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# Statistical mechanics: contributions to rigidity percolation and long range interacting systems

Julien Barre

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# UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS

Julien Barré

Habilitation à Diriger les Recherches

**Statistical mechanics: contributions to rigidity  
percolation  
and long range interacting systems**

le 2 février 2011

## Jury

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Rapporteur	Clément Sire	Université Paul Sabatier, Toulouse & CNRS
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Examineur	Thierry Dauxois	Ecole Normale Supérieure de Lyon & CNRS
Examineur	Stefano Ruffo	Université de Florence, Italie
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## Remerciements

Mes premiers remerciements vont à Leticia Cugliandolo, David Mukamel et Clément Sire, rapporteurs de cette habilitation : je suis très touché qu'ils aient accepté ce travail supplémentaire. J'exprime également ma gratitude à Alain Barrat, Yann Brenier, Thierry Dauxois, Stefano Ruffo et Emmanuel Trizac, qui ont bien voulu prendre le temps de participer au jury et d'écouter la présentation de mes travaux.

Les recherches présentées dans ce mémoire sont aussi celles de mes coauteurs; je profite de cette occasion pour les remercier des fructueux échanges que nous avons eus, et de tout ce que j'ai appris grâce à eux. Pour les mener à bien, j'ai bénéficié de conditions très favorables aussi bien au Laboratoire National de Los Alamos, qu'à l'Université de Nice au laboratoire Dieudonné et je remercie ceux qui m'ont permis de profiter de ces environnements scientifiques excellents.

Finalement, je ne peux pas conclure ce paragraphe sans un mot tendre pour Magali, Camille et Jeanne. Si le temps consacré à mes deux filles a parfois été en concurrence avec celui dédié à la recherche, elles ont fait de moi un chercheur heureux !

# Introduction to the dissertation

This dissertation presents my scientific activities since the end of my PhD in 2003. During this period, I spent two years as a post doc at the Los Alamos National Laboratory (USA), in the Center for Non Linear Studies (CNLS) and the Condensed Matter group. Then, in Fall 2005, I was recruited as a "Maître de Conférences" at the Nice-Sophia Antipolis University, in the Mathematics Department.

At first glance, my activity can be divided in two rather distinct themes: rigidity percolation on one side, and long range interacting systems on the other side, the latter regrouping a rather diverse body of works. However, my works in these two directions have several features in common. First, at the level of methods. In both cases, the problem is usually to understand some macroscopic properties starting from a microscopic modeling, using probabilistic tools: this might be a definition of the statistical mechanics endeavor in general. More precisely, tools from large deviation theory play an important role in many of the works presented here, both for rigidity percolation and long range interacting systems. Second, there are similarities in the strategies to attack the problems: I have often concentrated first on simple models, on which a detailed study is possible, before trying to extract a generic behavior.

Although the statistical mechanics of long range interacting systems was already the subject of my PhD thesis, done under the joint supervision of Thierry Dauxois and Stefano Ruffo, my research on the subject has followed new directions since 2003: my position in a mathematics department gave me the opportunity to start a mathematically oriented research project on kinetic limits for systems of interacting particles, with Pierre-Emmanuel Jabin and Maxime Hauray; at the same time, I started a collaboration with an experimental team on cold atoms in INLN (Institut Non Linéaire de Nice).

The dissertation is organized in two chapters: the first one is devoted to rigidity percolation, and the second one to long range interacting systems. In each case, I have tried to give a detailed and non technical introduction to the subject, to emphasize the motivations and questions behind my works, and

to present and summarize my contributions. I then append to each chapter a few articles which I consider to best illustrate my scientific activity since 2003.

There is one bibliography at the end of each chapter. Citations of the type [B\*] refer to works of which I am a coauthor. These references are gathered at the end of the document for clarity.

The works presented in this dissertation owe much to my numerous collaborators during these years; I take this opportunity to warmly thank all of them!

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# Chapter 1

## Contributions to rigidity percolation

Rigidity percolation is less known and much less studied than ordinary connectivity percolation. I will start this chapter by an introduction to the subject in 1.1, which is of course far from being exhaustive. I will then list some open problems in this field; some of them are motivated by purely theoretical considerations, see section 1.2, and others are related to the use of rigidity concepts in modeling real physical systems, notably network glasses, see section 1.3. In section 1.4, I introduce more precisely my contributions to the field, which are published in articles [B4,B5,B7,B15], and the book chapter [Bbook]. As this book chapter presents more technical details than [B5] and [B7], I have chosen to reproduce it in this report, as well as [B15].

My collaborators on this topic are Alan Bishop, Turab Lookman, Avadh Saxena and Olivier Rivoire. Although Olivier Rivoire did not participate to the writing of the book chapter, the work it reviews owes much to him.

### 1.1 Introduction to rigidity percolation

#### 1.1.1 Some definitions

Rigidity theory considers structures made of sites connected by links. Each link imposes a constraint, by prescribing the distance between the two sites it connects: if the actual distance between the two sites is different from the prescription, there is an associated energy cost. Rigidity theory deals with



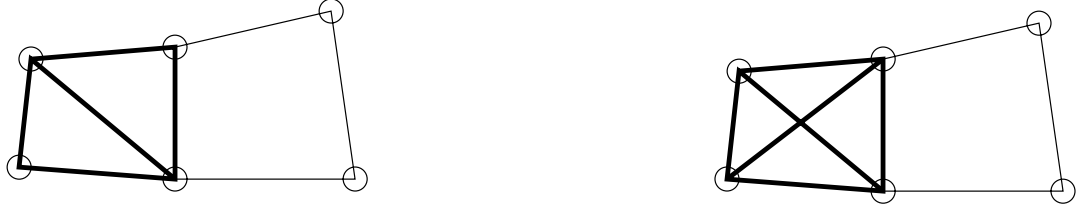


Figure 1.1: On the left hand side:  $N_s = 6$  sites,  $N_l = 8$  links; the bold links form a rigid but not overconstrained cluster; the total number of degrees of freedom of the network is  $N_{flop} = 4$ . Constraint counting is correct, as it gives  $N_{flop} = 2N_s - N_l = 4$ . On the right hand side: 6 sites, 9 links; the bold links form a rigid and overconstrained cluster; the total number of degrees of freedom of the network is still 4, since the added link is redundant. Constraint counting is not correct, as it would give  $N_{flop} = 3$ .

properties associated with the structure's topology, which do not depend on the physical nature of the constraints, nor on the precise form of the energy cost. Typical questions asked by rigidity theory are: how many degrees of freedom are left in the system? Is there a macroscopic cluster of sites rigidly connected one with the others? By contrast, questions related to the elastic properties of the structure do depend on the specification of the constraints. Examples of simple networks, with their number of degrees of freedom and decomposition in rigid clusters are presented in Fig. 1.1.

When the number of links in the structure is increased, the phenomenology is as follows. For a small enough number of links, that is a small mean connectivity of the structure, there are many more degrees of freedom than constraints; there is no macroscopic rigid cluster, and many degrees of freedom are left in the system: the system is said to be floppy. At large mean connectivity, there are many more constraints than degrees of freedom; there is a macroscopic rigid cluster, and many constraints cannot be satisfied: the system is said to be stressed rigid. In between these two phases takes place the rigidity percolation transition. There is a very simple way to estimate the number of degrees of freedom, or "floppy modes", left in a system: count the total number of degrees of freedom (for instance  $2N$  for a network with  $N$  sites in 2D), and subtract the total number of constraints (one constraint per link in the setting presented here). The percolation threshold corresponds to the point where the number of floppy modes vanishes. This procedure is usually called "Maxwell constraint counting"; it is clearly only approximate, as

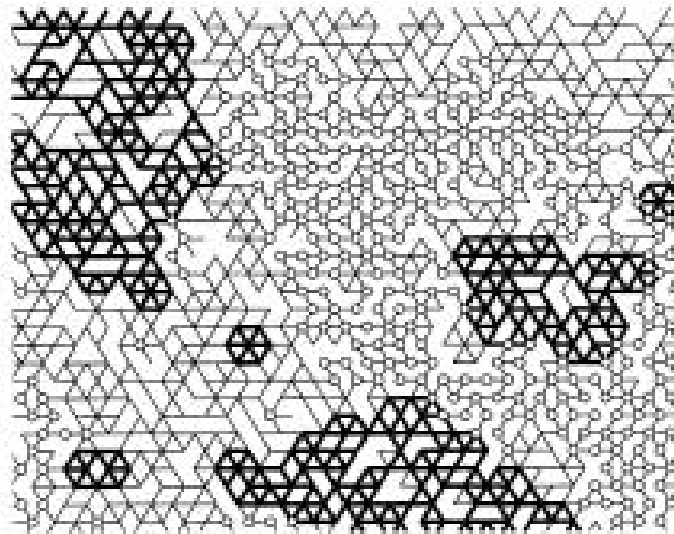


Figure 1.2: A region from a large generic triangular lattice; each bond is present with probability  $p = 0.6603$ , which is approximately the percolation threshold. Bold lines correspond to overconstrained bonds. Open circles represent sites acting as pivots between two or more rigid clusters. On this sample, there is a rigid cluster percolating from top to bottom and from left to right (the cluster containing no pivot); the overconstrained region does not percolate. This figure is drawn from the article [13], by D. Jacobs and M. Thorpe.

it forgets the redundant constraints below the rigidity percolation threshold, and the remaining floppy modes above the percolation threshold.

Fig. 1.2 shows a 2D triangular lattice close to the rigidity percolation threshold. In the literature, and in particular in the works I will present, the emphasis has been on bond percolation; however, a site rigidity percolation may also be defined, in analogy with ordinary percolation.

### 1.1.2 Generic rigidity

As explained above, some properties related to the rigidity of a network, such as the number of floppy modes, or the distribution of rigid clusters, depend only on the graph's topology. This means that they depend only on the network defined as a collection of sites and edges connecting the sites, and not

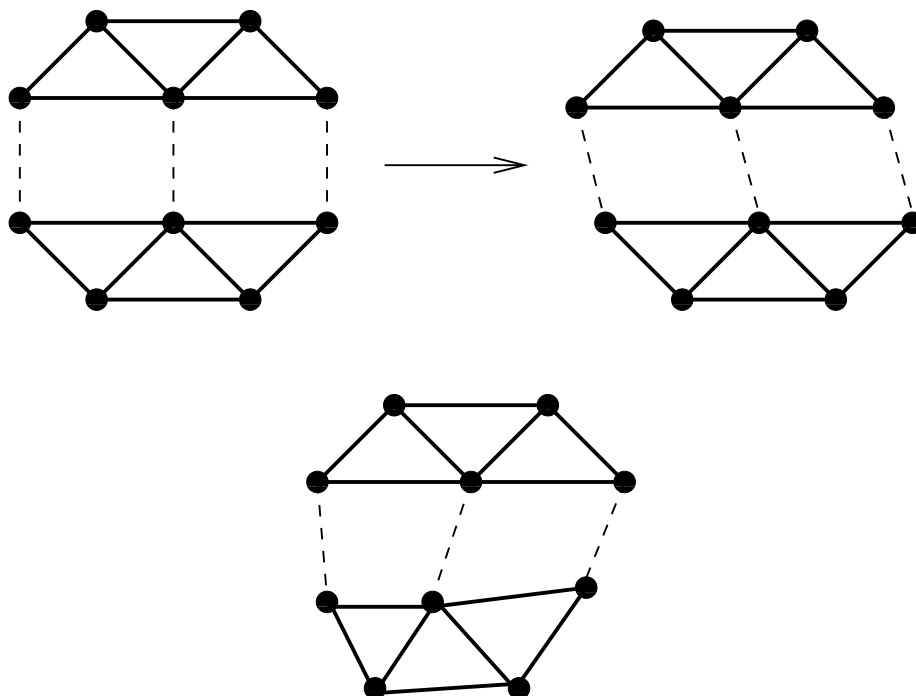


Figure 1.3: The graph on the upper row is non-generic, since for instance, the three dashed bonds are parallel. It is not rigid, as it may be “sheared”, as shown on the right. This “shearing” floppy mode is due to the parallelism of the three dashed bonds. In contrast, the lower graph is topologically equivalent to the upper one, but is generic; it is rigid.

on the precise embedding of the graph in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . The analogous statement for ordinary percolation is clear: the connectivity properties of a graph do not depend on its spatial embedding. In the case of rigidity percolation, it is less obvious, and it is actually true only for **generic** rigidity. To be a little bit more precise, an embedding of a graph in  $\mathbb{R}^2$  is said to be generic if sites and bonds do not present any “special” property, like two parallel bonds, or three aligned sites. Obviously, a slight perturbation of a non-generic embedding turns it into a generic one. For a clear and rigorous presentation of generic 2D rigidity, we refer to [10, 11]. To make the point clearer, Fig. 1.3 shows two embeddings of the same topological structure. One is generic and the other is not, and they do not have the same rigidity properties.

### 1.1.3 The pebble game algorithm

From now on, I will consider only generic rigidity. All properties considered then depend only on the topological structure of the graph. However, until the mid-nineties, it was not known how to compute these properties using only the network topology. Indeed, the basic method is as follows: consider an embedding of a graph; replace all bonds by springs; linearize the dynamics of the sites; then diagonalize the associated matrix, and count the number of zero eigenvalues. The diagonalization procedure may be slow, and, worse, it may be difficult to distinguish numerically between a mode with small frequency, and a genuine floppy mode with strictly zero frequency. Thus, this method led to slow algorithms and severe limitations on the size of the systems studied. A decisive theoretical progress was then made for 2D rigidity, with the introduction of combinatorial algorithms [12, 22], based on Laman’s theorem [16].

*Theorem :*

A graph  $G$  with  $N \geq 3$  sites and  $M = 2N - 3$  bonds, such that every subgraph  $S$  of  $G$  containing  $N_S$  sites contains at most  $2N_S - 3$  bonds is rigid.

Laman’s theorem essentially states that for 2D generic rigidity, Maxwell’s constraint counting is correct if done for each subgraph, and not only at the level of the whole graph. The algorithms in [12, 22] implement this idea in an efficient way, which avoids the combinatorial problem of having to deal with all subgraphs. These new algorithms allow for the study of much larger samples than before, as well as more precise estimates around the critical point [13]. The algorithm in [12] was called the “pebble game”; this is the one I have used in my own works.

Laman’s theorem is valid in two dimensions. Its generalization to three dimension would read (taking into account the fact that a rigid body in three dimensions has 6 degrees of freedom):

*A graph  $G$  with  $N \geq 3$  sites and  $M = 3N - 6$  bonds, such that every subgraph  $S$  of  $G$  containing  $N_S$  sites contains at most  $3N_S - 6$  bonds is rigid.*

Unfortunately, this statement is wrong! A counterexample, sometimes called in the literature the “double-banana” graph, is shown on Fig. 1.4.

The lack of a Laman’s like theorem in three dimension essentially prevents the generalization of the pebble-game algorithm. However, M. Thorpe and M. Chubynsky have recently shown that in some cases, the errors induced

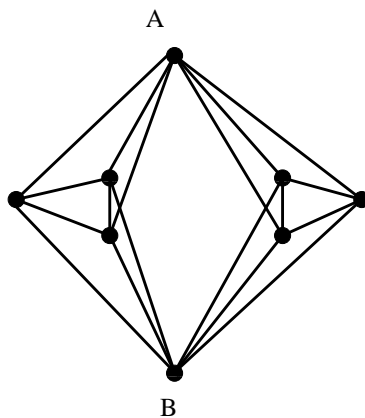


Figure 1.4: This graph has  $N = 8$  sites and  $M = 3N - 6 = 18$  bonds. One may check easily that each subgraph with  $N_S$  sites contains at most  $3N_S - 6$  bonds. However, both “bananas” may freely rotate around the axis defined by the sites labeled  $A$  and  $B$ . It is thus a counter example to the 3D generalization of Laman’s theorem.

by the dangerous configurations, such as the one in Fig. 1.4, are extremely small. In these cases, the pebble-game can be safely used. [5].

Up to now, I have presented only the standard rigidity percolation picture, where each bond brings only one constraint. It is usually called rigidity percolation in “central force” networks. In a variant, very important for modeling purposes 1.3, one considers that the angles between adjacent bonds are fixed, in addition to the bond’s lengths. Networks with this type of angular constraints are usually called “bond bending” networks. Although there is no proof of a “Laman like” theorem for bond-bending networks, it has been conjectured that a statement generalizing Laman’s theorem holds in 3D in this case. An associated pebble-game algorithm has thus been developed, and is conjectured to be exact [14]. As a final remark, let us mention that rigidity percolation on a 2D bond-bending network is actually equivalent to ordinary connectivity percolation.

## 1.2 Some analytical approaches and open questions

Since the introduction of the pebble game algorithm in 1995, numerical investigations of the rigidity percolation are much more precise. On the other hand, rigorous results, and even analytical results obtained with physicist's methods are still rare. I briefly review here the works I am aware of.

### 1.2.1 Some rigorous results

The problem of rigidity percolation, which mixes combinatorics and probabilities, has attracted the attention of mathematicians, at least since Laman [16]. Yet, the mathematical studies are much less numerous than in the field of ordinary percolation. Holroyd [11] has proved the existence of a critical probability  $p_r$  for bond rigidity percolation on the triangular lattice, with  $p_r < 1$ , and  $p_r$  strictly larger than the ordinary percolation critical probability, which is  $2 \sin \pi/18 \simeq 0.347$  on a triangular lattice. He also proved the uniqueness of the infinite rigid cluster above the threshold, a result extended for percolation in any dimension by Häggström [8]. Some results for Erdős-Rényi random graphs were recently obtained [29], but there is no rigorous proof yet of a threshold in this case.

In any case, I am not aware of any rigorous result precise enough to compute exactly a rigidity threshold, nor give insight into the order of the phase transition or the critical exponents.

### 1.2.2 Dimension two

For rigidity percolation on 2D regular lattices (typically, triangular), the scenario of a second order phase transition in a different universality class than ordinary percolation seems favored by the numerics, although there has been some debate on the subject [6]. To my knowledge, there has been no satisfactory analytical estimation of the critical exponents, which could be compared with the numerics. This is in sharp contrast with the ordinary percolation case, where the exponents are exactly known, and various approximation schemes have been used (series expansions, renormalization procedures,  $\epsilon$ -expansions...). We quote here a field theoretical attempt by Obukhov [24], which predicts a first order phase transition in 2D. This, in the light of the

numerics, seems to be a non generic feature. There has been also earlier attempts to study rigidity percolation through Position Space Renormalization Group [7, 15]; based on small renormalization cells, the associated predictions for the critical exponents are unprecise, and cannot discriminate between ordinary and rigidity percolation.

### 1.2.3 Dimension three

Central force rigidity percolation in 3D networks has been studied numerically using the pebble-game algorithm by M. Chubynsky and M. Thorpe [5] (in this paper, they first show that the error incurred by using the pebble game, which is not exact in this case, is negligible).

They present convincing numerical evidence that the transition is first order on a Face Centered Cubic (FCC) lattice, but second order on a Body Centered Cubic (BCC) lattice. This result, which is very surprising in views of the standard “universality” picture, has not yet received any explanation.

### 1.2.4 Models on trees and random graphs

The only known exactly solvable models (in a physicist’s sense) of rigidity percolation are models on trees, or random graphs having a locally tree-like topology [23, 31, 6, 2] (see section 1.4 for more details). However, in these types of models, the rigidity percolation transition is usually first order, whereas it is second order for generic 2D rigidity. Thus, they should be use with caution.

Despite the previous remark, notice that these models on random graphs may be used to “mimick” rigidity percolation in any dimension: although the graph has no dimension (and one may argue it has an effective infinite dimension), one chooses in the model the number of degrees of freedom  $f$  attributed to each site.  $f = 1$  corresponds to usual connectivity percolation;  $f = d > 1$  to rigidity in  $d$  dimensions <sup>1</sup>.

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<sup>1</sup>More generally, the dimension of the lattice on which one studies the percolation phenomenon needs not in principle be equal to the number of degrees of freedom of a single site.

## 1.3 Modeling issues

### 1.3.1 Network glasses

Despite its clearly mechanical origin, the problem of rigidity percolation has also attracted attention in the last 30 years because of its applications in understanding the properties of network forming glasses, like GeSe or GeAsSe alloys [25, 30]. In this case, the atomic bonds and the angles between adjacent atomic bonds should be considered as constraints. Such glasses are thus modeled as bond-bending networks.

It is interesting to perform the constraint counting approximation in this setting. Consider a Selenium atom, which has connectivity 2. It brings two length constraints, and one angular constraint. A Germanium atoms has connectivity 4, and brings 4 length constraint and 5 angular constraint. In general, an atom with connectivity  $k$  brings  $k$  length constraints and  $2k - 3$  angular constraints. The length constraints should be counted with weight one half, as they are shared between two neighboring atoms. Thus, calling  $r$  the mean connectivity of an atomic network, the number of constraints per atom is  $c = r/2 + 2r - 3$ . Maxwell's approximation predicts a rigidity transition when  $c \simeq 3$ , which translates in  $r \simeq 2.4$ . On the basis of this approach, one would then expect some changes in the properties of a glassy alloy  $\text{Ge}_x\text{Se}_{1-x}$  for  $x \simeq 0.2$ .

The modeling problems I have considered were related to this application to glasses, see section 1.4.2. Other models, such as cross linking stiff fibers forming random networks were shown to fall in the same universality class as central force 2D rigidity [17], and this type of system has been used to model network forming proteins [9].

### 1.3.2 Self-organization and the intermediate phase

About 10 years ago, Thorpe et al. have opened a new research direction by introducing the notion of network self-organization in rigidity percolation models [32]. This is a natural idea in the context of network glasses modeling: constraints that are not satisfied create stress and bear an energy cost; the network should then self-organize, i.e. tend to modify its structure, in order to minimize this energy cost. In [32], numerical simulations allowing self organization point to the existence of a new phase in between the usual floppy and rigid ones. As a function of the mean connectivity, the phase



diagram would then show two phase transitions instead of one. Around the same time, several experiments on network glasses drew a picture compatible with this predicted phenomenology [28, 33, 1]. A problem of the original simulations of [32] was their strongly out of equilibrium character; some theoretical studies improved on this, and seemed to confirm the “three phases” picture [20, 21].

## 1.4 Contributions

### 1.4.1 Using the cavity method [B4]

To go beyond the simple Maxwell “constraint counting” approximations, rigidity percolation models have been solved on trees; using the same method, called in the literature the “transfer matrix method”, various types of random graphs can be solved as well [23, 31, 6, 2]. The important point is that the graph should be locally tree-like. In other words, small loops should have a negligible effect. In this case, two sites that share a common neighbor may be considered practically statistically independent, once their common neighbor is removed. This independence hypothesis, which is fulfilled on a tree, and is a reasonable approximation in the absence of small loops, allows to solve the corresponding models. This is nothing but the traditional idea of solving models on Bethe lattices, adapted to rigidity percolation.

The cavity method was introduced in the 80’s in the context of spin glasses [18]. In the last 10 years, it was revisited, and proved extremely powerful to compute various quantities related to probabilistic and combinatorial models on random graphs [19], including in the event of replica symmetry breaking.

In this paragraph we introduce a combinatorial optimization problem, which we conjecture to be equivalent to rigidity percolation in the large  $N$  limit, in the case of random graphs. Let us consider first a network where each atom has  $d$  degrees of freedom, and each bond carries only one constraint (bond stretching). A bond removes one degree of freedom assigned to one of its neighboring atoms; we represent this by orienting the bond towards the atom which loses one degree of freedom. Once all bonds are oriented, one may count all the degrees of freedom (=floppy modes) remaining; this yields

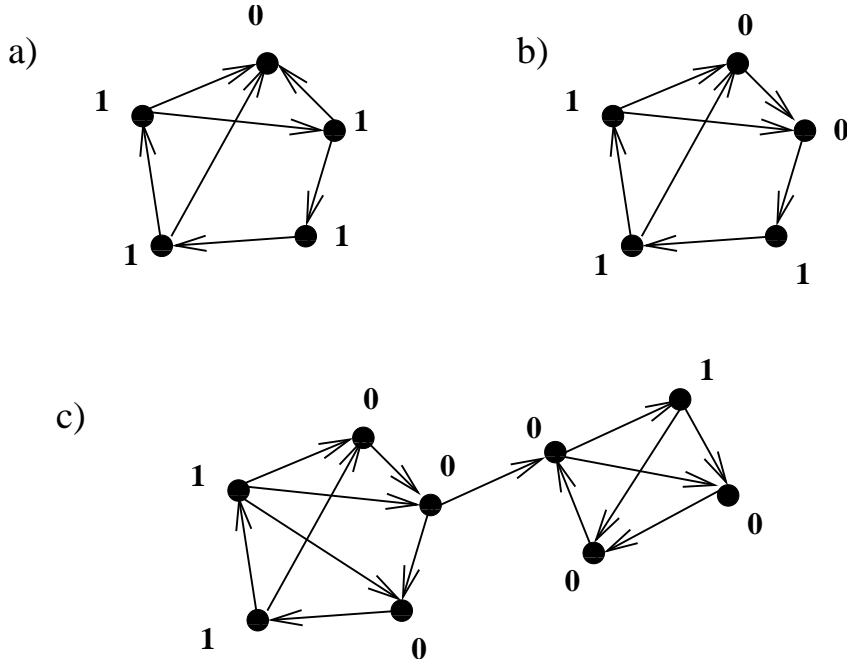


Figure 1.5: Upper-left oriented graph *a*):  $N_{flop}^\omega = 4$  but the bond orientation  $\omega$  is not optimal. Upper-right oriented graph *b*):  $N_{flop}^\omega = 3$  and the bond orientation  $\omega$  is optimal. Indeed, the graph *a*) and *b*) is rigid, and has three degrees of freedom. Lower oriented graph: this is an example where Eq. (1.2) does not yield the correct result. An optimal orientation  $\omega$  yields  $N_{flop} = 3$ , whereas the true result is  $N_{flop} = 5$ .

the following formula, for a given orientation of the bonds,  $\omega$ :

$$N_{flop}^\omega = \sum_{\text{atoms}} \max(0, d - d_i^{in}) , \quad (1.1)$$

where  $d_i^{in}$  is the number of bonds oriented towards atom  $i$ . The “max” expresses the fact that the number of degrees of freedom of an atom cannot be negative. Eq. (1.1) attributes a number of floppy modes to a given orientation of the bonds. The number of degrees of freedom of the network is the minimum of this quantity over all orientations of the network, that is,

$$N_{flop} = \min_{\omega \in \mathcal{O}} N_{flop}^\omega , \quad (1.2)$$

where  $\mathcal{O}$  represents all orientations of the network. Examples are given in Fig. 1.5.

Eqs. (1.1) and (1.2) recast rigidity percolation as finding the minimum of a function over an ensemble of  $2^{N_{bonds}}$  possible orientations. We stress once again that, in contrast to the combinatorial representation used in the pebble game algorithm, this combinatorial optimization formulation of rigidity

percolation, as given by Eqs. (1.1) and (1.2), is not strictly equivalent to the original problem. Fig. 1.5 provides an example of this fact. One sees that the discrepancy comes from the three global degrees of freedom that each graph always has: two translations and one rotation. Eqs. (1.1) and (1.2) wrongly allows the bonds to remove these degrees of freedom. The error incurred for one cluster is at most 3, which is negligible in the large  $N$  limit, as there is an extensive number of degrees of freedom. However, if there are many small rigid clusters, the errors may sum up. For a diluted random graph, one expects that this does not happen. Hence, we rely on the following conjecture, which seems reasonable but is unproved, despite the previous arguments:

*The error incurred in the computation of floppy modes by using Eq. 1.2 is subdominant in the large  $N$  limit for diluted random graphs.*

Recast in this combinatorial optimization form, the problem becomes perfectly suited for the cavity method. This remark, illustrated by an example of rigidity percolation in a small world chain was my first contribution to the rigidity percolation field [B4].

It turns out that the combinatorial optimization problem associated with rigidity percolation does not seem to present replica symmetry breaking, at least in all the forms it was studied so far. A priori, this makes the use of the cavity method only a minor technical improvement with respect to the traditional techniques used in models of rigidity percolation on trees. However, it is the basis for the use of the more sophisticated "large deviation cavity method", which gives access to informations not available with the traditional techniques (see 1.4.3).

### 1.4.2 A solvable model for the intermediate phase [B5]

The goal of [B5] was to get a clear picture of the effect of self-organization on rigidity percolation, by studying simple solvable models. Since the only solvable models at hand at this time were models on trees and random graphs, the starting idea was to introduce self-organization in a random graph model of rigidity percolation. We chose the following model. A random graph is made of  $Nx_1 = N_1$  sites with one bond and  $Nx_3 = N_3$  sites with 3 bonds, with  $x_1 + x_3 = 1$ . Each site originally carries 3 degrees of freedom. The length of the bonds are considered as constraints, as well as the angles between two adjacent bonds. The number of length constraints is then  $(N_1 + 3N_3)/2$ , and

the number of angles constraints is  $3N_3$  (each 3-coordinated site brings three angular constraints). When  $x_3$ , the fraction of 3-coordinated sites, increases, this model presents a rigidity percolation transition, which may be studied using for instance the cavity method introduced in section 1.4.1. This study assumes that the two types of sites are linked in the graph in a maximally random manner. Self-organization of the network now has a precise meaning: it corresponds to the building of correlations between neighboring sites, so that the random graph is not maximally random any more. A combinatorial entropy can be associated to these correlations, by counting the graph configurations.

We introduced an energy, which associates to each configuration of the graph the number of constraints that are not satisfied. Using combinatorial methods, we were also able to compute approximately the entropy of a given graph configuration: the higher the correlations between sites, the further is the configuration from the maximally random case, and the lower is the entropy. Putting everything together, we analytically predicted in this model two phase transitions, when  $x_3$  is increased. At low  $x_3$ , all constraints are easily satisfied, energy is zero, and the graph is maximally random; at intermediate  $x_3$ , the graph is still able to satisfy all constraints so that energy is zero, but at the cost of some self-organization: the entropy decreases; at high  $x_3$ , the self-organization is not sufficient to satisfy all constraints, and rigidity and stress percolate; energy is now positive.

All these findings were compared with Monte Carlo simulations, performed using an implementation of the pebble-game algorithm made available to us by M. Chubynsky. The pebble game allows us to compute the energy of a given graph configuration, that is the number of unsatisfied constraints. It is then coupled with a procedure of random rewiring of the network and a Metropolis algorithm to sample the different network's configurations. The results are in qualitative agreement with the theory, and the agreement improves when the approximation in the entropy computation is refined, see Fig 1.6.

The main advantage of this work is that it introduces simple models of rigidity percolation incorporating self-organization in an equilibrium statistical mechanics context; these models are amenable to analytical treatment. It also provides a picture of the intermediate phase as being exactly on the verge of rigidity percolation, in a way somewhat similar to self organized criticality. The main drawback is that this work is based on a random graph, which topology is far from that of the networks one would like to model. Chubyn-

## Intermediate

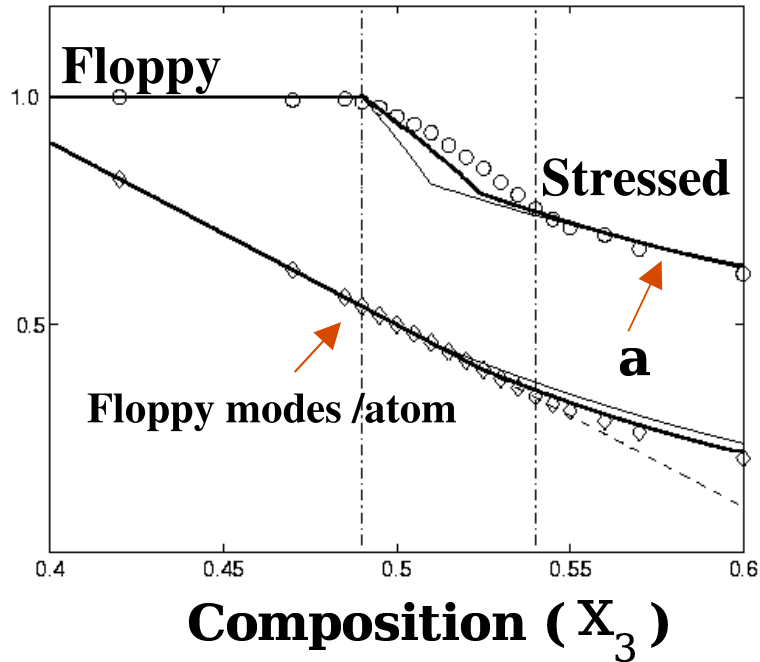


Figure 1.6: Lower curve: the number of floppy modes per site. The straight line at small connectivity (small  $x_3$ ) indicates the absence of overconstrained clusters. Upper curve: a parameter characterizing the departure from a maximally random graph, ie the self-organization.  $a = 1$  corresponds to the maximally random case. Symbols are from Monte-Carlo simulations, solid lines are approximate theoretical computations. The bold line takes better into account (although not exactly) the correlations between neighboring sites, and thus lies closer to the numerical data.

sky et al. then used an idea similar to the one above, using an underlying 2D lattice instead of a random graph, and working at zero temperature. Their study then had to be numerical. The results in [3, 4] are similar to the one in [B5]. One may wonder however if this phenomenology persists at non zero temperature.

### 1.4.3 Exact solution using the large deviation cavity method [B7,Bbook]

In [B5], the computation of the entropy of a graph configuration was done in an approximate way, with simple combinatorial methods. Computing exactly this entropy is equivalent to computing the probability for a random graph to have a given number of redundant constraints, or rather the large deviation function associated with this probability.

From the conjecture of section 1.4.1, the number of redundant constraints  $N_{red}$  of a graph  $g$  is the solution of an optimization problem, up to terms in  $o(N)$ , where  $N$  is the size of the graph: see Eqs. (1.1) and (1.2). The problem then boils down to computing a large deviation function with respect to the randomness of the graph  $g$  for the solution of a combinatorial optimization problem on this graph.

It turns out that this kind of questions was studied in 2005 by O. Rivoire, who introduced a refinement of the cavity method, called the "large deviation cavity method", which is well-suited to perform explicit computations in this case [27]. Using [B4] and [27], it is then possible to compute exactly the entropy of a self-organizing rigidity percolation model on a random graph.

This computation is performed in [B7], and the details are published in [Bbook]. For simplicity, we used an Erdős-Rényi random graph, with the connectivity  $\gamma$  as control parameter.

The results are qualitatively the same as in section 1.4.2: there are two true phase transitions, and an intermediate phase, when increasing  $\gamma$ . The results were again compared with numerics obtained from Monte Carlo simulations.

### 1.4.4 Hierarchical lattices [B15]

My most recent contribution to the field is the introduction of a class of solvable models of rigidity percolation, distinctly different from the class of

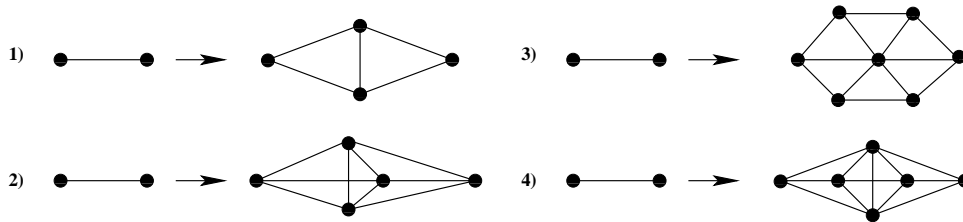


Figure 1.7: Four examples of hierarchical lattices. On the left, the graphs at step  $t = 0$ , with two sites and one bond; on the right, the graphs at  $t = 1$ . We call the  $t = 1$  graph the “elementary cell”. 1), 2), 3) and 4) have elementary cells with respectively 4 sites and 5 bonds, 5 sites and 9 bonds, 7 sites and 12 bonds, and 6 sites and 12 bonds. At step  $t$ , each bond of step  $t - 1$  is replaced by a subgraph identical to the  $t = 1$  graph.

”locally tree-like” models. The idea is to use hierarchical lattices, on which it is possible to define an exact renormalization transformation in the real space.

Hierarchical lattices are constructed as follows. We start from two sites, connected by one bond. The graph is then constructed iteratively; at each step, all bonds are replaced by a given elementary cell. Four examples of elementary cells, corresponding to four examples of hierarchical graphs, are given on Fig. 1.7.

In [B15], I show how to compute exactly the rigidity percolation threshold and the number of floppy modes for hierarchical lattices; I also analyze these models in relation with the ”intermediate phase” discussed above. Contrary to all trees and random graphs, rigidity percolation on these hierarchical lattices is a second order transition; this makes it closer to the case of a regular 2D lattice. However, the critical exponents strongly depend on the hierarchical lattice; it is then impossible to infer an estimation of the critical exponents of 2D generic rigidity percolation from the simple hierarchical lattices studied in [B15].

To this date and to my knowledge, there is still no analytical or semi-analytical theory for generic rigidity percolation on regular lattices, that would predict the critical exponents, and even the order of the transition (apparently second order in 2D, and first order in 3D). The hierarchical lattices may give a way to attack this problem. Indeed, in ordinary connectivity percolation, real space renormalization transformations on large elementary cells have been used to obtain fairly accurate estimates of the critical expo-

nents [26]. One could think of using a similar approach in the rigidity case, although it is not clear how to do it.



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## 1.5 Appendix

### 1.5.1 [B5]

## 1.5.2 [Bbook]

### 1.5.3 [B15]

# Chapter 2

## Long range interacting systems

### 2.1 Introduction

#### 2.1.1 Short introduction to long range interacting systems

The basic problem of statistical physics is to understand the macroscopic behavior of a system starting from its microscopic components and their interactions. The onset of a macroscopic collective behavior, like a phase transition for instance, is an especially fascinating topic. When the interactions between the microscopic components of a system are long range, that is, act on a length scale comparable to the size of the system, the collective effects in the system may be even more dramatic: gravitational collapse, large scale structure formation, clustering... Research on these long range interacting systems has been historically driven by its connection with astrophysics, plasma physics, and two dimensional fluid dynamics (vortices interact at long range).

The field has seen a lot of development recently, motivated by the intriguing theoretical properties of these systems, especially the possibility of negative specific heat, or long lived quasi stationary states, as well as by the wide range of potential applications.

In the astrophysics community, it has been known for a long time that equilibrium properties of self gravitating systems are rather peculiar: negative specific heat, canonical (fixed temperature) and microcanonical (fixed energy) ensembles are not equivalent... These anomalous equilibrium fea-

tures of long range interacting systems have been fully appreciated by other physicists' communities only later. For instance, in the seminal works [80, 42], Hertel and Thirring exactly solve very simple toy models exhibiting the above properties. Many contributions followed, and several reviews are available [62, 21]. In our opinion, equilibrium characteristics of long range interacting systems are now very well understood. The main remaining problem is the absence of a controllable experimental system showing the predicted peculiarities (see however [36], which claims the observation of a negative specific heat region in a small atomic cluster, a system which is not strictly speaking long range).

A large part of the research on long range interacting systems focuses on their relaxation to equilibrium. Indeed, these systems usually relax to equilibrium very slowly, over a time scale diverging with the number of particles. This makes statistical equilibrium techniques often irrelevant, and calls for a detailed dynamical understanding. This slow relaxation may be understood as follows: over short time scales, the dynamics is governed by the Vlasov equation, sometimes called collisionless Boltzmann equation, which does not relax towards statistical equilibrium. Instead, the system of particles is usually stuck for a long time (diverging with the number of particles) close to a stable stationary state of the Vlasov equation; this is called a Quasi Stationary State (QSS) for the discrete system.

In the astrophysical context, this phenomenology is known since the sixties [41, 55]. The approach of a QSS is in some cases called “violent relaxation” to make a difference with the collisional relaxation - to the “true” thermodynamical Boltzmann-Gibbs equilibrium- which occurs on much longer time scales. There is some numerical evidence of universality for these QSS (see [81, 64, 65] in a cosmological context). However, the analytical understanding of the strongly non linear physics involved in the formation of these quasi-equilibrium states is, despite the importance of the problem and many attempts to solve it, extremely limited. Among these attempts, the most noticeable is due to Lynden-Bell [55]. His theory is believed to be inadequate for gravity (see e.g. [3, 4]), but recent works indicate that it could give adequate quantitative results for some class of initial conditions [54], which encourages to search in this direction. Quite often, the concepts outlined above for gravitation are actually relevant for all types of long range interactions. For instance, the links between violent relaxation in astrophysics and the Landau damping phenomenon well known in plasma physics are developed in [47]. The Lynden-Bell theory has a direct equivalent in 2D fluid



dynamics: the Robert-Sommeria-Miller theory [59, 71]. This deep analogy has been emphasized in [25]. The direct connection with the physics of wave-particles interactions, crucial for instance in plasma physics and free electron lasers, has also been seen [34]. The growing recognition that all these systems share a lot of physical phenomena fostered the detailed study of toy models, notably the Hamiltonian Mean Field model: it is a simplified 1D self gravitating system, without the small scale singularity, and will be hereafter called HMF. The hope is that general theories may be developed and tested more easily in this simplified setting. There have been several developments in this context, related to kinetic theory [10], to tests of Lynden-Bell theory [2], or to Tsallis statistical mechanics [53, 28].

## 2.1.2 Short summary of my PhD work

My PhD work was concerned with statistical mechanics of long range interacting systems. It seems useful to quickly summarize here the results obtained, and then to emphasize the new research directions I have pursued since the end of my PhD.

The starting point of my PhD work was the recognition of deep analogies between very different systems, all sharing the long range character of the interactions. The general idea was then to point out and develop these analogies, and possibly to emphasize their limit.

A first part of my PhD thesis was devoted to equilibrium statistical mechanics of long range interacting systems. Starting from the study of exactly solvable toy models [BP3], we developed with Freddy Bouchet a classification of the peculiar phase transitions and inequivalence of ensemble situations [BP9]. In a second part, I studied the approach to equilibrium in toy models (violent relaxation, Quasi Stationary State, and slow relaxation). We illustrated precisely this behavior on the Hamiltonian Mean Field model [BP7] and used Lynden-Bell theory to approximately characterize the short time behavior of a wave-particle model used in Free Electron Laser physics [BP8]. A last part of my PhD was devoted to the detailed analysis of the formation of a complex structure in a toy model, a situation which does not fit in the standard theory [BP2,BP5,BP6]. It turns out that this route to complex structure formation should not be generic.

### 2.1.3 Research directions and contributions

I try to classify here the different aspects of my works on long range interacting systems since the end of my PhD. I precise that these works only deal with classical (non quantum) systems, including in the "cold atoms" section.

1. *Equilibrium statistical mechanics.* The equilibrium statistical mechanics of long range interacting systems is in my opinion well understood. Recently, I have been interested in pointing out some similarities between equilibrium statistical mechanics in presence of long range interactions and on random graphs (sparse or not). This implied the use of techniques from disordered statistical mechanics, see Sec. 2.2.

Although it is rather a dynamical problem, I have also included in this item the study of ergodicity breaking in long range interacting systems, because our strategy was actually based on equilibrium concepts, and on large deviation techniques.

2. *Rigorous results in kinetic theory.* This new direction is related to my present position in a mathematics department, and collaboration with P.E. Jabin and M. Hauray, see Sec. 2.3. The driving question here is: is it possible to derive the Vlasov equation for particles interacting through Coulombian or gravitational forces?
3. *Contributions to the old problem of Quasi Stationary States,* described in sec. 2.1. The main questions at stake here are: what is the long-time behavior of the Vlasov equation? what is the long-time behavior of a long-range interacting system of particles? The contributions described in Sec. 2.4 aim at very partial answers.
4. *Long range interactions and cold atoms.* I started to develop this new research axis very recently, thanks to a collaboration with experimentalists in the nearby physics laboratory INLN (Robin Kaiser, David Wilkowski, Maryvonne Chalony). The starting point is the existence of long range interactions between atoms trapped in a Magneto-Optical Trap, created by their interactions with the trap lasers. The system of atoms is however very different from those described above, at it involves non potential forces, stochastic forcing and dissipation.

The four sections of this chapter will be devoted to these four items.

## 2.2 Equilibrium statistical mechanics

### 2.2.1 Large deviations [B1]

As explained above, equilibrium statistical mechanics of long range interacting systems display some unusual phenomena, such as inequivalence between ensembles, and negative specific heat regions. It does not mean however that it is a difficult problem, in the physical or mathematical sense. One could actually argue just the contrary. In a long range interacting system, each particle, or site in a lattice, feels many other particles or sites; this implies that the statistical fluctuations of the potential or force that it feels are small. In physical terms, this means that a mean-field like approximation becomes exact in the large system size limit. The mathematical side of this remark is that many long range interacting systems are exactly solvable using large deviation techniques, once an appropriate scaling is chosen. A clear account of this general idea is given for instance in [33]. These large deviation techniques were already extensively used in my PhD thesis. In [B2], we give several simple physical applications of this technique. Since this work is rather directly related to my PhD thesis, I have chosen not to develop it here.

### 2.2.2 Metastable states and small systems [B3]

We consider a many-body Hamiltonian, with some conserved quantities, typically the energy per particle  $e$ . We describe the macroscopic state of this system by a few intensive parameters beyond the energy, typically the magnetization  $\mathbf{m}$  for a spin system. We define the entropy as a function of energy and the intensive parameters  $s(e, \mathbf{m})$  as the logarithm of the phase space volume compatible with these intensive parameters. The domain of definition of  $s(e, \cdot)$ , that is the region in the  $\mathbf{m}$  space where  $s(e, \mathbf{m})$  is not  $-\infty$ , is the accessible parameter region at energy  $e$ .

In short range interacting systems, this accessible region is always connected. Indeed, take two accessible points  $\mathbf{m}_1$  and  $\mathbf{m}_2$ ; then it is easy to show that any point in  $[\mathbf{m}_1, \mathbf{m}_2]$  is accessible, using the possibility of phase separation (this argument actually shows that the accessible region is convex). This is not necessarily true any more for long range interacting systems: the domain of definition of the entropy may not be connected. Indeed, this situation was pointed out and recognized as generic in [23], and studied in more

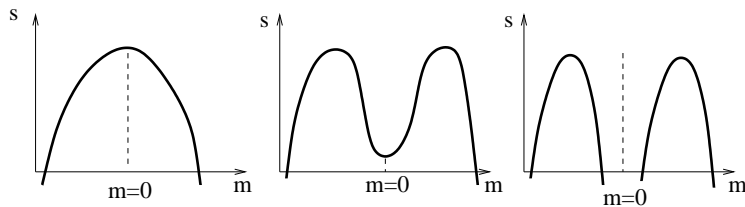


Figure 2.1: Sketches of the entropy curve as a function of an intensive parameter  $m$ , for various energies  $e$ . From left to right,  $e = e_1$ ,  $e = e_2$ ,  $e = e_3$ , see text. The accessible region in the  $m$  space becomes disconnected at  $e = e_c$ , with  $e_c$  between  $e_2$  and  $e_3$ .

details in [24, 63].

When the energy is varied, the region accessible to the system may change topology, from connected to disconnected. In [B3], we were interested in the properties of dynamical trajectories of a system close to the non-connectivity threshold described above. It turned out that large deviations techniques were again useful in this case, and allowed to extract some generic results.

Take for instance a model of a ferromagnet, with long range interactions, and consider the entropy as a function of magnetization and energy per spin  $s(e, \mathbf{m})$ . A typical situation is as follows, illustrated by Fig. 2.1:

1. for high energy  $e_1$ , the entropy has a single maximum at  $m = 0$  (Fig. 2.1, left panel), the phase space is connected. A typical dynamical trajectory spends most of its time close to  $m = 0$ .
2. For lower energy  $e_2$ ,  $s(e_2, \cdot)$  has two maxima at  $m = m^* \neq 0$ , and  $m = -m^*$  (Fig. 2.1, center panel); the phase space is still connected, but there is an "entropic bottleneck" between configurations around  $m = m^*$  and configurations around  $m = -m^*$ . We stress again that the total number of spins is finite. A typical dynamical trajectory switches randomly between configurations close to  $m = m^*$  and configurations close to  $m = -m^*$ . The waiting times between switches are expected to follow approximately an exponential distribution, with characteristic time  $\tau(e)$ .
3. For even lower energy  $e_3$ ,  $s(e_3, \cdot)$  has two maxima, and is not defined for  $m = 0$  (one might also say:  $s(e_3, 0) = -\infty$ ); the phase space is now

disconnected( Fig. 2.1, right panel). A typical dynamical trajectory spends most of its time close to  $m = m^*$  or close to  $m = -m^*$ , and no switch is possible.

The characteristic time  $\tau(e)$  diverges when the energy per spin  $e$  decreases towards  $e_c$ , where  $e_c$  is the energy at the non connectivity threshold. Numerical simulations showed a power law behavior:

$$\tau(e) \sim Cte (e - e_c)^{-\alpha(N)} \quad (2.1)$$

with  $\alpha(N)$  an exponent depending on the total number of spins  $N$ . More precisely, numerics showed  $\alpha(N) \simeq cN$ .

In [B3], we used large deviation techniques to understand the behavior of the entropy  $s$  close to the non connectivity threshold. This gives informations on  $\tau$  and the exponent  $\alpha$ , and explains the results of the numerical simulations. It is worth noticing that the large deviation results give valuable insight into simulations with a small number of spins, as small as  $N = 6$ .

The basic idea is to understand how a large deviation function for a sum of independent random variables becomes infinite. Consider  $N$  i.i.d. random variables  $X_1, \dots, X_N$ . Assume that their common density is  $\rho(x)$ , and that the support of  $\rho$  is included in  $[0, 1]$ . We call

$$m_N = \frac{X_1 + \dots + X_N}{N} \quad (2.2)$$

The behavior of  $\tau$  is related to the following question: what is the behavior of  $\mathbb{P}(m_N < \delta)$ , when  $\delta \rightarrow 0$  and  $N$  is fixed? In [B3], we give large deviation estimates for this quantity (detailed in an Appendix), and apply this idea to understand the numerical results for  $\tau$ . Since this idea is generic, it suggests that the results obtained in [B3] are valid beyond the specific model studied in this case.

### 2.2.3 Random graphs and cavity method [B9,B13]

In (macroscopic) short range interacting systems, entropy must be a concave function of an intensive parameter, say the energy per particle  $e$ . The argument for this is the usual "Maxwell construction", summarized in Fig.2.2. It rests on the fact that for any  $\lambda \in [0, 1]$ , it is possible to divide the system in two subsystems 1 and 2, of respective weights  $\lambda$  and  $1 - \lambda$ , such that the interaction energy between the subsystems is negligible. The total energy is

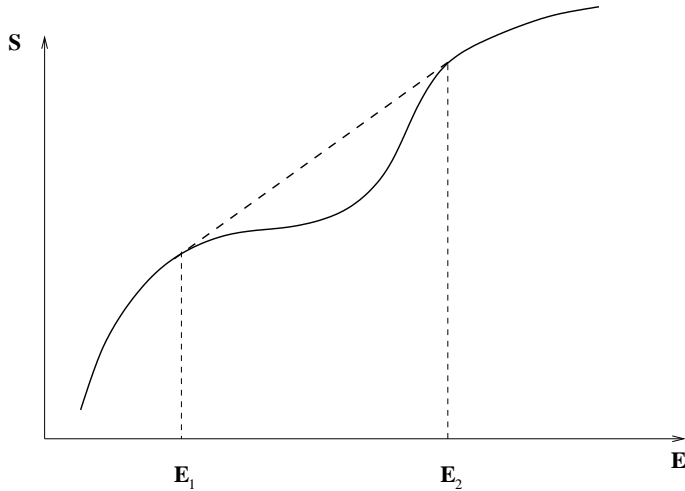


Figure 2.2: An example of Entropy ( $S$ ) vs Energy ( $E$ ) curve. If the system can phase separate, then the non concave part of the entropy has to be substituted by the dashed straight line, and the  $S(E)$  curve is concave.

then the sum of the energies of the two subsystems. This is of course wrong in the presence of long range interacting systems; there is then no reason for the entropy as a function of the energy (or another intensive parameter) to be concave, and inequivalence of ensembles is possible.

Random graphs of the Erdős-Rényi type, or random regular graphs, have a finite mean connectivity; in this respect, they look like short range interacting systems. However, they cannot be separated in two almost independent subsystems as a regular lattice for instance. From the previous discussion, one may expect that inequivalence of ensembles is possible for models on this type of random graphs. The goal of our work with Bruno Gonçalves was to show on a simple example that indeed it is the case, and that the same unusual phenomenology of microcanonical phase transitions observed in long range interacting systems should be expected for models on random graphs.

As already explained in the chapter on rigidity percolation, the cavity method is well suited to solve models on random graphs. When no disorder and no large deviation on the graph structure is involved, the cavity method is a rephrasing of the standard Bethe lattice solution, well suited for a microcanonical analysis. In [B9], we have solved in the canonical and microcanonical ensembles a 3-state Potts model on a regular random graph. We have found a region of negative specific heat in the microcanonical ensemble, and we have checked that this region is visible in Monte Carlo simulations,

using a Creutz algorithm for the microcanonical simulations. The article summarizing this work is appended at the end of this section.

[B13] is another work investigating the links between statistical mechanics with long range interactions and statistical mechanics on random graphs. The starting idea is to study spin models on random graphs with  $N$  sites and  $M = O(N^\gamma)$  links, with  $1 < \gamma < 2$ .  $\gamma = 2$  corresponds to the complete graph, that is a standard "mean-field" system where each site is coupled to all the others;  $\gamma = 1$  is the case of the Erdős-Rényi random graph. For  $1 < \gamma < 2$ , it is known that in the proper scaling limit, the  $\gamma = 2$  solution is exactly recovered [14]. However, the Metropolis Monte-Carlo simulations showed rather strong finite-size effects. It is thus important to understand the finite  $N$  corrections to the  $N \rightarrow \infty$  limit.

We have used two theoretical approaches to investigate these finite-size effects.

1. an expansion in powers in  $1/N$  followed by computations using the replica method
2. we have considered a graph with  $N$  sites and  $M = O(N^\gamma)$  links as a high connectivity Erdős-Rényi random graph, and used the cavity method

We have shown that the second option provides a much more accurate description of the finite-size effects. This work could be the starting point of a study a Quasi Stationary States for models defined on such random graphs interpolating between the complete graph and an Erdős-Rényi graph.

## 2.3 Kinetic theory, mathematical results

### 2.3.1 Introduction

As already stated in Sec.2.1, the dynamics of a long range interacting system is often described by a Vlasov equation in the adequate scaling limit and timescale, and this fact is of central importance in the traditional understanding of these systems; this point will be further developed in Sec. 2.4.

The classical proof of the Vlasov limit, valid for sufficiently regular interaction potentials [66, 32, 15], may be sketched as follows. Take two solutions  $f_1$  and  $f_2$  of the Vlasov equation. Then their distance increases at an exponential rate:  $d(f_1(t), f_2(t)) \leq e^{Ct}d(f_1(0), f_2(0))$ , where  $d$  is an adequate

distance to measure how close are two probability measures. Now take a sequence of discrete initial conditions with  $N$  particles

$$f_N(x, v, t = 0) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(0)) \delta(v - v_i(0))$$

indexed by  $N$ . The crucial point is that the discrete measures

$$f_N(x, v, t) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t)) \delta(v - v_i(t)) ,$$

which are built with the solutions  $(x_i(t), v_i(t))$  of the  $N$ -body dynamics with initial condition  $(x_i(0), v_i(0))$ , are actually solutions of the Vlasov equation.

Thus, if  $d(f_N(\cdot, \cdot, t = 0), f(\cdot, \cdot, t = 0))$  tends to 0 when  $N$  tends to infinity, then the discrete dynamics is adequately described by the Vlasov equation with initial condition  $f_0$ , on a time scale  $T(N)$  which depends on the rate of convergence of  $d(f_N, f_0)$ . Typically, if  $d(f_N, f_0) = O(1/N^\gamma)$ , then  $T(N) = O(\ln N)$ .

This strategy deals with smooth interactions ( $C^2$  potential for instance)[15, 66, 32]. It has proved very difficult to relax this regularity hypothesis down to the most physically interesting potentials, Coulomb and Newton. There are results using a short range regularization to the potential [87], and recently P.E. Jabin and M. Hauray [39] have introduced a new strategy and were able to prove the convergence of the finite  $N$  dynamics to the Vlasov dynamics for less regular potentials. Nevertheless, the paper [39] stays far from the Coulomb or gravitational potentials. This situation was the incentive for the contributions shortly described in the next two paragraphs, which do not solve the question of the Vlasov limit for Coulomb or Newton interactions.

### 2.3.2 Short range potentials [B11]

We have outlined the main ideas underlying the classical proofs of the rigorous results proving the convergence of the discrete dynamics to the Vlasov dynamics, and emphasized that these rigorous results do not hold for  $1/r$  potentials, because of the short range singularity.

The idea in [B12] was then to start from a potential with a strong short range singularity, and to try to control its effect on the large scale dynamics. For this purpose, the long range part of the interaction is then set to zero.



Strictly speaking, this work then deals with very short range interactions: the force between two particles vanishes if their distance is larger than  $R$ , with in dimension 3,  $NR^2 \ll 1$  ( $N$  is the number of particles). This is a simplification of the more usual (and more interesting) so called "dilute limit"  $NR^2 = \text{const}$ . With this scaling, one expects to obtain in the limit the Boltzmann equation; this is exactly what is proved by Lanford in the famous work [52], for hard spheres and small times.

In the scaling  $NR^2 \ll 1$ , one expects a free transport equation as kinetic limit. A rigorous proof for this is not so easy to obtain for general potentials (for hard spheres, Lanford's proof applies). In [B12], we prove this limit for general repulsive potentials, and for general attractive potentials, with the more stringent hypothesis  $R \ll N^{-3/5}$ . In this last case, we actually describe all possible collision sequences between particles. The core of the proof is to ensure that the build up of correlations between particles does not destroy the validity of simple scaling arguments.

### 2.3.3 Stability results for $N$ -particles dynamics with singular potential [B16]

We consider the Hamiltonian dynamics of  $N$  particles

$$\begin{cases} \dot{X}_i^N = V_i^N \\ \dot{V}_i^N = E_N(X_i^N) = \frac{1}{N} \sum_j K(X_i^N - X_j^N) \end{cases} \quad (2.3)$$

where for simplicity all the positions  $X_i^N$  belong to the torus  $\mathbb{T}^3$  ( $= [0, 1]^3$  with periodic boundary conditions) and all the velocities  $V_i^N$  belong to  $\mathbb{R}^3$ . We note  $Z^N = (X_1^N, \dots, X_N^N, V_1^N, \dots, V_N^N)$ ;  $\phi$  is the interaction potential, such that  $K = -\nabla\phi$ . The Hamiltonian is

$$H_N[Z^N] = \frac{1}{2} \sum_{i=1}^N (V_i^N)^2 + \frac{1}{N} \sum_{i \neq j} \phi(X_i^N - X_j^N)$$

The classical strategy to prove the Vlasov limit for a system of interacting particles deeply relies on a stability estimate: two solutions of the Vlasov equation with close initial conditions separate at a rate which is at most exponential in time. At the level of the  $N$ -particles dynamics, one may hope for estimates such as

$$\|Z^N(t) - Z^{N,\delta}(t)\|_1 \leq \|Z^N(0) - Z^{N,\delta}(0)\|_1 \exp(Ct), \quad (2.4)$$

where  $\delta$  is a shift in the initial conditions (so that  $Z^{N,\delta}(0) = Z^N(0) + \delta$ ),  $C$  is independent of the number of particles  $N$ , and where we have used the norm on  $\mathbb{T}^{3N} \times \mathbb{R}^{3N}$

$$\|Z\|_1 = \frac{1}{2N} \sum_{i=1}^N (|X_i| + |V_i|).$$

Estimates such as 2.4 are easy to obtain if the force  $K$  is regular enough. For instance, if  $K$  is  $C^2$ ,  $\nabla K$  is bounded and one can take  $C = 1 + \|\nabla K\|_{L^\infty}$ .

Our goal was to prove a stability result analogous to (2.4) for a singular two-body force  $K$  (say  $K(X) \sim C/|X|^\alpha$ ), which would be uniform in the number of particles. For this purpose, we adapt a method introduced in [31]. One cannot hope for a version of (2.4) valid for all initial conditions  $Z_0$ , so that we will use an average over the initial condition  $Z_0$ ; this average will be taken with respect to the canonical Gibbs measure  $\mu_N$ :

$$d\mu_N[Z^N] = \mathcal{N} e^{-\beta H_N[Z^N]}$$

where  $\mathcal{N}$  is a normalization factor. Likewise, an estimate like (2.4) cannot be true for all initial shifts  $\delta$ . Thus, we will also average over different  $\delta$ .

Finally, we study the quantity

$$Q(t) = \int d\mu_N(Z_0^N) \int_{\delta \in \mathbb{T}^{3N} \times \mathbb{R}^{3N}} \psi_N(Z_0^N, \delta) \ln \left( 1 + \frac{\|Z^N(t, Z_0^N) - Z^N(t, Z_0^N + \delta)\|_1}{\delta_N} \right) d\delta, \quad (2.5)$$

where  $\delta_N$  is a small parameter which will go slowly to zero when  $N$  goes to infinity and which gives the order of magnitude of the shift  $\delta$ . The function  $\psi_N$  is any non negative function representing the shift distribution. It is reasonable to impose that the initial shift does not change too much the energy, requiring for instance that for  $\delta$  in the support of  $\psi_N$ , and for some  $C$  independent of  $N$ , we have:

$$|H_N[Z_0^N + \delta] - H_N[Z_0^N]| \leq C$$

Beyond this “reasonable” assumption, which is already rather strong, the condition we have to impose on the shift distribution is actually a bit stronger.

The quantity  $Q(t)$  then measures the mean deviation between two trajectories in the configuration space starting very close one from the other: as

long as  $Q(t)$  remains of order 1,  $Z^N(t, Z_0^N) - Z^N(t, Z_0^N + \delta)$  remains of order  $\delta_N$  in some average sense.

The main result of the paper is a control on the growth on this quantity  $Q$ :

**Theorem 2.3.1** *Assume that  $\phi \geq 0$ , that is the potential is repulsive, and for some constant  $C$ , and for  $\alpha < 2$*

$$|\phi(x)| \leq \frac{C}{|x|^{\alpha-1}}, \quad |\nabla\phi| \leq \frac{C}{|x|^\alpha}, \quad |\nabla^2\phi| \leq \frac{C}{|x|^{\alpha+1}}.$$

*Then taking  $\delta_N = N^{-\varepsilon}$  for any  $\varepsilon < 1 - \alpha/3$  one has for  $N$  large enough*

$$Q(t) \leq C_{a,\alpha} (1 + c_\beta + c_\beta^a) t + Q(0) ,$$

*where  $c_\beta = (\beta^{3/2} e^{\beta e_0}) / (2\pi)^{3/2}$ ,  $a$  is any exponent strictly larger than  $2\alpha/3$  and  $C_\alpha$  is a constant depending on  $\alpha$  and  $a$  and blowing up as  $\alpha \rightarrow 2$  or  $a \rightarrow 2\alpha/3$ .*

The Coulombian case  $\alpha = 2$  is the limit case for the theorem, but is excluded. This is a result of stability "on average" for trajectories starting close one to another; the average over the initial condition is taken against the Gibbs measure. Note that the linear in time behavior of  $Q$  is the counterpart of the exponential separation of trajectories in Eq. (2.4). The proof deeply relies on the fact that the Gibbs measure is invariant by the flow, and on good estimates on its marginals: it is then not straightforward to extend the result.

## 2.4 Dynamics of $N$ particles with long range interactions; Quasi Stationary States [B1,B8,B12,B1

### 2.4.1 Introduction

I start this section by recalling the standard scenario for the relaxation to equilibrium of a long range interacting system, already sketched in section 2.1.

Over small enough time scales, we have seen that the dynamical evolution of such a system is governed by the Vlasov equation. This is the basis of the following scenario:

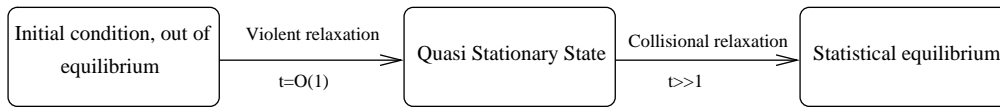


Figure 2.3: Illustration of a generic scenario for the relaxation to equilibrium of a long range interacting system of  $N$  particles. The violent relaxation time scale is of the order of the dynamical time scale, that is the time scale of the associated Vlasov equation. The slow "collisional" relaxation takes place on a time scale which diverges with the number of particles  $N$ , if the dynamical time scale is taken as fixed.

1. The Vlasov dynamics leads the  $N$ -particles system close to a stable stationary state of the Vlasov equation (there is an uncountable infinity of such states), in times of order 1, the time scale of the Vlasov equation. This state is then sometimes called a Quasi Stationary State (QSS in the following) for the discrete dynamics.
2. Then, the system does not evolve any more under the action of the mean field force; rather, it is driven by the finite  $N$  fluctuations. This evolution is slow, typically it takes place over times of order  $N$ , but this is not always the case; it is described by the Lenard-Balescu equation [8], or its simplified version the Landau equation.
3. This evolution under the action of the finite  $N$  fluctuations then drives the system to its final state, the Boltzmann-Gibbs equilibrium.

This scenario, illustrated in Fig. 2.3, is well established and well studied, both for the Coulomb and gravitational interactions. There are still however many questions related to it and to its extensions. I try to summarize below some of these questions, that drove my research in this field.

- *Selection of the quasi-stationary state among the infinite number of stable stationary states of the Vlasov equation.* The traditional answer is based on a maximum entropy criterion for the Lynden-Bell entropy (or equivalently the Robert-Sommeria-Miller entropy for 2D fluids). However, it has been known for a long time that this theory does not always provide a good answer (see the discussion and references in section 2.1). Trying to improve the Lynden-Bell criterion for the selection of the QSS is certainly an important, but difficult, question.

- *Understanding the behavior of small perturbations to a QSS* : this is a less ambitious, but still far from easy, challenge. In this case, the starting point is the linearized Vlasov equation. This is the realm for instance of Landau damping and much is known in this case in plasma physics, and more generally as soon as the QSS is homogeneous in space. When the QSS is not homogeneous, the technical difficulties are much greater. Because this is of central importance in many respects, there has been a lot of works in this direction despite the difficulties, both for self-gravitating systems and 2D fluids. However, there still are some open questions related to non homogeneous QSSs, see section 2.4.3.
- *The time scales for the relaxation to equilibrium of a QSS*. In this case also, there is a well established theory, based on the Landau, or Lenard-Balescu equations. On the basis of a formal  $1/N$  expansion, one would generically expect this time scale for relaxation  $t_R$  to scale as  $N$  for a regular enough interaction potential. For a gravitational interaction, this time is expected to be  $t_R \propto N/\ln N$ . There is an exception for homogeneous 1D systems, where the dominant relaxation term vanishes; one expects then  $t_R \gg N$ , and simulations suggest an exponential time [19]. When the system is brought in the course of the relaxation close to a neutrally stable state of the underlying Vlasov dynamics, the situation is less clear. Some simulations on the toy model HMF have yielded  $t_R \propto N^{1.7}$ , but there is no clear theory, nor indications that this exponent is universal.
- *Forced and dissipative systems*. Most of the literature on long range interacting systems deals with Hamiltonian systems. There is still much to understand when such systems undergo forcing and dissipation (see for instance [83] for an example of forced systems in astrophysics). The case of a simple friction and a white noise forcing has received some attention, notably by P.H. Chavanis (see [26, 27] and subsequent works). QSS have been investigated in a non Hamiltonian setting [7, 38]. Another interesting aspect is the possibility of out of equilibrium phase transitions, as in [12]. One of the basic building blocks to progress in this direction is the linearized Vlasov equation.

Several of my works try to answer some of the questions underlined above. My general approach is to use toy models as simple as possible to reduce the

technical difficulties. The hope is that the analytical and numerical studies can then be pushed further than is usually done, and that it could give the possibility to unveil generic phenomena. Articles [B8], [B12] and [B17] are along these lines. I have chosen to develop [B17], which is appended at the end of this section.

[B8] is an application of the Lynden-Bell theory to the violent relaxation of the HMF model. It is shown that the equilibrium statistical theory is qualitatively correct for the initial conditions used, and sometimes quantitatively. It is also well-known in plasma physics that the process of violent relaxation, or Landau damping when the initial perturbation is not too small, sometimes does not drive the system close to a stationary state of the Vlasov equation, but rather to a periodic solution. The situation is not as well studied for attractive long range interacting systems. In [B12], we derive and study numerically criteria for the appearance of these "quasi asymptotic periodic states" both for the repulsive (plasma like) and attractive HMF models, close to a spatially homogeneous stationary state.

### 2.4.2 “Quasi asymptotic periodic states” [B12]

The first stage of the relaxation of a long range interacting system follows a Vlasov evolution, and it usually settles after some time close to a stationary solution of the Vlasov equation. One may wonder however if the system of particles may approach instead a periodic solution of the Vlasov equation. In this case, one would expect an almost periodic behavior of the system, until finite size fluctuations drive it to the thermodynamical equilibrium: hence the formulation “quasi asymptotic periodic state”. Although the general problem of finding periodic solutions for the Vlasov equation is clearly of interest (and indeed has been the subject of several studies), I will concentrate on periodic solutions which are perturbations of a stationary state. In this case, one may hope to have a better analytical control on the dynamics, using the Vlasov equation linearized around the stationary state.

Indeed such a phenomenology is well known in plasma physics. In this case, BGK modes [9] and their non linear superposition [17] form periodic solutions of the Vlasov equation, and these periodic solutions have been shown to be possible attractors for the Vlasov dynamics, close to stationary solutions [56]. It is important to notice that a periodic solution of the Vlasov equation induces a periodic mean field; this creates a resonance phenomenon with particles moving along trajectories with the same frequency. Thus, the

solutions found in [17, 56] are genuinely non linear, despite the fact that they are small perturbations of a stationary state. In particular, they may exist only if the non linear effects are strong enough to counterbalance the Landau damping (see for instance [67, 50]).

Besides these non linear solutions, there may exist purely oscillatory linear modes close to a stationary state of the Vlasov equation, if no particle is resonant with the mode's frequency. This happens and indeed has been found numerically in 1D self gravitating systems [57, 84]

In [B12], we were interested in the existence of periodic attractors for attractive long range interacting systems, of the type studied in [56], thus genuinely non linear. On the example of the HMF model, we have found that indeed such periodic states and their associated clusters of particles may exist and be attractors for the  $N$ -body dynamics. However, the application of the standard criteria requiring strong enough non linear effects (or equivalently weak enough Landau damping) is much more restrictive in this case: thus, these periodic states do not seem as easy to reach in attractive systems.

One may wonder if these nonlinear periodic solutions may exist also close to non homogeneous stationary states of the Vlasov equation, which is clearly the relevant situation for astrophysics. Answering this question was one of the incentives for the work described in the following section.

### **2.4.3 Perturbation of inhomogeneous stationary states in the HMF model [B17]**

#### **Introduction**

Besides being a natural question, the understanding of the linearized Vlasov equation around a stationary state is also an important first step to address more difficult problems, such as the relaxation to the statistical equilibrium of a long range interacting Hamiltonian systems, or the response of a system to small forcing and dissipation.

Since the first study by Landau of the relaxation of a perturbation in a plasma [51], Landau damping is a basic concept in plasma physics. It is also very often invoked in astrophysics, where galaxies for instance are usually modeled as stable stationary states of the Vlasov equation; the linear response of a perturbed galaxy is then of course of central interest. However, in astrophysics, the stationary state is usually inhomogeneous. On the mathematical side, there has been several rigorous studies, restricted to homoge-

neous stationary states. [30] shows the exponential decay of a perturbation evolving through the linearized Vlasov equation, under precise conditions and in a certain sense. Glassey and Schaeffer [35] have given counter examples to this exponential decay, for non analytic stationary states, or unbounded systems. Finally, in a recent and impressive work [60], Mouhot and Villani have tackled the fully nonlinear problem, showing the exponential decay of a small enough perturbation evolving according to the Vlasov equation in a bounded domain.

A non homogeneous reference state adds a lot of technical difficulties to the linearization of the Vlasov equation and also, as we will see, qualitative changes with respect to the standard homogeneous case. The root of the difficulties is the following: when the background stationary state is homogeneous, its self consistent potential vanishes, so that the angle-action coordinates of a particle traveling in this potential are just the original position-velocity variables, and the trajectories are straight lines. This is not true anymore when the background stationary state is not homogeneous; one needs to transform the linearized Vlasov equation to angle-action variables. As a consequence, different Fourier modes (in the angle variables) are now usually coupled, and the "dispersion function"  $D(\omega)$  is the determinant of an infinite matrix<sup>1</sup>. Despite these difficulties, the computation of  $D(\omega)$  in the half plane  $Im(\omega) > 0$  has been done for self-gravitating disks [46] and spherically symmetric systems [70]. This provides information on the possible instability of the background state, and on the associated growth rate, given by the solution of  $D(\omega) = 0$ . However, studying the relaxation of a perturbation around a stable background requires the analytical continuation of  $D(\omega)$  to the lower half plane  $Im(\omega) \leq 0$ , or at least the understanding of its singularity on the real line. In particular, the gravitational analog of the Landau damping rate would be given by roots of the equation  $D(\omega) = 0$  in the lower half plane. In the general self-gravitating case, this analytical continuation is very difficult. As a consequence, despite the fact that "Landau damping" is a common concept in astrophysics, I am aware of only a few papers actually computing a Landau damping rate in this setting [85, 86]. These papers use the following approximate procedure: they fit the computed function  $D(\omega)$  in the upper half plane by a rational function, and continue this rational function into the lower half plane.

This exponential "Landau-like" damping may be expected to be relevant

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<sup>1</sup>Here,  $\omega$  is the argument of a Laplace transform in time.



at intermediate times. However, one rather expects asymptotically a power law decay for a solution of the linearized Vlasov equation around an inhomogeneous steady state. This question has apparently not been studied in the astrophysical literature, and I am aware of only one reference, for a particular class of Hamiltonians [77].

The linearization of the 2D Euler equation around a stationary vortex, or stationary shear flow shares many features with the similar problem for the Vlasov equation, and has been much more studied. To tackle the Vlasov problem, we have drawn ideas from the literature on the 2D Euler equation, such as [16, 13].

The basic idea of the work [B17] is to use a very simplified caricature of self-gravitating system. The hope was to reduce as much as possible the technical difficulties, to be able to push as far as possible both the analytical understanding and the numerical computations. The HMF model is a well suited toy model for this purpose.

Recently, the dynamical stability of inhomogeneous stationary state of a Vlasov equation has been studied in several papers, using toy models (HMF or others) to simplify the computations [43, 20, ?, 6]. However, the methods used in these papers, different from the “classical” Fourier-Laplace route we have followed, give access only to the stability threshold and the growth rate; the Landau damping rate and more generally the decay of a perturbation around a stable stationary state remain inaccessible.

I list here the specific goals and the results obtained in this joint work with A. Olivetti and Y. Yamaguchi:

1. We have computed the dispersion relation of the system around a non homogeneous stationary state. In particular, in view of the literature on the 2D Euler equation, branch points on the real axis were to be expected. Indeed we have found a (discrete) infinite number of branch points in the dispersion relation, and studied the precise form of the singularities.
2. We have computed explicitly for some families of stationary distributions the “Landau poles” in the lower half plane, from the analytic continuation of the dispersion relation, and exhibited quasi modes, that is perturbations with a very small Landau damping rate. We have performed these computations for “water-bags” stationary distributions (step profiles), statistical equilibrium distributions, and a

2-parameter family of distributions interpolating between the previous two 1-parameter families (sometimes called the Lynden-Bell distributions). We have obtained maps of Landau poles, showing some complicated crossing phenomena between poles.

3. We have looked for signatures of these Landau poles and quasi modes in direct numerical simulations of the  $N$ -body system. Although more systematic  $N$ -body simulations are needed, computations for waterbags and equilibrium stationary distributions have indeed shown such signatures, revealing an oscillating exponential decay in an intermediate time window, when a Landau pole exists.

The next logical goal, which is currently under study, is to understand in details the asymptotic behavior of a decaying perturbation. The basic idea is to characterize precisely the different singularities appearing in solving the linearized Vlasov equation by Fourier-Laplace transform. An algebraic decay, similar to what is obtained for the 2D Euler equation may be expected. Indeed such a decay was found in [77], but the exponent is probably not correct, since the perturbations used were not generic. Other questions are related to the phenomenon of "vorticity depletion" near the extrema of the angular frequency in phase space, as was found in [13] for 2D shear flows. Clearly, if HMF is a good starting point to perform explicit computations, it is important to understand which features can be valid in a more general setting, such as a 3D self gravitating system.

### A few technical details

I try in this paragraph to give a flavor of the technical part of the work done in [B17].

The HMF model is described by the following Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{1}{2N} \sum_{i \neq j} \cos(\theta_i - \theta_j). \quad (2.6)$$

where the position of particle  $i$  is given by  $\theta_i \in (-\pi, \pi]$ , and its conjugate momentum is  $p_i$ . The associated Vlasov equation is

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - M \sin(\theta - \psi) \frac{\partial f}{\partial p} = 0, \quad (2.7)$$

where the magnetization is defined as

$$\mathbf{M} = Me^{i\psi} = M_x + iM_y = \int_{-\infty}^{\infty} dp \int_{-\pi}^{\pi} d\theta f(\theta, p, t) e^{i\theta}, \quad (2.8)$$

We consider a stationary solution  $f_0(\theta, p)$  of Eq. (2.7).  $f_0(\theta, p)$  is actually a function of the action variable  $J(\theta, p)$  alone. With an abuse of notation, we write the stationary solution  $f_0(J)$ . We write  $\Omega(J)$  the angular frequency associated with the action  $J$ . We study a perturbation  $f_1$  around  $f_0$ , by linearizing Eq. (2.7). This equation is solved by a Fourier-Laplace transform. Calling  $\Sigma_C(\omega)$  and  $\Sigma_S(\omega)$  the Laplace transforms of  $M_x(t)$  and  $M_y(t)$  respectively, we obtain:

$$\Sigma_C(\omega) = \frac{G_C(\omega)}{\epsilon_C(\omega)} \quad (2.9)$$

$$\Sigma_S(\omega) = \frac{G_S(\omega)}{\epsilon_S(\omega)} \quad (2.10)$$

where  $G_{C,S}$  contain the initial condition;  $\epsilon_{C,S}$  are dispersion functions and read

$$\epsilon_C(\omega) = 1 + \frac{1}{2\pi} \sum_m \int \frac{m f'_0(J)}{m\Omega(J) - \omega} |c_m(J)|^2 dJ \quad (2.11)$$

$$\epsilon_S(\omega) = 1 + \frac{1}{2\pi} \sum_m \int \frac{m f'_0(J)}{m\Omega(J) - \omega} |s_m(J)|^2 dJ \quad (2.12)$$

In these expressions,  $c_m$  and  $s_m$  are complex functions related to the switching between Fourier basis in the original  $\theta$  variable and in the angle variable associated with the action  $J$ .  $\epsilon_{C,S}$  are a priori defined in the upper half  $\omega$ -plane, that is  $Im(\omega) > 0$ . Note that Eqs. (2.11)-(2.12) are not properly defined for  $\omega \in \mathbb{R}$ .

To get the time evolution of the magnetization, we now need to compute the inverse Laplace transform of (2.9)-(2.10). The standard procedure involves the deformation of the complex integration contour in the lower half  $\omega$ -plane, see Fig. 2.4. This requires the analytical continuation of (2.11)-(2.12). From (2.11)-(2.12), it can be shown that  $\epsilon_{C,S}$  have branch cuts singularities for  $\omega = m\omega_0$ , where  $\omega_0 = m\Omega(J=0)$ . More precisely, we have for the singular parts of  $\epsilon_{C,S}$ :

$$\epsilon_C^{sing}(\omega) \sim \text{const.} \times (\omega - m\omega_0)^m \ln(\omega - m\omega_0) \text{ if } m \text{ even,} \quad (2.13)$$

$$\epsilon_S^{sing}(\omega) \sim \text{const.} \times (\omega - m\omega_0)^m \ln(\omega - m\omega_0) \text{ if } m \text{ odd.} \quad (2.14)$$

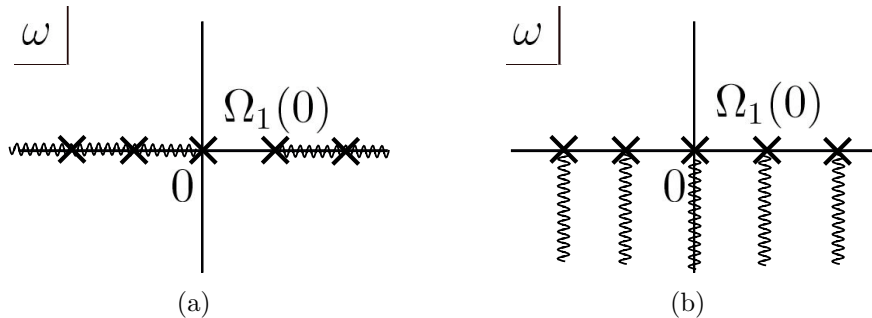


Figure 2.4: Two possible choices for the branch cuts in the  $\omega$ -plane. The integration path running from  $-\infty$  to  $+\infty$  to compute the inverse Laplace transform must be deformed avoiding the singularities.

Taking care of the branch cuts, we have performed the analytical continuation and computed the roots of  $\epsilon_{C,S}(\omega)$ . In the upper half-plane, they correspond to an exponential instability; in the lower half plane, they correspond to a Landau damping-like phenomenon.

## 2.5 Long range interactions and cold atoms

This axis of research started two years ago, thanks to the collaborations with experimental groups working with cold atoms in nearby INLN (in Sophia Antipolis).

One motivation for this move is that, despite the fact that many forces in the universe are long range, it is not easy to design a well controlled experimental system with long range interactions. Thus, it is not easy to test the different theories and predicted strange phenomena. There is a huge body of experiments on plasmas, but the long range force is in this case repulsive, precluding the observation of some of the most striking phenomena, like negative specific heat. There has been some proposals to remedy this situation (for instance [48, 18]), but the realization of a tabletop galaxy remains a dream. It is then tempting to look for signatures of long range interactions in experimental systems, and cold atoms are a possibility.

The experimental systems considered in the following are as follows. An atomic vapor is stored, either in a Magneto-Optical Trap (MOT) or in a dipolar trap; see Fig. 2.5 for a schematic picture of a MOT. When the atoms interact with a laser slightly red-detuned with respect to an atomic transition,

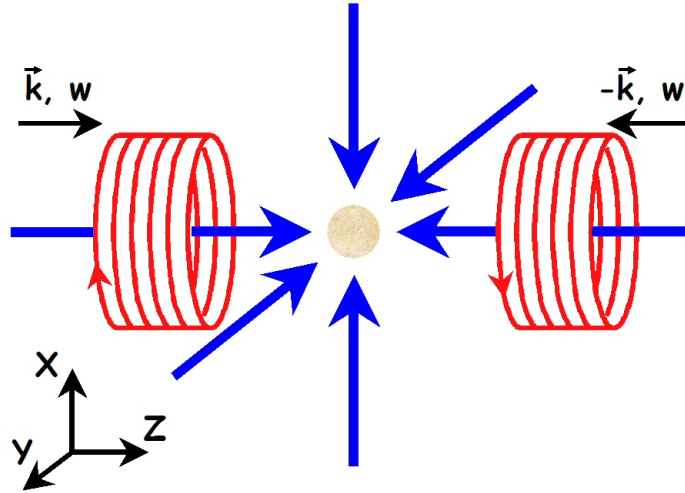


Figure 2.5: In the center, the atomic cloud. At INLN, there are experiments with Rubidium and Strontium. The blue arrows represent the 6 quasi resonant, but slightly red-detuned, laser beams. In red, the solenoids creating a static space dependent magnetic field. All forces acting on the atoms come from the complex photon/atom interactions.

several phenomena happen:

- the atoms are slowed down by Doppler effect, and feel a trapping force, in presence of a magnetic field gradient (these two effects are at the root of a MOT);
- the spontaneous emission of photons adds a stochastic noise on the atoms' velocities;
- the multiple scattering of photons by the atoms induces an effective repulsion between atoms;
- finally, the laser intensities decrease as the beams cross the atomic cloud, and this "shadow effect" induces an effective attractive force.

A complete description of all these effects would be very complicated. The following simplifications are standard, but not always completely justified:

1. We will assume that the trapping force induced by the magnetic field can be considered as an external harmonic potential. In the case of a dipolar trap (see 2.5.2), we will also take it as harmonic.

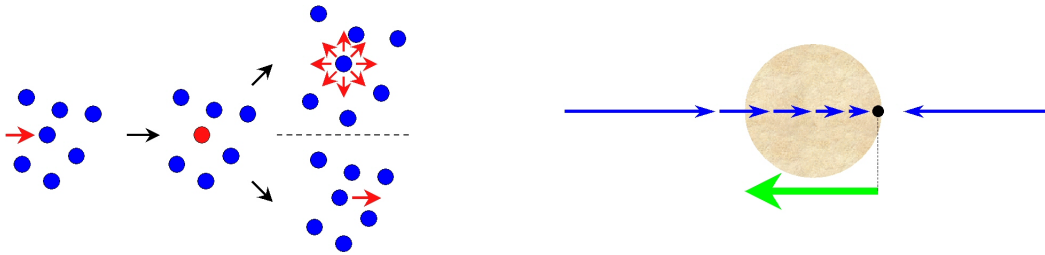


Figure 2.6: **Left:** a sketch to explain the effective repulsive force due to multiple scattering. An atom is excited by absorbing a photon coming from the left. It then goes back to its fundamental state by spontaneous emission of a photon, with isotropic probability. The re-emitted photon may then be absorbed again by another atom at distance  $r$ , with probability proportional to  $1/r^2$ ; the recoil effect due to this second absorption is equivalent to a repulsive force. **Right:** a sketch to explain the attractive force due to the "shadow effect". Without attenuation, the optical pressures due to the right and left lasers (blue arrows) cancel at any point in the cloud. However, each photon has a certain probability to be absorbed when crossing the cloud; then, the optical pressures do not balance any more, resulting in an effective net force directed towards the center of the cloud (green arrow). In the small optical width limit, it may be written as a two-body interaction force, directed along the laser axis and independent of the distance between atoms.

2. In principle, friction and diffusion depend on space and velocity, and even functionally on the whole one-particle distribution function; we will often assume a constant friction coefficient  $\kappa$  and a constant velocity diffusion coefficient  $D$ . We have to keep in mind that some experiments with large clouds seem to require a better modeling of friction [49, 69].
3. When the atomic cloud is not too optically thick, the repulsive force induced by multiple scattering of photons can be written as a long range two-body interacting force decaying as  $1/r^2$ : it is thus an effective “Coulomb” force [82].
4. Always in the small optical width limit, the attractive force induced by the shadow effect can also be written as a long range two-body interacting force. The divergence of this force is proportional to minus the atomic density, which makes this force similar to an effective “gravitation” [82]. However, it does not derive from a potential.

Making use of these approximations, the system may be modeled by a Vlasov-Fokker-Planck equation for the one-particle distribution  $f(\mathbf{r}, \mathbf{v}, t)$ :

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}} \cdot (\mathbf{v}f) + \mathbf{F}_{trap} \cdot \nabla_{\mathbf{v}} f + \mathbf{F}_{int}[f] \cdot \nabla_{\mathbf{v}} f = \nabla_{\mathbf{v}} \cdot (\kappa \mathbf{v}f + D \nabla_{\mathbf{v}} f) \quad (2.15)$$

where  $\mathbf{F}_{trap}(\mathbf{r})$  is a harmonic trapping force, and  $\mathbf{F}_{int}[f]$  is the interaction term given by:

$$\mathbf{F}_{int}[f](\mathbf{r}) = \int \mathbf{F}_{bin}(\mathbf{r}, \tilde{\mathbf{r}}) f(\tilde{\mathbf{r}}, \mathbf{v}, t) d\tilde{\mathbf{r}} d\mathbf{v}. \quad (2.16)$$

In the above equation,  $\mathbf{F}_{bin}$  is the binary long range interaction force. I stress again that the modeling of an atomic cloud with Eq. (2.15) already implies a number of approximations, not necessarily well controlled.

A further step in the approximations, which is often taken, is to replace the attractive non potential force by an effective gravitation [58]. Then Eq. (2.15) becomes an equation for Brownian particles with long range potential interactions, about which much is known: in particular, one may use equilibrium statistical mechanics to compute the stationary states, and  $H$ -functions to understand the dynamics.

However, at the level of Eq. (2.15), one may keep the more realistic non potential attractive interaction. We feel it is an incentive to develop an out-of-equilibrium thermodynamics for these long range interacting systems.

Part of Alain Olivetti's PhD thesis is devoted to the study of long range interactions in MOTs. Beyond him, the works described in the following were done in collaboration with Bruno Marcos, Robin Kaiser, Maryvonne Chalony, David Wilkowski and Freddy Bouchet.

### 2.5.1 Breathing dynamics [B14,B18]

The first axis in our collaboration with the group of R. Kaiser was to unveil phenomena which could characterize the interactions within the system, and be accessible both theoretically and experimentally. We decided to focus on the study of the oscillations of the system. Narrowing our study to the breathing dynamics, it turned out that a rather general study of these oscillations for trapped interacting particles was possible, encompassing a wide range of power law interactions, temperatures, at linear and non linear levels, in the underdamped regime. Although inspired by the modeling of Magneto-Optical Traps, this work is thus much more general.

We start from a Vlasov-Fokker-Planck equation like Eq. (2.15). The general idea to study the breathing dynamics is to assume the existence of a stationary state  $f_0$  solution of Eq. (2.15), say for instance thermal equilibrium, at a temperature given by the balance between friction and diffusion. If friction and diffusion vanish, Eq. (2.15) becomes the Vlasov equation, which has an infinite number of stable stationary solutions. We first consider this limit, and we now drastically simplify the dynamics by using a scaling ansatz which describes the breathing dynamics [37, 22, 45]:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\varphi(\mathbf{r}, \mathbf{v})), \quad (2.17)$$

with

$$\varphi(\mathbf{r}, \mathbf{v}) = (\mathbf{R} = \mathbf{r}/\lambda, \mathbf{V} = \lambda\mathbf{v} - \dot{\lambda}\mathbf{r}). \quad (2.18)$$

With this hypothesis all the time dependence in the dynamics is now included in the positive parameter  $\lambda$ . This ansatz is tailored to capture the radial dynamics; thus, we will not be able to look at higher order modes of the system, like quadrupole modes. Using moment equations up to second order, we then show that  $\lambda$  should satisfy the equation

$$\ddot{\lambda} + \left( \lambda - \frac{p}{\lambda^3} + \frac{p-1}{\lambda^k} \right) \omega_0^2 = 0 \quad (2.19)$$



where we have assumed that the trap has spherical symmetry, with frequency  $\omega_0$ , and that the binary interaction between particle is homogeneous (in the mathematical sense), with degree  $-k$ .  $p \sim (k_B T)/(E_{trap})$  is the ratio of thermal energy to the typical trap energy; this is a parameter characterizing the strength of the binary interaction in the cloud. We stress that Eq. (2.19) does not provide an exact solution, and that its validity must be limited in time.

In the  $T = 0$  limit with repulsive interactions (that is  $p = 0$  with the notation introduced above), the system is crystallized and the particles oscillate around their equilibrium positions. The breathing dynamics is in this case exactly described by (see [40] for a similar computation restricted to the linear regime):

$$\ddot{\lambda} + \kappa \dot{\lambda} + \left( \lambda - \frac{1}{\lambda^k} \right) \omega_0^2 = 0 \quad (2.20)$$

where we have assumed that the friction  $\kappa$  does not depend on space or velocity.

We consider now the following equation:

$$\ddot{\lambda} + \kappa \dot{\lambda} + \left( \lambda - \frac{p}{\lambda^3} + \frac{p-1}{\lambda^k} \right) \omega_0^2 = 0 \quad (2.21)$$

It coincides with Eqs (2.19) and (2.20) in the  $D = 0, \kappa = 0$  and the  $p = 0$  limits respectively. Extensive numerical simulations have shown that this equation describes with a very good approximation the frequency of the breathing oscillations of an  $N$ -particle systems in the underdamped regime. The agreement for the oscillation amplitude is not as good. Eq. (2.21) generalizes and puts in a common framework many results already present in the literature [1, 75, 76, 44, 79, 73, 68, 5, 40]

An account of this work is published in [B14], and a more precise discussion is given in [B18].

## 2.5.2 A 1D self gravitating system?

In a magneto-optical trap, if an atom absorbs one photon from a laser, this photon is not available any more to interact with atoms behind the first one. This is the "shadow effect" sketched on the right panel of Fig. 2.6. As explained above, it may be modeled in the small optical width limit as a long range attractive two-body interaction. In principle, it raises the possibility

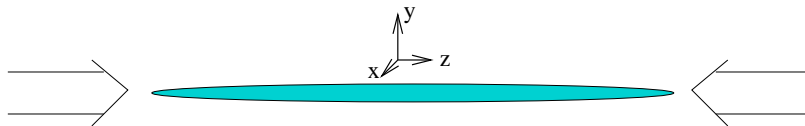


Figure 2.7: Sketch of the experiment. A cloud of Strontium is trapped in a very elongated dipolar trap (the dipolar trap is not represented). Two slightly red detuned laser beams are aligned along the  $z$ -axis; their frequency and intensity can be tuned. The shadow effect gives rise to an effective long-range attractive force within the cloud. The typical length of the cloud without the attraction is 0.5mm; a typical temperature is  $2\mu\text{K}$ .

to study in the lab an experimental system with attractive long range interactions. Unfortunately, in normal experimental conditions, this attractive interaction is dominated by the repulsive Coulomb-like interaction resulting from the multiple scattering of photons. However, in quasi one dimensional systems, such as atomic clouds trapped in very anisotropic external potentials, one may hope that multiple scattering becomes negligible.

Maryvonne Chalony and David Wilkowski have devised an experimental system to implement this idea, see Fig. 2.7.

To present as simply as possible the theoretical ideas, I use in this section a purely 1D model. In principle, one could start with a 3D model, and integrate over the small transverse directions to obtain a 1D model.

I write here more precisely the force exerted on an atom at position  $z$  and with velocity  $v_z$  by a laser coming from the left of intensity  $I_+(z)$  and wavenumber  $k$ :

$$F_+(z, v_z) = c \frac{I_+(z)}{\Gamma^2/4 + (\delta + kv_z)^2} \quad (2.22)$$

where  $c$  is a constant,  $\Gamma$  is the natural width of the atomic transition and  $\delta$  is the detuning of the laser with respect to this transition ( $\delta < 0$  as the lasers are slightly red-detuned). Similarly, one may write the force exerted by the laser coming from the right:

$$F_-(z, v_z) = -c \frac{I_-(z)}{\Gamma^2/4 + (\delta - kv_z)^2} \quad (2.23)$$

We linearize Eqs. (2.22)-(2.23) with respect to  $v_z$ ; then the total force  $F = F_+ + F_-$  can be written as a standard friction, linear in  $v_z$ , plus a space dependent force, linear in  $I_+(z) - I_-(z)$ . The atoms also feel an external trap, which we assume to be harmonic.

Calling  $f(z, v_z)$  the density of atoms at position  $z$  with velocity  $v_z$ , we can write a differential equation for the laser intensities

$$\frac{dI_+}{dz} = -c' \int \sigma(v_z) f(z, v_z) dv_z I_+(z) \quad (2.24)$$

$$\frac{dI_-}{dz} = c' \int \sigma(v_z) f(z, v_z) dv_z I_-(z) \quad (2.25)$$

where  $c'$  is a constant, and  $\sigma(v_z)$  is related to the probability of absorption of a photon by an atom with velocity  $v_z$ . Putting together Eqs. (2.22) to (2.25), and adding an equation for the density  $\rho(z) = \int f(z, v_z) dv$ , we obtain, for the stationary state and for reduced variables:

$$-\frac{L_i^2}{L_g^2} \bar{z} \bar{\rho} + \frac{1}{b} (\bar{I}_+ - \bar{I}_-) \bar{\rho} - \frac{d\bar{\rho}}{d\bar{z}} = 0 \quad (2.26)$$

$$\frac{d\bar{I}_+}{d\bar{z}} = -\frac{b}{2} \bar{I}_+ \bar{\rho} \quad (2.27)$$

$$\frac{d\bar{I}_-}{d\bar{z}} = \frac{b}{2} \bar{I}_- \bar{\rho} \quad (2.28)$$

We have introduced  $L_g$ , the characteristic size of the cloud when interactions are negligible;  $L_i$ , the characteristic size of the cloud when the trap is negligible;  $b$  the optical width of the cloud (it characterizes the probability that a photon is absorbed while crossing the cloud);  $\bar{z} = z/L_i$ ;  $\rho(z) = (N/L_i) \bar{\rho}(\bar{z})$  ( $N$  is the total number of atoms in the cloud);  $I_{\pm}(z) = I_0 \bar{I}_{\pm}(\bar{z})$  (where  $I_0$  is the laser intensities before they enter the cloud).

In the limit  $L_i/L_g \ll 1$  and  $b \ll 1$ , Eqs. (2.26)-(2.28) reduce to

$$-\frac{d\bar{\rho}}{d\bar{z}} + \frac{1}{2} \int_{-\infty}^{+\infty} F_G(s - \bar{z}) \bar{\rho}(s) ds = 0 \quad (2.29)$$

with  $F_G(u) = \text{sgn}(u)$ . This is exactly the equation for the stationary state of a 1D system of Brownian self gravitating particles [26, 27]. In these limits, the "shadow effect" acts as a 1D gravitation-like force between atoms. The solution of (2.29) is well known [72]:

$$\bar{\rho}(\bar{z}) = \frac{1}{8} \frac{1}{\cosh^2(\frac{\bar{z}}{4})} \quad (2.30)$$

Without the hypothesis  $b \ll 1$ , but keeping the hypothesis  $L_i/L_g \ll 1$ , we were still able to find an exact solution to Eqs. (2.26)-(2.28), generalizing

the solution (2.30):

$$\rho(\bar{z}) = \frac{2(1 - e^{-b/4})^2}{b^2} \frac{1 - \tanh^2 [(1 - e^{-b/2})\bar{z}/2b]}{1 - \frac{(1 - e^{-b/4})^2}{(1 + e^{-b/4})^2} \tanh^2 [(1 - e^{-b/2})\bar{z}/2b]} \quad (2.31)$$

A preliminary analysis of the experimental datas (taken by Maryvonne Chalony) has shown that the experimental parameters are not exactly in the range of validity of the "self-gravitating limit", but not far from it. As a consequence, it seems that some qualitative features predicted by the "self-gravitating model" are observed: effective attractive force within the cloud, density profile closer to a  $1/\cosh^2$  than to a gaussian. Unfortunately, more detailed comparisons between the "self-gravitating" theory and experiments seem precluded by the too many approximations made: the system cannot be really considered as a lab toy model of a 1D self-gravitating system.

This is still a work currently in progress. A more detailed account of the experimental results should be available soon.

### 2.5.3 Perspectives

This part of my research activity is quite recent. I try to list here some possible future directions.

1. A possible way to characterize the interactions in an atomic cloud inside a MOT is to study the spectrum of its oscillations. On the theoretical side, it requires to go beyond the breathing dynamics.
2. A transition to complex spatio-temporal dynamics has been observed in a numerical model of a large MOT [69]. This model includes a rather complete description of the complex photon-atom interaction. A natural question is: what are the minimal ingredients for the appearance of complex spatio temporal dynamics in such systems?
3. The 2D analog of the experiment described in Sec.sec:1D has several interests: first, in this case the shadow effect must be modeled as a non potential force; second, in 2D a collapse transition may be expected (see for instance [78]).
4. More generally, these systems are an incentive to develop the theory of out of equilibrium thermodynamics in long range interacting systems.

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## 2.6 Appendix

### 2.6.1 [B9]

## 2.6.2 [B12]

### 2.6.3 [B17]

## Articles written during the PhD

- [BP1] Julien Barré, Thierry Dauxois "Lyapunov exponent as a dynamical indicator of a phase transition", *Europhys. Lett.* **55**, 164 (2001).
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