R}^d} \xi_R(x)v(x)(-\Delta)^s(\phi)(x) dx \\ &= \int_{\mathbb{R}^d} \phi(x)\xi_R(x)(-\Delta)^s(v)(x) dx + \int_{\mathbb{R}^d} \phi(x)v(x)(-\Delta)^s(\xi_R)(x) dx \\ &\quad + 2C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) \frac{(v(x) - v(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy. \end{aligned} \quad (2.A.32)$$

Letting  $R \rightarrow \infty$ , the l.h.s. and the first term in the r.h.s. of (2.A.32) make no problem since  $v$  is locally regular and integrable at infinity with respect to the weight  $|x|^{-d-2s}$  (recall again the

hypothesis on  $\Lambda$ ), while  $(-\Delta)^s(v)$  is locally bounded. As concerns the second term, taking advantage of (2.A.5) (with  $\gamma = 0$ ) we obtain:

$$\int_{\mathbb{R}^d} |\phi(x)v(x)(-\Delta)^s(\xi_R)(x)| dx \leq \frac{\|(-\Delta)^s(\xi)\|_\infty}{R^{2s}} \int_{\mathbb{R}^d} |\phi(x)v(x)| dx. \quad (2.A.33)$$

It is then apparent that (2.A.33) vanishes as  $R \rightarrow \infty$ . In order to handle the last term in the r.h.s. of (2.A.32), we have to work a bit more. First of all let us introduce, for any  $q \in (1, \infty)$ , the following operator:

$$l_{q,s}(\phi)(x) := \int_{\mathbb{R}^d} \frac{|\phi(x) - \phi(y)|^q}{|x - y|^{d+qs}} dy \quad \forall x \in \mathbb{R}^d$$

(note that  $l_{2,s} = l_s$ ). As in Lemmas 2.A.2–2.A.4, it is easy to show that  $l_{q,s}(\xi)(x)$  is a continuous function which decays at least like  $|x|^{-d-qs}$  as  $|x| \rightarrow \infty$  and satisfies the scaling identities

$$l_{q,s}(\xi_R)(x) = R^{-qs} l_{q,s}(\xi)(x/R), \quad \|l_{q,s}(\xi_R)\|_\infty = R^{-qs} \|l_{q,s}(\xi)\|_\infty. \quad (2.A.34)$$

Proceeding in similar way as above, we can prove that also  $l_{p,s}(v)(x)$  is a locally bounded (actually continuous) function. In fact, for all  $R > 0$  and  $x \in B_R$ , the integral

$$\int_{B_{2R}} \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} dy$$

is bounded in modulus by a constant (depending on  $R$ ) times  $\|\nabla v\|_{L^\infty(B_{2R})}^p$ . Moreover,

$$\int_{B_{2R}^c} \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} dy \leq 2^{d-1+p(s+1)} \|v\|_{L^\infty(B_R)}^p \int_{B_{2R}^c} |y|^{-d-ps} dy + 2^{d-1+p(s+1)} \int_{B_{2R}^c} \frac{|v(y)|^p}{|y|^{d+ps}} dy,$$

and the r.h.s. is finite independently of  $x \in B_R$  (here we exploit the assumption  $\Lambda \geq -d-ps$ ). Now observe that, by writing

$$\frac{(v(x) - v(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} = \frac{(v(x) - v(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{\frac{d+ps}{p}} |x - y|^{\frac{d+p's}{p'}}}, \quad p' := \frac{p}{p-1},$$

applying Hölder's inequality w.r.t.  $dy$  and using (2.A.34), we get:

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \phi(x) \frac{(v(x) - v(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy &\leq \int_{\mathbb{R}^d} |\phi(x)| [l_{p,s}(v)(x)]^{\frac{1}{p}} [l_{p',s}(\xi_R)(x)]^{\frac{1}{p'}} dx \\ &\leq R^{-s} \|l_{p',s}(\xi)\|_\infty^{\frac{1}{p'}} \int_{\mathbb{R}^d} |\phi(x)| [l_{p,s}(v)(x)]^{\frac{1}{p}} dx. \end{aligned}$$

Letting  $R \rightarrow \infty$ , we then deduce that also the last term in the r.h.s. of (2.A.32) vanishes, so that (2.A.29) is finally proved.  $\square$

In the following, we shall mainly focus on weights  $\rho$  complying with (2.A.8) for *nonpositive*  $\lambda$  and  $\Lambda$ , though some of the results below can be extended to suitable positive exponents. Hence, for greater readability, we shall reformulate (2.A.8) as follows:

$$c|x|^{-\gamma_0} \leq \rho(x) \leq C|x|^{-\gamma_0} \quad \text{a.e. in } B_1 \quad \text{and} \quad c|x|^{-\gamma} \leq \rho(x) \leq C|x|^{-\gamma} \quad \text{a.e. in } B_1^c \quad (2.A.35)$$

for some positive constants  $c < C$  and nonnegative exponents  $\gamma_0, \gamma$ .

**Definition 2.A.8.** Let  $p \in (1, \infty)$  and  $p' := p/(p-1)$  be its conjugate exponent. Suppose that  $\rho$  satisfies (2.A.35) for some  $\gamma_0 \in [0, d)$  and  $\gamma \in [0, d+ps]$ . We denote as  $X_{p,s,\rho}$  the space of all functions  $v \in L^p_\rho(\mathbb{R}^d)$  such that  $(-\Delta)^s(v)$  (as a distribution) belongs to  $L^{p'}_{\rho'}(\mathbb{R}^d)$ , where  $\rho' := \rho^{-(p'-1)}$ . In the special case  $p = 2$ , we use the simplified notation  $X_{2,s,\rho} =: X_{s,\rho}$ .

According to the above definition, a function  $v \in L^p_\rho(\mathbb{R}^d)$  belongs to  $X_{p,s,\rho}$  if and only if there exists an element

$$\mathcal{V} \in L^{p'}_{\rho'}(\mathbb{R}^d)$$

such that

$$\int_{\mathbb{R}^d} v(x)(-\Delta)^s(\phi)(x) dx = \int_{\mathbb{R}^d} \mathcal{V}(x)\phi(x) dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \quad (2.A.36)$$

Notice that the assumptions  $\gamma_0 \in [0, d)$  and  $\gamma \in [0, d+ps]$  ensure that both the left and the right-hand side of (2.A.36) are actually distributions ( $v \in L^1_{\text{loc}}(\mathbb{R}^d) \cap L^1_{-d-2s}(B_1^c)$  and  $\mathcal{V} \in L^1_{\text{loc}}(\mathbb{R}^d)$ ).

**Lemma 2.A.9.** Let  $p, \rho, \rho'$  and  $X_{p,s,\rho}$  be as in Definition 2.A.8. Then the following properties hold true:

(a)  $\mathcal{D}(\mathbb{R}^d) \subset X_{p,s,\rho}$ ;

(b)  $X_{p,s,\rho}$ , endowed with the norm

$$\|v\|_{X_{p,s,\rho}} := \left( \|v\|_{p,\rho}^2 + \|(-\Delta)^s(v)\|_{p',\rho'}^2 \right)^{\frac{1}{2}} \quad \forall v \in X_{p,s,\rho},$$

is a Banach space (Hilbert if  $p = 2$ );

(c) the subspace  $C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$  is dense in  $X_{p,s,\rho}$ ;

(d) the map  $\mathcal{B} : X_{p,s,\rho} \times X_{p,s,\rho} \mapsto \mathbb{R}$ , defined as

$$\mathcal{B}(v, w) := \int_{\mathbb{R}^d} (-\Delta)^s(v)(x)w(x) dx \quad \forall v, w \in X_{p,s,\rho},$$

is a continuous bilinear form on  $X_{p,s,\rho}$ .

*Proof.* In order to prove (a) it is enough to check that, for any  $\phi \in \mathcal{D}(\mathbb{R}^d)$ , we have  $\phi \in L^p_\rho(\mathbb{R}^d)$  and  $(-\Delta)^s(\phi) \in L^{p'}_{\rho'}(\mathbb{R}^d)$ . But this is straightforward:  $\rho$  is locally integrable since  $\gamma_0 \in [0, d)$  and  $(-\Delta)^s(\phi)(x)$  is a regular function decaying at least like  $|x|^{-d-2s}$  as  $|x| \rightarrow \infty$ , which in particular implies that it belongs to  $L^{p'}_{\rho'}(\mathbb{R}^d)$  (recall that  $\gamma \in [0, d+ps]$ ).

As for (b), take a Cauchy sequence  $\{v_n\} \subset X_{p,s,\rho}$ . By the definition of  $\|\cdot\|_{X_{p,s,\rho}}$  and by the completeness of  $L^p_\rho(\mathbb{R}^d)$  and  $L^{p'}_{\rho'}(\mathbb{R}^d)$ , there exist  $v \in L^p_\rho(\mathbb{R}^d)$  and  $\mathcal{V} \in L^{p'}_{\rho'}(\mathbb{R}^d)$  such that

$$\lim_{n \rightarrow \infty} \left( \|v_n - v\|_{p,\rho}^2 + \|(-\Delta)^s(v_n) - \mathcal{V}\|_{p',\rho'}^2 \right)^{\frac{1}{2}} = 0.$$

In order to infer that  $X_{p,s,\rho}$  is complete we need to show that  $\mathcal{V} = (-\Delta)^s(v)$ , which is true if we can pass to the limit in the identity

$$\int_{\mathbb{R}^d} v_n(x)(-\Delta)^s(\phi)(x) dx = \int_{\mathbb{R}^d} (-\Delta)^s(v_n)(x)\phi(x) dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

This is indeed the case because  $(-\Delta)^s(\phi) \in L^{p'}_{\rho'}(\mathbb{R}^d)$  is equivalent to  $\rho^{-1}(-\Delta)^s(\phi) \in L^{p'}_\rho(\mathbb{R}^d)$ , and the same holds for  $(-\Delta)^s(v_n)$ . The fact that  $X_{2,s,\rho} = X_{s,\rho}$  is Hilbert just follows upon defining the scalar product

$$\langle v, w \rangle_{X_{s,\rho}} := \langle v, w \rangle_{L^2_\rho} + \langle (-\Delta)^s(v), (-\Delta)^s(w) \rangle_{L^2_{\rho'}}, \quad \forall v, w \in X_{s,\rho}.$$

Now let us prove (c). We shall exploit the crucial result provided by Theorem 2.A.5. Thanks to the latter, given any  $v \in X_{p,s,\rho}$ , its mollification  $v_\varepsilon$  belongs to  $C^\infty(\mathbb{R}^d) \cap L_\rho^p(\mathbb{R}^d)$  and converges to  $v$  in  $L_\rho^p(\mathbb{R}^d)$ . We claim that the fractional Laplacian of  $v_\varepsilon$ , which is well defined both in the classical and in the distributional sense in view of Lemma 2.A.7, is in fact the mollification of  $(-\Delta)^s(v)$ , that is

$$(-\Delta)^s(v_\varepsilon) = [(-\Delta)^s(v)]_\varepsilon. \quad (2.A.37)$$

This is straightforward to prove, since for any  $\phi \in \mathcal{D}(\mathbb{R}^d)$  the following identities hold true:

$$\begin{aligned} \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x)v_\varepsilon(x) dx &= \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(x) \left( \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)v(y) dy \right) dx \\ &= \int_{\mathbb{R}^d} \eta_\varepsilon(z) \left( \int_{\mathbb{R}^d} (-\Delta)^s(\phi)(z+y)v(y) dy \right) dz \\ &= \int_{\mathbb{R}^d} \eta_\varepsilon(z) \left( \int_{\mathbb{R}^d} \phi(z+y)(-\Delta)^s(v)(y) dy \right) dz \\ &= \int_{\mathbb{R}^d} \phi(x) \left( \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)(-\Delta)^s(v)(y) dy \right) dx \\ &= \int_{\mathbb{R}^d} \phi(x) [(-\Delta)^s(v)]_\varepsilon(x) dx. \end{aligned}$$

The changes of order of integration above are justified since the integrals

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Delta)^s(\phi)(x)\eta_\varepsilon(x-y)v(y)| dx dy \quad \text{and} \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\phi(x)\eta_\varepsilon(x-y)(-\Delta)^s(v)(y)| dx dy$$

are finite, a consequence of the fact that  $|v|_\varepsilon, \phi \in L_\rho^p(\mathbb{R}^d)$  and  $(-\Delta)^s(\phi), |(-\Delta)^s(v)|_\varepsilon \in L_{\rho'}^{p'}(\mathbb{R}^d)$ . Having established (2.A.37) and recalling again Theorem 2.A.5, we therefore deduce that  $v_\varepsilon \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$  and

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon - v\|_{X_{p,s,\rho}} = \lim_{\varepsilon \rightarrow 0} \left( \|v_\varepsilon - v\|_{p,\rho}^2 + \|[(-\Delta)^s(v)]_\varepsilon - (-\Delta)^s(v)\|_{p',\rho'}^2 \right)^{\frac{1}{2}} = 0,$$

which proves (c).

Finally, the only nontrivial point in (d) is the continuity of  $\mathcal{B}$ , which follows by a direct application of Hölder's inequality:

$$\begin{aligned} |\mathcal{B}(v, w)| &\leq \int_{\mathbb{R}^d} |(-\Delta)^s(v)(x)w(x)| dx \\ &\leq \left( \int_{\mathbb{R}^d} |(-\Delta)^s(v)(x)|^{p'} \rho'(x) dx \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}^d} |w(x)|^p \rho(x) dx \right)^{\frac{1}{p}} \\ &= \|(-\Delta)^s(v)\|_{p',\rho'} \|w\|_{p,\rho} \\ &\leq \|v\|_{X_{p,s,\rho}} \|w\|_{X_{p,s,\rho}} \end{aligned}$$

for all  $v, w \in X_{p,s,\rho}$ . □

**Lemma 2.A.10.** *Let  $p \in [2, \infty)$ . Suppose that  $\rho$  satisfies (2.A.35) for some  $\gamma_0 \in [0, d)$  and  $\gamma \in [0, d + 2s]$  if  $p = 2$ ,  $\gamma \in [0, d + ps)$  if  $p > 2$ . Let  $l_s, \xi$  and  $\xi_R$  be as in Lemmas 2.A.2, 2.A.3 and  $v_1, v_2 \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$ . Then the integral*

$$I_R(v_i) := \int_{\mathbb{R}^d} \xi_R^2(x) \left( \int_{\mathbb{R}^d} \frac{(v_i(x) - v_i(y))^2}{|x - y|^{d+2s}} dy \right) dx \quad i = 1, 2$$

is finite for all  $R > 0$ . Moreover, the following estimates hold true:

$$\int_{\mathbb{R}^d} |v_1(x)v_2(x)\xi_R(x)(-\Delta)^s(\xi_R)(x)| dx \leq K \|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right), \quad (2.A.38)$$

$$\int_{\mathbb{R}^d} |v_1(y)v_2(y)| \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x-y|^{d+2s}} dx \right) dy \leq K \|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right), \quad (2.A.39)$$

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x)v_1(x) \frac{(v_2(x) - v_2(y))(\xi_R(x) - \xi_R(y))}{|x-y|^{d+2s}} \right| dx dy \\ & \leq K^{\frac{1}{2}} \|v_1\|_{p,\rho} [I_R(v_2)]^{\frac{1}{2}} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.A.40)$$

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x)v_1(y) \frac{(v_2(x) - v_2(y))(\xi_R(x) - \xi_R(y))}{|x-y|^{d+2s}} \right| dx dy \\ & \leq K^{\frac{1}{2}} \|v_1\|_{p,\rho} [I_R(v_2)]^{\frac{1}{2}} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.A.41)$$

where  $K = K(d, s, p, \gamma_0, \gamma, c, \xi)$  is a suitable positive constant.

*Proof.* In order to prove that  $I_R(v_i)$  is finite for any  $R > 0$  note that, since  $\xi_R$  is supported in  $B_{2R}$ , it is enough to show that the function

$$h_i(x) := \int_{\mathbb{R}^d} \frac{(v_i(x) - v_i(y))^2}{|x-y|^{d+2s}} dy$$

is bounded as  $x$  varies in  $B_{2R}$ . We have:

$$\begin{aligned} \int_{B_{4R}} \frac{(v_i(x) - v_i(y))^2}{|x-y|^{d+2s}} dy & \leq \|\nabla v_i\|_{L^\infty(B_{4R})}^2 \int_{B_{4R}} |x-y|^{-d+2(1-s)} dy \\ & \leq \|\nabla v_i\|_{L^\infty(B_{4R})}^2 \int_{B_{6R}} |z|^{-d+2(1-s)} dz \end{aligned}$$

and

$$\begin{aligned} \int_{B_{4R}^c} \frac{(v_i(x) - v_i(y))^2}{|x-y|^{d+2s}} dy & \leq 2^{d+2s} \int_{B_{4R}^c} \frac{(v_i(x) - v_i(y))^2}{|y|^{d+2s}} dy \\ & \leq 2^{d+2s+1} \left( \|v_i\|_{L^\infty(B_{2R})}^2 \int_{B_{4R}^c} |y|^{-d-2s} dy + \int_{B_{4R}^c} \frac{v_i^2(y)}{|y|^{d+2s}} dy \right) \end{aligned} \quad (2.A.42)$$

for all  $x \in B_{2R}$ . As concerns the last term in the r.h.s. of (2.A.42), if  $p = 2$  and  $\gamma \in (0, d+2s]$  its finiteness is trivial; if instead  $p > 2$ , by means of Hölder's inequality we obtain

$$\int_{B_{4R}^c} \frac{v_i^2(y)}{|y|^{d+2s}} dy \leq \left( \int_{B_{4R}^c} \frac{|v_i(y)|^p}{|y|^\gamma} dy \right)^{\frac{2}{p}} \left( \int_{B_{4R}^c} |y|^{-\frac{p(d+2s)-2\gamma}{p-2}} dy \right)^{\frac{p-2}{p}},$$

so that the same holds true provided  $\gamma \in [0, d+ps)$ . We have therefore shown that  $h_i$  is locally bounded, whence the finiteness of  $I_R(v_i)$ .

## 2.A. Self-adjointness of a weighted fractional Laplacian operator

Now let us deal with estimates (2.A.38)–(2.A.41). Recalling that  $\|\xi\|_\infty = 1$ , using (2.A.4) with  $q = p/(p-2)$ , (2.A.5) with  $\gamma = 0$  and exploiting Hölder's (with three factors), we find:

$$\begin{aligned}
& \int_{\mathbb{R}^d} |v_1(x)v_2(x)\xi_R(x)(-\Delta)^s(\xi_R)(x)| dx \\
&= \int_{B_1} |v_1(x)v_2(x)| |(-\Delta)^s(\xi_R)(x)| \rho^{-1}(x)\rho(x) dx \\
&\quad + \int_{B_1^c} |v_1(x)v_2(x)| |(-\Delta)^s(\xi_R)(x)| \rho^{-1}(x)\rho(x) dx \\
&\leq \frac{\|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \|(-\Delta)^s(\xi)\|_\infty \left( \int_{B_1} |x|^{\frac{2\gamma_0}{p-2}} dx \right)^{\frac{p-2}{p}}}{c^{\frac{2}{p}} R^{2s}} + \frac{\|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \| |x|^\gamma (-\Delta)^s(\xi) \|_{\frac{p}{p-2}, -\gamma}}{c^{\frac{2}{p}} R^{2s-\gamma-\frac{(d-\gamma)(p-2)}{p}}}, \tag{2.A.43}
\end{aligned}$$

where for  $p = 2$  it is understood that  $p/(p-2) = \infty$ . Clearly (2.A.43) implies (2.A.38). Similarly (use (2.A.6) and (2.A.7)),

$$\begin{aligned}
& \int_{\mathbb{R}^d} |v_1(y)v_2(y)| \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x-y|^{d+2s}} dx \right) dy \\
&\leq \frac{\|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \|l_s(\xi)\|_\infty \left( \int_{B_1} |x|^{\frac{2\gamma_0}{p-2}} dx \right)^{\frac{p-2}{p}}}{c^{\frac{2}{p}} R^{2s}} + \frac{\|v_1\|_{p,\rho} \|v_2\|_{p,\rho} \| |x|^\gamma l_s(\xi) \|_{\frac{p}{p-2}, -\gamma}}{c^{\frac{2}{p}} R^{2s-\gamma-\frac{(d-\gamma)(p-2)}{p}}},
\end{aligned}$$

whence (2.A.39). As for (2.A.40), we can proceed as follows:

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x)v_1(x) \frac{(v_2(x) - v_2(y))(\xi_R(x) - \xi_R(y))}{|x-y|^{d+2s}} \right| dx dy \\
&\leq \left( \int_{\mathbb{R}^d} v_1^2(x) l_s(\xi_R)(x) dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R^2(x) \frac{(v_2(x) - v_2(y))^2}{|x-y|^{d+2s}} dx dy \right)^{\frac{1}{2}} \\
&= \left( \int_{B_1} v_1^2(x) l_s(\xi_R)(x) \rho^{-1}(x)\rho(x) dx + \int_{B_1^c} v_1^2(x) l_s(\xi_R)(x) \rho^{-1}(x)\rho(x) dx \right)^{\frac{1}{2}} [I_R(v_2)]^{\frac{1}{2}} \\
&\leq \|v_1\|_{p,\rho} \left( \frac{\|l_s(\xi)\|_\infty \left( \int_{B_1} |x|^{\frac{2\gamma_0}{p-2}} dx \right)^{\frac{p-2}{p}}}{c^{\frac{2}{p}} R^{2s}} + \frac{\| |x|^\gamma l_s(\xi) \|_{\frac{p}{p-2}, -\gamma}}{c^{\frac{2}{p}} R^{2s-\gamma-\frac{(d-\gamma)(p-2)}{p}}} \right)^{\frac{1}{2}} [I_R(v_2)]^{\frac{1}{2}}.
\end{aligned}$$

The proof of (2.A.41) is identical. □

**Lemma 2.A.11.** *Suppose that  $\rho$  satisfies (2.A.35) for some  $\gamma_0 \in [0, d)$  and  $\gamma \in [0, 2s)$ . Let  $p \in [2, \infty)$  if  $d \leq 2s$  and  $p \in [2, 2(d-\gamma)/(d-2s))$  if  $d > 2s$ . Then  $X_{p,s,\rho}$  is continuously embedded in  $\dot{H}^s(\mathbb{R}^d)$  and the following inequality holds true:*

$$\langle v, v \rangle_{\dot{H}^s} = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x-y|^{d+2s}} dx dy \leq \int_{\mathbb{R}^d} v(x)(-\Delta)^s(v)(x) dx \quad \forall v \in X_{p,s,\rho}, \tag{2.A.44}$$

where  $C_{d,s}$  is the same constant appearing in (2.A.28).

*Proof.* We shall first prove (2.A.44) for a sequence  $\{v_n\} \subset C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$  converging to  $v$  in  $X_{p,s,\rho}$  (recall Lemma 2.A.9-(c)), and then we shall pass to the limit as  $n \rightarrow \infty$ . To this end,



take a family of cut-off functions  $\{\xi_R\}$  as in Lemma 2.A.3. Since  $\xi_R v_n$  belongs to  $C_c^\infty(\mathbb{R}^d)$  for all  $R > 0$  and  $n \in \mathbb{N}$ , the identity

$$\langle \xi_R v_n, \xi_R v_n \rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R v_n)(x) dx \quad (2.A.45)$$

holds true. Using the product formula (2.A.31) (with  $v = v_n$ ) and plugging it in the r.h.s. of (2.A.45), we obtain:

$$\begin{aligned} & \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R v_n)(x) dx \\ &= \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx + \int_{\mathbb{R}^d} \xi_R(x) (-\Delta)^s (\xi_R)(x) v_n^2(x) dx \\ & \quad + 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(x) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy. \end{aligned} \quad (2.A.46)$$

As for the l.h.s. of (2.A.45), we have:

$$\begin{aligned} 2 \langle \xi_R v_n, \xi_R v_n \rangle_{\dot{H}^s} &= C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\xi_R(x) v_n(x) - \xi_R(y) v_n(y))^2}{|x - y|^{d+2s}} dx dy \\ &= C_{d,s} \int_{\mathbb{R}^d} \xi_R^2(x) \left( \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{d+2s}} dy \right) dx \\ & \quad + C_{d,s} \int_{\mathbb{R}^d} v_n^2(y) \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dx \right) dy \\ & \quad + 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy. \end{aligned} \quad (2.A.47)$$

Exploiting inequality (2.A.40) from Lemma 2.A.10, with the choices  $v_1 = v_2 = v_n$ , we can estimate the third term in the r.h.s. of (2.A.46) in the following way:

$$\begin{aligned} & 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(x) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy \\ & \leq 2 C_{d,s} K^{\frac{1}{2}} \|v_n\|_{p,\rho} [I_R(v_n)]^{\frac{1}{2}} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}} \\ & \leq \delta C_{d,s} I_R(v_n) + \delta^{-1} C_{d,s} K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right) \end{aligned} \quad (2.A.48)$$

for all  $\delta > 0$ . The same holds for the third term in the r.h.s. of (2.A.47) upon using (2.A.41):

$$\begin{aligned} & 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy \\ & \leq \delta C_{d,s} I_R(v_n) + \delta^{-1} C_{d,s} K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right) \end{aligned} \quad (2.A.49)$$

for all  $\delta > 0$ . Thanks to inequalities (2.A.38) and (2.A.39) (still with the choices  $v_1 = v_2 = v_n$ ), we can also estimate the second term in the r.h.s. of (2.A.46) and the second term in the r.h.s. of (2.A.47):

$$\int_{\mathbb{R}^d} \left| \xi_R(x) (-\Delta)^s (\xi_R)(x) v_n^2(x) \right| dx \leq K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right), \quad (2.A.50)$$

$$C_{d,s} \int_{\mathbb{R}^d} v_n^2(y) \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dx \right) dy \leq C_{d,s} K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right). \quad (2.A.51)$$

By combining (2.A.46), (2.A.48) and (2.A.50), we infer that

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R v_n)(x) dx - \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx \right| \\ & \leq \delta C_{d,s} I_R(v_n) + (\delta^{-1} C_{d,s} + 1) K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right) \end{aligned} \quad (2.A.52)$$

for all  $\delta > 0$ . Similarly, gathering (2.A.47), (2.A.49) and (2.A.51), we get:

$$\begin{aligned} & |2 \langle \xi_R v_n, \xi_R v_n \rangle_{\dot{H}^s} - C_{d,s} I_R(v_n)| \\ & \leq \delta C_{d,s} I_R(v_n) + (\delta^{-1} + 1) C_{d,s} K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right) \end{aligned} \quad (2.A.53)$$

for all  $\delta > 0$ . Hence, (2.A.45), (2.A.52) and (2.A.53) yield

$$\begin{aligned} C_{d,s} I_R(v_n) & \leq 2 \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx + |2 \langle \xi_R v_n, \xi_R v_n \rangle_{\dot{H}^s} - C_{d,s} I_R(v_n)| \\ & \quad + 2 \left| \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s (\xi_R v_n)(x) dx - \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx \right| \\ & \leq 2 \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx + 3\delta C_{d,s} I_R(v_n) \\ & \quad + (3\delta^{-1} C_{d,s} + C_{d,s} + 2) K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right), \end{aligned}$$

that is

$$\begin{aligned} (1 - 3\delta) C_{d,s} I_R(v_n) & \leq 2 \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (v_n)(x) dx + \\ & \quad + (3\delta^{-1} C_{d,s} + C_{d,s} + 2) K \|v_n\|_{p,\rho}^2 \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right). \end{aligned} \quad (2.A.54)$$

In view of the hypotheses on  $\gamma$  and  $p$ , which in particular ensure that

$$-2s + \gamma + \frac{(d-\gamma)(p-2)}{p} < 0,$$

for any (fixed)  $\delta \in (0, 1/3)$  we can pass to the limit in (2.A.54) as  $R \rightarrow \infty$  to get, by means *e.g.* of Fatou's Lemma and dominated convergence,

$$(1 - 3\delta) C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{d+2s}} dx dy \leq 2 \int_{\mathbb{R}^d} v_n(x) (-\Delta)^s (v_n)(x) dx. \quad (2.A.55)$$

Note that the r.h.s. of (2.A.55) is finite thanks to Lemma 2.A.9-(d). Letting  $\delta \rightarrow 0$  we end up with

$$C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y))^2}{|x - y|^{d+2s}} dx dy \leq 2 \int_{\mathbb{R}^d} v_n(x) (-\Delta)^s (v_n)(x) dx. \quad (2.A.56)$$

The fact that  $v_n$  actually belongs to  $\dot{H}^s(\mathbb{R}^d)$  is an immediate consequence of the boundedness of the family  $\{\xi_R v_n\}_{R \geq 1}$  in  $\dot{H}^s(\mathbb{R}^d)$ . Indeed, it is direct to check that the above estimates and (2.A.56) bound  $\langle \xi_R v_n, \xi_R v_n \rangle_{\dot{H}^s}$  uniformly w.r.t.  $R \geq 1$ . We can therefore rewrite the latter as

$$\langle v_n, v_n \rangle_{\dot{H}^s} \leq \int_{\mathbb{R}^d} v_n(x) (-\Delta)^s (v_n)(x) dx. \quad (2.A.57)$$

Since the sequence  $\{v_n\}$  converges to  $v$  in  $X_{p,s,\rho}$  and, by (2.A.57), it is bounded in  $\dot{H}^s(\mathbb{R}^d)$ , we deduce that  $v \in H^s(\mathbb{R}^d)$ ; hence, (2.A.44) just follows by letting  $n \rightarrow \infty$  in (2.A.57), exploiting Fatou's Lemma in the l.h.s. and the fact that the r.h.s. is a continuous bilinear form on  $X_{p,s,\rho}$  (recall again Lemma 2.A.9-(d)).  $\square$

We are now in position to prove our main “integration by parts” formula for functions belonging to  $X_{p,s,\rho}$ .

**Theorem 2.A.12.** *Suppose that  $\rho$  satisfies (2.A.35) for some  $\gamma_0 \in [0, d)$  and  $\gamma \in [0, 2s)$ . Let  $p \in [2, \infty)$  if  $d \leq 2s$  and  $p \in [2, 2(d - \gamma)/(d - 2s))$  if  $d > 2s$ . Then,*

$$\begin{aligned} \langle v, w \rangle_{\dot{H}^s} &= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2s}} dx dy = \int_{\mathbb{R}^d} v(x)(-\Delta)^s(w)(x) dx \\ &= \int_{\mathbb{R}^d} (-\Delta)^s(v)(x)w(x) dx \end{aligned} \quad (2.A.58)$$

for all  $v, w \in X_{p,s,\rho}$ .

*Proof.* We shall proceed as in Lemma 2.A.11, i.e. we start from the validity of the formula

$$\langle \xi_R v_n, \xi_R w_n \rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s(\xi_R w_n)(x) dx, \quad (2.A.59)$$

where  $\{v_n\}, \{w_n\} \subset C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$  are sequences converging to  $v$  and  $w$ , respectively, in  $X_{p,s,\rho}$ . Following the same lines of proof of (2.A.46) and (2.A.47), we obtain the identities

$$\begin{aligned} &\int_{\mathbb{R}^d} \xi_R(x) v_n(x) (-\Delta)^s(\xi_R w_n)(x) dx \\ &= \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s(w_n)(x) dx + \int_{\mathbb{R}^d} \xi_R(x) (-\Delta)^s(\xi_R(x) v_n(x) w_n(x)) dx \\ &\quad + 2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(x) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy \end{aligned} \quad (2.A.60)$$

and

$$\begin{aligned} 2 \langle \xi_R v_n, \xi_R w_n \rangle_{\dot{H}^s} &= C_{d,s} \int_{\mathbb{R}^d} \xi_R^2(x) \left( \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y)) (w_n(x) - w_n(y))}{|x - y|^{d+2s}} dy \right) dx \\ &\quad + C_{d,s} \int_{\mathbb{R}^d} v_n(y) w_n(y) \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dx \right) dy \\ &\quad + C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) v_n(y) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy \\ &\quad + C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi_R(x) w_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dx dy. \end{aligned} \quad (2.A.61)$$

Notice that the trivial inequality

$$C_{d,s} I_R(v) \leq 2 \langle v, v \rangle_{\dot{H}^s} \quad \forall v \in X_{p,s,\rho} \quad (2.A.62)$$

holds true, and in view of Lemma 2.A.11 we know that the r.h.s. is finite. Hence, arguing as in the proof of (2.A.48)–(2.A.51) and exploiting (2.A.62), we get:

$$\begin{aligned} &2 C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(x) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy \\ &\leq 2\sqrt{2} C_{d,s}^{\frac{1}{2}} K^{\frac{1}{2}} \|v_n\|_{p,\rho} \|w_n\|_{\dot{H}^s} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.A.63)$$

$$\begin{aligned}
 & C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) v_n(y) \frac{(w_n(x) - w_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy \\
 & \leq \sqrt{2} C_{d,s}^{\frac{1}{2}} K^{\frac{1}{2}} \|v_n\|_{p,\rho} \|w_n\|_{\dot{H}^s} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}},
 \end{aligned} \tag{2.A.64}$$

$$\begin{aligned}
 & C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \xi_R(x) w_n(y) \frac{(v_n(x) - v_n(y)) (\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} \right| dx dy \\
 & \leq \sqrt{2} C_{d,s}^{\frac{1}{2}} K^{\frac{1}{2}} \|w_n\|_{p,\rho} \|v_n\|_{\dot{H}^s} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}},
 \end{aligned} \tag{2.A.65}$$

$$\int_{\mathbb{R}^d} |\xi_R(x) (-\Delta)^s (\xi_R)(x) v_n(x) w_n(x)| dx \leq K \|v_n\|_{p,\rho} \|w_n\|_{p,\rho} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right), \tag{2.A.66}$$

$$\begin{aligned}
 & C_{d,s} \int_{\mathbb{R}^d} |v_n(y) w_n(y)| \left( \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x - y|^{d+2s}} dx \right) dy \\
 & \leq C_{d,s} K \|v_n\|_{p,\rho} \|w_n\|_{p,\rho} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right).
 \end{aligned} \tag{2.A.67}$$

Gathering (2.A.59)–(2.A.61) and (2.A.63)–(2.A.67), we deduce that

$$\begin{aligned}
 & \left| C_{d,s} \int_{\mathbb{R}^d} \xi_R^2(x) \left( \int_{\mathbb{R}^d} \frac{(v_n(x) - v_n(y)) (w_n(x) - w_n(y))}{|x - y|^{d+2s}} dy \right) dx \right. \\
 & \quad \left. - 2 \int_{\mathbb{R}^d} \xi_R^2(x) v_n(x) (-\Delta)^s (w_n)(x) dx \right| \\
 & \leq \sqrt{2} C_{d,s}^{\frac{1}{2}} K^{\frac{1}{2}} \left( 5 \|v_n\|_{p,\rho} \|w_n\|_{\dot{H}^s} + \|w_n\|_{p,\rho} \|v_n\|_{\dot{H}^s} \right) \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{1}{2}} \\
 & \quad + (C_{d,s} + 2) K \|v_n\|_{p,\rho} \|w_n\|_{p,\rho} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right).
 \end{aligned}$$

Letting  $R \rightarrow \infty$ , we find the identity

$$\langle v_n, w_n \rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} v_n(x) (-\Delta)^s (w_n)(x) dx. \tag{2.A.68}$$

Now we can let  $n \rightarrow \infty$  and pass to the limit in (2.A.68): in fact, thanks to Lemmas 2.A.9 and 2.A.11, both in the left and in the right-hand side we have continuous bilinear forms on  $X_{p,s,\rho}$ . This entails

$$\langle v, w \rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} v(x) (-\Delta)^s (w)(x) dx \quad \forall v, w \in X_{p,s,\rho}.$$

The last equality in (2.A.58) just follows by swapping  $v$  and  $w$ .  $\square$

We finally focus on the case  $p = 2$ , showing that Theorem 2.A.12 implies the self-adjointness of the operator  $\rho^{-1}(-\Delta)^s$  in  $L_\rho^2(\mathbb{R}^d)$  (and related consequences).

**Theorem 2.A.13.** *Suppose that  $\rho$  satisfies (2.A.35) for some  $\gamma_0 \in [0, d]$  and  $\gamma \in [0, 2s)$ . Let us define the linear operator  $A : D(A) := X_{s,\rho} \subset L_\rho^2(\mathbb{R}^d) \mapsto L_\rho^2(\mathbb{R}^d)$  as follows:*

$$A(f) := \rho^{-1}(-\Delta)^s(f) \quad \forall f \in D(A).$$

*Then,  $A$  is a densely defined, nonnegative self-adjoint operator on  $L_\rho^2(\mathbb{R}^d)$ , whose quadratic form is*

$$Q(v, v) := \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x - y|^{d+2s}} dx dy,$$

with domain  $D(Q) = L^2_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ . Moreover,  $Q$  is a Dirichlet form and  $A$  generates a Markov semigroup  $S_2(t)$  on  $L^2_\rho(\mathbb{R}^d)$ . In particular, for all  $p \in [1, \infty]$  there exists a contraction semigroup  $S_p(t)$  on  $L^p_\rho(\mathbb{R}^d)$ , consistent with  $S_2(t)$  on  $L^2_\rho(\mathbb{R}^d) \cap L^p_\rho(\mathbb{R}^d)$ , which is furthermore analytic with a suitable angle  $\theta_p > 0$  for all  $p \in (1, \infty)$ .

*Proof.* It is direct to check that  $A$  actually acts from  $D(A)$  to  $L^2_\rho(\mathbb{R}^d)$ , since  $A(f) \in L^2_\rho(\mathbb{R}^d)$  is equivalent to  $(-\Delta)^s(f) \in L^2_{\rho^{-1}}(\mathbb{R}^d)$ , which is true by definition of  $X_{s,\rho}$ . The fact that  $A$  is densely defined follows from the inclusion  $\mathcal{D}(\mathbb{R}^d) \subset X_{s,\rho}$  (Lemma 2.A.9-(a)). The nonnegativity of  $A$  is a trivial consequence of Theorem 2.A.12 (take  $v = w = f$  in (2.A.58)).

In order to prove that  $A$  is a symmetric operator, we can again take advantage of (2.A.58): for all  $f, g \in D(A)$  there holds

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)A(g)(x) \rho(x)dx &= \int_{\mathbb{R}^d} f(x)(-\Delta)^s(g)(x) dx = \int_{\mathbb{R}^d} (-\Delta)^s(f)(x)g(x) dx \\ &= \int_{\mathbb{R}^d} A(f)(x)g(x) \rho(x)dx, \end{aligned}$$

so that  $D(A) \subset D(A^*)$  and  $A = A^*$  on  $D(A)$ . Hence, proving that  $A$  is self-adjoint amounts to establish the inclusion  $D(A^*) \subset D(A)$ . By definition of  $D(A^*)$ , we know that  $h \in D(A^*)$  if and only if  $h \in L^2_\rho(\mathbb{R}^d)$  and there exists a positive constant  $M_h$  such that

$$\int_{\mathbb{R}^d} h(x)A(g)(x) \rho(x)dx \leq M_h \|g\|_{2,\rho} \quad \forall g \in D(A). \quad (2.A.69)$$

As recalled above,  $\mathcal{D}(\mathbb{R}^d) \subset D(A)$ , so that in particular (2.A.69) holds true for all  $g = \phi \in \mathcal{D}(\mathbb{R}^d)$ . Since  $\mathcal{D}(\mathbb{R}^d)$  is dense in  $L^2_\rho(\mathbb{R}^d)$ , thanks to (2.A.69) and to Riesz representation Theorem we infer the existence of a unique element  $\mathcal{E} \in L^2_\rho(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} h(x)A(\phi)(x) \rho(x)dx = \int_{\mathbb{R}^d} h(x)(-\Delta)^s(\phi)(x) dx = \int_{\mathbb{R}^d} \mathcal{E}(x)\phi(x) \rho(x)dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \quad (2.A.70)$$

But the last equality in (2.A.70) does mean that  $\rho\mathcal{E} = (-\Delta)^s(h)$  and therefore  $h \in X_{s,\rho} = D(A)$  (note that  $\|(-\Delta)^s(h)\|_{2,\rho^{-1}} = \|\mathcal{E}\|_{2,\rho} < \infty$ ). We have then established the inclusion  $D(A^*) \subset D(A)$  and so the self-adjointness of  $A$ .

Let us finally deal with the quadratic form  $Q$  associated to  $A$ . Thanks to Theorem 2.A.12, we have that

$$Q(v, v) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x - y|^{d+2s}} dx dy \quad (2.A.71)$$

for all  $v \in D(A)$ . As it is well known (see *e.g.* [59]), the domain  $D(Q)$  of  $Q$  is just the closure of  $D(A)$  w.r.t. the norm

$$\|v\|_Q^2 := \|v\|_{2,\rho^{-1}}^2 + Q(v, v) = \|v\|_{2,\rho^{-1}}^2 + \|v\|_{\dot{H}^s}^2.$$

It is straightforward to see that such closure is nothing but  $L^2_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$  and that the quadratic form on  $D(Q) = L^2_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$  is still represented by (2.A.71).

By classical results (we refer again to [59]), proving that  $A$  generates a Markov semigroup is equivalent to proving that if  $v$  belongs to  $D(Q)$  then both  $v \vee 0$  and  $v \wedge 1$  belong to  $D(Q)$  and satisfy

$$Q(v \vee 0, v \vee 0) \leq Q(v, v), \quad Q(v \wedge 1, v \wedge 1) \leq Q(v, v); \quad (2.A.72)$$

but (2.A.72) is an immediate consequence of the characterization of  $Q$  given above. The last assertion of the statement follows from the general theory of symmetric Markov semigroups (see in particular [59, Theorems 1.4.1, 1.4.2]).  $\square$

**Remark 2.A.14.** *A posteriori*, for  $\rho$  and  $p$  complying with the hypotheses of Theorem 2.A.12, the subspace  $\mathcal{D}(\mathbb{R}^d)$  is dense in  $X_{p,s,\rho}$ . In fact, in view of Lemma 2.A.9-(c), it is enough to show that for any  $\varphi \in C^\infty(\mathbb{R}^d) \cap X_{p,s,\rho}$  the cut-off family  $\{\xi_R\varphi\}$  converges to  $\varphi$  in  $X_{p,s,\rho}$  as  $R \rightarrow \infty$ . The only nontrivial point is the convergence of  $\{(-\Delta)^s(\xi_R\varphi)\}$  to  $(-\Delta)^s(\varphi)$  in  $L_{\rho'}^p(\mathbb{R}^d)$ . We have:

$$\begin{aligned}
 & \int_{\mathbb{R}^d} |(-\Delta)^s(\xi_R\varphi)(x) - (-\Delta)^s(\varphi)(x)|^{p'} \rho'(x) dx \\
 \leq & C' \left[ \int_{\mathbb{R}^d} |\xi_R(x) - 1|^{p'} |(-\Delta)^s(\varphi)(x)|^{p'} \rho'(x) dx + \int_{\mathbb{R}^d} |(-\Delta)^s(\xi_R)(x)|^{p'} |\varphi(x)|^{p'} \rho'(x) dx \right. \\
 & \left. + \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{(\varphi(x) - \varphi(y))(\xi_R(x) - \xi_R(y))}{|x - y|^{d+2s}} dy \right|^{p'} \rho'(x) dx \right] \\
 \leq & C' \left[ \int_{\mathbb{R}^d} |\xi_R(x) - 1|^{p'} |(-\Delta)^s(\varphi)(x)|^{p'} \rho'(x) dx + \|\varphi\|_{p,\rho}^{\frac{p}{p-1}} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{p}{p-1}} \right. \\
 & \left. + \|\varphi\|_{\dot{H}^s}^{\frac{p}{p-1}} \left( R^{-2s} + R^{-2s+\gamma+\frac{(d-\gamma)(p-2)}{p}} \right)^{\frac{p}{2(p-1)}} \right],
 \end{aligned}$$

where  $C' = C'(d, s, p, \gamma_0, \gamma, c, \xi)$  is a positive constant, possibly changing from line to line. Letting  $R \rightarrow \infty$  we get the claimed result.



**Optimal functions for Caffarelli-Kohn-Nirenberg  
inequalities and weighted fast diffusion**

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**3.1 Introduction**

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This chapter is devoted to a special class of Caffarelli-Kohn-Nirenberg interpolation inequalities that were first introduced in [41] and can be written as

$$\|w\|_{2p,\gamma} \leq C_\gamma \|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta} \quad \forall w \in \mathcal{D}(\mathbb{R}^d), \quad (3.1.1)$$

where by  $C_\gamma > 0$  we denote the *best constant* in the inequality. The spatial dimension  $d$  and the parameters  $p, \gamma, \vartheta$  are assumed to satisfy

$$d \geq 3, \quad \gamma \in (0, 2), \quad p \in \left(1, \frac{2_\gamma^*}{2}\right) \quad \text{with} \quad 2_\gamma^* := 2 \frac{d-\gamma}{d-2}, \quad (3.1.2)$$

$$\vartheta := \frac{2_\gamma^*(p-1)}{2p(2_\gamma^* - p - 1)} = \frac{(d-\gamma)(p-1)}{p(d+2-2\gamma-p(d-2))}. \quad (3.1.3)$$

The norms are defined by

$$\|w\|_q := \|w\|_{q,0} \quad \text{with} \quad \|w\|_{q,\gamma} := \left( \int_{\mathbb{R}^d} |w(x)|^q |x|^{-\gamma} dx \right)^{\frac{1}{q}}.$$

When  $\gamma = 0$  the corresponding inequalities become a particular subfamily of the well-known Gagliardo-Nirenberg inequalities introduced in [100, 142]. In that case, optimal functions have completely been characterized in [62]. These inequalities have attracted lots of interest in the recent years: see for instance [86] and references therein, or [14].

There are several reasons to consider the problem of finding functions which attain the best constant in (3.1.1) (in this regard, see also the forthcoming Subsection 3.1.1). First of all, optimality *among radial functions* is achieved by

$$\mathbf{b}_\gamma(x) := \left(1 + |x|^{2-\gamma}\right)^{-\frac{1}{p-1}} \quad (3.1.4)$$



up to a multiplication by a constant and a scaling. In fact, in the case of radial functions, it is not very difficult to reduce the problem with  $\gamma > 0$  to the case  $\gamma = 0$ ; see Proposition 3.3.14 below and [25] for more details. However, without such a radial symmetry, it is remarkable that the function  $\mathfrak{b}_\gamma$  clearly departs from standard optimal functions that are usually characterized using the conformal invariance properties of the sphere and the stereographic projection, like for instance in [14, Section 6.10]. Even with a radial weight of the form  $|x|^{-\gamma}$ , there is no simple symmetry result that would allow us to identify the optimal functions in terms of  $\mathfrak{b}_\gamma$ . In other words, upon defining the quotient

$$\mathcal{Q}[w] := \frac{\|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta}}{\|w\|_{2p,\gamma}},$$

it is not known whether equality holds in

$$\inf_{w \in \mathcal{D}^*(\mathbb{R}^d) \setminus \{0\}} \mathcal{Q}[w] =: (\mathbf{C}_\gamma^*)^{-1} = \mathcal{Q}[\mathfrak{b}_\gamma] \geq (\mathbf{C}_\gamma)^{-1} = \inf_{w \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}} \mathcal{Q}[w], \quad (3.1.5)$$

where we denote as  $\mathcal{D}^*(\mathbb{R}^d)$  the subset of  $\mathcal{D}(\mathbb{R}^d)$  which is spanned by radial smooth functions, *i.e.* functions which depend only on  $|x|$ . The main result we shall prove here is a first step in this direction.

**Theorem 3.1.1.** *Let  $d \geq 3$ . For any  $p \in (1, d/(d-2))$  there exists a positive  $\gamma^*$  such that equality holds in (3.1.5) for all  $\gamma \in (0, \gamma^*)$ .*

We remark that optimal functions for  $\mathcal{Q}$  (which can be assumed to be nonnegative) satisfy, up to a multiplication by a constant and a scaling, the semilinear elliptic equation

$$-\Delta w + \frac{w^p - w^{2p-1}}{|x|^\gamma} = 0 \quad \text{in } \mathbb{R}^d; \quad (3.1.6)$$

this will be discussed in detail in Subsection 3.2.1. However, the standard Gidas-Ni-Nirenberg result in [102] does not allow us to decide whether a positive solution of (3.1.6) is forced to be radially symmetric. To the best of our knowledge, even for a minimizer of  $\mathcal{Q}$ , it is not known yet if the result can be deduced from a symmetrization method either. We shall then say that *symmetry breaking* occurs if  $\mathbf{C}_\gamma^* < \mathbf{C}_\gamma$ . Whether this occurs for some  $\gamma \in (0, 2)$  and  $p$  in the appropriate range, or not, is an open question: our result ensures that symmetry breaking is ruled out in a right-hand neighbourhood of  $\{\gamma = 0\}$ .

The proof of Theorem 3.1.1 is based on a perturbation method (sketched in Subsection 3.1.2) which relies on the fact that, thanks to the results established in [62] by J. Dolbeault and M. Del Pino, the optimal functions in the case  $\gamma = 0$  are radial up to translations. Our strategy is adapted from [79], except that we have no Emden-Fowler type transformation that would allow us to get rid of the weights. This has the unpleasant consequence that a fully developed analysis of the convergence of the optimal functions for (3.1.1) is needed, based on a concentration-compactness technique, as  $\gamma \downarrow 0$ . We prove that the limit is in fact the radial solution to the limit problem, namely the only one centred at the origin, although such problem is translation invariant. Then we are able to prove that the optimal functions are themselves radially symmetric for  $\gamma > 0$  sufficiently small. As a consequence,  $\mathbf{C}_\gamma^* = \mathbf{C}_\gamma$  for any such  $\gamma$ , and the optimal functions are all given by (3.1.4) up to a multiplication by a constant and a scaling.

Symmetry issues in Caffarelli-Kohn-Nirenberg inequalities are well known to be a difficult problem. While symmetry breaking is usually proved by means of a spherical harmonics expansion as in [47, 63, 94], symmetry results are harder. The best established method is

based on a perturbation argument that has been used in [79, 80, 126, 165], although in some cases symmetry has also been proved by direct estimates, see for instance [78, 81]. We refer the reader to [47, 75] for results on the existence of optimal functions in the *Hardy-Sobolev* case  $p = (d - \gamma)/(d - 2)$ , to [74] for an overview of symmetry breaking issues and to [76, 77] for some additional numerical investigations. It is worth pointing out that inequality (3.1.1) has three endpoints where symmetry is known: the case of the *Gagliardo-Nirenberg inequalities* corresponding to  $\gamma = 0$ , the case of the *Hardy-Sobolev inequalities* with  $p = (d - \gamma)/(d - 2)$  and  $\vartheta = 1$ , and finally the case of *Hardy's inequality* with  $(p, \gamma) = (1, 2)$  (see *e.g.* [79]). Note that, if  $\vartheta = 1$ , moving plane methods as in [53] or symmetrization techniques as in [81, 109] can indeed be applied to establish that the optimal functions are given by (3.1.4), still up to a multiplication by a constant and a scaling.

In [14, Section 6.10] the problem of finding the best constant for (3.1.1), in the case  $\gamma = 0$ , has been reduced to the question of the optimality of the Aubin-Talenti functions (*i.e.* (3.1.4) with  $\gamma = 0$  and  $p = d/(d - 2)$ , named after [11, 167]) for the Sobolev inequality, in higher dimension, within a class of functions having some special symmetry properties. However, such a method does not seem to apply to the case  $\gamma > 0$ , at least directly, for the simple reason that  $|x|^{2-\gamma}$  cannot be decomposed into a sum of terms involving only some of the coordinates of  $x$ . It could well be that a more clever change of variables solves the issue, but this remains an open question so far.

### 3.1.1 Connections with power-weighted Euclidean fast diffusion equations

The symmetry result of Theorem 3.1.1 has very interesting consequences. In this regard, let us consider the following power-weighted *fast diffusion equation*:

$$u_t + |x|^\gamma \operatorname{div} [u \nabla (u^{m-1})] = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \quad (3.1.7)$$

with initial condition  $u(\cdot, 0) = u_0 \geq 0$  and  $m \in (m_1, 1)$ , where

$$m_1 := \frac{2d - \gamma - 2}{2(d - \gamma)}.$$

From the point of view of the well-posedness of the Cauchy problem and of the long-time behaviour of solutions, such an equation, in the porous medium case (namely for  $m > 1$ ), has thoroughly been studied in [163, 164]. As for the fast diffusion regime we consider here, analogous issues will be addressed in detail in [25]. In particular, it can be shown that the mass  $M := \int_{\mathbb{R}^d} u(x, t) |x|^{-\gamma} dx$  is preserved along the evolution. In order to better understand the asymptotics of (3.1.7), let us introduce the time-dependent rescaling

$$u(x, t) = R^{\gamma-d}(t) v \left( x/R(t), (2 - \gamma)^{-1} \log R(t) \right), \quad (3.1.8)$$

with  $R = R(t)$  defined to be the solution of

$$\frac{dR}{dt} = (2 - \gamma) R^{(1-m)(d-\gamma)-1}, \quad R(0) = 1. \quad (3.1.9)$$

Integrating (3.1.9), we find that such solution is explicit and given by

$$R(t) = [1 + (2 - \gamma)(d - \gamma)(m - m_c) t]^{\frac{1}{(d-\gamma)(m-m_c)}}, \quad (3.1.10)$$

where

$$m_c := \frac{d - \gamma - 2}{d - \gamma}.$$

We point out that, even though (3.1.7) is a fast diffusion equation, one can show that there is *no extinction* in the range  $m \in (m_c, 1)$ , in contrast with the situation in bounded Euclidean domains (see the Introduction to Chapter 4).

After changing variables in agreement with (3.1.8), we infer that the rescaled function  $v$  solves the Fokker-Planck-type equation

$$v_t + |x|^\gamma \operatorname{div} \left[ v \nabla \left( v^{m-1} - |x|^{2-\gamma} \right) \right] = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \quad (3.1.11)$$

with initial datum  $v(\cdot, 0) = u_0$ . Accordingly, the convergence of  $u$  towards a self-similar solution of Barenblatt type as  $t \rightarrow \infty$  (see [23, 62] in the non-weighted case) is replaced by the convergence of  $v$  towards a stationary solution of (3.1.11) given by

$$\mathfrak{B}(x) := \left( C + |x|^{2-\gamma} \right)^{-\frac{1}{1-m}}, \quad (3.1.12)$$

where  $C = C(M) > 0$  is uniquely determined by the condition

$$\int_{\mathbb{R}^d} \mathfrak{B}(x) |x|^{-\gamma} dx = M.$$

It is straightforward to check that the *free energy*

$$\mathcal{F}[v(t)] := -\frac{1}{1-m} \int_{\mathbb{R}^d} \left[ v^m(x, t) - \mathfrak{B}^m(x) - m\mathfrak{B}^{m-1}(x) (v(x, t) - \mathfrak{B}(x)) \right] |x|^{-\gamma} dx$$

is nonnegative and satisfies

$$\frac{d}{dt} \mathcal{F}[v(t)] = -\mathcal{I}[v(t)],$$

where the *Fisher information* is defined by

$$\mathcal{I}[v(t)] := \frac{m}{1-m} \int_{\mathbb{R}^d} v(x, t) \left| \nabla \left( v^{m-1} \right) (x, t) - (2-\gamma) \frac{x}{|x|^\gamma} \right|^2 dx.$$

An elementary computation shows that, as a consequence of Theorem 3.1.1,

$$(2-\gamma)^2 \mathcal{F}[v] \leq \mathcal{I}[v] \quad (3.1.13)$$

provided  $\gamma \in (0, \gamma^*)$ . More precisely, the above inequality is exactly equivalent to the fact that  $\mathfrak{B}^{m-1/2}$  is an optimal function for (3.1.1), as it can be verified by setting  $w = v^{m-1/2}$ ,  $p = 1/(2m-1)$  and performing a well-chosen scaling. This is consistent with the fact that  $\mathfrak{B}^{m-1/2}$  is equal to  $\mathfrak{b}_\gamma$  up to a scaling and a multiplication by a constant.

In view of the above discussion, we can then generalize to  $\gamma \in (0, \gamma^*)$  the results obtained in [62] for equation (3.1.11) in the case  $\gamma = 0$ .

**Corollary 3.1.2.** *Let  $d \geq 3$ ,  $m \in (1 - 1/d, 1)$  and  $\gamma \in (0, \gamma^*)$ , where  $\gamma^*$  is defined as in Theorem 3.1.1 with  $p = 1/(2m-1)$ . If  $v$  is the solution to (3.1.11) corresponding to a nonnegative initial datum  $u_0$  such that  $M = \int_{\mathbb{R}^d} u_0(x) |x|^{-\gamma} dx$  and  $\int_{\mathbb{R}^d} u_0^m(x) |x|^{-\gamma} dx + \int_{\mathbb{R}^d} u_0(x) |x|^{2-2\gamma} dx$  are finite, then*

$$\mathcal{F}[v(t)] \leq \mathcal{F}[u_0] e^{-(2-\gamma)^2 t} \quad \forall t > 0. \quad (3.1.14)$$

Estimate (3.1.14) is sharp in the sense that it is actually equivalent to (3.1.13) and henceforth to (3.1.1) as it can be checked by computing  $\frac{d}{dt} \mathcal{F}[v(t)]$  at  $t = 0$ . Asymptotic decay rates can be obtained for  $\gamma \geq \gamma^*$  as well: we refer to [25] for results in this direction. The free energy  $\mathcal{F}[v]$  is in fact a measure of the distance between  $v$  and the Barenblatt-type solutions

(3.1.12). We can then undo the change of variables (3.1.8) and write an intermediate asymptotics estimate based on a Csiszár-Kullback inequality which was established in [86, Theorem 4]. Taking advantage of a time shift  $\delta$  as in [84], which amounts to an adjustment of the scaling at lower order or, equivalently, to the introduction of a time rescaling which ensures that the solution  $u$  and the Barenblatt-type solution with the same mass also have the same second moment (*i.e.*  $\int_{\mathbb{R}^d} u(x, t) |x|^{2-2\gamma} dx$ ), we can deduce the following result.

**Corollary 3.1.3.** *Let  $R = R(t)$  be given by (3.1.10). Under the same assumptions as in Corollary 3.1.2, if  $u$  is a solution to (3.1.7) corresponding to a nonnegative initial datum  $u_0$  such that  $M = \int_{\mathbb{R}^d} u_0(x) |x|^{-\gamma} dx$  and  $\int_{\mathbb{R}^d} u_0^m(x) |x|^{-\gamma} dx + \int_{\mathbb{R}^d} u_0(x) |x|^{2-2\gamma} dx$  are finite, then there exists a positive constant  $\delta$  such that*

$$\left\| u(\cdot, t + \delta) - R^{\gamma-d}(t) \mathfrak{B}(\cdot/R(t)) \right\|_{1,\gamma} \leq K R^{-\frac{2-\gamma}{2}}(t) \quad \forall t > 0$$

for some  $K > 0$  depending only on  $M$ ,  $\int_{\mathbb{R}^d} u_0^m(x) |x|^{-\gamma} dx$  and  $\int_{\mathbb{R}^d} u_0(x) |x|^{2-2\gamma} dx$ .

### 3.1.2 Outline of the proof of the main result

In the following, we give a hint of the strategy we develop throughout Sections 3.2 and 3.3 to prove Theorem 3.1.1. Before going on, let us introduce some basic notations.

Given  $p, q \geq 1$ , we define the Banach spaces  $L_\gamma^q(\mathbb{R}^d)$  and  $\mathcal{H}_\gamma(\mathbb{R}^d)$  as the space of all measurable functions  $w$  such that  $\|w\|_{q,\gamma}$  is finite and the space of all measurable functions  $w$ , with  $\nabla w$  measurable, such that  $\|w\|_{\mathcal{H}_\gamma} := \|w\|_{p+1,\gamma} + \|\nabla w\|_2$  is finite, respectively. If  $\Omega \subset \mathbb{R}^d$ , the spaces  $L_\gamma^q(\Omega)$  and  $\mathcal{H}_\gamma(\Omega)$  are understood likewise. In addition, we denote as  $\mathcal{H}_\gamma^*(\mathbb{R}^d)$  the subspace of  $\mathcal{H}_\gamma(\mathbb{R}^d)$  spanned by radial functions. A simple density argument shows, in particular, that  $\mathcal{D}(\mathbb{R}^d)$  is dense in  $\mathcal{H}_\gamma(\mathbb{R}^d)$  (see Proposition 3.2.3 below), so that inequality (3.1.1) holds for true any function in  $\mathcal{H}_\gamma(\mathbb{R}^d)$  and, as a consequence,  $\mathcal{H}_\gamma(\mathbb{R}^d)$  is continuously embedded in  $L_\gamma^{2p}(\mathbb{R}^d)$ . For the sake of greater readability, in almost all the integrals that will appear below we shall drop the explicit dependence of functions on the variable of integration and use the notations  $L_\gamma^q$  and  $\mathcal{H}_\gamma$  in place of  $L_\gamma^q(\mathbb{R}^d)$  and  $\mathcal{H}_\gamma(\mathbb{R}^d)$ , respectively.

In order to avoid possible ambiguity with previous chapters, we remark that here by  $\|\cdot\|_{q,\gamma}$  we denote the  $L^q$  norm with respect to the *negative* power  $|x|^{-\gamma}$  (in contrast with Chapter 2, Paragraph 2.1.1.1). As for terminology, below we shall refer to functions of the type of (3.1.4), for simplicity, as Barenblatt-type functions, though they are not compactly supported (see the Introduction to Chapter 1). Note that, in the case  $\gamma = 0$ , such functions are also referred to in the literature as Aubin-Talenti-type functions.

We are now ready to outline the main steps of our approach.

1. In Subsection 3.2.2 we establish compactness of the embedding of  $\mathcal{H}_\gamma(\mathbb{R}^d)$  in  $L_\gamma^{2p}(\mathbb{R}^d)$ , which implies existence of an optimal function for (3.1.1); the latter satisfies (3.1.6) up to a multiplication by a constant and a scaling. Afterwards, in Subsection 3.2.3, we prove some integrability and regularity estimates for solutions to (3.1.6). We point out that  $C^{1,\alpha}$  regularity can be expected only for  $\gamma \in (0, 1)$ , as it can easily be guessed by considering the function  $\mathfrak{b}_\gamma$  (see Remark 3.2.11).
2. We work with the non-scale-invariant form of inequality (3.1.1) which can be written as

$$\mathcal{E}_\gamma[w] := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \frac{|w|^{p+1}}{|x|^\gamma} dx - I_\gamma^1 \left( \int_{\mathbb{R}^d} \frac{|w|^{2p}}{|x|^\gamma} dx \right)^{\theta_\gamma} \geq 0, \quad (3.1.15)$$

where  $I_\gamma^1$  denotes another optimal constant and  $\theta_\gamma \in (0, 1)$  is a suitable exponent that depends regularly on  $\gamma$ . A simple scaling argument, which we describe in Subsection 3.2.1, shows that (3.1.15) and (3.1.1) are equivalent and relates  $I_\gamma^1$  to  $C_\gamma$  explicitly.

3. If we denote as  $w_\gamma$  any optimal function corresponding to some  $\gamma \geq 0$ , so that  $\mathcal{E}_\gamma[w] \geq \mathcal{E}_\gamma[w_\gamma] = 0$ , then we readily deduce that

$$\mathcal{E}_\gamma[w_\gamma] - \mathcal{E}_0[w_\gamma] \leq \mathcal{E}_\gamma[w_0] - \mathcal{E}_0[w_0]. \quad (3.1.16)$$

The concentration-compactness analysis we perform in Subsection 3.3.1 shows that, up to subsequences,  $\lim_{\gamma \downarrow 0} w_\gamma(\cdot + y_\gamma) = w_0$ , where  $\{y_\gamma\}$  is a suitable sequence of translations. Dividing (3.1.16) by  $\gamma$  and passing to the limit one can prove, after several simplifications, that

$$\limsup_{\gamma \downarrow 0} \int_{\mathbb{R}^d} \left( -\frac{w_\gamma^{p+1}}{p+1} + \frac{w_\gamma^{2p}}{2p} |x|^{-\gamma} \right) \log |x| \, dx \leq \lim_{\gamma \downarrow 0} \frac{\mathcal{E}_\gamma[w_0] - \mathcal{E}_0[w_0]}{\gamma}. \quad (3.1.17)$$

The fact that the right-hand side of (3.1.17) is explicit, and in particular finite, entails that  $\{y_\gamma\}$  is bounded, whence  $\lim_{\gamma \downarrow 0} w_\gamma = w_0$ . This is proved rigorously in Subsection 3.3.2.

4. Let  $w_b$  be the unique minimizer of  $\mathcal{E}_0$ , with a suitable mass  $\int_{\mathbb{R}^d} w_b^{2p} |x|^{-\gamma} dx$ , which is radial about the origin. In fact  $w_b$  coincides with  $b_0$  up to a multiplication by a constant and a scaling (see formula (3.2.4) below), in view of the results in [62]. As mentioned above, any other minimizer  $w_0$  having the same mass can be written as  $w_b(\cdot - y)$  for some  $y \in \mathbb{R}^d$ . Inspired by selection principles in  $\Gamma$ -convergence as in [7], and exploiting the fact that in the r.h.s. of (3.1.17) we are actually allowed to pick any such  $w_0$ , we infer that  $\lim_{\gamma \downarrow 0} w_\gamma = w_b(\cdot - \bar{y})$  for some  $\bar{y} \in \mathbb{R}^d$  which minimizes the function

$$y \mapsto \int_{\mathbb{R}^d} \left( -\frac{w_b^{p+1}(x)}{p+1} + \frac{w_b^{2p}(x)}{2p} \right) \log |x + y| \, dx.$$

The minimum turns out to be attained exactly at  $y = 0$ , so that  $\{w_\gamma\}$  converges to  $w_b$ . The detailed analysis is carried out in Subsection 3.3.3.

5. Finally, in Subsection 3.3.4, we prove Theorem 3.1.1 (more precisely, its equivalent formulation given by Theorem 3.2.1 below) arguing by contradiction. The idea is to use a method similar to the one of [79, 80] and consider *angular derivatives* of  $w_\gamma$ , which are nontrivial if  $w_\gamma$  is supposed not to be radial. In fact, differentiating the Euler-Lagrange equation (3.1.6), one finds that the angular derivatives of  $w_\gamma$  belong to the kernel of the linear operator associated with a suitable functional. Passing to the limit as  $\gamma \downarrow 0$  and taking advantage of the previous convergence analysis, we get a contradiction with the fact that the linear operator associated with the limit functional enjoys a spectral gap property and its kernel is known to coincide with the *directional derivatives* of  $b_0$ .

## 3.2 Existence and elementary properties of optimal functions

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In this section we establish some basic properties of the space  $\mathcal{H}_\gamma(\mathbb{R}^d)$  introduced above, which will permit us to establish existence of optimal functions for (3.1.1) (Subsection 3.2.2). Integrability and regularity properties of such functions are then discussed in Subsection 3.2.3. In order to do so, proceeding along the lines of [62, Section 2], it is convenient to work with a non-scale-invariant minimization problem related to (3.1.1) (which we introduce in the next subsection), so that finding the corresponding minimizers is equivalent to finding optimal functions for (3.1.1).

## 3.2.1 Reformulating the problem in a non-scale-invariant form

To begin with, let us consider the following functional:

$$\mathcal{G}_\gamma[w] := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \frac{|w|^{p+1}}{|x|^\gamma} dx \quad \forall w \in \mathcal{H}_\gamma(\mathbb{R}^d).$$

By means of standard scaling arguments, it turns out that looking for nontrivial optimal functions for (3.1.1) is equivalent to solving, for any fixed mass  $M > 0$ , the following constrained minimization problem:

$$\begin{aligned} & \text{Find } w_\gamma \in \mathcal{H}_\gamma(\mathbb{R}^d) \text{ such that} \\ & \|w_\gamma\|_{2p,\gamma}^{2p} = M \quad \text{and} \quad \mathcal{G}_\gamma[w_\gamma] = \inf_{w \in \mathcal{H}_\gamma(\mathbb{R}^d): \|w\|_{2p,\gamma}^{2p} = M} \mathcal{G}_\gamma[w]. \end{aligned} \quad (\text{P})$$

In fact, multiplying any solution to (P) by an arbitrary constant  $\alpha \in \mathbb{R}$  and letting  $M$  and  $\alpha$  vary, we obtain the whole family of optimal functions for (3.1.1). Hence, as a necessary preliminary result, one needs to show that problem (P) admits solutions (Proposition 3.2.6). Actually, with respect to the non-weighted case, this can be proved in an easier way: as long as  $\gamma$  is strictly positive the embedding of  $\mathcal{H}_\gamma(\mathbb{R}^d)$  in  $L_\gamma^{2p}(\mathbb{R}^d)$  is *compact* (Proposition 3.2.5).

Before proving their existence, let us discuss some important properties of solutions to problem (P). First of all, since  $\mathcal{G}_\gamma[w_\gamma] = \mathcal{G}_\gamma[|w_\gamma|]$  and  $|w_\gamma| \in \mathcal{H}_\gamma(\mathbb{R}^d)$ , to our purposes we can and shall assume that  $w_\gamma$  is nonnegative. By classical variational arguments  $w_\gamma$  must satisfy, in distributional (and pointwise) sense, the semilinear Euler-Lagrange equation

$$-\Delta w_\gamma + \frac{w_\gamma^p}{|x|^\gamma} = \mu \frac{w_\gamma^{2p-1}}{|x|^\gamma} \quad \text{in } \mathbb{R}^d$$

for some positive parameter  $\mu = \mu(M)$ , namely the *Lagrange multiplier*. By scaling arguments, if  $w_\gamma$  is a minimizer of  $\mathcal{G}_\gamma$  under the constraint  $\|w_\gamma\|_{2p,\gamma}^{2p} = M$ , then the function  $\widehat{w}_\gamma$  defined as (let  $\beta > 0$ )

$$\widehat{w}_\gamma(x) := \beta^{\frac{2-\gamma}{p-1}} w_\gamma(\beta x) \quad (3.2.1)$$

is also a minimizer of  $\mathcal{G}_\gamma$  under the constraint

$$\|\widehat{w}_\gamma\|_{2p,\gamma}^{2p} = \beta^{\frac{d-\gamma+p(4-\gamma-d)}{p-1}} M.$$

*In particular, we can always fix the mass  $M$  in order to make  $\mu$  equal to 1: from here on, by  $w_\gamma$ , we shall implicitly mean any solution to (P) satisfying*

$$-\Delta w_\gamma + \frac{w_\gamma^p}{|x|^\gamma} = \frac{w_\gamma^{2p-1}}{|x|^\gamma} \quad \text{in } \mathbb{R}^d. \quad (3.2.2)$$

We point out that the Barenblatt-type function that solves (3.2.2), which also minimizes  $\mathcal{G}_\gamma$  in  $\mathcal{H}_\gamma^*(\mathbb{R}^d)$  (see Proposition 3.3.14), is the following:

$$w_{\mathfrak{b},\gamma}(x) := \left( \frac{a_\gamma}{b_\gamma + |x|^{2-\gamma}} \right)^{\frac{1}{p-1}}, \quad (3.2.3)$$

where

$$a_\gamma := (2-\gamma) \frac{d-\gamma-p(d-2)}{(p-1)^2}, \quad b_\gamma := \frac{(d-\gamma-p(d-2))^2}{p(p-1)^2}.$$

Recalling (3.1.4), observe that the identity  $w_{\mathbf{b},\gamma}(x) = \alpha \mathbf{b}_\gamma(\beta x)$  holds true with the choices

$$\alpha = (a_\gamma/b_\gamma)^{\frac{1}{p-1}}, \quad \beta = b_\gamma^{-\frac{1}{2-\gamma}}.$$

On the other hand, the Barenblatt-type function that solves (3.2.2) for  $\gamma = 0$ , which minimizes  $\mathcal{G}_0$  in  $\mathcal{H}_0(\mathbb{R}^d)$  thanks to the results in [62], is

$$w_{\mathbf{b}}(x) := \left( \frac{a}{b + |x|^2} \right)^{\frac{1}{p-1}}, \quad a := 2 \frac{d - p(d-2)}{(p-1)^2}, \quad b := \frac{(d - p(d-2))^2}{p(p-1)^2}. \quad (3.2.4)$$

In fact, in this case, *all* minimizers of  $\mathcal{G}_0$  whose mass is such that they solve (3.2.2) are of the form  $w_{\mathbf{b}}(\cdot - y_0)$  for some  $y_0 \in \mathbb{R}^d$  (see in particular [62, Theorem 4]).

According to the above notations, we can then reformulate our main result as follows.

**Theorem 3.2.1** (Equivalent to Theorem 3.1.1). *Let  $d \geq 3$ . For any  $p \in (1, d/(d-2))$  there exists  $\gamma^* = \gamma^*(p, d) \in (0, d - (d-2)p)$  such that  $w_\gamma = w_{\mathbf{b},\gamma}$  for all  $\gamma \in (0, \gamma^*)$ .*

**Remark 3.2.2.** We shall prove Theorem 3.2.1 in Subsection 3.3.4, after several intermediate results which are established below. As already mentioned, throughout we shall tacitly work with nonnegative optimal functions. This is not restrictive: in fact Theorem 3.2.1 also holds true for signed solutions to (P), since it holds true for their modulus.

The scaling properties of  $\mathcal{G}_\gamma$  entail that any solution to (P) is actually a solution of an explicit *unconstrained* minimization problem. In fact, consider the infimum

$$I_\gamma^1 := \inf_{w \in \mathcal{H}_\gamma(\mathbb{R}^d) \setminus \{0\}} \frac{\frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} |w|^{p+1} |x|^{-\gamma} dx}{\left( \int_{\mathbb{R}^d} |w|^{2p} |x|^{-\gamma} dx \right)^{\theta_\gamma}}, \quad (3.2.5)$$

where, for all  $d, \gamma$  and  $p$  complying with (3.1.2), we set

$$\theta_\gamma := \frac{d + 2 - 2\gamma - p(d-2)}{d - \gamma - p(d + \gamma - 4)} \in (0, 1). \quad (3.2.6)$$

Let us denote as  $I_\gamma^M$  the value of the infimum appearing in problem (P). By exploiting the rescaling (3.2.1), it is straightforward to check that the following identity holds true:

$$I_\gamma^M = M^{\theta_\gamma} I_\gamma^1 \quad \forall M > 0. \quad (3.2.7)$$

From (3.2.7) we deduce that  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$  attains  $I_\gamma^1$  if and only if it is a solution to problem (P) for some  $M > 0$ . By the definition of  $I_\gamma^1$  there holds

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \frac{|w|^{p+1}}{|x|^\gamma} dx \geq I_\gamma^1 \left( \int_{\mathbb{R}^d} \frac{|w|^{2p}}{|x|^\gamma} dx \right)^{\theta_\gamma}$$

for all  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$ , and the two sides are equal if and only if  $w \equiv 0$  or  $w$  is a minimizer of (3.2.5). Recalling the functional  $\mathcal{E}_\gamma$  introduced in (3.1.15), this means that

$$\inf_{w \in \mathcal{H}_\gamma(\mathbb{R}^d)} \mathcal{E}_\gamma[w] = 0,$$

and such infimum is attained if and only if  $w \equiv 0$  or  $w$  is a minimizer of (3.2.5), that is a solution to problem (P) for some  $M > 0$ . Note that, if one replaces  $I_\gamma^1$  with  $I > I_\gamma^1$  in (3.1.15), then the infimum of the corresponding functional becomes  $-\infty$  (it suffices to pick a sequence

of minimizers of (3.2.5) with increasing mass). Conversely, if  $I < I_\gamma^1$  then the infimum is still 0, but it is attained only by  $w \equiv 0$ .

The fact that, in particular,  $w_\gamma$  is also a minimizer of  $\mathcal{E}_\gamma$  over  $\mathcal{H}_\gamma(\mathbb{R}^d)$  provides us with an explicit relation between its mass  $M_\gamma := \int_{\mathbb{R}^d} w_\gamma^{2p} |x|^{-\gamma} dx$  and  $I_\gamma^1$ . Indeed, differentiating  $\mathcal{E}_\gamma[w]$  in  $w = w_\gamma$  and recalling the Euler-Lagrange equation (3.2.2), we obtain the identity

$$2pI_\gamma^1 \theta_\gamma M_\gamma^{\theta_\gamma - 1} = 1, \quad (3.2.8)$$

which can equivalently be written as

$$\theta_\gamma \left( \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w_\gamma|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \frac{w_\gamma^{p+1}}{|x|^\gamma} dx \right) = \frac{1}{2p} \int_{\mathbb{R}^d} \frac{w_\gamma^{2p}}{|x|^\gamma} dx. \quad (3.2.9)$$

Note that, performing simple but tedious computations (one picks  $w_\alpha(x) := \alpha w(x)$  and optimizes the ratio in (3.2.5) w.r.t.  $\alpha > 0$ ), we can express  $I_\gamma^1$  in terms of  $C_\gamma$ :

$$I_\gamma^1 = C_\gamma^{-2p\theta_\gamma} \frac{p-1}{p+1-2p\theta_\gamma} \left( \frac{2p\theta_\gamma-2}{p+1-2p\theta_\gamma} \right)^{-\frac{2p\theta_\gamma-2}{p-1}} 2^{-\frac{p+1-2p\theta_\gamma}{p-1}} (p+1)^{-\frac{2p\theta_\gamma-2}{p-1}}. \quad (3.2.10)$$

### 3.2.2 Basic properties of $\mathcal{H}_\gamma$ and existence of minimizers

We now prove some elementary results concerning  $\mathcal{H}_\gamma(\mathbb{R}^d)$ , which will be crucial in order to establish existence of solutions to (P) (or minimizers of (3.2.5)).

**Proposition 3.2.3.** *Let (3.1.2) hold true. Then  $C_c^\infty(\mathbb{R}^d \setminus \{0\})$  is dense in  $\mathcal{H}_\gamma(\mathbb{R}^d)$ .*

*Proof.* We have to prove that, for any function  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$ , there exists a sequence  $\{\varphi_n\} \subset C_c^\infty(\mathbb{R}^d \setminus \{0\})$  such that

$$\lim_{n \rightarrow \infty} \|w - \varphi_n\|_{p+1, \gamma} = 0, \quad \lim_{n \rightarrow \infty} \|\nabla w - \nabla \varphi_n\|_2 = 0. \quad (3.2.11)$$

If  $w$  is compactly supported in  $\mathbb{R}^d \setminus \{0\}$  such a sequence does exist by standard mollification arguments (the weight  $|x|^{-\gamma}$  is, locally in  $\mathbb{R}^d \setminus \{0\}$ , bounded and bounded away from zero). It is therefore enough to prove that, for any  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$ , one can pick a sequence of functions  $\{w_n\} \subset \mathcal{H}_\gamma(\mathbb{R}^d)$  having compact support in  $\mathbb{R}^d \setminus \{0\}$  and such that (3.2.11) holds true with  $\varphi_n = w_n$ . In view of another classical density argument, it is not restrictive to assume in addition that  $w \in L^\infty(\mathbb{R}^d)$ . Consider then the following sequence (let  $n \geq 2$ ):

$$w_n(x) := \left[ \chi_{\{|x| < 2\}}(x) (1 - \xi(nx)) + \chi_{\{|x| \geq 2\}}(x) \xi(x/n) \right] v(x),$$

where  $\xi$  is a smooth function such that

$$0 \leq \xi(x) \leq 1 \quad \forall x \in \mathbb{R}^d, \quad \xi(x) = 1 \quad \forall x \in B_1, \quad \xi(x) = 0 \quad \forall x \in B_2^c.$$

By dominated convergence, it is apparent that

$$\lim_{n \rightarrow \infty} \|w - w_n\|_{p+1, \gamma} = 0.$$

As for the convergence of  $\{\nabla w_n\}$  to  $\nabla w$  in  $[L^2(\mathbb{R}^d)]^d$ , first notice that the gradient of  $w_n$  reads

$$\begin{aligned} \nabla w_n(x) = & \underbrace{\left[ -n \chi_{\{\frac{1}{n} \leq |x| \leq \frac{2}{n}\}}(x) \nabla \xi(nx) + n^{-1} \chi_{\{n \leq |x| \leq 2n\}}(x) \nabla \xi(x/n) \right]}_{F_n(x)} w(x) \\ & + \underbrace{\left[ \chi_{\{|x| < 2\}}(x) (1 - \xi(nx)) + \chi_{\{|x| \geq 2\}}(x) \xi(x/n) \right]}_{G_n(x)} \nabla w(x). \end{aligned}$$



Since  $\{G_n\}$  converges to  $\nabla w$  in  $[L^2(\mathbb{R}^d)]^d$  (*e.g.* again by dominated convergence), we only need to prove that

$$\lim_{n \rightarrow \infty} \|F_n\|_2 = 0. \quad (3.2.12)$$

To this end, let  $M > 0$  be any constant such that  $\|\nabla \xi\|_\infty \leq M$  and  $\|w\|_\infty \leq M$ . We have:

$$\|F_n\|_2^2 \leq CM^4 n^2 \int_{\frac{1}{n}}^{\frac{2}{n}} r^{d-1} dr + \frac{M^2}{n^2} \int_{n \leq |x| \leq 2n} w^2 dx, \quad (3.2.13)$$

where  $C := |\mathbb{S}_{d-1}|$ . Since  $d \geq 3$ , the first term in the r.h.s. of (3.2.13) vanishes as  $n \rightarrow \infty$ . As for the second term, we have to work a bit more. Exploiting Hölder's inequality and the fact that  $|x|^{-\gamma}$  is decreasing in  $|x|$ , we get:

$$\begin{aligned} \int_{n \leq |x| \leq 2n} w^2 dx &\leq 2^{\frac{2\gamma}{p+1}} n^{\frac{2\gamma}{p+1}} \int_{n \leq |x| \leq 2n} w^2 |x|^{-\frac{2\gamma}{p+1}} dx \\ &\leq C^{\frac{p-1}{p+1}} 2^{\frac{2\gamma}{p+1}} n^{\frac{2\gamma}{p+1}} \left( \int_{n \leq |x| \leq 2n} |w|^{p+1} |x|^{-\gamma} dx \right)^{\frac{2}{p+1}} \left( \int_n^{2n} r^{d-1} dr \right)^{\frac{p-1}{p+1}}. \end{aligned} \quad (3.2.14)$$

Since  $w \in L_\gamma^{p+1}(\mathbb{R}^d)$ , from (3.2.14) we deduce that the second term in the r.h.s. of (3.2.13) goes to zero as  $n \rightarrow \infty$  provided

$$\frac{2\gamma}{p+1} + d \frac{p-1}{p+1} \leq 2,$$

that is  $p \leq (d-2\gamma+2)/(d-2)$ , which clearly holds true in view of (3.1.2). We have therefore proved (3.2.12) and the result follows.  $\square$

Using a standard notation, from here on we shall denote by  $\dot{H}^1(\mathbb{R}^d)$  the closure of  $\mathcal{D}(\mathbb{R}^d)$  w.r.t. the norm  $\|\nabla(\cdot)\|_2$  (see also Chapter 2, Paragraph 2.1.1.3 for the definition of the analogous space in the fractional framework). It is well known that for all  $\gamma \in [0, 2]$  there exists a suitable positive constant  $C_* = C_*(\gamma, d)$  such that the Hardy-Sobolev inequality

$$\|w\|_{2_\gamma^*} \leq C_* \|\nabla w\|_2 \quad \forall w \in \dot{H}^1(\mathbb{R}^d) \quad (3.2.15)$$

holds true, where  $2_\gamma^*$  is defined in (3.1.2) and we set  $2^* := 2_0^*$ . Note that for  $\gamma = 0$  we recover the standard Sobolev inequality

$$\left( \int_{\mathbb{R}^d} |w|^{2^*} dx \right)^{1/2^*} \leq C_S \|\nabla w\|_2 \quad \forall w \in \dot{H}^1(\mathbb{R}^d), \quad (3.2.16)$$

while for  $\gamma = 2$  we obtain Hardy's inequality:

$$\left( \int_{\mathbb{R}^d} w^2 |x|^{-2} dx \right)^{1/2} \leq C_H \|\nabla w\|_2 \quad \forall w \in \dot{H}^1(\mathbb{R}^d). \quad (3.2.17)$$

On the other hand, (3.2.15) follows by interpolation between (3.2.17) and (3.2.16):

$$\|w\|_{2_\gamma^*} \leq \|w\|_{2,2}^{\frac{\gamma(d-2)}{2(d-\gamma)}} \|w\|_{2^*}^{\frac{d(2-\gamma)}{2(d-\gamma)}} \leq C_H^{\frac{\gamma(d-2)}{2(d-\gamma)}} C_S^{\frac{d(2-\gamma)}{2(d-\gamma)}} \|\nabla w\|_2 \quad \forall w \in \dot{H}^1(\mathbb{R}^d), \quad (3.2.18)$$

although the *best constant*  $C_*$  is not the one appearing in (3.2.18). For best constants in Sobolev inequalities we refer the reader to the pioneering papers [11, 167] and *e.g.* to [109]. Performing another interpolation and using (3.2.18), we actually recover the Caffarelli-Kohn-Nirenberg inequality (3.1.1):

$$\|w\|_{2p,\gamma} \leq C_H^{\frac{\gamma(p-1)(d-2)}{2p[2(d-\gamma)-(p+1)(d-2)]}} C_S^{\frac{d(p-1)(2-\gamma)}{2p[2(d-\gamma)-(p+1)(d-2)]}} \|\nabla w\|_2^\vartheta \|w\|_{p+1,\gamma}^{1-\vartheta} \quad (3.2.19)$$

for all  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$ , where  $\vartheta$  is as in (3.1.3). Again, we do not claim that the constant that appears in the r.h.s. of (3.2.19) is the best one for which the inequality holds true, namely  $C_\gamma$ . However, it provides us with an explicit upper bound for the latter, which will be fundamental for the concentration-compactness analysis of Subsection 3.3.1 to work.

**Lemma 3.2.4.** *Let  $d \geq 3$  and  $\gamma \in [0, 2)$ . Then  $\dot{H}^1(\mathbb{R}^d)$  is locally compactly embedded in  $L_\gamma^q(\mathbb{R}^d)$  for all  $q \in [1, 2_\gamma^*)$ .*

*Proof.* First of all we remark that it is enough to establish the result for all  $q \in (2, 2_\gamma^*)$ . Since  $2_\gamma^* \leq 2^*$ , in view of the well-known local compactness of subcritical Sobolev embeddings (see e.g. [103, Theorem 7.22]) and thanks to the local boundedness and boundedness away from zero of the weight  $|x|^{-\gamma}$ , it is plain that  $\dot{H}^1(\mathbb{R}^d)$  is compactly embedded in  $L_\gamma^q(B_R \setminus B_r)$  for all  $R > r > 0$ . We are then left with proving an equi-integrability property. That is, we need to show that

$$\int_{B_r} |w|^q |x|^{-\gamma} dx \rightarrow 0$$

as  $r \rightarrow 0$ , uniformly as  $w$  varies in a bounded subset of  $\dot{H}^1(\mathbb{R}^d)$ . But this readily follows by interpolation, as a direct consequence of (3.2.15) and (3.2.17):

$$\begin{aligned} \int_{B_r} |w|^q |x|^{-\gamma} dx &\leq \left( \int_{B_r} w^2 |x|^{-\gamma} dx \right)^{\frac{2_\gamma^* - q}{2_\gamma^* - 2}} \left( \int_{B_r} |w|^{2_\gamma^*} |x|^{-\gamma} dx \right)^{\frac{q-2}{2_\gamma^* - 2}} \\ &\leq r^{(2-\gamma)\frac{2_\gamma^* - q}{2_\gamma^* - 2}} C_H^{2\frac{2_\gamma^* - q}{2_\gamma^* - 2}} C_*^{2_\gamma^* \frac{q-2}{2_\gamma^* - 2}} \|\nabla w\|_2^q. \end{aligned}$$

□

Thanks to the above results, we are in position to prove a key compact embedding for the space  $\mathcal{H}_\gamma(\mathbb{R}^d)$ .

**Proposition 3.2.5.** *Let (3.1.2) hold true. Then  $\mathcal{H}_\gamma(\mathbb{R}^d)$  is compactly embedded in  $L_\gamma^{2p}(\mathbb{R}^d)$ .*

*Proof.* Recalling the definitions of  $\dot{H}^1(\mathbb{R}^d)$  and  $\mathcal{H}_\gamma(\mathbb{R}^d)$ , from Proposition 3.2.3 we immediately deduce that  $\mathcal{H}_\gamma(\mathbb{R}^d)$  is continuously embedded in  $\dot{H}^1(\mathbb{R}^d)$ . Hence, using Lemma 3.2.4 and noticing that  $2p < 2_\gamma^*$ , we deduce that  $\mathcal{H}_\gamma(\mathbb{R}^d)$  is *locally* compactly embedded in  $L_\gamma^{2p}(\mathbb{R}^d)$ . In order to prove *global* compactness, we have to establish an equi-integrability property at infinity (as we did in the proof of Lemma 3.2.4). To this end, by means of Hölder's inequality we get, for all  $w \in \mathcal{H}_\gamma(\mathbb{R}^d)$ :

$$\begin{aligned} \int_{B_R^c} \frac{|w|^{2p}}{|x|^\gamma} dx &\leq \frac{1}{R^{\frac{\gamma(q-1)}{q}}} \left( \int_{B_R^c} \frac{|w|^{2p\alpha q}}{|x|^\gamma} dx \right)^{1/q} \left( \int_{B_R^c} |w|^{2p(1-\alpha)q'} dx \right)^{1/q'} \\ &\forall R > 0, \quad \forall q, q' : 1/q + 1/q' = 1, \quad q > 1, \quad \forall \alpha \in (0, 1). \end{aligned} \quad (3.2.20)$$

Choosing

$$q = \frac{2^* - p - 1}{2^* - 2p}, \quad \alpha = \frac{(p+1)(2^* - 2p)}{2p(2^* - p - 1)},$$

inequality (3.2.20) becomes

$$\int_{B_R^c} \frac{|w|^{2p}}{|x|^\gamma} dx \leq \frac{1}{R^{\frac{\gamma(q-1)}{q}}} \left( \int_{B_R^c} \frac{|w|^{p+1}}{|x|^\gamma} dx \right)^{1/q} \left( \int_{B_R^c} |w|^{2^*} dx \right)^{1/q'}. \quad (3.2.21)$$

Thanks to the standard Sobolev inequality (3.2.16) and to the definition of  $\|\cdot\|_{\mathcal{H}_\gamma}$  (see Subsection 3.1.2), from (3.2.21) we infer that

$$\int_{B_R^c} \frac{|w|^{2p}}{|x|^\gamma} dx \leq \frac{C_S^{\frac{2^*(q-1)}{q}}}{R^{\frac{\gamma(q-1)}{q}}} \|w\|_{\mathcal{H}_\gamma}^{2p} \quad \forall R > 0, \quad \forall w \in \mathcal{H}_\gamma(\mathbb{R}^d). \quad (3.2.22)$$

In particular, (3.2.22) shows the equi-integrability at infinity of  $|w(x)|^{2p} |x|^{-\gamma}$  as  $w$  varies in a bounded subset of  $\mathcal{H}_\gamma(\mathbb{R}^d)$ , and permits us therefore to conclude that the embedding of  $\mathcal{H}_\gamma(\mathbb{R}^d)$  in  $L^{2p}(\mathbb{R}^d)$  is compact.  $\square$

As an immediate consequence of Proposition 3.2.5 we have the claimed result about existence of solutions to (P).

**Proposition 3.2.6.** *Let (3.1.2) hold true. Then for any  $M > 0$  problem (P) admits (at least) one solution.*

*Proof.* It is direct to check that the functional  $\mathcal{G}_\gamma$  is coercive and weakly lower semi-continuous on  $\mathcal{H}_\gamma(\mathbb{R}^d)$ . The latter being a reflexive Banach space, the existence of a minimizer  $w_\gamma \in \mathcal{H}_\gamma(\mathbb{R}^d)$  for  $\mathcal{G}_\gamma$  complying with the constraint  $\|w_\gamma\|_{2p,\gamma}^{2p} = M$  follows by means of the compact embedding ensured by Proposition 3.2.5.  $\square$

**Remark 3.2.7.** Under assumptions (3.1.2), the inclusion

$$\mathcal{H}_0(\mathbb{R}^d) \subset \mathcal{H}_\gamma(\mathbb{R}^d) \quad (3.2.23)$$

holds true. Indeed, using (3.2.15) and the fact that  $\mathcal{H}_0(\mathbb{R}^d)$  is continuously embedded in  $\dot{H}^1(\mathbb{R}^d)$ , we get:

$$\int_{\mathbb{R}^d} |w|^q |x|^{-\gamma} dx \leq \left( \int_{B_1} |w|^{2_\gamma^*} |x|^{-\gamma} dx \right)^{\frac{q}{2_\gamma^*}} \left( \int_{B_1} |x|^{-\gamma} dx \right)^{\frac{2_\gamma^* - q}{2_\gamma^*}} + \int_{B_1^c} |w|^q dx \quad (3.2.24)$$

for all  $w \in \mathcal{H}_0(\mathbb{R}^d)$  and  $q \in [p+1, 2_\gamma^*]$ . Picking  $q = p+1$  in (3.2.24), property (3.2.23) readily follows.

### 3.2.3 *A priori* estimates for minimizers

The aim of this subsection is to provide some estimates over  $L^q$  and Hölder norms of  $w_\gamma$ , by exploiting the semilinear Euler-Lagrange equation (3.2.2) solved by it. Such estimates will turn out to be fundamental in order to prove convergence results and justify passages to the limit in Section 3.3.

**Lemma 3.2.8.** *Let  $d \geq 3$ ,  $p \in (1, 2^*/2)$  and  $\gamma \in [0, d - (d-2)p)$ . Then*

$$\|w_\gamma\|_q \leq C \quad \forall q \in [2^*, \infty], \quad (3.2.25)$$

where  $C = C(\gamma, \|\nabla w_\gamma\|_2, p, d)$  is a positive constant which depends continuously on  $\gamma \in [0, d - (d-2)p)$  and  $\|\nabla w_\gamma\|_2$ .

*Proof.* We can proceed by means of a standard Moser iterative method, in the spirit of the pioneering works [134, 137]. That is, let  $A > 0$  and set

$$\varepsilon_0 := 2_\gamma^* - 2p. \quad (3.2.26)$$

### 3.2. Existence and elementary properties of optimal functions

Multiplying the Euler-Lagrange equation (3.2.2) by the test function  $(w_\gamma \wedge A)^{1+\varepsilon_0}$  and integrating by parts in  $\mathbb{R}^d$ , we get:

$$\int_{\mathbb{R}^d} \langle \nabla(w_\gamma \wedge A)^{1+\varepsilon_0}, \nabla w_\gamma \rangle dx + \int_{\mathbb{R}^d} (w_\gamma \wedge A)^{1+\varepsilon_0} w_\gamma^p |x|^{-\gamma} dx = \int_{\mathbb{R}^d} (w_\gamma \wedge A)^{1+\varepsilon_0} w_\gamma^{2p-1} |x|^{-\gamma} dx,$$

that is

$$\begin{aligned} & \frac{4(1+\varepsilon_0)}{(2+\varepsilon_0)^2} \int_{\mathbb{R}^d} |\nabla(w_\gamma \wedge A)^{1+\frac{\varepsilon_0}{2}}|^2 dx + \int_{\mathbb{R}^d} (w_\gamma \wedge A)^{1+\varepsilon_0} w_\gamma^p |x|^{-\gamma} dx \\ &= \int_{\mathbb{R}^d} (w_\gamma \wedge A)^{1+\varepsilon_0} w_\gamma^{2p-1} |x|^{-\gamma} dx. \end{aligned} \quad (3.2.27)$$

Letting  $A \rightarrow +\infty$  in (3.2.27) we obtain the following identity:

$$\frac{4(1+\varepsilon_0)}{(2+\varepsilon_0)^2} \int_{\mathbb{R}^d} \left| \nabla \left( w_\gamma^{1+\frac{\varepsilon_0}{2}} \right) \right|^2 dx + \int_{\mathbb{R}^d} w_\gamma^{p+1+\varepsilon_0} |x|^{-\gamma} dx = \int_{\mathbb{R}^d} w_\gamma^{2p+\varepsilon_0} |x|^{-\gamma} dx. \quad (3.2.28)$$

Applying the weighted Sobolev inequality (3.2.15) to the function  $w = w_\gamma^{1+\varepsilon_0/2}$  in the l.h.s. of (3.2.28) and using (3.2.26), we deduce that

$$\|w_\gamma\|_{2_\gamma^*(1+\varepsilon_0/2), \gamma}$$

is bounded from above by

$$\|w_\gamma\|_{2_\gamma^*, \gamma}.$$

Now let us define the sequence  $\{\varepsilon_n\}$  through the recursive equation

$$\varepsilon_{n+1} := \frac{2_\gamma^*}{2} \varepsilon_n + 2_\gamma^* - 2p \quad \forall n \in \mathbb{N}. \quad (3.2.29)$$

Performing the same computations as above, with  $\varepsilon_0$  replaced by  $\varepsilon_n$ , it is apparent that we end up with a bound over the  $L_\gamma^{2p+\varepsilon_{n+1}}$  norm of  $w_\gamma$  in terms of the  $L_\gamma^{2p+\varepsilon_n}$  norm of  $w_\gamma$ . More precisely, we obtain the following recursive inequality:

$$\|w_\gamma\|_{2p+\varepsilon_{n+1}, \gamma} \leq \left[ \frac{C_*^2 (2+\varepsilon_n)^2}{4(1+\varepsilon_n)} \right]^{\frac{2_\gamma^*}{2(2p+\varepsilon_{n+1})}} \|w_\gamma\|_{2p+\varepsilon_n, \gamma}^{\frac{2_\gamma^*(2p+\varepsilon_n)}{2(2p+\varepsilon_{n+1})}}. \quad (3.2.30)$$

Note that from (3.2.26) and (3.2.29) we can derive an explicit expression for  $\{\varepsilon_n\}$ :

$$\varepsilon_n = \frac{2_\gamma^* - 2p}{2_\gamma^* - 2} \left[ 2_\gamma^* \left( \frac{2_\gamma^*}{2} \right)^n - 2 \right] \quad \forall n \in \mathbb{N}. \quad (3.2.31)$$

By iterating (3.2.30) and exploiting (3.2.31) it is not difficult to deduce the validity of the estimate

$$\|w_\gamma\|_{2p+\varepsilon_{n+1}, \gamma} \leq D \|w_\gamma\|_{2_\gamma^*, \gamma}^{\left( \frac{2_\gamma^*}{2} \right)^{n+1} \frac{2_\gamma^*}{2p+\varepsilon_{n+1}}}, \quad (3.2.32)$$

for a suitable positive constant  $D = D(\gamma, p, d)$  which depends continuously on

$$\gamma \in [0, d - (d-2)p)$$

and actually blows up as  $\gamma \uparrow d - (d-2)p$ . Letting  $n \rightarrow \infty$  in (3.2.32) we thus get the  $L^\infty$  bound

$$\|w_\gamma\|_\infty \leq D \|w_\gamma\|_{2_\gamma^*, \gamma}^{(2_\gamma^*-2)/(2_\gamma^*-2p)}. \quad (3.2.33)$$

Finally, in order to prove (3.2.25), it suffices to use the standard interpolation inequality

$$\|w_\gamma\|_q \leq \|w_\gamma\|_{2_\gamma^*}^{\frac{2_\gamma^*}{q}} \|w_\gamma\|_\infty^{1-\frac{2_\gamma^*}{q}}$$

together with (3.2.33) and the Sobolev inequalities (3.2.15), (3.2.16).  $\square$

**Lemma 3.2.9.** *Let  $d \geq 3$ ,  $p \in (1, 2^*/2)$  and  $q \in [1, \infty)$ . There exist  $\bar{\gamma} = \bar{\gamma}(q, p, d) \in (0, d - (d - 2)p)$  and  $C = C(q, p, d) > 0$  such that the estimate*

$$\|\Delta w_\gamma\|_{L^q(\Omega)} \leq C \left( \|w_\gamma\|_{L^{(2p-1)q}(\Omega)}^{2p-1} + \|w_\gamma\|_{L^{pq}(\Omega)}^p + \|w_\gamma\|_{L^{2pq}(\Omega)}^{2p-1} + \|w_\gamma\|_{L^{2pq}(\Omega)}^p \right) \quad (3.2.34)$$

holds true for all  $\gamma \in [0, \bar{\gamma})$  and any  $\Omega \subset \mathbb{R}^d$ .

*Proof.* From the Euler-Lagrange equation (3.2.2), we obtain:

$$\begin{aligned} \|\Delta w_\gamma\|_{L^q(\Omega)}^q &\leq 2^{q-1} \left( \int_\Omega w_\gamma^{(2p-1)q} dx + \int_\Omega w_\gamma^{pq} dx \right) + 2^{q-1} \left( \int_{B_1} |x|^{-\gamma 2pq} dx \right)^{\frac{1}{2p}} \left( \int_\Omega w_\gamma^{2pq} dx \right)^{\frac{2p-1}{2p}} \\ &\quad + 2^{q-1} \left( \int_{B_1} |x|^{-\gamma 2q} dx \right)^{\frac{1}{2}} \left( \int_\Omega w_\gamma^{2pq} dx \right)^{\frac{1}{2}}. \end{aligned} \quad (3.2.35)$$

Let  $\bar{\gamma} \in (0, d - (d - 2)p)$  be any number such that  $\bar{\gamma} < d/2pq$ . In view of (3.2.35) it is clear that, under the assumption  $\gamma \in [0, \bar{\gamma})$ , for a suitable positive constant  $C = C(q, p, d)$  the estimate

$$\|\Delta w_\gamma\|_{L^q(\Omega)}^q \leq C \left( \|w_\gamma\|_{L^{(2p-1)q}(\Omega)}^{(2p-1)q} + \|w_\gamma\|_{L^{pq}(\Omega)}^{pq} + \|w_\gamma\|_{L^{2pq}(\Omega)}^{(2p-1)q} + \|w_\gamma\|_{L^{2pq}(\Omega)}^{pq} \right)$$

holds true, whence (3.2.34) upon relabelling  $C$ .  $\square$

**Lemma 3.2.10.** *Let  $d \geq 3$ ,  $p \in (1, 2^*/2)$  and  $q \in (1, \infty)$ . There exists a positive constant  $C = C(q, d)$  such that  $w_\gamma \in W^{2,q}(B_1(x_0))$  and the estimate*

$$\|w_\gamma\|_{W^{2,q}(B_1(x_0))} \leq C \left( \|w_\gamma\|_{L^q(B_2(x_0))} + \|\Delta w_\gamma\|_{L^q(B_2(x_0))} \right) \quad (3.2.36)$$

holds true for any  $x_0 \in \mathbb{R}^d$  and all  $\gamma \in [0, \bar{\gamma})$ , where  $\bar{\gamma} = \bar{\gamma}(q, p, d)$  is as in Lemma 3.2.9. In particular, if  $q \in (d, \infty)$  then there exists another positive constant  $\tilde{C} = \tilde{C}(q, d)$  such that the estimate

$$\|w_\gamma\|_{C^{1,1-d/q}(\bar{B}_1(x_0))} \leq \tilde{C} \left( \|w_\gamma\|_{L^q(B_2(x_0))} + \|\Delta w_\gamma\|_{L^q(B_2(x_0))} \right) \quad (3.2.37)$$

holds true for any  $x_0 \in \mathbb{R}^d$  and all  $\gamma \in [0, \bar{\gamma})$ .

*Proof.* Thanks to (3.2.25) and (3.2.34), we have that  $w_\gamma \in L^q(B_2(x_0))$  and  $\Delta w_\gamma \in L^q(B_2(x_0))$  for all  $\gamma \in [0, \bar{\gamma})$ . On the other hand, by the Calderón-Zygmund theory, we know that there exists a positive constant  $C = C(q, d)$  such that, for any function  $w \in W_{\text{loc}}^{2,q}(B_2(x_0)) \cap L^q(B_2(x_0))$  with  $\Delta w \in L^q(B_2(x_0))$ , the inequality

$$\|w\|_{W^{2,q}(B_1(x_0))} \leq C \left( \|w\|_{L^q(B_2(x_0))} + \|\Delta w\|_{L^q(B_2(x_0))} \right) \quad (3.2.38)$$

holds true. This is the content (in a particular case) of [103, Theorem 9.11]. *A priori* we do not know whether  $w_\gamma \in W_{\text{loc}}^{2,q}(B_2(x_0))$ . However, one can consider the standard mollification  $w_{\gamma,\varepsilon} := w_\gamma * \rho_\varepsilon$  (let  $\{\rho_\varepsilon\}$  be a regular mollifier), apply (3.2.38) to  $w_{\gamma,\varepsilon}$  and then pass to the limit as  $\varepsilon \rightarrow 0$  to get (3.2.36), recalling that  $w_{\gamma,\varepsilon} \rightarrow w_\gamma$  and  $\Delta w_{\gamma,\varepsilon} \rightarrow \Delta w_\gamma$  in  $L^q(B_2(x_0))$ .

Finally, estimate (3.2.37) is just a consequence of (3.2.36) and standard Sobolev embeddings (see *e.g.* [103, Theorem 7.26]).  $\square$

**Remark 3.2.11.** Thanks to the  $L^\infty$  bound provided by Lemma 3.2.8, the Euler-Lagrange equation (3.2.2) implies that  $|x|^\gamma \Delta w_\gamma \in L^\infty(\mathbb{R}^d)$ , for all  $\gamma$  and  $p$  complying with (3.1.2). Arguing as in the proof of Lemma 3.2.10, we then deduce that  $w_\gamma \in W_{\text{loc}}^{2,q}(\mathbb{R}^d)$  for all  $q \in [1, d/\gamma)$ . In particular, if  $\gamma \in (0, 1)$  then  $w_\gamma \in C^{1,\alpha}$  for all  $\alpha < 1 - \gamma$ , while if  $\gamma \in [1, 2)$  then  $w_\gamma \in C^{0,\alpha}$  for all  $\alpha < 2 - \gamma$ .

We point out that this is the “almost” optimal regularity for solutions to (3.2.2). In fact, the Barenblatt-type function  $w_{\mathfrak{b},\gamma}$  defined in (3.2.3) is a solution to (3.2.2) whose regularity, because of the presence of the singularity  $|x|^{-\gamma}$ , is precisely  $C^{1,1-\gamma}$  for  $\gamma \in (0, 1)$  and  $C^{0,2-\gamma}$  for  $\gamma \in [1, 2)$ .

### 3.3 Convergence as $\gamma \downarrow 0$ and radial symmetry of optimal functions

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Having at our disposal optimal functions for (3.1.1) (or, equivalently, minimizers of problem (P)) and associated *a priori* estimates, we are in position to analyse their asymptotic behaviour as the parameter  $\gamma$  tends to zero. Our strategy, which has been sketched in Subsection 3.1.2, will finally allow us to prove Theorem 3.1.1 (*i.e.* Theorem 3.2.1), namely radial symmetry of such optimal functions for any  $\gamma$  lying in some (unknown) right-hand neighbourhood of  $\{0\}$ .

#### 3.3.1 Concentration-compactness analysis

In this subsection we shall make use of a suitable variant of the concentration-compactness principle originally introduced in [127, 128] to prove that the solutions  $w_\gamma$  to problem (P) satisfying (3.2.2) approximate, as  $\gamma \downarrow 0$ , the Barenblatt-type function  $w_{\mathfrak{b}}$  (see (3.2.4)) *up to translations*. Afterwards, taking advantage of the estimates of Subsection 3.2.3, we shall refine a little bit this result.

**Lemma 3.3.1.** *Let  $p \in (1, 2^*/2)$  and  $\{\gamma_n\} \subset (0, d - (d - 2)p)$  be a decreasing sequence such that  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . Let  $M_{\gamma_n} := \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx$ . Then, up to subsequences:*

–  $\{M_{\gamma_n}\}$  tends to the mass of  $w_{\mathfrak{b}}$ :

$$\lim_{n \rightarrow \infty} M_{\gamma_n} = \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{2p} dx; \quad (3.3.1)$$

– there exist  $\bar{y} \in \mathbb{R}^d$  and a sequence of translations  $\{y_n\} \subset \mathbb{R}^d$  such that

$$v_n \rightarrow w_{\mathfrak{b}}(\cdot - \bar{y}) \text{ strongly in } \dot{H}^1(\mathbb{R}^d), \quad (3.3.2)$$

where  $v_n := w_{\gamma_n}(\cdot + y_n)$ . Moreover, either  $\{y_n\} = \{0\}$  or  $|y_n| \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} |y_n|^{\gamma_n} = 1. \quad (3.3.3)$$

As a consequence,

$$\lim_{\gamma \rightarrow 0} I_\gamma^1 = I_0^1, \quad (3.3.4)$$

where  $I_\gamma^1$  is the value of the infimum appearing in (3.2.5).

*Proof.* Let  $\varrho_n(x) := w_{\gamma_n}^{2p}(x) |x|^{-\gamma_n}$ . Our first aim is to prove that the sequence  $M_{\gamma_n} = \int_{\mathbb{R}^d} \varrho_n(x) dx$  converges (up to subsequences) to a positive number  $\bar{M}$ . To this end, a crucial step consists in establishing the boundedness and the boundedness away from zero of the sequence  $\{I_{\gamma_n}^1\}$ . The former is straightforward. In fact, by the definition of  $I_\gamma^1$ , we have:

$$I_{\gamma_n}^1 \leq \frac{\frac{1}{2} \int_{\mathbb{R}^d} |\nabla w_{\mathfrak{b}}|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{p+1} |x|^{-\gamma_n} dx}{\left( \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}}},$$

which yields

$$\limsup_{n \rightarrow \infty} I_{\gamma_n}^1 \leq I_0^1. \quad (3.3.5)$$

In view of (3.2.10), the boundedness away from zero of  $\{I_{\gamma_n}^1\}$  is equivalent to the boundedness of  $\{\mathcal{C}_{\gamma_n}\}$ , and the latter is a direct consequence, for instance, of (3.2.19). Summing up, we have proved that

$$0 < \liminf_{n \rightarrow \infty} I_{\gamma_n}^1 \leq \limsup_{n \rightarrow \infty} I_{\gamma_n}^1 < \infty,$$

that is, up to subsequences,

$$\lim_{n \rightarrow \infty} I_{\gamma_n}^1 =: I \in (0, \infty). \quad (3.3.6)$$

Recalling (3.2.8), we finally obtain (again, up to subsequences)

$$\lim_{n \rightarrow \infty} M_{\gamma_n} = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varrho_n \, dx = \lim_{n \rightarrow \infty} \left(2p I_{\gamma_n}^1 \theta_{\gamma_n}\right)^{\frac{1}{1-\theta_{\gamma_n}}} = (2p I \theta_0)^{\frac{1}{1-\theta_0}} =: \overline{M}. \quad (3.3.7)$$

Thanks to (3.3.6), (3.3.7) and to the scaling identity (3.2.7), we have:

$$\lim_{n \rightarrow \infty} \mathcal{G}_{\gamma_n}[w_{\gamma_n}] = \lim_{n \rightarrow \infty} M_{\gamma_n}^{\theta_{\gamma_n}} I_{\gamma_n}^1 = \overline{M}^{\theta_0} I. \quad (3.3.8)$$

Having established (3.3.7), we can apply the classical concentration-compactness criterion [127, Lemma I.1] to the sequence  $\{\varrho_n\} \subset L^1(\mathbb{R}^d)$ . In fact, only three scenarios are admissible for  $\{\varrho_n\}$  (up to subsequences): vanishing, dichotomy or compactness. We shall rule out vanishing and dichotomy, and thus infer compactness. Afterwards we shall see how compactness entails the convergence of  $\{w_{\gamma_n}\}$  to  $w_b$  up to translations.

**No vanishing.** Suppose, by contradiction, that there exists  $R > 0$  such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{y+B_R} \varrho_n \, dx = 0. \quad (3.3.9)$$

Let  $f_n(x) := w_{\gamma_n}(x) |x|^{-\frac{\gamma_n}{2p}}$ , so that (3.3.9) reads

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{y+B_R} f_n^{2p} \, dx = 0. \quad (3.3.10)$$

For any given  $\epsilon_2 > \epsilon_1 > 0$ , consider the function  $\tilde{f}_n(x) := \phi(x) f_n(x)$ , where  $\phi$  is a *fixed* smooth cut-off function such that  $0 \leq \phi \leq 1$  in  $\mathbb{R}^d$ ,  $\phi \equiv 0$  in  $B_{\epsilon_1}$  and  $\phi \equiv 1$  in  $B_{\epsilon_2}^c$ . It is easy to check that one can choose  $\epsilon_2$  (and  $\epsilon_1$  accordingly) so small that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} \tilde{f}_n^{2p} \, dx > 0. \quad (3.3.11)$$

Indeed, if this were not possible, then there would exist a subsequence  $\{f_{n_k}\}$  and a sequence  $\epsilon_k \rightarrow 0$  such that

$$\lim_{k \rightarrow \infty} \int_{B_{\epsilon_k}} f_{n_k}^{2p} \, dx = \lim_{k \rightarrow \infty} \int_{B_{\epsilon_k}} \varrho_{n_k} \, dx = \overline{M};$$

but this means that the sequence  $\{\varrho_{n_k}\}$  concentrates about  $x = 0$ , which is impossible in view of the locally compact embeddings provided by Lemma 3.2.4. Our aim is now to apply [128, Lemma I.1] to the sequence  $\{\tilde{f}_n\}$ . First of all note that, if (3.3.10) holds, then trivially

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{y+B_R} \tilde{f}_n^{2p} \, dx = 0. \quad (3.3.12)$$

Moreover,  $\{\tilde{f}_n\}$  is bounded in  $L^{2p}(\mathbb{R}^d)$  (recall (3.3.7)). In order to use [128, Lemma I.1], we need to show that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla \tilde{f}_n|^2 \, dx < \infty.$$

By differentiating  $\tilde{f}_n$ , we get:

$$\nabla \tilde{f}_n(x) = \frac{w_{\gamma_n}(x)}{|x|^{\frac{\gamma_n}{2p}}} \nabla \phi(x) - \gamma_n \frac{w_{\gamma_n}(x)\phi(x)}{2p|x|^{\frac{\gamma_n}{2p}+2}} x + \frac{\phi(x)}{|x|^{\frac{\gamma_n}{2p}}} \nabla w_{\gamma_n}(x).$$

The sequence

$$\int_{\mathbb{R}^d} |\nabla \phi|^2 w_{\gamma_n}^2 |x|^{-\frac{\gamma_n}{p}} dx$$

is bounded since  $\nabla \phi$  is bounded and supported in  $\overline{B_{\epsilon_2}} \setminus B_{\epsilon_1}$ , where the weight  $|x|^{-\frac{\gamma_n}{p}}$  is regular, and we can bound  $\int_{B_{\epsilon_2} \setminus B_{\epsilon_1}} w_{\gamma_n}^2 dx$  by  $\|w_{\gamma_n}\|_{2^*}$ , which is in turn bounded because so is  $\{\mathcal{G}_{\gamma_n}[w_{\gamma_n}]\}$  (recall (3.2.16) and (3.3.8)). The sequence

$$\int_{\mathbb{R}^d} |\nabla w_{\gamma_n}|^2 \phi^2 |x|^{-\frac{\gamma_n}{p}} dx$$

is also bounded because the same holds true for  $\{\|\nabla w_{\gamma_n}\|_2\}$  and  $\phi$  is bounded and supported in  $B_{\epsilon_1}^c$ . As for the sequence

$$\frac{\gamma_n^2}{4p^2} \int_{\mathbb{R}^d} \frac{\phi^2 w_{\gamma_n}^2}{|x|^{\frac{\gamma_n}{p}+2}} dx,$$

we can proceed in this way:

$$\frac{\gamma_n^2}{4p^2} \int_{\mathbb{R}^d} \frac{\phi^2 w_{\gamma_n}^2}{|x|^{\frac{\gamma_n}{p}+2}} dx \leq \frac{\gamma_n^2}{4p^2 \epsilon_1^{\gamma_n/p}} \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^2}{|x|^2} dx \leq \frac{\gamma_n^2 C_H^2}{4p^2 \epsilon_1^{\gamma_n/p}} \int_{\mathbb{R}^d} |\nabla w_{\gamma_n}|^2 dx,$$

where in the last computation we used Hardy's inequality (3.2.17). Hence,

$$\{\tilde{f}_n\} \text{ is bounded in } L^{2p}(\mathbb{R}^d)$$

and

$$\{\nabla \tilde{f}_n\} \text{ is bounded in } [L^2(\mathbb{R}^d)]^d;$$

this means that if (3.3.12) holds true, then by [128, Lemma I.1]

$$\lim_{n \rightarrow \infty} \|\tilde{f}_n\|_q = 0 \quad \forall q \in (2p, 2^*). \quad (3.3.13)$$

Let us show that (3.3.13) contradicts (3.3.11). By standard interpolation inequalities, we have:

$$\|\tilde{f}_n\|_{2p} \leq \|\tilde{f}_n\|_r^{\frac{r(q-2p)}{2p(q-r)}} \|\tilde{f}_n\|_q^{\frac{q(2p-r)}{2p(q-r)}} \quad (3.3.14)$$

for all  $r \in (p+1, 2p)$  and  $q \in (2p, 2^*)$ . It is possible to prove that, for a suitable choice of  $r \in (p+1, 2p)$ , the sequence  $\{\tilde{f}_n\}$  is bounded in  $L^r$ . In fact, by means of Hölder's inequality, we get:

$$\int_{\mathbb{R}^d} w_{\gamma_n}^r |x|^{-\gamma_n \frac{2^*-r}{2^*-p-1}} dx \leq \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} |x|^{-\gamma_n} dx \right)^{\frac{2^*-r}{2^*-p-1}} \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2^*} dx \right)^{\frac{r-p-1}{2^*-p-1}}. \quad (3.3.15)$$

Moreover,

$$\|\tilde{f}_n\|_r^r = \int_{\mathbb{R}^d} \phi^r w_{\gamma_n}^r |x|^{-\gamma_n \frac{r}{2p}} dx \leq \epsilon_1^{-\gamma_n \left( \frac{r}{2p} - \frac{2^*-r}{2^*-p-1} \right)} \int_{\mathbb{R}^d} w_{\gamma_n}^r |x|^{-\gamma_n \frac{2^*-r}{2^*-p-1}} dx \quad (3.3.16)$$

provided

$$\frac{r}{2p} \geq \frac{2^* - r}{2^* - p - 1}, \quad (3.3.17)$$



which is true for all  $r$  close enough to  $2p$ . Since  $\{\|w_{\gamma_n}\|_{p+1, \gamma_n}\}$  and  $\{\|w_{\gamma_n}\|_{2^*}\}$  are bounded (recall (3.3.8)), from (3.3.15) and (3.3.16) we deduce the boundedness of  $\{\tilde{f}_n\}$  in  $L^r$  for any  $r \in (p+1, 2p)$  complying with (3.3.17). But then (3.3.13) and (3.3.14) yield

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \tilde{f}_n^{2p} dx = 0,$$

which is in contradiction with (3.3.11). Hence, (3.3.10) cannot hold and vanishing is ruled out.

**No dichotomy.** By [127, Lemma I.1], if dichotomy occurs then there exists a number  $\lambda \in (0, \bar{M})$  such that, for every  $\epsilon > 0$ , it is possible to choose  $R_0 > 0$  (arbitrarily large), a monotone sequence  $R_n \uparrow \infty$  and a sequence  $\{y_n\} \subset \mathbb{R}^d$  such that

$$\int_{y_n + B_{R_0}} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \geq \lambda - \epsilon \quad \text{and} \quad \int_{y_n + B_{2R_n}} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \leq \lambda + \epsilon \quad (3.3.18)$$

for all  $n$  large enough. Let  $\xi$  and  $\varphi$  be two regular functions complying with the following conditions:

$$0 \leq \xi, \varphi \leq 1 \quad \text{in } \mathbb{R}^d, \quad \xi \equiv 1 \quad \text{in } B_1, \quad \xi \equiv 0 \quad \text{in } B_2^c, \quad \varphi \equiv 0 \quad \text{in } B_1, \quad \varphi \equiv 1 \quad \text{in } B_2^c.$$

Using the same strategy as in [128, Theorem I.2], we define the cut-off functions  $\xi_n, \varphi_n$  as

$$\xi_n(x) := \xi((x - y_n)/R_0), \quad \varphi_n(x) := \varphi((x - y_n)/R_n)$$

and set

$$w_{1,n} := \xi_n w_{\gamma_n}, \quad w_{2,n} := \varphi_n w_{\gamma_n}.$$

With no loss of generality, we can and shall suppose that  $n$  is large enough (*i.e.* such that  $R_n > 2R_0$ ), so that in particular  $w_{1,n}$  and  $w_{2,n}$  have disjoint supports. From (3.3.18) and the definitions of  $M_{\gamma_n}$ ,  $w_{1,n}$  and  $w_{2,n}$  we deduce that

$$\lambda - \epsilon \leq \int_{\mathbb{R}^d} w_{1,n}^{2p} |x|^{-\gamma_n} dx \leq \lambda + \epsilon \quad (3.3.19)$$

and

$$M_{\gamma_n} - \lambda - \epsilon \leq \int_{\mathbb{R}^d} w_{2,n}^{2p} |x|^{-\gamma_n} dx \leq M_{\gamma_n} - \lambda + \epsilon. \quad (3.3.20)$$

Now we aim at comparing  $\mathcal{G}_{\gamma_n}[w_{\gamma_n}]$  with  $\mathcal{G}_{\gamma_n}[w_{1,n}] + \mathcal{G}_{\gamma_n}[w_{2,n}]$ . By construction and by the fact that  $w_{1,n}$  and  $w_{2,n}$  have disjoint supports, we have:

$$\int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} |x|^{-\gamma_n} dx \geq \int_{\mathbb{R}^d} w_{1,n}^{p+1} |x|^{-\gamma_n} dx + \int_{\mathbb{R}^d} w_{2,n}^{p+1} |x|^{-\gamma_n} dx. \quad (3.3.21)$$

Comparing the  $L^2$  norms of the gradients requires a little more care. First of all, the square of the  $L^2$  norm of  $\nabla w_{1,n}$  reads

$$\int_{\mathbb{R}^d} |\nabla w_{1,n}|^2 dx = \int_{\mathbb{R}^d} \xi_n^2 |\nabla w_{\gamma_n}|^2 dx + \int_{\mathbb{R}^d} w_{\gamma_n}^2 |\nabla \xi_n|^2 dx + 2 \int_{\mathbb{R}^d} w_{\gamma_n} \xi_n \langle \nabla w_{\gamma_n}, \nabla \xi_n \rangle dx. \quad (3.3.22)$$

We shall prove that the last two terms in the r.h.s. of (3.3.22) are smaller than  $\epsilon$  provided one picks  $R_0$  large enough. We have:

$$|\nabla \xi_n(x)| \leq CR_0^{-1} \chi_{\{R_0 \leq |x - y_n| \leq 2R_0\}}(x) \quad \forall x \in \mathbb{R}^d, \quad (3.3.23)$$

### 3.3. Convergence as $\gamma \downarrow 0$ and radial symmetry of optimal functions

where  $C := \|\nabla \xi\|_\infty$ . Let us fix  $\widehat{R} > 0$  so that the left-hand inequality in (3.3.18) holds true, with  $R_0 = \widehat{R}$ , for all  $n$  large enough. It is then possible to prove that

$$|y_n|^{\gamma_n} \leq \widehat{C} \quad \forall n \in \mathbb{N} \quad (3.3.24)$$

for another positive constant  $\widehat{C} = \widehat{C}(\widehat{R}, \lambda, \epsilon)$ . This is actually a direct consequence of the left-hand inequality in (3.3.18). In fact, with no loss of generality, we can suppose that  $|y_n| \geq 2\widehat{R}$ , so that

$$|x + y_n| \geq |y_n|/2 \quad \forall x \in B_{\widehat{R}}.$$

Hence,

$$\lambda - \epsilon \leq \int_{B_{\widehat{R}}} \frac{w_{\gamma_n}^{2p}(x + y_n)}{|x + y_n|^{\gamma_n}} dx \leq \frac{2^{\gamma_n}}{|y_n|^{\gamma_n}} \int_{y_n + B_{\widehat{R}}} w_{\gamma_n}^{2p}(x) dx \leq \frac{2^{\gamma_n}}{|y_n|^{\gamma_n}} \|w_{\gamma_n}\|_{2^*}^{2p} |B_{\widehat{R}}|^{\frac{2^* - 2p}{2^*}},$$

which reads

$$|y_n|^{\gamma_n} \leq \frac{2^{\gamma_n} \|w_{\gamma_n}\|_{2^*}^{2p} |B_{\widehat{R}}|^{\frac{2^* - 2p}{2^*}}}{\lambda - \epsilon},$$

namely (3.3.24) recalling once again the boundedness of  $\{\|w_{\gamma_n}\|_{2^*}\}$ . Collecting (3.3.23) and (3.3.24), we obtain:

$$\begin{aligned} \int_{\mathbb{R}^d} w_{\gamma_n}^2 |\nabla \xi_n|^2 dx &\leq C^2 R_0^{-2} \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} |x|^{-\gamma_n} dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^d} |x|^{\gamma_n \frac{2}{p-1}} \chi_{R_0 \leq |x-y_n| \leq 2R_0} dx \right)^{\frac{p-1}{p+1}} \\ &\leq D R_0^{d \frac{p-1}{p+1} - 2} \left( |y_n|^{\gamma_n \frac{2}{p+1}} + R_0^{\gamma_n \frac{2}{p+1}} \right) \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} |x|^{-\gamma_n} dx \right)^{\frac{2}{p+1}} \\ &\leq D R_0^{d \frac{p-1}{p+1} - 2} \left( \widehat{C}^{\frac{2}{p+1}} + R_0^{\gamma_n \frac{2}{p+1}} \right) \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} |x|^{-\gamma_n} dx \right)^{\frac{2}{p+1}}, \end{aligned} \quad (3.3.25)$$

for another suitable positive constant  $D$  depending on  $C$ ,  $p$  and  $d$ . Since  $\{\|w_{\gamma_n}\|_{p+1, \gamma_n}\}$  is bounded,  $\gamma_n \downarrow 0$  and

$$d \frac{p-1}{p+1} - 2 < 0,$$

in view of (3.3.25) we can make the second term in the r.h.s. of (3.3.22) smaller than  $\epsilon$  (for all  $n$  large enough) upon choosing  $R_0 = R_0(\widehat{R}, \lambda, \epsilon, p, d)$  sufficiently large. Due to the boundedness of  $\{\|\nabla w_{\gamma_n}\|_2\}$  and to Cauchy-Schwarz inequality, the same is true for the last term in the r.h.s. of (3.3.22). Noticing that  $\xi_n$  is supported in  $y_n + B_{2R_0}$  and  $0 \leq \xi_n \leq 1$ , from (3.3.22) we then deduce that for  $R_0$  as above and all  $n$  large enough there holds

$$\int_{\mathbb{R}^d} |\nabla w_{1,n}|^2 dx \leq \int_{y_n + B_{2R_0}} |\nabla w_{\gamma_n}|^2 dx + 2\epsilon. \quad (3.3.26)$$

As concerns  $w_{2,n}$ , one can reason in a similar way: it suffices to replace  $\xi_n$  with  $\varphi_n$  and  $R_0$  with  $R_n$  in (3.3.25) to find that

$$\int_{\mathbb{R}^d} |\nabla w_{2,n}|^2 dx \leq \int_{y_n + B_{R_n}^c} |\nabla w_{\gamma_n}|^2 dx + 2\epsilon \quad (3.3.27)$$

for all  $n$  large enough. By adding up (3.3.21) (multiplied by  $1/(p+1)$ ), (3.3.26) and (3.3.27) (multiplied by  $1/2$ ), we obtain:

$$\mathcal{G}_{\gamma_n}[w_{\gamma_n}] \geq \mathcal{G}_{\gamma_n}[w_{1,n}] + \mathcal{G}_{\gamma_n}[w_{2,n}] - 2\epsilon. \quad (3.3.28)$$

Let us set

$$\alpha_n := \int_{\mathbb{R}^d} w_{1,n}^{2p} |x|^{-\gamma_n} dx, \quad \beta_n := \int_{\mathbb{R}^d} w_{2,n}^{2p} |x|^{-\gamma_n} dx.$$

Recalling the definitions of  $w_{\gamma_n}$  and  $I_{\gamma_n}^M$ , from (3.3.28) we infer the following inequality:

$$I_{\gamma_n}^{M\gamma_n} = \mathcal{G}_{\gamma_n}[w_{\gamma_n}] \geq \mathcal{G}_{\gamma_n}[w_{1,n}] + \mathcal{G}_{\gamma_n}[w_{2,n}] - 2\epsilon \geq I_{\gamma_n}^{\alpha_n} + I_{\gamma_n}^{\beta_n} - 2\epsilon,$$

which, thanks to the scaling identity (3.2.7), becomes

$$M_{\gamma_n}^{\theta_{\gamma_n}} I_{\gamma_n}^1 \geq \alpha_n^{\theta_{\gamma_n}} I_{\gamma_n}^1 + \beta_n^{\theta_{\gamma_n}} I_{\gamma_n}^1 - 2\epsilon.$$

Dividing by  $I_{\gamma_n}^1$ , passing to the limit as  $n \rightarrow \infty$  and using (3.3.6), (3.3.19) and (3.3.20), we deduce that

$$\overline{M}^{\theta_0} \geq \overline{\alpha}^{\theta_0} + \overline{\beta}^{\theta_0} - 2\epsilon/I \tag{3.3.29}$$

for some  $\overline{\alpha} \in [\lambda - \epsilon, \lambda + \epsilon]$  and  $\overline{\beta} \in [\overline{M} - \lambda - \epsilon, \overline{M} - \lambda + \epsilon]$ . Since we can establish (3.3.29) for arbitrarily small  $\epsilon > 0$ , letting  $\epsilon \rightarrow 0$  we get that

$$\overline{M}^{\theta_0} \geq \lambda^{\theta_0} + (\overline{M} - \lambda)^{\theta_0},$$

which is absurd since  $\lambda \in (0, \overline{M})$  and

$$\theta_0 = \frac{d+2-p(d-2)}{d-p(d-4)} < 1.$$

**Compactness.** Vanishing and dichotomy having been ruled out, only compactness can occur. That is, there exists a sequence of translations  $\{y_n\} \subset \mathbb{R}^d$  such that, for any  $\epsilon > 0$ , one can choose  $R_0 > 0$  so that

$$\int_{y_n + B_{R_0}^c} \varrho_n dx = \int_{y_n + B_{R_0}^c} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \leq \epsilon. \tag{3.3.30}$$

There are two possibilities: either the sequence  $\{y_n\}$  is bounded or it is unbounded. In the first case, let  $B_{R_1}$  be a fixed ball in which  $\{y_n\}$  lies. If we set  $R_2 := R_0 + R_1$ , then from (3.3.30) we get that

$$\int_{B_{R_2}^c} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \leq \epsilon. \tag{3.3.31}$$

Since  $\{\mathcal{G}_{\gamma_n}[w_{\gamma_n}]\}$  is bounded, we can assume that  $\{w_{\gamma_n}\}$  converges, up to subsequences, pointwise and weakly in  $\dot{H}^1(\mathbb{R}^d)$ , to some limit  $\overline{w}$ . By weak lower semi-continuity, Fatou's Lemma, (3.2.7), (3.3.5) and (3.3.7), we deduce that

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \overline{w}|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \overline{w}^{p+1} dx \leq \liminf_{n \rightarrow \infty} \mathcal{G}_{\gamma_n}[w_{\gamma_n}] = \liminf_{n \rightarrow \infty} I_{\gamma_n}^{M\gamma_n} \leq I_0^{\overline{M}}. \tag{3.3.32}$$

Recalling that  $\dot{H}^1(\mathbb{R}^d)$  is locally compactly embedded in  $L_{\gamma}^{2p}(\mathbb{R}^d)$  for any  $\gamma \in [0, 2)$  (Lemma 3.2.4), we can pass to the limit in (3.3.31) to obtain

$$\int_{\mathbb{R}^d} \overline{w}^{2p} dx \geq \int_{B_{R_2}} \overline{w}^{2p} dx \geq \overline{M} - \epsilon. \tag{3.3.33}$$

Furthermore, Fatou's Lemma yields  $\int_{\mathbb{R}^d} \overline{w}^{2p} dx \leq \overline{M}$ , whence, letting  $\epsilon \rightarrow 0$  in (3.3.33),

$$\int_{\mathbb{R}^d} \overline{w}^{2p} dx = \overline{M}. \tag{3.3.34}$$

### 3.3. Convergence as $\gamma \downarrow 0$ and radial symmetry of optimal functions

Gathering (3.3.32) and (3.3.34) we deduce that  $\bar{w}$  is a solution to problem (P) for  $\gamma = 0$  and  $M = \bar{M}$ . This means that

$$\mathcal{G}_0[\bar{w}] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \bar{w}|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \bar{w}^{p+1} dx = I_0^{\bar{M}}, \quad (3.3.35)$$

so that in particular

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla w_{\gamma_n}|^2 dx = \int_{\mathbb{R}^d} |\nabla \bar{w}|^2 dx, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^{p+1}}{|x|^{\gamma_n}} dx = \int_{\mathbb{R}^d} \bar{w}^{p+1} dx. \quad (3.3.36)$$

Thanks to (3.3.36), the convergence of  $\{w_{\gamma_n}\}$  to  $\bar{w}$  is actually *strong* in  $\dot{H}(\mathbb{R}^d)$ , so that (3.3.2) holds true with  $\{y_n\} = \{0\}$ . In the end of the proof we shall see that in fact  $\bar{w} = w_b(\cdot - \bar{y})$  for some  $\bar{y} \in \mathbb{R}^d$ . Suppose now that  $\{y_n\}$  is unbounded. Let  $v_n := w_{\gamma_n}(\cdot + y_n)$ , so that (3.3.30) reads

$$\int_{B_{R_0}^c} v_n^{2p} |x + y_n|^{-\gamma_n} dx \leq \epsilon$$

or, equivalently,

$$M_{\gamma_n} - \epsilon \leq \int_{B_{R_0}} v_n^{2p} |x + y_n|^{-\gamma_n} dx \leq M_{\gamma_n}. \quad (3.3.37)$$

Up to a subsequence, we can assume that  $|y_n| \rightarrow \infty$  and that there exists

$$\lim_{n \rightarrow \infty} |y_n|^{\gamma_n} =: L \in [1, \infty]. \quad (3.3.38)$$

Using local compactness as above (note that  $\|\nabla v_n\|_2 = \|\nabla w_{\gamma_n}\|_2$ ), we can pass to the limit in (3.3.37) to get

$$\bar{M} - \epsilon \leq \frac{1}{L} \int_{B_{R_0}} \bar{w}^{2p} dx \leq \bar{M}, \quad (3.3.39)$$

where now  $\bar{w}$  denotes the pointwise and weak limit of  $\{v_n\}$  in  $\dot{H}(\mathbb{R}^d)$ . Since we can assume that  $\epsilon < \bar{M}$  and we know (*e.g.* by local embeddings) that  $\int_{B_{R_0}} \bar{w}^{2p} dx < \infty$ , (3.3.39) implies  $L < \infty$ . Moreover, as  $R_0$  is arbitrarily large and  $\epsilon$  is arbitrarily small, still (3.3.39) yields the identity

$$\frac{1}{L} \int_{\mathbb{R}^d} \bar{w}^{2p} dx = \bar{M}. \quad (3.3.40)$$

Again, by weak lower semi-continuity, Fatou's Lemma, (3.2.7), (3.3.5), (3.3.7) and (3.3.38), we obtain:

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \bar{w}|^2 dx + \frac{1}{(p+1)L} \int_{\mathbb{R}^d} \bar{w}^{p+1} dx \leq \liminf_{n \rightarrow \infty} I_{\gamma_n}^{M_{\gamma_n}} \leq I_0^{\bar{M}}. \quad (3.3.41)$$

We shall prove that actually  $L = 1$ . To this end, let us set  $g := \bar{w}/L^{\frac{1}{2p}}$ . Using (3.3.40) and (3.3.41), it is straightforward to check that  $g$  satisfies  $\int_{\mathbb{R}^d} g^{2p} dx = \bar{M}$  and

$$\frac{L^{\frac{1}{2}}}{2} \int_{\mathbb{R}^d} |\nabla g|^2 dx + \frac{1}{(p+1)L^{\frac{p-1}{2p}}} \int_{\mathbb{R}^d} g^{p+1} dx \leq I_0^{\bar{M}}.$$

Hence, the family of rescaled functions

$$g_\sigma(x) := \sigma^{\frac{d}{2p}} g(\sigma x), \quad \sigma \in \mathbb{R}^+$$

satisfies

$$\int_{\mathbb{R}^d} g_\sigma^{2p} dx = \bar{M} \quad (3.3.42)$$

and

$$\frac{L^{\frac{1}{p}}}{2\sigma^{\frac{d-p(d-2)}{p}}} \int_{\mathbb{R}^d} |\nabla g_\sigma|^2 dx + \frac{\sigma^{\frac{d(p-1)}{2p}}}{(p+1)L^{\frac{p-1}{2p}}} \int_{\mathbb{R}^d} g_\sigma^{p+1} dx \leq I_0^{\overline{M}}. \quad (3.3.43)$$

If we choose  $\sigma = \sigma_L := L^{\frac{1}{d-p(d-2)}}$ , then (3.3.43) becomes

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla g_{\sigma_L}|^2 dx + \frac{L^{\frac{p-1}{2p}(\frac{d}{d-p(d-2)}-1)}}{p+1} \int_{\mathbb{R}^d} g_{\sigma_L}^{p+1} dx \leq I_0^{\overline{M}}. \quad (3.3.44)$$

Since  $p \in (1, d/(d-2))$ , we have that

$$\frac{p-1}{2p} \left( \frac{d}{d-p(d-2)} - 1 \right) > 0.$$

As a consequence, if  $L > 1$  then (3.3.44) would imply that

$$\mathcal{G}_0[g_{\sigma_L}] < I_0^{\overline{M}},$$

which is in contradiction with (3.3.42) and the definition of  $I_0^{\overline{M}}$ . We therefore deduce that  $L = 1$ , so that (3.3.3) holds true and  $\bar{w}$  is a solution to problem (P) for  $\gamma = 0$  and  $M = \overline{M}$ . Because (3.3.41) is in fact an identity, that is

$$\mathcal{G}_0[\bar{w}] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \bar{w}|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^d} \bar{w}^{p+1} dx = I_0^{\overline{M}}, \quad (3.3.45)$$

reasoning exactly as in the case where  $\{y_n\}$  is bounded we infer that  $\{v_n\}$  converges strongly in  $\dot{H}^1(\mathbb{R}^d)$  to  $\bar{w}$ .

Finally, from (3.3.32) and (3.3.35) (or from (3.3.41) and (3.3.45)) we get that

$$\lim_{n \rightarrow \infty} I_{\gamma_n}^{M\gamma_n} = I_0^{\overline{M}} \quad (3.3.46)$$

up to subsequences. Using (3.2.7), (3.3.7), (3.3.46) and the fact that we can repeat the same argument along *any* sequence  $\gamma_n \downarrow 0$ , we end up with (3.3.4).

It remains to prove (3.3.1) and (3.3.2). Due to the characterization of the minimizers of  $\mathcal{G}_0$  provided in [62], the fact that  $\bar{w} = w_b(\cdot - \bar{y})$  for some  $\bar{y} \in \mathbb{R}^d$  is equivalent to

$$2pI_0^1\theta_0\overline{M}^{\theta_0-1} = 1,$$

which follows by (3.3.7) and (3.3.46). □

**Corollary 3.3.2.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Then, up to subsequences:*

$$\lim_{n \rightarrow \infty} \|v_n - w_b(\cdot - \bar{y})\|_q = 0 \quad \forall q \in [2^*, \infty) \quad (3.3.47)$$

and

$$\lim_{n \rightarrow \infty} \|v_n - w_b(\cdot - \bar{y})\|_{C^{1,\alpha}(\mathbb{R}^d)} = 0 \quad \forall \alpha \in (0, 1). \quad (3.3.48)$$

*Proof.* By interpolation, we have

$$\|v_n - w_b(\cdot - \bar{y})\|_q \leq \|v_n - w_b(\cdot - \bar{y})\|_{2^*}^{\frac{2^*}{q}} \|v_n - w_b(\cdot - \bar{y})\|_\infty^{1-\frac{2^*}{q}}. \quad (3.3.49)$$

### 3.3. Convergence as $\gamma \downarrow 0$ and radial symmetry of optimal functions

From Lemma 3.3.1 and the Sobolev inequality (3.2.16) we know that  $\{v_n\}$  converges to  $w_b(\cdot - \bar{y})$  in  $L^{2^*}(\mathbb{R}^d)$ . Moreover, Lemma 3.2.8 and again Lemma 3.3.1 ensure that the sequence  $\{\|v_n\|_\infty\}$  is bounded, while  $w_b \in L^\infty(\mathbb{R}^d)$ . These properties and (3.3.49) yield (3.3.47).

In order to prove (3.3.48), we shall make use of the following elementary inequalities:

$$\|w\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq 4 \sup_{x_0 \in \mathbb{R}^d} \|w\|_{C^{1,\alpha}(\bar{B}_1(x_0))}, \quad (3.3.50)$$

$$\sup_{x_0 \in B_R} \|w\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \leq 2(1 + 2^\alpha) \sup_{x_0 \in B_{R+1}} \|w\|_{C^{1,\alpha}(\bar{B}_{1/2}(x_0))}, \quad (3.3.51)$$

where  $R > 0$  and  $w$  is any function such that the right-hand sides of (3.3.50) and (3.3.51) are finite. Clearly there exists a suitable number  $N_R \in \mathbb{N}$  and a set of points  $\{y_k\}_{k=1 \dots N_R} \subset B_{R+1}$  such that, for every  $x_0 \in B_{R+1}$ ,

$$\bar{B}_{1/2}(x_0) \subset \bar{B}_1(y_k)$$

for some  $k \in \{1, \dots, N_R\}$  depending on  $x_0$ . From (3.3.51) we thus get

$$\sup_{x_0 \in B_R} \|w\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \leq 2(1 + 2^\alpha) \max_{k=1 \dots N_R} \|w\|_{C^{1,\alpha}(\bar{B}_1(y_k))}. \quad (3.3.52)$$

Gathering (3.3.50) and (3.3.52) we deduce the validity of the following estimate:

$$\|w\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq 4 \max \left\{ 2(1 + 2^\alpha) \max_{k=1 \dots N_R} \|w\|_{C^{1,\alpha}(\bar{B}_1(y_k))}, \sup_{x_0 \in \bar{B}_R^c} \|w\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \right\}. \quad (3.3.53)$$

Given  $\alpha \in (0, 1)$ , let  $q_\alpha := d/(1 - \alpha)$ . In view of Lemmas 3.2.9 and 3.2.10 (see in particular (3.2.34) and (3.2.37)), we have that for a suitable  $\bar{\gamma} = \bar{\gamma}(\alpha, p, d) > 0$  and all  $n$  such that  $\gamma_n < \bar{\gamma}$  there holds

$$\begin{aligned} & \|v_n\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \\ & \leq C \left( \|v_n\|_{L^{q_\alpha}(B_2(x_0))} + \|v_n\|_{L^{(2p-1)q_\alpha}(B_2(x_0))}^{2p-1} + \|v_n\|_{L^{pq_\alpha}(B_2(x_0))}^p + \|v_n\|_{L^{2pq_\alpha}(B_2(x_0))}^{2p-1} + \|v_n\|_{L^{2pq_\alpha}(B_2(x_0))}^p \right) \end{aligned} \quad (3.3.54)$$

for all  $x_0 \in \mathbb{R}^d$  and a positive constant  $C = C(\alpha, p, d)$ . Thanks to (3.3.47) and (3.3.54), for every  $\varepsilon > 0$  there exist  $R_0 > 0$  and  $n_0 \in \mathbb{N}$  such that

$$\sup_{x_0 \in \bar{B}_{R_0}^c} \|v_n\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \leq \varepsilon \quad \forall n \geq n_0.$$

This is apparent if  $q_\alpha \in [2^*, \infty) \cap (d, \infty)$ . In case  $q_\alpha \in (d, 2^*)$  (which can occur only if  $d = 3$ ), it is enough to control the norms appearing in the r.h.s. of (3.3.54) with  $\|v_n\|_{L^{\bar{q}}(B_2(x_0))}$  for some  $\bar{q} \in [2^*, \infty)$  large enough, and then use (3.3.47) with  $q = \bar{q}$ . Recalling the explicit profile (3.2.4) we can assume that, still for  $R_0$  large enough,

$$\sup_{x_0 \in \bar{B}_{R_0}^c} \|w_b(\cdot - \bar{y})\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \leq \varepsilon,$$

so that

$$\sup_{x_0 \in \bar{B}_{R_0}^c} \|v_n - w_b(\cdot - \bar{y})\|_{C^{1,\alpha}(\bar{B}_1(x_0))} \leq 2\varepsilon \quad \forall n \geq n_0. \quad (3.3.55)$$

Now let  $N_{R_0} \in \mathbb{N}$  and  $\{y_k\}_{k=1 \dots N_{R_0}} \subset B_{R_0+1}$  as above. Fix  $\alpha_0 \in (\alpha, 1)$ . Exploiting (3.3.54) with  $\alpha = \alpha_0$ , we infer that  $\{v_n\}$  is (eventually) bounded in  $C^{1,\alpha_0}(\bar{B}_1(x_0))$  for all  $x_0 \in \mathbb{R}^d$ . As a consequence,

$$\lim_{n \rightarrow \infty} \|v_n - w_b(\cdot - \bar{y})\|_{C^{1,\alpha}(\bar{B}_1(y_k))} = 0 \quad \forall k = 1 \dots N_{R_0}. \quad (3.3.56)$$

Using (3.3.53) with  $w = v_n - w_{\mathfrak{b}}(\cdot - \bar{y})$  and  $R = R_0$  and recalling (3.3.55), we obtain:

$$\|v_n - w_{\mathfrak{b}}(\cdot - \bar{y})\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq 24 \max \left\{ \max_{k=1 \dots N_{R_0}} \|v_n - w_{\mathfrak{b}}(\cdot - \bar{y})\|_{C^{1,\alpha}(\bar{B}_1(y_k))}, \varepsilon \right\}$$

for all  $n \geq n_0$ . Hence, in view of (3.3.56),

$$\limsup_{n \rightarrow \infty} \|v_n - w_{\mathfrak{b}}(\cdot - \bar{y})\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq 24\varepsilon. \quad (3.3.57)$$

Since  $\varepsilon$  is arbitrarily small, from (3.3.57) we finally deduce (3.3.48).  $\square$

Thanks to Lemma 3.3.1 and Corollary 3.3.2 we can provide an upper bound for  $v_n$ , which we shall often exploit below.

**Lemma 3.3.3.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Then the estimate*

$$v_n(x) \leq C (1 + |x|)^{-\frac{2-\gamma_n}{p-1}} \quad \forall x \in \mathbb{R}^d, \quad \forall n \geq \tilde{n}, \quad (3.3.58)$$

holds true for a suitable  $\tilde{n} \in \mathbb{N}$  and a positive constant  $C$  independent of  $n$ .

*Proof.* First notice that, in view of (3.2.2),  $v_n$  solves the following equation:

$$-\Delta v_n = \frac{-v_n^p + v_n^{2p-1}}{|x + y_n|^{\gamma_n}} \quad \text{in } \mathbb{R}^d.$$

In view of (3.3.48), and recalling the explicit profile (3.2.4) of  $w_{\mathfrak{b}}$ , we infer that there exist  $R_0 > 0$  and  $n_0 \in \mathbb{N}$  (independent of  $n$ ) such that

$$v_n(x) \leq 2^{-\frac{1}{p-1}} \quad \forall x \in B_{R_0}^c, \quad \forall n \geq n_0.$$

In particular,

$$-\Delta v_n \leq -\frac{v_n^p}{2|x + y_n|^{\gamma_n}} \quad \text{in } B_{R_0}^c, \quad \forall n \geq n_0. \quad (3.3.59)$$

One can prove that there exist  $R_1 > 0$  and  $n_1 \in \mathbb{N}$  (independent of  $n$ ) such that

$$|x + y_n|^{\gamma_n} \leq 2|x|^{\gamma_n} \quad \forall x \in B_{R_1}^c, \quad \forall n \geq n_1. \quad (3.3.60)$$

If  $\{y_n\} = \{0\}$  then (3.3.60) is trivial. On the other hand, if  $|y_n| \rightarrow \infty$  then (3.3.60) follows from (3.3.3) and the inequality

$$|x + y_n|^{\gamma_n} \leq |y_n|^{\gamma_n} \left( \frac{|x|}{|y_n|} + 1 \right)^{\gamma_n}.$$

Let  $R_2 := R_0 \vee R_1$  and  $n_2 := n_0 \vee n_1$ . Combining (3.3.59) and (3.3.60) we infer that  $v_n$  satisfies

$$-\Delta v_n \leq -\frac{v_n^p}{4|x|^{\gamma_n}} \quad \text{in } B_{R_2}^c, \quad \forall n \geq n_2. \quad (3.3.61)$$

It is readily seen that the function

$$\hat{v}_n(x) := \hat{C} |x|^{-\frac{2-\gamma_n}{p-1}} \quad (3.3.62)$$

is a supersolution to (3.3.61), namely it satisfies

$$-\Delta \hat{v}_n \geq -\frac{\hat{v}_n^p}{4|x|^{\gamma_n}} \quad \text{in } B_{R_2}^c, \quad \forall n \geq n_2, \quad (3.3.63)$$

provided  $\widehat{C}$  is large enough (independently of  $n$ ), that is

$$4 \frac{2 - \gamma_n}{p - 1} \left( \frac{2 - \gamma_n}{p - 1} + 2 - d \right) \leq \widehat{C}^{p-1}. \quad (3.3.64)$$

Still from (3.3.48) (or directly from (3.2.25)), we know that there exist  $n_3 \in \mathbb{N}$  and a positive constant  $D$  such that

$$\|v_n\|_{C^0(\mathbb{R}^d)} \leq D \quad \forall n \geq n_3. \quad (3.3.65)$$

Due to (3.3.65), if we choose  $\widehat{C}$  so large that, in addition to (3.3.64), it also satisfies

$$\widehat{C} \geq DR_2^{\frac{2-\gamma_n}{p-1}},$$

then

$$\widehat{v}_n(x) \geq v_n(x) \quad \forall x \in \partial B_{R_2}, \quad \forall n \geq n_3. \quad (3.3.66)$$

Let  $\tilde{n} := n_2 \vee n_3$ . Thanks to (3.3.61), (3.3.63) and (3.3.66) we can apply the comparison principle for weak sub- and supersolutions to the equation

$$-\Delta w = -\frac{w^p}{4|x|^{\gamma_n}} \quad \text{in } B_{R_2}^c,$$

which entails

$$v_n(x) \leq \widehat{v}_n(x) \quad \forall x \in B_{R_2}^c, \quad \forall n \geq \tilde{n}. \quad (3.3.67)$$

Estimate (3.3.58) is then a consequence of (3.3.62), (3.3.65) and (3.3.67).  $\square$

### 3.3.2 Removing translations

Lemma 3.3.1 establishes the convergence of  $\{w_{\gamma_n}\}$  to  $w_b(\cdot - \bar{y})$  up to a sequence of translations  $\{y_n\}$ . Proceeding in the spirit of [7], here we show that  $\{y_n\}$  is necessarily bounded, which entails convergence of  $\{w_{\gamma_n}\}$  to  $w_b(\cdot - \bar{y})$ .

In the following,  $\mathcal{E}_\gamma$  is the functional introduced in (3.1.15), whose minimizers, we recall, are all solutions to problem (P) for some  $M > 0$ . By  $w_{b[M]}$  we denote the unique minimizer of  $\mathcal{E}_0$ , with mass  $M$ , which is radial about the origin. In agreement with the discussion in Subsection 3.2.1 (see in particular (3.2.1) for  $\gamma = 0$ ), we know that

$$w_{b[M]}(x) = \beta_M^{2/(p-1)} w_b(\beta_M x), \quad \beta_M := \left( M \|w_b\|_{2p}^{-2p} \right)^{\frac{p-1}{d+p(4-d)}}. \quad (3.3.68)$$

**Lemma 3.3.4.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Then the sequence of translations  $\{y_n\}$  is bounded.*

*Proof.* In view of the upper barrier (3.3.58) provided by Lemma 3.3.3, it is plain that  $w_{\gamma_n} \in \mathcal{H}_0(\mathbb{R}^d)$  for all  $n$  large enough. Furthermore, from Remark 3.2.7 we know that  $\mathcal{H}_0(\mathbb{R}^d) \subset \mathcal{H}_{\gamma_n}(\mathbb{R}^d)$ . Hence, both  $w_{\gamma_n}$  and  $w_{b[M]}$  belong to  $\mathcal{H}_0(\mathbb{R}^d) \cap \mathcal{H}_{\gamma_n}(\mathbb{R}^d)$ . Since  $w_{\gamma_n}$  and  $w_{b[M]}$  minimize  $\mathcal{E}_{\gamma_n}$  over  $\mathcal{H}_{\gamma_n}(\mathbb{R}^d)$  and  $\mathcal{E}_0$  over  $\mathcal{H}_0(\mathbb{R}^d)$ , respectively, we readily deduce the validity of the following inequalities:

$$\mathcal{E}_{\gamma_n}[w_{b[M]}] - \mathcal{E}_0[w_{b[M]}] \geq \mathcal{E}_{\gamma_n}[w_{\gamma_n}] - \mathcal{E}_0[w_{b[M]}] \geq \mathcal{E}_{\gamma_n}[w_{\gamma_n}] - \mathcal{E}_0[w_{\gamma_n}],$$



which can be written as

$$\begin{aligned}
& \frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathfrak{b}[M]}^{p+1} (|x|^{-\gamma_n} - 1) dx + (I_0^1 - I_{\gamma_n}^1) \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[M]}^{2p} dx \right)^{\theta_0} \\
& + I_{\gamma_n}^1 \left[ \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[M]}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[M]}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \right] \\
& \geq \frac{1}{p+1} \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} (|x|^{-\gamma_n} - 1) dx + (I_0^1 - I_{\gamma_n}^1) \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} dx \right)^{\theta_0} \\
& + I_{\gamma_n}^1 \left[ \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \right].
\end{aligned} \tag{3.3.69}$$

Since the rate at which  $\{I_{\gamma_n}^1\}$  converges to  $I_0^1$  is *a priori* unknown, it is convenient to set

$$\int_{\mathbb{R}^d} w_{\mathfrak{b}[M]}^{2p} dx = M = \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} dx =: \mathcal{M}_n,$$

so that the terms multiplied by  $I_0^1 - I_{\gamma_n}^1$  cancel out in (3.3.69) and the latter reads

$$\begin{aligned}
& \frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{p+1} (|x|^{-\gamma_n} - 1) dx + I_{\gamma_n}^1 \left[ \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \right] \\
& \geq \frac{1}{p+1} \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} (|x|^{-\gamma_n} - 1) dx + I_{\gamma_n}^1 \left[ \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \right].
\end{aligned} \tag{3.3.70}$$

Performing some first-order developments in (3.3.70), we get:

$$\begin{aligned}
& \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \\
& = \theta_0 \widetilde{\mathcal{M}}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} (1 - |x|^{-\gamma_n}) dx \\
& + \log \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} dx \right) \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} dx \right)^{\widetilde{\theta}_{\mathfrak{b},n}} (\theta_0 - \theta_{\gamma_n})
\end{aligned} \tag{3.3.71}$$

and

$$\begin{aligned}
& \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} dx \right)^{\theta_0} - \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \right)^{\theta_{\gamma_n}} \\
& = \theta_0 \widetilde{M}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} (1 - |x|^{-\gamma_n}) dx + \log \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \right) \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} dx \right)^{\widetilde{\theta}_n} (\theta_0 - \theta_{\gamma_n}),
\end{aligned} \tag{3.3.72}$$

where  $\widetilde{M}_n$  lies between  $\int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} dx$  and  $\int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} dx = \mathcal{M}_n$ , while  $\widetilde{\theta}_{\mathfrak{b},n}$  lies between  $\theta_0$  and  $\theta_{\gamma_n}$ . The same holds true for the corresponding quantities appearing in (3.3.72) (namely  $\widetilde{M}_n$  and  $\widetilde{\theta}_n$ ) upon replacing  $w_{\mathfrak{b}[\mathcal{M}_n]}$  with  $w_{\gamma_n}$ .

It is now crucial to provide bounds, with respect to  $\gamma$ , over the difference  $1 - |x|^{-\gamma}$  appearing in the above integrals. Because for any fixed  $|x| \neq 0$  the function  $\lambda \mapsto |x|^\lambda$  is convex on  $\mathbb{R}$ , its graph lies above the tangent line at  $\lambda = 0$ . Using this property at  $\lambda = -\gamma$  and  $\lambda = \gamma$ , we deduce the validity of the following numerical inequalities:

$$1 - |x|^{-\gamma} \leq \gamma \log |x| \quad \text{and} \quad 1 - |x|^{-\gamma} \geq \gamma |x|^{-\gamma} \log |x|. \tag{3.3.73}$$

Gathering (3.3.70)–(3.3.73) (with  $\gamma = \gamma_n$ ) and dividing by  $\gamma_n$ , we obtain:

$$\begin{aligned}
 & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{p+1} |x|^{-\gamma_n} \log |x| \, dx + I_{\gamma_n}^1 \theta_0 \widetilde{\mathcal{M}}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} \log |x| \, dx \\
 & + I_{\gamma_n}^1 \log \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} \, dx \right) \left( \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} \, dx \right)^{\widetilde{\theta}_{\mathfrak{b},n}} \frac{\theta_0 - \theta_{\gamma_n}}{\gamma_n} \\
 \geq & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} \log |x| \, dx + I_{\gamma_n}^1 \theta_0 \widetilde{M}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \log |x| \, dx \\
 & + I_{\gamma_n}^1 \log \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \, dx \right) \left( \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \, dx \right)^{\widetilde{\theta}_n} \frac{\theta_0 - \theta_{\gamma_n}}{\gamma_n}.
 \end{aligned} \tag{3.3.74}$$

Let  $\{y_n\}$  be the sequence of translations given by Lemma 3.3.1, so that  $\{v_n\} = \{w_{\gamma_n}(\cdot + y_n)\}$  converges in  $\dot{H}^1(\mathbb{R}^d)$  to  $w_{\mathfrak{b}}(\cdot - \bar{y})$  for some  $\bar{y} \in \mathbb{R}^d$ . The idea is that, if  $|y_n| \rightarrow \infty$ , then we get a contradiction with (3.3.74). To this end, a key property to be exploited is the following:

$$\lim_{n \rightarrow \infty} \mathcal{M}_n = \lim_{n \rightarrow \infty} M_{\gamma_n} = \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \, dx = \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{2p} \, dx =: \overline{M}. \tag{3.3.75}$$

This is a direct consequence of the definition of  $\mathcal{M}_n$  and of fact that, thanks to Lemmas 3.3.1 and 3.3.3,  $\{v_n\}$  converges to  $w_{\mathfrak{b}}(\cdot - \bar{y})$  also in  $L^{2p}(\mathbb{R}^d)$ . In order to get the claimed contradiction, we first prove that all the terms in the l.h.s. of (3.3.74) stay bounded as  $n \rightarrow \infty$ . Taking advantage of the explicit formula (3.2.4) for  $w_{\mathfrak{b}}$ , (3.3.68) and (3.3.75), it is straightforward to check that also

$$\left\{ \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} |x|^{-\gamma_n} \, dx \right\}$$

converges to  $\overline{M}$ . In view of (3.2.6), it is apparent that  $\theta_{\gamma_n} \rightarrow \theta_0$  and that  $(\theta_0 - \theta_{\gamma_n})/\gamma_n \rightarrow -\theta'_0$ . In particular,  $\widetilde{\theta}_{\mathfrak{b},n} \rightarrow \theta_0$ . As a consequence, recalling that  $I_{\gamma_n}^1 \rightarrow I_0^1$ , the last term in the l.h.s. of (3.3.74) stays bounded as  $n \rightarrow \infty$ . As for the first two terms, upon noticing that  $\widetilde{\mathcal{M}}_n \rightarrow \overline{M}$ , we only need to show that

$$\int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{p+1} |x|^{-\gamma_n} \log |x| \, dx \quad \text{and} \quad \int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} \log |x| \, dx$$

stay bounded. Recalling (3.3.68) and (3.3.75), we have:

$$\int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{p+1} |x|^{-\gamma_n} \log |x| \, dx = \beta_{\mathcal{M}_n}^{\frac{2(p+1)}{p-1}} \beta_{\mathcal{M}_n}^{\gamma_n-d} \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{p+1} |y|^{-\gamma_n} \log \left( \beta_{\mathcal{M}_n}^{-1} |y| \right) \, dy \tag{3.3.76}$$

and

$$\int_{\mathbb{R}^d} w_{\mathfrak{b}[\mathcal{M}_n]}^{2p} \log |x| \, dx = \beta_{\mathcal{M}_n}^{\frac{4p}{p-1}} \beta_{\mathcal{M}_n}^{-d} \int_{\mathbb{R}^d} w_{\mathfrak{b}}^{2p} \log \left( \beta_{\mathcal{M}_n}^{-1} |y| \right) \, dy. \tag{3.3.77}$$

Since  $\gamma_n \rightarrow 0$  and  $w_{\mathfrak{b}}$  is a regular function behaving like  $|x|^{-2/(p-1)}$  as  $|x| \rightarrow \infty$ , it is plain that the integrals (3.3.76) and (3.3.77) stay bounded as  $n \rightarrow \infty$ , and so the l.h.s. of (3.3.74). Let us then consider the r.h.s. of (3.3.74). To prove boundedness of the last term, one can proceed exactly as above. Hence, we get a contradiction as soon as we prove that  $|y_n| \rightarrow \infty$  implies

$$-\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} \log |x| \, dx + I_{\gamma_n}^1 \theta_0 \widetilde{M}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \log |x| \, dx \rightarrow +\infty. \tag{3.3.78}$$

Recalling the definition of  $v_n$ , we can rewrite the l.h.s. of (3.3.78) as follows:

$$\begin{aligned} & \log |y_n| \left( -\frac{1}{p+1} \int_{\mathbb{R}^d} v_n^{p+1} dx + I_{\gamma_n}^1 \theta_0 \widetilde{M}_n^{\theta_0-1} \int_{\mathbb{R}^d} v_n^{2p} dx \right) \\ & - \frac{1}{p+1} \int_{\mathbb{R}^d} v_n^{p+1} (\log |x+y_n| - \log |y_n|) dx \\ & + I_{\gamma_n}^1 \theta_0 \widetilde{M}_n^{\theta_0-1} \int_{\mathbb{R}^d} v_n^{2p} (|x+y_n|^{-\gamma_n} \log |x+y_n| - \log |y_n|) dx. \end{aligned} \quad (3.3.79)$$

Lemmas 3.3.1 and 3.3.3 ensure that, in particular,  $\{v_n\}$  converges to  $w_{\mathbf{b}}(\cdot - \bar{y})$  in  $L^{p+1}(\mathbb{R}^d) \cap L^{2p}(\mathbb{R}^d)$ . In particular, the factor between brackets in the first line of (3.3.79) converges to

$$-\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathbf{b}}^{p+1} dx + I_0^1 \theta_0 \overline{M}^{\theta_0-1} \int_{\mathbb{R}^d} w_{\mathbf{b}}^{2p} dx, \quad (3.3.80)$$

which, thanks to (3.2.8) at  $\gamma = 0$ , is equal to

$$-\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\mathbf{b}}^{p+1} dx + \frac{1}{2p} \int_{\mathbb{R}^d} w_{\mathbf{b}}^{2p} dx. \quad (3.3.81)$$

Exploiting (3.2.9) and the identity

$$\int_{\mathbb{R}^d} |\nabla w_\gamma|^2 dx + \int_{\mathbb{R}^d} w_\gamma^{p+1} |x|^{-\gamma} dx = \int_{\mathbb{R}^d} w_\gamma^{2p} |x|^{-\gamma} dx,$$

which can be obtained by multiplying the Euler-Lagrange equation (3.2.2) by  $w_\gamma$  and integrating in  $\mathbb{R}^d$ , we get:

$$\frac{1}{p+1} \int_{\mathbb{R}^d} w_\gamma^{p+1} |x|^{-\gamma} dx = \frac{2(p\theta_\gamma - 1)}{\theta_\gamma(p-1)} \frac{1}{2p} \int_{\mathbb{R}^d} w_\gamma^{2p} |x|^{-\gamma} dx. \quad (3.3.82)$$

Due to (3.3.82) evaluated at  $\gamma = 0$ , we have that (3.3.80) and (3.3.81) coincide with

$$\left[ 1 - \frac{2(p\theta_0 - 1)}{\theta_0(p-1)} \right] \frac{1}{2p} \int_{\mathbb{R}^d} w_{\mathbf{b}}^{2p} dx,$$

a strictly positive term. Recalling (3.3.79), in order to come to a contradiction it is therefore enough to prove that

$$\left| \int_{\mathbb{R}^d} v_n^{p+1} \left( \frac{\log |x+y_n|}{\log |y_n|} - 1 \right) dx \right| + \left| \int_{\mathbb{R}^d} v_n^{2p} \left( |x+y_n|^{-\gamma_n} \frac{\log |x+y_n|}{\log |y_n|} - 1 \right) dx \right|$$

converges to zero as  $n \rightarrow \infty$ . Since  $\{v_n\}$  converges in  $L^{p+1}(\mathbb{R}^d) \cap L^{2p}(\mathbb{R}^d)$  and by Lemma 3.3.1 we know that in case  $|y_n| \rightarrow \infty$  then  $|y_n|^{\gamma_n} \rightarrow 1$ , it is clear that *locally* in  $\mathbb{R}^d$  such integrals vanish. Hence, we are left with proving that for any  $\varepsilon > 0$  there exist  $R_\varepsilon > 0$  and  $n_\varepsilon \in \mathbb{N}$  such that

$$\left| \int_{B_{R_\varepsilon}^c} v_n^{p+1} \frac{\log |x+y_n|}{\log |y_n|} dx \right| + \left| \int_{B_{R_\varepsilon}^c} v_n^{2p} |x+y_n|^{-\gamma_n} \frac{\log |x+y_n|}{\log |y_n|} dx \right| \leq 2\varepsilon \quad (3.3.83)$$

for all  $n \geq n_\varepsilon$ . For simplicity we consider only the first integral in the l.h.s. of (3.3.83), but note that the second one can be dealt with similarly. We have:

$$\begin{aligned} & \left| \int_{B_{R_\varepsilon}^c} v_n^{p+1} \frac{\log |x+y_n|}{\log |y_n|} dx \right| \\ & \leq \left| \int_{B_1(-y_n)} v_n^{p+1} \frac{\log |x+y_n|}{\log |y_n|} dx \right| + \left| \int_{B_{R_\varepsilon}^c \cap B_1^c(-y_n)} v_n^{p+1} \frac{\log |x+y_n|}{\log |y_n|} dx \right|. \end{aligned} \quad (3.3.84)$$

As for the first integral in the r.h.s. of (3.3.84), there holds

$$\left| \int_{B_1(-y_n)} v_n^{p+1} \frac{\log|x+y_n|}{\log|y_n|} dx \right| \leq C^{p+1} \left| \int_{B_1(-y_n)} \frac{\log|x+y_n|}{\log|y_n|} dx \right| = \frac{C^{p+1}}{\log|y_n|} \left| \int_{B_1} \log|y| dy \right| \quad (3.3.85)$$

for all  $n \geq \tilde{n}$ , where  $C$  and  $\tilde{n}$  are as in the statement of Lemma 3.3.3 (assuming in addition that  $|y_n| > 1$  for all  $n \geq \tilde{n}$ ). If  $|y_n| \rightarrow \infty$ , then clearly (3.3.85) ensures the existence of  $n_1 \geq \tilde{n}$  such that

$$\left| \int_{B_1(-y_n)} v_n^{p+1} \frac{\log|x+y_n|}{\log|y_n|} dx \right| \leq \frac{\varepsilon}{2} \quad (3.3.86)$$

for all  $n \geq n_1$ . In order to handle the second integral in the r.h.s. of (3.3.84), first observe that

$$|\log|x+y_n|| = \log|y_n| + \log|x/y_n + y_n/y_n| \leq \log|y_n| + \log(1+|x|) \quad (3.3.87)$$

for all  $x \in B_1^c(-y_n)$  and  $n \geq n_1$ . Using (3.3.87) and again estimate (3.3.58) we end up with

$$\begin{aligned} & \left| \int_{B_{R_\varepsilon}^c \cap B_1^c(-y_n)} v_n^{p+1} \frac{\log|x+y_n|}{\log|y_n|} dx \right| \\ & \leq C^{p+1} \int_{B_{R_\varepsilon}^c \cap B_1^c(-y_n)} [1 + \log(1+|x|)] (1+|x|)^{-\frac{(2-\gamma_n)(p+1)}{p-1}} dx \end{aligned} \quad (3.3.88)$$

for all  $n \geq \tilde{n} \vee n_2$ , where  $n_2$  is such that  $\log|y_n| \geq 1$  for all  $n \geq n_2$ . Since  $\gamma_n \rightarrow 0$  and  $2(p+1)/(p-1) > d$ , it is plain that the r.h.s. of (3.3.88) can be made smaller than  $\varepsilon/2$ , independently of  $n \geq n_3$ , provided  $R_\varepsilon$  is taken large enough. Hence,

$$\left| \int_{B_{R_\varepsilon}^c \cap B_1^c(-y_n)} v_n^{p+1} \frac{\log|x+y_n|}{\log|y_n|} dx \right| \leq \frac{\varepsilon}{2} \quad (3.3.89)$$

for all  $n \geq \tilde{n} \vee n_2 \vee n_3$ . In view of (3.3.86) and (3.3.89), we can then assert that the first integral in the l.h.s. of (3.3.83) is smaller than  $\varepsilon$  for all  $n \geq n_\varepsilon := n_1 \vee n_2 \vee n_3$ .  $\square$

**Corollary 3.3.5.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Then,*

$$\lim_{n \rightarrow \infty} \|w_{\gamma_n} - w_{\mathfrak{b}}(\cdot - \bar{y})\|_q = 0 \quad \forall q \in \left(d \frac{p-1}{2}, \infty\right), \quad (3.3.90)$$

$$\lim_{n \rightarrow \infty} \|w_{\gamma_n} - w_{\mathfrak{b}}(\cdot - \bar{y})\|_{C^{1,\alpha}(\mathbb{R}^d)} = 0 \quad \forall \alpha \in (0, 1) \quad (3.3.91)$$

and the estimate

$$w_{\gamma_n}(x) \leq C (1+|x|)^{-\frac{2-\gamma_n}{p-1}} \quad \forall x \in \mathbb{R}^d, \quad \forall n \geq \tilde{n}, \quad (3.3.92)$$

holds true for a suitable  $\tilde{n} \in \mathbb{N}$  and a positive constant  $C$  independent of  $n$ .

*Proof.* By Lemmas 3.3.1 and 3.3.4 we deduce that  $\{y_n\} = \{0\}$ . As a consequence, (3.3.91) and estimate (3.3.92) follow directly from (3.3.48) and (3.3.58), respectively. The validity of (3.3.90) is then inherited from (3.3.47): in order to prove it for  $q \in (d(p-1)/2, 2^*)$  as well, one just exploits the fact that the r.h.s. of (3.3.92) eventually belongs to  $L^q(\mathbb{R}^d)$  for all such  $q$ .  $\square$

### 3.3.3 Identification of the limit minimizer

In this subsection we shall prove that in fact  $\{w_{\gamma_n}\}$  converges to  $w_b$ , or equivalently that  $\bar{y} = 0$ , according to the notations of the previous Subsections 3.3.1 and 3.3.2. This means that, among all the solutions of problem (P) for  $\gamma = 0$ , the sequence  $\{w_{\gamma_n}\}$  *selects the one centred at zero*. We shall proceed by means of a selection principle argument, inspired again by some original ideas introduced in the paper [7].

**Lemma 3.3.6.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Then,*

$$\bar{y} \in \operatorname{argmin}_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left[ -\frac{1}{p+1} w_b^{p+1}(x-y) + \frac{1}{2p} w_b^{2p}(x-y) \right] \log |x| \, dx. \quad (3.3.93)$$

*Proof.* In order to prove the assertion we can proceed along the proof of Theorem 3.3.4, upon noticing that in place of  $w_{b[\mathcal{M}_n]}$  we can pick  $w_{b[\mathcal{M}_n]}(\cdot - y)$  for *any* fixed  $y \in \mathbb{R}^d$  (recall that  $\mathcal{E}_0$  is translation invariant). Hence, for possibly different  $\tilde{\mathcal{M}}_n$  and  $\tilde{\theta}_{b,n}$  having the same properties as in the mentioned proof, we obtain the analogue of (3.3.74):

$$\begin{aligned} & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_{b[\mathcal{M}_n]}^{p+1}(x-y) |x|^{-\gamma_n} \log |x| \, dx + I_{\gamma_n}^1 \theta_0 \tilde{\mathcal{M}}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{b[\mathcal{M}_n]}^{2p}(x-y) \log |x| \, dx \\ & + I_{\gamma_n}^1 \log \left( \int_{\mathbb{R}^d} \frac{w_{b[\mathcal{M}_n]}^{2p}(x-y)}{|x|^{\gamma_n}} \, dx \right) \left( \int_{\mathbb{R}^d} \frac{w_{b[\mathcal{M}_n]}^{2p}(x-y)}{|x|^{\gamma_n}} \, dx \right)^{\tilde{\theta}_{b,n}} \frac{\theta_0 - \theta_{\gamma_n}}{\gamma_n} \\ \geq & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_{\gamma_n}^{p+1} \log |x| \, dx + I_{\gamma_n}^1 \theta_0 \tilde{\mathcal{M}}_n^{\theta_0-1} \int_{\mathbb{R}^d} w_{\gamma_n}^{2p} |x|^{-\gamma_n} \log |x| \, dx \\ & + I_{\gamma_n}^1 \log \left( \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^{2p}}{|x|^{\gamma_n}} \, dx \right) \left( \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^{2p}}{|x|^{\gamma_n}} \, dx \right)^{\tilde{\theta}_n} \frac{\theta_0 - \theta_{\gamma_n}}{\gamma_n}. \end{aligned} \quad (3.3.94)$$

Passing to the limit as  $n \rightarrow \infty$  in (3.3.94), which is feasible *e.g.* thanks to Lemma 3.3.1 and Corollary 3.3.5, we deduce:

$$\begin{aligned} & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_b^{p+1}(x-y) \log |x| \, dx + I_0^1 \theta_0 \bar{M}^{\theta_0-1} \int_{\mathbb{R}^d} w_b^{2p}(x-y) \log |x| \, dx \\ & - I_0^1 \log \left( \int_{\mathbb{R}^d} w_b^{2p}(x-y) \, dx \right) \left( \int_{\mathbb{R}^d} w_b^{2p}(x-y) \, dx \right)^{\theta_0} \theta_0' \\ \geq & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_b^{p+1}(x-\bar{y}) \log |x| \, dx + I_0^1 \theta_0 \bar{M}^{\theta_0-1} \int_{\mathbb{R}^d} w_b^{2p}(x-\bar{y}) \log |x| \, dx \\ & - I_0^1 \log \left( \int_{\mathbb{R}^d} w_b^{2p}(x-\bar{y}) \, dx \right) \left( \int_{\mathbb{R}^d} w_b^{2p}(x-\bar{y}) \, dx \right)^{\theta_0} \theta_0'. \end{aligned}$$

Recalling the identities  $\int_{\mathbb{R}^d} w_b^{2p} \, dx =: \bar{M}$  and  $I_0^1 \theta_0 \bar{M}^{\theta_0-1} = 1/2p$ , the above inequality reads

$$\begin{aligned} & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_b^{p+1}(x-y) \log |x| \, dx + \frac{1}{2p} \int_{\mathbb{R}^d} w_b^{2p}(x-y) \log |x| \, dx \\ \geq & -\frac{1}{p+1} \int_{\mathbb{R}^d} w_b^{p+1}(x-\bar{y}) \log |x| \, dx + \frac{1}{2p} \int_{\mathbb{R}^d} w_b^{2p}(x-\bar{y}) \log |x| \, dx. \end{aligned} \quad (3.3.95)$$

Since  $y$  is arbitrary, (3.3.95) is equivalent to (3.3.93).  $\square$

By a change of variables, we have that the function to be minimized in (3.3.93) coincides with

$$F(y) := \int_{\mathbb{R}^d} \left[ -\frac{1}{p+1} w_b^{p+1}(x) + \frac{1}{2p} w_b^{2p}(x) \right] \log|x+y| dx \quad \forall y \in \mathbb{R}^d.$$

Our aim is therefore to prove that  $F$  has a unique minimum at  $y = 0$ . Since both  $w_b$  and  $\log|x|$  are *radial* about the origin, it is straightforward to check that also  $F$  enjoys such property. This is a general fact: the convolution between two radial functions is still radial. Hence, letting

$$\mathcal{K} := -\frac{1}{p+1} w_b^{p+1} + \frac{1}{2p} w_b^{2p}, \quad (3.3.96)$$

we have that minimizing  $F$  w.r.t.  $y \in \mathbb{R}^d$  is equivalent to minimizing

$$G(R) := \int_{\mathbb{R}^d} \mathcal{K}(x) \log|x + \sqrt{R} e_1| dx \quad (3.3.97)$$

w.r.t.  $R \in \mathbb{R}^+$ , where  $e_1$  is *e.g.* the first versor of the canonical basis of  $\mathbb{R}^d$ .

**Lemma 3.3.7.** *The function  $G$  defined in (3.3.97) can be rewritten as*

$$G(R) = \frac{1}{4} \int_{\mathbb{R}^d} \mathcal{K}(x) \log \left[ (R + |x|^2)^2 - 4R x_1^2 \right] dx \quad \forall R \in \mathbb{R}^+. \quad (3.3.98)$$

*Proof.* By splitting the integral in the regions  $\{x_1 \geq 0\}$  and  $\{x_1 < 0\}$ , we obtain:

$$\begin{aligned} G(R) &= \frac{1}{2} \int_{\mathbb{R}^d \cap \{x_1 \geq 0\}} \mathcal{K}(x) \log \left[ (x_1 + \sqrt{R})^2 + x_2^2 + \dots + x_d^2 \right] dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \cap \{x_1 < 0\}} \mathcal{K}(x) \log \left[ (-x_1 + \sqrt{R})^2 + x_2^2 + \dots + x_d^2 \right] dx, \end{aligned} \quad (3.3.99)$$

where in the region  $\{x_1 < 0\}$  we used the change of variables

$$(x_1, x_2, \dots, x_d) \mapsto (-x_1, x_2, \dots, x_d)$$

and exploited the radially of  $\mathcal{K}$ . Since

$$(\pm x_1 + \sqrt{R})^2 + x_2^2 + \dots + x_d^2 = R + |x|^2 \pm 2\sqrt{R} x_1,$$

we deduce that (3.3.99) coincides with

$$\frac{1}{2} \int_{\mathbb{R}^d \cap \{x_1 \geq 0\}} \mathcal{K}(x) \log \left[ (R + |x|^2 + 2\sqrt{R} x_1) (R + |x|^2 - 2\sqrt{R} x_1) \right] dx,$$

namely (3.3.98) because the integrand is symmetric w.r.t.  $\{x_1 = 0\}$ . □

In the sequel we shall prove that  $\frac{dG(R)}{dR}$  is actually *strictly positive*, whence the fact that  $G(R)$  has a unique minimum at  $R = 0$ . It is readily seen that  $G(R)$  is continuous. Moreover, taking the derivative w.r.t.  $R$  in (3.3.98), we obtain:

$$\frac{dG(R)}{dR} = \frac{1}{4} \int_{\mathbb{R}^d} \mathcal{K}(x) \frac{2(R + |x|^2) - 4x_1^2}{(R + |x|^2)^2 - 4R x_1^2} dx \quad \forall R \in \mathbb{R}^+. \quad (3.3.100)$$

Note that, since  $d \geq 3$  and  $\mathcal{K}(x)$  is a regular function decaying fast enough as  $|x| \rightarrow \infty$ , we are allowed to take derivatives under the integral sign in (3.3.98) and therefore get (3.3.100).

Moreover, we have that also  $\frac{dG(R)}{dR}$  is continuous, so that  $G \in C^1(\mathbb{R}^+)$ . To our purposes, it is convenient to write the integral in (3.3.100) in spherical coordinates:

$$\frac{dG(R)}{dR} = C \int_0^{+\infty} \mathcal{K}(r) r^{d-1} \int_0^\pi \frac{2(R+r^2) - 4r^2 \cos^2(\theta)}{(R+r^2)^2 - 4Rr^2 \cos^2(\theta)} \sin^{d-2}(\theta) d\theta dr \quad (3.3.101)$$

for all  $R \in \mathbb{R}^+$ , where  $C$  is a positive constant depending only on  $d$  and, with a slight abuse of notation, we denote by  $\mathcal{K}(r)$  the function  $\mathcal{K}(x)$  evaluated at any point  $x$  such that  $|x| = r$ .

**Lemma 3.3.8.** *Let  $R > 0$ . Then the function*

$$\ell_R(r) := \int_0^\pi \frac{2(R+r^2) - 4r^2 \cos^2(\theta)}{(R+r^2)^2 - 4Rr^2 \cos^2(\theta)} \sin^{d-2}(\theta) d\theta \quad \forall r \in \mathbb{R}^+ \quad (3.3.102)$$

*is continuous, positive and decreasing.*

*Proof.* It is straightforward to check that  $\ell_R(r) = (2/R) \hat{\ell}(r^2/R)$  for all  $r \in \mathbb{R}^+$ , where

$$\hat{\ell}(s) := \int_0^{\frac{\pi}{2}} \left[ 1 + \frac{2(1+s) - (1+s)^2}{(1+s)^2 - 4s \cos^2(\theta)} \right] \sin^{d-2}(\theta) d\theta \quad \forall s \in \mathbb{R}^+. \quad (3.3.103)$$

Proving our statement is therefore equivalent to proving that  $\hat{\ell}(s)$  is continuous, positive and decreasing. It is apparent that  $\lim_{s \rightarrow +\infty} \hat{\ell}(s) = 0$ , from which, if we establish that  $\hat{\ell}(s)$  is *strictly decreasing*, then it is necessarily positive. Since the denominator of the integrand in (3.3.103) vanishes if and only if  $s = 1$  and  $\theta = 0$ , it is immediate to see that  $\hat{\ell}(s)$  is continuous in  $[0, 1) \cup (1, +\infty)$ . In order to prove continuity at  $s = 1$  as well, we must show that

$$\lim_{s \rightarrow 1} \int_0^{\frac{\pi}{2}} \frac{2(1+s) - (1+s)^2}{(1+s)^2 - 4s \cos^2(\theta)} \sin^{d-2}(\theta) d\theta = 0.$$

Recalling that  $d \geq 3$  and  $\theta \in [0, \pi/2]$ , we have:

$$\left| \frac{2(1+s) - (1+s)^2}{(1+s)^2 - 4s \cos^2(\theta)} \sin^{d-2}(\theta) \right| \leq \frac{|1-s|}{1+s - 2\sqrt{s} \cos(\theta)} \sin(\theta), \quad (3.3.104)$$

so that by calculating the exact integral of the r.h.s. of (3.3.104), we get:

$$\left| \int_0^{\frac{\pi}{2}} \frac{2(1+s) - (1+s)^2}{(1+s)^2 - 4s \cos^2(\theta)} \sin^{d-2}(\theta) d\theta \right| \leq \frac{|1-s|}{2\sqrt{s}} \log \left[ \frac{1+s}{(1-\sqrt{s})^2} \right],$$

which actually vanishes as  $s \rightarrow 1$ . We are left with proving that  $\hat{\ell}(s)$  is strictly decreasing. In order to do so, it is enough to show that

$$\frac{d\hat{\ell}(s)}{ds} < 0 \quad \forall s \in (0, 1) \cup (1, +\infty). \quad (3.3.105)$$

Let

$$m(s) := \int_0^{\frac{\pi}{2}} \frac{2(1+s) - (1+s)^2}{(1+s)^2 - 4s \cos^2(\theta)} \sin^{d-2}(\theta) d\theta \quad \forall s \in \mathbb{R}^+$$

and  $C := \int_0^{\frac{\pi}{2}} \sin^{d-2}(\theta) d\theta$ . It is readily seen that

$$\hat{\ell}(s) = C + m(s) = C - m(s^{-1}) \quad \forall s > 0. \quad (3.3.106)$$

Thanks to (3.3.106), it is enough to establish (3.3.105) *e.g.* only for  $s > 1$ . To this end, we shall first consider the case  $d = 3$  and then infer from it the result for all  $d > 3$ . Let then  $d = 3$ , so that we can compute  $\hat{\ell}(s)$  explicitly:

$$\hat{\ell}(s) = 1 + (1+s)(1-s) \int_0^1 \frac{1}{(1+s)^2 - 4sy^2} dy = 1 + \frac{1-s}{2\sqrt{s}} \log \left( \frac{1+\sqrt{s}}{|1-\sqrt{s}|} \right).$$

In order to prove (3.3.105) for all  $s > 1$ , we can set  $s = t^2$ ,

$$\hat{\ell}(t) := \frac{t^2 - 1}{2t} \log \left( 1 - \frac{2}{t+1} \right) \quad \forall t > 1$$

and show that  $\frac{d\hat{\ell}(t)}{dt} < 0$  for all  $t > 1$ . Since

$$\frac{d\hat{\ell}(t)}{dt} = \frac{(t^2 + 1) \log [1 - 2/(t+1)] + 2t}{2t^2} \quad \forall t > 1,$$

we need to prove that

$$(t^2 + 1) \log [1 - 2/(t+1)] + 2t < 0 \quad \forall t > 1. \quad (3.3.107)$$

To this end we can exploit the fact that  $\log(1-x)$ , for every  $x \in (0, 1)$ , is smaller than its Taylor expansion (at any order) centred at  $x = 0$ . In particular,

$$\log [1 - 2/(t+1)] < -2(t+1)^{-1} - 2(t+1)^{-2} - (8/3)(t+1)^{-3} \quad \forall t > 1.$$

Therefore, (3.3.107) is implied by

$$(t^2 + 1) \left[ -2(t+1)^{-1} - 2(t+1)^{-2} - (8/3)(t+1)^{-3} \right] + 2t < 0 \quad \forall t > 1. \quad (3.3.108)$$

After straightforward computations, one finds that (3.3.108) is equivalent to  $-8t^2 - 12t - 20 < 0$  for all  $t > 1$ , which is trivially true. Hence, we have proved (3.3.105) in the case  $d = 3$ . Let us show that the validity of (3.3.105) for  $d = 3$  ensures the validity of the same inequality for  $d > 3$ . As above, it is enough to prove it for  $s > 1$  only. We have:

$$\frac{d\hat{\ell}(s)}{ds} = \int_0^{\pi/2} \frac{-2(1+s)^2 + 4(1+s^2)\cos^2(\theta)}{\underbrace{[(1+s)^2 - 4s\cos^2(\theta)]^2}_{f(s,\theta)}} \sin^{d-2}(\theta) d\theta. \quad (3.3.109)$$

Note that taking derivatives under the integral sign is safe because for  $s > 1$  the denominator of the integrand is bounded away from zero as  $\theta \in [0, \pi/2]$ . It is readily seen that

$$f(s, \theta) \geq 0 \quad \forall \theta \in [0, \hat{\theta}(s)], \quad f(s, \theta) < 0 \quad \forall \theta \in (\hat{\theta}(s), \pi/2], \quad (3.3.110)$$

where, for  $s > 1$ , we set

$$\hat{\theta}(s) := \arccos \left[ (1+s)/\sqrt{2(1+s^2)} \right] \in (0, \pi/2).$$

The validity of (3.3.105) in the case  $d = 3$  entails that

$$\int_0^{\pi/2} f(s, \theta) \sin(\theta) d\theta < 0. \quad (3.3.111)$$



In view of (3.3.109)–(3.3.111), for any  $d > 3$  we obtain:

$$\begin{aligned} \frac{d\hat{\ell}(s)}{ds} &= \int_0^{\hat{\theta}(s)} f(s, \theta) \sin(\theta) \sin^{d-3}(\theta) d\theta + \int_{\hat{\theta}(s)}^{\frac{\pi}{2}} f(s, \theta) \sin(\theta) \sin^{d-3}(\theta) d\theta \\ &\leq \sin^{d-3}(\hat{\theta}(s)) \int_0^{\hat{\theta}(s)} f(s, \theta) \sin(\theta) d\theta + \sin^{d-3}(\hat{\theta}(s)) \int_{\hat{\theta}(s)}^{\frac{\pi}{2}} f(s, \theta) \sin(\theta) d\theta \\ &= \sin^{d-3}(\hat{\theta}(s)) \int_0^{\frac{\pi}{2}} f(s, \theta) \sin(\theta) d\theta < 0. \end{aligned}$$

Hence,  $\hat{\ell}(s)$  is strictly decreasing and the proof is completed.  $\square$

We are now in position to prove the following result.

**Proposition 3.3.9.** *The function  $G(R)$  defined in (3.3.98) is strictly increasing. In particular, it attains its unique minimum at  $R = 0$ .*

*Proof.* In order to prove the assertion it suffices to show that  $\frac{dG(R)}{dR} > 0$  for all  $R > 0$ , which, due to (3.3.101) and (3.3.102), is equivalent to proving that

$$\int_0^{+\infty} \mathcal{K}(r) r^{d-1} \ell_R(r) dr > 0 \quad \forall R > 0.$$

Recalling the strict positivity of (3.3.81) and the definition (3.3.96) of  $\mathcal{K}$ , we have:

$$\int_0^{+\infty} \mathcal{K}(r) r^{d-1} dr > 0. \quad (3.3.112)$$

Thanks to the explicit profile (3.2.4) of  $w_b$ , it is immediate to check that there exists  $\hat{R} > 0$  such that

$$\mathcal{K}(r) \geq 0 \quad \forall r \in [0, \hat{R}], \quad \mathcal{K}(r) < 0 \quad \forall r > \hat{R}. \quad (3.3.113)$$

Exploiting (3.3.112), (3.3.113) and Lemma 3.3.8, we obtain:

$$\begin{aligned} \int_0^{+\infty} \mathcal{K}(r) r^{d-1} \ell_R(r) dr &\geq \ell_R(\hat{R}) \int_0^{\hat{R}} \mathcal{K}(r) r^{d-1} dr + \ell_R(\hat{R}) \int_{\hat{R}}^{+\infty} \mathcal{K}(r) r^{d-1} dr \\ &= \ell_R(\hat{R}) \int_0^{+\infty} \mathcal{K}(r) r^{d-1} dr > 0. \end{aligned}$$

$\square$

We can finally state the fundamental consequence of Lemma 3.3.6, Proposition 3.3.9 and the relation between  $F$  and  $G$ .

**Corollary 3.3.10.** *Corollary 3.3.5 holds true with  $\bar{y} = 0$ .*

### 3.3.4 Optimal functions are radial for $\gamma$ small

This final subsection is devoted to the proof of Theorem 3.2.1. We shall exploit the convergence results of the previous subsections and perform an argument by contradiction, which involves *angular derivatives* of possibly non-radial optimal functions.

Given a nontrivial antisymmetric matrix  $A$  and a differentiable function  $f$ , we define the angular derivative of  $f$  with respect to  $A$  as follows:

$$f_A(x) := x^T A \nabla f(x) \quad \forall x \in \mathbb{R}^d,$$

where  $T$  denotes transposition. Antisymmetric matrices are deeply linked with infinitesimal rotations. In fact, take a curve of rotations about the origin  $\{R(t)\}$  (let  $t \geq 0$ ) in the Euclidean space, such that  $R(0) = I$  and  $R'(0) = A^T$ . A straightforward computation shows that  $f_A$  represents the first-order variation of  $f$  along  $\{R(t)\}$  at  $t = 0$ , that is

$$\left. \frac{d[f(R(t)x)]}{dt} \right|_{t=0} = x^T \underbrace{[R'(0)]^T}_A \nabla f(x) \quad \forall x \in \mathbb{R}^d.$$

Note that it is always possible to construct such a curve: this is a consequence of the well-known fact that the tangent space to the rotation group at  $R = I$  coincides with the space of antisymmetric matrices.

The term ‘‘angular derivative’’ is readily justified. To fix ideas, let  $d = 3$ . Here the usual change of variables in spherical coordinates reads

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

where  $r \in [0, \infty)$  is the radius,  $\theta \in [0, \pi]$  is the polar angle and  $\phi \in [0, 2\pi)$  is the azimuthal angle. The derivative of a differentiable function  $f$  w.r.t. the azimuthal angle is  $f_\phi = -yf_x + xf_y$ , namely  $f_\phi = f_{A_\phi}$  with the choice

$$A_\phi := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Such property can easily be generalized to the case  $d > 3$  by considering the last angle in spherical coordinates (the only one ranging between 0 and  $2\pi$ ). Note that this is false for the polar angle  $\theta$ :  $d\theta$  is not an infinitesimal rotation.

As a consequence of the above discussion, it is apparent that if  $f$  is a radial function then necessarily  $f_A \equiv 0$  for any antisymmetric matrix  $A$ . It is not difficult to prove (e.g. by contradiction) that also the vice versa holds true: if  $f_A \equiv 0$  for every antisymmetric matrix  $A$ , then  $f$  is a radial function. Our aim is to prove that  $w_\gamma$  is radial, at least for  $\gamma$  small. Hence, if it is not, there must exist an antisymmetric matrix  $A_\gamma$  such that  $(w_\gamma)_{A_\gamma} \not\equiv 0$ . In the following, we shall prove some crucial properties of the angular derivatives of  $w_\gamma$ , which are basically inherited from the Euler-Lagrange equation.

**Lemma 3.3.11.** *Assume the same hypotheses as Lemma 3.3.1 (and use notations accordingly). Given any antisymmetric matrix  $A$ , set  $\omega_{n[A]} := (w_{\gamma_n})_A$ . Then there exists  $\tilde{n} \in \mathbb{N}$ , independent of  $A$ , such that  $\omega_{n[A]}$  belongs to  $\dot{H}^1(\mathbb{R}^d)$  and satisfies*

$$-\Delta \omega_{n[A]} + p \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} \omega_{n[A]} = (2p-1) \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} \omega_{n[A]} \quad \text{in } \mathbb{R}^d \quad (3.3.114)$$

for all  $n \geq \tilde{n}$ . Moreover, the following identities hold true:

$$\int_{\mathbb{R}^d} |\nabla \omega_{n[A]}|^2 dx + p \int_{\mathbb{R}^d} \omega_{n[A]}^2 \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} dx = (2p-1) \int_{\mathbb{R}^d} \omega_{n[A]}^2 \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx, \quad (3.3.115)$$

$$\int_{\mathbb{R}^d} \omega_{n[A]} w_{\gamma_n}^{2p-1} |x|^{-\gamma_n} dx = 0. \quad (3.3.116)$$

*Proof.* The validity of (3.3.114) can be proved just by plugging  $\varphi_A$  as a test function in the weak formulation of (3.2.2), where  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ :

$$-\int_{\mathbb{R}^d} w_{\gamma_n} \Delta \varphi_A dx + \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^p}{|x|^{\gamma_n}} \varphi_A dx = \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^{2p-1}}{|x|^{\gamma_n}} \varphi_A dx. \quad (3.3.117)$$

In fact, noticing that

$$\Delta\varphi_A = (A^T x) \cdot \nabla(\Delta\varphi) + \underbrace{\Delta(A^T x) \cdot \nabla\varphi + 2A^T : \nabla^2\varphi}_{=0} = (\Delta\varphi)_A$$

and

$$\int_{\mathbb{R}^d} f \varphi_A dx = \int_{\mathbb{R}^d} \langle f A^T x, \nabla\varphi \rangle dx = - \int_{\mathbb{R}^d} \operatorname{div}(f A^T x) \varphi dx = - \int_{\mathbb{R}^d} f_A \varphi dx$$

for all  $f$  regular enough (recall the antisymmetry of  $A$  and the symmetry of  $\nabla^2\varphi$ ), from (3.3.117) we infer

$$- \int_{\mathbb{R}^d} (w_{\gamma_n})_A \Delta\varphi dx + \int_{\mathbb{R}^d} (w_{\gamma_n}^p |x|^{-\gamma_n})_A \varphi dx = \int_{\mathbb{R}^d} (w_{\gamma_n}^{2p-1} |x|^{-\gamma_n})_A \varphi dx,$$

namely (3.3.114) since  $(|x|^{-\gamma_n})_A = 0$ . As a consequence of Lemma 3.2.10, we have that  $w_{\gamma_n} \in H^2(B_r)$  for any  $r > 0$  and all  $n$  sufficiently large (it is enough that  $\gamma_n < \bar{\gamma}(2, p, d)$ ). In particular,  $\omega_{n[A]} \in H^1(B_r)$  and equation (3.3.114) also holds in the  $H^1$  (local) weak sense. In order to prove (3.3.115), let us then multiply (3.3.114) by the test function  $\varphi = \xi_R \omega_{n[A]}$  and integrate by parts, where  $\xi_R(x) := \xi(x/R)$  and  $\xi$  is a regular cut-off function (see *e.g.* after formula (3.3.18)). We obtain:

$$\begin{aligned} & \int_{\mathbb{R}^d} \xi_R |\nabla\omega_{n[A]}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla\xi_R, \nabla(\omega_{n[A]}^2) \rangle dx + p \int_{\mathbb{R}^d} \xi_R \omega_{n[A]}^2 \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} dx \\ & = (2p-1) \int_{\mathbb{R}^d} \xi_R \omega_{n[A]}^2 \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx. \end{aligned} \quad (3.3.118)$$

Since  $w_{\gamma_n} \in \dot{H}^1(\mathbb{R}^d)$ , by the definition of  $\omega_{n[A]}$  we have that  $\omega_{n[A]} \in L^2_2(\mathbb{R}^d)$ , namely

$$\int_{\mathbb{R}^d} \omega_{n[A]}^2 |x|^{-2} dx < \infty.$$

We can therefore estimate the second term in the l.h.s. of (3.3.118) as follows:

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \langle \nabla\xi_R, \nabla(\omega_{n[A]}^2) \rangle dx \right| &= \left| \int_{\mathbb{R}^d} \Delta\xi_R \omega_{n[A]}^2 dx \right| \leq \frac{\|\Delta\xi\|_\infty}{R^2} \left| \int_{B_{2R} \setminus B_R} \omega_{n[A]}^2 dx \right| \\ &\leq 4 \|\Delta\xi\|_\infty \left| \int_{B_{2R} \setminus B_R} \frac{\omega_{n[A]}^2}{|x|^2} dx \right|, \end{aligned}$$

so that

$$\lim_{R \rightarrow \infty} \left| \int_{\mathbb{R}^d} \langle \nabla\xi_R, \nabla(\omega_{n[A]}^2) \rangle dx \right| = 0 \quad (3.3.119)$$

in view of the just recalled integrability properties of  $\omega_{n[A]}$ . Now notice that

$$\int_{\mathbb{R}^d} \omega_{n[A]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx < \infty. \quad (3.3.120)$$

This is still a consequence of the fact that  $\omega_{n[A]} \in L^2_2(\mathbb{R}^d)$  and of the uniform boundedness of the function  $|x|^{2-\gamma_n} w_{\gamma_n}^{2(p-1)}$ , which in turn follows by the estimate from above (3.3.92) of Corollary 3.3.5 (here we need to assume  $n \geq \tilde{n}$ ). Using (3.3.119) and (3.3.120) we can then pass the limit in (3.3.118) to obtain

$$\int_{\mathbb{R}^d} |\nabla\omega_{n[A]}|^2 dx + p \int_{\mathbb{R}^d} \omega_{n[A]}^2 \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} dx \leq (2p-1) \int_{\mathbb{R}^d} \omega_{n[A]}^2 \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx. \quad (3.3.121)$$

Since (3.3.121) implies that both terms in the l.h.s. are finite, passing to the limit once again in (3.3.118) we deduce the validity of (3.3.115).

We still have to show that  $\omega_{n[A]} \in \dot{H}^1(\mathbb{R}^d)$ , namely that there exists a sequence  $\{\varphi_k\} \subset \mathcal{D}(\mathbb{R}^d)$  such that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla \omega_{n[A]} - \nabla \varphi_k|^2 dx = 0.$$

By standard mollification arguments, it is enough to prove that the same holds true for the sequence  $\{\xi_k \omega_{n[A]}\}$ , and this is indeed the case:

$$\int_{\mathbb{R}^d} |\nabla \omega_{n[A]} - \nabla(\xi_k \omega_{n[A]})|^2 dx \leq 2 \int_{\mathbb{R}^d} (1 - \xi_k)^2 |\nabla \omega_{n[A]}|^2 dx + 8 \|\nabla \xi\|_\infty^2 \int_{B_{2k} \setminus B_k} \omega_{n[A]}^2 |x|^{-2} dx,$$

where the r.h.s. vanishes as  $k \rightarrow \infty$  since  $\nabla \omega_{n[A]} \in [L^2(\mathbb{R}^d)]^d$  and  $\omega_{n[A]} \in L^2_2(\mathbb{R}^d)$ . Let us finally prove (3.3.116). This is equivalent to proving that

$$\int_{\mathbb{R}^d} (w_{\gamma_n}^{2p} |x|^{-\gamma_n})_A dx = 0. \quad (3.3.122)$$

In order to establish (3.3.122), we can exploit once again a cut-off argument:

$$\left| \int_{\mathbb{R}^d} \xi_R (w_{\gamma_n}^{2p} |x|^{-\gamma_n})_A dx \right| = \left| \int_{\mathbb{R}^d} (\xi_R)_A \frac{w_{\gamma_n}^{2p}}{|x|^{\gamma_n}} dx \right| \leq \|\xi_A\|_\infty \int_{B_{2R} \setminus B_R} \frac{w_{\gamma_n}^{2p}}{|x|^{\gamma_n}} dx. \quad (3.3.123)$$

To complete the proof, just recall (3.3.120) and that  $w_{\gamma_n} \in L^2_{\gamma_n}(\mathbb{R}^d)$ , use the bound

$$\int_{\mathbb{R}^d} |\omega_{n[A]}| \frac{w_{\gamma_n}^{2p-1}}{|x|^{\gamma_n}} dx \leq \left( \int_{\mathbb{R}^d} \omega_{n[A]}^2 \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \frac{w_{\gamma_n}^{2p}}{|x|^{\gamma_n}} dx \right)^{\frac{1}{2}}$$

and let  $R \rightarrow \infty$  in (3.3.123). □

**Lemma 3.3.12.** *Assume the same hypotheses of Lemma 3.3.11 (and use notations accordingly). If  $\{w_{\gamma_n}\}$  is eventually not radial (up to subsequences), then there exists a corresponding sequence of antisymmetric matrices  $\{A_n\}$  such that*

$$\int_{\mathbb{R}^d} \omega_{n[A_n]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx = 1 \quad (3.3.124)$$

*eventually. Moreover,  $\{\omega_{n[A_n]}\}$  converges weakly in  $\dot{H}^1(\mathbb{R}^d)$ , up to subsequences, to a non-trivial function  $\bar{\omega}$  satisfying the inequality*

$$\int_{\mathbb{R}^d} |\nabla \bar{\omega}|^2 dx + p \int_{\mathbb{R}^d} \bar{\omega}^2 w_b^{p-1} dx \leq (2p-1) \int_{\mathbb{R}^d} \bar{\omega}^2 w_b^{2(p-1)} dx \quad (3.3.125)$$

*and the identities*

$$\int_{\mathbb{R}^d} \bar{\omega} w_b^{2p-1} dx = 0, \quad (3.3.126)$$

$$\int_{\mathbb{R}^d} \frac{x_i}{|x|^2} (w_b^p - w_b^{2p-1}) \bar{\omega} dx = 0 \quad \forall i = 1 \dots d. \quad (3.3.127)$$

*Proof.* As recalled in the beginning of this subsection, if  $w_{\gamma_n}$  is not radial then there exists an antisymmetric matrix  $A_n$  such that  $\omega_{n[A_n]} \not\equiv 0$ . In particular,

$$\int_{\mathbb{R}^d} \omega_{n[A_n]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx > 0. \quad (3.3.128)$$

In fact, thanks to the regularity of  $w_{\gamma_n}$  ensured by Lemma 3.2.10, we know that  $\omega_{n[A_n]}$  is eventually continuous. Since  $\omega_{n[A_n]}$  is by hypothesis nontrivial, there exist  $x_0 \in \mathbb{R}^d$  and  $r > 0$  such that  $\omega_{n[A_n]}^2 > 0$  in  $\overline{B_r}(x_0)$ . Therefore,  $w_{\gamma_n}$  cannot be identically zero in  $B_r(x_0)$ : this means that

$$\int_{B_r(x_0)} \omega_{n[A_n]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx > 0,$$

which proves (3.3.128). Hence, in order to get (3.3.124), it is enough to replace the matrix  $A_n$  with

$$\tilde{A}_n := \left( \int_{\mathbb{R}^d} \omega_{n[A_n]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx \right)^{-\frac{1}{2}} A_n.$$

For notational simplicity, we shall keep denoting  $\tilde{A}_n$  as  $A_n$ . In view of (3.3.124), the identity (3.3.115) reads

$$\int_{\mathbb{R}^d} |\nabla \omega_{n[A_n]}|^2 dx + p \int_{\mathbb{R}^d} \omega_{n[A_n]}^2 w_{\gamma_n}^{p-1} |x|^{-\gamma_n} dx = 2p - 1. \quad (3.3.129)$$

Thanks to (3.3.129), the sequence  $\{\omega_{n[A_n]}\}$  is then bounded in  $\dot{H}^1(\mathbb{R}^d)$ ; we can therefore assume that, up to subsequences, it converges in  $\dot{H}^1(\mathbb{R}^d)$  and pointwise to some function  $\bar{\omega}$ . First of all, let us show that

$$1 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \omega_{n[A_n]}^2 w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n} dx = \int_{\mathbb{R}^d} \bar{\omega}^2 w_b^{2(p-1)} dx. \quad (3.3.130)$$

Exploiting the just mentioned convergence properties of  $\{\omega_{n[A_n]}\}$  and the Sobolev inequality, we infer that  $\{\omega_{n[A_n]}^2\}$  converges weakly in  $L^{d/(d-2)}(\mathbb{R}^d)$  to  $\bar{\omega}^2$ , up to subsequences. In order to establish (3.3.130), it is then enough to prove that the sequence

$$w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n}$$

converges strongly in  $L^{d/2}(\mathbb{R}^d)$  to  $w_b^{2(p-1)}$ . But this is a direct consequence of Corollaries 3.3.5 and 3.3.10. In fact, from (3.3.91) such convergence is trivially true in  $B_1$ . In  $B_1^c$  one can use estimate (3.3.92), which in particular ensures that

$$w_{\gamma_n}^{2(p-1)}(x) |x|^{-\gamma_n} \leq C (1 + |x|)^{-4+\gamma_n} \quad \forall x \in B_1^c \quad (3.3.131)$$

for all  $n$  large enough and a positive constant  $C$  independent of  $n$ . Since  $\gamma_n \downarrow 0$ , the r.h.s. of (3.3.131) is eventually dominated by a function belonging to  $L^{d/2}(\mathbb{R}^d)$ . Hence, by dominated convergence, we get that

$$w_{\gamma_n}^{2(p-1)} |x|^{-\gamma_n}$$

does converge to  $w_b^{2(p-1)}$  strongly in  $L^{d/2}(\mathbb{R}^d)$ , and so (3.3.130) holds true. At this point, estimate (3.3.125) follows by passing to the limit in (3.3.129) as  $n \rightarrow \infty$ : in the l.h.s. we use weak lower semi-continuity and in the r.h.s. we use (3.3.130).

We are left with proving the validity of (3.3.126) and (3.3.127). As for the first one, it is enough to note that the sequences

$$\omega_{n[A_n]} w_{\gamma_n}^{p-1} |x|^{-\gamma_n/2} \quad \text{and} \quad w_{\gamma_n}^p |x|^{-\gamma_n/2}$$

converge strongly in  $L^2(\mathbb{R}^d)$  to  $\bar{\omega} w_b^{p-1}$  and  $w_b^p$ , respectively. This allows us to pass to the limit in (3.3.116) (with  $A = A_n$ ) to get (3.3.126). Let us finally prove (3.3.127). To begin with, for all  $i = 1 \dots d$ , we consider the equation solved by the *directional derivatives* of  $w_{\gamma_n}$ , namely  $v_{n[i]} := (w_{\gamma_n})_{x_i}$ . Proceeding as in the proof of (3.3.114), we get that  $v_{n[i]} \in \dot{H}^1(\mathbb{R}^d)$

and it solves exactly the same equation as  $\omega_{n[A_n]}$  except for a forcing term coming up from the directional derivative of the weight  $|x|^{-\gamma_n}$ :

$$-\Delta v_{n[i]} + p \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} v_{n[i]} = (2p-1) \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} v_{n[i]} + \gamma_n \frac{x_i}{|x|^{\gamma_n+2}} (w_{\gamma_n}^p - w_{\gamma_n}^{2p-1}) \quad (3.3.132)$$

in  $\mathbb{R}^d$ . Thanks to (3.3.114) (with  $A = A_n$ ) and (3.3.132) we deduce the identities

$$\int_{\mathbb{R}^d} \langle \nabla \omega_{n[A_n]}, \nabla f \rangle dx + p \int_{\mathbb{R}^d} \omega_{n[A_n]} f \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} dx = (2p-1) \int_{\mathbb{R}^d} \omega_{n[A_n]} f \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx \quad (3.3.133)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle \nabla v_{n[i]}, \nabla g \rangle dx + p \int_{\mathbb{R}^d} v_{n[i]} g \frac{w_{\gamma_n}^{p-1}}{|x|^{\gamma_n}} dx \\ & = (2p-1) \int_{\mathbb{R}^d} v_{n[i]} g \frac{w_{\gamma_n}^{2(p-1)}}{|x|^{\gamma_n}} dx + \gamma_n \int_{\mathbb{R}^d} \frac{x_i}{|x|^{\gamma_n+2}} (w_{\gamma_n}^p - w_{\gamma_n}^{2p-1}) g dx \end{aligned} \quad (3.3.134)$$

for all  $f, g \in \dot{H}^1(\mathbb{R}^d)$ . Picking  $f = v_{n[i]}$ ,  $g = \omega_{n[A_n]}$  and subtracting (3.3.134) from (3.3.133), we end up with the key identity

$$\int_{\mathbb{R}^d} \frac{x_i}{|x|^{\gamma_n+2}} (w_{\gamma_n}^p - w_{\gamma_n}^{2p-1}) \omega_{n[A_n]} dx = 0. \quad (3.3.135)$$

The validity of (3.3.127) is then just a matter of passing to the limit in (3.3.135) as  $n \rightarrow \infty$ . This is feasible since, for instance,  $\{\omega_{n[A_n]}\}$  converges to  $\bar{\omega}$  weakly in  $L_2^2(\mathbb{R}^d)$  and the sequence

$$x_i |x|^{-\gamma_n} (w_{\gamma_n}^p - w_{\gamma_n}^{2p-1})$$

converges to  $x_i (w_b^p - w_b^{2p-1})$  strongly in  $L_2^2(\mathbb{R}^d)$ . □

The next proposition recalls a fundamental spectral gap property, which is a consequence of the results in [24, 65, 66].

**Proposition 3.3.13.** *Consider the functional*

$$\mathcal{J}_b(w) := \int_{\mathbb{R}^d} |\nabla w|^2 dx + p \int_{\mathbb{R}^d} w^2 w_b^{p-1} dx - (2p-1) \int_{\mathbb{R}^d} w^2 w_b^{2(p-1)} dx,$$

where  $w \in \dot{H}^1(\mathbb{R}^d)$ . Then,  $\mathcal{J}_b(w) \geq 0$  for all  $w \in \dot{H}^1(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} w w_b^{2p-1} dx = 0, \quad (3.3.136)$$

and equality is attained if and only if there exists a vector  $\mathbf{a} \in \mathbb{R}^d$  such that

$$w(x) = \mathbf{a}^T \nabla w_b(x) \quad \forall x \in \mathbb{R}^d. \quad (3.3.137)$$

*Proof.* As a particular case of [24, Theorem 2] (let  $\alpha = -2p/(p-1) < -d$  there), we have that the Hardy-Poincaré inequality

$$\frac{4p}{a(p-1)} \int_{\mathbb{R}^d} f^2 w_b^{3p-1} dx \leq \int_{\mathbb{R}^d} |\nabla f|^2 w_b^{2p} dx \quad (3.3.138)$$

(let  $a$  be as in (3.2.4)) holds true for all  $f$  belonging to the closure of  $\mathcal{D}(\mathbb{R}^d)$  w.r.t. the norm appearing in the r.h.s. of (3.3.138) and such that

$$\int_{\mathbb{R}^d} f w_b^{3p-1} dx = 0. \quad (3.3.139)$$

Moreover, nontrivial optimal functions for (3.3.138) subject to (3.3.139) are explicit: there holds equality if and only if

$$f(x) = \mathbf{b}^T x \quad \forall x \in \mathbb{R}^d \quad (3.3.140)$$

for some  $\mathbf{b} \in \mathbb{R}^d$ . We point out that these results had already been obtained by Denzler and McCann in [65, 66], but in a different functional setting. We refer the reader to [24, pp. 16462-16463] for the details.

At this point the proof of our lemma consists in plugging the function  $f = ww_{\mathbf{b}}^{-p}$  in (3.3.138)–(3.3.140) and performing some integrations by parts. In order to justify this rigorously, we shall use compactly supported functions. That is, if  $w \in \mathcal{D}(\mathbb{R}^d)$  and satisfies (3.3.136), then  $f = ww_{\mathbf{b}}^{-p}$  is an eligible function for (3.3.138), so that integrating by parts in the r.h.s. we find that  $\mathcal{J}_{\mathbf{b}}(w) \geq 0$ . Given any  $w \in \dot{H}^1(\mathbb{R}^d)$ , take a sequence  $\{w_n\} \subset \mathcal{D}(\mathbb{R}^d)$  converging to  $w$  in  $\dot{H}^1(\mathbb{R}^d)$ , having the property that  $\int_{\mathbb{R}^d} w_n w_{\mathbf{b}}^{2p-1} dx = 0$  for all  $n \in \mathbb{N}$ . Passing to the limit in the inequality  $\mathcal{J}_{\mathbf{b}}(w_n) \geq 0$  we deduce that  $\mathcal{J}_{\mathbf{b}}(w) \geq 0$ . We are left with proving that  $\mathcal{J}_{\mathbf{b}}(w) = 0$  if and only if (3.3.137) holds true. Again, the idea is to take a sequence  $\{w_n\}$  as above. Since  $\mathcal{J}_{\mathbf{b}}(w_n) \rightarrow 0$  ( $\mathcal{J}_{\mathbf{b}}$  is continuous in  $\dot{H}^1(\mathbb{R}^d)$ ), we infer that  $f_n := w_n w_{\mathbf{b}}^{-p}$  satisfies

$$\frac{4p}{a(p-1)} \int_{\mathbb{R}^d} f_n^2 w_{\mathbf{b}}^{3p-1} dx = \int_{\mathbb{R}^d} |\nabla f_n|^2 w_{\mathbf{b}}^{2p} dx + \varepsilon_n \quad (3.3.141)$$

and  $\int_{\mathbb{R}^d} f_n w_{\mathbf{b}}^{3p-1} dx = 0$ , where  $\varepsilon_n \rightarrow 0$ . To conclude that  $f = ww_{\mathbf{b}}^{-p}$  is an optimal function for (3.3.138), we just need to pass to the limit in (3.3.141). The only nontrivial part is to show that we can pass to the limit in the right-hand side. Noticing that

$$\nabla f_n = w_{\mathbf{b}}^{-p} \nabla w_n - p w_n w_{\mathbf{b}}^{-p-1} \nabla w_{\mathbf{b}} \quad \text{and} \quad \nabla f = w_{\mathbf{b}}^{-p} \nabla w - p w w_{\mathbf{b}}^{-p-1} \nabla w_{\mathbf{b}},$$

this amounts to proving that

$$\int_{\mathbb{R}^d} |\nabla w_n - \nabla w|^2 dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |w_n - w|^2 w_{\mathbf{b}}^{-2} |\nabla w_{\mathbf{b}}|^2 dx \rightarrow 0. \quad (3.3.142)$$

But (3.3.142) is just a consequence of the convergence of  $\{w_n\}$  to  $w$  in  $\dot{H}^1(\mathbb{R}^d)$ . In fact, by Hardy's inequality, the latter implies convergence in  $L_2^2(\mathbb{R}^d)$ , and it is immediate to check that

$$w_{\mathbf{b}}^{-2}(x) |\nabla w_{\mathbf{b}}(x)|^2 \leq C (1 + |x|^2)^{-1} \quad \forall x \in \mathbb{R}^d$$

for a suitable positive constant  $C$ . Passing to the limit in (3.3.141) we then infer that  $f = ww_{\mathbf{b}}^{-p}$  optimizes (3.3.138) subject to (3.3.139). Hence, (3.3.140) implies that  $ww_{\mathbf{b}}^{-p} = \mathbf{b}^T x$  for some  $\mathbf{b} \in \mathbb{R}^d$ , and (3.3.137) follows by noting that  $\nabla w_{\mathbf{b}}$  is proportional to  $xw_{\mathbf{b}}^p$ .  $\square$

Now we show that the optimal functions for (3.1.1), *when restricting to the radial setting*, are Barenblatt-type profiles.

**Proposition 3.3.14.** *Let (3.1.2) hold true. Then the solution to problem (P) restricted to  $\mathcal{H}_{\gamma}^*(\mathbb{R}^d)$  is unique and explicit. If the mass  $M$  is chosen so that it satisfies the Euler-Lagrange equation (3.2.2), it coincides with  $w_{\mathbf{b},\gamma}$  as in (3.2.3).*

*Proof.* Since every (suitably rescaled) solution to problem (P) restricted to  $\mathcal{H}_{\gamma}^*(\mathbb{R}^d)$  is a solution to (3.2.2), it is enough to establish uniqueness for nonnegative, nontrivial solutions to (3.2.2) belonging to  $\mathcal{H}_{\gamma}^*(\mathbb{R}^d)$  (hereafter we shall denote such solutions as  $w_{\gamma}^*$ ). Actually, this follows from the results of [152]. In fact, there it is proved uniqueness for the non-weighted semilinear equation

$$-\Delta w + w^p = w^{2p-1} \quad \text{in } \mathbb{R}^d \quad (3.3.143)$$

### 3.3. Convergence as $\gamma \downarrow 0$ and radial symmetry of optimal functions

among radial solutions which are nontrivial, nonnegative, belong to  $C^1(\mathbb{R}^d)$  and vanish at infinity. Nevertheless, for radial functions, one can always perform the change of variables

$$v(s) = w_\gamma^*(c s^{2/(2-\gamma)}) \quad \forall s \in \mathbb{R}^+, \quad c := [(2-\gamma)/2]^{2/(2-\gamma)},$$

where, with a slight abuse of notation, it is understood that  $w_\gamma^*(|x|) = w_\gamma^*(x)$ . Lengthy but elementary computations show that  $v$  solves

$$v'' + \frac{d_\gamma - 1}{s} v' + v^p = v^{2p-1} \quad \text{in } \mathbb{R}^+, \quad (3.3.144)$$

where  $d_\gamma := 2(d-\gamma)/(2-\gamma)$ . Now notice that (3.3.144) is just the same equation as (3.3.143) but for radial functions and in the “real” dimension  $d_\gamma$ . One can then apply the uniqueness results of [152] (see Theorem 2 in Section 2 there) to  $v$  and so recover uniqueness for  $w_\gamma^*$ . However, such results hold true provided  $v \in C^1$ . But this is indeed the case. In fact, from the  $L^\infty$  bound (3.2.25) and from the Calderón-Zygmund theory (see the proof of Lemma 3.2.10), we easily get that  $w_\gamma^* \in L^\infty(\mathbb{R}^d) \cap C^1(\mathbb{R}^d \setminus \{0\})$ . The latter property ensures that  $v \in C^1$ : this is carefully justified in [152, Section 6, Remark 3].

Finally, since  $w_{\mathbf{b},\gamma}$  as in (3.2.3) belongs to  $\mathcal{H}_\gamma^*(\mathbb{R}^d)$  and solves (3.2.2), the last statement is a trivial consequence of the just proved uniqueness result.  $\square$

Thanks to Lemma 3.3.12 and Propositions 3.3.13, 3.3.14, we are in position to prove our main symmetry result.

*Proof of Theorem 3.2.1.* We argue by contradiction. By Proposition 3.3.14, if the assertion is false then there exist a sequence  $\{\gamma_n\} \subset (0, d - (d-2)p)$  such that  $\gamma_n \downarrow 0$  and a corresponding sequence of minimizers  $\{w_{\gamma_n}\}$  such that  $w_{\gamma_n}$  is *not radial* for all  $n \in \mathbb{N}$ . In view of Lemma 3.3.12, this entails the existence of a nontrivial function  $\bar{w} \in \dot{H}^1(\mathbb{R}^d)$  satisfying (3.3.125), (3.3.126) and (3.3.127): but then Proposition 3.3.13 forces (3.3.125) to be an equality and  $\bar{w}$  to have the explicit form  $\bar{w} = \mathbf{a}^T \nabla w_{\mathbf{b}}$  for some  $\mathbf{a} \neq 0$ . Combining this information with (3.3.127), we infer that

$$\sum_{j=1}^d \mathbf{a}_j \int_{\mathbb{R}^d} \frac{x_i}{|x|^2} (w_{\mathbf{b}}^p - w_{\mathbf{b}}^{2p-1}) (w_{\mathbf{b}})_{x_j} dx = 0 \quad \forall i = 1 \dots d,$$

that is

$$\sum_{j=1}^d \mathbf{a}_j \int_{\mathbb{R}^d} \frac{x_i}{|x|^2} \left( \frac{1}{p+1} w_{\mathbf{b}}^{p+1} - \frac{1}{2p} w_{\mathbf{b}}^{2p} \right)_{x_j} dx = 0 \quad \forall i = 1 \dots d. \quad (3.3.145)$$

Integrating by parts in (3.3.145), we obtain:

$$\sum_{j=1}^d \mathbf{a}_j \int_{\mathbb{R}^d} \left( \frac{\delta_{ij}}{|x|^2} - 2 \frac{x_i x_j}{|x|^4} \right) \left( -\frac{1}{p+1} w_{\mathbf{b}}^{p+1} + \frac{1}{2p} w_{\mathbf{b}}^{2p} \right) dx = 0 \quad \forall i = 1 \dots d, \quad (3.3.146)$$

where  $\delta_{ij}$  denotes Kronecker’s delta. Since  $w_{\mathbf{b}}$  is radial, it is apparent that for all  $j \neq i$  the integrals appearing in the summation (3.3.146) are identically zero, so that the latter amounts to

$$\mathbf{a}_i \int_{\mathbb{R}^d} \left( \frac{1}{|x|^2} - 2 \frac{x_i^2}{|x|^4} \right) \left( -\frac{1}{p+1} w_{\mathbf{b}}^{p+1} + \frac{1}{2p} w_{\mathbf{b}}^{2p} \right) dx = 0 \quad \forall i = 1 \dots d. \quad (3.3.147)$$



The radially of  $w_{\mathbf{b}}$  also implies that the integral in (3.3.147) is independent of  $i$ :

$$\begin{aligned} & \int_{\mathbb{R}^d} \left( \frac{1}{|x|^2} - 2 \frac{x_i^2}{|x|^4} \right) \left( -\frac{1}{p+1} w_{\mathbf{b}}^{p+1} + \frac{1}{2p} w_{\mathbf{b}}^{2p} \right) dx \\ &= \frac{d-2}{d} \int_{\mathbb{R}^d} \frac{1}{|x|^2} \left( -\frac{1}{p+1} w_{\mathbf{b}}^{p+1} + \frac{1}{2p} w_{\mathbf{b}}^{2p} \right) dx \quad \forall i = 1 \dots d. \end{aligned} \tag{3.3.148}$$

Since  $\mathbf{a} \neq 0$ , from (3.3.147) and (3.3.148) we deduce that

$$\int_{\mathbb{R}^d} \frac{1}{|x|^2} \left( -\frac{1}{p+1} w_{\mathbf{b}}^{p+1} + \frac{1}{2p} w_{\mathbf{b}}^{2p} \right) dx = 0. \tag{3.3.149}$$

Actually (3.3.149) cannot hold. In fact, repeating exactly the same proof as Proposition 3.3.9, with  $\ell_R(r)$  replaced by  $1/r^2$ , we get that the integral in (3.3.149) is strictly positive. The contradiction is then established, and the proof is complete.  $\square$

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## Fast diffusion on hyperbolic space: sharp asymptotics of radial solutions

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### 4.1 Introduction and main results

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In this chapter we study the asymptotic behaviour of nonnegative solutions to the following *fast diffusion equation* on  $N$ -dimensional (let  $N \geq 2$ ) hyperbolic space  $\mathbb{H}^N$ :

$$\begin{cases} u_t = \Delta(u^m) & \text{on } \mathbb{H}^N \times (0, T), \\ u = u_0 & \text{on } \mathbb{H}^N \times \{0\}, \end{cases} \quad (4.1.1)$$

where  $\Delta$  is the Riemannian Laplacian,  $m \in (m_s, 1)$  and  $m_s$  is the *critical exponent*

$$m_s := \frac{N-2}{N+2}.$$

The initial datum  $u_0$  is assumed to be nonnegative and *radial* in the sense that it depends only on the geodesic distance  $r$  from a given point  $o \in \mathbb{H}^N$  (which we shall denote as  $d(x, o)$ ), namely the *pole*, which has to be considered as fixed. Solutions to the fast diffusion equation corresponding to radial data are of course themselves radial at any time. In (4.1.1) the parameter  $T = T(u_0)$  denotes, for an appropriate class of data, the *extinction time* of the solution  $u$ , namely the smallest positive time  $t$  at which  $u(t) \equiv 0$ . In fact, the results of [29] show that, in a class of Cartan-Hadamard manifolds which includes  $\mathbb{H}^N$ , the extinction time exists finite for initial data which belong to  $L^q(\mathbb{H}^N)$ , where  $q > 1 \vee N(1-m)/2$ . We refer to such paper also for the relevant existence and uniqueness results for solutions to (4.1.1) provided there (see however an alternative approximation procedure sketched in Subsection 4.2.1).

It is well known that the fine asymptotics of solutions to the fast diffusion equation posed in the whole Euclidean space  $\mathbb{R}^N$  is governed by (suitable rescalings of) Barenblatt, or pseudo-Barenblatt, solutions. A huge literature on the topic has been produced in the last decade, so

we limit ourselves to quote the monographs [172,173] and, without any claim of completeness, the papers [23, 24, 30, 43–46, 58, 62, 64, 66, 85, 122, 132] and references therein (see also the Introduction to Chapter 3). We point out that in this context extinction in finite time takes place, for suitable classes of data, only for  $m < (N - 2)/N$  (provided *e.g.*  $N \geq 3$ ).

The situation on negatively curved manifolds is different from the Euclidean one since solutions vanish in finite time not only for  $m$  small enough, but for all  $m < 1$ . In fact, this is somewhat similar to what happens in the case of the homogeneous Dirichlet problem in *bounded* Euclidean domains  $\Omega$ , which was deeply investigated *e.g.* in [19, 31, 73, 93, 166]. In such papers, at various levels of detail, it is shown that the asymptotics of suitable classes of solutions can be described in terms of *separable solutions* of the form  $(1 - t/T)^{1/(1-m)} S^{1/m}$ , where  $S$  is a positive solution to the elliptic problem  $-\Delta S = c S^{1/m}$  in  $\Omega$ , with  $S = 0$  on  $\partial\Omega$ , for a suitable constant  $c > 0$  (in principle depending on the initial datum). No positive solution to the above elliptic problem exists in the whole Euclidean space  $\mathbb{R}^N$ . However, on  $\mathbb{H}^N$ , the fact that the bottom of the  $L^2$  spectrum of  $-\Delta$  is strictly positive points towards existence of such a solution. Indeed, this result was proved in [129]. More precisely, the authors establish there that, given any  $c > 0$  and  $m \in (m_s, 1)$ , the equation

$$-\Delta V = c V^{\frac{1}{m}} \quad \text{on } \mathbb{H}^N \tag{4.1.2}$$

admits strictly positive solutions  $V$  belonging to the Sobolev space  $H^1(\mathbb{H}^N)$  (what we call *energy solutions*), which are necessarily *radial* with respect to some pole  $o \in \mathbb{H}^N$ , the latter thus being the only free parameter characterizing such family of solutions. Notice that solutions to (4.1.2) corresponding to different values of  $c$  are related by scaling, namely they are all multiples of the solution with  $c = 1$ . The asymptotics of  $V = V(r)$  as  $r \rightarrow \infty$  was studied as well in [129] and slightly improved in [26]: the main result is the existence of a constant  $l = l(c, m, N) > 0$  such that, for all  $k \in \mathbb{N} \cup \{0\}$ ,

$$\lim_{r \rightarrow \infty} e^{(N-1)r} \frac{d^k V}{dr^k}(r) = (-1)^k (N-1)^k l. \tag{4.1.3}$$

Actually, in [129] and [26], formula (4.1.3) is proved for  $k = 0, 1$  only, but by exploiting (4.1.2) it is straightforward to deduce its validity for all  $k \in \mathbb{N}$ . Infinitely many other positive solutions to (4.1.2) exist, but none of them belongs to  $H^1(\mathbb{H}^N)$  and their behaviour as  $r \rightarrow \infty$  is of power type (see again [26]).

As concerns asymptotics, here we shall prove that the behaviour of a given solution  $u$  to (4.1.1) with a nonnegative radial initial datum  $u_0$  is related to the energy solution  $V$  of (4.1.2) having the same pole  $o$  as  $u_0$  and corresponding to a value of  $c$  that depends on  $u_0$  itself via the extinction time  $T$  of  $u$ , namely the one that satisfies

$$-\Delta V = \frac{1}{(1-m)T} V^{\frac{1}{m}} \quad \text{on } \mathbb{H}^N. \tag{4.1.4}$$

More precisely, our main result is the following.

**Theorem 4.1.1** (Convergence in relative error and convergence of derivatives). *Let  $u$  be the solution to the fast diffusion equation (4.1.1) corresponding to a non-identically zero initial datum  $u_0 \geq 0$ , which is supposed to be radial w.r.t.  $o \in \mathbb{H}^N$  and to belong to  $L^q(\mathbb{H}^N)$  for some  $q > N(1-m)/2$ , with  $q \geq 1$ . If  $T > 0$  is the extinction time of  $u$  and  $V$  is the unique positive energy solution, with pole  $o$ , to the elliptic problem (4.1.4), then*

$$\lim_{t \rightarrow T^-} \left\| \frac{u(t)}{\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}} - 1 \right\|_{\infty} = 0. \tag{4.1.5}$$

Moreover, for all  $k \in \mathbb{N}$  there holds

$$\lim_{t \rightarrow T^-} \left| \frac{u(t)}{\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}} \right|_{C^k(\mathbb{R}^+)} = 0, \quad (4.1.6)$$

where

$$|\phi|_{C^k(\mathbb{R}^+)} := \left\| \frac{\partial^k \phi}{\partial r^k} \right\|_{\infty}$$

for any regular radial function  $\phi$ . As a consequence, for all  $k \in \mathbb{N}$  there exists a smooth function  $F_k(r)$ , having the property

$$\lim_{r \rightarrow \infty} F_k(r) = 1,$$

such that

$$\lim_{t \rightarrow T^-} \left\| \frac{\frac{\partial^k u(t)}{\partial r^k}}{(-1)^k \left(\frac{N-1}{m}\right)^k \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} V^{\frac{1}{m}}} - F_k \right\|_{\infty} = 0. \quad (4.1.7)$$

**Remark 4.1.2.** Estimates (4.1.3) and (4.1.7) (together with identity (4.4.41) below) imply that, given any  $k \in \mathbb{N}$ , for all  $\epsilon > 0$  there exist  $t_\epsilon \in (0, T)$  and  $r_\epsilon > 0$  such that

$$1 - \epsilon \leq \frac{\frac{\partial^k u}{\partial r^k}(r, t)}{\left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} \frac{d^k V^{\frac{1}{m}}}{dr^k}(r)} \leq 1 + \epsilon \quad \forall r \geq r_\epsilon, \quad \forall t \in [t_\epsilon, T].$$

Notice that (4.1.7) bears some similarity with some of the results established in [124] in the Euclidean case and for  $m > 1$  (porous medium equation). See also [123] for analogous results for the Euclidean  $p$ -Laplacian driven evolution.

The method proof of Theorem 4.1.1 and the known behaviour at infinity of  $V$  and its derivatives allow us to state a *global Harnack principle* and give upper and lower bounds on derivatives of the solution.

**Theorem 4.1.3** (Global Harnack principle and bounds for derivatives). *Let the assumptions of Theorem 4.1.1 be satisfied. Then for all  $\varepsilon > 0$  there exist positive constants  $c_1 = c_1(u_0, m, N, \varepsilon)$  and  $c_2 = c_2(u_0, m, N, \varepsilon)$  such that the bound*

$$c_1 \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} e^{-\frac{N-1}{m}r} \leq u(r, t) \leq c_2 \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} e^{-\frac{N-1}{m}r} \quad (4.1.8)$$

holds true for all  $r \geq 0$  and  $t \in [\varepsilon, T)$ . Moreover, for all  $k \in \mathbb{N}$  there hold

$$\left| \frac{\partial^k u}{\partial r^k}(r, t) \right| \leq C_{1,k} \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} e^{-\frac{N-1}{m}r}, \quad (4.1.9)$$

$$\left| \frac{\partial^k u}{\partial t^k}(r, t) \right| \leq C_{2,k} \frac{e^{k(N-1)\left(\frac{1}{m}-1\right)r}}{\left(1 - \frac{t}{T}\right)^k} e^{-\frac{N-1}{m}r} \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}}, \quad (4.1.10)$$

$\forall r \geq 0, \quad \forall t \in [\varepsilon, T)$

for suitable positive constants  $C_{1,k} = C_{1,k}(u_0, m, N, \varepsilon)$  and  $C_{2,k} = C_{2,k}(u_0, m, N, \varepsilon)$ . In addition, for any  $k \in \mathbb{N}$  there exist  $\underline{t} \in (0, T)$ ,  $\underline{r} > 0$  and a suitable positive constant  $C'_k = C'_k(u_0, m, N)$  such that

$$\left| \frac{\partial^k u}{\partial r^k}(r, t) \right| \geq C'_k \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} e^{-\frac{N-1}{m}r} \quad \forall r \geq \underline{r}, \quad \forall t \in [\underline{t}, T). \quad (4.1.11)$$

Notice that our global Harnack principle (as well as the corresponding bounds for derivatives) is in the spirit of the one proved by E. DiBenedetto, Y. Kwong and V. Vespri in the fundamental paper [73] for the fast diffusion equation in bounded domains of  $\mathbb{R}^N$ , and of similar results proved by M. Bonforte and J. L. Vázquez in [34] for the fast diffusion equation in  $\mathbb{R}^N$ : in the latter case solutions can in fact be bounded above and below by special solutions of Barenblatt type. The concept of convergence in relative error was first introduced in [170], still as concerns the fast diffusion equation in  $\mathbb{R}^N$ . Later on, M. Bonforte, G. Grillo and J. L. Vázquez proved in [31] that convergence in relative error to a separable solution occurs in the case of bounded domains, thus improving on the results of [73].

It is worth pointing out that the techniques of proof of Theorems 4.1.1 and 4.1.3 can be used to capture the *spatial* behaviour of solutions, at any *fixed*  $t > 0$ , to the fast diffusion equation on  $\mathbb{H}^N$  in the *subcritical* range  $m \in (0, m_s]$  as well. Indeed, the following result holds true.

**Theorem 4.1.4** (Spatial bounds for subcritical  $m$ ). *Let  $N \geq 3$  and the assumptions of Theorem 4.1.1 be satisfied. Suppose that  $m$  lies in the subcritical range  $(0, m_s]$ . Then for any fixed  $t \in (0, T)$  there exist positive constants  $c_1 = c_1(t, u_0, m, N)$  and  $c_2 = c_2(t, u_0, m, N)$  such that the bound*

$$c_1 e^{-\frac{N-1}{m}r} \leq u(r, t) \leq c_2 e^{-\frac{N-1}{m}r} \quad (4.1.12)$$

holds true for all  $r \geq 0$ . Moreover, for all  $k \in \mathbb{N}$  the bounds

$$\left| \frac{\partial^k u}{\partial r^k}(r, t) \right| \leq C_{1,k} e^{-\frac{N-1}{m}r}, \quad (4.1.13)$$

$$\left| \frac{\partial^k u}{\partial t^k}(r, t) \right| \leq C_{2,k} e^{k(N-1)(\frac{1}{m}-1)r} e^{-\frac{N-1}{m}r}, \quad (4.1.14)$$

$$\forall r \geq 0, \quad \forall t \in (0, T)$$

hold true for suitable positive constants  $C_{1,k} = C_{1,k}(t, u_0, m, N)$  and  $C_{2,k} = C_{2,k}(t, u_0, m, N)$ .

Some words have to be said about the assumption of radiality that we require on the initial data, which is related to several technical issues. First of all, we need some *a priori* decay properties for the solution in order to exploit suitable *barrier arguments*. Such decay properties are straightforward for radial functions in  $H^1(\mathbb{H}^N)$  but need not be valid for general solutions. In the second place, it is not obvious that the solution (suitably rescaled in time) corresponding to a non-radial datum selects a *unique* asymptotic profile  $V$  along subsequences (recall the degree of freedom given by the pole  $o$ ). This was proved in [93] in the Euclidean case (bounded domains) but it is not known in the present context. Besides, in the proof of the key Lemma 4.2.3 below, the compactness of the embedding (see the forthcoming Subsection 4.1.1 for notations)

$$H_{\text{rad}}^1(\mathbb{H}^N) \hookrightarrow L_{\text{rad}}^{1+1/m}(\mathbb{H}^N)$$

is exploited in a crucial way, and such property fails in the non-radial framework. We remark that it is not even clear how to consider data which are not radial but bounded above and below by some radial data, since the extinction times of the corresponding solutions are in principle different. The existence of *ordered* radial data whose solutions have the *same* extinction time  $T > 0$  is an open problem. Should such a construction be possible, our methods of proof would readily give convergence in relative error to the separable solution extinguishing at time  $T$  also for non-radial data in between.

Finally, let us mention the very recent paper [169]. In such work, the author studies the asymptotic behaviour of solutions to the porous medium equation on  $\mathbb{H}^N$ , obtaining sharp

results for radial solutions, which show that they approach a Barenblatt-type function whose support grows in time with a logarithmic rate (in contrast with Euclidean power rate). Hence, it is remarkable that for  $m > 1$  equation (4.1.1), to some extent, is closer to the same equation in Euclidean space rather than in bounded domains.

**Remark 4.1.5.** Keeping the fundamental hypothesis of radiality, our results hold in somewhat more general geometric frameworks, but we preferred to work in the explicit case of  $\mathbb{H}^N$  to avoid bothering the reader with heavier notations and technicalities. In fact one could consider *Riemannian models* (see [20, 104] as general references and [18] for the analysis of Lane-Emden-Fowler equations in such context) whose metric is defined, in spherical coordinates about a pole  $o$ , by  $ds^2 = dr^2 + \psi^2(r)d\Theta^2$ , where  $\Theta \in \mathbb{S}^{N-1}$ ,  $\psi \in C^2([0, \infty))$  with  $\psi(0) = \psi''(0) = 0$ ,  $\psi'(0) = 1$ ,  $\psi'(r) > 0$  for every  $r > 0$  and  $\lim_{r \rightarrow \infty} \psi'(r)/\psi(r) \in (0, \infty)$ . Note that sectional curvatures at a point  $P$  tend, as the geodesic distance  $d(o, P)$  tends to  $\infty$ , to a strictly positive constant. In such a kind of manifolds a radial energy solution having the properties of the present solution  $V$  was shown to exist in [18].

### 4.1.1 Notations and preliminaries

As for the initial datum  $u_0 = u_0(r)$  in (4.1.1), in addition to its nonnegativity, in principle, we should also assume that it is bounded and such that  $u_0^m \in H_{\text{rad}}^1(\mathbb{H}^N)$ , where

$$H_{\text{rad}}^1(\mathbb{H}^N) := \left\{ v \text{ radial: } \|v\|_{H^1}^2 := \int_0^\infty v^2(s) (\sinh s)^{N-1} ds + \int_0^\infty [v'(s)]^2 (\sinh s)^{N-1} ds < \infty \right\}.$$

Notice that  $H_{\text{rad}}^1(\mathbb{H}^N)$  is precisely the space of radial functions w.r.t.  $o$  which belong to  $H^1(\mathbb{H}^N)$ . By *energy solutions* to (4.1.1) one should mean those corresponding to data  $u_0$  with  $u_0^m \in H_{\text{rad}}^1(\mathbb{H}^N)$  ( $u_0^m \in H^1(\mathbb{H}^N)$  if  $u_0$  is not radial), but in fact the results of [29] show that the solution  $u$  corresponding to an initial datum which complies with the integrability conditions of Theorem 4.1.1 automatically satisfies  $u^m(\varepsilon) \in H_{\text{rad}}^1(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$  for all  $\varepsilon > 0$ . This is stated in [29] for  $N \geq 3$ , but it holds true for  $N = 2$  as well because the methods of proof exploited in [29] only rely on the validity of a suitable Sobolev inequality (to some extent, in the same spirit as Chapter 1), which is *a fortiori* satisfied in  $H^1(\mathbb{H}^2)$ .

Now let us see what problem (4.1.1) looks like for radial solutions. Recall that the Riemannian Laplacian on hyperbolic space, for a radial function  $v = v(r)$ , reads

$$\Delta v(r) = \frac{1}{(\sinh r)^{N-1}} \left[ (\sinh r)^{N-1} v'(r) \right]' = v''(r) + (N-1)(\coth r)v'(r), \quad (4.1.15)$$

where the superscript  $'$  means derivation with respect to  $r$ . From (4.1.15) we have that studying energy solutions to (4.1.1) for radial initial data is equivalent to studying energy solutions to the problem

$$\begin{cases} u_t = (u^m)'' + (N-1)(\coth r)(u^m)' & \text{in } (0, \infty) \times (0, T), \\ (u^m)' = 0 & \text{on } \{0\} \times (0, T), \\ u = u_0 & \text{on } [0, \infty) \times \{0\}. \end{cases} \quad (4.1.16)$$

The fact that there exists a *finite* extinction time  $T > 0$  is a straightforward consequence of the validity in  $H_{\text{rad}}^1(\mathbb{H}^N)$  (in fact also in  $H^1(\mathbb{H}^N)$ ) of both a Poincaré and a Sobolev inequality (see *e.g.* [173, Section 5.10] and [26, Section 3]), that is

$$\|v\|_2 \leq C_P \|v'\|_2, \quad \|v\|_{\frac{2N}{N-2}} \leq C_S \|v'\|_2 \quad (4.1.17)$$

for all  $v \in H_{\text{rad}}^1(\mathbb{H}^N)$  and suitable positive constants  $C_P = C_P(N)$ ,  $C_S = C_S(N)$ , where

$$\|v\|_p^p := \int_0^\infty |v(s)|^p (\sinh s)^{N-1} ds, \quad L_{\text{rad}}^p(\mathbb{H}^N) := \{v \text{ radial: } \|v\|_p < \infty\}.$$

Moreover, one can prove [21, Theorem 3.1] that the embedding of  $H_{\text{rad}}^1(\mathbb{H}^N)$  in  $L_{\text{rad}}^p(\mathbb{H}^N)$  is *compact* for all  $p \in (2, 2N/(N-2))$ . Notice that, since  $m \in (m_s, 1)$ , this means in particular that

$$H_{\text{rad}}^1(\mathbb{H}^N) \xhookrightarrow{c} L_{\text{rad}}^{1+1/m}(\mathbb{H}^N), \quad (4.1.18)$$

a crucial fact that we shall exploit in the next section. Recall however that such compact embedding *fails* in  $H^1(\mathbb{H}^N)$ , another nontrivial issue that points out the advantage of working in the radial framework.

In the sequel, for notational simplicity, we shall write  $L^p$  instead of  $L^p(\mathbb{H}^N)$  and do the same with all the other functional spaces involved. Also, since we shall mostly deal with radial functions, in the discussion below we shall often replace *e.g.*  $\mathbb{H}^N$  with  $[0, \infty)$ ,  $B_r$  with  $[0, r)$  or  $o$  with 0, as long as there is no risk of ambiguity.

### 4.1.2 Outline of the chapter

The above results will be proved through several intermediate steps. Local uniform convergence of  $u^m/(1-t/T)^{m/(1-m)}$  to the energy solution  $V$  of the elliptic problem (4.1.4) is established in Section 4.2. In particular, in Subsection 4.2.1 we prove convergence on compact subsets of  $\mathbb{H}^N \setminus \{o\}$ , while in Subsection 4.2.2 we extend such result to neighbourhoods of  $\{o\}$ . Given the strict positivity of  $V$ , this ensures that (4.1.5) is satisfied at least locally. Afterwards, in Section 4.3, we prove the global Harnack principle (4.1.8), by means of suitable barriers at infinity which provide us with the correct upper and lower bounds (Subsections 4.3.1 and 4.3.2, respectively). In Section 4.4 we then deal with the proof of Theorem 4.1.1. The convergence in relative error (4.1.5) is established in Subsections 4.4.1 (where we first write the differential problem satisfied by the relative error and see it as a limit of suitable approximate problems) and 4.4.2, in fact by taking advantage of the global Harnack principle and exploiting a further delicate barrier argument. Finally, as concerns derivatives, estimates (4.1.9)–(4.1.11) and the  $C^k$  convergence results (4.1.6), (4.1.7) and are dealt with in Subsection 4.4.3. As already mentioned, Theorem 4.1.4 is just a direct consequence of the methods of proof of Theorems 4.1.1 and 4.1.3 (in this regard, see Remarks 4.3.8, 4.3.14 and 4.4.7 below).

## 4.2 Local uniform convergence to the separable solution

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As previously mentioned, each solution to (4.1.1) extinguishes in a finite time  $T > 0$ . Therefore the asymptotic behaviour of  $u$  is, from this point of view, trivial: the solution goes to zero as  $t \uparrow T$ . In order to study finer properties of  $u$  it is very useful to look for *separable solutions* to (4.1.1) (if any), so that their asymptotic behaviour might unveil at least the expected order of convergence to zero of a generic solution. To this end, let us set  $u(x, t) = g(t) V^{1/m}(x)$ . After some straightforward computations one gets that  $u$  is a solution to (4.1.1) for some  $u_0 \geq 0$  (not identically zero) if and only if

$$g(t) = \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} \quad \forall t \in [0, T] \quad (4.2.1)$$

and  $V$  is a positive solution to the elliptic problem (4.1.4) for some parameter  $T > 0$  (the extinction time). When  $m \in (m_s, 1)$  existence, uniqueness and radiality of such a  $V$  is guaranteed by compactness and by a moving plane method (in this regard, see the fundamental

paper [129]). Local regularity and strict positivity of  $V$  are then a consequence of standard elliptic arguments. So the rate of convergence to zero as  $t \uparrow T$  for separable solutions is given by (4.2.1). This suggests that, in order to analyse a nontrivial asymptotics, it is convenient to study the behaviour of the rescaled solution  $u(r, t)/g(t)$ . Notice that, if  $u(r, t) = g(t) V^{1/m}(r)$ , then such rescaled solution trivially coincides with  $V^{1/m}$ . For a generic  $u$  this is of course not true: however,  $V^{1/m}$  seems to naturally maintain the role of an attractor for  $u/g$ .

Motivated by the above discussion, given the extinction time  $T > 0$  associated with the solution  $u$  of (4.1.1), let us consider the corresponding rescaled solution  $w$  defined as

$$w(r, \tau) := \left(\frac{T}{T-t}\right)^{\frac{1}{1-m}} u(r, t) = e^{\frac{\tau}{(1-m)T}} u\left(r, T - Te^{-\frac{\tau}{T}}\right), \quad \tau := T \log\left(\frac{T}{T-t}\right), \quad (4.2.2)$$

$$\forall r \in (0, \infty), \quad \forall t \in (0, T), \quad \forall \tau \in (0, \infty).$$

Straightforward computations show that  $w$  solves the following problem:

$$\begin{cases} w_\tau = \Delta(w^m) + \frac{1}{(1-m)T} w & \text{in } (0, \infty) \times (0, \infty), \\ (w^m)' = 0 & \text{on } \{0\} \times (0, \infty), \\ w = u_0 & \text{on } [0, \infty) \times \{0\}. \end{cases} \quad (4.2.3)$$

The aim of this section is to prove that  $w^m(\tau)$  converges *locally uniformly* in  $\{r \in [0, \infty)\}$  to  $V$  as  $\tau \rightarrow \infty$  (since  $V$  is positive, this is clearly equivalent to claiming that  $w(\tau)$  converges locally uniformly to  $V^{1/m}$ ). The basic estimates one needs to exploit in order to prove such kind of result were obtained in a celebrated paper [19] by J. G. Berryman and C. J. Holland, though for regular solutions to the fast diffusion equation in regular bounded domains of  $\mathbb{R}^N$ . Here, first we shall only point out how their techniques, with minor modifications, can be applied to this framework as well. This will ensure local uniform convergence at least away from  $\{r = 0\}$  (Subsection 4.2.1). Some further work will then be required to extend the result to neighbourhoods of the pole  $o$  (Subsection 4.2.2).

#### 4.2.1 Convergence away from the pole

To our purposes, it is convenient to see  $u$  as a monotone increasing limit of the sequence of solutions  $\{u_n\}$  (with extinction times  $\{T_n\}$ ) to the problems

$$\begin{cases} (u_n)_t = \Delta(u_n^m) & \text{in } (0, n) \times (0, T_n), \\ u_n = 0 & \text{on } \{n\} \times (0, T_n), \\ (u_n^m)' = 0 & \text{on } \{0\} \times (0, T_n), \\ u_n = u_{0n} & \text{on } [0, n] \times \{0\}, \end{cases} \quad (4.2.4)$$

where  $\{u_{0n}\}$  is a sequence of regular data such that  $u_{0n}(n) = 0$ ,  $(u_{0n}^m)'(0) = 0$ ,  $u_{0n} \leq u_0$ , which suitably approximates  $u_0$ , and  $T_n \uparrow T$ . We shall identify  $u_n(\cdot, t)$  as functions in the whole of  $[0, \infty)$  by setting them to zero outside  $[0, n]$ . Notice that (4.2.4) corresponds to the radial fast diffusion equation with homogeneous Dirichlet boundary conditions posed on the ball of radius  $n$  of  $\mathbb{H}^N$  centred at  $x = o$ .

**Lemma 4.2.1.** *There exists a positive constant  $C = C(m, N)$  such that*

$$C(T-t)^{\frac{1}{1-m}} \leq \|u(t)\|_{m+1} \leq \left(1 - \frac{t}{T}\right)^{\frac{1}{1-m}} \|u_0\|_{m+1} \quad \forall t \in (0, T). \quad (4.2.5)$$



Moreover, the ratio

$$t \mapsto \frac{\|(u^m)'(t)\|_2}{\|u(t)\|_{m+1}^m} \quad (4.2.6)$$

is nonincreasing along the evolution.

*Proof.* The left inequality in (4.2.5) can be proved exactly as in [19, Lemma 1] using the identity

$$\frac{d}{dt} \int_0^\infty u^{m+1}(s, t) (\sinh s)^{N-1} ds = -(m+1) \int_0^\infty [(u^m)'(s, t)]^2 (\sinh s)^{N-1} ds \quad (4.2.7)$$

and the Poincaré-Sobolev inequalities in (4.1.17). To justify the the other statements we proceed as in [19, Lemma 2], outlining the main steps only. At a formal level, we have:

$$\frac{\int_0^\infty [(u^m)'(s, t)]^2 (\sinh s)^{N-1} ds}{\int_0^\infty u^{m+1}(s, t) (\sinh s)^{N-1} ds} \leq \frac{\int_0^\infty u^{m-1}(s, t) [\Delta(u^m)(s, t)]^2 (\sinh s)^{N-1} ds}{\int_0^\infty [(u^m)'(s, t)]^2 (\sinh s)^{N-1} ds}, \quad (4.2.8)$$

$$\frac{d}{dt} \int_0^\infty [(u^m)'(s, t)]^2 (\sinh s)^{N-1} ds = -2m \frac{\int_0^\infty u^{m-1}(s, t) [\Delta(u^m)(s, t)]^2 (\sinh s)^{N-1} ds}{\int_0^\infty [(u^m)'(s, t)]^2 (\sinh s)^{N-1} ds}, \quad (4.2.9)$$

where (4.2.8) follows from integration by parts and Cauchy-Schwarz inequality. From (4.2.7)–(4.2.9) one easily shows, exactly as in [19, Lemma 2], that the ratio in (4.2.6) is nonincreasing. Thanks to this, the right inequality in (4.2.5) also follows as in [19, Lemma 2, formula (14)].

To justify such steps it is convenient to exploit the approximate solutions  $\{u_n\}$ . Indeed the proof of [19, Lemma 2] requires the finiteness of the quantity

$$\int_0^\infty u^{m-1}(s, t) [\Delta(u^m)(s, t)]^2 (\sinh s)^{N-1} ds$$

for all  $t \in (0, T)$ , which *a priori* may not hold here. However, one obtains (4.2.5) and (4.2.6) for  $u_n$  (we can assume that  $u_n^m$  is regular enough up to the boundary) and then passes to the limit as  $n \rightarrow \infty$ . This is feasible since  $\{u_n^m(t)\}$  converges weakly in  $H_{\text{rad}}^1$  to  $u^m(t)$ , and by monotonicity  $\{u_n(t)\}$  converges to  $u(t)$  in  $L_{\text{rad}}^{m+1}$  and  $\{T_n\}$  converges to  $T$ .  $\square$

The next result is key in order to establish the mentioned convergence of  $w^m$  to the stationary profile  $V$ .

**Lemma 4.2.2.** *The following inequality holds true for all  $\tau \in (0, \infty)$ :*

$$\begin{aligned} & \int_0^\infty \left[ \frac{1}{2} [(w^m)'(s, \tau)]^2 - \frac{m}{(1-m^2)T} w^{m+1}(s, \tau) \right] (\sinh s)^{N-1} ds \\ & + m \int_0^\tau \int_0^\infty w^{m-1}(s, \sigma) [w_\tau(s, \sigma)]^2 (\sinh s)^{N-1} ds d\sigma \\ & \leq \int_0^\infty \left[ \frac{1}{2} [(u_0^m)'(s)]^2 - \frac{m}{(1-m^2)T} u_0^{m+1}(s) \right] (\sinh s)^{N-1} ds. \end{aligned} \quad (4.2.10)$$

*Proof.* For the smooth rescaled solutions  $w_n$  inequality (4.2.10) is in fact an equality, since by straightforward computations one verifies that

$$\begin{aligned} & \frac{d}{d\tau} \int_0^\infty \left[ \frac{1}{2} [(w_n^m)'(s, \tau)]^2 - \frac{m}{(1-m^2)T_n} w_n^{m+1}(s, \tau) \right] (\sinh s)^{N-1} ds \\ & = -m \int_0^\infty w_n^{m-1}(s, \tau) [(w_n)_\tau(s, \tau)]^2 \chi_{(0, n)}(s) (\sinh s)^{N-1} ds. \end{aligned} \quad (4.2.11)$$

In order to get estimate (4.2.10) it suffices to integrate (4.2.11) from 0 to  $\tau$  and let  $n \rightarrow \infty$ : to the first integral on the l.h.s. of (4.2.10) we can apply the weak convergence of  $\{u_n^m(t)\}$  to  $u^m(t)$  in  $H_{\text{rad}}^1$  and the strong convergence of  $\{u_n(t)\}$  to  $u(t)$  in  $L_{\text{rad}}^{m+1}$ , while the second integral is handled by means of Fatou's Lemma (thanks to local regularity we can assume that, up to subsequences,  $\{(u_n)_t\}$  converges pointwise to  $u_t$ ) or by the fact that

$$u_n^{\frac{m+1}{2}} \rightarrow u^{\frac{m+1}{2}} \quad \text{weakly in } H^1((0, T); L_{\text{rad}}^2(\mathbb{H}^N)).$$

□

We can now prove the claimed local uniform convergence away from  $o$ .

**Lemma 4.2.3.** *Let  $w$  be the rescaled solution (4.2.2) to (4.2.3) and  $V$  be the radial, positive energy solution to the elliptic problem (4.1.4). Then*

$$\lim_{\tau \rightarrow \infty} \|w^m(\tau) - V\|_{L_{\text{loc}}^\infty((0, \infty))} = 0, \quad (4.2.12)$$

that is  $w^m(\tau)$  converges uniformly to  $V$  in any compact set  $K \Subset (0, \infty)$  as  $\tau \rightarrow \infty$ .

*Proof.* We adapt the proof of [19, Theorem 2]. First of all notice that, from (4.2.10), one deduces the existence of a sequence  $\{\tau_n\} \rightarrow \infty$  such that

$$\int_0^\infty w^{m-1}(s, \tau_n) [w_\tau(s, \tau_n)]^2 (\sinh s)^{N-1} ds \rightarrow 0. \quad (4.2.13)$$

To show this fact notice that, thanks to (4.2.5) and (4.2.2),  $\|w(\tau)\|_{m+1}$  is bounded as a function of  $\tau$ , hence the first integral on the l.h.s. of (4.2.10) is bounded from below. Moreover, the r.h.s. does not depend on  $\tau$ , therefore the integral

$$\int_0^\infty \int_0^\infty w^{m-1}(s, \sigma) [w_\tau(s, \sigma)]^2 (\sinh s)^{N-1} ds d\sigma$$

must be finite. Still from (4.2.10) and (4.2.5) one gets the boundedness of  $\|w^m(\tau_n)\|_{H^1}$ ; hence, up to subsequences,  $\{w^m(\tau_n)\}$  converges weakly in  $H_{\text{rad}}^1$  to a certain function  $R$ . From the compact embedding (4.1.18), such convergence is in fact *strong* in  $L_{\text{rad}}^{1+1/m}$ . In particular,  $R$  is a nonnegative non-identically zero function (indeed (4.2.5) prevents  $\|w^m(\tau_n)\|_{1+1/m}$  from going to zero) belonging to  $H_{\text{rad}}^1$ .

The next step is to show that  $R$  solves (4.1.4). To this end, take any test function  $\phi : [0, \infty) \mapsto \mathbb{R}$  with compact support in  $[0, \infty)$ , multiply by it the first equation in (4.2.3) (evaluated at  $\tau = \tau_n$ ) and integrate by parts in  $[0, \infty)$ . This leads to the identity

$$\begin{aligned} \int_0^\infty w_\tau(s, \tau_n) \phi(s) (\sinh s)^{N-1} ds &= - \int_0^\infty (w^m)'(s, \tau_n) \phi'(s) (\sinh s)^{N-1} ds \\ &+ \int_0^\infty \frac{1}{(1-m)T} w(s, \tau_n) \phi(s) (\sinh s)^{N-1} ds. \end{aligned} \quad (4.2.14)$$

The two integrals in the r.h.s. of (4.2.14) are stable under passage to the limit as  $n \rightarrow \infty$ : indeed  $\{w^m(\tau_n)\}$  converges weakly in  $H_{\text{rad}}^1$  to  $R$  and  $\{w(\tau_n)\}$  converges strongly in  $L_{\text{rad}}^{m+1}$  to  $R^{1/m}$  (and so also in  $L_{\text{rad}}^1$  locally). Finally, the left-hand side goes to zero since its modulus is bounded by

$$\left( \int_0^\infty w^{1-m}(s, \tau_n) \phi^2(s) (\sinh s)^{N-1} ds \right)^{\frac{1}{2}} \left( \int_0^\infty w^{m-1}(s, \tau_n) [w_\tau(s, \tau_n)]^2 (\sinh s)^{N-1} ds \right)^{\frac{1}{2}},$$

which goes to zero thanks to (4.2.13) and to the boundedness of  $\|w(\tau_n)\|_{m+1}$ . From the arbitrariness of  $\phi$  and the uniqueness of energy solutions to (4.1.4) we infer that  $R$  must coincide with  $V$ . Moreover, the convergence of  $\{w^m(\tau_n)\}$  to  $R = V$  in  $H_{\text{rad}}^1$  is also strong. To prove that, just replace  $\phi$  with  $w^m(\tau_n)$  in the computations above to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty [(w^m)'(s, \tau_n)]^2 (\sinh s)^{N-1} ds &= \int_0^\infty \frac{1}{(1-m)T} V^{\frac{m+1}{m}}(s) (\sinh s)^{N-1} ds \\ &= \int_0^\infty [V'(s)]^2 (\sinh s)^{N-1} ds. \end{aligned}$$

Hence, weak convergence plus convergence of the norms in  $H_{\text{rad}}^1$  gives the claimed strong convergence. Since  $H_{\text{rad}}^1$  is continuously embedded in  $L_{\text{loc}}^\infty$  (see *e.g.* Lemma 4.3.2 below), we have proved (4.2.12) along the special sequence  $\{\tau_n\}$ . To prove that such convergence takes place along any other subsequence, one can argue by contradiction. That is, suppose there exists a sequence  $\{\tau_k\}$  such that  $\{w^m(\tau_k)\}$  does not converge strongly in  $H_{\text{rad}}^1$  to  $V$ . By (4.2.10) we can assume that, up to subsequences,  $\{w^m(\tau_k)\}$  converges weakly in  $H_{\text{rad}}^1$  to a certain function  $Q$ . Now note that, again from (4.2.5) and (4.2.10) (up to a shift of the time origin), both

$$\tau \mapsto \|w(\tau)\|_{m+1}^{m+1}$$

and

$$\tau \mapsto \frac{1}{2} \|(w^m)'(\tau)\|_2^2 - \frac{m}{(1-m^2)T} \|w(\tau)\|_{m+1}^{m+1}$$

are nonincreasing functions of  $\tau$ . In particular,

$$\|Q\|_{\frac{m+1}{m}} = \lim_{k \rightarrow \infty} \|w(\tau_k)\|_{m+1}^m = \lim_{n \rightarrow \infty} \|w(\tau_n)\|_{m+1}^m = \|V\|_{\frac{m+1}{m}}^m \quad (4.2.15)$$

and

$$\|Q'\|_2^2 \leq \liminf_{k \rightarrow \infty} \left( \|(w^m)'(\tau_k)\|_2^2 \pm \frac{m}{(1-m^2)T} \|w(\tau_k)\|_{m+1}^{m+1} \right) = \lim_{n \rightarrow \infty} \|(w^m)'(\tau_n)\|_2^2 = \|V'\|_2^2. \quad (4.2.16)$$

Since  $V$  is the *unique* minimizer of  $\|v'\|_2^2$  among all functions  $v$  with prescribed  $L_{\text{rad}}^{1+1/m}$  norm (see [129]), (4.2.15) and (4.2.16) necessarily imply that  $Q = V$ . *Strong* convergence of  $\{w^m(\tau_k)\}$  to  $Q = V$  is then a consequence of (4.2.16), which leads to a contradiction.  $\square$

#### 4.2.2 Convergence in a neighbourhood of the pole

We are left with proving that the local uniform convergence (4.2.12) takes place also down to  $r = 0$ . In order to do that, we shall use Lemma 4.2.3 and the following two lemmas, which show how positivity and boundedness of  $w$  can be extended to a neighbourhood of  $o$ .

**Lemma 4.2.4.** *For any  $\epsilon > 0$  there exist  $r_\epsilon > 0$  small enough and  $\tau_\epsilon > 0$  large enough such that*

$$w^m(r, \tau) \geq V(0) - \epsilon \quad \forall (r, \tau) \in [0, r_\epsilon] \times [\tau_\epsilon, \infty). \quad (4.2.17)$$

*Proof.* We can adapt the techniques of proof of [73, Lemma 6.2]. First of all recall that, thanks to the local uniform convergence to the stationary profile (4.2.12), we have uniform boundedness away from zero in any compact set which does not contain  $o$ . In particular, consider a point  $x_0 \in \mathbb{H}^N$  such that  $r_0 := d(x_0, o) \in (0, 1/2)$ . For a given  $\tau_0 > 0$ , set

$$k := \inf_{(x, \tau): \tau \geq \tau_0, x \in B_{r_0/2}(x_0)} w(d(x, o), \tau), \quad (4.2.18)$$

where  $B_{r_0/2}(x_0)$  is the hyperbolic ball of radius  $r_0/2$  centred at  $x_0$ . Thanks to the observations above,  $k > 0$  provided  $\tau_0$  is sufficiently large. Let us consider equation (4.2.3) (more precisely, its interpretation as a differential equation on  $\mathbb{H}^N$ ) centred at  $x_0$  in place of  $o$ . To avoid confusion, we shall call  $\rho$  the radial coordinate about such  $x_0$ . Upon defining

$$\widetilde{N} := 1 + (N - 1) \sup_{\rho \in (0,1)} \rho \coth(\rho),$$

for any function  $f = f(\rho)$  such that  $f'(\rho) \leq 0$  we have

$$\Delta f(\rho) = f''(\rho) + (N - 1) \coth(\rho) f'(\rho) \geq f''(\rho) + \frac{\widetilde{N} - 1}{\rho} f'(\rho) \quad \forall \rho \in (0, 1), \quad (4.2.19)$$

where the term in the r.h.s. of (4.2.19) is the Euclidean Laplacian of  $f$  associated with the ‘‘artificial’’ dimension  $\widetilde{N}$ . In order to seek a *subsolution*  $\psi(\rho, \tau)$  to (4.2.3) centred at  $x_0$  it is therefore enough to require that (we keep denoting as  $'$  derivative w.r.t.  $\rho$ )

$$\psi_\tau \leq (\psi^m)'' + \frac{\widetilde{N} - 1}{\rho} (\psi^m)', \quad \psi \geq 0, \quad (\psi^m)' \leq 0, \quad (4.2.20)$$

as long as  $\rho$  varies in  $(0, 1)$ . The proof of Lemma 6.2 of [73] ensures that the function

$$\psi(\rho, \tau) = k \frac{(1 - \rho^\beta)^{\frac{2}{m}}}{\left(1 + k^{1-m} \frac{b\rho^2}{\tau - \tau_0}\right)^{\frac{\theta}{1-m}}}$$

satisfies (4.2.20) in the region

$$\left\{ (\rho, \tau) \in \left(\frac{r_0}{2}, 1\right) \times \left(\tau_0, \tau_0 + \frac{r_0^2}{4}\right) \right\}$$

upon choosing appropriately the positive parameters  $\beta = \beta(m, \widetilde{N})$ ,  $\theta = \theta(m, \widetilde{N})$  and  $b = b(m, \widetilde{N}, k)$ ,  $k$  being as in (4.2.18). Let us check conditions on the parabolic boundary. For  $\tau = \tau_0$  and for  $\rho = 1$  we have, by construction,  $\psi \equiv 0$  and so trivially  $\psi \leq w$ . For  $\rho = r_0/2$  (actually for any  $\rho \in (0, 1)$ ) there holds  $\psi \leq k$ , from which  $\psi \leq w$  by definition of  $k$ . Hence, by comparison,

$$\begin{aligned} w(d(x, o), \tau_0 + r_0^2/4) &\geq \psi(d(x, x_0), \tau_0 + r_0^2/4) \geq \psi(3r_0/2, \tau_0 + r_0^2/4) = k \frac{\left(1 - \left(\frac{3r_0}{2}\right)^\beta\right)^{\frac{2}{m}}}{\left(1 + 9bk^{1-m}\right)^{\frac{\theta}{1-m}}} \\ &=: C_0 > 0 \end{aligned}$$

$$\forall x \in \mathbb{H}^N : \frac{r_0}{2} \leq d(x, x_0) \leq \frac{3r_0}{2}.$$

In particular we obtain the existence of a radius  $r_1 > 0$  and a time  $\tau_1 > 0$  such that

$$w(r, \tau) \geq C_0 > 0 \quad \forall (r, \tau) \in [0, r_1] \times \{\tau_1\}. \quad (4.2.21)$$

Indeed (4.2.21) holds for *all*  $\tau$  larger than  $\tau_1$ , rather than only for  $\tau = \tau_1$ . This is a trivial consequence of the fact that  $C_0$  is a subsolution to (4.2.3) (the comparison condition on the lateral boundary  $\{r_1\} \times (\tau_1, \infty)$  is satisfied provided  $C_0$  is small enough, again as a consequence of the local uniform convergence (4.2.12)).

Finally, we need to refine estimate (4.2.21). To this end, just observe that the function

$$g(\tau) := C_0 e^{\frac{\tau - \tau_*}{(1-m)T}}$$

is a solution to the differential equation in (4.2.3) for any  $\tau_* > 0$ . Still from the local uniform convergence (4.2.12) (and from the fact that  $V(\cdot)$  is decreasing) we have that, for any  $\epsilon > 0$ , we can choose  $r_2 = r_2(\epsilon) < r_1$  and  $\tau_2 = \tau_2(\epsilon) > \tau_1$  such that  $w^m(r_2, \tau) \geq V(0) - \epsilon$  for all  $\tau \geq \tau_2$ . Therefore  $g(\tau)$ , with the choice  $\tau_* = \tau_2$ , is a subsolution to (4.2.3) in the region

$$\{(r, \tau) \in (0, r_2) \times (\tau_2, \tau_3)\},$$

where  $\tau_3$  is the time at which  $g^m(\tau_3) = V(0) - \epsilon$ . Since the constant  $(V(0) - \epsilon)^{1/m}$  is then a subsolution in  $\{(r, \tau) \in (0, r_2) \times (\tau_3, \infty)\}$ , estimate (4.2.17) follows.  $\square$

Now we prove the analogue of (4.2.17) from above.

**Lemma 4.2.5.** *For any  $\epsilon > 0$  there exist  $r_\epsilon > 0$  sufficiently small and  $\tau_\epsilon > 0$  sufficiently large such that*

$$w^m(r, \tau) \leq V(0) + \epsilon \quad \forall (r, \tau) \in [0, r_\epsilon] \times [\tau_\epsilon, \infty). \quad (4.2.22)$$

*Proof.* Again, we shall proceed by constructing a proper *supersolution* to (4.2.3). To begin with, let  $\alpha$  and  $\varepsilon$  two small positive parameters. Our aim is first to obtain a suitable estimate for  $\Delta(V(r/\alpha))$  in the region  $\{r \leq \alpha\varepsilon\}$ . We have:

$$\Delta(V(r/\alpha)) = -\frac{1}{\alpha^2(1-m)T} V^{1/m}(r/\alpha) + \frac{N-1}{\alpha^2} (\alpha \coth(r) - \coth(r/\alpha)) V'(r/\alpha). \quad (4.2.23)$$

The function  $h(r) := r \coth(r)$  is regular and satisfies

$$h(r) = 1 + h'(0)r + q(r)r^2, \quad h(r/\alpha) = 1 + h'(0)\frac{r}{\alpha} + q(r/\alpha)\left(\frac{r}{\alpha}\right)^2, \quad (4.2.24)$$

where both  $|q(r)|$  and  $|q(r/\alpha)|$  can be bounded by

$$Q := \max_{s \in [0,1]} \frac{h''(s)}{2}$$

provided  $\alpha$  and  $\varepsilon$  are smaller than 1. In order to control the right term in the r.h.s. of (4.2.23), we use (4.2.24):

$$\begin{aligned} \left| \frac{\alpha \coth(r) - \coth(r/\alpha)}{\alpha^2} \right| &= \left| \frac{r \coth(r) - \frac{r}{\alpha} \coth(r/\alpha)}{\alpha r} \right| = \left| \frac{h'(0)}{\alpha} + q(r)\frac{r}{\alpha} - \frac{h'(0)}{\alpha^2} - q(r/\alpha)\frac{r}{\alpha^3} \right| \\ &\leq \frac{C}{\alpha^2} \end{aligned}$$

for a suitable constant  $C > 0$  independent of  $\alpha$  and  $\varepsilon$ . Notice that, since  $V$  is regular and  $V'(0) = 0$ , there exists  $D > 0$  (independent of  $\alpha, \varepsilon$ ) such that  $|V'(r/\alpha)| \leq D\varepsilon$  for all  $r \leq \alpha\varepsilon$ . Hence,

$$\left| \frac{N-1}{\alpha^2} (\alpha \coth(r) - \coth(r/\alpha)) V'(r/\alpha) \right| \leq \frac{(N-1)CD\varepsilon}{\alpha^2}. \quad (4.2.25)$$

Then recall that  $V(\cdot)$  is decreasing and  $V(0) > 0$ , so that from (4.2.23) and (4.2.25) we can claim that there exists a constant  $E > 0$  such that for any  $\varepsilon > 0$  sufficiently small (depending only on  $V, m$  and  $N$ ) there holds

$$\Delta(V(r/\alpha)) \leq -\frac{E}{\alpha^2 T} \quad \forall r \in (0, \alpha\varepsilon). \quad (4.2.26)$$

From  $L^q$ - $L^\infty$  smoothing effects (see [73, Lemma 6.1] or [29, Theorem 4.1], together with (4.2.5)) we know that there exist  $A > 0$  and  $\tau_0 > 0$  such that  $w(r, \tau) \leq A$  for all  $(r, \tau) \in (0, \infty) \times (\tau_0, \infty)$ . Let  $A_1 > A_2$  be two given positive constants and let  $\varepsilon > 0$  be so small that (4.2.26) holds. For a fixed  $\tau_* \geq \tau_0 + 1$  set  $f(\tau) := (\tau - \tau_0)/(\tau_* - \tau_0)$ . First we shall prove that if  $\alpha > 0$  is small enough then the function

$$\varphi(r, \tau) := [A_1(1 - f(\tau)) + A_2f(\tau)]^{\frac{1}{m}} V(r/\alpha)^{\frac{1}{m}}$$

is a *supersolution* to (4.2.3) in the region

$$\{(r, \tau) \in (0, \alpha\varepsilon) \times (\tau_0, \tau_*)\}. \quad (4.2.27)$$

To this end, notice that

$$\varphi(r, \tau_0) = A_1^{\frac{1}{m}} V^{\frac{1}{m}}(r/\alpha) \geq A_1^{\frac{1}{m}} V^{\frac{1}{m}}(\varepsilon) \quad \forall r \in (0, \alpha\varepsilon), \quad \varphi(\alpha\varepsilon, \tau) \geq A_2^{\frac{1}{m}} V^{\frac{1}{m}}(\varepsilon) \quad \forall \tau \in (\tau_0, \tau_*), \quad (4.2.28)$$

while derivatives of  $\varphi$  entail

$$\begin{aligned} \varphi_\tau(r, \tau) &= -\frac{1}{m} f'(\tau) V^{\frac{1}{m}}(r/\alpha) (A_1 - A_2) [A_1(1 - f(\tau)) + A_2f(\tau)]^{\frac{1}{m}-1} \\ &\geq -\frac{1}{m} (A_1 - A_2) A_1^{\frac{1}{m}-1} V^{\frac{1}{m}}(0), \end{aligned} \quad (4.2.29)$$

$$\Delta(\varphi^m)(r, \tau) = [A_1(1 - f(\tau)) + A_2f(\tau)] \Delta(V(r/\alpha)) \leq -\frac{A_2E}{\alpha^2T}$$

and

$$\varphi(r, \tau) \leq A_1^{\frac{1}{m}} V^{\frac{1}{m}}(0). \quad (4.2.30)$$

In view of (4.2.29) and (4.2.30), we get that for  $\varphi$  to be a supersolution in the region (4.2.27) it is enough to require

$$-\frac{1}{m} (A_1 - A_2) A_1^{\frac{1}{m}-1} V^{\frac{1}{m}}(0) \geq -\frac{A_2E}{\alpha^2T} + \frac{A_1^{\frac{1}{m}} V^{\frac{1}{m}}(0)}{(1 - m)T}, \quad (4.2.31)$$

which is achieved by choosing  $\alpha = \alpha(V, T, m, N, A_1, A_2)$  sufficiently close to zero.

Now fix  $\varepsilon > 0$  small enough. Set  $A_1 = A^m/V(\varepsilon)$  and  $A_2 = V(0)/V(\varepsilon)$ , where we assume without loss of generality that  $A^m > V(0)$  and pick  $\alpha = \alpha(V, T, m, N, A, \varepsilon)$  complying with (4.2.31). Thanks to (4.2.28) we have

$$\varphi(r, \tau_0) \geq A \quad \forall r \in [0, \alpha\varepsilon], \quad \varphi(\alpha\varepsilon, \tau) \geq V^{\frac{1}{m}}(0) \quad \forall \tau \in [\tau_0, \tau_*]. \quad (4.2.32)$$

From the fact that  $V(\cdot)$  is decreasing and from the local uniform convergence (4.2.12), we can take  $\tau_0$  so large that  $w^m(\alpha\varepsilon, \tau) \leq V(0)$  for all  $\tau \geq \tau_0$  (notice that  $\varphi$  is a supersolution independently of  $\tau_0, \tau_*$  provided  $\tau_* - \tau_0 \geq 1$ ). Since (4.2.32) holds we can conclude, by comparison, that  $w \leq \varphi$  in the region (4.2.27). In particular,

$$w^m(r, \tau_*) \leq \frac{V^2(0)}{V(\varepsilon)} \quad \forall r \in [0, \alpha\varepsilon].$$

By the remarks above this last result is actually valid for *all*  $\tau_* \geq \tau_0 + 1$ . Hence, since  $V(\varepsilon) \rightarrow V(0) > 0$  as  $\varepsilon \rightarrow 0$ , we conclude that for any  $\epsilon > 0$  there exist  $r_\epsilon$  so small and  $\tau_\epsilon$  so large that (4.2.22) holds true.  $\square$

Lemmas 4.2.4 and 4.2.5 then allow us extend the result of Lemma 4.2.3 down to  $r = 0$ .

**Proposition 4.2.6.** *Let  $w$  be the rescaled solution (4.2.2) to (4.2.3) and  $V$  be the radial, positive energy solution to the elliptic problem (4.1.4). Then*

$$\lim_{\tau \rightarrow \infty} \|w^m(\tau) - V\|_{L^\infty_{\text{loc}}([0, \infty))} = 0, \quad (4.2.33)$$

that is  $w^m(\tau)$  converges uniformly to  $V$  in any compact set  $K \Subset [0, \infty)$  as  $\tau \rightarrow \infty$ .

### 4.3 The global Harnack principle

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The goal of this section is to bound the ratio  $w^m(\tau)/V$  in  $L^\infty((0, \infty))$  (and not only in  $L^\infty_{\text{loc}}([0, \infty))$  as we did in Section 4.2), namely to prove the global Harnack principle (4.1.8), a crucial ingredient in the proof of Theorem 4.1.1. Since  $V(r)$  behaves like  $e^{-(N-1)r}$  at infinity (see (4.1.3) or Lemma 4.3.3 below), it will be enough to give bounds over

$$\frac{w^m(r, \tau)}{e^{-(N-1)r}}. \quad (4.3.1)$$

#### 4.3.1 Estimates from above

Our first aim is to obtain the correct estimate from above for  $w$ .

**Proposition 4.3.1.** *Let  $w$  be the rescaled solution corresponding to a nonnegative energy solution  $u$  to (4.1.16), as in (4.2.2). Then, for any  $\varepsilon > 0$ , there exists a positive constant  $Q' = Q'(u_0, m, N, \varepsilon)$  such that*

$$w(r, \tau) \leq Q' e^{-\frac{N-1}{m}r} \quad \forall (r, \tau) \in [0, \infty) \times [\varepsilon, \infty). \quad (4.3.2)$$

In order to prove Proposition 4.3.1, we need some preliminary lemmas.

**Lemma 4.3.2.** *Let  $v \in H^1_{\text{rad}}$ . For any  $r_0 > 0$  there exists a positive constant  $C(r_0, N)$  such that*

$$v(r) \leq C(r_0, N) \|v\|_{H^1} e^{-\frac{N-1}{2}r} \quad \forall r \in [r_0, \infty).$$

*Proof.* Consider the function  $z(r) := (\sinh r)^{N-1} v^2(r)$ . We have:

$$z'(r) = (N-1)(\cosh r)(\sinh r)^{N-2} v^2(r) + 2(\sinh r)^{N-1} v(r)v'(r);$$

integrating between  $r$  and  $r_0$  we end up with

$$\begin{aligned} z(r) &= (\sinh r_0)^{N-1} v^2(r_0) + (N-1) \int_{r_0}^r v^2(s) (\cosh s) (\sinh s)^{N-2} ds \\ &\quad + 2 \int_{r_0}^r v(s)v'(s) (\sinh s)^{N-1} ds. \end{aligned} \quad (4.3.3)$$

Since  $r_0 > 0$  and the behaviour at infinity of  $\sinh r$  and  $\cosh r$  is the same, we can control the last two terms in the r.h.s. of (4.3.3) with a constant (depending on  $r_0$  and  $N$ ) times  $\|v\|_{H^1}^2$ . As for the first term, notice that  $H^1((r_0/2, 3r_0/2))$  is continuously embedded in  $L^\infty((r_0/2, 3r_0/2))$  and  $H^1_{\text{rad}}$  is in turn continuously embedded in  $H^1((r_0/2, 3r_0/2))$  (again, through constants depending on  $r_0$  and  $N$ ). Hence, there exists  $C(r_0, N) > 0$  such that

$$z(r) \leq C^2(r_0, N) \|v\|_{H^1}^2 \quad \forall r \in [r_0, \infty),$$

which yields the claimed result since  $\sinh r \approx e^r/2$  for  $r$  large.  $\square$

**Lemma 4.3.3.** *For any  $m_0 \in (m_s, 1)$  there exists a solution  $V$  to*

$$-\Delta V(r) = V^{\frac{1}{m_0}}(r) \quad \forall r \in (0, \infty) \quad (4.3.4)$$

*which is smooth, strictly positive, belongs to  $H_{\text{rad}}^1$  and satisfies*

$$A^{-1} e^{-(N-1)r} \leq V(r) \leq A e^{-(N-1)r} \quad \forall r \in [0, \infty) \quad (4.3.5)$$

*for some positive constant  $A = A(m_0)$ .*

*Proof.* As recalled in the Introduction, see e.g. [129] and [26].  $\square$

**Lemma 4.3.4.** *Let  $u$  be a bounded, nonnegative energy solution to (4.1.16). There exists a positive constant  $C_0 = C_0(u_0, m, N)$  such that*

$$u(r, t) \leq C_0 e^{-\frac{N-1}{2m}r} \quad \forall (r, t) \in [0, \infty) \times (0, T). \quad (4.3.6)$$

*Proof.* It is a matter of straightforward computations to show that

$$-\Delta \left( e^{-\frac{N-1}{2}r} \right) \geq 0 \quad \forall r \in (0, \infty). \quad (4.3.7)$$

Thanks to Lemma 4.3.2, to the fact that  $u_0^m \in H_{\text{rad}}^1$  and to the boundedness of  $u$ , one can choose  $C_0$  so large that  $C_0 e^{-\frac{N-1}{2m}r}$  is above  $u$  on a parabolic boundary of the type  $[r_0, \infty) \times \{0\} \cup \{r_0\} \times (0, T)$ , for a given  $r_0 \in (0, \infty)$ . The conclusion then follows from (4.3.7), the comparison principle (which is standard here) and again the boundedness of  $u$ .  $\square$

We are now ready to prove a better (spatial) estimate from above for nonnegative energy solutions to (4.1.16).

**Lemma 4.3.5.** *Let  $u$  be a nonnegative energy solution to (4.1.16). For any  $t_* \in (0, T)$  there exists a positive constant  $Q = Q(t_*, u_0, m, N)$  such that*

$$u(r, t_*) \leq Q e^{-\frac{N-1}{m}r} \quad \forall r \in [0, \infty). \quad (4.3.8)$$

*Proof.* We shall construct a proper barrier. In particular, we shall prove that for a suitable choice of the parameter  $\xi > 0$ , the following function is a supersolution to (4.1.16) in the parabolic domain  $(\xi, \infty) \times (0, t_*)$ :

$$\bar{u}(r, t) := C_0 \left[ A e^{\frac{N-1}{2}\xi} V(r) f(t) + e^{-\frac{N-1}{2}r} (1 - f(t)) \right]^{\frac{1}{m}}, \quad (4.3.9)$$

where  $V$  is the solution to (4.3.4) associated with a fixed  $m_0 \in (2m/(1+m), 1)$  ( $A$  being the corresponding constant that appears in (4.3.5)) and  $f(t) : [0, t_*] \mapsto [0, 1]$  is a regular increasing function such that  $f(0) = 0$  and  $f(t_*) = 1$ , which we shall define later. The constant  $C_0$  is the one that appears in (4.3.6): indeed, thanks to smoothing effects (recall the brief discussion in Subsection 4.1.1), there is no loss of generality in assuming that  $u_0$  is bounded. Since

$$\bar{u}(r, 0) = C_0 e^{-\frac{N-1}{2m}r} \geq u_0(r) \quad \forall r \in [0, \infty)$$

and

$$\bar{u}(\xi, t) = C_0 \left[ A e^{\frac{N-1}{2}\xi} V(\xi) f(t) + e^{-\frac{N-1}{2}\xi} (1 - f(t)) \right]^{\frac{1}{m}} \geq C_0 e^{-\frac{N-1}{2m}\xi} \geq u(\xi, t) \quad \forall t \in [0, t_*],$$

in order to prove that  $u(r, t) \leq \bar{u}(r, t)$  for all  $(r, t) \in [\xi, \infty) \times [0, t_*]$  we are left with showing that, by suitably choosing  $\xi$ , there holds

$$\bar{u}_t(r, t) \geq \Delta(\bar{u}^m)(r, t) \quad \forall (r, t) \in (\xi, \infty) \times (0, t_*).$$



We have:

$$\begin{aligned}\bar{u}_t(r, t) &= C_0 f'(t) \left[ A e^{\frac{N-1}{2}\xi} V(r) - e^{-\frac{N-1}{2}r} \right] \frac{1}{m} \left[ A e^{\frac{N-1}{2}\xi} V(r) f(t) + e^{-\frac{N-1}{2}r} (1 - f(t)) \right]^{\frac{1}{m}-1} \\ &\geq -C_0 \frac{1}{m} f'(t) e^{-\frac{N-1}{2}r} \left[ A^2 e^{\frac{N-1}{2}\xi} e^{-(N-1)r} f(t) + e^{-\frac{N-1}{2}r} (1 - f(t)) \right]^{\frac{1}{m}-1}.\end{aligned}$$

Moreover, thanks to (4.3.4),

$$\begin{aligned}\Delta(\bar{u}^m)(r, t) &= -C_0^m \left[ A e^{\frac{N-1}{2}\xi} V^{\frac{1}{m_0}}(r) f(t) + \frac{(N-1)^2}{4} e^{-\frac{N-1}{2}r} (2 \coth r - 1) (1 - f(t)) \right] \\ &\leq -C_0^m \left[ A^{1-\frac{1}{m_0}} e^{\frac{N-1}{2}\xi} e^{-\frac{N-1}{m_0}r} f(t) + \frac{(N-1)^2}{4} e^{-\frac{N-1}{2}r} (1 - f(t)) \right].\end{aligned}$$

In particular, there exist two positive constants  $B_0(m, m_0)$  and  $B_1(m_0, N)$  such that

$$\begin{aligned}-\bar{u}_t(r, t) &\leq C_0 B_0 f'(t) \left[ e^{\frac{(N-1)(1-m)}{2m}\xi} e^{-\frac{(N-1)(2-m)}{2m}r} f^{\frac{1}{m}-1}(t) + e^{-\frac{N-1}{2m}r} (1 - f(t))^{\frac{1}{m}-1} \right], \\ -\Delta(\bar{u}^m)(r, t) &\geq C_0^m B_1 \left[ e^{\frac{N-1}{2}\xi} e^{-\frac{N-1}{m_0}r} f(t) + e^{-\frac{N-1}{2}r} (1 - f(t)) \right].\end{aligned}$$

Hence it is enough to show that, if  $\xi$  is properly chosen, the following inequality holds in  $(\xi, \infty) \times (0, t_*)$ :

$$\begin{aligned}C_0^{1-m} B_* f'(t) &\left[ e^{\frac{(N-1)(1-m)}{2m}\xi} e^{-\frac{(N-1)(2-m)}{2m}r} f^{\frac{1}{m}-1}(t) + e^{-\frac{N-1}{2m}r} (1 - f(t))^{\frac{1}{m}-1} \right] \\ &\leq e^{\frac{N-1}{2}\xi} e^{-\frac{N-1}{m_0}r} f(t) + e^{-\frac{N-1}{2}r} (1 - f(t)),\end{aligned}\tag{4.3.10}$$

where  $B_* = B_*(m, m_0, N)$  is another suitable positive constant. Exploiting the change of variable  $\rho = r - \xi$  and using the fact that  $m_0 < 1$  we infer that (4.3.10) is implied by

$$\begin{aligned}C_0^{1-m} B_* f'(t) e^{-\frac{N-1}{2m}\xi} &\left[ e^{-\frac{(N-1)(2-m)}{2m}\rho} f^{\frac{1}{m}-1}(t) + e^{-\frac{N-1}{2m}\rho} (1 - f(t))^{\frac{1}{m}-1} \right] \\ &\leq e^{-\frac{(N-1)(2-m_0)}{2m_0}\xi} \left[ e^{-\frac{N-1}{m_0}\rho} f(t) + e^{-\frac{N-1}{2}\rho} (1 - f(t)) \right]\end{aligned}\tag{4.3.11}$$

for all  $(\rho, t) \in (0, \infty) \times (0, t_*)$ . Since we choose  $m_0$  to lie in the interval  $(2m/(1+m), 1)$  we have that

$$\epsilon := (N-1) \left( \frac{1}{2m} - \frac{1}{m_0} + \frac{1}{2} \right) > 0;$$

therefore (4.3.11) reads

$$C_0^{1-m} B_* e^{-\epsilon\xi} f'(t) \underbrace{\frac{e^{-\frac{(N-1)(2-m)}{2m}\rho} f^{\frac{1}{m}-1}(t) + e^{-\frac{N-1}{2m}\rho} (1 - f(t))^{\frac{1}{m}-1}}{e^{-\frac{N-1}{m_0}\rho} f(t) + e^{-\frac{N-1}{2}\rho} (1 - f(t))}}_{l(\rho, t)} \leq 1.\tag{4.3.12}$$

Our aim is now to show that for a suitable choice of  $f(t)$  the function  $l(\rho, t)$  stays bounded in  $(0, \infty) \times (0, t_*)$ . To this end, let us set  $y = e^{-(N-1)\rho}$  and  $f(t) = h(t/t_*)$ ,  $h$  being a function to be defined which has the same properties as  $f$  but in the interval  $[0, 1]$  instead of  $[0, t_*]$ . The boundedness of  $l(\rho, t)$  is implied by the boundedness of the ratio (recall that  $m_0 > 2m/(1+m)$ )

$$h'(\tau) \frac{y^{\left(\frac{1}{m}-\frac{1}{2}\right)} h^{\left(\frac{1}{m}-1\right)}(\tau) + y^{\frac{1}{2m}} (1 - h(\tau))^{\left(\frac{1}{m}-1\right)}}{y^{\left(\frac{1}{2m}+\frac{1}{2}\right)} h(\tau) + y^{\frac{1}{2}} (1 - h(\tau))} = h'(\tau) \frac{y^\alpha h^\alpha(\tau) + y^{\frac{\alpha}{2}} (1 - h(\tau))^\alpha}{y^{\frac{\alpha+1}{2}} h(\tau) + 1 - h(\tau)},\tag{4.3.13}$$

$$\alpha := \frac{1}{m} - 1,$$

for all  $(y, \tau) \in (0, 1) \times (0, 1)$ . If  $\alpha \geq 1$ , which corresponds to  $m \leq 1/2$ , the numerator in (4.3.13) is always smaller than or equal to the denominator. Therefore we are left with the case  $\alpha \in (0, 1)$ , that is  $m \in (1/2, 1)$ . Here it is convenient to pick  $h$  as

$$h(\tau) = 1 - (1 - \tau)^{\frac{1}{\alpha}}.$$

In this way, performing the change of variable  $\sigma = 1 - \tau$ , (4.3.13) becomes

$$\frac{1}{\alpha} \sigma^{\frac{1}{\alpha}-1} \frac{y^\alpha \left(1 - \sigma^{\frac{1}{\alpha}}\right)^\alpha + y^{\frac{\alpha}{2}} \sigma}{y^{\frac{\alpha+1}{2}} \left(1 - \sigma^{\frac{1}{\alpha}}\right) + \sigma^{\frac{1}{\alpha}}} \quad (4.3.14)$$

for  $(y, \sigma) \in (0, 1) \times (0, 1)$ . Now notice that, for  $\sigma$  varying in  $[1/2, 1)$ , (4.3.14) is bounded by a constant that depends only on  $\alpha$  (recall that  $\alpha \in (0, 1)$ ). Otherwise, if  $\sigma$  varies in  $(0, 1/2)$ , the boundedness of (4.3.14) is equivalent to the boundedness of

$$\frac{y^\alpha \sigma^{-1} + y^{\frac{\alpha}{2}}}{y^{\frac{\alpha+1}{2}} \sigma^{-\frac{1}{\alpha}} + 1} \leq \frac{y^\alpha \sigma^{-1}}{y^{\frac{\alpha+1}{2}} \sigma^{-\frac{1}{\alpha}} + 1} + 1. \quad (4.3.15)$$

For any fixed  $y \in (0, 1)$ , the maximum of the r.h.s. of (4.3.15) as  $\sigma \in (0, \infty)$  can be computed explicitly, and it is equal to

$$y^{\frac{\alpha}{2}(1-\alpha)} \alpha^\alpha (1 - \alpha)^{1-\alpha} + 1.$$

Summing up, we have proved that for a suitable choice of  $f(t)$  (depending on whether  $m \leq 1/2$  or  $m \in (1/2, 1)$ ) the function  $l(\rho, t)$  in (4.3.12) is bounded by a positive constant  $K = K(t_*, m)$  as  $(\rho, t)$  varies in  $(0, \infty) \times (0, t_*)$ . This means that if we set

$$\xi \geq \frac{1}{\epsilon} \log \left( C_0^{1-m} B_* K \right)$$

we ensure that  $\bar{u}(r, t)$  as in (4.3.9) is a supersolution to (4.1.16) in the region  $(\xi, \infty) \times (0, t_*)$ . By the comparison principle, in particular,

$$u(r, t_*) \leq \bar{u}(r, t_*) \leq C_0 \frac{2}{m} e^{\frac{N-1}{2m}\xi} e^{-\frac{N-1}{m}r} \quad \forall r \in (\xi, \infty).$$

Since  $u(\cdot, t_*)$  is also bounded in  $(0, \xi]$ , this gives (4.3.8).  $\square$

The result just proved is not enough in order to bound from above the ratio  $w^m(r, \tau)/V(r)$  because it only establishes such boundedness at any *fixed*  $\tau \in (0, \infty)$ : indeed, recall that  $u(\cdot, t)$  is bounded (so is  $w(\cdot, \tau)$ ),  $V(r)$  is locally bounded away from zero and its behaviour at infinity is the same as  $e^{-(N-1)r}$ . What estimate (4.3.8) lacks is a decay rate of order  $(T - t_*)^{1/(1-m)}$  in the r.h.s., which we shall now provide.

**Lemma 4.3.6.** *Let  $w$  be the rescaled solution corresponding to a bounded, nonnegative energy solution  $u$  to (4.1.16). There exists a constant  $C_1 = C_1(u_0, m, N)$  such that*

$$w(r, \tau) \leq C_1 e^{-\frac{N-1}{2m}r} \quad \forall (r, \tau) \in [0, \infty) \times (0, \infty). \quad (4.3.16)$$

*Proof.* The fact that

$$\|u(t)\|_\infty \leq D (T - t)^{\frac{1}{1-m}} \quad \forall t \in (0, T) \quad (4.3.17)$$

for a suitable positive constant  $D = D(u_0, m, N)$  can be proved exactly as in [73, Sections 5, 6] (see also the results in [29]). Then notice that Lemma 4.2.1 in particular yields

$$\|(w^m)'(\tau)\|_2 \leq \|(u_0^m)'\|_2 \quad \forall \tau \in (0, \infty). \quad (4.3.18)$$

The Poincaré inequality in (4.1.17), Lemma 4.3.2 and (4.3.18) finally entail the following bound:

$$w^m(r, \tau) \leq D'(r_0, N) \|(u_0^m)'\|_2 e^{-\frac{N-1}{2}r} \quad \forall (r, \tau) \in [r_0, \infty) \times (0, \infty),$$

which, together with (4.3.17), gives the claimed estimate (4.3.16).  $\square$

Notice that we proved (4.3.16) under the hypothesis that  $u$  is a bounded, nonnegative energy solution. However, thanks to the aforementioned smoothing effects, it also holds for any solution corresponding to data as in the hypothesis of Theorem 4.1.1, provided one starts from  $\tau = \varepsilon > 0$  rather than  $\tau = 0$ .

The bound (4.3.16) for  $w$  is the exactly the same as (4.3.6) for  $u$ , which was a key starting point in order to prove the claim of Lemma 4.3.5. Indeed, as we shall see now, the barrier exploited in the proof of such Lemma also works for the equation solved by  $w$ . This allows us to obtain our estimate from above.

*Proof of Proposition 4.3.1.* As just remarked, we need only prove that for a suitable choice of the parameter  $\xi > 0$  the barrier constructed in the proof of Lemma 4.3.5 still works. For a given  $\tau_* \in (\varepsilon, \infty)$ , consider again the function

$$\bar{w}(r, \tau) = C_1 \left[ A e^{\frac{N-1}{2}\xi} V(r) f(\tau) + e^{-\frac{N-1}{2}r} (1 - f(\tau)) \right]^{\frac{1}{m}}, \quad (4.3.19)$$

where  $\xi$ ,  $V$ ,  $A$ ,  $f$  are as in (4.3.9) (just replace  $t$  with  $\tau$  and let  $f(\varepsilon) = 0$ ) and  $C_1$  is the constant appearing in (4.3.16). First of all, since  $f(\cdot)$  is always included in  $[0, 1]$ ,  $m < 1$  and  $m_0 \in (2m/(1+m), 1)$ , we can bound the reaction term in (4.2.3) (what actually makes (4.2.3) different from (4.1.1)) in the following way:

$$\begin{aligned} \frac{1}{(1-m)T} \bar{w}(\rho + \xi, \tau) &\leq C_1 B_2 e^{-\frac{N-1}{2m}\xi} \left[ e^{-\frac{N-1}{m}\rho} f^{\frac{1}{m}}(\tau) + e^{-\frac{N-1}{2m}\rho} (1 - f(\tau))^{\frac{1}{m}} \right] \\ &\leq C_1 B_2 e^{-\frac{N-1}{2m}\xi} \left[ e^{-\frac{N-1}{m_0}\rho} f(\tau) + e^{-\frac{N-1}{2}\rho} (1 - f(\tau)) \right], \end{aligned}$$

where  $B_2$  is a positive constant depending only on  $T$ ,  $m$ ,  $m_0$  and we performed the usual change of space variable  $\rho = r - \xi$ . The equivalent of (4.3.11) then reads

$$\begin{aligned} &C_1 B_0 f'(\tau) e^{-\varepsilon\xi} \left[ e^{-\frac{(N-1)(2-m)}{2m}\rho} f^{\frac{1}{m}-1}(\tau) + e^{-\frac{N-1}{2m}\rho} (1 - f(\tau))^{\frac{1}{m}-1} \right] \\ &\leq \left( C_1^m B_1 - C_1 B_2 e^{-\varepsilon\xi} \right) \left[ e^{-\frac{N-1}{m_0}\rho} f(\tau) + e^{-\frac{N-1}{2}\rho} (1 - f(\tau)) \right] \\ &\quad \forall (\rho, \tau) \in (0, \infty) \times (\varepsilon, \tau_*). \end{aligned}$$

By elementary computations one gets that

$$\frac{C_1^m B_1}{2} \leq C_1^m B_1 - C_1 B_2 e^{-\varepsilon\xi}$$

provided

$$\xi \geq \frac{1}{\varepsilon} \log \left( \frac{2C_1^{1-m} B_2}{B_1} \right). \quad (4.3.20)$$

Therefore, under assumption (4.3.20), we can repeat the same proof of Lemma 4.3.5 starting from (4.3.12) (one replaces  $B_*$  with  $2B_0/B_1$  and  $C_0$  with  $C_1$ ). Hence we deduce the existence of a positive parameter  $\xi = \xi(C_1(u_0, m, N, \varepsilon), T, \varepsilon)$  such that

$$w(r, \tau) \leq \bar{w}(r, \tau) \quad \forall (r, \tau) \in [\xi, \infty) \times [\varepsilon, \tau_*]. \quad (4.3.21)$$

The validity of (4.3.2) is then a consequence of (4.3.21), (4.3.19) (evaluated at  $\tau = \tau_*$ ) and (4.3.17).  $\square$

**Remark 4.3.7.** In the proofs of Lemmas 4.3.4, 4.3.5 and Proposition 4.3.1 we applied the comparison principle in parabolic regions of the form  $(\xi, \infty) \times (0, t_*)$ , neglecting the  $\{r = \infty\}$  side of the parabolic boundary. However, this technical issue is easily solvable by approximating  $u$  with the solutions  $u_n$  to (4.2.4), applying comparison between  $u_n$  and  $\bar{u}$  in  $(\xi, n) \times (0, t_*)$  and passing to the limit as  $n \rightarrow \infty$  (using also the fact that  $T_n \uparrow T$ ).

**Remark 4.3.8.** The estimate from above provided by Lemma 4.3.5 actually holds for all  $m \in (0, 1)$ , since its method of proof only requires the existence of a solution to (4.3.4) satisfying (4.3.5) for a value of  $m_0$  which can be taken as close to 1 as necessary.

### 4.3.2 Estimates from below

We now aim to bound the ratio  $w^m(\tau)/V$  from below. Again, thanks to (4.3.5), this is equivalent to establishing a lower bound for (4.3.1). The result we shall prove is the following.

**Proposition 4.3.9.** *Let  $u$  be a nonnegative nonzero energy solution to (4.1.16). For any  $\varepsilon \in (0, T)$  there exists a positive constant  $P' = P'(u_0, m, N, \varepsilon)$  such that*

$$u(r, t_*) \geq P' e^{-\frac{N-1}{m}r} (T - t_*)^{\frac{1}{1-m}} \quad \forall (r, t_*) \in [0, \infty) \times [\varepsilon, T]. \quad (4.3.22)$$

To this end, we first need a preliminary step.

**Lemma 4.3.10.** *Let  $u$  be a nonnegative nonzero energy solution to (4.1.16). For any given  $\alpha > N - 1$  and  $t_* \in (0, T)$  there exists a positive constant  $P = P(t_*, \alpha, u_0, m, N)$  such that*

$$u(r, t_*) \geq P e^{-\frac{\alpha}{m}r} \quad \forall r \in [0, \infty). \quad (4.3.23)$$

*Proof.* We shall prove that for a suitable choice of the positive parameters  $\mu_0$  and  $\xi$  the following function is a subsolution to (4.1.16) in the parabolic region  $(\xi, \infty) \times (t_*/2, t_*)$ :

$$\underline{u}(r, t) := \mu_0 \left[ \left(1 + e^{-\alpha(r-\xi)}\right) f(t) - 1 \right]_+^{\frac{1}{m}}, \quad (4.3.24)$$

where  $f : [t_*/2, t_*] \mapsto [1/2, 1]$  is an increasing function such that  $f(t_*/2) = 1/2$  and  $f(t_*) = 1$ , to be defined later. We have:

$$\underline{u}_t(r, t) = \mu_0 \frac{1}{m} f'(t) \left(1 + e^{-\alpha(r-\xi)}\right) \left[ \left(1 + e^{-\alpha(r-\xi)}\right) f(t) - 1 \right]_+^{\frac{1}{m}-1}$$

and

$$\Delta(\underline{u}^m)(r, t) = \mu_0^m f(t) e^{-\alpha(r-\xi)} \left( \alpha^2 - (N-1) \alpha \coth r \right)$$

in the region where  $(1 + e^{-\alpha(r-\xi)})f(t) - 1$  is nonnegative (below, we shall work tacitly in such region), while both  $\underline{u}_t$  and  $\Delta(\underline{u}^m)$  are zero outside it. Let us check conditions on the parabolic boundary. On  $[\xi, \infty) \times \{t_*/2\}$   $\underline{u}$  satisfies

$$\underline{u}(r, t_*/2) = \mu_0 \left[ \frac{1}{2} \left(1 + e^{-\alpha(r-\xi)}\right) - 1 \right]_+^{\frac{1}{m}} = 0$$

and on  $\{\xi\} \times (t_*/2, t_*)$  there holds

$$\underline{u}(\xi, t) = \mu_0 [2f(t) - 1]_+^{\frac{1}{m}} \leq \mu_0.$$

Therefore, in order to have  $\underline{u} \leq u$  on  $[\xi, \infty) \times \{t_*/2\} \cup \{\xi\} \times (t_*/2, t_*)$ , we need only require

$$\mu_0 \leq \inf_{t \in (t_*/2, t_*)} u(\xi, t) =: \lambda(\xi, t_*, u_0) > 0,$$

the last inequality following from standard positivity results (if  $t_*$  is close enough to  $T$ , one can also exploit the results of Section 4.2). Now let us check the differential equation. By means of the usual change of spatial variable  $\rho = r - \xi$ ,

$$\underline{u}_t(r, t) \leq \Delta(\underline{u}^m)(r, t) \quad \forall (r, t) \in (\xi, \infty) \times \left(\frac{t_*}{2}, t_*\right)$$

reads

$$\begin{aligned} & \mu_0 \frac{1}{m} f'(t) (1 + e^{-\alpha\rho}) \left[ (1 + e^{-\alpha\rho}) f(t) - 1 \right]^{\frac{1}{m}-1} \\ & \leq \mu_0^m f(t) e^{-\alpha\rho} \left( \alpha^2 - (N-1) \alpha \coth(\rho + \xi) \right) \quad \forall (\rho, t) \in (0, \infty) \times \left(\frac{t_*}{2}, t_*\right). \end{aligned} \quad (4.3.25)$$

Upon choosing  $\xi = \xi(\alpha, N)$  so large that, for instance,

$$\alpha^2 - (N-1) \alpha \coth \xi \geq \frac{1}{2} \left( \alpha^2 - (N-1) \alpha \right) =: C_1(\alpha, N) > 0, \quad (4.3.26)$$

we get that (4.3.25) is implied by

$$\mu_0^{1-m} \frac{1}{m C_1} f'(t) \frac{(1 + e^{-\alpha\rho}) \left[ (1 + e^{-\alpha\rho}) f(t) - 1 \right]^{\frac{1}{m}-1}}{f(t) e^{-\alpha\rho}} \leq 1 \quad \forall (\rho, t) \in (0, \infty) \times \left(\frac{t_*}{2}, t_*\right),$$

which is in turn implied by (recall that  $1/2 \leq f(t) \leq 1$ )

$$\underbrace{\mu_0^{1-m} \frac{4}{m C_1} f'(t) e^{\alpha\rho} \left[ (1 + e^{-\alpha\rho}) f(t) - 1 \right]^{\frac{1}{m}-1}}_{L(\rho, t)} \leq 1 \quad \forall (\rho, t) \in (0, \infty) \times \left(\frac{t_*}{2}, t_*\right). \quad (4.3.27)$$

If  $m \leq 1/2$  the function  $L$  in (4.3.27) is bounded from above (take  $f$  regular enough) by a constant that depends only on  $t_*$ ,  $\alpha$ ,  $m$  and  $N$ . On the other hand, if  $m \in (1/2, 1)$  this is in general false, unless one chooses  $f$  carefully. To this end, consider the function

$$h(\tau) := 1 - \frac{1}{2} (1 - \tau)^{\frac{m}{1-m}} \quad \forall \tau \in [0, 1]$$

and set

$$f(t) = h\left(2 \frac{t}{t_*} - 1\right) \quad \forall t \in \left[\frac{t_*}{2}, t_*\right].$$

Elementary computations (one can find the exact maximum of  $L(\rho, t)$ , in the region where  $(1 + e^{-\alpha\rho})f(t) \geq 1$ , at any given  $t$ ) show that

$$L(\rho, t) \leq \frac{C_2}{t_*} \quad \forall (\rho, t) \in (0, \infty) \times \left(\frac{t_*}{2}, t_*\right)$$

for a suitable positive constant  $C_2 = C_2(\alpha, m, N)$  (which we assume to work for the case  $m \leq 1/2$  as well). Hence, we proved that  $\underline{u}$  as in (4.3.24) is indeed a subsolution to (4.1.16) providing that

$$\mu_0 \leq \min \left\{ \lambda(\xi(\alpha, N), t_*, u_0), \left(\frac{t_*}{C_2}\right)^{\frac{1}{1-m}} \right\}.$$

In particular, at  $t = t_*$  there holds

$$u(r, t_*) \geq \underline{u}(r, t_*) = \mu_0(t_*, \alpha, u_0, m, N) e^{\frac{\alpha}{m}\xi(\alpha, N)} e^{-\frac{\alpha}{m}r} \quad \forall r \in (\xi, \infty),$$

which yields the assertion together with the local positivity of  $u(\cdot, t_*)$  in  $(0, \xi)$ .  $\square$

The result provided by Lemma 4.3.10 ensures that at any given  $t_* \in (0, T(u_0))$  all non-negative nonzero energy solutions to (4.1.16) go to infinity (as  $r \rightarrow \infty$ ) slower than  $e^{-\frac{\alpha}{m}r}$ , for any  $\alpha > N - 1$ . Exploiting such property, we are then able to prove Proposition 4.3.9.

*Proof of Proposition 4.3.9.* Once again we construct a lower barrier which has the desired behaviour as in (4.3.22). Indeed, given  $t_* \in (2\varepsilon, T)$  (the final result will follow just by replacing  $\varepsilon$  with  $\varepsilon/2$ ), consider the function

$$\underline{u}(r, t) := \mu_0 \left[ \left( \left( e^{-\beta(r-\xi)} + A^{-1}V(r)e^{(N-1)\xi} \right) f(t) - e^{-\beta(r-\xi)} \right)_+ + e^{-\alpha(r-\xi)} \right]^{\frac{1}{m}}, \quad (4.3.28)$$

where  $\alpha = \alpha(m, N)$  and  $\beta = \beta(m, N)$  are fixed parameters such that

$$\alpha > N - 1, \quad \beta < N - 1, \quad \alpha \leq \beta + (N - 1) \frac{1 - m}{m}, \quad (4.3.29)$$

$f : [\varepsilon, t_*] \mapsto [0, 1]$  is a regular increasing function that satisfies  $f(\varepsilon) = 0$ ,  $f(t_*) = 1$  and  $V$  is the solution to (4.3.4) corresponding to  $m_0 = m$  ( $A$  being the related constant appearing in (4.3.5)). We want to prove that, if one chooses the positive parameters  $\xi$  and  $\mu_0$  properly, then  $\underline{u}$  is a subsolution to (4.1.16) in the parabolic region  $(\xi, \infty) \times (\varepsilon, t_*)$ . By (4.3.28), we have

$$\underline{u}(r, t_0) = \mu_0 e^{\frac{\alpha}{m}\xi} e^{-\frac{\alpha}{m}r} \quad \forall r \in [\xi, \infty)$$

and

$$\underline{u}(\xi, t) \leq \mu_0 \left[ A^{-1}V(\xi)e^{(N-1)\xi} + 1 \right]^{\frac{1}{m}} \leq \mu_0 2^{\frac{1}{m}} \quad \forall t \in (\varepsilon, t_*).$$

Therefore  $\underline{u}$  and  $u$  are correctly ordered on the parabolic boundary  $[\xi, \infty) \times \{\varepsilon\} \cup \{\xi\} \times (\varepsilon, t_*)$  provided (recall (4.3.23))

$$\mu_0 \leq e^{-\frac{\alpha}{m}\xi} P(\varepsilon, \alpha, u_0, m, N) \quad \text{and} \quad \mu_0 \leq 2^{-\frac{1}{m}} \inf_{t \in (\varepsilon, t_*)} u(\xi, t) =: \lambda(\xi, t_*, u_0, m, N, \varepsilon) > 0,$$

that is

$$\mu_0 \leq \min \left\{ e^{-\frac{\alpha}{m}\xi} P, \lambda \right\}. \quad (4.3.30)$$

Now let us compute derivatives of  $\underline{u}(r, t)$ :

$$\begin{aligned} \underline{u}_t(r, t) &= \mu_0 \frac{1}{m} f'(t) \left( e^{-\beta(r-\xi)} + A^{-1}V(r)e^{(N-1)\xi} \right) \text{sign}_+ q(r, t) \\ &\quad \times \left[ \left( \left( e^{-\beta(r-\xi)} + A^{-1}V(r)e^{(N-1)\xi} \right) f(t) - e^{-\beta(r-\xi)} \right)_+ + e^{-\alpha(r-\xi)} \right]^{\frac{1}{m}-1}, \end{aligned}$$

$$\begin{aligned} \Delta(\underline{u}^m)(r, t) &= \mu_0^m \left[ \left( (f(t) - 1) (\beta^2 - (N - 1) \beta \coth r) e^{-\beta(r-\xi)} - A^{-1}V^{\frac{1}{m}}(r)e^{(N-1)\xi} f(t) \right) \text{sign}_+ q(r, t) \right. \\ &\quad \left. + (\alpha^2 - (N - 1) \alpha \coth r) e^{-\alpha(r-\xi)} \right], \end{aligned}$$

where for the sake of notational convenience we set

$$q(r, t) := \left( e^{-\beta(r-\xi)} + A^{-1}V(r)e^{(N-1)\xi} \right) f(t) - e^{-\beta(r-\xi)}.$$

If we take  $\xi = \xi(\alpha, N)$  so large that (4.3.26) holds, set  $\rho = r - \xi$  and use (4.3.5), we obtain:

$$\underline{u}_t(\rho + \xi, t) \leq \mu_0 \frac{1}{m} f'(t) \left( e^{-\beta\rho} + e^{-(N-1)\rho} \right) \left[ \left( \left( e^{-\beta\rho} + e^{-(N-1)\rho} \right) f(t) - e^{-\beta\rho} \right)_+ + e^{-\alpha\rho} \right]^{\frac{1}{m}-1},$$

$$\Delta(\underline{u}^m)(\rho + \xi, t) \geq \mu_0^m \left[ -A^{\frac{1-m}{m}} e^{-(N-1)\frac{1-m}{m}\xi} f(t) e^{-\frac{N-1}{m}\rho} + C_1 e^{-\alpha\rho} \right]$$

$$\forall(\rho, t) \in (0, \infty) \times (\varepsilon, t_*).$$

Hence once we choose  $\xi = \xi(\alpha, m, N)$  such that, in addition to (4.3.26), it also satisfies

$$A^{\frac{1-m}{m}} e^{-(N-1)\frac{1-m}{m}\xi} \leq \frac{C_1}{2},$$

we deduce that in order to have  $\underline{u} \leq u$  in  $(\xi, \infty) \times (\varepsilon, t_*)$  it is enough to require (recall that  $f(t) \in [0, 1]$ )

$$\mu_0 \frac{1}{m} f'(t) \left( e^{-\beta\rho} + e^{-(N-1)\rho} \right) \left[ e^{-(N-1)\rho} + e^{-\alpha\rho} \right]^{\frac{1}{m}-1} \leq \mu_0^m \frac{C_1}{2} e^{-\alpha\rho} \quad (4.3.31)$$

$$\forall(\rho, t) \in (0, \infty) \times (\varepsilon, t_*).$$

Upon setting  $f(t) = h((t - \varepsilon)/(t_* - \varepsilon))$ ,  $h : [0, 1] \mapsto [0, 1]$  being a given regular, increasing function such that  $h(0) = 0$  and  $h(1) = 1$ , (4.3.31) reads

$$\underbrace{\mu_0^{1-m} \frac{2}{m C_1 (t_* - \varepsilon)} h' \left( \frac{t - \varepsilon}{t_* - \varepsilon} \right) \left( e^{(\alpha-\beta)\rho} + e^{(\alpha-N+1)\rho} \right) \left[ e^{-(N-1)\rho} + e^{-\alpha\rho} \right]^{\frac{1}{m}-1}}_{L(\rho, t)} \leq 1 \quad (4.3.32)$$

$$\forall(\rho, t) \in (0, \infty) \times (\varepsilon, t_*).$$

Thanks to (4.3.29) and to the fact that  $t_* \in (2\varepsilon, T)$ , the function  $L(\rho, t)$  in (4.3.32) is bounded in  $(0, \infty) \times (\varepsilon, t_*)$  by a positive constant  $C_2 = C_2(T, m, N, \varepsilon)$ . Therefore  $\underline{u}(r, t)$  is a subsolution to (4.1.16) providing that (recall (4.3.30))

$$\mu_0 \leq \min \left\{ e^{-\frac{\alpha}{m}\xi} P, C_2^{-\frac{1}{1-m}}, 2^{-\frac{1}{m}} \inf_{t \in (\varepsilon, t_*)} u(\xi, t) \right\},$$

which is implied by

$$\mu_0 \leq C_3 \min \left\{ 1, \inf_{t \in (\varepsilon, t_*)} u(\xi, t) \right\} \quad (4.3.33)$$

for a suitable positive constant  $C_3 = C_3(\xi, u_0, m, N, \varepsilon)$ . Taking advantage of Proposition 4.2.6 we can give a quantitative lower bound for the r.h.s. of (4.3.33). Indeed (4.2.33) yields (in particular) the existence, for any given  $\xi > 0$ , of a time  $\hat{t} = \hat{t}(\xi, u_0)$  such that

$$u(\xi, t) \geq \frac{V^{\frac{1}{m}}(\xi)}{2} (T - t)^{\frac{1}{1-m}} \quad \forall t \in (\hat{t}, T). \quad (4.3.34)$$

From standard positivity results we also know that, in case  $\varepsilon < \hat{t}$ ,  $u(\xi, t)$  is still positive between  $\varepsilon$  and  $\hat{t}$ ; this fact and (4.3.34) (together with the local positivity of  $V$ ) ensure the existence of a positive constant  $C_4 = C_4(\xi, u_0, m, N, \varepsilon)$  such that

$$u(\xi, t) \geq C_4 (T - t)^{\frac{1}{1-m}} \quad \forall t \in (\varepsilon, T),$$

which gives

$$\inf_{t \in (\varepsilon, t_*)} u(\xi, t) \geq C_4 (T - t_*)^{\frac{1}{1-m}}. \quad (4.3.35)$$

Combining (4.3.35) and (4.3.33) we infer that (4.3.33) is implied by

$$\mu_0 \leq C_5 (T - t_*)^{\frac{1}{1-m}} \quad (4.3.36)$$

for another positive constant  $C_5 = C_5(\xi, u_0, m, N, \varepsilon)$ . The validity of (4.3.22) for  $r$  varying in  $(\xi, \infty)$  then follows by choosing  $\mu_0$  equal to the r.h.s. of (4.3.36) and evaluating the subsolution  $\underline{u}(r, t)$  at  $t = t_*$ , recalling (4.3.5). On the other hand, the validity of (4.3.22) as  $r$  varies in  $(0, \xi]$  is a direct consequence of (4.2.33) and the local positivity of  $u$ .  $\square$

It is apparent that Proposition 4.3.9 can be reformulated as follows.

**Corollary 4.3.11.** *Let  $w$  be the rescaled solution corresponding to a positive energy solution  $u$  to (4.1.16). For any  $\tau_0 > 0$  there exists a positive constant  $P'' = P''(\tau_0, u_0, m, N)$  such that*

$$w(r, \tau) \geq P'' e^{-\frac{N-1}{m}r} \quad \forall (r, \tau) \in [0, \infty) \times [\tau_0, \infty).$$

Thanks to Propositions 4.3.1 and 4.3.9 (or Corollary 4.3.11) we then deduce formula (4.1.8) in Theorem 4.1.3, namely the *global Harnack principle*.

**Remark 4.3.12.** Both in the proofs of Lemma 4.3.10 and Proposition 4.3.9, when computing the hyperbolic Laplacian of the barriers (4.3.24) and (4.3.28), we neglected Dirac terms coming up from second derivatives of positive parts. However, this is not an issue since it is easy to check that such terms are nonnegative.

**Remark 4.3.13.** When we applied the comparison principle to  $\underline{u}$  and  $u$ , both in the proofs of Lemma 4.3.10 and Proposition 4.3.9, we did not take into account  $\{r = \infty\}$  as a side of the parabolic boundary. In order to justify more rigorously those computations, it is enough to consider the following family of modified barriers:

$$\underline{u}_\epsilon := [\underline{u}^m - \epsilon]_+^{\frac{1}{m}} \leq \underline{u}, \quad \epsilon > 0,$$

where  $\underline{u}$  is either (4.3.24) or (4.3.28). Straightforward computations show that  $\underline{u}_\epsilon$  is a subsolution to (4.1.16) as long as  $\underline{u}$  is. Moreover, for any fixed  $\epsilon > 0$ ,  $\underline{u}_\epsilon(\cdot, t)$  is zero outside a compact set of the form  $[\xi, R(\epsilon)] \subset [\xi, \infty)$ . One then applies the comparison principle in  $(\xi, R(\epsilon)) \times (\varepsilon, t_*)$  (let  $\varepsilon = t_*/2$  when  $\underline{u}$  is as in (4.3.24)) to get  $\underline{u}_\epsilon \leq u$  and lets  $\epsilon \rightarrow 0$ .

**Remark 4.3.14.** The estimate from below (4.3.22) holds true for all  $m \in (0, 1)$ , even though one has to let the constant  $P'$  in there, for  $m \in (0, m_s]$ , depend on  $t^*$  as well. Indeed, in the proof of Proposition 4.3.9, for simplicity we exploited the existence of a solution  $V$  to (4.3.4) satisfying (4.3.5) for  $m_0 = m$ . However, if  $m \in (0, m_s]$ , such a solution does not exist. Nevertheless, it is easy to see that the choice of any  $m_0 \in (m_s, 1)$  instead of  $m$  would have sufficed (provided one requires in addition that  $\alpha < (N - 1)/m_0$ ).

This result and the one discussed in Remark 4.3.8 prove the bounds (4.1.12) of Theorem 4.1.4.

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## 4.4 Convergence in relative error

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In this section we shall prove the claimed *uniform convergence in relative error*, that is (4.1.5), which can equivalently be rewritten in terms of the rescaled solution  $w$  as

$$\lim_{\tau \rightarrow \infty} \left\| \frac{w^m(\tau)}{V} - 1 \right\|_\infty = 0. \quad (4.4.1)$$

Since  $V$  is strictly positive in any compact subset of  $\mathbb{H}^N$ , as a direct consequence of Proposition 4.2.6 (formula (4.2.33)) we already know that (4.4.1) holds true *locally* on  $\mathbb{H}^N$ . The nontrivial point is to prove that (4.2.33) holds up to  $r = \infty$ , what we are concerned with in the next two subsections. Finally, in Subsection 4.4.3 we shall deal with estimates for derivatives and improved convergence in  $C^k$  spaces, namely with the proofs of (4.1.9)–(4.1.11), (4.1.6) and (4.1.7).



### 4.4.1 The relative error and its equation

To begin with, let us write the equation solved by the *relative error*  $\phi := w^m/V - 1$ . Using the fact that  $V$  satisfies (4.1.4), we get:

$$\frac{1}{m} (1 + \phi)^{\frac{1}{m}-1} \phi_\tau = V^{1-\frac{1}{m}} \Delta \phi + 2 \frac{V' \phi'}{V^{\frac{1}{m}}} + \frac{1}{(1-m)T} \left[ (1 + \phi)^{\frac{1}{m}} - (1 + \phi) \right], \quad (4.4.2)$$

where, as remarked above, the superscript  $'$  stands for derivation w.r.t.  $r$ . We need also to consider the equation solved by *minus* the relative error  $\psi := 1 - w^m/V$ , that is

$$\frac{1}{m} (1 - \psi)^{\frac{1}{m}-1} \psi_\tau = V^{1-\frac{1}{m}} \Delta \psi + 2 \frac{V' \psi'}{V^{\frac{1}{m}}} + \frac{1}{(1-m)T} \left[ (1 - \psi) - (1 - \psi)^{\frac{1}{m}} \right].$$

In order to prove (4.4.1) we shall construct suitable upper barriers both for  $\phi$  and  $\psi$ , following the approach of [31].

The next lemma shows a good approximation property enjoyed by  $V$ , which will turn out to be very useful to overcome some technical difficulties related to the upper barrier for  $\psi$ .

**Lemma 4.4.1.** *There exists a sequence of positive, regular radial solutions  $\{V_n\}$  to*

$$\begin{cases} -\Delta V_n = \frac{1}{(1-m)T_n} V_n^{\frac{1}{m}} & \text{on } B_n, \\ V_n = 0 & \text{on } \partial B_n, \end{cases} \quad (4.4.3)$$

where  $B_n$  is the hyperbolic ball of radius  $n$  centred at  $o$ , such that:

- $V_n(r) \leq V(r)$  for all  $r \geq 1$  and  $V_n \rightarrow V$  pointwise;
- $T_n \rightarrow T$ ;
- for  $\epsilon > 0$  arbitrarily small, one can choose  $n_\epsilon$  and  $r_\epsilon$  so large that the following inequality holds true:

$$V'_n(r) \leq (1 - \epsilon) V'(r) \quad \forall r \in (r_\epsilon, n), \quad \forall n \geq n_\epsilon. \quad (4.4.4)$$

*Proof.* For any given  $n \in \mathbb{N}$ , let  $X_n$  be the set of all functions  $v \in H_{\text{rad}}^1$  such that  $v(r) = 0$  for all  $r \geq n$  and  $\|v\|_{1+1/m} = \|V\|_{1+1/m}$ . Consider the following minimization problem:

$$\text{Find } v_n \in X_n : \|v'_n\|_2 = \min_{v \in X_n} \|v'\|_2. \quad (4.4.5)$$

Thanks to standard arguments, the solution  $v_n$  of (4.4.5) exists, is unique, strictly positive on  $B_n$  and solves (4.4.3) for some  $T_n > 0$ . Since  $X_n \subset X_{n+1}$ , the sequence  $\{\|v'_n\|_2\}$  is nonincreasing. This implies that  $\{T_n\}$  is nondecreasing: indeed, multiplying (4.4.3) by  $v_n$  (upon replacing  $V_n$  by  $v_n$ ) and integrating by parts we obtain

$$T_n = \frac{\|v_n\|_{\frac{m+1}{m}}^{\frac{m+1}{m}}}{(1-m) \|v'_n\|_2^2} = \frac{\|V\|_{\frac{m+1}{m}}^{\frac{m+1}{m}}}{(1-m) \|v'_n\|_2^2}.$$

Let us denote as  $T_\ell$  the limit of  $\{T_n\}$  as  $n \rightarrow \infty$ . Because  $\{\|v'_n\|_2\}$  is bounded,  $\{v_n\}$  (along subsequences) converges weakly in  $H_{\text{rad}}^1$  and therefore strongly in  $L_{\text{rad}}^{1+1/m}$  to a certain function  $V_\ell$ . Passing to the limit in the (weak formulation of the) equation solved by  $v_n$ , we get:

$$-\Delta V_\ell = \frac{1}{(1-m)T_\ell} V_\ell^{\frac{1}{m}} \quad \text{on } \mathbb{H}^N. \quad (4.4.6)$$

First of all note that  $T_\ell$  cannot be infinity: if it were, then from the Poincaré inequality in (4.1.17)  $V_\ell$  would be zero, while we know that

$$\|V_\ell\|_{\frac{m+1}{m}} = \lim_{n \rightarrow \infty} \|v_n\|_{\frac{m+1}{m}} = \|V\|_{\frac{m+1}{m}} > 0.$$

So  $V_\ell$  is a positive, energy solution to (4.4.6) for some  $T_\ell < \infty$ , having the same  $L_{\text{rad}}^{1+1/m}$  norm as  $V$ : by the uniqueness result recalled in the beginning of Section 4.2, it necessarily coincides with  $V$  (and then  $T_n \uparrow T$ ).

The next step is to prove that  $\{v_n\}$  converges to  $V$  also in  $L_{\text{rad}}^{1/m}$ . Since  $\{v_n\}$  converges pointwise to  $V$  (up to subsequences), and locally  $L_{\text{rad}}^{1+1/m}$  is continuously embedded in  $L_{\text{rad}}^{1/m}$ , we only need to dominate  $\{v_n\}$  outside a compact set with a function that belongs to  $L_{\text{rad}}^{1/m}$ . To this end note that, from Lemma 4.3.2 and from the fact that  $\{\|v'_n\|_2\}$  is nonincreasing, there exists a constant  $K > 0$  such that

$$v_n(r) \leq K e^{-\frac{N-1}{2}r} \quad \forall r \in [1, \infty), \quad \forall n \in \mathbb{N}. \quad (4.4.7)$$

When  $m$  is greater than or equal to  $1/2$ , the function in the r.h.s. of (4.4.7) does not belong to  $L_{\text{rad}}^{1/m}$ . We have therefore to improve on (4.4.7) using the equation solved by  $v_n$ : integrating it between 0 and  $r$  we obtain the identity

$$-(\sinh r)^{N-1} v'_n(r) = \int_0^r \frac{v_n^{\frac{1}{m}}(s)}{T_n} (\sinh s)^{N-1} ds. \quad (4.4.8)$$

Suppose now that (4.4.7) holds with a generic exponent  $-(N-1)a$  (let  $a > 0$ ,  $a \neq m$ ) in place of  $-(N-1)/2$ . Exploiting the corresponding analogues of (4.4.7) and (4.4.8), after some straightforward computations we get:

$$-v'_n(r) \leq Q \frac{m}{m-a} e^{-(N-1)r} \left( e^{(N-1)\frac{m-a}{m}r} - 1 \right) \quad \forall r \in [1, \infty), \quad (4.4.9)$$

where  $Q > 0$  is a suitable positive constant that does not depend on  $n$  (which may change from line to line). Since  $v_n(n) = 0$ , we have:

$$v_n(r) = \int_r^n -v'_n(s) ds \quad \forall r \in [1, n). \quad (4.4.10)$$

If  $a > m$  (4.4.9) gives  $-v'_n(r) \leq Q e^{-(N-1)r}$ , in which case an integration of (4.4.10) entails

$$v_n(r) \leq Q e^{-(N-1)r} \quad \forall r \in [1, \infty). \quad (4.4.11)$$

On the other hand, if  $a < m$  then (4.4.9) entails  $-v'_n(r) \leq Q e^{-(N-1)\frac{a}{m}r}$ ; integrating again (4.4.10) one gets

$$v_n(r) \leq Q e^{-(N-1)\frac{a}{m}r} \quad \forall r \in [1, \infty)$$

(the case  $a = m$  can be dealt with similarly). It is plain that starting from  $a = 1/2$  and proceeding in this way, after a finite number of steps we obtain (4.4.11). Since  $e^{-(N-1)r}$  belongs to  $L_{\text{rad}}^{1/m}$ , we have our dominating function and the convergence of  $\{v_n\}$  to  $V$  also takes place in  $L_{\text{rad}}^{1/m}$ . Such convergence is crucial because it ensures that (4.4.4) holds for the sequence  $\{v_n\}$ . Indeed, applying (4.4.8) to  $v_n = V$  we deduce that

$$\lim_{r \rightarrow \infty} -(\sinh r)^{N-1} V'(r) = \int_0^\infty \frac{V^{\frac{1}{m}}(s)}{T} (\sinh s)^{N-1} ds =: C_V > 0; \quad (4.4.12)$$

for a given  $\epsilon > 0$ , take  $r_\epsilon$  so large that

$$\int_{r_\epsilon}^\infty \frac{V^{\frac{1}{m}}(s)}{T} (\sinh s)^{N-1} ds \leq C_V \frac{\epsilon}{2}.$$

In view of the just proved convergence of  $\{v_n\}$  to  $V$  in  $L_{\text{rad}}^{1/m}$  and to the fact that  $T_n \uparrow T$ , there exists  $n = n(r_\epsilon, \epsilon)$ , large enough, such that

$$\begin{aligned} -(\sinh r)^{N-1} v'_n(r) &= \int_0^r \frac{v_n^{\frac{1}{m}}(s)}{T_n} (\sinh s)^{N-1} ds \geq \int_0^{r_\epsilon} \frac{v_n^{\frac{1}{m}}(s)}{T_n} (\sinh s)^{N-1} ds \\ &\geq \int_0^{r_\epsilon} \frac{V^{\frac{1}{m}}(s)}{T} (\sinh s)^{N-1} ds - C_V \frac{\epsilon}{2} \\ &\geq C_V(1 - \epsilon) \end{aligned}$$

for all  $r \in (r_\epsilon, n)$  and  $n \geq n(r_\epsilon, \epsilon)$ . Hence, there holds

$$\frac{-v'_n(r)}{-V'(r)} = \frac{-(\sinh r)^{N-1} v'_n(r)}{-(\sinh r)^{N-1} V'(r)} \geq \frac{C_V(1 - \epsilon)}{C_V} \geq 1 - \epsilon \quad \forall r \in (r_\epsilon, n), \quad \forall n \geq n(r_\epsilon, \epsilon),$$

that is (4.4.4) for the sequence  $\{v_n\}$ .

In general it is not for granted that  $v_n(r) \leq V(r)$  for all  $r \in [1, \infty)$ . Therefore, in order to conclude our proof, it is necessary to modify  $\{v_n\}$ . To this aim, consider the following sequence:

$$V_n = \lambda_n v_n,$$

where, for any fixed  $n$ ,  $\lambda_n$  is the *largest* number belonging to  $(0, 1]$  for which  $\lambda_n v_n(r) \leq V(r)$  for all  $r \in [1, \infty)$ . We can assume that  $\lambda_n < 1$  eventually, otherwise there is nothing to prove since along a subsequence  $\{V_n\}$  has all the properties claimed in the statement of the Lemma. But if  $\lambda_n$  is strictly smaller than 1 then  $V_n(r)$  necessarily touches  $V(r)$  at some point  $r = \xi_n \in [1, \infty)$  (see Figures 4.1 and 4.2 below), otherwise  $\lambda_n$  would not be the largest number in  $(0, 1)$  for which  $\lambda_n v_n(r) \leq V(r)$  for all  $r \in [1, \infty)$  (recall that each  $v_n$  is compactly supported,  $V$  is strictly positive and both  $v_n$  and  $V$  are continuous). Now there are two possibilities: either the sequence  $\{\xi_n\}$  stays bounded or it is unbounded. In the first case,  $\{\xi_n\}$  converges along subsequences to a certain number  $\bar{\xi} \in [1, \infty)$ . Since  $\{v_n\}$  also converges *locally uniformly* in  $[1, \infty)$  to  $V$  (see Remark 4.4.2 below), then

$$v_n(\xi_n) \rightarrow V(\bar{\xi}) > 0;$$

however, by definition of  $\xi_n$ ,

$$\lambda_n v_n(\xi_n) = V(\xi_n) \rightarrow V(\bar{\xi}),$$

whence  $\lambda_n \rightarrow 1$ . In the case  $\xi_n \rightarrow \infty$  (again, along subsequences), clearly each  $\xi_n$  lies in the interior of  $[1, n)$  eventually. Therefore, in addition to  $V_n(\xi_n) = V(\xi_n)$ , we also have  $V'_n(\xi_n) = V'(\xi_n)$ , so that

$$-\lambda_n (\sinh \xi_n)^{N-1} v'_n(\xi_n) = -(\sinh \xi_n)^{N-1} V'_n(\xi_n) = -(\sinh \xi_n)^{N-1} V'(\xi_n) \rightarrow C_V > 0$$

and

$$-(\sinh \xi_n)^{N-1} v'_n(\xi_n) = \int_0^{\xi_n} \frac{v_n^{\frac{1}{m}}(s)}{T_n} (\sinh s)^{N-1} ds \rightarrow \int_0^\infty \frac{V^{\frac{1}{m}}(s)}{T} (\sinh s)^{N-1} ds = C_V,$$

where we have used once again the fact that  $\{v_n\}$  converges to  $V$  in  $L_{\text{rad}}^{1/m}$  and  $T_n \uparrow T$ . This means that in this case as well  $\{\lambda_n\}$  is forced to go to 1 and so  $\{V_n\}$  is indeed a sequence which has all the required properties (no subsequence is needed since the ongoing proof actually holds along any subsequence). Just note that the parameter  $T_n$  appearing in (4.4.3) here is actually  $T_n \lambda_n^{1/m-1}$ .  $\square$

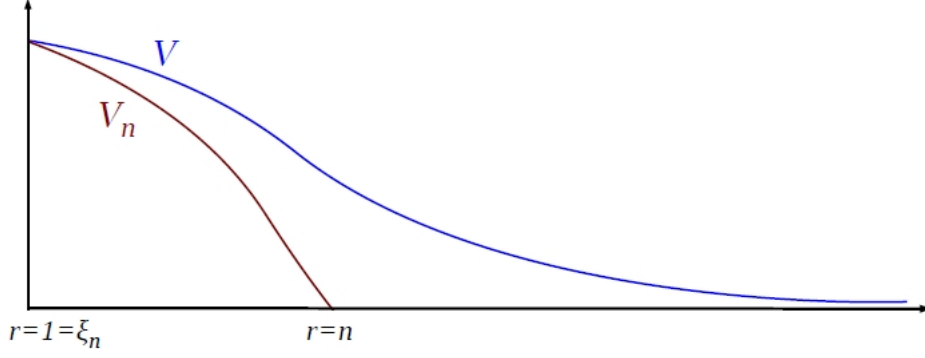


Figure 4.1:  $V_n$  touches  $V$  at  $\xi_n = 1$ .

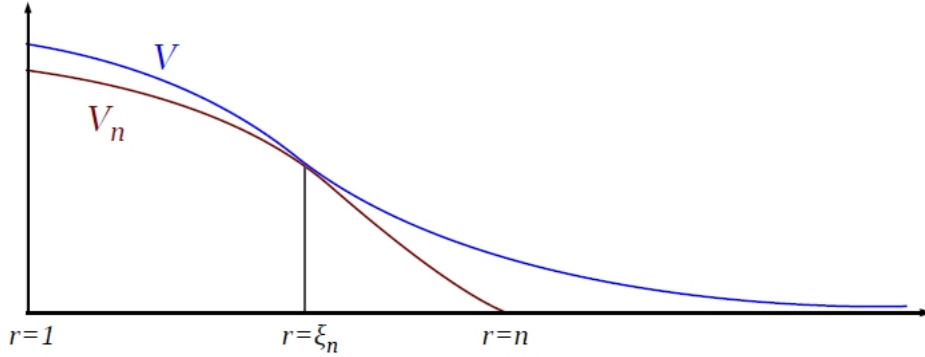


Figure 4.2:  $V_n$  touches  $V$  at  $\xi_n > 1$ .

**Remark 4.4.2.** As mentioned in the proof above, convergence of the sequence  $\{v_n\}$  solving (4.4.5) to  $V$  also occurs *locally uniformly* in  $[1, \infty)$ . This fact can be proved in several ways. One of them is the following: *a priori*  $\{v_n\}$  converges weakly in  $H_{\text{rad}}^1$  to  $V$  (and so strongly in  $L_{\text{rad}}^{1+1/m}$ ), but using the equation solved by each  $v_n$  and the fact that  $T_n \rightarrow T$  one gets  $\|v_n'\|_2 \rightarrow \|V'\|_2$ . Therefore such convergence is actually strong. Since  $H_{\text{rad}}^1$  is locally (in  $(0, \infty)$ ) continuously embedded in  $L^\infty$ , the assertion follows.

For reasons that will become clearer later (see the proof of Lemma 4.4.4 below), instead of studying  $\psi$  it is convenient to work with its natural approximation  $\psi_n := 1 - w^m/V_n$ , where  $\{V_n\}$  is the sequence constructed in Lemma 4.4.1. It is straightforward to check that  $\psi \geq \psi_n$  and that the equation solved by  $\psi_n$  is the following:

$$\frac{1}{m} (1 - \psi_n)^{\frac{1}{m}-1} \psi_{n,\tau} = V_n^{1-\frac{1}{m}} \Delta \psi_n + 2 \frac{V_n' \psi_n'}{V_n^{\frac{1}{m}}} + \frac{1}{(1-m)} \left[ \frac{1}{T_n} (1 - \psi_n) - \frac{1}{T} (1 - \psi_n)^{\frac{1}{m}} \right], \quad (4.4.13)$$

in the region  $\{(r, \tau) \in (0, n) \times (0, \infty)\}$ .

#### 4.4.2 Barriers and global convergence in $L^\infty$

We first provide a suitable family of supersolutions to (4.4.13), in the spirit of [31].

**Lemma 4.4.3.** *Let  $\tau_0 > 0$ . Consider the function*

$$\Psi(r, \tau) := C - \frac{B}{r} - A(\tau - \tau_0). \quad (4.4.14)$$

If the positive parameters  $A, B, C, \bar{r}$  comply with the condition

$$BK_\epsilon e^{(N-1)(1-\epsilon)(\frac{1}{m}-1)\bar{r}} \geq \frac{A}{m} + \frac{2}{(1-m)T}, \quad (4.4.15)$$

where  $\epsilon > 0$  is a fixed number small enough ( $\epsilon = 1/5$  will work) and  $K_\epsilon$  is a positive constant depending on  $\epsilon$ , then  $\Psi$  is a supersolution to (4.4.13) in the region

$$\{(r, \tau) : \Psi(r, \tau) \in [0, 1)\} \cap \{(r, \tau) \in (\hat{r}, n) \times (\tau_0, \infty)\},$$

independently of  $n$ , provided

$$\hat{r} \geq \bar{r} \vee r^*, \quad n \geq n^*,$$

$r^*$  and  $n^*$  being positive numbers depending only on  $\epsilon, m$  and  $N$ .

*Proof.* Let us compute the derivatives of  $\Psi$ :

$$\Psi_\tau = -A, \quad \Psi' = \frac{B}{r^2},$$

$$\Delta\Psi = -B\Delta\left(\frac{1}{r}\right) = -B\left(\frac{2}{r^3} - (N-1)\frac{\coth(r)}{r^2}\right).$$

In order to ensure that  $\Psi$  is a supersolution to (4.4.13) it is enough to require that

$$\frac{1}{m}(1-\Psi)^{\frac{1}{m}-1}\Psi_\tau \geq V_n^{1-\frac{1}{m}}\Delta\Psi + 2\frac{V'_n\Psi'}{V_n^{\frac{1}{m}}} + \frac{1}{(1-m)T_n}(1-\Psi),$$

namely

$$-\frac{A}{m}(1-\Psi)^{\frac{1}{m}-1} \geq \underbrace{-\frac{B}{V_n^{\frac{1}{m}}}\left[V_n\left(\frac{2}{r^3} - (N-1)\frac{\coth(r)}{r^2}\right) - 2\frac{V'_n}{r^2}\right]}_{\mathcal{R}} + \frac{1}{(1-m)T_n}(1-\Psi), \quad (4.4.16)$$

where we have neglected the nonpositive term  $-(1-\Psi)^{1/m}$ . Now we need to suitably estimate from above the r.h.s. of (4.4.16). To this end, take  $\epsilon > 0$  sufficiently small. Moreover, take  $n_0$  and  $r_0$  so large that  $T_n \geq T/2$  for all  $n \geq n_0$  and  $\coth(r) \leq 1 + \epsilon$  for all  $r \geq r_0$ . Thanks to (4.4.4) and to the fact that  $V_n \leq V$  in  $[1, \infty)$ , we have:

$$\begin{aligned} \mathcal{R} &\leq \frac{B}{V_n^{\frac{1}{m}}}\left[(N-1)(1+\epsilon)\frac{V_n}{r^2} + 2\frac{V'_n}{r^2}\right] + \frac{2}{(1-m)T}(1-\Psi) \\ &\leq \frac{B}{r^2 V_n^{\frac{1}{m}}}\left[(N-1)(1+\epsilon)V + 2(1-\epsilon)V'\right] + \frac{2}{(1-m)T}(1-\Psi) \end{aligned}$$

$$\forall (r, \tau) \in (r_0 \vee r_\epsilon \vee 1, n) \times (\tau_0, \infty), \quad \forall n \geq n_0 \vee n_\epsilon.$$

Recall that  $V$  satisfies (4.4.12): in particular, by (4.1.3), there exists  $r_1$  so large that

$$V'(r) \leq -(N-1)(1-\epsilon)V(r) \quad \forall r \geq r_1.$$

Hence,

$$\mathcal{R} \leq -(N-1)\frac{BV}{r^2 V_n^{\frac{1}{m}}}\left[-(1+\epsilon) + 2(1-\epsilon)^2\right] + \frac{2}{(1-m)T}(1-\Psi) \quad (4.4.17)$$

$$\forall(r, \tau) \in (r_0 \vee r_1 \vee r_\epsilon \vee 1, n) \times (\tau_0, \infty), \quad \forall n \geq n_0 \vee n_\epsilon.$$

Since  $\epsilon$  is small (it is enough that  $-(1 + \epsilon) + 2(1 - \epsilon)^2 > 0$ ) and  $V_n \leq V$  in  $[1, \infty)$ , we can replace  $V_n$  with  $V$  in (4.4.17), which yields

$$\mathcal{R} \leq -\frac{B}{r^2 V^{\frac{1}{m}-1}} \underbrace{(N-1) [-(1+\epsilon) + 2(1-\epsilon)^2]}_{K_\epsilon} + \frac{2}{(1-m)T} (1-\Psi) \quad (4.4.18)$$

$$\forall(r, \tau) \in (r_0 \vee r_1 \vee r_\epsilon \vee 1, n) \times (\tau_0, \infty), \quad \forall n \geq n_0 \vee n_\epsilon.$$

Collecting (4.4.16) and (4.4.18), we deduce that a necessary condition for  $\Psi$  to be a supersolution to (4.4.13) in the region

$$\{(r, \tau) : \Psi(r, \tau) \in [0, 1)\} \cap \{(r, \tau) \in (r_0 \vee r_1 \vee r_\epsilon \vee 1, n) \times (\tau_0, \infty)\}$$

is

$$\frac{B}{r^2 V^{\frac{1}{m}-1}} K_\epsilon \geq \frac{A}{m} (1-\Psi)^{\frac{1}{m}-1} + \frac{2}{(1-m)T} (1-\Psi). \quad (4.4.19)$$

Still from (4.4.12) and (4.1.3) we infer that

$$\frac{K_\epsilon}{r^2 V^{\frac{1}{m}-1}} \geq K_\epsilon e^{(N-1)(1-\epsilon)(\frac{1}{m}-1)r} \quad \forall r \geq r_2$$

for another positive constant  $K_\epsilon$ , provided  $r_2$  is large enough. The final condition on  $\bar{r}$  is then

$$BK_\epsilon e^{(N-1)(1-\epsilon)(\frac{1}{m}-1)\bar{r}} \geq \frac{A}{m} + \frac{2}{(1-m)T}. \quad (4.4.20)$$

Summing up, if one fixes the positive parameters  $\epsilon$  (we can take *e.g.*  $\epsilon = 1/5$ ),  $A, B, C$ , picks  $r^* = r_0 \vee r_1 \vee r_2 \vee r_\epsilon \vee 1$  and  $\bar{r}$  so large that (4.4.20) is satisfied, then the function  $\Psi$  as in (4.4.14) is a supersolution to (4.4.13) in the region

$$\{(r, \tau) : \Psi(r, \tau) \in [0, 1)\} \cap \{(r, \tau) \in (\hat{r}, n) \times (\tau_0, \infty)\}$$

for all  $\hat{r} \geq \bar{r} \vee r^*$  and  $n \geq n_0 \vee n_\epsilon = n^*$ .  $\square$

Note that (4.4.14) is a family of supersolutions in the sense that the initial time  $\tau_0 > 0$  is a free parameter. In fact, in the next lemma, proceeding along the lines of the proof of [31, Theorem 2.1], we shall show how to use such barriers in order to prove that  $\psi$  becomes small as  $\tau \rightarrow \infty$ . To this aim it is crucial to recall the global Harnack principle (4.1.8), which entails the existence of a time  $\tau_w > 0$  and two positive constants  $c_0, c_1$  such that

$$c_0 \leq \frac{w^m(r, \tau)}{V(r)} \leq c_1 \quad \forall(r, \tau) \in [0, \infty) \times [\tau_w, \infty), \quad c_0 < 1 < c_1. \quad (4.4.21)$$

**Lemma 4.4.4.** *Let  $\psi := 1 - w^m/V$ . There holds*

$$\limsup_{\tau \rightarrow \infty} \sup_{r \in [0, \infty)} \psi(r, \tau) \leq 0.$$

*Proof.* First of all consider the barrier  $\Psi$  as in (4.4.14), with the choices  $A = B = 1$ . Then set  $C = 1 - c_0/2$  and assume that  $\hat{r}$  is greater than  $(2/c_0) \vee \bar{r} \vee r^*$ , where  $\bar{r}$  is taken large enough so as to satisfy

$$K_\epsilon e^{(N-1)(1-\epsilon)(\frac{1}{m}-1)\bar{r}} \geq \frac{1}{m} + \frac{2}{(1-m)T},$$

that is (4.4.15) with  $A = B = 1$ . By construction, such a  $\Psi$  is always lower than 1. Therefore, from Lemma 4.4.3, it is a supersolution to (4.4.13) in the region  $(\hat{r}, n) \times (\tau_0, \infty)$  (let  $\tau_0 \geq \tau_w$ ) for all  $n \geq n^*$  as long as it is greater than or equal to zero. Since we want to compare  $\Psi$  with the solutions  $\psi_n$ , let us check conditions on a parabolic boundary of the form

$$[\hat{r}, n] \times \{\tau_0\} \cup \{\hat{r}\} \times (\tau_0, \tau_1) \cup \{n\} \times (\tau_0, \tau_1),$$

where  $\tau_1 > \tau_0$  is a positive time to be chosen later. In view of the above choice of  $C$  and (4.4.21), on the bottom we have that

$$\Psi(r, \tau_0) = C - \frac{1}{r} \geq 1 - c_0 \geq \psi(r, \tau_0) \geq \psi_n(r, \tau_0) \quad \forall r \in [\hat{r}, n],$$

the last inequality following from the fact that  $V \geq V_n$  for  $r \geq 1$ . On the inner lateral boundary there holds

$$\Psi(\hat{r}, \tau) = C - \frac{1}{\hat{r}} - (\tau - \tau_0) = 1 - \frac{c_0}{2} - \frac{1}{\hat{r}} - (\tau - \tau_0) \quad \forall \tau \in (\tau_0, \tau_1).$$

Now let us fix a small  $\varepsilon > 0$  (e.g. with  $\varepsilon \leq c_0/2$ ). Assume that  $\hat{r}$  is also larger than  $2/\varepsilon$ . Exploiting the local uniform convergence of  $\psi$  to zero (Proposition 4.2.6), we know that if  $\tau_0 \geq \tau_0(\hat{r}, \varepsilon)$  then

$$\psi(\hat{r}, \tau) \leq \frac{\varepsilon}{2} \quad \forall \tau \geq \tau_0.$$

Hence, once  $\tau$  satisfies

$$\tau \leq \tau_1 := \tau_0 + 1 - \frac{c_0}{2} - \frac{1}{\hat{r}} - \frac{\varepsilon}{2} \geq \tau_0 + 1 - c_0 > \tau_0,$$

there holds

$$\Psi(\hat{r}, \tau) \geq \Psi(\hat{r}, \tau_1) = \frac{\varepsilon}{2} \geq \psi(\hat{r}, \tau) \geq \psi_n(\hat{r}, \tau) \quad \forall \tau \in (\tau_0, \tau_1).$$

On the outer lateral boundary  $\{n\} \times (\tau_0, \tau_1)$  it is plain that  $\Psi(n, \tau) > \psi_n(n, \tau) = -\infty$ . In fact this is the reason why we needed to suitably approximate  $V$  from below with the sequence  $\{V_n\}$ , otherwise we would have not been able to compare  $\Psi$  with  $\psi$  in an outer lateral boundary. By comparison we then infer that  $\Psi \geq \psi_n$  in the region  $[\hat{r}, n] \times [\tau_0, \tau_1]$ . In particular,

$$\psi_n(r, \tau_1) \leq \Psi(r, \tau_1) \leq \varepsilon \quad \forall r \in [\hat{r}, n]. \quad (4.4.22)$$

Passing to the limit in (4.4.22) as  $n \rightarrow \infty$  yields

$$\psi(r, \tau_1) \leq \Psi(r, \tau_1) \leq \varepsilon \quad \forall r \in [\hat{r}, \infty). \quad (4.4.23)$$

Since  $\tau_1 = \tau_0 + 1 - c_0/2 - 1/\hat{r} - \varepsilon/2$  and (4.4.23) holds true for all  $\tau_0 \geq \tau_0(\hat{r}, \varepsilon)$ , we have:

$$\psi(r, \tau) \leq \varepsilon \quad \forall (r, \tau) \in [\hat{r}, \infty) \times [\bar{\tau}, \infty), \quad (4.4.24)$$

where  $\bar{\tau} := \tau_0(\hat{r}, \varepsilon) + 1 - c_0/2 - 1/\hat{r} - \varepsilon/2$ . Thanks to (4.4.24) and to the local uniform convergence of  $\psi$  to zero, we end up with

$$\limsup_{\tau \rightarrow \infty} \sup_{r \in [0, \infty)} \psi(r, \tau) \leq \varepsilon.$$

As  $\varepsilon > 0$  is arbitrarily small, the proof is complete.  $\square$

The final step consists in proving an analogous result for  $\phi := w^m/V - 1$ . Since the arguments follow closely the ones carried out in the proofs of Lemmas 4.4.3 and 4.4.4, we shall be concise.

**Lemma 4.4.5.** *Let  $\tau_0 > 0$ . Consider the function*

$$\Phi(r, \tau) := C - \frac{B}{r} - A(\tau - \tau_0). \quad (4.4.25)$$

*If the positive parameters  $A, B, C, \bar{r}$  comply with the condition*

$$BK_\epsilon e^{(N-1)(1-\epsilon)(\frac{1}{m}-1)\bar{r}} \geq (1+C)^{\frac{1}{m}-1} \left[ \frac{A}{m} + \frac{2(1+C)}{(1-m)T} \right], \quad (4.4.26)$$

*where  $\epsilon > 0$  is a fixed number small enough ( $\epsilon = 1/5$  will work) and  $K_\epsilon$  is a positive constant depending on  $\epsilon$ , then  $\Phi$  is a supersolution to (4.4.2) in the region*

$$\{(r, \tau) : \Phi(r, \tau) > -1\} \cap \{(r, \tau) \in (\hat{r}, \infty) \times (\tau_0, \infty)\}$$

*provided  $\hat{r} \geq \bar{r} \vee r_*$ , where  $r_* = r_*(\epsilon, m, N) > 0$ .*

*Proof.* Condition (4.4.26) follows by reasoning as in the proof of Lemma 4.4.3. In fact it is even easier to deduce an inequality like (4.4.19) with  $(1-\Psi)^{1/m-1}$  replaced by  $(1+\Phi)^{1/m-1}$  and  $(1-\Psi)$  replaced by  $(1+\Phi)^{1/m}$ , since here we do not deal with the approximate sequence  $\{V_n\}$ . Hence, in order to obtain (4.4.26), one just exploits the fact that  $\Phi \leq C$  by construction.  $\square$

Thanks to the barriers (4.4.25), we can now prove the analogue of Lemma 4.4.4 for  $\phi$ .

**Lemma 4.4.6.** *Let  $\phi := w^m/V - 1$ . There holds*

$$\limsup_{\tau \rightarrow \infty} \sup_{r \in [0, \infty)} \phi(r, \tau) \leq 0.$$

*Proof.* Again, we proceed along the lines of the proof of Lemma 4.4.4. Set  $A = B = 1$ ,  $C = c_1 - 1/2$ ,  $\epsilon < 1/2$  and  $\hat{r}$  so large that it satisfies  $\hat{r} \geq \bar{r} \vee r_* \vee 2/\epsilon$ , where  $\bar{r}$  complies with (4.4.26). These choices ensure that  $\Phi$  is a supersolution to (4.4.2) in the region  $(\hat{r}, \infty) \times (\tau_0, \tau_1)$ , with

$$\tau_1 := \tau_0 + c_1 - \frac{1}{2} - \frac{1}{\hat{r}} - \frac{\epsilon}{2}.$$

Then, one takes  $\tau_0 = \tau_0(\hat{r}, \epsilon)$  so that  $\phi(\hat{r}, \tau) \leq \epsilon/2$  for all  $\tau \geq \tau_0(\hat{r}, \epsilon)$ ; in this way  $\Phi$  and  $\phi$  are correctly ordered on  $[\hat{r}, \infty) \times \{\tau_0\} \cup \{\hat{r}\} \times (\tau_0, \tau_1)$ . However, we have no clue about their relation in an outer lateral boundary of the form  $\{n\} \times (\tau_0, \tau_1)$ , for  $n > \hat{r}$ . In order to overcome this difficulty it is enough to replace  $w$  with  $w_n \leq w$ , the latter being the sequence of rescaled solutions corresponding to the approximate solutions  $u_n$  of (4.2.4), which vanish on  $\{n\} \times (\tau_0, \tau_1)$ . By construction,  $\phi_n := w_n^m/V - 1 \leq \phi$  and the equation solved by  $\phi_n$  is basically the same as (4.4.2) (just replace  $(1+\phi)^{1/m}/T$  with  $(1+\phi_n)^{1/m}/T_n$ ), so that for  $n$  large enough (that is,  $T_n$  close to  $T$ )  $\Phi$  is a supersolution to such equation as well. From here on one can proceed exactly as in the proof of Lemma 4.4.4.  $\square$

Thanks to Lemmas 4.4.4 and 4.4.6 we finally deduce (4.4.1). Going back to the original variables  $t$  and  $u(r, t)$ , we obtain convergence of  $u^m(r, t)$  to  $(1 - t/T)^{m/(1-m)}V(r)$  in relative error, which is clearly equivalent to (4.1.5).

### 4.4.3 Bounds for derivatives and global convergence in $C^k$

This subsection is devoted to proving the claimed results of Theorems 4.1.1 and 4.1.3 which deal with derivatives. To this end, we shall exploit a useful change of spatial variable and rescaling techniques in the spirit of [73].



Let

$$s = s(r) := \int_r^\infty \frac{1}{(\sinh \sigma)^{N-1}} d\sigma, \quad (4.4.27)$$

so that

$$\frac{d}{ds} = -(\sinh r)^{N-1} \frac{d}{dr}.$$

Notice that  $s \downarrow 0$  as  $r \uparrow \infty$ , while  $s \uparrow \infty$  as  $r \downarrow 0$ . Hence, the hyperbolic Laplacian of a regular function  $f = f(r) = f(r(s))$  reads

$$\Delta f = (\sinh r)^{-2(N-1)} \frac{d^2 f}{ds^2}.$$

Let  $w$  be the solution to (4.2.3). Take  $\tau_0 \geq 1$ ,  $r_0 \geq R$  with  $R$  large enough (to be chosen later) and let  $s_0 = s(r_0)$ . Consider the rescaled function

$$\mathcal{W}(y, z) := (\sinh r_0)^{\frac{N-1}{m}} w \left( r \left( s_0 + y (\sinh r_0)^{-(N-1)} \right), \tau_0 + z (\sinh r_0)^{-(N-1)(\frac{1}{m}-1)} \right), \quad (4.4.28)$$

where  $(y, z)$  varies in the square  $(-\alpha, \alpha)^2$ ,  $\alpha > 0$  being a fixed number that we shall pick later. After straightforward computations one finds that  $\mathcal{W}$  satisfies the following equation:

$$\mathcal{W}_z = \underbrace{\frac{(\sinh r_0)^{2(N-1)}}{(\sinh r)^{2(N-1)}}}_{a(r, r_0)} (\mathcal{W}^m)_{yy} + \underbrace{\frac{(\sinh r_0)^{-(N-1)(\frac{1}{m}-1)}}{(1-m)T}}_{b(r_0)} \mathcal{W}. \quad (4.4.29)$$

From here on we shall fix  $\alpha$  to be independent of  $r_0$  and  $\tau_0$  and so small that  $s$  is forced not to leave the interval  $(s_0/2, 3s_0/2)$  as  $y$  varies in the interval  $(-\alpha, \alpha)$ . This is feasible. Indeed, first note that from (4.4.27) one has

$$\lim_{r \rightarrow \infty} \frac{s(r)}{(\sinh r)^{-(N-1)}} =: C(N) > 0. \quad (4.4.30)$$

Moreover,  $s$  varies in the interval

$$(s_0 - \alpha(\sinh r_0)^{-(N-1)}, s_0 + \alpha(\sinh r_0)^{-(N-1)}).$$

Hence, if  $r_0 \geq R$  (namely  $s_0 \leq s(R)$ ) with  $R = R(N)$  large enough, a proper choice of  $\alpha = \alpha(N)$  (small enough) ensures that  $s$  does not leave the interval  $(s_0/2, 3s_0/2)$  as  $y \in (-\alpha, \alpha)$  and that  $\tau \geq 1/2$  as  $z \in (-\alpha, \alpha)$ . Still from (4.4.30) one deduces that

$$\lim_{s \downarrow 0} r(s/2) - r(s) =: \underline{C}, \quad \lim_{s \downarrow 0} r(3s/2) - r(s) =: \overline{C} \quad (4.4.31)$$

for suitable constants  $\underline{C} = \underline{C}(N) > 0$  and  $\overline{C} = \overline{C}(N) < 0$ . The above choice of  $\alpha$  and (4.4.31) imply that, for  $r_0 = r(s_0)$  large enough (again, greater than or equal to a given number  $R$  that depends only on  $N$ ), there holds

$$\frac{1}{C_1} \sinh r_0 \leq \sinh r \leq C_1 \sinh r_0 \quad \forall r \in (r(s_0/2), r(3s_0/2)), \quad (4.4.32)$$

$C_1$  being another positive constant depending only on  $N$ . Gathering all this information, let us go back to equation (4.4.29). First, from the global Harnack principle (4.1.8) and from (4.4.32), one deduces that

$$|\mathcal{W}(y, z)| \leq M \quad \forall (y, z) \in (-\alpha, \alpha)^2 \quad (4.4.33)$$

for a positive constant  $M = M(u_0, m, N)$  that does not depend on  $r_0$  and  $\tau_0$ . Moreover, recalling (4.4.28),

$$\frac{d}{dy} = - \frac{(\sinh r)^{N-1}}{(\sinh r_0)^{N-1}} \frac{d}{dr}, \quad (4.4.34)$$

where, with some abuse of notation, it is understood that

$$r(y) = r(s(y)) = r\left(s_0 + y(\sinh r_0)^{-(N-1)}\right).$$

Hence, (4.4.29), (4.4.34) and the inequality

$$\frac{1}{C_1} \cosh r_0 \leq \cosh r \leq C_1 \cosh r_0 \quad \forall r \in (r(s_0/2), r(3s_0/2)),$$

which can be proved as (4.4.32) (for a possibly different  $C_1 = C_1(N) > 0$ ), yield

$$\frac{1}{P_k} \leq \left| \frac{d^k a(r(y), r_0)}{dy^k} \right| \leq P_k, \quad |b(r_0)| \leq Q \quad \forall (y, z) \in (-\alpha, \alpha)^2, \quad \forall k \in \mathbb{N} \cup \{0\}, \quad (4.4.35)$$

again for positive constants  $P_k = P_k(N)$  and  $Q = Q(T(u_0), m, N)$  that do not depend on  $r_0$  and  $\tau_0$ . The bounds (4.4.33) and (4.4.35) permit us to conclude, as in the proofs of Theorem 1.1 and Lemmas 3.1, 3.2 of [73] (one exploits *e.g.* the regularity results of [97] and [51] for bounded solutions to nonlinear parabolic equations like (4.4.29)), that the following estimates hold true:

$$\left| \frac{\partial^k \mathcal{W}}{\partial y^k}(0, 0) \right| \leq M_{1,k}, \quad \left| \frac{\partial^k \mathcal{W}}{\partial z^k}(0, 0) \right| \leq M_{2,k} \quad \forall k \in \mathbb{N}, \quad (4.4.36)$$

$M_{1,k}$  and  $M_{2,k}$  being suitable positive constants depending only on  $M$ ,  $\alpha$ ,  $T$ ,  $m$  and  $N$ . Going back to the original variables, we infer the existence of positive constants  $A_{1,k} = A_{1,k}(u_0, m, N, \varepsilon)$  and  $A_{2,k} = A_{2,k}(u_0, m, N, \varepsilon)$  such that

$$\left| \frac{\partial^k w}{\partial r^k}(r_0, \tau_0) \right| \leq A_{1,k} e^{-\frac{N-1}{m} r_0}, \quad (4.4.37)$$

$$\left| \frac{\partial^k w}{\partial \tau^k}(r_0, \tau_0) \right| \leq A_{2,k} e^{k(N-1)(\frac{1}{m}-1)r_0} e^{-\frac{N-1}{m} r_0}, \quad (4.4.38)$$

$$\forall r_0 \geq 0, \quad \forall \tau_0 \geq \varepsilon, \quad \forall k \in \mathbb{N}.$$

In fact, in order to prove (4.4.37) and (4.4.38) in the region  $\{r_0 \geq R, \tau_0 \geq 1\}$ , one can use (4.4.28), (4.4.34), (4.4.36) and the identity

$$d\tau = (\sinh r_0)^{-(N-1)(\frac{1}{m}-1)} dy$$

recursively. To extend such bounds to  $r_0 \in (0, R)$  it is enough to apply the aforementioned regularity results of [51, 97] to  $w$  directly, since in this region  $w$  is bounded away from zero. Finally, they also hold for  $\tau_0 \geq \varepsilon$  upon letting the constants  $A_{1,k}$  and  $A_{2,k}$  depend on  $\varepsilon$  as well through  $\alpha$  and  $M$  (as a consequence of the global Harnack principle (4.1.8)).

Recalling that the solution  $V(r)$  to (4.1.4) and any of its derivatives  $V^{(k)}(r)$  behave like  $e^{-(N-1)r}$  at infinity (see (4.1.3) and formula (4.4.41) below), it is only a matter of tedious computations to check that estimates (4.4.37) imply that

$$\left\| \frac{\partial^k \varphi}{\partial r^k}(\tau) \right\|_{\infty} = |\varphi(\tau)|_{C^k(\mathbb{R}^+)} \leq A_{1,k} \quad \forall \tau \geq \varepsilon, \quad \forall k \in \mathbb{N} \quad (4.4.39)$$

up to possibly different constants that we keep denoting as  $A_{1,k}$ , where  $\varphi$  is the relative error  $\varphi(r, \tau) := w(r, \tau)/V^{1/m}(r) - 1$  (note that it is slightly different from the relative error  $\phi$  we introduced in Subsection 4.4.1). Taking advantage of the interpolation inequalities (let  $\varphi(-r, \tau) = \varphi(r, \tau)$  for all  $r > 0$ )

$$|\varphi(\tau)|_{C^k(\mathbb{R})} \leq C_{j,k} |\varphi(\tau)|_{C^j(\mathbb{R})}^{k/j} \|\varphi(\tau)\|_{\infty}^{1-k/j} \quad \forall k, j \in \mathbb{N} : k < j$$

(we refer again to [142, pp. 125-126], [30, Appendix A.3] or to [100, 101]), (4.4.1) and the bounds (4.4.39), we deduce that in fact

$$\lim_{\tau \rightarrow \infty} |\varphi(\tau)|_{C^k(\mathbb{R}^+)} = 0 \quad \forall k \in \mathbb{N}. \quad (4.4.40)$$

In view of (4.2.2), this is equivalent to (4.1.6) in Theorem 4.1.1. As for estimates (4.1.9) and (4.1.10) in Theorem 4.1.3, just notice that they can be readily deduced from (4.4.37) and (4.4.38).

Let us investigate what (4.4.40) means in terms of the spatial derivatives of  $w$  (or  $u$ ). That is, we are going to prove (4.1.7) in Theorem 4.1.1. First of all we can show, by induction, that

$$\lim_{r \rightarrow \infty} \frac{d^k V^p}{dr^k}(r) = (-p(N-1))^k \quad \forall p \neq 0, \quad \forall k \in \mathbb{N}. \quad (4.4.41)$$

Indeed, the existence of the limit in (4.4.41) is a consequence of the chain rule and (4.1.3). Then, suppose that (4.4.41) holds for a given  $k$ . Consider the identities

$$\lim_{r \rightarrow \infty} \frac{d^{k+1} V^p}{dr^{k+1}}(r) = \frac{1}{p} \lim_{r \rightarrow \infty} \frac{d^{k+1} V^p}{dr^{k+1}}(r) \frac{V(r)}{V'(r)} = \frac{1}{-p(N-1)} \lim_{r \rightarrow \infty} \frac{d^{k+1} V^p}{dr^{k+1}}(r). \quad (4.4.42)$$

Since, in view of (4.1.3), the last limit in the r.h.s. of (4.4.42) exists, by de l'Hôpital's Theorem it coincides with

$$-p(N-1) \lim_{r \rightarrow \infty} \frac{d^k V^p}{dr^k}(r),$$

and this proves the inductive step. As (4.4.41) is trivially valid for  $k = 0$ , we conclude that it holds for all  $k \in \mathbb{N}$ . Now note that, for the  $k$ -th spatial derivative of  $\varphi$ , we can exploit the binomial formula

$$\frac{\partial^k \varphi}{\partial r^k} = \sum_{j=0}^k \binom{k}{j} w^{(j)} \left(V^{-\frac{1}{m}}\right)^{(k-j)}, \quad (4.4.43)$$

where, for greater readability, we only use superscripts to denote derivatives w.r.t.  $r$ . We shall establish that, for all  $k \in \mathbb{N}$ , the ratio  $w^{(k)}(r, \tau)/V^{1/m}(r, \tau)$  converges in  $L^\infty$  to a smooth function  $G_k(r)$  such that

$$\lim_{r \rightarrow \infty} G_k(r) = (-1)^k \left(\frac{N-1}{m}\right)^k. \quad (4.4.44)$$

This is clearly equivalent to proving (4.1.7). For  $k = 0$  (4.4.44) is exactly (4.1.5), with the choice  $G_0 = 1$ . As for further derivatives, we proceed by induction. Indeed, suppose the result is true for all  $j \leq k-1$ . From (4.4.40) and (4.4.43) we get

$$\lim_{\tau \rightarrow \infty} \left\| \frac{w^{(k)}(\tau)}{V^{\frac{1}{m}}} - \sum_{j=0}^{k-1} \binom{k}{j} w^{(j)}(\tau) \left(V^{-\frac{1}{m}}\right)^{(k-j)} \right\|_{\infty} = 0. \quad (4.4.45)$$

Thanks to the inductive hypothesis and (4.4.41), (4.4.45) implies

$$\lim_{\tau \rightarrow \infty} \left\| \frac{w^{(k)}(\tau)}{V^{\frac{1}{m}}} - \sum_{j=0}^{k-1} -\binom{k}{j} G_j \frac{(V^{-\frac{1}{m}})^{(k-j)}}{V^{-\frac{1}{m}}} \right\|_{\infty} = 0.$$

So we are left with proving that the function

$$G_k(r) := - \sum_{j=0}^{k-1} \binom{k}{j} G_j(r) \frac{(V^{-\frac{1}{m}})^{(k-j)}(r)}{V^{-\frac{1}{m}}(r)}$$

complies with (4.4.44). From (4.4.41) and the inductive hypothesis, this is indeed the case provided

$$- \left( \frac{N-1}{m} \right)^k \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j = (-1)^k \left( \frac{N-1}{m} \right)^k,$$

an identity which is a direct consequence of Newton's binomial formula.

Finally, we point out that the estimate from below (4.1.11) is an immediate corollary of (4.1.7).

**Remark 4.4.7.** In order to obtain the bounds (4.1.13) and (4.1.14) of Theorem 4.1.4 (subcritical  $m$ ), just note the constants  $c_1(t, \cdot)$  and  $c_2(t, \cdot)$  appearing in (4.1.12) can be taken to be continuous functions of  $t \in (0, T)$ , as a consequence of the method of proof of such inequalities (see Section 4.3). This allows us to prove that there exists a constant  $M$  (here depending on  $\tau_0$  as well) such that (4.4.33) holds true, which is enough in order to proceed as above and get the claimed estimates.



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