



HAL
open science

Study of the discrete spectrum of complex perturbations of operators from mathematical physics

Clément Dubuisson

► **To cite this version:**

Clément Dubuisson. Study of the discrete spectrum of complex perturbations of operators from mathematical physics. Spectral Theory [math.SP]. Université Bordeaux, 2014. English. NNT : . tel-01260464

HAL Id: tel-01260464

<https://hal.science/tel-01260464>

Submitted on 22 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Étude du spectre discret de perturbations d'opérateurs de la physique mathématique

Study of the discrete spectrum of complex perturbations of operators from
mathematical physics

THÈSE

soutenue publiquement le 20 Novembre 2014

pour l'obtention du

Doctorat de l'Université de Bordeaux
(spécialité mathématiques pures)

par

Clément DUBUISSON

Rapporteurs :

M. JOYE Alain
M. LAPTEV Ari

Prof. Univ. Grenoble 1
Prof. Imperial College London

Composition du jury :

M. BOUCLET Jean-Marc
M. GOLÉNIA Sylvain
M. HARTMANN Andreas
M. KUPIN Stanislas

Prof. Univ. Toulouse 3
MdC Univ. Bordeaux
Prof. Univ. Bordeaux
Prof. Univ. Bordeaux (Directeur)

**Study of the discrete spectrum of complex
perturbations of operators from mathematical
physics**

Clément Dubuisson

Thèse de Mathématiques pures
Novembre 2014

Abstract

The topic of this thesis concern the discrete spectrum of non-selfadjoint operators defined by relatively compact perturbation of selfadjoint operators. These selfadjoint operators are chosen among classical operators of quantum mechanics. These are the Dirac operator, the Klein-Gordon operator, and the fractional laplacian who generalize the Schrödinger operator usually studied for such issues. The main method is based on a theorem of complex analysis which gives Blaschke-type condition on the zeros of a holomorphic function on the unit disc. This Blaschke condition gives informations on the behaviour of eigenvalues of the perturbed operator by mean of Lieb-Thirring-type inequalities. Another method using functional analysis is also used to obtain these kind of inequalities and both methods are compared to each other.

Keywords : Discrete spectrum, Lieb-Thirring-type inequalities, conformal mappings, Dirac operator, Klein-Gordon operator, fractional Schrödinger operator.

Résumé

L'objet de cette thèse est d'obtenir des informations sur le spectre discret d'opérateurs non auto-adjoints définis par des perturbations relativement compactes d'opérateurs auto-adjoints. Ces opérateurs auto-adjoints sont choisis parmi les opérateurs classiques de mécanique quantique. Il s'agit des opérateurs de Dirac, de Klein-Gordon et le laplacien fractionnaire qui généralise l'étude de l'opérateur de Schrödinger habituellement considéré pour de tels problèmes. La principale méthode utilisée ici relève d'un théorème d'analyse complexe donnant une condition de type Blaschke sur les zéros d'une fonction holomorphe du disque unité. Cette condition traduit le comportement des valeurs propres de l'opérateur perturbé sous forme d'inégalités de type Lieb-Thirring. Une autre méthode venant d'analyse fonctionnelle a été employée pour obtenir de telles inégalités et les deux méthodes sont comparées entre elles.

Mots-clés : Spectre discret, inégalités de type Lieb-Thirring, transformations conformes, opérateur de Dirac, opérateur de Klein-Gordon, opérateur de Schrödinger fractionnaire.

Institut de Mathématiques de Bordeaux, UMR 5251
351, cours de la Libération - F 33405 TALENCE cedex

À mes parents.

*"Ce n'est qu'en essayant continuellement qu'on finit par réussir.
Autrement dit : plus ça rate, plus on a de chances que ça marche."
Proverbe Shadoks*

Remerciements

Je tiens à remercier en premier lieu mon directeur de thèse, Stanislas Kupin, de m'avoir soutenu pour obtenir cette bourse de thèse d'abord, puis tout au long de ces trois années de doctorat (et maintenant l'ATER). Je suis impressionné par le calme et la patience dont il a fait preuve à mon égard et je lui en suis très reconnaissant. Je remercie les membres du jury d'avoir accepté d'en faire partie. Je remercie tout particulièrement les rapporteurs MM. Joye et Laptev pour leurs commentaires instructifs et M. Joye d'avoir pris le temps de m'aider à corriger certains défauts du mémoire.

Je remercie ensuite mes parents qui m'ont permis d'arriver à ce niveau d'étude. Décrire, témoigner ou expliquer tout ce qu'ils m'apportent depuis si longtemps serait une gageure tant ils m'ont accompagné dans mes diverses activités.

Dans mon parcours en mathématiques, j'ai une pensée émue pour mes professeurs du secondaire : Mme Guine, Mme Laubal et M. Luc, qui m'ont fait apprécier les mathématiques; une pensée nostalgique pour MM. Cozar et Soladié, mes professeurs de CPGE qui m'ont fait découvrir les mathématiques dans leur rigueur et leur beauté; une pensée pleine d'admiration pour MM. Cassou et Galusinski pour la magie de leurs enseignements. Je remercie également MM. Matignon, Deville et Fresnel pour leur implication dans l'agrégation et pour avoir toujours été disponibles pour répondre à mes questions. Tous ces professeurs sont des sources d'inspiration tant en mathématiques qu'en enseignement.

Pour avoir fait partie de divers conseils, je ne peux pas ne pas remercier les professeurs qui ont permis à ma génération de suivre une formation de qualité à l'université de Bordeaux (1) jusqu'à l'agrégation : MM. Fresnel, Jaulent et Matignon pour les noms que je connais et tous les professeurs qui y ont participé, sans oublier le personnel administratif qui fait preuve d'une belle efficacité pour tenir ces formations. Je tiens à leur témoigner toute ma reconnaissance d'autant plus par les temps actuels. Je remercie aussi tout le personnel BIATSS qui me (nous) permet de travailler dans d'excellentes conditions. Je remercie M. Ruch, directeur de l'UFR puis UF durant mon mandat d'élus étudiant pour m'avoir reçu et écouté à de nombreuses reprises.

Sur le plan professionnel, je remercie M. Boyer pour la confiance qu'il m'a accordée pour le tutorat, M. Golénia pour son aide depuis le début de mon premier article et M. Golinskii pour ses commentaires mathématiques très instructifs.

Enfin, je remercie les doctorants avec qui j'ai passé de bons moments : Giovanni, Stéphanie, Diomba, Samuel, Bruno, Nicola D.P., Nicolas De., Nicolas Du., Pierre, Youssef, Marouane, Aurélien, Nicolas M., Fanilo, Jocelyn, Jean-Baptiste, etc.

J'espère que ceux que j'ai oubliés ne m'en tiendront pas rigueur...

Je remercie ma famille et mes proches pour leur présence tout au long de ces années. J'ai une pensée particulière pour Philippe T. qui aurait dû suivre un parcours analogue au mien et qui, au lieu de cela, doit se battre contre la maladie.

Last but not least, Alice, depuis notre rencontre et encore aujourd'hui dans la vie de tous les jours, j'admire ton intelligence, ta vivacité d'esprit et tes passions. Merci d'être à mes côtés tous les jours, malgré mon caractère...

Contents

1	Introduction	1
1.1	General setting of the thesis	1
1.2	An overview of the topic	2
1.3	Main results and methods of proofs	4
1.4	Possible directions for the future work	8
2	Preliminaries	9
2.1	Generalities: definitions and notation	9
2.1.1	General operator theory	9
2.1.2	Spectrum of operators and Weyl's Theorem	11
2.2	Schatten-von Neumann classes and perturbation determinant	13
2.2.1	Definitions and properties	13
2.2.2	A regularized perturbation determinant	14
2.3	Complex analysis results	16
2.3.1	Borichev-Golinskii-Kupin Theorem	16
2.3.2	Useful inequalities	17
2.3.3	Conformal mappings	18
2.4	Birman-Solomyak inequality	18
3	The Dirac operator : results and proofs	21
3.1	Definition	21
3.2	Results and comparisons	23
3.3	Proof in dimension greater than 2	26
3.3.1	Conformal map for the Dirac operator	26
3.3.2	Bound on the resolvent of the unperturbed Dirac operator	30
3.3.3	Bound on the resolvent of the perturbed Dirac operator	32
3.3.4	Proof of Theorem 3.2.1 in dimension greater than 2	34
3.4	The dimension 1 case	36
3.4.1	Upper bound of the resolvent	36
3.4.2	Proof in dimension 1	39
3.5	The massless case	40
3.5.1	Conformal map and bound of the resolvent	40
3.5.2	Proof of Theorem 3.2.3	42
3.6	The Klein-Gordon operator	43
3.6.1	Definition and results	43
3.6.2	Proof of results	46
4	The fractional Laplacian operator : results and proof	53

4.1	Definition	53
4.2	Results and proofs	54
4.2.1	Conformal mapping for the fractional Laplacian	56
4.2.2	Upper bound for fractional Laplacian	56
4.2.3	Upper bound of the perturbed operator	59
4.2.4	Proof of Theorem 4.2.2	60
5	Lieb-Thirring-type Inequalities with Hansmann's results	69
5.1	Hansmann's Theorem and results	69
5.1.1	Hansmann's Theorem	69
5.1.2	Results and comparisons	70
5.1.3	Distortion for resolvent	72
5.2	Proof for the fractional Laplacian	74
5.3	Proof for the Klein-Gordon operator	76
6	Appendix	79
6.1	Integration of the Lieb-Thirring-type inequality in the case of massless Dirac operator	79
6.2	Glossary of the notation	82
	Bibliography	83
	Index	87

1 Introduction

Table of Contents

1.1	General setting of the thesis	1
1.2	An overview of the topic	2
1.3	Main results and methods of proofs	4
1.4	Possible directions for the future work	8

1.1 General setting of the thesis

¹ We start briefly presenting the general setting of the problem considered in this work. Most notation and terms will be given later; a glossary of the notation can be found in the appendix at the end of the thesis.

Let H_0 be a “well known” self-adjoint operator, satisfying $\sigma(H_0) = \sigma_{\text{ess}}(H_0)$, where σ_{ess} denote the *essential spectrum* of the operator. For instance, we can choose H_0 to be the Laplacian

$$\Delta f = \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}.$$

Then, add a *potential* V to H_0 ; recall that the potential acts as a *multiplicative operator* by the function V . This *perturbed* operator is denoted by

$$H = H_0 + V. \tag{1.1.1}$$

It acts on some dense subspace of $L^2(\mathbb{R}^d, \mathbb{C}^n)$ where $n = 1$ for the scalar case and $n > 1$ in the vectorial case.

In the case where $H_0 = -\Delta$, we call H *Schrödinger operator*.

We also consider the cases where H_0 is either the *Dirac operator* D_m (Chapter 3), or the *Klein-Gordon operator* K_m (same Chapter, last Section), or the *fractional Laplacian* $(-\Delta)^s$, $s > 0$ (Chapter 4).

The choice of the potential V , taken in a space L^p , is made so that the *essential spectrum* of the operator H is the same as the essential spectrum of unperturbed operator H_0 , that is

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0).$$

¹This version of the manuscript is the updated version of august 2015. The major modifications with respect to the original manuscript concern the presentation of the results in the introduction and some misprints.

Since we know the essential spectrum of H_0 , the problem we are interested in is to “determine” the *discrete spectrum* $\sigma_d(H)$ of H , which is the set of eigenvalues of H of finite algebraic multiplicity.

In this study, we obtain the information on the discrete spectrum of H under the form of the so called *Lieb-Thirring-type inequalities*, written as:

$$\sum_{\lambda \in \sigma_d(H_0)} \frac{d(\lambda, \sigma_{\text{ess}}(H))^\alpha}{d(\lambda, E)^\beta (1 + |\lambda|)^\gamma} \leq C \cdot \|V\|_{L^p}^p,$$

where E is the set of complex number which are the “edges” of $\sigma_{\text{ess}}(H)$ (except for ∞), the positive parameters α, β, γ bear the information on the distribution of the discrete spectrum $\sigma_d(H)$, and C is a constant (to be discussed in more detail later).

Before going to an overview of recent results on the subject, we recall that the *classical Lieb-Thirring inequality* is the following one, [LT76]: for real potentials V such that the integral below is convergent, we have

$$\sum_j (\lambda_j)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2} dx \quad (1.1.2)$$

where $L_{\gamma,d}$ is the sharp constant in (1.1.2), $x_- = \max\{0, -x\}$, and either $\gamma \geq \frac{1}{2}$ for $d = 1$, or $\gamma > 0$ for $d = 2$, or $\gamma \geq 0$ for $d \geq 3$. The quest for the sharp constant $L_{\gamma,d}$ is a interesting subject on its own, see [LW00], Theorem 12.4 in [LL01] and the remarks therein. This issue will not be discussed here.

Originally, the Lieb-Thirring inequalities were obtained in articles [LT76] and [LT75], which were devoted to the search for a new proof of “stability of matter”. For more details on this subject see the book by Lieb and Seiringer [LS09b] and the book by Balinsky and Evans [BE11].

The present work is structured as follow:

1. In the Introduction (Chapter 1), we present the general setting of the topic of this thesis, give its brief overview. We continue with new results and we discuss the methods applied in the proofs.
2. Chapter 2 deals with the preliminaries. We recall some definitions and propositions of general operator and spectral theory as well as more specific results we use later.
3. Chapters 3, 4, and 5 are devoted to Dirac operator, fractional Laplacian, and the application of Hansmann’s Theorem (including proofs and corresponding discussion), respectively. The results of Chapters 3 and 4 are obtained with the help of Borichev-Golinskii-Kupin Theorem 2.3.1, whereas the results of the Chapter 5 follow from results of Hansmann (Theorem 5.1.1).

1.2 An overview of the topic

As we said before, the problem we are interested in here, is to obtain some information on the discrete spectrum of the operator H .

First, let us focus on the Schrödinger operator

$$H = -\Delta + V. \quad (1.2.1)$$

When the potential V is real-valued, H is (essentially) self-adjoint on $L^2(\mathbb{R}^d, \mathbb{R})$, which allow us to perform the analysis described in the previous section. The main interest of this work is to deal with complex-valued potentials, so the perturbed operator is *a priori* non-self-adjoint.

At the beginning of the 2000's, Abramov, Aslanyan et Davies proved, [AAD01, Theorem 4], that the eigenvalues λ of the Schrödinger operator acting in $L^2(\mathbb{R})$ with a complex-valued potential $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfy

$$\forall \lambda \in \mathbb{C} \setminus \mathbb{R}^+, \quad |\lambda| \leq \frac{1}{4} \|V\|_{L^1}^2. \quad (1.2.2)$$

The proof was based on Birman-Schwinger principle and evaluating some integral kernel.

This inequality clearly gives a localization of the discrete spectrum of H since it lies in a disc centered at the origin 0 with radius $\frac{1}{4} \|V\|_{L^1}^2$. The evaluating the number of eigenvalues proved to be a more difficult task, and the authors proved a finiteness result [AAD01, Theorem 5] when the potential V satisfies $\forall \gamma \in \mathbb{R}, \|Ve^{\gamma x}\|_{L^1} < \infty$ and $d = 1$.

This article initiated a large number of papers on the subject; though, it is commonly admitted ([BO08], [GK07], [BGK09], [DHK09], etc.) that the paper which gave the second birth to the study of Lieb-Thirring-type inequalities is [FLLS06]. The point of the latter work was to consider a complex-valued potential instead of a real-valued one. Let us quote [CLN96] for the Dirac operator, [HS02] for the Jacobi matrices, or in [FLS08] for the fractional Schrödinger operator as connected to [FLLS06] works.

In the paper [FLLS06], the authors study the Schrödinger operator $H = -\Delta + V$, acting on $L^2(\mathbb{R}^d, \mathbb{C})$. They explain that, following a question from Davies, the aim was to find a similar inequality to (1.2.2) in dimension $d \geq 2$. They did not succeed to bound the modulus of the eigenvalues but they obtained a Lieb-Thirring-type inequality [FLLS06, Theorem 1]:

Theorem 1.2.1. *Let $d \geq 1$ and $\gamma \geq 1$.*

1. *For eigenvalues with non-positive real part*

$$\sum_{\operatorname{Re}(\lambda_j) < 0} |\operatorname{Re}(\lambda_j)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} \operatorname{Re}(V(x))_-^{\gamma + \frac{d}{2}} dx.$$

2. *If $\varkappa > 0$, then the eigenvalues inside the cone $\{|\operatorname{Im}(z)| < \varkappa \operatorname{Re}(z)\}$ verify*

$$\sum_{|\operatorname{Im}(\lambda_j)| \geq \varkappa \operatorname{Re}(\lambda_j)} |\lambda_j|^\gamma \leq C_{\gamma,d}(\varkappa) \int_{\mathbb{R}^d} |V(x)|^{\gamma + \frac{d}{2}} dx.$$

Here $C_{\gamma,d}(\varkappa) = 2^{1+\gamma/2+d/4} \left(1 + \frac{2}{\varkappa}\right)^{\gamma+d/2} L_{\gamma,d}$ and $L_{\gamma,d}$ is the Lieb-Thirring constant given in (1.1.2).

The rise of the interest to the topic resulted in many articles on Lieb-Thirring-type inequalities for different unperturbed operators. We quote for instance the papers by Bruneau and Ouhabaz ([BO08]), Borichev, Golinskii, and Kupin ([BGK09]), Laptev and Safronov ([LS09a]), Demuth, Hansmann, and Katriel ([DHK09, DHK13]), Golinskii and Kupin ([GK12, GK13]), Hansmann and Katriel ([HK11]), Hansmann ([Han11, Han13]).

Finally, concerning a counterpart of (1.2.2) in dimension $d \geq 2$ (the question posed by Davies), Frank showed ([Fra11, Theorem 1]) a similar inequality for $|\lambda|^\gamma$ with $0 < \gamma \leq \frac{1}{2}$.

Until now, we essentially spoke about the Schrödinger operator. This is not the only operator from the mathematical physics which have been studied recently: the Klein-Gordon operator, the Dirac operator, and more recently the fractional Laplacian also attracted some attention. Here are some references on the corresponding works.

For the Dirac operator, we mention the book by Thaller [Tha91], and we refer to the papers [CLN96], [CLT14], and [FS11].

For the Klein-Gordon operator, sometimes called *relativistic Schrödinger operator* as well as for the fractional Laplacian, we refer to [FLS08] or [LS09b].

It should be mentioned that these operators have also been studied in the presence of a magnetic field, see for instance the article by Sambou [Sam14] for the case of magnetic Schrödinger operators.

We finish this overview with a few words on Jacobi matrices. Since their study appeal to the same methods as Schrödinger operator, we quote the following articles: [GK07] - application of the idea in [FLLS06] to Jacobi matrices, [BGK09] - as application of Borichev-Golinskii-Kupin Theorem 2.3.1, [HK11] - with an improvement of the previous theorem, [FS11] - with a gap in the spectrum and the method of the article are also used to periodic Schrödinger operator and Dirac operator in dimension 1, and [Han11] - as application of Theorem 5.1.1.

1.3 Main results and methods of proofs

We comment on the methods used in the present work. Then we give the results in a simplified form (*i.e.*, the auxiliary constants are not made explicit).

The main method developed in this work is based on a theorem from complex analysis; we call it Borichev-Golinskii-Kupin Theorem 2.3.1, and it is proved in [BGK09]. In particular, this theorem is applied to (complex) Jacobi matrices, by Demuth, Hansmann, and Katriel in [DHK09] to Schrödinger operator, and by Sambou in [Sam14] to magnetic Schrödinger operator in odd dimension.

This theorem gives a Blaschke-type condition for the zeros of a holomorphic function on the unit disc satisfying the growth hypothesis (2.3.1). We consider as holomorphic function f a variation of the regularized perturbation determinant such that the zeros of f are exactly the eigenvalues of the perturbed operator H . Then we know how to bound this function with respect to the L^p -norm of the potential V and the norm of the resolvent of H_0 . Composing with a conformal map, this function f defined on $\mathbb{C} \setminus \sigma_{\text{ess}}(H)$ and its bound are transferred to the unit disc. We then apply Borichev-Golinskii-Kupin Theorem to find our estimate on the zeros of the composed function in the unit disc. At least, we

transfer it back in \mathbb{C} with the help of the inverse conformal map, and we obtain an estimate on the eigenvalues of H which is the Lieb-Thirring-type inequality we sought for.

Another method comes from a result of Hansmann 5.1.1, proved in [Han11]. This result gives a bound in terms of the L_p -norm of the potential for the sum of distance between the eigenvalues of the perturbed operator and the numerical range of the unperturbed operator. It is presented as a variation of a theorem from Kato ([Han11, Theorem 1.1]). Since the result is given for bounded operators, we apply it to the resolvents of the operators H and H_0 . Hence we need to bound the norm of the resolvent (this is the same computation as in the previous method), and to bound from below the distance between a point of the complex plan and the spectrum of the resolvent. These two calculations lead us to the Lieb-Thirring-type inequality.

Hansmann used this theorem to obtain new estimates for complex Jacobi matrices and Schrödinger operator.

Here is the list of our main results. The first result concerns the Dirac operator $D_m, m \geq 0$. We put $D = D_m + V$. In addition to its importance in physics, the particularity (and the difficulty) of this operator is that it is not semi-bounded, unlike the Schrödinger operator.

Theorem 1.3.1 (case $m > 0$). *Let D_m be the Dirac operator defined in (3.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ with $p > d$ and $0 < \tau \leq p - d$. Then the discrete spectrum $\sigma_d(D)$ of D satisfies the following Lieb-Thirring-type inequalities.*

For $d \geq 2$,

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_m))^{p+\tau}}{|\lambda - m| \cdot |\lambda + m| (1 + |\lambda|)^{2p-2+2\tau}} \leq C \cdot \|V\|_{L^p}^p,$$

with C depending on various parameters.

When $d = 1$, we have

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_m))^{p+1+\tau}}{|\lambda + m| \cdot |\lambda - m| \cdot (1 + |\lambda|)^{2p+2\tau}} \leq C \cdot \|V\|_{L^p}^p,$$

where C depends on various parameters.

In the case where the mass m is null, the result is the following:

Theorem 1.3.2 (case $m = 0$). *Let D_0 be the Dirac operator defined in (3.1.1) with $m = 0$ and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then, for $0 < \tau \leq p - d$,*

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{2(p+\tau)}} \leq C \cdot \|V\|_{L^p}^p,$$

with C depending on various parameters.

Using appropriately a method first developed for semi-bounded operator, we succeed to improve this last result. We then obtain the next result which is proved in Appendix.

Theorem 1.3.3 (case $m = 0$). *With the above notation, we have, for $0 < \tau \leq p - d$,*

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{d+2\tau}} \leq C \cdot \|V\|_{L^p}^p,$$

where C depends on various parameters.

With the help of the auxiliary results proved for the Dirac operator, we also obtain Lieb-Thirring-type inequality for the Klein-Gordon operator $K_m, m \geq 0$. We put $K = K_m + V$.

Theorem 1.3.4 (case $m > 0$). *Let K_m be the Klein-Gordon operator defined in (3.6.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_\ell(\mathbb{C}))$. We assume $p > d$ and let $\tau > 0$ be small enough.*

For $d \geq 2$, we have the following estimate:

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(1 + |\lambda - m|)^{d-1+2\tau}} \leq C \cdot \|V\|_{L^p}^p.$$

where C depends on various parameters.

In dimension $d = 1$, we have:

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda - m|(1 + |\lambda - m|)^{1+2\tau}} \leq C \cdot \|V\|_{L^p}^p,$$

with C depending on various parameters.

If the mass is null, we refer to the fractional Laplacian with $s = \frac{1}{2}$. It should be noted a difference at the point m of the spectrum when $m > 0$ or $m = 0$. The explanation comes from the different bound for the norm of the resolvent (compare the Proposition 3.3.6 and 4.2.5).

Concerning the fractional Laplacian $H_0 = -(\Delta)^s$, we obtain a Lieb-Thirring-type inequality for all $s > 0$ and not just for $s < \min\{1; \frac{d}{2}\}$, as in [FLS08]. The result is the following:

Theorem 1.3.5. *Let $H = H_0 + V$ be the fractional Schrödinger operator defined by (4.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > \max\{1; \frac{d}{2s}\}$. We take $\tau > 0$ small enough.*

For $0 < s \leq \frac{d}{2}$, the following inequality is satisfied

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^\alpha (1 + |\lambda|)^\beta} \leq C \cdot \|V\|_{L^p}^p,$$

where C depends on various parameters, and the powers verify

1. $\alpha = \min\{\frac{p+\tau}{2}; \frac{d}{2s}\}$,
2. $\beta = 2\tau + \frac{1}{2}(\frac{d}{s} - p - \tau)_+$.

For $s > \frac{d}{2}, p > 1$, we have

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p - \frac{d}{2s} + 1 + \tau}}{|\lambda|^\alpha (1 + |\lambda|)^\beta} \leq C \cdot \|V\|_{L^p}^p,$$

where C depends on various parameters and the powers verify

1. $\alpha = \frac{1}{2} + \frac{1}{2} \min\{p - \frac{d}{2s} + \tau; 1\}$,
2. $\beta = 2\tau + \frac{1}{2}(\frac{d}{2s} - p + 1 - \tau)_+$.

Finally, applying Hansmann's Theorem, we obtain the following results for the Klein-Gordon operator and for the fractional Laplacian.

Theorem 1.3.6. *Let K_m be the Klein-Gordon operator with $m > 0$ defined in (3.6.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then, for $\tau > 0$, the Lieb-Thirring-type inequality looks as follow*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^p}{(1 + |\lambda|)^{d + \tau}} \leq C \cdot \|V\|_{L^p}^p,$$

where the constant C depends on various parameters.

Concerning the fractional Laplacian we have the following inequality:

Theorem 1.3.7. *Let $H = H_0 + V$ be the Schrödinger operator defined in (4.1.1) for $s > 0$ and $p > \max\{1; \frac{d}{2s}\}$ with $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$. Then, for $\tau > 0$, the following inequality holds true*

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(1 + |\lambda|)^{\frac{d}{2s} + \tau}} \leq C \cdot \|V\|_{L^p}^p,$$

with C depending on various parameters.

Let us summarise the chronology of the previous results. Demuth, Hansmann, and Katriel worked on the Schrödinger operator, or more generally on a perturbation of a positive self-adjoint operator. We wonder then if the method works with a non-semi-bounded operator which lead us to the Dirac operator. Following the Dirac operator, we deduce results for the Klein-Gordon operator who appears as “ half ” a Dirac operator. Finally we interested ourself with the fractional Laplacian as a generalisation of the Schrödinger and Klein-Gordon (with $m = 0$) operators.

From the various results we obtain for a perturbed operator $H = H_0 + V$, we can imagine the following picture for the behaviour of the eigenvalues of H and their rate of convergence to the essential spectrum of H which is here the spectrum of H_0 . Indeed it seems that three cases occur was whether the limit is a (finite) real edge of the spectrum, or the point at infinity, or a point of the spectrum distinct from the edges (finite or not). Since the results are limited by the fact that such sequences of eigenvalues exist or not, we only give a few examples of possible conclusions deduce from the Lieb-Thirring-type inequality of a perturbation of the Dirac operator, after remark 3.2.4. The comparison between the Lieb-Thirring-type inequalities obtained from Borichev, Golinskii, and Kupin Theorem, and the one obtained from Hansmann Theorem for a same operator can be found in Subsection 5.1.2.

1.4 Possible directions for the future work

The present work gives rise to following natural problems we plan to work on in the future:

- An improvement (or a variation) of the Borichev-Golinskii-Kupin Theorem allowing to consider more peculiar growth condition near the boundary.
- If the unperturbed operator is $H_0 = f(-\Delta)$ with f a Bernstein function, for instance, under which conditions are we able to find a Lieb-Thirring-type inequality with the help of the methods described above ?
- In the spirit of [GK13], is it possible to obtain the above Lieb-Thirring-type inequalities for perturbations of self-adjoint perturbation, like the Stark operator or a Schrödinger operator with a Coulomb or Hardy potential?
- We quote the Birman-Schwinger principle after (1.2.2) which is often used to localize the discrete spectrum. Concerning the Dirac operator, the dimension $d = 1$ has been studied by Cuenin, Laptev, and Tretter in [CLT14] for the Dirac operator. So the question is still open for dimension $d \geq 2$.

The study of perturbations of operators is not constrained to operators acting on Hilbert space; in different setting, we sometimes have to restart from the beginning.

- We also may consider Banach spaces. In some Banach spaces the notion of determinant is well understood. However many points remain to clear up in order to apply results on the zeros of a holomorphic function. Results similar to those of this dissertation have been obtained by Demuth and Hanauska in [DH13].

2 Preliminaries

Table of Contents

2.1	Generalities: definitions and notation	9
2.1.1	General operator theory	9
2.1.2	Spectrum of operators and Weyl's Theorem	11
2.2	Schatten-von Neumann classes and perturbation determinant	13
2.2.1	Definitions and properties	13
2.2.2	A regularized perturbation determinant	14
2.3	Complex analysis results	16
2.3.1	Borichev-Golinskii-Kupin Theorem	16
2.3.2	Useful inequalities	17
2.3.3	Conformal mappings	18
2.4	Birman-Solomyak inequality	18

2.1 Generalities: definitions and notation

2.1.1 General operator theory

We only consider *complex separable Hilbert space* \mathcal{H} in this dissertation.

Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces, we define an *operator* from \mathcal{H}_1 to \mathcal{H}_2 as a linear application A defined on a dense subspace \mathcal{D} of \mathcal{H}_1 , called *domain*, with values in \mathcal{H}_2 . Often $\mathcal{H}_1 = \mathcal{H}_2$.

If A is defined on the whole \mathcal{H}_1 , we say that A is *bounded* on \mathcal{H}_1 if there exists a constant $C \in \mathbb{R}$ such that $\forall x \in \mathcal{H}, \|Ax\| \leq C \cdot \|x\|$. We note $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the set of bounded operators on \mathcal{H}_1 with values in \mathcal{H}_2 . If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we simply write $\mathcal{L}(\mathcal{H})$.

Generally, since the operators we consider come from differential equations, they will be defined on a dense subspace \mathcal{D} of a Hilbert space \mathcal{H} and are not bounded. We call them *unbounded operators*.

Fortunately working with a dense subspace \mathcal{D} in analysis has the advantage that we can take the closure $\overline{\mathcal{D}}$ of this subspace to “extend” known properties on \mathcal{D} to $\overline{\mathcal{D}}$. Especially we would like to define the notion of *adjoint* for an unbounded operator. But for that we need to “extend” the operator from \mathcal{D} to \mathcal{H} .

The following notion of *closable operator* is important to work with unbounded operator. We first need to define the *extension* of an operator.

Definition 2.1.1 (extension of an operator). Let A and B be two operators on \mathcal{H} . We say that B is an *extension* of A , and we write $A \subset B$, if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $\forall x \in \mathcal{D}(A)$, $Ax = Bx$.

Definition 2.1.2 (closed operator). We say that the operator $A : \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H}$ is *closed* if its *graph* $\Gamma(A)$ defined by $\Gamma(A) := \{\langle x; Ax \rangle, x \in \mathcal{D}\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$ endowed with the scalar product $\langle (x_1; y_1); (x_2; y_2) \rangle = \langle x_1; x_2 \rangle + \langle y_1; y_2 \rangle$.

We now define the notion of *closable operator*.

Definition 2.1.3 (closable operator and closure). An operator A is called *closable* if it has a closed extension.

Every closable operator has a smallest closed extension, called its *closure*. We denote it by \overline{A} .

There are operators which are not closable, but the self-adjoint operators or the essentially self-adjoints operators are. We now define the *adjoint* of an operator.

Definition 2.1.4 (adjoint operator). Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a densely defined operator with domain $\mathcal{D}(A)$. We denote

$$\mathcal{D}(A^*) := \{x \in \mathcal{H}, \exists y \in \mathcal{H}, \forall z \in \mathcal{D}(A), \langle Az; x \rangle = \langle z; y \rangle\}.$$

For each $x \in \mathcal{D}(A^*)$, we define $A^*x = y$. We call the operator A^* the *adjoint* of A .

See [Sch12, p.8] for a definition with $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$.

Proposition 2.1.5 (Theorem VIII.1 [RS80]). *Let A be a densely defined operator in \mathcal{H} . Then*

1. A^* is closed.
2. A is closable if and only if $\mathcal{D}(A^*)$ is dense, in which case $\overline{A} = A^{**}$.
3. If A is closable then $(\overline{A})^* = A^*$.

Definition 2.1.6 (self-adjoint, essentially self-adjoint operator). Let A be a densely defined operator in \mathcal{H} .

1) We say that A is *self-adjoint* if $A = A^*$, that is, if $\mathcal{D}(A) = \mathcal{D}(A^*)$ and for all $x \in \mathcal{D}(A)$, $Ax = A^*x$.

2) We say that A is *essentially self-adjoint* if \overline{A} is self-adjoint.

Remark 2.1.7. If A is essentially self-adjoint, then it has one and only one self-adjoint extension: \overline{A} . In the sequel we denote the operator and its closure by the same symbol.

We end this section with the notion of *compact operator*.

Definition 2.1.8 (compact operator). Let $A \in \mathcal{L}(\mathcal{H})$. We say that A is *compact* if for each bounded subset E , the image $A(E)$ is relatively compact in \mathcal{H} (i.e. the closure of $A(E)$ is compact in \mathcal{H}). We denote the space of compact operator on \mathcal{H} by \mathbf{S}_∞ (see Section 2.2).

We recall the Riesz-Schauder Theorem (see [RS80, Theorem VI.15] or [EE89, Theorem I.1.9])

Theorem 2.1.9 (Riesz-Schauder Theorem). *Let A be a compact operator on \mathcal{H} . Then the spectrum $\sigma(A)$ of A is discrete with eventually $\lambda = 0$ as unique accumulation point. Moreover any nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite geometric multiplicity.*

Remark 2.1.10. One can show that the algebraic multiplicity is also finite ([EE89, Remark I.1.19]). Those notions will be defined in the next subsection.

2.1.2 Spectrum of operators and Weyl's Theorem

We now turn our attention to the notion of *spectrum* of an operator.

Definition 2.1.11 (spectrum and resolvent). Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a closed operator in \mathcal{H} defined on \mathcal{D} .

We say that $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(A)$ of A if the operator $(\lambda\text{Id} - A)$ is a bijection from \mathcal{D} to \mathcal{H} with bounded inverse.

In this case, we call $(\lambda\text{Id} - A)^{-1}$ the *resolvent* of A at λ .

If $\lambda \notin \rho(A)$, we say that λ belongs to the *spectrum* of A and we note $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Usually we write $(\lambda - A)^{-1}$ instead of $(\lambda\text{Id} - A)^{-1}$.

Definition 2.1.12 (eigenvalue). We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists $x \in \mathcal{H} \setminus \{0\}$ such that $Ax = \lambda x$. We then call x an *eigenvector* of A . The set of eigenvalues of A is denoted by $\sigma_p(A)$ and called *point spectrum*.

If λ is an eigenvalue of A then $\lambda\text{Id} - A$ is not injective hence $\lambda \in \sigma(A)$. In other terms the kernel of $\lambda\text{Id} - A$, denoted by $\text{Ker}(\lambda\text{Id} - A)$, is not reduce to the null vector. In this case, the dimension of $\text{Ker}(\lambda\text{Id} - A)$ is called *geometric multiplicity* of λ .

Let $\lambda \in \sigma_p(A)$ and consider $\mathcal{R}_\lambda := \{x \in \mathcal{H}, \exists n \in \mathbb{N}^*, (\lambda\text{Id} - A)^n x = 0\}$ which is a linear subspace of \mathcal{H} . We call *algebraic multiplicity* of the eigenvalue λ , the dimension of \mathcal{R}_λ . Clearly the geometric multiplicity is less or equal to the algebraic multiplicity.

For more details about geometric and algebraic multiplicities see the book by Edmunds and Evans [EE89].

The next proposition is [EE89, Theorem IX.2.3].

Proposition 2.1.13 (spectrum of the resolvent). *Let A be a closed operator in \mathcal{H} and $\mu \in \rho(A)$. Then, for $\lambda \neq \mu$,*

1. $\lambda \in \rho(A)$ if and only if $(\mu - \lambda)^{-1} \in \rho((\mu\text{Id} - A)^{-1})$,
2. λ is an eigenvalue of A if and only if $(\mu - \lambda)^{-1}$ is an eigenvalue of $(\mu\text{Id} - A)^{-1}$ and in this case the algebraic and geometric multiplicities are the same,
3. and λ is in the essential spectrum of A (see definition below) if and only if $(\mu - \lambda)^{-1}$ belongs to the essential spectrum of $(\mu\text{Id} - A)^{-1}$.

Remark 2.1.14. Since the operators A we will consider are unbounded then, in particular, for each $\mu \in \rho(A)$, $0 \in \sigma((\mu\text{Id} - A)^{-1})$ ([Dav07, Lemma 8.1.9]).

Definition 2.1.15 (discrete spectrum). Let A be a closed operator in \mathcal{H} . We call *discrete spectrum* of A and denote it by $\sigma_d(A)$ the set of eigenvalues of A with finite algebraic multiplicity.

We can now define the *essential spectrum* in the context of a Hilbert space.

Let us first recall there are different definitions for the essential spectrum of a closed operator defined on a dense subspace of a Banach space. A detailed discussion can be found at the beginning of [EE89, Section IX.1]. Theorem 1.6 in [EE89, p.417] tells us that those definitions coincide when the operator is self-adjoint in a Hilbert space. For a definition in the case of Hilbert space, see [RS80, Subsection VII.3]. In both cases we can define the essential spectrum in the following way.

Definition 2.1.16 (essential spectrum). Let A be a self-adjoint operator in \mathcal{H} . Then the *essential spectrum* of A is defined as the complementary set in $\sigma(A)$ of the discrete spectrum, *i.e.*

$$\sigma_{\text{ess}}(A) = \sigma(A) \setminus \sigma_d(A).$$

We now formulate Weyl's Theorem (see [RS78, Theorem XIII.14] or [EE89, Theorem IX.2.1]).

Theorem 2.1.17 (Weyl's Theorem). *Let A be a self-adjoint operator and B be a closed operator such that*

1. *For some $\lambda \in \rho(B) \cap \rho(A)$, $(A - \lambda)^{-1} - (B - \lambda)^{-1}$ is compact.*
2. *and either*
 - a) *$\sigma(A) \neq \mathbb{R}$ and $\rho(B) \neq \emptyset$,*
 - or*
 - b) *there are elements of $\rho(B)$ in both the upper and lower half-planes.*

Then $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.

To satisfy the first condition of Weyl's Theorem, we will assume that, for suitable λ , $V(\lambda - H_0)^{-1}$ is compact since we know that $(\lambda - H)^{-1} - (\lambda - H_0)^{-1} = (\lambda - H)^{-1}V(\lambda - H_0)^{-1}$. This is the point of the next definition.

Definition 2.1.18 (relatively compact). The potential V is a *relatively compact perturbation* of H_0 if $\text{dom}(H_0) \subset \text{dom}(V)$ and for $\lambda \in \rho(H_0)$, $V(\lambda - H_0)^{-1}$ is compact.

The first condition of Weyl's Theorem is then satisfied. The condition 2a) will be verified for the Dirac operator with positive mass, the Klein-Gordon operator and the fractional Laplacian, and 2b) will be verified for the massless Dirac operator.

We will assume more than the relative compactness of V . In fact $V(\lambda - H_0)^{-1}$ will belong to some subspaces of the space of compact operators, the Schatten-von Neumann spaces \mathbf{S}_p . These spaces are defined in the next section. We introduce the following definition.

Definition 2.1.19 (relatively Schatten-von Neumann). We say that V is a *relatively Schatten-von Neumann perturbation* of H_0 if V is a relatively compact perturbation of H_0 such that, for some $p > 1$,

$$\text{for } \lambda \in \rho(H_0), V(\lambda - H_0)^{-1} \in \mathbf{S}_p. \quad (2.1.1)$$

With this hypothesis in the hand, we may apply the Birman-Solomyak inequality (see (2.4.2)) and also define a perturbation determinant. These tools are introduced in the next sections.

2.2 Schatten-von Neumann classes and perturbation determinant

2.2.1 Definitions and properties

For this subsection we refer to the monographs by Gohberg and Krein [GK69] and by Simon [Sim05], and the paper [Sim77].

Recall that the space of compact operators on \mathcal{H} is denoted by \mathbf{S}_∞ .

Definition 2.2.1 (Schatten-von Neumann spaces). The *Schatten-von Neumann spaces* \mathbf{S}_p , $1 \leq p < \infty$, are the subspaces of compact operators satisfying

$$\mathbf{S}_p := \{A \in \mathbf{S}_\infty, \|A\|_{\mathbf{S}_p}^p := \sum_{n=1}^{+\infty} s_n(A)^p < +\infty\},$$

where $(s_n(A))_{\mathbb{N}}$ are the *singular values* of A (or *s-numbers*), that is, the eigenvalues of $(AA^*)^{1/2}$ enumerated in decreasing order.

We recall ([GK69, Theorem III.7.1]) that the spaces \mathbf{S}_p , $p \geq 1$, are two-sided ideals of $\mathcal{L}(\mathcal{H})$ and have the structure of a complete algebra for the norm $\|\cdot\|_{\mathbf{S}_p}$. Furthermore we have the following inclusion: for $p \leq q$, $\mathbf{S}_p \subset \mathbf{S}_q$.

Definition 2.2.2 (regularized determinant). Let $A \in \mathbf{S}_n$, $n \in \mathbb{N}^*$, we define the *regularized determinant*

$$\det_n(\text{Id} - A) := \prod_{k=1}^{+\infty} \left[(1 - \lambda_k) \exp \left(\sum_{j=1}^{n-1} \frac{\lambda_k^j}{j} \right) \right],$$

where $(\lambda_k)_k$ is the sequence of eigenvalues of A .

This determinant satisfy the following properties:

1. $\det_n(\text{Id}) = 1$.
2. Let $A \in \mathbf{S}_n$, $\text{Id} - A$ is invertible if and only if $\det_n(\text{Id} - A) \neq 0$ ([Sim77, Cor.6.3]).
3. For all $A, B \in \mathcal{L}(\mathcal{H})$ satisfying $AB, BA \in \mathbf{S}_n$, $\det_n(\text{Id} - AB) = \det_n(\text{Id} - BA)$.

4. If $A(\cdot)$ is a holomorphic function on a region Ω with values in \mathbf{S}_p , then the function $\det_n(\text{Id} - A(\cdot))$ is holomorphic on Ω ([Sim77, Lemma 6.1] and [GK69, 8. p.163]).
5. Let $A \in \mathbf{S}_p$ for some real $p \geq 1$. Then clearly $A \in \mathbf{S}_{\lceil p \rceil}$, where $\lceil p \rceil$ is defined by $\min\{n \in \mathbb{N}, n \geq p\}$, and verify the following property

$$|\det_{\lceil p \rceil}(\text{Id} - A)| \leq \exp\left(\Gamma_p \|A\|_{\mathbf{S}_p}^p\right), \quad (2.2.1)$$

with Γ_p a positive constant depending only on p ([Sim77, Theorem 6.4]).

The next definition comes from [GK69, Subsection IV.3.1].

Definition 2.2.3 (regularized perturbation determinant). Let A and B be bounded operator in \mathcal{H} with $B - A \in \mathbf{S}_p$. Let μ be so that $\text{Id} - \mu A$ is invertible, then, we call *regularized perturbation determinant* of A by $B - A \in \mathbf{S}_p$, the determinant defined by

$$d(\mu) := \det_{\lceil p \rceil}((\text{Id} - \mu B)(\text{Id} - \mu A)^{-1})$$

The following property links the zeros of the regularized perturbation determinant and the eigenvalues of the perturbed operator.

Proposition 2.2.4 (Zeros of the determinant). *If $\mu \in \rho(A) \cap \sigma_p(B)$ has finite algebraic multiplicity then the order of μ^{-1} as zero of d is equal to the algebraic multiplicity of μ^{-1} as eigenvalue of B .*

For the proof, we refer to [GK69, Subsection IV.3.4] - case $p = 1$ - or [GGK00, Theorem XII.2.3], or [Han10, Proposition 1.5.8].

2.2.2 A regularized perturbation determinant

This subsection deals with the construction of a regularized perturbation determinant needed in our proofs. The content is inspired by the work done in the paper [DHK09] in Section 3.1.1. with the results in [GK69, Section IV.3].

We consider the following general setting: H_0 is a self-adjoint operator and we assume that $\sigma(H_0) = \sigma_{\text{ess}}(H_0)$. Then $H = H_0 + V$ with $V(\lambda - H_0)^{-1} \in \mathbf{S}_p$ for some $p > 1$ and for all $\lambda \in \rho(H_0)$. Hence, by Weyl's Theorem, $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$.

We show the following property.

Proposition 2.2.5. *With the above hypothesis, for $-a \in \rho(H)$, we define a function F in $\mathbb{C} \setminus \sigma(H_0)$ with values in \mathbf{S}_p by*

$$F(\lambda) = (\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1}, \quad (2.2.2)$$

and a function $f : \mathbb{C} \setminus \sigma(H_0) \rightarrow \mathbb{C}$ by

$$f(\lambda) := \det_{\lceil p \rceil}(\text{Id} - F(\lambda)). \quad (2.2.3)$$

We call the function f regularized perturbation determinant of H_0 by V . It verify the following properties: f is holomorphic on $\mathbb{C} \setminus \sigma(H_0)$, the zeros of f are the eigenvalues of H , and the order of the zero is equal to the algebraic multiplicity of the corresponding eigenvalue.

Proof. Let $\lambda \in \rho(H_0)$ and $-a \in \rho(H)$ (in particular $-a \in \rho(H_0)$). Denoting $\mu = \lambda + a$, $A = (a + H_0)^{-1}$ and $B = (a + H)^{-1}$, we have

$$\begin{aligned} (\text{Id} - \mu B)(\text{Id} - \mu A)^{-1} &= \text{Id} - \text{Id} + (\text{Id} - \mu B)(\text{Id} - \mu A)^{-1} \\ &= \text{Id} - [(\text{Id} - \mu A) - (\text{Id} - \mu B)](\text{Id} - \mu A)^{-1} \\ &= \text{Id} - \mu(B - A)(\text{Id} - \mu A)^{-1}. \end{aligned}$$

We recall that for all $\lambda \in \rho(H_0)$, $\mu^{-1} \in \rho(A)$, by Proposition 2.1.13, which justifies the invertibility of $\text{Id} - \mu A$. In fact we have the following equivalence:

$$\begin{aligned} \text{Id} - (\lambda + a)(a + H_0)^{-1} \text{ invertible} &\iff [a + H_0 - \lambda - a](a + H_0)^{-1} \text{ invertible} \\ &\iff (\lambda - H_0) \text{ invertible} \\ &\iff \lambda \in \rho(H_0). \end{aligned} \tag{2.2.4}$$

We describe the above formula in terms of a , λ , H_0 , and H . That is, we have the following resolvent relation

$$\begin{aligned} B - A &= (a + H)^{-1} - (a + H_0)^{-1} \\ &= (a + H)^{-1} [(a + H_0) - (a + H)] (a + H_0)^{-1} \\ &= -(a + H)^{-1} V (a + H_0)^{-1}, \end{aligned} \tag{2.2.5}$$

so, substituting in the previous relation, we obtain, for $\lambda \in \rho(H_0)$,

$$\begin{aligned} \text{Id} - \mu(B - A)(\text{Id} - \mu A)^{-1} &= \\ &= \text{Id} + (\lambda + a)(a + H)^{-1} V (a + H_0)^{-1} [\text{Id} - (\lambda + a)(a + H_0)^{-1}]^{-1} \\ &= \text{Id} + (\lambda + a)(a + H)^{-1} V [(\text{Id} - (\lambda + a)(a + H_0)^{-1}) (a + H_0)]^{-1} \\ &= \text{Id} + (\lambda + a)(a + H)^{-1} V [-\lambda + H_0]^{-1} \\ &= \text{Id} - F(\lambda), \end{aligned}$$

where F is defined in (2.2.2) and $F(\lambda) \in \mathbf{S}_p$ since $V(\lambda - H_0)^{-1} \in \mathbf{S}_p$.

Now we show that the function $f : \mathbb{C} \setminus \sigma(H_0) \rightarrow \mathbb{C}$ defined in (2.2.3) verifies the wanted properties.

First of all the map $\lambda \mapsto F(\lambda)$ is well defined on $\rho(H_0)$ with values in \mathbf{S}_p hence f is well defined, holomorphic on $\rho(H_0)$, and $f(-a) = \det_{[\rho]}(\text{Id}) = 1$.

We now show that $\text{Id} - F(\lambda)$ is non invertible if and only if $\lambda \in \sigma_d(H)$. We know the following resolvent equation from (2.2.5)

$$(a + H_0)^{-1} - (a + H)^{-1} = (a + H)^{-1} V (a + H_0)^{-1},$$

consequently

$$\begin{aligned} \text{Id} - (\lambda + a)(a + H)^{-1} &= \\ &= \text{Id} - (\lambda + a) [(a + H_0)^{-1} - (a + H)^{-1} V (a + H_0)^{-1}] \\ &= \text{Id} - (\lambda + a)(a + H_0)^{-1} + (\lambda + a)(a + H)^{-1} V (a + H_0)^{-1}, \end{aligned}$$

hence, since $\lambda \in \rho(H_0)$,

$$\begin{aligned}
& (\text{Id} - (\lambda + a)(a + H)^{-1}) [\text{Id} - (\lambda + a)(a + H_0)^{-1}]^{-1} = \\
& = (\text{Id} - (\lambda + a)(a + H_0)^{-1} + (\lambda + a)(a + H)^{-1}V(a + H_0)^{-1}) \times \\
& \quad \times [\text{Id} - (\lambda + a)(a + H_0)^{-1}]^{-1} \\
& = \text{Id} + (\lambda + a)(a + H)^{-1}V(a + H_0)^{-1} [\text{Id} - (\lambda + a)(a + H_0)^{-1}]^{-1} \\
& = \text{Id} - F(\lambda).
\end{aligned}$$

So $\text{Id} - F(\lambda)$ is non invertible if and only if $\text{Id} - (\lambda + a)(a + H)^{-1}$ is non invertible. But, using the same argument as in (2.2.4) and knowing that $\lambda \notin \sigma(H_0) = \sigma_{\text{ess}}(H)$, we have

$$\begin{aligned}
\text{Id} - (\lambda + a)(a + H)^{-1} \text{ non invertible} & \iff \lambda \in \sigma(H) \\
& \iff \lambda \in \sigma_d(H).
\end{aligned}$$

Finally, by Proposition 2.2.4,

$$f(\lambda) = 0 \iff \lambda \in \sigma_d(H),$$

and the order of the zero for f is equal to the algebraic multiplicity of the eigenvalue λ for H . \square

2.3 Complex analysis results

2.3.1 Borichev-Golinskii-Kupin Theorem

The next theorem, proved in [BGK09, Theorem 0.3], gives a bound of the zeros of a holomorphic function on the unit disc $\mathbb{D} := \{|z| < 1\}$ according to its growth near the boundary $\mathbb{T} := \{|z| = 1\}$. The important aspect of this result for the sequel is that we can consider points (ζ_j) on the unit circle where the growth increase.

Theorem 2.3.1 (Borichev, Golinskii, Kupin). *Let h be a holomorphic function on \mathbb{D} with $h(0) = 1$. We assume that h satisfies the following inequality*

$$|h(z)| \leq \exp \left(\frac{K}{(1 - |z|)^\alpha} \prod_{j=1}^N \frac{1}{|z - \zeta_j|^{\beta_j}} \right), \quad (2.3.1)$$

where $K > 0$, $|\zeta_j| = 1$ and $\alpha, \beta_j \geq 0$, $j = 1, \dots, N$.

Then, for all $\tau > 0$, the zeros of h verify the following Blaschke-type inequality

$$\sum_{h(z)=0} (1 - |z|)^{\alpha+1+\tau} \prod_{j=1}^N |z - \zeta_j|^{(\beta_j-1+\tau)_+} \leq C \cdot K,$$

where C depends on α, β_j, ζ_j , and τ .

Above, $x_+ = \max\{x, 0\}$.

The above Blaschke-type inequality shows that the zeros of h may accumulate ‘‘slower’’ to the ζ_j -points than elsewhere on the unit circle.

Remark 2.3.2. In the applications, the points ζ_j correspond (in general) to the edges of the essential spectrum of the perturbed operator. But we shall be careful that a modification of the power β_j of the factor $|z - \zeta_j|$ does not only imply a modification of the corresponding edge of the essential spectrum. Indeed, we can verify this fact with the proofs in this thesis, but we also have to remember that we are working with a holomorphic function h on \mathbb{D} and so, a modification of the behaviour of h in the neighbourhood of some point in the disc induce a modification in other regions in the disc.

For instance, in the case of the fractional Schrödinger operator, $0 \in \mathbb{C}$ is the image of $-1 \in \mathbb{D}$ via the conformal map φ_a defined in (4.2.6), and Formula (4.2.9) with $|1 + z| = \frac{2\sqrt{|\lambda|}}{|\sqrt{\lambda} + i\sqrt{a}|}$ for $z \in \mathbb{D}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and a large enough, shows that a modification of the power in the factor $|1 + z|$ implies the factor $|\lambda|$ (behaviour at 0) and $a + |\lambda|$ (behaviour at ∞).

Remark 2.3.3. Since for $\beta \in \mathbb{R}$ we have $(\beta_+ - 1 + \tau)_+ = (\beta - 1 + \tau)_+$, instead of resolving in the first place $\beta \geq 0$, we will apply Theorem 2.3.1 directly when the inequality 2.3.1 occurs and we distinguish afterwards the different cases $\beta_j - 1 + \tau \geq 0$.

This result has been improved in some sense by Demuth and Hansmann: [HK11, Theorem 4]. But in our cases when we may apply the latter (massless Dirac, Klein-Gordon, fractional Laplacian), it brings no difference with Theorem 2.3.1 (see also remark 3.6.9).

In particular, Favorov and Golinski discuss different investigations around the previous theorem (see for instance [FG09]).

2.3.2 Useful inequalities

For two nonnegative functions f, g defined on a domain Ω of the complex plane \mathbb{C} , we write $f(\lambda) \approx g(\lambda)$ if there are constants $C_1, C_2 > 0$ so that $C_1 f(\lambda) \leq g(\lambda) \leq C_2 f(\lambda)$ for all $\lambda \in \Omega$. And we write $f(\lambda) \lesssim g(\lambda)$ if there is a positive constant C such that $f(\lambda) \leq C \cdot g(\lambda)$ for $\lambda \in \Omega$. The choice of the domain Ω will be clear from the context.

We used repeatedly the next inequalities (in particular to bound integrals).

Lemma 2.3.4. 1. Let $a, b \geq 0$, and $p \geq 1$, then

$$a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p).$$

2. Let $a, b \geq 0$, then

$$\sqrt{a^2 + b^2} \leq a + b \leq \sqrt{2}\sqrt{a^2 + b^2}.$$

3. Let $0 < \alpha < 1$ and $a, b \geq 0$. Then

$$2^{\alpha-1}(a^\alpha + b^\alpha) \leq (a + b)^\alpha \leq a^\alpha + b^\alpha.$$

2.3.3 Conformal mappings

The conformal maps appear in our work for two reasons: the first to send the domain of a map into \mathbb{D} in order to apply Theorem 2.3.1 of Borichev, Golinskii, and Kupin. The other to work with Hansmann's Theorem 5.1.2 (or 5.1.1). In both case this is a *distortion* problem, that is, to compare the distance between a point and the boundary of a region and the distance of the image of this point to the image of the boundary by a conformal map. We denote the *distance between z and A* by $d(z, A) := \inf_{w \in A} |z - w|$.

This allow on one hand to send our study of the discrete spectrum from \mathbb{C} to a study in \mathbb{D} so that we can apply Theorem 2.3.1 of Borichev, Golinskii, and Kupin, or on the other hand to transfer a problem with an unbounded operator to a problem with its resolvent so that we can apply the results of Hansmann 5.1.2 or 5.1.1.

Theorem 2.3.5 (Koebe distortion). *Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be injective. Then*

$$\forall z \in \mathbb{D}, \frac{1}{4}(1 - |z|^2)|f'(z)| \leq d(f(z), \partial f(\mathbb{D})) \leq (1 - |z|^2)|f'(z)|.$$

For a proof we refer to [Pom92, Corollary 1.4].

In the sequel we will use the following writing of the above inequality with the introduction of the distance between z and the unit circle \mathbb{T} . Since for all $z \in \mathbb{D}$, $1 \leq 1 + |z| \leq 2$, we have

$$\forall z \in \mathbb{D}, \frac{1}{4} d(z, \mathbb{T}) |f'(z)| \leq d(f(z), \partial f(\mathbb{D})) \leq 2 d(z, \mathbb{T}) |f'(z)|. \quad (2.3.2)$$

2.4 Birman-Solomyak inequality

The next result is a generalization of Theorem 4.1 in [Sim05] to vector-valued function.

We put $\mathcal{M}_n(\mathbb{C})$ to be the space of square complex matrices of order n . For $p \geq 1$, we consider the space of measurable functions on \mathbb{R}^d with values in $\mathcal{M}_n(\mathbb{C})$ defined by

$$L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C})) = \left\{ V : \|V\|_{L^p}^p = \int_{\mathbb{R}^d} \|V(x)\|_F^p dx \right\},$$

where $\|\cdot\|_F$ is called *Frobenius norm*,

$$\|V(x)\|_F = \left(\sum_{i,j=1,\dots,n} |(V(x))_{i,j}|^2 \right)^{1/2}. \quad (2.4.1)$$

The choice of the Frobenius norm is important for the proof of the following proposition.

Proposition 2.4.1. *Let f and $g \in L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))$ with $1 < p < \infty$. Then we have the following inequality*

$$\|f \cdot g(-i\nabla)\|_{\mathfrak{S}_p}^p \leq (2\pi)^{-d} \|f\|_{L^p}^p \cdot \|g\|_{L^p}^p. \quad (2.4.2)$$

Proof. The proof closely follows [Sim05, Theorem 4.1], the main modifications being the use of the Froebenius norm and relation (2.4.3) for matrix-valued integral operators. To stress the differences of the matrix-valued case as compared to the scalar one, we give the argument in a somewhat more detailed form than the quoted theorem from [Sim05].

All operators considered in this proposition act on $L^2(\mathbb{R}^d; \mathbb{C}^n)$. Let B be a bounded operator given, for $f \in L^2(\mathbb{R}^d; \mathbb{C}^n)$, by

$$(Bf)(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy,$$

where the kernel $K(\cdot, \cdot)$ is a $\mathcal{M}_n(\mathbb{C})$ -valued measurable function. A familiar result from [GK69] or [Sim05] says that

$$\|B\|_{\mathfrak{S}_2}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|K(x, y)\|_F^2 dx dy. \quad (2.4.3)$$

Now, denote by $A := f(x)g(-i\nabla)$ the integral operator associated to the kernel

$$(2\pi)^{-d/2} f(x) \check{g}(x - y),$$

where \check{g} is the inverse Fourier transform of g .

Suppose that f and g are in $L^2(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$. Recalling (2.4.1) and the fact that the norm is submultiplicative, we obtain that

$$\begin{aligned} \|A\|_{\mathfrak{S}_2}^2 &= \|f(x)g(-i\nabla)\|_{\mathfrak{S}_2}^2 = (2\pi)^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|f(x) \check{g}(x - y)\|_F^2 dx dy \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|f(x)\|_F^2 \|\check{g}(x - y)\|_F^2 dx dy \\ &\leq (2\pi)^{-d} \|f\|_{L^2}^2 \cdot \|g\|_{L^2}^2, \end{aligned}$$

where we used Fubini and Fourier-Parseval Theorems. So, the integral operator A lies in \mathfrak{S}_2 (i.e., it is Hilbert-Schmidt), and we have bound (2.4.2) for $p = 2$. In particular, A is a compact operator.

Recall that L^∞ is the space endowed with the norm

$$\|f\|_{L^\infty} := \text{ess-sup}_{x \in \mathbb{R}^d} \|f(x)\|_F.$$

Let us take two test functions ϕ, ψ from $L^2(\mathbb{R}^d; \mathbb{C}^n)$ such that

$$\|\phi\|_{L^2}^2 = \int_{\mathbb{R}^d} \|\phi(x)\|_2^2 dx = \int_{\mathbb{R}^d} \left(\sum_{i=1}^n |\phi_i(x)|^2 \right) dx \leq 1$$

and $\|\psi\|_{L^2}^2 \leq 1$. We are to prove that

$$\|A\| = \|A\|_{\mathfrak{S}_\infty} = \sup_{\phi, \psi} |(\phi, A\psi)| \leq \|f\|_\infty \cdot \|g\|_\infty, \quad (2.4.4)$$

where $f, g \in L^2 \cap L^\infty$. Indeed,

$$\begin{aligned} |(\phi, A\psi)| &= |(\phi, f(g\hat{\psi}))| = |(f^* \phi, (g\hat{\psi})^\sim)| \\ &\leq \|f^* \phi\|_{L^2} \|(g\hat{\psi})^\sim\|_{L^2}. \end{aligned}$$

Then

$$\|f^* \phi\|_{L^2}^2 = \int_{\mathbb{R}^d} \|f^* \phi\|_2^2 dx \leq \int_{\mathbb{R}^d} \|f\|_F^2 \|\phi\|_2^2 dx \leq \|f\|_\infty^2 \|\phi\|_{L^2}^2,$$

and, similarly,

$$\|g \hat{\psi}\|_{L^2}^2 \leq \|g\|_\infty^2 \|\hat{\psi}\|_{L^2}^2 = \|g\|_\infty^2 \|\psi\|_{L^2}^2.$$

Hence, (2.4.4) is proved for all f, g in $L^2 \cap L^\infty$. Then the standard complex interpolation argument yields

$$\|f(x)g(-i\nabla)\|_{\mathfrak{S}_p}^p \leq (2\pi)^{-d} \|f\|_{L^p}^p \cdot \|g\|_{L^p}^p,$$

for all $2 \leq p < \infty$. The same result for indices $1 < p \leq 2$ follows by duality. □

3 The Dirac operator : results and proofs

Table of Contents

3.1	Definition	21
3.2	Results and comparisons	23
3.3	Proof in dimension greater than 2	26
3.3.1	Conformal map for the Dirac operator	26
3.3.2	Bound on the resolvent of the unperturbed Dirac operator	30
3.3.3	Bound on the resolvent of the perturbed Dirac operator	32
3.3.4	Proof of Theorem 3.2.1 in dimension greater than 2	34
3.4	The dimension 1 case	36
3.4.1	Upper bound of the resolvent	36
3.4.2	Proof in dimension 1	39
3.5	The massless case	40
3.5.1	Conformal map and bound of the resolvent	40
3.5.2	Proof of Theorem 3.2.3	42
3.6	The Klein-Gordon operator	43
3.6.1	Definition and results	43
3.6.2	Proof of results	46

3.1 Definition

According to Dirac's theory (see for instance [Tha91, Section 1]) the relativistic motion of a particle with spin $\frac{1}{2}$ - electron or positron for examples in the case of a positive mass $m > 0$ - (without electromagnetic field) is describe by the Dirac operator. Because of spin structure, the configuration space of the particle takes values in \mathbb{C}^n , where $n = 2^\nu$ with $\nu \geq 1$. The movement of the free particle of mass m is given by the Dirac equation,

$$i\hbar \frac{\partial \varphi}{\partial t} = D_m \varphi,$$

where $\varphi \in L^2(\mathbb{R}^d; \mathbb{C}^n)$ with $d \in \{1, \dots, n-1\}$, if $m > 0$ and $d \in \{1, \dots, n\}$ otherwise. The Dirac operator is defined as

$$D_m := -i\hbar \alpha \cdot \nabla + mc^2\beta = -i\hbar \sum_{k=1}^d \alpha_k \frac{\partial}{\partial x_k} + mc^2\beta. \quad (3.1.1)$$

Here c is the speed of light, and \hbar is the reduced Planck constant. We renormalize and consider $\hbar = c = 1$. Here we set $\alpha := (\alpha_1, \dots, \alpha_d)$ and $\beta := \alpha_{d+1}$. The matrices α_i are $d+1$ linearly independent self-adjoint linear maps, acting in \mathbb{C}^n , satisfying the following anti-commutation relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{i,j} \text{Id},$$

for $i, j = 1, \dots, d+1$. For instance, on \mathbb{R}^3 , one can choose the Pauli-Dirac representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \text{Id}_{\mathbb{C}^2} & 0 \\ 0 & -\text{Id}_{\mathbb{C}^2} \end{pmatrix},$$

for $i = 1, 2, 3$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the general case, the $n \times n$ -matrices α_j are constructed as special elements of the so-called Clifford algebra (see [Obo98, Chapter 1]). Without any loss of generality we take

$$\beta := \begin{pmatrix} \text{Id}_{\mathbb{C}^{n/2}} & 0 \\ 0 & -\text{Id}_{\mathbb{C}^{n/2}} \end{pmatrix}.$$

Mimicking the proofs of Section 1.1 to Section 1.4 of [Tha91, Section 1] it is easy to check that the operator D_m is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{C}^n)$ and the domain of its closure is $\mathcal{H}^1(\mathbb{R}^d; \mathbb{C}^n)$, the Sobolev space of order 1 with values in \mathbb{C}^n . The closure of the operator is denoted with the same symbol D_m . With the help of the Fourier transform, it is easy to prove that D_m is unitarily equivalent to

$$\begin{pmatrix} \sqrt{-\Delta_{\mathbb{R}^d} + m^2} \times \text{Id}_{\mathbb{C}^{n/2}} & 0 \\ 0 & -\sqrt{-\Delta_{\mathbb{R}^d} + m^2} \times \text{Id}_{\mathbb{C}^{n/2}} \end{pmatrix}. \quad (3.1.2)$$

Hence the spectrum of D_m is absolutely continuous and given by $] -\infty, -m] \cup [m, +\infty[$.

We recall that the function V is identified with the operator of multiplication by itself. Assuming that $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ and $p > d$, we show (Proposition 2.4.1) that the multiplication by V is a relatively Schatten-von Neumann perturbation of D_m (see definition 2.1.19). We consider the perturbed operator

$$D = D_m + V. \quad (3.1.3)$$

By Weyl's Theorem 2.1.17

$$\sigma_{\text{ess}}(D) = \sigma_{\text{ess}}(D_m) = \sigma(D_m) =] -\infty, -m] \cup [m, +\infty[.$$

3.2 Results and comparisons

Our main results for the Dirac operator are the following. We first give the results when the mass m is positive and then when $m = 0$. We compare the first theorem with previously known results.

Let us give some precisions about the constants involved in the results and who will be introduced in the following sections. The constant b is a real number, larger than 1, in order to define the resolvent of the perturbed operator D at the point ib (see Lemma 3.3.7), and Z_b is a constant from the distortion calculus.

Theorem 3.2.1 (case $m > 0$). *Let D be the perturbed Dirac operator defined in (3.1.3) with $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ with $p > d$. Then the discrete spectrum $\sigma_d(D)$ verify the following Lieb-Thirring-type inequalities.*

If $d \geq 2$ and $0 < \tau \leq p - d$ then

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_m))^{p+\tau}}{|\lambda - m| \cdot |\lambda + m| (1 + |\lambda|)^{2p-2+2\tau}} \leq C \cdot K_4 \cdot \|V\|_{L^p}^p, \quad (3.2.1)$$

with

1. $K_4 = K_1 b^p 2^{5p+d-2+3\tau} \cdot \frac{(1+m)^{3p+d-3+2\tau}}{m^{3p+d-2+\tau}} \cdot Z_b^{12p+2d-13+3\tau}$,
 - a) where $K_1 = \Gamma_p 2^{d-2} (2\pi)^{-d} s_{d-1} n^{\frac{p}{2}} M (1+2m)^{d-1}$,
 - b) and M, s_{d-1} are defined in Proposition 3.3.6;
2. and C depends on d, p, m , and τ .

When $d = 1$, for $\tau > 0$, the inequality becomes

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_m))^{p+1+\tau}}{|\lambda + m| \cdot |\lambda - m| \cdot (1 + |\lambda|)^{2p+2\tau}} \leq C \cdot K_3 \cdot \|V\|_{L^p}^p, \quad (3.2.2)$$

with

1. $K_3 = n^{\frac{p}{2}} s_0 \cdot \frac{\Gamma_p}{2\pi} \cdot b^p \cdot M (1+2m) 2^{5p+4+3\tau} \cdot \frac{(1+m)^{3p+1+\tau}}{m^{3p+2+\tau}} \cdot Z_b^{12p+1+3\tau}$,
2. the constants M and s_0 are defined in Proposition 3.4.1,
3. and C depends on p, m , and τ .

Remark 3.2.2. The above constants C come from the application of Theorem 2.3.1 of Borichev-Golinskii-Kupin and are not explicit. But the constants K_4 and K_3 are computable from the proof of the theorems. This remark is also valid for the other theorems deduce from Theorem 2.3.1.

We compare these inequalities with result from Cancelier, Lévy-Bruhl, and Nourrigat ([CLN96]) in dimension $d = 3$, and another from Frank and Simon ([FS11]) in dimension $d = 1$. In these articles the authors worked with self-adjoint perturbations (the potential V being real), and in these cases, the relations (3.2.1) and (3.2.2) can be written in a simpler form. In fact the discrete spectrum $\sigma_d(D)$ is then real and hence is contained in $] - m; m[$. We denote $E_m = \{\pm m\}$ and we have $d(\lambda, \sigma(D_m)) = d(\lambda, E_m)$ for $\lambda \in \sigma_d(D)$. A simple computation shows that (3.2.1) becomes

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^{p-1+\tau} \lesssim \|V\|_{L^p}^p, \quad (3.2.3)$$

and (3.2.2) becomes

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^{p+\tau} \lesssim \|V\|_{L^p}^p. \quad (3.2.4)$$

In [CLN96], $d = 3$ and, with our notation, one of their results (see [CLN96, Corollaire 1.3]) says that

$$\forall \gamma > 0, \quad \sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^\gamma \leq C_p \left(\|V\|_{L^{\gamma+3}}^{\gamma+3} + \|V\|_{L^{\gamma+3/2}}^{\gamma+3/2} \right),$$

for real potentials such that the right-hand side is convergent. Taking $\gamma = p - 3$, *i.e.* $p > d = 3$, we find

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^{p-3} \leq C_p (\|V\|_{L^p}^p + \|V\|_{L^{p-3/2}}^{p-3/2}), \quad (3.2.5)$$

where $p > 3$ and $V \in L^p(\mathbb{R}^3; \mathbb{R}) \cap L^{p-3/2}(\mathbb{R}^3; \mathbb{R})$.

In [FS11], $d = 1$ and they proved (see [FS11, Theorem 7.1]), for $\gamma \geq \frac{1}{2}$ and $V \in L^{\gamma+1}(\mathbb{R}; \mathbb{R}) \cap L^{\gamma+1/2}(\mathbb{R}; \mathbb{R})$.

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^\gamma \leq C_{1,p} \|V\|_{L^{\gamma+1}}^{\gamma+1} + C_{2,p,m} \|V\|_{L^{\gamma+1/2}}^{\gamma+1/2}.$$

Taking $\gamma = p - \frac{1}{2}$ with $p \geq 1$, we get to

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, E_m)^{p-\frac{1}{2}} \leq C_{1,p} \|V\|_{L^{p+\frac{1}{2}}}^{p+\frac{1}{2}} + C_{2,p,m} \|V\|_{L^p}^p, \quad (3.2.6)$$

and $V \in L^p(\mathbb{R}; \mathbb{R}) \cap L^{p+1/2}(\mathbb{R}; \mathbb{R})$. Clearly the inequalities (3.2.5) and (3.2.6) are respectively better than (3.2.3) and (3.2.4). On the other hand, even for real-valued case, the inequality (3.2.3) is valid for larger classes of potentials. The point is that (3.2.1) remains true for complex-valued perturbations. As often happens, the strength of the method we use is indivisible from its weakness, *i.e.*, being very general and rather powerful, it does not go ultimately far in exploiting the specifics of operators under consideration. Consequently the bounds on the discrete spectrum it produces are expected to be improvable at least in some special cases.

In the case $m = 0$ we obtain the next result.

Theorem 3.2.3 (case $m = 0$). *Let D be the Dirac operator defined in (3.1.3) with $m = 0$ and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then, for $0 < \tau \leq p - d$, we have*

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{2(p+\tau)}} \leq C \cdot K_1 \cdot \|V\|_{L^p}^p, \quad (3.2.7)$$

with C depending on p, d , and τ , and $K_1 = \Gamma_p 2^{2p-1} b^{p+d+\tau} K$, K is defined in Proposition 3.5.2.

Remark 3.2.4. In Appendix, we improve this result using the integration seen for operator bounded from below. We obtain the following result (see Theorem 6.1.1):

Theorem 3.2.5 (case $m = 0$). *With the above hypothesis, for $0 < \tau \leq p - d$, we have*

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{d+2\tau}} \lesssim \|V\|_{L^p}^p.$$

Theorems 3.2.1 and 3.2.3 provide quantitative estimates for the convergence of sequences of eigenvalues $(\lambda_n) \subset \sigma_d(D)$ to $\sigma_{\text{ess}}(D)$ for $V \in L^p(\mathbb{R}^d)$. To illustrate, we fix $m > 0$ and consider sequences $(\lambda_n) \subset \sigma_d(D)$ converging to a point λ chosen in three different ways. Suppose that $\text{Im}(\lambda_n) > 0$.

1. Let $\lambda = \pm m$ and assume there is a constant C strictly positive such that $|\text{Re}(\lambda_n \mp m)| \leq C |\text{Im}\lambda_n|$. Then

$$d(\lambda_n, \sigma(D_m)) \approx |\lambda_n \mp m|, \quad |\lambda_n \pm m| \approx \text{const}, \quad 1 + |\lambda_n| \approx \text{const},$$

and relation (3.2.1) implies that

$$\sum_{n=1}^{\infty} |\lambda_n \mp m|^{p-1+\tau} < +\infty.$$

2. Let $\lambda = \infty$ and $|\text{Im}(\lambda_n)| \leq C$. Then

$$d(\lambda_n, \sigma(D_m)) \approx |\text{Im}(\lambda_n)|, \quad |\lambda_n + m| \cdot |\lambda_n - m| \approx |\lambda_n|^2, \quad 1 + |\lambda_n| \approx |\lambda_n|,$$

and relation (3.2.1) implies that

$$\sum_{n=1}^{\infty} \frac{|\text{Im}(\lambda_n)|^{p+\tau}}{|\lambda_n|^{2p+2\tau}} < +\infty.$$

3. If $\lambda \in]m; \infty[$, then

$$d(\lambda_n, \sigma(D_m)) \approx |\text{Im}(\lambda_n)|, \quad |\lambda_n + m| \cdot |\lambda_n - m| \approx \text{const}, \quad 1 + |\lambda_n| \approx \text{const},$$

and relation (3.2.1) implies that

$$\sum_{n=1}^{\infty} |\text{Im}(\lambda_n)|^{p+\tau} < +\infty.$$

Concerning the proofs, we start with the proof of the case $m > 0$ with $d \geq 2$ and then with $d = 1$. We prove then the case $m = 0$.

3.3 Proof in dimension greater than 2

In this section and the next one ($d = 1$), we assume $m > 0$.

3.3.1 Conformal map for the Dirac operator

We have to send the resolvent set $\rho(D_m) = \mathbb{C} \setminus \{] - \infty, -m] \cup [m, +\infty[\}$ in the unit disc \mathbb{D} via a conformal map. The mappings we need are the next ones :

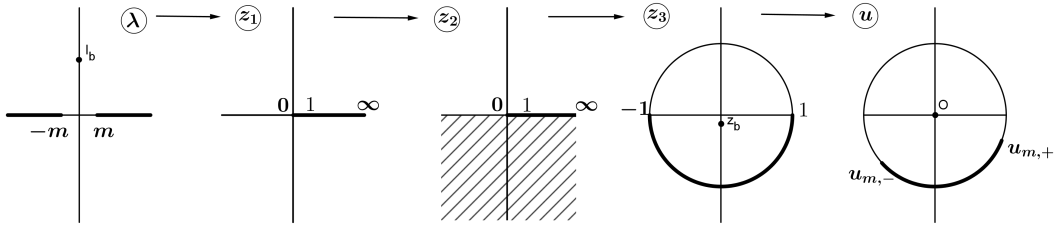


Figure 3.1: Map's domains

1. $z_1 = \frac{\lambda - m}{\lambda + m} : \mathbb{C} \setminus \sigma(D_m) \rightarrow \mathbb{C} \setminus [0, +\infty[$. The inverse mapping is given by $\lambda = m \frac{1 + z_1}{1 - z_1}$.
2. $z_2 = \sqrt{z_1} : \mathbb{C} \setminus [0, +\infty[\rightarrow \{ \text{Im}(z) > 0 \}$. The inverse mapping is $z_1 = z_2^2$.
3. $z_3 = \frac{z_2 - i}{z_2 + i} : \{ \text{Im}(z) > 0 \} \rightarrow \mathbb{D}$. The inverse map is $z_2 = i \frac{1 + z_3}{1 - z_3}$.
4. The normalization is operated by

$$u = e^{i\theta} \frac{z_3 - z_b}{1 - \bar{z}_b z_3} : \mathbb{D}_{z_3} \rightarrow \mathbb{D}_u,$$

where $z_b = -ib/(|m + ib| + m)$ is the image of ib by the three first conformal mappings. We recall that b is a real number needed to define the resolvent of D at ib . As above, we sometimes label the unit disc \mathbb{D} by the corresponding variable to avoid misunderstanding. We put furthermore

$$u_{m,+} := u(1), \quad u_{m,-} := u(-1).$$

The inverse map is $z_3 = \frac{u + e^{i\theta} z_b}{e^{i\theta} + u \bar{z}_b}$.

Notice that the conformal mapping u will serve to match the normalization $h(0) = 1$ from Theorem 2.3.1. The following conformal maps

$$\begin{aligned}\psi &= (z_3 \circ z_2 \circ z_1)^{-1} : \mathbb{D}_{z_3} \rightarrow \mathbb{C} \setminus \sigma(D_m), \\ \varphi &= (u \circ z_3 \circ z_2 \circ z_1)^{-1} : \mathbb{D}_u \rightarrow \mathbb{C} \setminus \sigma(D_m)\end{aligned}\tag{3.3.1}$$

will be important for the sequel.

The map ψ is easy to compute,

$$\lambda = \psi(z_3) = -2m \frac{z_3}{1 + z_3^2}.\tag{3.3.2}$$

The next propositions deals with the computation of distortion. We prove them with the help of the Koebe distortion Theorem 2.3.5.

Remark 3.3.1 (Notation). To simplify the following relations we note $Z_b = \frac{1 + |z_b|}{1 - |z_b|}$.

Lemma 3.3.2. *With the notation above we have*

1. $(2Z_b)^{-1} d(u, \mathbb{T}) \leq d(z_3, \mathbb{T}) \leq 2Z_b d(u, \mathbb{T})$.
2. $Z_b^{-1} |u - u(a)| \leq |z_3 - a| \leq Z_b |u - u(a)|$, with $a \in \{1, -1, i, -i\}$.

Proof. To simplify the reading we note in this proof z instead of z_3 . Notice that $\forall z \in \mathbb{D}$, $\frac{1}{2}(1 - |z|^2) \leq d(z, \mathbb{T}) \leq 1 - |z|^2$, since for $z \in \mathbb{D}$, $1 \leq 1 + |z| \leq 2$.

Let $z \in \mathbb{D}_z$, then

$$\begin{aligned}1 - |z|^2 &= \frac{|e^{i\theta} + u\bar{z}_b|^2 - |u + e^{i\theta}z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \\ &= \frac{1 - |u|^2 + |uz_b|^2 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} = \frac{(1 - |u|^2)(1 - |z_b|^2)}{|e^{i\theta} + u\bar{z}_b|^2}.\end{aligned}$$

Hence

$$\frac{1}{2} \frac{1 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \cdot d(u, \mathbb{T}) \leq d(z, \mathbb{T}) \leq 2 \frac{1 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \cdot d(u, \mathbb{T}).\tag{3.3.3}$$

See the remark next to the proof for the use of Koebe distortion Theorem.

Let us prove the second point. Recall that $u_{m,+} = u(1) = e^{i\theta} \frac{1 - z_b}{1 - \bar{z}_b}$ and that $u_{m,-} = u(-1) = -e^{i\theta} \frac{1 + z_b}{1 + \bar{z}_b}$.

In the case $a = 1$, we have

$$\begin{aligned}|z - 1| &= \left| \frac{u + e^{i\theta}z_b - (e^{i\theta} + u\bar{z}_b)}{e^{i\theta} + u\bar{z}_b} \right| = \left| \frac{u(1 - \bar{z}_b) + e^{i\theta}(z_b - 1)}{e^{i\theta} + u\bar{z}_b} \right| \\ &= \left| \frac{1 - \bar{z}_b}{e^{i\theta} + u\bar{z}_b} \right| \cdot \left| u + e^{i\theta} \frac{z_b - 1}{1 - \bar{z}_b} \right| = \frac{|1 - \bar{z}_b|}{|e^{i\theta} + u\bar{z}_b|} \cdot |u - u_{m,+}|.\end{aligned}\tag{3.3.4}$$

In the same way we obtain

$$|z + 1| = \frac{|1 + \bar{z}_b|}{|e^{i\theta} + u\bar{z}_b|} \cdot |u - u_{m,-}|, \quad |z + i| = \frac{|1 + i\bar{z}_b|}{|e^{i\theta} + u\bar{z}_b|} \cdot |u - u(-i)|,$$

and $|z - i| = \frac{|1 - i\bar{z}_b|}{|e^{i\theta} + u\bar{z}_b|} \cdot |u - u(i)|.$

To explicit the constant depending on b , we have

$$1 - |z_b| \leq 1 - |u\bar{z}_b| \leq |e^{i\theta} + u\bar{z}_b| \leq 1 + |z_b|,$$

so

$$\frac{1 - |z_b|}{1 + |z_b|} \leq \frac{1 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \leq \frac{1 + |z_b|}{1 - |z_b|},$$

then, with $a \in \{1, -1, i, -i\}$,

$$\frac{1 - |z_b|}{1 + |z_b|} \leq \frac{|1 + a\bar{z}_b|}{|e^{i\theta} + u\bar{z}_b|} \leq \frac{1 + |z_b|}{1 - |z_b|}. \quad (3.3.5)$$

Finally with the notation in Remark 3.3.1 we get the lemma. \square

Remark 3.3.3. We may also use (2.3.2). We would obtain

$$\frac{1}{4} \frac{1 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \cdot d(u, \mathbb{T}) \leq d(z, \mathbb{T}) \leq 2 \frac{1 - |z_b|^2}{|e^{i\theta} + u\bar{z}_b|^2} \cdot d(u, \mathbb{T}).$$

In one hand, the function here is simple, hence the computations are easy; on the other hand, it gives an example where the constants in the inequalities (2.3.2) are not sharp. But it is interesting to note that the constant depending on b is the same with both methods. However to find the distortion at the points -1 and 1 , the Koebe distortion Theorem does not help.

Proposition 3.3.4 (Distortion between $\mathbb{C} \setminus \sigma(D_m)$ and \mathbb{D}). *The distortion via the map φ defined in (3.3.1) between $\mathbb{C} \setminus \sigma(D_m)$ and \mathbb{D} is the following :*

$$d(\lambda, \sigma(D_m)) \geq \frac{m}{2} Z_b^{-7} \frac{|u - u_{m,+}| \cdot |u - u_{m,-}|}{|u - u(i)|^2 |u - u(-i)|^2} d(u, \mathbb{T}), \quad (3.3.6)$$

$$d(\lambda, \sigma(D_m)) \leq 4m Z_b^7 \frac{|u - u_{m,+}| \cdot |u - u_{m,-}|}{|u - u(i)|^2 |u - u(-i)|^2} d(u, \mathbb{T}), \quad (3.3.7)$$

and

$$d(u, \mathbb{T}) \leq 16Z_b(1 + m) \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda + m| \cdot |\lambda - m|)^{\frac{1}{2}} (1 + |\lambda|)}, \quad (3.3.8)$$

$$d(u, \mathbb{T}) \geq \frac{mZ_b^{-1}}{8(1 + m)} \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda + m| \cdot |\lambda - m|)^{\frac{1}{2}} (1 + |\lambda|)}. \quad (3.3.9)$$

Proof. Since $\psi'(z) = -2m \frac{1-z^2}{(1+z^2)^2}$, we obtain by (2.3.2)

$$\frac{m}{2} \cdot \frac{|1-z_3^2|}{|1+z_3^2|^2} d(z_3, \mathbb{T}) \leq d(\lambda, \sigma(D_m)) \leq 4m \cdot \frac{|1-z_3^2|}{|1+z_3^2|^2} d(z_3, \mathbb{T}). \quad (3.3.10)$$

Now

$$|1-z_3^2| = |1-z_3| \cdot |1+z_3| \text{ and } |1+z_3^2| = |z_3-i| \cdot |z_3+i|,$$

then we use the previous lemma to find (3.3.6) and (3.3.7).

To prove (3.3.8) and (3.3.9), we start from (3.3.10) to obtain

$$\frac{1}{4m} \cdot \frac{d(\lambda, \sigma(D_m)) |1+z_3^2|^2}{|1-z_3^2|} \leq 1 - |z_3| \leq \frac{2}{m} \cdot \frac{d(\lambda, \sigma(D_m)) |1+z_3^2|^2}{|1-z_3^2|}. \quad (3.3.11)$$

The definitions of $z_i, i = 1, 2, 3$ give the following relations

$$1 - z_3^2 = \frac{4i\sqrt{z_1}}{(\sqrt{z_1}+i)^2} \text{ and } 1 + z_3^2 = \frac{2z_1 - 2}{(\sqrt{z_1}+i)^2}, \quad (3.3.12)$$

hence

$$\frac{|1+z_3^2|^2}{|1-z_3^2|} = \frac{|z_1-1|^2}{\sqrt{|z_1|} \cdot |\sqrt{z_1}+i|^2}$$

and

$$1 + |z_1| \leq |\sqrt{z_1}+i|^2 \leq 2(1+|z_1|), \quad (3.3.13)$$

since $\text{Im}(\sqrt{z_1}) = \text{Im}(z_2) > 0$. Then

$$|z_1-1| = \frac{2m}{|\lambda+m|}, \quad |\sqrt{z_1}| = \left| \frac{\lambda-m}{\lambda+m} \right|^{\frac{1}{2}}, \quad \frac{1}{1+|z_1|} = \frac{|\lambda+m|}{|\lambda+m|+|\lambda-m|}. \quad (3.3.14)$$

On the other side the inequalities in Lemma 2.3.4 give

$$\begin{aligned} |\lambda+m| + |\lambda-m| &\leq \sqrt{2} \sqrt{|\lambda+m|^2 + |\lambda-m|^2} = 2\sqrt{|\lambda|^2 + m^2} \\ &\leq 2(|\lambda|+m) \leq 2(1+m)(1+|\lambda|) \end{aligned} \quad (3.3.15)$$

and

$$\begin{aligned} |\lambda+m| + |\lambda-m| &\geq \sqrt{|\lambda+m|^2 + |\lambda-m|^2} = \sqrt{2} \sqrt{|\lambda|^2 + m^2} \\ &\geq |\lambda| + m \geq \frac{m}{1+m} (1+|\lambda|). \end{aligned} \quad (3.3.16)$$

Putting all this together in (3.3.11), we obtain

$$d(z_3, \mathbb{T}) \leq 8(1+m) \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda+m| \cdot |\lambda-m|)^{\frac{1}{2}} (1+|\lambda|)}$$

and

$$d(z_3, \mathbb{T}) \geq \frac{m}{4(1+m)} \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda+m| \cdot |\lambda-m|)^{\frac{1}{2}} (1+|\lambda|)}$$

We finish the proof with Lemma 3.3.2. \square

3.3.2 Bound on the resolvent of the unperturbed Dirac operator

We apply Proposition 2.4.1 to the D_m operator by putting, for $x \in \mathbb{R}^d$,

$$f(x) = V(x), \quad g(x) = (\lambda - \mu_m(x))^{-1} \times \text{Id}_{\mathbb{C}^n},$$

with $\mu_m(x) := \sqrt{|x|^2 + m^2} \times \text{Id}_{\mathbb{C}^n}$. We obtain the next proposition

Proposition 3.3.5. *Let $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$, and $\lambda \in \rho(D_m)$. Then $V(\lambda - D_m)^{-1} \in \mathcal{S}_p$, and*

$$\|V(\lambda - D_m)^{-1}\|_{\mathcal{S}_p}^p \leq (2\pi)^{-d} \|V\|_{L^p}^p \cdot \|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p}^p$$

if $\text{Re}(\lambda) \geq 0$ and

$$\|V(\lambda - D_m)^{-1}\|_{\mathcal{S}_p}^p \leq (2\pi)^{-d} \|V\|_{L^p}^p \cdot \|(\lambda + \mu_m(\cdot))^{-1}\|_{L^p}^p$$

if $\text{Re}(\lambda) \leq 0$.

We find then a bound of the expression $\|(\lambda \pm \mu_m(\cdot))^{-1}\|_{L^p}$ appearing in the above inequality for $d \geq 2$. The case $d = 1$ is treated in the corresponding section.

Proposition 3.3.6. *Assume $d \geq 2$. Let $\lambda = \lambda_0 + i\lambda_1$ and $p > d$. Then*

$$\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p}^p \leq 2^{d-2} s_{d-1} n^{\frac{p}{2}} \cdot M \cdot \frac{(|\lambda - m| + m)^{d-1}}{d(\lambda, \sigma(D_m))^{p-1}}$$

for $\lambda_0 \geq 0$, and

$$\|(\lambda + \mu_m(\cdot))^{-1}\|_{L^p}^p \leq 2^{d-2} s_{d-1} n^{\frac{p}{2}} \cdot M \cdot \frac{(|\lambda + m| + m)^{d-1}}{d(\lambda, \sigma(D_m))^{p-1}}$$

for $\lambda_0 \leq 0$.

Above, $M = \max\{M_1; M_2\}$ where

$$1. \quad M_1 = 2^{\frac{d-1}{2}} \times \max \left\{ (1 + 2^{d-2}) \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}; 2^{d-2} \int_0^{+\infty} \frac{t^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right\},$$

$$2. \quad \text{and } M_2 = 2 \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}},$$

and $s_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ is the total mass of the invariant measure on the unit sphere of dimension $d - 1$, Γ is Euler's function.

Proof. First of all, recall that $(\lambda - \mu_m(\cdot))^{-1} = (\lambda - \mu_m(\cdot))^{-1} \times \text{Id}_{\mathbb{C}^n}$, hence $\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))}^p = n^{p/2} \|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p(\mathbb{R}^d, \mathbb{C})}^p$.

The case $\pm\lambda_0 \geq 0$ being similar, we give the proof for $\lambda_0 \geq 0$ (and $\lambda_1 > 0$) only. After a polar change of variable,

$$\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p(\mathbb{R}^d, \mathbb{C})}^p = s_{d-1} \int_0^{+\infty} \frac{r^{d-1}}{|\sqrt{r^2 + m^2} - \lambda|^p} dr,$$

where $s_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$. We are then reduced to bound

$$I = \int_0^{+\infty} \frac{r^{d-1}}{|\sqrt{r^2 + m^2} - \lambda|^p} dr.$$

We write $|\sqrt{r^2 + m^2} - \lambda|^p = ((\sqrt{r^2 + m^2} - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}$ and make the change of variable $s = \sqrt{r^2 + m^2} - m$. Hence,

$$I = \int_0^{+\infty} \frac{((s+m)^2 - m^2)^{\frac{d-2}{2}} (s+m)}{((s+m - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}} ds. \quad (3.3.17)$$

We now distinguish the cases $m \leq \lambda_0$ and $0 \leq \lambda_0 < m$. For $m \leq \lambda_0$, we observe that $d(\lambda, \sigma(D_m)) = |\lambda_1|$. On pose $\beta = \lambda_0 - m \geq 0$ et (comme $d \geq 2$) on utilise l'inégalité $\sqrt{(s+m)^2 - m^2} \leq s+m$, d'où

$$I \leq \int_0^{+\infty} \frac{(s+m)^{d-1}}{((s-\beta)^2 + \lambda_1^2)^{\frac{p}{2}}} ds.$$

Since $m \leq \lambda_0$ and $\lambda \notin \sigma(D_m)$, we have $|\lambda_1| > 0$, and

$$\begin{aligned} \int_0^{+\infty} \frac{(s+m)^{d-1}}{((s-\beta)^2 + \lambda_1^2)^{\frac{p}{2}}} &= \frac{1}{|\lambda_1|^p} \int_0^\beta \frac{(s+m)^{d-1}}{\left(\left(\frac{s-\beta}{\lambda_1}\right)^2 + 1\right)^{\frac{p}{2}}} ds \\ &+ \frac{1}{|\lambda_1|^p} \int_\beta^{+\infty} \frac{(s+m)^{d-1}}{\left(\left(\frac{s-\beta}{\lambda_1}\right)^2 + 1\right)^{\frac{p}{2}}} ds. \end{aligned} \quad (3.3.18)$$

In the right hand-side of (3.3.18), we make the change of variable $t = \frac{\beta - s}{\lambda_1}$ in the first

integral and $t = \frac{s - \beta}{\lambda_1}$ in the second one (we assume $\lambda_1 > 0$). Then we apply the inequality $(a+b)^{d-1} \leq 2^{d-2}(a^{d-1} + b^{d-1})$ for $a, b \geq 0$. This leads to the bounds

$$\begin{aligned} I \leq \frac{2^{d-2}}{|\lambda_1|^{p-1}} &\left(\int_0^{\frac{\beta}{\lambda_1}} \frac{(\beta - \lambda_1 t)^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} + \int_0^{\frac{\beta}{\lambda_1}} \frac{m^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right. \\ &\left. + \int_0^{+\infty} \frac{(\beta + \lambda_1 t)^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} + \int_0^{+\infty} \frac{m^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right). \end{aligned}$$

For $0 \leq t \leq \frac{\beta}{\lambda_1}$, $\beta - \lambda_1 t \leq \beta$ since $\lambda_1 > 0$, and recalling that $p > d$, we can bound in the following way

$$I \leq \frac{2^{d-2}}{|\lambda_1|^{p-1}} \left[((1 + 2^{d-2})\beta^{d-1} + 2m^{d-1}) \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}} + 2^{d-2} |\lambda_1|^{d-1} \int_0^{+\infty} \frac{t^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right].$$

Moreover, using Lemma 2.3.4,

$$\beta^{d-1} + |\lambda_1|^{d-1} \leq (\lambda_0 - m + \lambda_1)^{d-1} \leq \left(\sqrt{2} \sqrt{(\lambda_0 - m)^2 + \lambda_1^2} \right)^{d-1},$$

we obtain, for $m \leq \lambda_0$,

$$I \leq \frac{2^{d-2}}{d(\lambda, \sigma(D_m))^{p-1}} (M_1 |\lambda - m|^{d-1} + M_2 m^{d-1}),$$

with $M_1 = 2^{\frac{d-1}{2}} \max \left\{ (1 + 2^{d-2}) \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}; 2^{d-2} \int_0^{+\infty} \frac{t^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right\}$ and

$$M_2 = 2 \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}.$$

We now turn to the case $0 \leq \lambda_0 < m$. We see $d(\lambda, \sigma(D_m)) = |\lambda - m|$; going back to (3.3.17), we use the inequality $(s + m - \lambda_0)^2 + \lambda_1^2 \geq s^2 + |\lambda - m|^2$. Consequently

$$I \leq \frac{1}{|\lambda - m|^p} \int_0^{+\infty} \frac{(s + m)^{d-1}}{\left(\left(\frac{s}{|\lambda - m|} \right)^2 + 1 \right)^{\frac{p}{2}}} ds.$$

Doing the change of variable $t = \frac{s}{|\lambda - m|}$ and bounding as in the first part of the computation, we come to, for $0 \leq \lambda_0 < m$,

$$I \leq \frac{2^{d-2}}{d(\lambda, \sigma(D_m))^{p-1}} \left(\tilde{M}_1 |\lambda - m|^{d-1} + \tilde{M}_2 m^{d-1} \right),$$

with $\tilde{M}_1 = \int_0^{+\infty} \frac{t^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}}$ et $\tilde{M}_2 = \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}$. The proposition is proved. \square

3.3.3 Bound on the resolvent of the perturbed Dirac operator

The next lemma shows that, for b large enough, $ib \in \rho(H)$. Although it is not an obligation, it is useful to bound $\|(-ib + D)^{-1}\|$ by 1. We refer to [Han10, Lemma 3.3.4] for a more general statement.

Lemma 3.3.7. *Let $p > d \geq 2$. Then for b large enough, we know that*

1. the operator $-ib + D$ is invertible, that is $ib \in \rho(D)$,

2. and $\|(-ib + D)^{-1}\| \leq 1$.

We use the bound in Proposition 3.3.6, hence we assume $d \geq 2$.

Proof. First notice that inequality $\|V(ib - D_m)^{-1}\| < 1$ implies that the operator $(-ib + D)$ is invertible.

Indeed, the inequality $\|V(ib - D_m)^{-1}\| < 1$ implies that $\text{Id} - V(ib - D_m)^{-1}$ is invertible, and we have

$$\begin{aligned} \text{Id} - V(ib - D_m)^{-1} &= (ib - D_m)(ib - D_m)^{-1} - V(ib - D_m)^{-1} \\ &= (ib - D_m - V)(ib - D_m)^{-1} \\ &= (ib - D)(ib - D_m)^{-1}. \end{aligned}$$

Second, we show that we have $\|V(ib - D_m)^{-1}\| < 1$ for b large enough. Since $\|A\| \leq \|A\|_{\mathfrak{S}_p}$, for all operators A , Propositions 2.4.1 and 3.3.5 entail

$$\begin{aligned} \|V(ib - D_m)^{-1}\|^p &\leq \|V(ib - D_m)^{-1}\|_{\mathfrak{S}_p}^p \\ &\leq K \|V\|_{L^p}^p \frac{(1 + |ib - m|^{d-1})}{|ib - m|^{p-1}}, \end{aligned} \tag{3.3.19}$$

where K does not depend on b . It is convenient to put

$$C(b)^p = K \|V\|_{L^p}^p \frac{(1 + |ib - m|^{d-1})}{|ib - m|^{p-1}}.$$

The right-hand side of inequality (3.3.19) trivially goes to zero when b goes to infinity, and so $\|V(ib - D_m)^{-1}\| \leq C(b) < 1$ for b large enough.

Now we prove that $\|(-ib + D)^{-1}\| \leq 1$ for b large enough. Using the resolvent identity, we get

$$\begin{aligned} \|(-ib + D)^{-1}\| &\leq \|(-ib + D_m)^{-1}\| + \|(-ib + D)^{-1}\| \cdot \|V(-ib + D_m)^{-1}\| \\ &\leq \|(-ib + D_m)^{-1}\| + \|(-ib + D)^{-1}\| \cdot \|V(-ib + D_m)^{-1}\|_{\mathfrak{S}_p}. \end{aligned}$$

Since $D_m^* = D_m$,

$$\|(-ib + D_m)^{-1}\| = \frac{1}{d(ib, \sigma(D_m))} = \frac{1}{|ib - m|},$$

and, as above, we obtain

$$\|(-ib + D)^{-1}\| \leq \frac{1}{|ib - m|} + C(b) \|(-ib + D)^{-1}\|.$$

Resolving this inequality with respect to $\|(-ib + D)^{-1}\|$, we get the claim of the lemma. \square

3.3.4 Proof of Theorem 3.2.1 in dimension greater than 2

Recall from (2.2.3) that $f(\lambda) = \det_{[\mathfrak{p}]}(\text{Id} - F(\lambda))$, with

$$F(\lambda) := (\lambda - ib)(-ib + D)^{-1}V(\lambda - D_m)^{-1} \in \mathbf{S}_p.$$

We have by the property of the regularized determinant

$$|f(\lambda)| \leq \exp\left(\Gamma_p \|(\lambda - ib)(D - ib)^{-1}V(\lambda - D_m)^{-1}\|_{\mathbf{S}_p}^p\right). \quad (3.3.20)$$

Applying Propositions 3.3.5 and 3.3.6, we get, for $\lambda \in \mathbb{C} \setminus \sigma(D_m)$, to

$$\begin{aligned} \|V(\lambda - D_m)^{-1}\|_{\mathbf{S}_p}^p &\leq (2\pi)^{-d} \|V\|_{L^p}^p \|(\lambda \pm \mu_m(\cdot))^{-1}\|_{L^p}^p \\ &\leq (2\pi)^{-d} 2^{d-2} s_{d-1} n^{\frac{p}{2}} M \cdot \|V\|_{L^p}^p \cdot \frac{(|\lambda \pm m| + m)^{d-1}}{d(\lambda, \sigma(D_m))^{p-1}} \\ &\leq (2\pi)^{-d} 2^{d-2} s_{d-1} n^{\frac{p}{2}} M (1 + 2m)^{d-1} \|V\|_{L^p}^p \frac{(1 + |\lambda|)^{d-1}}{d(\lambda, \sigma(D_m))^{p-1}} \end{aligned}$$

with M defined in Proposition 3.3.6.

We apply Lemme 3.3.7 and the previous inequality to (3.3.20), and we obtain

$$\log |f(\lambda)| \leq K_1 \|V\|_{L^p}^p \frac{|\lambda - ib|^p (1 + |\lambda|)^{d-1}}{d(\lambda, \sigma(D_m))^{p-1}},$$

for $\lambda \in \mathbb{C} \setminus \sigma(D_m)$ and with $K_1 = \Gamma_p 2^{d-2} (2\pi)^{-d} s_{d-1} n^{\frac{p}{2}} M (1 + 2m)^{d-1}$. Since $|\lambda - ib|^p \leq b^p (1 + |\lambda|)^p$, we have

$$\log |f(\lambda)| \leq K_1 b^p \|V\|_{L^p}^p \frac{(1 + |\lambda|)^{p+d-1}}{d(\lambda, \sigma(D_m))^{p-1}}. \quad (3.3.21)$$

We now have to go in \mathbb{D} in order to apply Borichev-Golinskii-Kupin Theorem 2.3.1. For this, we consider the function $g(u) = f \circ \varphi(u)$, where φ is defined in (3.3.1); the function g is holomorphic on \mathbb{D}_u . By inequalities (3.3.16) and (3.3.14), we have

$$\begin{aligned} 1 + |\lambda| &\leq \frac{1 + m}{m} (|\lambda + m| + |\lambda - m|) \\ &\leq \frac{1 + m}{m} \cdot \frac{1 + |z_1|}{|z_1 - 1|}. \end{aligned}$$

But $z_1 = z_2^2 = -\left(\frac{1 + z_3}{1 - z_3}\right)^2$, that is, $z_1 - 1 = -2\frac{1 + z_3^2}{(1 - z_3)^2}$ and $1 + |z_1| = 2\frac{1 + |z_3|^2}{|1 - z_3|^2}$, so $\frac{1 + |z_1|}{|z_1 - 1|} = \frac{1 + |z_3|^2}{|1 + z_3|^2} \leq \frac{2}{|1 + z_3|^2}$. Consequently, with the help of Lemma 3.3.2, we obtain

$$1 + |\lambda| \leq 2 \frac{1 + m}{m} \cdot \frac{Z_b^2}{|u - u(i)||u - u(-i)|} \quad (3.3.22)$$

From the lower bound of $d(\lambda, \sigma(D_m))$ in Proposition 3.3.4 and the previous inequality, we bound (3.3.21) by

$$\log |g(u)| \leq K_2 \cdot \|V\|_{L^p}^p \frac{|u - u(i)|^{p-d-1} |u - u(-i)|^{p-d-1}}{d(u, \mathbb{T})^{p-1} |u - u_{m,+}|^{p-1} |u - u_{m,-}|^{p-1}},$$

with $K_2 = K_1 b^p \left(2Z_b^2 \frac{1+m}{m}\right)^{p+d-1} \left(\frac{2Z_b^7}{m}\right)^{p-1}$.

We apply Theorem 2.3.1; we get, for $\tau > 0$,

$$\begin{aligned} \sum_{g(u)=0} (1 - |u|)^{p+\tau} (|u - u_{m,+}| |u - u_{m,-}|)^{p-2+\tau} (|u - u(i)| |u - u(-i)|)^{(d-p+\tau)+} \\ \leq C K_2 \cdot \|V\|_{L^p}^p, \end{aligned}$$

with C depending on $d, p, \{u_{m,+}; u_{m,-}; u(i); u(-i)\}$, and τ and K_2 is described above.

By hypothesis, we always have $p > d \geq 2$, hence, for all $0 < \tau < p - d$, we obtain

$$\sum_{g(u)=0} (1 - |u|)^{p+\tau} |u - u_{m,+}|^{p-2+\tau} |u - u_{m,-}|^{p-2+\tau} \leq C K_2 \cdot \|V\|_{L^p}^p. \quad (3.3.23)$$

The last step of the proof consists in transferring relation (3.3.23) back to $\rho(D_m) = \mathbb{C} \setminus \sigma(D_m)$. Remind that we have by Lemma 3.3.2 and Proposition 3.3.4

$$1 - |u| = d(u, \mathbb{T}) \geq \frac{mZ_b^{-1}}{8(1+m)} \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda + m| \cdot |\lambda - m|)^{\frac{1}{2}} (1 + |\lambda|)},$$

and, by Lemma 3.3.2 and the (in)equalities (3.3.12) to (3.3.15), we have

$$\begin{aligned} |u - u_{m,+}| \cdot |u - u_{m,-}| &\geq Z_b^{-2} |z_3 - 1| \cdot |z_3 + 1| = 4Z_b^{-2} \frac{|z_1|^{\frac{1}{2}}}{|\sqrt{z_1} + i|^2} \\ &\geq 2Z_b^{-2} \frac{|z_1|^{\frac{1}{2}}}{1 + |z_1|} = 2Z_b^{-2} \frac{(|\lambda - m| |\lambda + m|)^{\frac{1}{2}}}{|\lambda + m| + |\lambda - m|} \\ &\geq \frac{Z_b^{-2}}{(1+m)} \cdot \frac{(|\lambda - m| \cdot |\lambda + m|)^{1/2}}{1 + |\lambda|}. \end{aligned}$$

Hence, for $p > d \geq 2$ and τ small enough, we get to

$$(1 - |u|)^{p+\tau} (|u - u_{m,+}| \cdot |u - u_{m,-}|)^{p-2+\tau} \geq \frac{K_3 d(\lambda, \sigma(D_m))^{p+\tau}}{|\lambda + m| \cdot |\lambda - m| (1 + |\lambda|)^{2(p+\tau-1)}},$$

with $K_3 = \left(\frac{mZ_b^{-1}}{2^3(1+m)}\right)^{p+\tau} \left(\frac{Z_b^{-2}}{1+m}\right)^{p-2+\tau}$. We deduce the results of Theorem 3.2.1,

and, in particular, the constant is

$$\begin{aligned}
K_4 &= K_2 \times K_3^{-1} \\
&= K_1 b^p \left(2Z_b^2 \frac{1+m}{m} \right)^{p+d-1} \left(\frac{2Z_b^7}{m} \right)^{p-1} \left(\frac{2^3(1+m)}{mZ_b^{-1}} \right)^{p+\tau} \times \\
&\quad \times \left(\frac{1+m}{Z_b^{-2}} \right)^{p-2+\tau} \\
&= K_1 b^p 2^{5p+d-2+3\tau} \cdot \frac{(1+m)^{3p+d-3+2\tau}}{m^{3p+d-2+\tau}} \cdot Z_b^{12p+2d-13+3\tau}.
\end{aligned}$$

This ends the proof. \square

Remark 3.3.8. Of course we can wonder what happens to (3.3.23) when $\tau \geq p - d$. We can verify that Theorem 2.3.1 still apply but, as expected, the resulting inequality is then weaker than (3.3.23). Indeed, the additional factor $|u - u(i)| \cdot |u - u(-i)|$ gives a contribution as $\frac{1}{|\lambda+m|+|\lambda-m|}$ and after computations, we find

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(D_m))^{p+\tau}}{|\lambda+m| \cdot |\lambda-m|(1+|\lambda|)^{p+d+3\tau}} \lesssim \|V\|_{L^p}^p. \quad (3.3.24)$$

The comparison between (3.2.1) and (3.3.24) gives

$$\frac{(1+|\lambda|)^{2p-2+2\tau}}{(1+|\lambda|)^{d+p+3\tau}} = \frac{1}{(1+|\lambda|)^{d-p+2+\tau}} \leq 1$$

since $d - p + 2 > 1$.

3.4 The dimension 1 case

We start by bounding the resolvent norm $\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p}^p$.

3.4.1 Upper bound of the resolvent

Proposition 3.4.1. *Let $\lambda \in \mathbb{C} \setminus \sigma(D_m)$, $m > 0$, and $p > d = 1$. Then*

$$\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p}^p \leq n^{p/2} s_0 M \cdot \frac{d(\lambda; \sigma(D_m)) + m}{d(\lambda, \sigma(D_m))^p},$$

with $M = \max \left\{ 1; \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}} \right\}$ and $s_0 = 2$.

Proof. First $\int_{\mathbb{R}} \frac{dx}{|\sqrt{x^2 + m^2} - \lambda|^p} = s_0 n^{p/2} \int_0^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p}$.

We put $\lambda = \lambda_0 + i\lambda_1$. We assume $0 \leq \lambda_0 < m$, i.e. $d(\lambda, \sigma(D_m)) = |\lambda - m| > 0$, and $\lambda_1 \geq 0$, we split the integral

$$I = \int_0^{|\lambda-m|} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} + \int_{|\lambda-m|}^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p}. \quad (3.4.1)$$

and we recall that $|\sqrt{r^2 + m^2} - \lambda|^p = ((\sqrt{r^2 + m^2} - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}$.

For $0 \leq r \leq |\lambda - m|$, we have $0 < m - \lambda_0 \leq \sqrt{r^2 + m^2} - \lambda_0$ so

$$\begin{aligned} \int_0^{|\lambda-m|} \frac{dr}{((\sqrt{r^2 + m^2} - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}} &\leq \int_0^{|\lambda-m|} \frac{dr}{((m - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}} \\ &\leq \int_0^{|\lambda-m|} \frac{dr}{|\lambda - m|^p} = \frac{|\lambda - m|}{|\lambda - m|^p}. \end{aligned}$$

For $r \geq |\lambda - m| > 0$, we make the change of variable $s = \sqrt{r^2 + m^2} - m$, $0 < m - \lambda_0 \leq \sqrt{r^2 + m^2} - \lambda_0$ and use the inequality $\sqrt{(s + m)^2 - m^2} \geq s$ and $(s + m - \lambda_0)^2 + \lambda_1^2 \geq s^2 + |\lambda - m|^2$ to obtain

$$\begin{aligned} \int_{|\lambda-m|}^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} &= \int_A^{+\infty} \frac{(s + m) ds}{((s + m - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}} \sqrt{(s + m)^2 - m^2}} \\ &\leq \int_A^{+\infty} \frac{(s + m) ds}{(s^2 + |\lambda - m|^2)^{\frac{p}{2}} \sqrt{(s + m)^2 - m^2}}, \end{aligned}$$

with $A = \sqrt{|\lambda - m|^2 + m^2} - m$.

The function $f : x \mapsto \frac{x + m}{\sqrt{(x + m)^2 - m^2}}$ is decreasing on $[A; +\infty[$ (the derivative is non-positive on this interval), so $\forall x \geq A, f(x) \leq f(A) = \frac{\sqrt{|\lambda - m|^2 + m^2}}{|\lambda - m|} \leq \frac{|\lambda - m| + m}{|\lambda - m|}$.

Then we bound and use the change of variable $s = |\lambda - m|t$, so we find

$$\begin{aligned} \int_{|\lambda-m|}^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} &\leq \frac{|\lambda - m| + m}{|\lambda - m|^{p+1}} \int_A^{+\infty} \frac{ds}{\left(\left(\frac{s}{|\lambda - m|} \right)^2 + 1 \right)^{\frac{p}{2}}} \\ &\leq \frac{|\lambda - m| + m}{|\lambda - m|^p} \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}. \end{aligned}$$

Hence $I \leq N \cdot \frac{|\lambda - m| + m}{|\lambda - m|^p}$ with $N = \max \left\{ 1; \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}} \right\}$. This finishes the case $0 \leq \lambda_0 < m$.

We now suppose that $\lambda_0 \geq m$. In this case we always have $\lambda_1 \neq 0$. One takes $\lambda_1 > 0$ (else $-\lambda_1 > 0$), and we split the integral in two

$$I = \int_0^{\lambda_1} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} + \int_{\lambda_1}^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p}. \quad (3.4.2)$$

On one hand

$$\begin{aligned} \int_0^{\lambda_1} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} &= \int_0^{\lambda_1} \frac{dr}{((\sqrt{r^2 + m^2} - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}} \\ &\leq \int_0^{\lambda_1} \frac{dr}{\lambda_1^p} = \frac{\lambda_1}{\lambda_1^p}. \end{aligned}$$

and for $r \geq \lambda_1$, we make the change of variable $s + m = \sqrt{r^2 + m^2}$, hence, putting $A = \sqrt{\lambda_1^2 + m^2} - m$,

$$\int_{\lambda_1}^{+\infty} \frac{dr}{((\sqrt{r^2 + m^2} - \lambda_0)^2 + \lambda_1^2)^{\frac{p}{2}}} = \int_A^{+\infty} \frac{(s + m)ds}{|(s + m - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}} \sqrt{(s + m)^2 - m^2}}.$$

As above, the function $f : x \mapsto \frac{x + m}{\sqrt{(x + m)^2 - m^2}}$ is decreasing on $[A; +\infty[$, so $\forall x \geq A, f(x) \leq f(A) = \frac{\sqrt{\lambda_1^2 + m^2}}{\lambda_1} \leq \frac{\lambda_1 + m}{\lambda_1}$. Hence

$$\begin{aligned} \int_A^{+\infty} \frac{\left(\sqrt{(s + m)^2 - m^2}\right)^{-1} ds}{|(s + m - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}}} &\leq \frac{\lambda_1 + m}{\lambda_1} \int_A^{+\infty} \frac{ds}{|(s + m - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}}} \\ &\leq \frac{\lambda_1 + m}{\lambda_1^{p+1}} \int_A^{+\infty} \frac{ds}{\left(\left(\frac{s + m - \lambda_0}{\lambda_1}\right)^2 + 1\right)^{\frac{p}{2}}}. \end{aligned}$$

We make the change of variable $t = \frac{s + m - \lambda_0}{\lambda_1}$, so

$$\int_A^{+\infty} \frac{ds}{\left(\left(\frac{s + m - \lambda_0}{\lambda_1}\right)^2 + 1\right)^{\frac{p}{2}}} = \lambda_1 \int_B^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}} \leq \lambda_1 \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}.$$

Consequently, we find

$$\int_{\lambda_1}^{+\infty} \frac{dr}{|\sqrt{r^2 + m^2} - \lambda|^p} \leq \frac{\lambda_1 + m}{\lambda_1^p} \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}},$$

hence $I \leq \frac{M}{\lambda_1^p}(\lambda_1 + m)$, with $M = \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}$. We deduce the proposition. \square

3.4.2 Proof in dimension 1

We go back to inequality (3.3.20) and apply Propositions 3.3.5 and 3.4.1, hence

$$\begin{aligned}
\log |f(\lambda)| &\leq \Gamma_p \|(\lambda - ib)(D - ib)^{-1}V(\lambda - D_m)^{-1}\|_{\mathfrak{S}_p}^p \\
&\leq \frac{\Gamma_p}{2\pi} s_0 n^{p/2} M |\lambda - ib|^p \|(D - ib)^{-1}\|^p \|V\|_{L^p}^p \cdot \frac{d(\lambda; \sigma(D_m)) + m}{d(\lambda, \sigma(D_m))^p} \\
&\leq \frac{\Gamma_p}{2\pi} s_0 n^{p/2} b^p M \|(D - ib)^{-1}\|^p \|V\|_{L^p}^p (1 + |\lambda|)^p \cdot \frac{|\lambda - m| + m}{d(\lambda, \sigma(D_m))^p} \\
&\leq K_1 b^p \|(D - ib)^{-1}\|^p \|V\|_{L^p}^p \cdot \frac{(1 + |\lambda|)^{p+1}}{d(\lambda, \sigma(D_m))^p},
\end{aligned}$$

with $K_1 = \frac{\Gamma_p}{2\pi} \cdot s_0 n^{p/2} M (1 + 2m)$.

In the proof of Lemma 3.3.7, $C(b)$ becomes

$$C(b)^p = \|V\|_{L^p}^p K \cdot \frac{m + |ib - m|}{|ib - m|^p}$$

which tends to zero when b tends to $+\infty$. Hence the lemma still apply for $p > 1$. Consequently, for b large enough, $\|(D - ib)^{-1}\| \leq 1$.

Hence, for all $\lambda \in \mathbb{C} \setminus \sigma(D_m)$,

$$\log |f(\lambda)| \leq K_1 \cdot \|V\|_{L^p}^p \cdot \frac{(1 + |\lambda|)^{p+1}}{d(\lambda, \sigma(D_m))^p},$$

with K_1 defined above.

Recall that, by (3.3.22) and (3.3.6), we have

$$\begin{aligned}
1 + |\lambda| &\leq 2 \frac{1 + m}{m} \cdot \frac{Z_b^2}{|u - u(i)||u - u(-i)|} \\
\text{and } d(\lambda, \sigma(D_m)) &\geq \frac{m}{2} Z_b^{-7} \frac{|u - u_{m,+}| \cdot |u - u_{m,-}|}{|u - u(i)|^2 |u - u(-i)|^2} d(u, \mathbb{T}),
\end{aligned}$$

so,

$$\begin{aligned}
\log |g(u)| &\leq K_1 b^p \left(2 \frac{1 + m}{m} Z_b^2\right)^{p+1} \left(\frac{2Z_b^7}{m}\right)^p \|V\|_{L^p}^p \frac{|u - u(i)|^{p-1} |u - u(-i)|^{p-1}}{d(u, \mathbb{T})^p |u - u_{m,+}|^p |u - u_{m,-}|^p} \\
&\leq K_2 \cdot \|V\|_{L^p}^p \cdot \frac{|u - u(i)|^{p-1} |u - u(-i)|^{p-1}}{d(u, \mathbb{T})^p |u - u_{m,+}|^p |u - u_{m,-}|^p},
\end{aligned}$$

with $K_2 = K_1 \cdot b^p 2^{2p+1} Z_b^{9p+2} \cdot \frac{(1 + m)^{p+1}}{m^{2p+1}}$. Since $p - 1 > 0$, applying Theorem 2.3.1, we find

$$\sum_{g(u)=0} d(u, \mathbb{T})^{p+1+\tau} (|u - u_{m,+}| \cdot |u - u_{m,-}|)^{p-1+\tau} \leq C \cdot K_2 \cdot \|V\|_{L^p}^p,$$

where C depends on $p, \{u_{m,+}; u_{m,-}; u(i); u(-i)\}$, and τ . But we know that

$$|u - u_{m,+}| \cdot |u - u_{m,-}| \geq \frac{Z_b^{-2}}{(1+m)} \cdot \frac{(|\lambda - m| \cdot |\lambda + m|)^{1/2}}{1 + |\lambda|}$$

$$\text{and } d(u, \mathbb{T}) \geq \frac{mZ_b^{-1}}{8(1+m)} \cdot \frac{d(\lambda, \sigma(D_m))}{(|\lambda + m| \cdot |\lambda - m|)^{\frac{1}{2}}(1 + |\lambda|)},$$

hence

$$d(u, \mathbb{T})^{p+1+\tau} (|u - u_{m,+}| \cdot |u - u_{m,-}|)^{p-1+\tau} \geq \frac{m^{p+1+\tau} Z_b^{-3p+1-3\tau}}{2^{3(p+1+\tau)}(1+m)^{2p+2\tau}} \times \\ \times \frac{d(\lambda, \sigma(D_m))^{p+1+\tau}}{|\lambda + m| \cdot |\lambda - m| \cdot (1 + |\lambda|)^{2p+2\tau}}$$

Finally

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_m))^{p+1+\tau}}{|\lambda + m| \cdot |\lambda - m| \cdot (1 + |\lambda|)^{2p+2\tau}} \leq C \cdot K_3 \cdot \|V\|_{L^p}^p,$$

$$\text{with } K_3 = K_1 \cdot b^p 2^{5p+4+3\tau} \cdot \frac{(1+m)^{3p+1+\tau}}{m^{3p+2+\tau}} \cdot Z_b^{12p+1+3\tau}.$$

3.5 The massless case

We prove in this section Theorem 3.2.3. The method is the same but the spectrum of D_0 is the whole real line : $\sigma(D_0) = \mathbb{R}$. The small differences with the case $m > 0$ come from the study of the conformal map and the upper bound of the Schatten norm of the resolvent in the expression $V(\lambda - D_0)^{-1}, \lambda \in \rho(D_0)$.

3.5.1 Conformal map and bound of the resolvent

Concerning the conformal map, we recall that $\rho(D_0) = \mathbb{C}^+ \cup \mathbb{C}^-$, where \mathbb{C}^\pm are the sets $\{\lambda \in \mathbb{C} : \pm \text{Im}(\lambda) > 0\}$. So we can compute the contributions of the discrete spectrum $\sigma_d(D) \cap \mathbb{C}^\pm$ to (3.2.7) and then add them up. That is why, without loss of generality, we discuss the case of $\lambda \in \sigma_d(D) \cap \mathbb{C}^+$, and the case of $\sigma_d(D) \cap \mathbb{C}^-$ is treated similarly. The conformal map φ we are interested in, is particularly simple,

$$\lambda = \varphi(z) = ib \cdot \frac{1+z}{1-z} : \mathbb{D} \rightarrow \mathbb{C}^+, \\ z = \varphi^{-1}(\lambda) = \frac{\lambda - ib}{\lambda + ib} : \mathbb{C}^+ \rightarrow \mathbb{D}.$$

with $b \geq 1$.

Proposition 3.5.1 (distortion between \mathbb{C}^+ and \mathbb{D}). *For $b \geq 1$, the distortion between \mathbb{C}^+ and \mathbb{D} are*

$$\frac{b}{2} \cdot \frac{d(z, \mathbb{T})}{|z-1|^2} \leq d(\lambda, \sigma(D_0)) \leq 4b \cdot \frac{d(z, \mathbb{T})}{|z-1|^2},$$

and

$$\frac{1}{b} \cdot \frac{d(\lambda, \sigma(D_0))}{(1 + |\lambda|)^2} \leq d(z, \mathbb{T}) \leq 16b \cdot \frac{d(\lambda, \sigma(D_0))}{(1 + |\lambda|)^2}.$$

Proof. For the first set of inequalities, we apply (2.3.2) to φ , with $\varphi'(z) = \frac{2ib}{(z-1)^2}$.

The second part result from the first $|z-1|^2 = \frac{4b^2}{|\lambda+ib|^2}$. Since $b \geq 1$, we bound by $|\lambda+ib|^2 \leq (|\lambda|+b)^2 \leq b^2(1+|\lambda|)^2$ and we bound from below by $|\lambda+ib|^2 \geq |\lambda|^2 + b^2 \geq \frac{1}{2}(|\lambda|+b)^2 \geq \frac{1}{2}(1+|\lambda|)^2$. Hence

$$\frac{1}{b^2(1+|\lambda|)^2} \leq \frac{1}{|\lambda+ib|^2} \leq \frac{2}{(1+|\lambda|)^2},$$

and we conclude. \square

Proposition 3.5.2. *Let $p > d$. Then*

$$\|V(\lambda - D_0)^{-1}\|_{\mathcal{S}_p}^p \leq K \|V\|_{L^p}^p \frac{|\lambda|^{d-1}}{d(\lambda, \sigma(D_0))^{p-1}},$$

with

1. if $d \geq 2$,

$$a) K = n^{\frac{p}{2}} s_{d-1} (2\pi)^{-d} 2^{\frac{d-1}{2}} M$$

$$b) \text{ where } M = \max \left\{ (1 + 2^{d-2}) \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}; 2^{d-2} \int_0^{+\infty} \frac{t^{d-1} dt}{(t^2 + 1)^{\frac{p}{2}}} \right\}.$$

2. Else $d = 1$ and we have $K = n^{\frac{p}{2}} \frac{2}{\pi} \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}$.

Proof. For $\lambda = \lambda_0 + i\lambda_1 \in \mathbb{C}^+$, from the inequality 2.4.2, we bound of the norm resolvent with

$$\|V(\lambda - D_0)^{-1}\|_{\mathcal{S}_p}^p \leq (2\pi)^{-d} n^{\frac{p}{2}} \|V\|_{L^p}^p \|(\lambda - \mu_0(x))^{-1}\|_{L^p(\mathbb{R}^d, \mathbb{C})}^p,$$

where $\mu_0(x) = |x|$. Then we follow the proof of Proposition 3.3.6.

For $d \geq 2$, we start with the polar change of variable and we have to bound the integral

$$I = \int_0^{+\infty} \frac{r^{d-1}}{|r - \lambda|^p} dr.$$

Remind that here $d(\lambda, \sigma(D_0)) = \text{Im}(\lambda) = \lambda_1 > 0$.

$$\begin{aligned} I &= \int_0^{+\infty} \frac{r^{d-1}}{|(r - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}}} dr \\ &= \frac{1}{|\lambda_1|^p} \int_0^{+\infty} \frac{r^{d-1}}{\left| \left(\frac{r - \lambda_0}{\lambda_1} \right)^2 + 1 \right|^{\frac{p}{2}}} dr \end{aligned}$$

We make the change of variable $s = \frac{r - \lambda_0}{\lambda_1}$, hence, using the inequalities of Lemma 2.3.4, we obtain

$$\begin{aligned}
I &= \frac{1}{|\lambda_1|^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 s + \lambda_0)^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds \\
&= \frac{1}{|\lambda_1|^{p-1}} \left(\int_{-\frac{\lambda_0}{\lambda_1}}^0 \frac{(\lambda_1 s + \lambda_0)^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds + \int_0^{+\infty} \frac{(\lambda_1 s + \lambda_0)^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds \right) \\
&= \frac{1}{|\lambda_1|^{p-1}} \left(\int_0^{\frac{\lambda_0}{\lambda_1}} \frac{(-\lambda_1 s + \lambda_0)^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds + \int_0^{+\infty} \frac{(\lambda_1 s + \lambda_0)^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds \right) \\
&\leq \frac{1}{|\lambda_1|^{p-1}} \left(\int_0^{+\infty} \frac{|\lambda_0|^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds + 2^{d-2} \int_0^{+\infty} \frac{|\lambda_1 s|^{d-1} + |\lambda_0|^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds \right) \\
&\leq \frac{1}{|\lambda_1|^{p-1}} \left((1 + 2^{d-2}) |\lambda_0|^{d-1} \int_{\mathbb{R}^+} \frac{ds}{(s^2 + 1)^{\frac{p}{2}}} + 2^{d-2} |\lambda_1|^{d-1} \int_{\mathbb{R}^+} \frac{s^{d-1} ds}{(s^2 + 1)^{\frac{p}{2}}} \right).
\end{aligned}$$

Again with Lemma 2.3.4, we have

$$|\lambda_0|^{d-1} + |\lambda_1|^{d-1} \leq (|\lambda_0| + \lambda_1)^{d-1} \leq \left(\sqrt{2} \sqrt{\lambda_0^2 + \lambda_1^2} \right)^{d-1} = 2^{\frac{d-1}{2}} |\lambda|^{d-1}.$$

Putting $M = \max \left\{ (1 + 2^{d-2}) \int_0^{+\infty} \frac{1}{(s^2 + 1)^{\frac{p}{2}}} ds; 2^{d-2} \int_0^{+\infty} \frac{s^{d-1}}{(s^2 + 1)^{\frac{p}{2}}} ds \right\}$, we get to

$$I \leq \frac{2^{\frac{d-1}{2}} M}{d(\lambda, \sigma(D_0))^{p-1}} \cdot |\lambda|^{d-1}$$

and the Proposition is proved.

The bound for $d = 1$ is almost done after the second change of variable so the proof is omitted. \square

3.5.2 Proof of Theorem 3.2.3

By the properties of the regularized determinant \mathbf{S}_p and Proposition 3.5.2, we have

$$\log |f(\lambda)| \leq \Gamma_p K \|(ib - D)^{-1}\|^p \|V\|_{L^p}^p \cdot \frac{|\lambda - ib|^p |\lambda|^{d-1}}{d(\lambda; \sigma(D_0))^{p-1}},$$

where f is defined in (2.2.3) and F is the same as (2.2.2) but with $m = 0$. As before, we have $\|(ib - D)^{-1}\| \leq 1$ for b large enough. Putting $\lambda = \varphi(z)$ and $g = f \circ \varphi$, we obtain, thank to the results of Proposition 3.5.1,

$$\begin{aligned}
\log |g(z)| &\leq K_1 b^d \|V\|_{L^p}^p \cdot \frac{|z|^p |1 + z|^{d-1}}{|1 - z|^{d+p-1}} \cdot \frac{|1 - z|^{2(p-1)}}{d(z, \mathbb{T})^{p-1}} \\
&\leq K_1 b^d \|V\|_{L^p}^p \cdot \frac{|z|^p |1 + z|^{d-1}}{|1 - z|^{d-p+1} d(z, \mathbb{T})^{p-1}},
\end{aligned}$$

where $K_1 = \Gamma_p 2^{2p-1} K$ and K is defined in Proposition 3.5.2. We apply Theorem 2.3.1 to the function g to get, for all $\tau > 0$,

$$\sum_{g(z)=0} d(z, \mathbb{T})^{p+\tau} |z - 1|^{(d-p+\tau)_+} \leq C \cdot K_1 \cdot b^d \cdot \|V\|_{L^p}^p, \quad (3.5.1)$$

with C depending on $p, d, \{1\}$, and τ , and K_1 is given above. Since $p > d$, for $0 < \tau < p - d$, we have $d - p + \tau < 0$; And from the distortion inequalities, $d(z, \mathbb{T}) \geq \frac{1}{b} \cdot \frac{d(\lambda, \sigma(D_0))}{(1 + |\lambda|)^2}$, hence

$$\sum_{\lambda \in \sigma(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{2(p+\tau)}} \leq C \cdot b^{p+d+\tau} K_1 \cdot \|V\|_{L^p}^p.$$

The proof is done. □

3.6 The Klein-Gordon operator

3.6.1 Definition and results

Another operator close to the Dirac operator is the following defined by [Tha91, Formula (1.3)]

$$K_m = \sqrt{-\Delta_{\mathbb{R}^d} + m^2} \times \text{Id}_{\mathbb{C}^\ell} \quad (3.6.1)$$

with $m \geq 0$ et $\ell \geq 1$. We call it *Klein-Gordon operator* but there are other possible conventions; it is frequently called *relativistic Schrödinger operator* sometimes labelled “quasi” ([BE11]). In [LS09b] (or [Sei10]), the authors replace the study of the Klein-Gordon operator by the study of the “ultra-relativistic” Schrödinger operator $(-\Delta)^{1/2}$ (see reference therein).

As Thaller indicates in [Tha91, Section 1.1] the Klein-Gordon equation [Tha91, Formula (1.3)] does not satisfy some properties of the quantum mechanics (for example, the time derivative should be of order 1).

Remark 3.6.1. For more details about the problem encounter with this operator see Chapter VI of the book by Holstein *Topics in quantum mechanics*.

This time, unlike the Dirac operator, the dimension ℓ is not related to the dimension d . This operator is used as model for a particle of mass m without spin (see [BE11, Section 2.2]); In the view of (3.1.2), it is a “half” of the Dirac operator. As the Dirac operator, it is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{C}^\ell)$, the domain of its closure is $\mathcal{H}^1(\mathbb{R}^d; \mathbb{C}^\ell)$, and its spectrum is absolutely continuous and equals $[m, +\infty[$.

We consider the perturbed Klein-Gordon operator

$$K = K_m + V,$$

with $V \in L^p(\mathbb{R}^d; \mathcal{M}_\ell(\mathbb{C}))$ and $p > d$. As before, we have by Weyl’s Theorem 2.1.17,

$$\sigma_{\text{ess}}(K) = \sigma_{\text{ess}}(K_m) = [m; +\infty[.$$

One advantage of the Klein-Gordon operator is that it is bounded from below. Using this fact we are able to “integrate” according to the idea of Demuth, Hansmann, and Katriel.

This gives the next result

Theorem 3.6.2. *Let K be as above and $V \in L^p(\mathbb{R}^d; \mathcal{M}_\ell(\mathbb{C}))$. We assume $p > d$. The constants ω and C_ω are defined in Proposition 3.6.7. Then, for $d \geq 2$ and τ small enough, we have*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(1 + |\lambda - m|)^{d-1+2\tau}} \leq C \cdot \frac{K_3}{I} \cdot \frac{\omega^{d-1+\tau} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p. \quad (3.6.2)$$

where

$$1. K_3 = \frac{\Gamma_p}{(2\pi)^d} \cdot \ell^{p/2} s_{d-1} M (1+m)^{d-1} 2^\delta,$$

with a) M defined in Proposition 3.3.6,

$$b) \delta = 2p + d - \max\{\frac{p-\tau}{2}; d\} - 1 + \tau,$$

$$2. I = \int_0^{+\infty} \frac{t^{p+(\frac{p}{2}-d-1-\frac{\tau}{2})_+}}{(t+1)^{p+d+2\tau+(\frac{p}{2}-d-1-\frac{\tau}{2})_+}} dt,$$

3. and finally C depends on ℓ, d, p, m , and τ .

In dimension $d = 1$, with $0 < \tau \leq p - 1$, we find

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda - m|(1 + |\lambda - m|)^{1+2\tau}} \leq C \cdot \frac{K_3}{I} \cdot \frac{\omega^{1+\tau} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p, \quad (3.6.3)$$

with

$$1. K_3 = \frac{\Gamma_p}{2\pi} \cdot \ell^{\frac{p}{2}} s_0 M (1+m) 2^{\frac{1}{2}(7p+9+3\tau)}, M \text{ is defined in Proposition 3.4.1,}$$

$$2. I = \int_0^{+\infty} \frac{t^{p+\frac{1}{2}(p-3-\tau)_+}}{(t+1)^{p+2+2\tau+\frac{1}{2}(p-3-\tau)_+}} dt,$$

3. and C depending on ℓ, p, m , and τ .

We recall that the constant C comes from Theorem 2.3.1 (see Remark 3.2.2).

We recall the result we obtained in [Dub14] using the computations done for the perturbed Dirac operator.

Theorem 3.6.3 (case $m > 0$). *With the notation above, we have*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(1 + |\lambda|)^{p+\max\{\frac{p}{2}; d\}-1+2\tau}} \lesssim \|V\|_{L^p}^p. \quad (3.6.4)$$

In dimension $d = 1$, we have

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda - m|(1 + |\lambda|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} \lesssim \|V\|_{L^p}^p. \quad (3.6.5)$$

We notice that, for $m = 0$, the operator is simply

$$K = (-\Delta)^{1/2} + V,$$

that is the fractional Schrödinger operator with $s = \frac{1}{2}$ (see next Chapter). We observe that a non-trivial degeneration of a bound on the resolvent of K_0 takes place in this case and the inequality of Theorem 3.6.2 can be refined in the following way (see Theorem 4.2.1) :

Theorem 3.6.4 (case $m = 0$). *Let K be the perturbed Klein-Gordon operator defined above with $m = 0$ and $V \in L^p(\mathbb{R}^d; \mathcal{M}_\ell(\mathbb{C}))$, $p > d$. Then, for $\tau > 0$ small enough, we have*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_0))^{p+\tau}}{|\lambda|^{\min\{(p+\tau)/2, d\}} (1 + |\lambda|)^{\frac{1}{2}(2d-p-\tau)_++2\tau}} \lesssim \|V\|_{L^p}^p. \quad (3.6.6)$$

As we said before, the massless Klein-Gordon operator is also the fractional Laplacian $(-\Delta)^s$ with $s = \frac{1}{2}$. For the self-adjoint case, an account on Lieb-Thirring-type inequalities can be found in Frank, Lieb and Seiringer ([FLS08]) and Lieb and Seiringer ([LS09b, Chapter 4]).

It is convenient to compare Theorem 3.6.4 with these results for $s = 1/2$. Of course, the discrete spectrum $\sigma_d(K)$ lies on the negative real half-axis in this case. In our notation Theorem 2.1 from [FLS08] says that

$$\sum_{\lambda \in \sigma_d(K)} |\lambda|^{p-d} \leq C_{p,d} \|V_-\|_{L^p}^p, \quad (3.6.7)$$

where $p > d$ and $V_- = \min\{V, 0\}$. Since $d(\lambda, \sigma(K_0)) = |\lambda|$, bound (3.6.6) looks like

$$\sum_{\lambda \in \sigma_d(K)} \frac{|\lambda|^{\max\{(p+\tau)/2, p+\tau-d\}}}{(1 + |\lambda|)^{\frac{1}{2}(2d-p-\tau)_++2\tau}} \leq C \|V\|_{L^p}^p, \quad (3.6.8)$$

which is slightly weaker than (3.6.7) as we will see immediately. Indeed, if we have $p + \tau - d > (p + \tau)/2$ (or, equivalently, $p/2 \geq d$), the left hand-side of (3.6.8) is

$$\frac{|\lambda|^{p+\tau-d}}{(1 + |\lambda|)^{2\tau}} \leq |\lambda|^{p-d} \cdot \left(\frac{|\lambda|}{(1 + |\lambda|)^2} \right)^\tau \leq |\lambda|^{p-d}.$$

If $p/2 < d$, we have for the left hand-side of (3.6.8)

$$\begin{aligned} \frac{|\lambda|^{(p+\tau)/2}}{(1 + |\lambda|)^{d-(p+\tau)/2}} &\leq \frac{|\lambda|^{p+\tau-d} \cdot |\lambda|^{-(p+\tau)/2+d}}{(1 + |\lambda|)^{d-(p+\tau)/2}} \\ &= |\lambda|^{p-d} \cdot \left(\frac{|\lambda|}{1 + |\lambda|} \right)^{d-(p+\tau)/2} \cdot \frac{|\lambda|^\tau}{(1 + |\lambda|)^{2\tau}}, \end{aligned}$$

and a simple bound using $d - (p + \tau)/2 > 0$ yields that the second factor in the above formula is less or equal to one.

3.6.2 Proof of results

We start with the computation of the distortion.

We consider the function $\varphi_a : \mathbb{D} \rightarrow \mathbb{C} \setminus \mathbb{R}^+$ defined by $\varphi_a(z) = -a \left(\frac{z+1}{z-1} \right)^2$ (see (4.2.6)) and the function $\psi_a(z) = m + \varphi_a(z)$. We put $\lambda_1 = \varphi_a(z)$ et $\lambda = \lambda_1 + m = \psi_a(z)$.

The distortions for φ_a are known by Proposition 4.2.3, hence we easily deduce the one for ψ_a .

Proposition 3.6.5 (Distortion between $\sigma(K_m)$ and \mathbb{D}). *The distortion via ψ_a behaves the following way :*

$$a \cdot d(z, \mathbb{T}) \frac{|z+1|}{|z-1|^3} \leq d(\lambda, \sigma(K_m)) \leq 8a \cdot d(z, \mathbb{T}) \frac{|1+z|}{|1-z|^3}, \quad (3.6.9)$$

and

$$\frac{\sqrt{a}}{4} \cdot \frac{d(\lambda, \sigma(K_m))}{\sqrt{|\lambda-m|(a+|\lambda-m|)}} \leq d(z, \mathbb{T}) \leq 4\sqrt{a} \frac{d(\lambda, \sigma(K_m))}{\sqrt{|\lambda-m|(a+|\lambda-m|)}}. \quad (3.6.10)$$

Remark 3.6.6. In the article [Dub14], we used a less precise version of (3.6.10) :

$$\frac{1}{4\sqrt{a}} \cdot \frac{d(\lambda, \sigma(K_m))}{\sqrt{|\lambda-m|(1+|\lambda-m|)}} \leq d(z, \mathbb{T}) \leq 4\sqrt{a} \frac{d(\lambda, \sigma(K_m))}{\sqrt{|\lambda-m|(1+|\lambda-m|)}}. \quad (3.6.11)$$

Proof. We refer to the proof of Proposition 4.2.3.

For (3.6.9), it suffices to see that $\psi'_a(z) = \varphi'_a(z)$ and use (4.2.7).

For (3.6.10), it is just a change of variable $\lambda_1 = \lambda - m$ in (4.2.8). \square

As we said before, the Klein-Gordon operator is bounded from below, hence we have the following proposition (see also [Han10, Lemma 3.3.4]).

Proposition 3.6.7. *Let $p > d$, there is $\omega \geq 1$ depending on d, ℓ, p, m , and V , satisfying $-\omega + m \in \rho(K_m)$ such that*

$$\forall a - m > \omega, \|(-a + m - K)^{-1}\| \leq \frac{C_\omega}{a - \omega}, \quad (3.6.12)$$

with $C_\omega = (1 - \|V(-\omega + m - K_m)\|)^{-1}$.

Proof. For convenience, some details will be proved in Proposition 4.2.7.

We have $\|V(\lambda - K_m)^{-1}\| < 1 \Rightarrow \lambda \in \rho(K_m)$ (see Proposition 4.2.7). Take $\lambda = -a + m$ and let us show there is $\omega \geq 1$ such that for all $a > \omega + m$, $\|V(-a + m - K_m)^{-1}\| < 1$. Indeed, we have

$$\begin{aligned} \|V(-a + m - K_m)^{-1}\| &\leq \|V(-a + m - K_m)^{-1}\|_{\mathbf{s}_p} \\ &\leq (2\pi)^{-d/p} \|V\|_{L^p} \cdot \|(-a + m - \mu_m(x))^{-1}\| \end{aligned}$$

and, by the bound obtained in Proposition 3.3.6 (and 3.4.1 for dimension $d = 1$) for the Dirac operator when $\operatorname{Re}(\lambda) \geq 0$, the right hand-side of the above inequality tends to 0 when a tends to $+\infty$.

Now, for all $a - m \geq \omega \geq 1$, we have $\|(-a + m - K_m)^{-1}\| \leq \frac{C_\omega}{a - \omega}$. Indeed, from the relation 4.2.15, with $\lambda = -a + m$ we have

$$\begin{aligned} \|(-a + m - K)^{-1}\| &\leq \frac{1}{d(-a + m; \sigma(K_m))} \cdot \frac{1}{1 - \|V(-a + m - K_m)^{-1}\|} \\ &\leq \frac{C_\omega}{a} \leq \frac{C_\omega}{a - \omega}, \end{aligned}$$

since $d(-a + m, \sigma(K_m)) = a \geq a - \omega$ and $\|V(-a + m - K_m)^{-1}\|$ tends to 0 when a tends to $+\infty$. \square

Remark 3.6.8. In the article [Dub14], as for the distortion, we limited ourself to the use of results obtained with the Dirac operator, hence, we bounded $\|(-a + m - K)^{-1}\|$ by 1. Following the ideas of Demuth, Hansmann, and Katriel ([DHK09] or [DHK13], thanks to the bound above, we obtain better estimations (see Remark 3.6.9).

Proof of Theorem 3.6.2. We go back to the expression of F defined in (2.2.2) with, this time, the operator $K_m : F(\lambda) = (\lambda + a - m)(-a + m - K)^{-1}V(\lambda - K_m)^{-1}$, and f is defined as in (2.2.3) for $\lambda \in \mathbb{C} \setminus \sigma(K_m)$.

We start with the case $d \geq 2$. Since $\|(-a + m - K)^{-1}\| \leq \frac{C_\omega}{|a - \omega|}$, for $a \geq \omega$, then, from above and Proposition 3.3.6 for $d \geq 2$,

$$\log(f(\lambda)) \leq \frac{\Gamma_p C_\omega^p}{|a - \omega|^p} \cdot |\lambda + a - m|^p \|V(\lambda - K_m)^{-1}\|_{\mathfrak{S}_p}^p \quad (3.6.13)$$

$$\begin{aligned} &\leq \frac{K_1}{|a - \omega|^p} \cdot \|V\|_{L^p}^p \cdot |\lambda + a - m|^p \cdot \frac{(|\lambda - m| + m)^{d-1}}{d(\lambda, \sigma(K_m))^{p-1}} \\ &\leq \frac{K_1(1 + m)^{d-1}}{|a - \omega|^p} \cdot \|V\|_{L^p}^p |\lambda_1 + a|^p \cdot \frac{(1 + |\lambda_1|)^{d-1}}{d(\lambda, \sigma(K_m))^{p-1}}, \end{aligned} \quad (3.6.14)$$

where, by Proposition 3.3.6 for $d \geq 2$, $K_1 = \frac{\Gamma_p}{(2\pi)^d} C_\omega^p 2^{d-2} \ell^{\frac{p}{2}} s_{d-1} M$. By definition

$$|\lambda_1| = a \left| \frac{z + 1}{z - 1} \right|^2, \text{ hence}$$

$$|\lambda + a - m| = |\lambda_1 + a| = \frac{4a|z|}{|z - 1|^2} \text{ and } 1 + |\lambda_1| = 2a \frac{(1 + |z|^2)}{|z - 1|^2} \leq \frac{4a}{|z - 1|^2}.$$

We express the inequality (3.6.14) in terms of $z \in \mathbb{D}$, so

$$\begin{aligned} \log |g(z)| &\leq \frac{K_1(1 + m)^{d-1}}{|a - \omega|^p} \cdot \|V\|_{L^p}^p \cdot \frac{(4a|z|)^p \cdot (4a)^{d-1} \cdot a^{1-p} |z - 1|^{3p-3}}{|z - 1|^{2p} \cdot |z - 1|^{2d-2} \cdot d(z, \mathbb{T})^{p-1} |z + 1|^{p-1}} \\ &\leq K_2 \cdot \frac{a^d}{|a - \omega|^p} \cdot \frac{\|V\|_{L^p}^p}{d(z, \mathbb{T})^{p-1} |z + 1|^{p-1} |z - 1|^{2d-p+1}}, \end{aligned}$$

with $K_2 = K_1(1 + m)^{d-1} 4^{p+d-1}$. We apply Theorem 2.3.1, hence, for all $\tau > 0$,

$$\sum_{g(z)=0} d(z, \mathbb{T})^{p+\tau} |z - 1|^{(2d-p+\tau)_+} |z + 1|^{p-2+\tau} \leq C \cdot K_2 \cdot \frac{a^d}{|a - \omega|^p} \cdot \|V\|_{L^p}^p \quad (3.6.15)$$

where C depends on $d, p, \{1, -1\}$, and τ , and K_2 is described above. Since $p > d \geq 2$, we always have $p - 2 > 0$, we have to distinguish $d < p \leq 2d$ and $p > 2d$.

We recall the following inequalities:

we have $z = \frac{\sqrt{\lambda - m} - i\sqrt{a}}{\sqrt{\lambda - m} + i\sqrt{a}}$, so $z - 1 = \frac{-2i\sqrt{a}}{\sqrt{\lambda - m} + i\sqrt{a}}$ and $z + 1 = \frac{2\sqrt{\lambda - m}}{\sqrt{\lambda - m} + i\sqrt{a}}$.
On the other hand $|\sqrt{\lambda - m} + i\sqrt{a}|^2 = a + |\lambda - m|$, hence

$$|z - 1| \geq \frac{\sqrt{2a}}{(a + |\lambda - m|)^{1/2}}, \quad |z + 1| \geq \frac{\sqrt{2|\lambda - m|}}{(a + |\lambda - m|)^{1/2}}. \quad (3.6.16)$$

If $p > 2d$ then

$$\begin{aligned} d(z, \mathbb{T})^{p+\tau} |z + 1|^{p-2+\tau} &\geq \left(\frac{\sqrt{a}}{4}\right)^{p+\tau} 2^{\frac{1}{2}(p-2+\tau)} \cdot \frac{d(\lambda, \sigma(K_m))^{p+\tau} |\lambda - m|^{\frac{p+\tau}{2}-1}}{|\lambda - m|^{\frac{p+\tau}{2}} (a + |\lambda - m|)^{p+\tau+\frac{p+\tau}{2}-1}} \\ &\geq \frac{a^{\frac{p+\tau}{2}}}{2^{\frac{3}{2}(p+\tau)+1}} \cdot \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(a + |\lambda - m|)^{3\frac{p+\tau}{2}-1}}, \end{aligned}$$

hence

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(a + |\lambda - m|)^{3\frac{p+\tau}{2}-1}} \lesssim K_2 \cdot 2^{\frac{3}{2}(p+\tau)+1} \cdot \frac{a^{d-\frac{p+\tau}{2}}}{|a - \omega|^p} \cdot \|V\|_{L^p}^p. \quad (3.6.17)$$

Similarly, if $d < p \leq 2d$, then

$$d(z, \mathbb{T})^{p+\tau} |z - 1|^{2d-p+\tau} |z + 1|^{p-2+\tau} \geq \frac{a^{d+\tau}}{2^{2p-d+1+\tau}} \cdot \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(a + |\lambda - m|)^{p+d-1+2\tau}}.$$

hence

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(a + |\lambda - m|)^{p+d-1+2\tau}} \lesssim K_2 \cdot 2^{2p-d+1+\tau} \cdot \frac{a^{-\tau}}{|a - \omega|^p} \cdot \|V\|_{L^p}^p. \quad (3.6.18)$$

Remark 3.6.9. A very interesting idea of Demuth, Hansmann, and Katriel is to see the inequalities of type (3.6.17) and (3.6.18) as a family of inequalities depending on the parameter a . Their idea is to consider an ‘‘average’’ of these inequalities by integrating with respect to a between ω and $+\infty$.

We follow the proof of Theorem 4.4.6 from [DHK13], but the difference here is that we do use the Theorem of Borichev, Golinskii, and Kupin instead of Theorem 3.3.5 from [DHK13] (although based on the same theorem) which gives no difference here except maybe for the constant in front of $\|V\|_{L^p}$. The other difference concern the integration of the left-hand side member : the inequality given before Formula (4.4.19) page 27 in [DHK13] miss one case : when $\varphi_1 - 1 + \tau < 0$. Fortunately, by an obvious lower bound, we find the same power for the factor $|\lambda| + \omega$, but the constant is again different.

We start from the above cases. We precise that the interchanging of the sum and the integral is valid, by Fubini Theorem, since all functions are positive.

When $p > 2d$ then, from (3.6.17), we find

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|} \int_{\omega}^{+\infty} \frac{a^{\frac{p+\tau}{2}-d-1-\tau} |a - \omega|^p}{(a + |\lambda - m|)^{3\frac{p+\tau}{2}-1}} da \lesssim \int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

The integral in the right-hand side equals $\frac{1}{\tau\omega^\tau}$. In the left-hand side, we make the change of variable $t = \frac{a - \omega}{|\lambda - m| + \omega}$, and, if $\frac{p}{2} - d - 1 > 0$, we bound from below $(|\lambda - m| + \omega)t + \omega$ by $(|\lambda - m| + \omega)t$, else we bound from above $(|\lambda - m| + \omega)t + \omega$ by $(|\lambda - m| + \omega)(t + 1)$, hence, if $\frac{p}{2} - d - 1 > 0$, we have

$$\begin{aligned} & \int_{\omega}^{+\infty} \frac{a^{\frac{p-\tau}{2}-d-1} |a - \omega|^p}{(a + |\lambda - m|)^{3\frac{p+\tau}{2}-1}} da \\ &= \frac{(|\lambda - m| + \omega)^{p+1}}{(|\lambda - m| + \omega)^{\frac{3}{2}(p+\tau)-1}} \int_{\mathbb{R}^+} \frac{[(|\lambda - m| + \omega)t + \omega]^{\frac{p-\tau}{2}-d-1} t^p}{(t + 1)^{\frac{3}{2}(p+\tau)-1}} dt \\ &\geq \frac{(|\lambda - m| + \omega)^{\frac{p-\tau}{2}-d-1}}{(|\lambda - m| + \omega)^{\frac{p}{2}-2+\frac{3\tau}{2}}} \int_0^{+\infty} \frac{t^{\frac{3p}{2}-d-1-\frac{\tau}{2}}}{(t + 1)^{\frac{3}{2}(p+\tau)-1}} dt \\ &\geq \frac{1}{(|\lambda - m| + \omega)^{d-1+2\tau}} \int_0^{+\infty} \frac{t^{\frac{3p}{2}-d-1-\frac{\tau}{2}}}{(t + 1)^{\frac{3}{2}(p+\tau)-1}} dt, \end{aligned}$$

which gives, for $2d < p$ and $\frac{p}{2} - d - 1 > 0$,

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(\omega + |\lambda - m|)^{d-1+2\tau}} \lesssim \frac{K_2}{I} \cdot 2^{\frac{3}{2}(p+\tau)+1} \frac{1}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p,$$

with $I = \int_0^{+\infty} \frac{t^{\frac{3p}{2}-d-1-\frac{\tau}{2}}}{(t + 1)^{\frac{3}{2}(p+\tau)-1}} dt$.

And if $\frac{p}{2} - d - 1 \leq 0$ then

$$\int_{\omega}^{+\infty} \frac{a^{\frac{p-\tau}{2}-d-1} |a - \omega|^p}{(a + |\lambda - m|)^{3\frac{p+\tau}{2}-1}} da \geq \frac{1}{(|\lambda - m| + \omega)^{d-1+2\tau}} \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+d+2\tau}}.$$

Hence, for $2d < p$ and $\frac{p}{2} - d - 1 \leq 0$

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(\omega + |\lambda - m|)^{d-1+2\tau}} \lesssim \frac{K_2}{I} \cdot 2^{\frac{3}{2}(p+\tau)+1} \cdot \frac{1}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p,$$

with $I = \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+d+2\tau}}$.

Now when $d < p \leq 2d$ then, from (3.6.18), we obtain

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|} \int_{\omega}^{+\infty} \frac{a^{-1} |a - \omega|^p}{(a + |\lambda - m|)^{p+d-1+2\tau}} da \lesssim \int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

Here, we bound first from below a^{-1} by $(a + |\lambda - m|)^{-1}$, and then doing the change of variable $t = \frac{a - \omega}{|\lambda - m| + \omega}$, we find

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{a^{-1}|a - \omega|^p}{(a + |\lambda - m|)^{p+d-1+2\tau}} &\geq \int_{\omega}^{+\infty} \frac{|a - \omega|^p}{(a + |\lambda - m|)^{p+d+2\tau}} da \\ &\geq \frac{(|\lambda - m| + \omega)^{p+1}}{(|\lambda - m| + \omega)^{p+d+2\tau}} \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+d+2\tau}}. \end{aligned}$$

Hence, for $d < p \leq 2d$,

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+\tau}}{|\lambda - m|(\omega + |\lambda - m|)^{d-1+2\tau}} \lesssim \frac{K_2}{I} \cdot 2^{2p-d+1+\tau} \cdot \frac{1}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p,$$

with $I = \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+d+2\tau}}$.

Summarising the different cases, we obtain the claimed inequality for $d \geq 2$.

Remark 3.6.10. One can wonder what happens if, in the right-hand side of (3.6.18), instead of bounding by the integral $\int_{\omega}^{+\infty} \frac{1}{a^{1+\tau}} da$, we bound with $\int_{\omega}^{+\infty} \frac{1}{a^2} da$ for instance. In this case, the power in the numerator of the left-hand side is smaller and consequently the power of the quotient is higher. But in order to obtain the best estimate possible, it has to be the smallest possible.

Now we consider the dimension $d = 1$.

Recall the result of Proposition 3.4.1 :

$$\|(\lambda - \mu_m(\cdot))^{-1}\|_{L^p}^p \leq \ell^{p/2} s_0 M \cdot \frac{d(\lambda; \sigma(K_m)) + m}{d(\lambda, \sigma(K_m))^p},$$

with M only depending on p , and we bound $d(\lambda; \sigma(K_m))$ by $|\lambda - m|$. We still have $\|(a - m - K)^{-1}\| \leq \frac{C_\omega}{|a - \omega|}$, for $a \geq \omega$, so, from (3.6.13),

$$\begin{aligned} \log(f(\lambda)) &\leq \frac{\Gamma_p C_\omega^p}{|a - \omega|^p} \cdot |\lambda + a - m|^p \|V(\lambda - K_m)^{-1}\|_{\mathbf{S}_p}^p \\ &\leq \frac{K_1}{|a - \omega|^p} \cdot \|V\|_{L^p}^p |\lambda + a - m|^p \cdot \frac{|\lambda - m| + m}{d(\lambda, \sigma(K_m))^p} \\ &\leq \frac{K_1(1 + m)}{|a - \omega|^p} \cdot \|V\|_{L^p}^p |\lambda_1 + a|^p \cdot \frac{|\lambda_1| + m}{d(\lambda, \sigma(K_m))^p}, \end{aligned}$$

with $K_1 = \frac{\Gamma_p}{2\pi} \cdot \ell^{\frac{p}{2}} s_0 M C_\omega^p$, M is defined in Proposition 3.4.1. Then

$$\begin{aligned} \log |g(z)| &\leq \frac{K_1(1 + m)}{|a - \omega|^p} \|V\|_{L^p}^p \frac{(4a|z|)^p}{|z - 1|^{2p}} \cdot \frac{4a}{|z - 1|^2} \cdot \frac{a^{-p}|z - 1|^{3p}}{d(z, \mathbb{T})^p |z + 1|^p} \\ &\leq K_2 \frac{a}{|a - \omega|^p} \cdot \|V\|_{L^p}^p \cdot \frac{|z|^p}{d(z, \mathbb{T})^p |z - 1|^{2-p} |z + 1|^p}, \end{aligned}$$

with $K_2 = K_1(1+m)4^{p+1}$. We apply Theorem 2.3.1 and we find

$$\sum_{g(z)=0} d(z, \mathbb{T})^{p+1+\tau} |z+1|^{p-1+\tau} |z-1|^{(1-p+\tau)_+} \leq C \cdot K_2 \cdot \frac{a}{|a-\omega|^p} \cdot \|V\|_{L^p}^p,$$

where K_2 is describe above and C is a constant depending on $p, \{1; -1\}$, and τ .

Since $p > 1$ then, for all $0 < \tau < p-1$, we have $1-p+\tau < 0$. Then

$$d(z, \mathbb{T}) \geq \frac{\sqrt{a}}{4} \cdot \frac{d(\lambda, \sigma(K_m))}{\sqrt{|\lambda-m|(1+|\lambda-m|)}} \text{ and } |z+1| \geq \left(\frac{2|\lambda-m|}{a+|\lambda-m|} \right)^{\frac{1}{2}},$$

hence

$$\begin{aligned} d(z, \mathbb{T})^{p+1+\tau} |z+1|^{p-1+\tau} &\geq \frac{a^{\frac{1}{2}(p+1+\tau)}}{2^{2(p+1+\tau)-\frac{1}{2}(p-1+\tau)}} \cdot \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda-m|(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} \\ &\geq \frac{a^{\frac{1}{2}(p+1+\tau)}}{2^{\frac{3}{2}(p+\tau)+\frac{5}{2}}} \cdot \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda-m|(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}}. \end{aligned}$$

Consequently,

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda-m|(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} \lesssim \frac{a^{-\frac{1}{2}(p-1+\tau)}}{|a-\omega|^p} \cdot \|V\|_{L^p}^p,$$

It remains to integrate the previous inequality with respect to the parameter a between ω and $+\infty$. We have to bound from below the next integral

$$\int_{\omega}^{+\infty} \frac{|a-\omega|^p a^{\frac{1}{2}(p-1+\tau)-1-\tau}}{(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} da = \int_{\omega}^{+\infty} \frac{|a-\omega|^p a^{\frac{1}{2}(p-3-\tau)}}{(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} da.$$

We distinguish if $p > 3$ or $1 < p \leq 3$ and make the change of variable $t = \frac{a-\omega}{|\lambda-m|+\omega}$.

We start with $p > 3$, so we bound $(|\lambda-m|+\omega)t + \omega$ from below by $(|\lambda-m|+\omega)t$ hence

$$\begin{aligned} &\int_{\omega}^{+\infty} \frac{|a-\omega|^p a^{\frac{1}{2}(p-3-\tau)}}{(a+|\lambda-m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} da = \\ &= \frac{(|\lambda-m|+\omega)^{p+1}}{(|\lambda-m|+\omega)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} \int_0^{+\infty} \frac{t^p [(|\lambda|+\omega)t + \omega]^{\frac{1}{2}(p-3-\tau)}}{(t+1)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} dt \\ &\geq \frac{(|\lambda-m|+\omega)^{\frac{1}{2}(p-3-\tau)}}{(|\lambda-m|+\omega)^{\frac{1}{2}(p-1+3\tau)}} \int_0^{+\infty} \frac{t^{\frac{1}{2}(3p-3-\tau)}}{(t+1)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} dt. \end{aligned}$$

So, for $p > 3$, with $K_3 = K_2 2^{\frac{3}{2}(p+\tau)+\frac{5}{2}}$,

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda-m|(\omega+|\lambda-m|)^{1+2\tau}} \lesssim \left(\int_0^{+\infty} \frac{t^{\frac{1}{2}(3p-3-\tau)}}{(t+1)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} dt \right)^{-1} \frac{K_3}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p.$$

And if $1 < p \leq 3$ then we first bound a from above by $a + |\lambda - m|$ and then

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{|a - \omega|^p a^{\frac{1}{2}(p-3-\tau)} da}{(a + |\lambda - m|)^{\frac{3}{2}(p+\tau)+\frac{1}{2}}} &\geq \int_{\omega}^{+\infty} \frac{|a - \omega|^p da}{(a + |\lambda - m|)^{p+2+2\tau}} \\ &\geq \frac{(|\lambda - m| + \omega)^{p+1}}{(|\lambda - m| + \omega)^{p+2+2\tau}} \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+2+2\tau}} dt. \end{aligned}$$

Hence, for $1 < p \leq 3$, with K_3 as above,

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^{p+1+\tau}}{|\lambda - m|(\omega + |\lambda - m|)^{1+2\tau}} \lesssim \left(\int_0^{+\infty} \frac{t^p dt}{(t + 1)^{p+2+2\tau}} dt \right)^{-1} \frac{K_3}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p.$$

This finishes the proof of Theorem 3.6.3. □

The case $m = 0$ is a particular case of the fractional Laplacian with $s = \frac{1}{2}$, hence we refer to the next chapter for the proof. However we precise that the proof of Lieb-Thirring-type inequalities for the fractional Laplacian was done in the paper [Dub15] after the article [Dub14], for which we made the computation in the particular case of $s = \frac{1}{2}$.

4 The fractional Laplacian operator : results and proof

Table of Contents

4.1	Definition	53
4.2	Results and proofs	54
4.2.1	Conformal mapping for the fractional Laplacian	56
4.2.2	Upper bound for fractional Laplacian	56
4.2.3	Upper bound of the perturbed operator	59
4.2.4	Proof of Theorem 4.2.2	60

4.1 Definition

The referee of the paper [Dub14] make us aware of the existence of Lieb-Thirring-type inequalities for fractional Laplacian in [LS09b] (for instance), generalising the Klein-Gordon massless case. We define for $s > 0$ the *fractional Laplacian* $(-\Delta)^s$ with the help of the functional calculus applied to the positive operator $-\Delta$. Hence $(-\Delta)^s$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d; \mathbb{C}^n)$ and the domain of its closure is the fractional Sobolev space $W^{2,2s}(\mathbb{R}^d, \mathbb{C}^n) := \{f, \int_{\mathbb{R}^d} (1 + |\zeta|^2)^s \|\widehat{f}(\zeta)\|^2 d\zeta < +\infty\}$, where \widehat{f} is the Fourier transform of f . By the spectral mapping Theorem the spectrum of $(-\Delta)^s$ is $[0; +\infty[$.

Let us denote by $H_0 := (-\Delta)^s, s > 0$ and consider the perturbed operator

$$H = H_0 + V \tag{4.1.1}$$

where V is the operator of multiplication by V . We call it *fractional Schrödinger operator*. Here again we assume that V is a relatively Schatten-von Neumann perturbation of H_0 (see definition 2.1.19).

The potential V belongs to $L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C})) = \left\{ V : \int_{\mathbb{R}^d} \|V(x)\|_F^p dx < +\infty \right\}$, where $\|\cdot\|_F$ is the Frobenius norm,

$$\|V(x)\|_F = \left(\sum_{i,j=1,\dots,n} |(V(x))_{i,j}|^2 \right)^{1/2}.$$

If $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ and $p > \max\{1; \frac{d}{2s}\}$ then the hypothesis (2.1.1) is satisfied (see Proposition 2.4.1).

By Weyl's Theorem 2.1.17 on the essential spectrum

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = [0, +\infty[.$$

4.2 Results and proofs

The following Lieb-Thirring-type inequalities are obtained with the help of Theorem 2.3.1 of Borichev, Golinskii, and Kupin, and using the integration method of Demuth, Hansmann, and Katriel.

Theorem 4.2.1. *Let H be the fractional Schrödinger operator defined by (4.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > \max\{1; \frac{d}{2s}\}$. We take τ small enough. The constants ω and C_ω are defined in Proposition 4.2.7.*

For $0 < s \leq \frac{d}{2}$, the next inequality is verified

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^\alpha (1 + |\lambda|)^\beta} \leq C \cdot \frac{K_1}{I} \cdot \frac{\omega^{\beta-\tau} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.1)$$

where

1. the powers satisfy

- a) $\alpha = \min\{\frac{p+\tau}{2}; \frac{d}{2s}\}$,
- b) $\beta = 2\tau + \frac{1}{2}(\frac{d}{s} - p - \tau)_+$,
- c) $\delta = \frac{7p}{2} + \frac{3\tau}{2} + \min\{p; \frac{d}{s}\} - \frac{d}{2s}$ (see K_1),

2. C depends on d, p, s , and τ ,

$$3. K_1 = \frac{\Gamma_p}{(2\pi)^d} \cdot 2^\delta \cdot \frac{n^{p/2} s_{d-1}}{2s} \cdot M_1, \quad M_1 \text{ is defined in Proposition 4.2.5,}$$

$$4. \text{ and } I = \int_0^{+\infty} \frac{t^{p+\frac{1}{2}(p-\frac{d}{s}-2-\tau)_+}}{(t+1)^{p+1+2\tau+\frac{1}{2}\max\{\frac{d}{s}-p-2\tau; 0; p-\frac{d}{s}-2-\tau\}}} dt.$$

For $s > \frac{d}{2}$ and $p > 1$, we have

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^\alpha (1 + |\lambda|)^\beta} \leq C \cdot \frac{K_4}{I} \cdot \frac{\omega^{\beta-\tau} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.2)$$

where

1. the powers satisfy

- a) $\alpha = \frac{1}{2} + \frac{1}{2} \min\{p - \frac{d}{2s} + \tau; 1\}$,
- b) $\beta = 2\tau - \frac{1}{2} \min\{0; p - \frac{d}{2s} - 1 + \tau\}$,
- c) $\delta = 2(2p + 1 - \frac{d}{2s} + \tau) - \frac{1}{2} \max\{p - \frac{d}{2s} - 1 + \tau; 0; \frac{3d}{2s} - p - 1 + \tau\}$ (see K_4),

2. C depends on $d, p, s,$ and $\tau,$

3. $K_4 = \frac{\Gamma_p}{(2\pi)^d} 2^\delta \cdot \frac{n^{p/2} s_{d-1}}{2s} \cdot N_1, N_1$ is defined in Proposition 4.2.5,

4. and $I = \int_0^{+\infty} \frac{t^{p+\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)_+}}{(t+1)^{p+1+2\tau+\frac{1}{2}\max\{p-\frac{3d}{2s}-1-\tau; 0; \frac{d}{2s}+1-p-\tau\}}} dt.$

We give here the result obtained without integration.

Theorem 4.2.2. *Let H be the fractional Schrödinger operator defined by (4.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ with $p > \max\{1; \frac{d}{2s}\}.$ Let $\tau > 0$ be small enough.*

For $0 < s \leq \frac{d}{2},$ the next inequality is satisfied

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^\alpha (1+|\lambda|)^\beta} \lesssim \|V\|_{L^p}^p, \quad (4.2.3)$$

where

1. $\alpha = \min\{\frac{p+\tau}{2}; \frac{d}{2s}\},$
2. $\beta = \frac{p}{2} + \max\{p; \frac{d}{s}\} - \frac{d}{2s} + 2\tau.$

For $s > \frac{d}{2}, p > 1,$ we have

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^\alpha (1+|\lambda|)^\beta} \lesssim \|V\|_{L^p}^p, \quad (4.2.4)$$

where

1. $\alpha = \frac{1}{2} + \frac{1}{2} \min\{p - \frac{d}{2s} + \tau; 1\},$
2. $\beta = p - \frac{d}{2s} + 1 + \frac{3\tau}{2} + \frac{1}{2} \max\{p - \frac{d}{2s} - 1; 0; \frac{3d}{2s} - p - 1\}.$

In the self-adjoint setting we refer to Theorem 2.1 in [FLS08]. In this case, $\sigma_d(H)$ lies on the negative real half-axis and $0 < s < \min\{1; \frac{d}{2}\}.$ In our notation Formula (5.11) from [FLS08] says

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^{p-\frac{d}{2s}} \leq C_{p,d} \|V_-\|_{L^p}^p, \quad (4.2.5)$$

where $p > \frac{d}{2s}$ and $V_- = \min\{V, 0\}.$ For $0 < s < \min\{1; \frac{d}{2}\},$ A simple computation proves that (4.2.1) and (5.1.1) are weaker than (4.2.5).

4.2.1 Conformal mapping for the fractional Laplacian

Let φ_a be a map sending \mathbb{D} to the resolvent set $\mathbb{C} \setminus \{\mathbb{R}^+\}$ of H_0 . It is defined for $a > 0$ by

$$\varphi_a : z \mapsto \lambda := -a \left(\frac{z+1}{z-1} \right)^2, \quad (4.2.6)$$

and the inverse map defined in $\bar{\mathbb{C}} \setminus \mathbb{R}^+$ is

$$\varphi_a^{-1} : \lambda \mapsto z := \frac{\sqrt{\lambda} - i\sqrt{a}}{\sqrt{\lambda} + i\sqrt{a}}.$$

Proposition 4.2.3 (Distortion between $\mathbb{C} \setminus \mathbb{R}^+$ and \mathbb{D}). *The distortion via φ_a behave the following way :*

$$a \cdot d(z, \mathbb{T}) \frac{|z+1|}{|z-1|^3} \leq d(\lambda, \mathbb{R}^+) \leq 8a \cdot d(z, \mathbb{T}) \frac{|1+z|}{|1-z|^3}, \quad (4.2.7)$$

and

$$\frac{\sqrt{a}}{4} \cdot \frac{d(\lambda, \mathbb{R}^+)}{\sqrt{|\lambda|(a+|\lambda|)}} \leq d(z, \mathbb{T}) \leq 4\sqrt{a} \frac{d(\lambda, \mathbb{R}^+)}{\sqrt{|\lambda|(a+|\lambda|)}}. \quad (4.2.8)$$

Proof. The first distortion directly follows from (2.3.2) to the function φ_a with, for $z \in \mathbb{D}$, $\varphi_a'(z) = 4a \frac{z+1}{(z-1)^3}$.

For the second, we have

$$|1+z| = \frac{2\sqrt{|\lambda|}}{|\sqrt{\lambda} + i\sqrt{a}|} \text{ and } |1-z| = \frac{2\sqrt{a}}{|\sqrt{\lambda} + i\sqrt{a}|}. \quad (4.2.9)$$

On the other hand $|\sqrt{\lambda} + i\sqrt{a}|^2 = |\lambda| + a + 2\sqrt{a} \operatorname{Im}(\sqrt{\lambda})$, and, since $\operatorname{Im}(\sqrt{\lambda}) \geq 0$, we obtain

$$|\lambda| + a \leq |\sqrt{\lambda} + i\sqrt{a}|^2 \leq \left(|\sqrt{\lambda}| + |\sqrt{a}| \right)^2 \leq 2(a + |\lambda|).$$

Going back to the inequalities (4.2.7), we obtain (4.2.8). \square

4.2.2 Upper bound for fractional Laplacian

From Proposition 2.4.1 we have

Proposition 4.2.4. *Let $V \in L^p(\mathbb{R}^d; \mathbb{C}^n)$, $p > \max\{1; \frac{d}{2s}\}$ and $\lambda \in \rho(H_0)$. Then $V(\lambda - H_0)^{-1} \in \mathcal{S}_p$, and*

$$\|V(\lambda - H_0)^{-1}\|_{\mathcal{S}_p}^p \leq (2\pi)^{-d} \|V\|_{L^p}^p \cdot \|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))}^p.$$

We are going to bound the expression $\|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p}$ appearing in the previous proposition. We recall that $s_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ for $d \geq 2$ and it is convenient to put $s_0 = 2$ for $d = 1$.

Proposition 4.2.5. Let $\lambda = \lambda_0 + i\lambda_1 \in \mathbb{C} \setminus \mathbb{R}^+$ and $p > \max\{1; \frac{d}{2s}\}$. We put in this proposition $\delta = \frac{d}{2s} - 1$.

For $0 < s \leq \frac{d}{2}$, we have

$$\|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p}^p \leq \frac{n^{\frac{p}{2}} s_{d-1}}{2s} \cdot M_1 \cdot \frac{|\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}}, \quad (4.2.10)$$

with $M_1 = \max\left\{K_2; \int_0^{+\infty} \frac{t^\delta dt}{(t^2+1)^{\frac{p}{2}}}\right\}$, and K_2 , defined in the proof, depends on d, p , and s .

And for $s > \frac{d}{2}$, we have

$$\|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p}^p \leq \frac{n^{\frac{p}{2}} s_{d-1}}{2s} \cdot \frac{N_1}{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}}}, \quad (4.2.11)$$

where $N_1 = \max\left\{\int_0^{+\infty} \frac{t^\delta}{(t^2+1)^{\frac{p}{2}}}; \int_0^1 t^\delta dt + 2 \int_0^{+\infty} \frac{dt}{(t^2+1)^{\frac{p}{2}}}\right\}$.

Proof. As for the Dirac operator, we start with

$$\begin{aligned} \|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))}^p &= n^{\frac{p}{2}} \|(\lambda - |\cdot|^{2s})^{-1}\|_{L^p(\mathbb{R}^d, \mathbb{C})}^p \\ &= n^{\frac{p}{2}} s_{d-1} I, \end{aligned}$$

with s_{d-1} known for $d \geq 2$, $s_0 = 2$ for $d = 1$ and where

$$I = \int_0^{+\infty} \frac{r^{d-1}}{|r^{2s} - \lambda|^p} dr = \int_0^{+\infty} \frac{r^{d-1}}{|(r^{2s} - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}}} dr. \quad (4.2.12)$$

We assume first that $\lambda_0 < 0$, that is $d(\lambda, \sigma(H_0)) = |\lambda|$. In (4.2.12), we use $(r^{2s} - \lambda_0)^2 \geq r^{4s} + \lambda_0^2$, and we make the change of variable $t = \frac{r^{2s}}{|\lambda|}$, so

$$\begin{aligned} I &\leq \int_0^{+\infty} \frac{r^{d-1}}{(r^{4s} + |\lambda|^2)^{\frac{p}{2}}} dr \\ &\leq \frac{1}{2s} \cdot \frac{|\lambda|^{\frac{d}{2s}}}{|\lambda|^p} \int_0^{+\infty} \frac{t^{\frac{d}{2s}-1}}{(t^2+1)^{\frac{p}{2}}} dt. \end{aligned} \quad (4.2.13)$$

The integral in (4.2.13) converges since $p > \frac{d}{2s} > 0$.

Now we assume $\lambda_0 \geq 0$ and $\lambda_1 > 0$ since $\|(\lambda - |\cdot|^{2s})^{-1}\| = \|(\bar{\lambda} - |\cdot|^{2s})^{-1}\|$. In (4.2.12), doing the change of variable $t = \frac{r^{2s} - \lambda_0}{\lambda_1}$, we obtain

$$\begin{aligned} I &= \frac{1}{2s\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d-1}{2s}} (\lambda_1 t + \lambda_0)^{\frac{1}{2s}-1}}{(t^2+1)^{\frac{p}{2}}} dt \\ &= \frac{1}{2s\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2+1)^{\frac{p}{2}}} dt. \end{aligned} \quad (4.2.14)$$

If $s = \frac{d}{2}$, the bound is obvious.

Now assume that $0 < s < \frac{d}{2}$. Since $\lambda_1 > 0$, $-\frac{\lambda_0}{\lambda_1} < 0$, we have

$$\int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt = \int_{-\frac{\lambda_0}{\lambda_1}}^0 \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt + \int_0^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt.$$

In the first integral of the right-hand side of the above equality, we use $\lambda_1 t + \lambda_0 \leq \lambda_0$, and in the second $(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1} \leq C_{d,s} \left((\lambda_1 t)^{\frac{d}{2s}-1} + \lambda_0^{\frac{d}{2s}-1} \right)$, with $C_{d,s} = \max\{1; 2^{\frac{d}{2s}-2}\}$. Hence, putting $\delta = \frac{d}{2s} - 1$, we have

$$\begin{aligned} I &\leq \frac{1}{2s\lambda_1^{p-1}} \left[\lambda_0^\delta \int_{-\frac{\lambda_0}{\lambda_1}}^0 \frac{1}{(t^2 + 1)^{\frac{p}{2}}} dt + \right. \\ &\quad \left. + C_{d,s} \lambda_1^\delta \int_0^{+\infty} \frac{t^\delta}{(t^2 + 1)^{\frac{p}{2}}} dt + C_{d,s} \lambda_0^\delta \int_0^{+\infty} \frac{1}{(t^2 + 1)^{\frac{p}{2}}} dt \right] \\ &\leq \frac{K_1}{2s\lambda_1^{p-1}} \cdot [\lambda_0^\delta + \lambda_1^\delta], \end{aligned}$$

with $K_1 = \max \left\{ (1 + C_{d,s}) \int_0^{+\infty} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}}; C_{d,s} \int_0^{+\infty} \frac{t^\delta dt}{(t^2 + 1)^{\frac{p}{2}}} \right\}$. Then, putting $C'_{d,s} = \max\{1; 2^{1-\delta}\}$, we have

$$\begin{aligned} I &\leq \frac{K_1}{2s\lambda_1^{p-1}} \cdot C'_{d,s} (\lambda_0 + \lambda_1)^\delta \\ &\leq \frac{K_1}{2s\lambda_1^{p-1}} \cdot C'_{d,s} (\sqrt{2})^\delta |\lambda|^\delta. \end{aligned}$$

Consequently, for $\lambda_0 \geq 0$, $I \leq \frac{K_2}{2s} \cdot \frac{|\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}}$ with $K_2 = K_1 \cdot C'_{d,s} (\sqrt{2})^\delta$.

This ends the case $s \leq \frac{d}{2}$.

Now we consider the case $s > \frac{d}{2}$ and we suppose again $\lambda_1 > 0$. From (4.2.14), we know that

$$I = \frac{1}{2s\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^\delta}{(t^2 + 1)^{\frac{p}{2}}} dt.$$

The difficulty here is that $-1 < \delta = \frac{d}{2s} - 1 < 0$. Making the change of variable $u = t + \frac{\lambda_0}{\lambda_1}$, we obtain

$$\begin{aligned} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^\delta}{(t^2 + 1)^{\frac{p}{2}}} dt &= \lambda_1^\delta \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(t + \frac{\lambda_0}{\lambda_1})^\delta}{(t^2 + 1)^{\frac{p}{2}}} dt \\ &= \lambda_1^\delta \int_0^{+\infty} \frac{u^\delta}{\left((u - \frac{\lambda_0}{\lambda_1})^2 + 1 \right)^{\frac{p}{2}}} du. \end{aligned}$$

We bound the last integral in the following way :

$$\begin{aligned} \int_0^{+\infty} \frac{u^\delta}{\left((u - \frac{\lambda_0}{\lambda_1})^2 + 1\right)^{\frac{p}{2}}} du &\leq \int_0^1 u^\delta du + \int_1^{+\infty} \frac{1}{\left((u - \frac{\lambda_0}{\lambda_1})^2 + 1\right)^{\frac{p}{2}}} du \\ &\leq \int_0^1 u^\delta du + \int_{\mathbb{R}} \frac{1}{(u^2 + 1)^{\frac{p}{2}}} du. \end{aligned}$$

Indeed, in the first inequality, we use $(u - \frac{\lambda_0}{\lambda_1})^2 + 1 \geq 1$ when $0 \leq u \leq 1$, and $u^\delta \leq 1$ when $u \geq 1$ (since $\delta < 0$). Hence, for $\lambda_0 \geq 0$, $I \leq \frac{K_3}{2s} \cdot \frac{\lambda_1^\delta}{\lambda_1^{p-1}}$, with $K_3 = \int_0^1 u^\delta du + 2 \int_0^{+\infty} \frac{du}{(u^2 + 1)^{\frac{p}{2}}}$. This finishes the proof. \square

Remark 4.2.6. When $0 < s \leq d/2$ we cannot factorize by λ_1 in the computation for $\lambda_0 \geq 0$, because we cannot bound the same way on $[1; +\infty[$.

4.2.3 Upper bound of the perturbed operator

To bound $\|(-a - H)^{-1}\|$ we may use exactly the same proof as for the perturbed Dirac operator (see Theorem 4.2.2). However, as for the Klein-Gordon operator, we can use the following bound of $\|(-a - H)^{-1}\|$ to obtain a better Lieb-Thirring-type inequality (Theorem 4.2.1). Again we refer to [Han10, Lemma 3.3.4] for a more general statement.

Proposition 4.2.7. *Let $p > \max\{1; \frac{d}{2s}\}$, there is $\omega \geq 1$, depending on n, d, p, s , and V , so that $-\omega \in \rho(H)$ and*

$$\forall a \geq \omega, \|(-a - H)^{-1}\| \leq \frac{C_\omega}{|\omega - a|}.$$

Proof. We use the following relation

$$\forall \lambda \in \rho(H_0), \lambda - H = (\text{Id} - V(\lambda - H_0)^{-1})(\lambda - H_0). \quad (4.2.15)$$

Since $\sigma(H_0) = \sigma_{\text{ess}}(H)$, we deduce that $\lambda \in \rho(H)$ if and only if $\text{Id} - V(\lambda - H_0)^{-1}$ is invertible. But we know that $\|V(\lambda - H_0)^{-1}\| < 1$ implies that $\text{Id} - V(\lambda - H_0)^{-1}$ is invertible hence $\|V(\lambda - H_0)^{-1}\| < 1$ implies $\lambda \in \rho(H)$.

Taking $\lambda = -a$, we show that, for all $a \geq \omega$ where $\omega \in \mathbb{R}^+$ is large enough, the inequality $\|V(-a - H_0)^{-1}\| < 1$ is satisfy. For that we use the Birman-Solomyak inequality (2.4.2). Indeed

$$\begin{aligned} \|V(-a - H_0)^{-1}\| &\leq \|V(-a - H_0)^{-1}\|_{\mathbf{s}_p} \\ &\leq (2\pi)^{-d/p} \|V\|_{L^p} \cdot \|(-a - |x|^{2s})^{-1}\|_{L^p}. \end{aligned}$$

Hence, by Proposition 4.2.5, we find

$$\|V(-a - H_0)^{-1}\| \leq K'(d, n, p, s) \|V\|_{L^p} \cdot a^{\frac{1}{p}(\frac{d}{2s} - p)}.$$

The right-hand side member in the above inequality tends to 0 when a tends to $+\infty$, hence there is $\omega \geq 1$, depending on n, d, p, s , and V , such that $\forall a > \omega$, $\|V(-a - H_0)^{-1}\| < 1$.

We show now that, for $a > \omega$, we have $\|(-a - H)^{-1}\| \leq \frac{C(\omega)}{|\omega - a|}$. From relation (4.2.15), we have for $\lambda \in \rho(H)$,

$$\begin{aligned} \|(\lambda - H)^{-1}\| &\leq \|(\lambda - H_0)^{-1}\| \cdot \|(\text{Id} - V(\lambda - H_0)^{-1})^{-1}\| \\ &\leq \frac{1}{d(\lambda, \sigma(H_0))} \cdot \frac{1}{1 - \|V(\lambda - H_0)^{-1}\|}. \end{aligned}$$

For $\lambda = -a$, $d(-a, \sigma(H_0)) = a \geq |\omega - a|$, and since $\lim_{a \rightarrow +\infty} \|V(-a - H_0)^{-1}\| = 0$, there is $\omega \geq 1$ so that for all $a > \omega$,

$$\|(-a - H)^{-1}\| \leq \frac{C_\omega}{a} \leq \frac{C_\omega}{|\omega - a|}.$$

□

Remark 4.2.8. See the proof of Lemma 3.3.4 in [Han10] for a more general setting. It proves the existence of ω such that in $R(\omega) = \{\lambda \in \rho(H_0), \text{Re}(\lambda) \leq -\omega\}$, we have $\forall \lambda \in R(\omega)$, $\|(\lambda - H)^{-1}\| \leq \frac{C_\omega}{|\text{Re}(\lambda) + \omega|}$.

4.2.4 Proof of Theorem 4.2.2

Remember from (2.2.3) that $f(\lambda) = \det_{[p]}(\text{Id} - F(\lambda))$, with

$$F(\lambda) := (\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1} \in \mathbf{S}_p, p \geq 1.$$

By properties of the regularized determinant we know that

$$\log(|f(\lambda)|) \leq \Gamma_p \|(\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1}\|_{\mathbf{S}_p}^p.$$

From Proposition 4.2.7, for $p > \max\{1; \frac{d}{2s}\}$ and $a \geq \omega$, we have $\|(a + H)^{-1}\| \leq \frac{C(\omega)}{|\omega - a|}$, and from Proposition 2.4.1 we obtain the next inequality

$$\log |f(\lambda)| \leq \frac{\Gamma_p}{(2\pi)^d} \cdot \frac{C_\omega^p}{|\omega - a|^p} \cdot |\lambda + a|^p \|V\|_{L^p}^p \|(\lambda - |x|^{2s})^{-1}\|_{L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))}^p, \quad (4.2.16)$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$.

Case $0 < s \leq \frac{d}{2}$. From (4.2.10), we have

$$\log |f(\lambda)| \leq \frac{K_1}{|\omega - a|^p} \cdot \|V\|_{L^p}^p \cdot \frac{|\lambda + a|^p |\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}}, \quad (4.2.17)$$

where we put $K_1 = \frac{\Gamma_p}{(2\pi)^d} \cdot C_\omega^p \cdot \frac{n^{p/2} s_{d-1}}{2s} \cdot M_1$ and M_1 is defined in (4.2.10).

We now transfer the above inequality back on \mathbb{D} in order to apply Theorem 2.3.1. That is, we consider the function $g(z) = f \circ \varphi_a(z)$, where φ_a is defined by (4.2.6); it is clearly holomorphic on \mathbb{D} . By definition (4.2.6) we have $|\lambda + a| = \frac{4a|z|}{|1 - z|^2}$. By the previous relation and Proposition 4.2.3 applied to the previous inequality we obtain

$$\begin{aligned} \log |g(z)| &\leq \frac{K_1}{|\omega - a|^p} \cdot \|V\|_{L^p}^p \cdot \frac{(4a)^p |z|^p a^{\frac{d}{2s}-1} |z + 1|^{\frac{d}{s}-2} |z - 1|^{3(p-1)}}{|z - 1|^{2p} |z - 1|^{\frac{d}{s}-2} a^{p-1} d(z, \mathbb{T})^{p-1} |1 + z|^{p-1}} \\ &\leq \frac{K_2 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p \cdot \frac{|z|^p}{d(z, \mathbb{T})^{p-1} |1 + z|^{p-\frac{d}{s}+1} |1 - z|^{\frac{d}{s}-p+1}}, \end{aligned}$$

with $K_2 = 4^p K_1$.

We apply then Theorem 2.3.1, hence, for all $\tau > 0$, we have

$$\sum_{g(z)=0} (1 - |z|)^{p+\tau} |z - 1|^{(\frac{d}{s}-p+\tau)_+} |z + 1|^{(p-\frac{d}{s}+\tau)_+} \lesssim \frac{K_2 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \quad (4.2.18)$$

with K_2 defined above.

Consider first the case $\frac{d}{2s} < p < \frac{d}{s}$. Hence, for $0 < \tau < \frac{d}{s} - p$, (4.2.18) becomes

$$\sum_{g(z)=0} (1 - |z|)^{p+\tau} |z - 1|^{\frac{d}{s}-p+\tau} \leq C \cdot \frac{K_2 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \quad (4.2.19)$$

where C depends on $d, p, s, \{-1; 1\}$, and τ .

In the second case, we have $p = \frac{d}{s}$. We obtain

$$\sum_{g(z)=0} (1 - |z|)^{p+\tau} |z - 1|^\tau |1 + z|^\tau \leq C \cdot \frac{K_2 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \quad (4.2.20)$$

where C depends on $d, p, s, \{-1; 1\}$, and τ .

The third and last case is $\frac{d}{s} < p$, and (4.2.18) becomes, for $0 < \tau < p - \frac{d}{s}$,

$$\sum_{g(z)=0} (1 - |z|)^{p+\tau} |z + 1|^{p-\frac{d}{s}+\tau} \leq C \cdot \frac{K_2 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \quad (4.2.21)$$

where C depends on $d, p, s, \{-1; 1\}$, and τ .

The next step of the proof consist in transferring relations (4.2.19), (4.2.20) and (4.2.21) back to $\rho(H_0) = \mathbb{C} \setminus \mathbb{R}^+$. Remind that we have, by Proposition 4.2.3,

$$\begin{aligned} 1 - |z| = d(z, \mathbb{T}) &\geq \frac{\sqrt{a}}{4} \cdot \frac{d(\lambda, \sigma(H_0))}{|\lambda|^{1/2}(a + |\lambda|)}, \\ |1 + z|^2 &\geq \frac{2|\lambda|}{a + |\lambda|}, \quad \text{and} \quad |1 - z|^2 \geq \frac{2a}{a + |\lambda|}. \end{aligned}$$

From equation (4.2.19) we get

$$(1 - |z|)^{p+\tau} |z - 1|^{\frac{d}{s} - p + \tau} \geq \frac{a^{\frac{d}{2s} + \tau}}{2^{\frac{5p}{2} + \frac{3\tau}{2} - \frac{d}{2s}}} \cdot \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}} (a + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + \frac{3\tau}{2}}},$$

so

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}} (a + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + \frac{3\tau}{2}}} \lesssim a^{-\tau} \cdot \frac{K_2}{|\omega - a|^p} \cdot 2^{\frac{5p}{2} + \frac{3\tau}{2} - \frac{d}{2s}} \cdot \|V\|_{L^p}^p.$$

From equation (4.2.20) we obtain

$$(1 - |z|)^{p+\tau} |z - 1|^\tau |z + 1|^\tau \geq \frac{a^{\frac{p}{2} + \tau}}{2^{2p+\tau}} \cdot \frac{d(\lambda, \sigma_d(H_0))^{p+\tau}}{|\lambda|^{\frac{p}{2}} (a + |\lambda|)^{p+2\tau}},$$

so

$$\sum_{\lambda \in \sigma(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p}{2}} (a + |\lambda|)^{p+2\tau}} \lesssim a^{-\tau} \cdot \frac{K_2}{|\omega - a|^p} \cdot 2^{2p+\tau} \cdot \|V\|_{L^p}^p.$$

Finally from equation (4.2.21) we get

$$(1 - |z|)^{p+\tau} |z + 1|^{p - \frac{d}{s} + \tau} \geq \frac{a^{\frac{p+\tau}{2}}}{2^{\frac{3}{2}(p+\tau) + \frac{d}{2s}}} \cdot \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{d}{2s}} (a + |\lambda|)^{\frac{3p}{2} - \frac{d}{2s} + \frac{3\tau}{2}}},$$

so

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{d}{2s}} (a + |\lambda|)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} \lesssim a^{\frac{d}{2s} - \frac{p+\tau}{2}} \cdot \frac{K_2}{|\omega - a|^p} \cdot 2^{\frac{3}{2}(p+\tau) + \frac{d}{2s}} \cdot \|V\|_{L^p}^p.$$

We can now proceed to the integration with respect to $a \in [\omega; +\infty[$ (see Remark 3.6.9).

We start from the above cases. We precise that the interchanging of the sum and the integral is valid thank to Fubini Theorem, since the functions are positive.

If $\frac{d}{2s} < p < \frac{d}{s}$ then,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}}} \int_{\omega}^{+\infty} \frac{|\omega - a|^p a^{-1}}{(a + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + \frac{3\tau}{2}}} da \lesssim \int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

On one hand, $\int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} = \frac{1}{\tau \omega^\tau}$, and in the left-hand side member, we make the change

of variable $t = \frac{a - \omega}{|\lambda| + \omega}$ hence, since $a \leq a + |\lambda|$,

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{|\omega - a|^p a^{-1}}{(a + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + \frac{3\tau}{2}}} da &\geq \int_{\omega}^{+\infty} \frac{|\omega - a|^p}{(a + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + 1 + \frac{3\tau}{2}}} da \\ &\geq \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{\frac{d}{2s} + \frac{p}{2} + 1 + \frac{3\tau}{2}}} \int_0^{+\infty} \frac{t^p dt}{(t + 1)^{\frac{d}{2s} + \frac{p}{2} + 1 + \frac{3\tau}{2}}}. \end{aligned}$$

So, for $\frac{d}{2s} < p < \frac{d}{s}$,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p+\tau}{2}} (\omega + |\lambda|)^{\frac{d}{2s} - \frac{p}{2} + \frac{3\tau}{2}}} \lesssim \frac{K_2}{I_1} \cdot \frac{2^{\frac{5p}{2} + \frac{3\tau}{2} - \frac{d}{2s}}}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.22)$$

$$\text{with } I_1 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{\frac{d}{2s} + \frac{p}{2} + 1 + \frac{3\tau}{2}}}.$$

If $p = \frac{d}{s}$, the proof is the same, and we find

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{p}{2}} (\omega + |\lambda|)^{2\tau}} \lesssim \frac{K_2}{I_2} \cdot \frac{2^{2p+\tau}}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.23)$$

$$\text{with } I_2 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{p+1+2\tau}}.$$

If $p > \frac{d}{s}$ then,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{d}{2s}}} \int_\omega^{+\infty} \frac{|\omega - a|^p a^{\frac{p+\tau}{2} - \frac{d}{2s} - 1 - \tau}}{(a + |\lambda|)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} da \lesssim \int_\omega^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

As previously, we do the change of variable $t = \frac{a - \omega}{|\lambda| + \omega}$ so, if $\frac{1}{2}(p - \frac{d}{s} - 2 - \tau) \geq 0$ then we bound from below $(|\lambda| + \omega)t + \omega$ by $(|\lambda| + \omega)t$, and if $\frac{1}{2}(p - \frac{d}{s} - 2 - \tau) < 0$ we bound from above $(|\lambda| + \omega)t + \omega$ by $(|\lambda| + \omega)(t + 1)$. We present the case $\frac{1}{2}(p - \frac{d}{s} - 2 - \tau) \geq 0$, the second being the same as above.

$$\begin{aligned} & \int_\omega^{+\infty} \frac{|\omega - a|^p a^{\frac{p}{2} - \frac{d}{2s} - 1 - \frac{\tau}{2}}}{(a + |\lambda|)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} da = \\ &= \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} \int_0^{+\infty} \frac{t^p [(|\lambda| + \omega)t + \omega]^{\frac{p}{2} - \frac{d}{2s} - 1 - \frac{\tau}{2}}}{(t+1)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} dt \\ &\geq \frac{(|\lambda| + \omega)^{\frac{p}{2} - \frac{d}{2s} - 1 - \frac{\tau}{2}}}{(|\lambda| + \omega)^{\frac{p}{2} - \frac{d}{2s} - 1 + \frac{3\tau}{2}}} \int_0^{+\infty} \frac{t^{\frac{3p}{2} - \frac{d}{2s} - 1 - \frac{\tau}{2}}}{(t+1)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} dt, \end{aligned}$$

hence, when $\frac{1}{2}(p - \frac{d}{s} - 2 - \tau) > 0$, we obtain

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{d}{2s}} (\omega + |\lambda|)^{2\tau}} \lesssim \frac{K_2}{I_3} \cdot \frac{2^{\frac{3}{2}(p+\tau) + \frac{d}{2s}}}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.24)$$

with $I_3 = \int_0^{+\infty} \frac{t^{\frac{3p}{2} - \frac{d}{2s} - 1 - \frac{\tau}{2}}}{(t+1)^{\frac{3}{2}(p+\tau) - \frac{d}{2s}}} dt$. When $\frac{1}{2}(p - \frac{d}{s} - 2 - \tau) < 0$, we obtain

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\frac{d}{2s}} (\omega + |\lambda|)^{2\tau}} \lesssim \frac{K_2}{I_4} \cdot \frac{2^{\frac{3}{2}(p+\tau) + \frac{d}{2s}}}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.25)$$

where we put $I_4 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{p+1+2\tau}}$.

Summarizing (4.2.22) to (4.2.25), we find (4.2.3). This ends the case $0 < s \leq \frac{d}{2}$.

Case $s > \frac{d}{2}$.

Using (4.2.11) and Proposition 4.2.7, the inequality (4.2.16) becomes

$$\log |f(\lambda)| \leq \frac{K_4}{|\omega - a|^p} \cdot \|V\|_{L^p}^p \cdot \frac{|\lambda + a|^p}{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}}},$$

where we put $K_4 = \frac{\Gamma_p}{(2\pi)^d} \cdot C_\omega^p \cdot \frac{n^{p/2} s_{d-1}}{2s} \cdot N_1$, with N_1 depending only on d, n, p , and s . Then, as before, we have

$$\log |g(z)| \leq \frac{K_4}{|\omega - a|^p} \cdot \|V\|_{L^p}^p \cdot \frac{4^p a^{\frac{d}{2s}} |z|^p}{d(z, \mathbb{T})^{p-\frac{d}{2s}} |1 - z|^{\frac{3d}{2s}-p} |1 + z|^{p-\frac{d}{2s}}}.$$

We put now $K_5 = 4^p K_4$. Applying Theorem 2.3.1, we have

$$\sum_{g(z)=0} (1 - |z|)^{p-\frac{d}{2s}+1+\tau} |z - 1|^{(\frac{3d}{2s}-p-1+\tau)_+} |z + 1|^{(p-\frac{d}{2s}-1+\tau)_+} \lesssim \frac{K_5 a^{\frac{d}{2s}}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p.$$

The possible cases with respect to p and $\frac{d}{2s}$ are clear from the following drawing (Figure 4.1). The x-axis represents p and the y-axis represents $\frac{d}{2s}$. There are 4 lines of the equations $y = 1$, $x - y - 1 = 0$, $-x + 3y - 1 = 0$, and $x - 3y - 1 = 0$. This last equation will be useful during the integration of the inequalities.

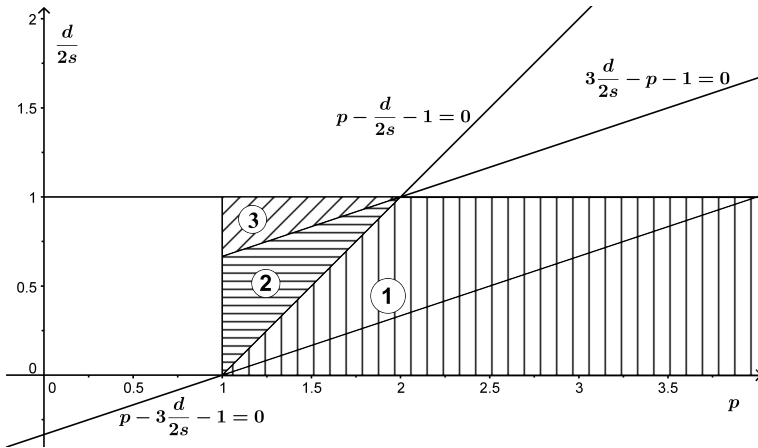


Figure 4.1: The different cases for $s > \frac{d}{2}$.

Consequently, there are three cases to consider :

case 1. The part 1 : $p - \frac{d}{2s} - 1 \geq 0$ and $\frac{3d}{2s} - p - 1 < 0$.

case 2. The part 2: $p - \frac{d}{2s} - 1 < 0$ and $\frac{3d}{2s} - p - 1 < 0$.

case 3. The part 3: $p - \frac{d}{2s} - 1 < 0$ and $\frac{3d}{2s} - p - 1 \geq 0$.

The computations are similar to the case $s \leq \frac{d}{2}$. We recall that

$$1 - |z| = d(z, \mathbb{T}) \geq \frac{\sqrt{a}}{4} \cdot \frac{d(\lambda, \sigma(H_0))}{|\lambda|^{1/2}(a + |\lambda|)},$$

$$|1 + z|^2 \geq \frac{2|\lambda|}{a + |\lambda|}, \quad \text{and} \quad |1 - z|^2 \geq \frac{2a}{a + |\lambda|}.$$

In the first case, we have

$$\begin{aligned} d(z, \mathbb{T})^{p+1-\frac{d}{2s}+\tau} |z + 1|^{p-\frac{d}{2s}-1+\tau} &\geq \left(\frac{\sqrt{a}}{4}\right)^{p+1-\frac{d}{2s}+\tau} \cdot 2^{\frac{1}{2}(p-\frac{d}{2s}-1+\tau)} \times \\ &\times \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau} |\lambda|^{\frac{p-1+\tau}{2}-\frac{d}{4s}}}{|\lambda|^{\frac{p+1+\tau}{2}-\frac{d}{4s}} (a + |\lambda|)^{p+1-\frac{d}{2s}+\tau+\frac{p-1+\tau}{2}-\frac{d}{4s}}} \\ &\geq \frac{a^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)}}{2^{\frac{3}{2}(p+\tau)+\frac{5}{2}-\frac{3d}{4s}}} \cdot \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|(a + |\lambda|)^{3\frac{p+\tau}{2}+\frac{1}{2}-\frac{3d}{4s}}}, \end{aligned}$$

so

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|(a + |\lambda|)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} \lesssim \frac{a^{-\frac{1}{2}(p+1-\frac{3d}{2s}+\tau)}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p. \quad (4.2.26)$$

In the second case, we have

$$d(z, \mathbb{T})^{p+1-\frac{d}{2s}+\tau} \geq \left(\frac{\sqrt{a}}{4}\right)^{p+1-\frac{d}{2s}+\tau} \cdot \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|^{\frac{p+1+\tau}{2}-\frac{d}{4s}} (a + |\lambda|)^{p+1-\frac{d}{2s}+\tau}},$$

so

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)} (a + |\lambda|)^{p+1-\frac{d}{2s}+\tau}} \lesssim \frac{a^{-\frac{1}{2}(p+1-\frac{3d}{2s}+\tau)}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p. \quad (4.2.27)$$

In the last case, we have

$$\begin{aligned} d(z, \mathbb{T})^{p+1-\frac{d}{2s}+\tau} |z - 1|^{\frac{3d}{2s}-p-1+\tau} &\geq \\ &\geq \left(\frac{\sqrt{a}}{4}\right)^{p+1-\frac{d}{2s}+\tau} (\sqrt{2a})^{\frac{3d}{2s}-p-1+\tau} \cdot \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|^{\frac{p+1+\tau}{2}-\frac{d}{4s}} (a + |\lambda|)^{p+1-\frac{d}{2s}+\tau+\frac{3d}{4s}-\frac{p+1-\tau}{2}}} \\ &\geq \frac{a^{\frac{d}{2s}+\tau}}{2^{\frac{5}{2}(p+1)-\frac{7d}{4s}+\frac{3\tau}{2}}} \cdot \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)} (a + |\lambda|)^{\frac{p+1}{2}+\frac{d}{4s}+\frac{3\tau}{2}}} \end{aligned}$$

so

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^{\frac{p+1+\tau}{2}-\frac{d}{4s}}(a+|\lambda|)^{\frac{p+1}{2}+\frac{d}{4s}+\frac{3\tau}{2}}} \lesssim \frac{a^{-\tau}}{|\omega-a|^p} \cdot \|V\|_{L^p}^p. \quad (4.2.28)$$

This give the second part of Theorem 4.2.2.

It remains to integrate with respect to a on $[\omega; +\infty[$. We do it the same way as $0 < s \leq \frac{d}{2}$.

In the first case : $p - \frac{d}{2s} - 1 \geq 0$ and $\frac{3d}{2s} - p - 1 < 0$, we obtain from (4.2.26)

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|} \int_{\omega}^{+\infty} \frac{|\omega-a|^p a^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a+|\lambda|)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} da \lesssim \int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

On one hand $\int_{\omega}^{+\infty} \frac{da}{a^{1+\tau}} = \frac{1}{\tau\omega^\tau}$, and in the left-hand side member, if $p - \frac{3d}{2s} - 1 > 0$ we make the change of variable $t = \frac{a-\omega}{|\lambda|+\omega}$, and if $p - \frac{3d}{2s} - 1 \leq 0$ we bound first from above a by $a+|\lambda|$, (see the figure 4.1 for the draw of the line $p - \frac{3d}{2s} - 1 = 0$) hence, if $p - \frac{3d}{2s} - 1 > 0$, for τ small enough,

$$\begin{aligned} & \int_{\omega}^{+\infty} \frac{|\omega-a|^p a^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a+|\lambda|)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} da = \\ &= \frac{(|\lambda|+\omega)^{p+1}}{(|\lambda|+\omega)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} \int_0^{+\infty} \frac{t^p [(|\lambda|+\omega)t + \omega]^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(t+1)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} dt \\ &\geq \frac{(|\lambda|+\omega)^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(|\lambda|+\omega)^{\frac{p}{2}-\frac{3d}{4s}-\frac{1}{2}+\frac{3\tau}{2}}} \int_0^{+\infty} \frac{t^{\frac{1}{2}(3p-\frac{3d}{2s}-1-\tau)}}{(t+1)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} dt. \end{aligned}$$

Hence, if $p - \frac{3d}{2s} - 1 > 0$,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|(\omega+|\lambda|)^{2\tau}} \lesssim \frac{K_5}{I_5} \cdot \frac{2^{\frac{1}{2}(3(p+\tau)+5-\frac{3d}{2s})}}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.29)$$

with $I_5 = \int_0^{+\infty} \frac{t^{\frac{1}{2}(3p-\frac{3d}{2s}-1-\tau)}}{(t+1)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} dt$. And if $p - \frac{3d}{2s} - 1 \leq 0$,

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{|\omega-a|^p a^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a+|\lambda|)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} da &\geq \int_{\omega}^{+\infty} \frac{|\omega-a|^p (a+|\lambda|)^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a+|\lambda|)^{3\frac{p+\tau}{2}-\frac{3d}{4s}+\frac{1}{2}}} da \\ &\geq \frac{(|\lambda|+\omega)^{p+1}}{(|\lambda|+\omega)^{p+1+2\tau}} \int_0^{+\infty} \frac{t^p dt}{(t+1)^{p+1+2\tau}}. \end{aligned}$$

Hence, if $p - \frac{3d}{2s} - 1 \leq 0$,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|(\omega + |\lambda|)^{2\tau}} \lesssim \frac{K_5}{I_6} \cdot \frac{2^{\frac{1}{2}(3(p+\tau)+5-\frac{3d}{2s})}}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.30)$$

with $I_6 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{p+1+2\tau}}$.

In the second case, we have $p - \frac{d}{2s} - 1 < 0$ and $\frac{3d}{2s} - p - 1 < 0$. We integrate the inequality (4.2.27), which gives

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)}} \int_\omega^{+\infty} \frac{|\omega - a|^p a^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a + |\lambda|)^{p+1-\frac{d}{2s}+\tau}} da \lesssim \int_\omega^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

We have $p - \frac{3d}{2s} - 1 \leq 0$, hence we bound from above a by $a + |\lambda|$ and then with the same change of variable, we find

$$\begin{aligned} \int_\omega^{+\infty} \frac{|\omega - a|^p a^{\frac{1}{2}(p-\frac{3d}{2s}-1-\tau)}}{(a + |\lambda|)^{p+1-\frac{d}{2s}+\tau}} da &\geq \int_\omega^{+\infty} \frac{|\omega - a|^p}{(a + |\lambda|)^{\frac{1}{2}(p+\frac{d}{2s}+3+3\tau)}} da \\ &\geq \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{\frac{1}{2}(p+\frac{d}{2s}+3+3\tau)}} \int_{\mathbb{R}^+} \frac{t^p dt}{(t+1)^{\frac{1}{2}(p+\frac{d}{2s}+3+3\tau)}} \\ &\geq \frac{1}{(|\lambda| + \omega)^{\frac{1}{2}(\frac{d}{2s}-p+1+3\tau)}} \int_{\mathbb{R}^+} \frac{t^p dt}{(t+1)^{\frac{1}{2}(p+\frac{d}{2s}+3+3\tau)}}, \end{aligned}$$

and we know that $\frac{d}{2s} - p + 1 > 0$ by hypothesis hence

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)} (\omega + |\lambda|)^{\frac{1}{2}(\frac{d}{2s}-p+1+3\tau)}} \lesssim \frac{K_5}{I_7} \cdot \frac{2^{2(p+1-\frac{d}{2s}+\tau)}}{\tau\omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.31)$$

with $I_7 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{\frac{1}{2}(p+\frac{d}{2s}+3+3\tau)}}$. We recall here that $0 < p - \frac{d}{2s} < 1$, hence $\frac{d}{2s} - p + 1 > 0$.

Finally the last case : $p - \frac{d}{2s} - 1 < 0$ and $\frac{3d}{2s} - p - 1 \geq 0$. Integrating the inequality (4.2.28) we find

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}+1+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)}} \int_\omega^{+\infty} \frac{|\omega - a|^p a^{-1}}{(a + |\lambda|)^{\frac{p+1}{2} + \frac{d}{4s} + \frac{3\tau}{2}}} da \lesssim \int_\omega^{+\infty} \frac{da}{a^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

Bounding from above a by $a + |\lambda|$, and then with the change of variable, we obtain

$$\begin{aligned} \int_\omega^{+\infty} \frac{|\omega - a|^p a^{-1}}{(a + |\lambda|)^{\frac{p+1}{2} + \frac{d}{4s} + \frac{3\tau}{2}}} da &\geq \int_\omega^{+\infty} \frac{|\omega - a|^p}{(a + |\lambda|)^{\frac{1}{2}(p+3+\frac{d}{2s}+3\tau)}} da \\ &\geq \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{\frac{1}{2}(p+3+\frac{d}{2s}+3\tau)}} \int_0^{+\infty} \frac{t^p dt}{(t+1)^{\frac{1}{2}(p+3+\frac{d}{2s}+3\tau)}}. \end{aligned}$$

So

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p+1-\frac{d}{2s}+\tau}}{|\lambda|^{\frac{1}{2}(p+1-\frac{d}{2s}+\tau)}(\omega + |\lambda|)^{\frac{1}{2}(\frac{d}{2s}-p+1+3\tau)}} \lesssim \frac{K_5}{I_8} \cdot \frac{2^{\frac{1}{2}(5(p+1)-\frac{7d}{2s}+3\tau)}}{\tau \omega^\tau} \cdot \|V\|_{L^p}^p, \quad (4.2.32)$$

with $I_8 = \int_0^{+\infty} \frac{t^p dt}{(t+1)^{\frac{1}{2}(p+3+\frac{d}{2s}+3\tau)}}$. As before $0 < p - \frac{d}{2s} < 1$, hence $\frac{d}{2s} - p + 1 > 0$.

To clarify the statement of the theorem, in the case 2., we bound from below $\frac{1}{(1+|\lambda|)^\tau}$

by $\frac{1}{(1+|\lambda|)^{\frac{3\tau}{2}}}$, so this gives β .

The proof of Theorem 4.2.1 is finished. \square

5 Lieb-Thirring-type Inequalities with Hansmann's results

Table of Contents

5.1	Hansmann's Theorem and results	69
5.1.1	Hansmann's Theorem	69
5.1.2	Results and comparisons	70
5.1.3	Distortion for resolvent	72
5.2	Proof for the fractional Laplacian	74
5.3	Proof for the Klein-Gordon operator	76

5.1 Hansmann's Theorem and results

5.1.1 Hansmann's Theorem

The following theorem is the main ingredient of the proofs of Theorems 5.2.2 and 5.3.1. It is proved in the article [Han11].

Theorem 5.1.1. *Let A be a normal bounded operator and B be a bounded operator such that $B - A \in \mathbf{S}_p$ for some $p \geq 1$. We assume further that $\sigma(A)$ is convex, then*

$$\sum_{\lambda \in \sigma_d(B)} d(\lambda, \sigma(A))^p \leq \|B - A\|_{\mathbf{S}_p}^p.$$

In [Han13, Cor. 1], the author generalize the previous theorem in the following way.

Theorem 5.1.2 (Hansmann). *Let A be a bounded self-adjoint operator and B be a bounded operator such that $B - A \in \mathbf{S}_p$ for some $p > 1$. Then the following inequality holds :*

$$\sum_{\lambda \in \sigma_d(B)} d(\lambda, \sigma(A))^p \leq C_p \|B - A\|_{\mathbf{S}_p}^p,$$

where C_p depends only on p .

The constant C_p can be describe in the following form : we have

$$C_p = \begin{cases} 12^{2-p} \Gamma_p & \text{if } 1 < p < 2 \\ 2 & \text{if } p = 2 \\ 2^{p/2-1} 4^{p-2} \Gamma_p & \text{if } p > 2 \end{cases},$$

where Γ_p is a constant given before Theorem 1 in [Han13], and satisfy $\Gamma_p \geq 2$ pour tout $p > 1$.

We notice first that the first theorem is valid for $p \geq 1$ and the second consider $p > 1$, Also the constant C_p in Theorem 5.1.2 is greater than 1, hence, in the case where both theorems apply (for instance when the spectrum of the unperturbed operator is convex), Theorem 5.1.1 is more interesting. For more details, we refer to the articles [Han11], [Han13] and [DHK13].

in [Han11], the author applied Theorem 5.1.1 to the Schrödinger and Jacobi operators, and in [Han13], to the case of a unitary operator, but also on zero-sets of Cauchy transforms of a finite, complex Borel measure on the unit circle.

Theorem 5.1.2 was used by Golinskii and Kupin in [GK13] for a perturbation of a self-adjoint Schrödinger operator, and by Sambou in [Sam14] for magnetic Schrödinger operator.

More generally, we saw how to bound the resolvent norm of the perturbed operator (see Proposition 4.2.7) when the unperturbed operator is bounded from below. We can then use Theorem 5.3.3 in [DHK13] who use an integration with respect to the parameter a , useful here to define the resolvent of the operators, and give better result than the simple application of Theorems 5.1.1 and 5.1.2 alone. In the next results the only constants we did not compute are ω and C_ω .

5.1.2 Results and comparisons

We saw in Proposition 4.2.7 the existence of $\omega \geq 1$, depending on n, d, p, s , and V , such that

$$\forall a > \omega, \|(-a - H)^{-1}\| \leq \frac{C_\omega}{|\omega - a|}.$$

In this case, the obtained result for the fractional Schrödinger operator is

Theorem 5.1.3. *Let H be the fractional Schrödinger operator defined in (4.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > \max\{1; \frac{d}{2s}\}$ with $s > 0$.*

Then, for $\tau > 0$, the following inequality holds

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(1 + |\lambda|)^{\frac{d}{2s} + \tau}} \leq \frac{K_2}{I} \cdot \frac{\omega^{\frac{d}{2s}} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p, \quad (5.1.1)$$

with

1. $K_2 = (2\sqrt{5})^p \cdot \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot M_1$ if $0 < s \leq \frac{d}{2}$,
2. $K_2 = (2\sqrt{5})^p \cdot \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot N_1$ if $s > \frac{d}{2}$,
3. and $I = \int_0^{+\infty} \frac{t^{p+(p-\frac{d}{2s}-1-\tau)_+} dt}{(t+1)^{p+\frac{d}{2s}+1+\tau+(p-\frac{d}{2s}-1-\tau)_+}}$,

where M_1 and N_1 are defined in Proposition 4.2.5.

To compare Theorems 4.2.1 and 5.1.3, consider a sequence of eigenvalues $(\lambda_n)_{\mathbb{N}}$, $\lambda_n \in \sigma_d(H)$, converging to $\lambda \in \sigma_{\text{ess}}(H) = \mathbb{R}^+$.

Without loss of generality, we assume $d(\lambda, \sigma(H_0)) \leq 1$.

We start with the simplest case $\lambda \in]0; +\infty[$. In this case, (5.1.1) is better than (4.2.1) or (4.2.2).

The case $\lambda = \infty$ is simple too. We distinguish the different cases with respect to p and $\frac{d}{s}$, and after computation the term in (4.2.1) becomes $\frac{d(\lambda_n, \sigma(H_0))^{p+\tau}}{|\lambda_n|^{\frac{d}{2s}+2\tau}}$ and the term in (5.1.1) becomes $\frac{d(\lambda_n, \sigma(H_0))^p}{|\lambda_n|^{\frac{d}{2s}+\tau}}$. We have $\frac{d(\lambda_n, \sigma(H_0))^\tau}{|\lambda_n|^\tau} \leq 1$. Hence (5.1.1) gives a better estimate than (4.2.1). The same reasoning is valid to compare (5.1.1) and (4.2.2), and give the same conclusion.

The “integration” method introduced by Demuth, Hansmann, and Katriel is useful here since, without it, the inequality (4.2.1) give a better estimate at ∞ than (5.2.1).

When $\lambda = 0$, the situation is more delicate: If $d(\lambda_n, \sigma(H_0)) \approx |\lambda_n|$, *i.e.*, $\text{Re}(\lambda_n) \leq 0$ or $0 \leq \text{Re}(\lambda_n) \leq C|\text{Im}(\lambda_n)|$ with $C > 0$; then (4.2.1) give a better result than (5.1.3). Else, when $d(\lambda_n, \sigma(H_0)) = |\text{Im}(\lambda)|$ and $\text{Re}(\lambda_n) \geq |\text{Im}(\lambda_n)|$, there are two cases. Indeed, the

term in (4.2.1) becomes $\frac{|\text{Im}(\lambda)|^{p+\tau}}{|\lambda_n|^{\min\{(p+\tau)/2; d/2s\}}}$ and the term in (5.1.3) becomes $|\text{Im}(\lambda)|^p$.

Hence the quotient is $\frac{|\lambda_n|^{\min\{(p+\tau)/2; d/2s\}}}{|\text{Im}(\lambda)|^\tau} \approx \frac{\text{Re}(\lambda_n)^{\min\{(p+\tau)/2; d/2s\}}}{|\text{Im}(\lambda)|^\tau}$. The case to distinguish are then clear. The same hold with (4.2.2). Finally when $\lambda = 0$, (4.2.1) or (4.2.2) have a small advantage over (5.1.1).

In the same way for the Klein-Gordon operator, there is $\omega \geq 1$, depending on n, d, p, m et V , so that

$$\forall a \geq \omega, \|(-a - K)^{-1}\| \leq \frac{C_\omega}{|\omega - a|}. \quad (5.1.2)$$

Hence we have

Theorem 5.1.4. *Let $K = K_m + V$ be the perturbed Klein-Gordon operator $m > 0$, defined in 3.6.1 and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then, for $\tau > 0$, we have the following inequality*

$$\sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^p}{(1 + |\lambda|)^{d+\tau}} \leq \frac{K_3 \cdot (\omega + m)^d}{I \cdot \tau} \cdot \|V\|_{L^p}^p, \quad (5.1.3)$$

with $K_3 = (2\sqrt{5})^p \cdot \frac{n^{p/2} s_{d-1}}{(2\pi)^d} \cdot C_\omega^p \cdot 2^\delta M$, where

1. $\delta = 2d - 3$ and $M = \max\{M_1; M_2\}$ if $d \geq 2$ (see Proposition 3.3.6),
2. else $\delta = 1$ and M is defined in Proposition 3.4.1 if $d = 1$,

$$\text{and } I = \int_0^{+\infty} \frac{t^{2p-d}}{(t+1)^{2p+1+\tau}} dt.$$

We compare the results (3.6.2) obtained with Borichev-Golsinkii-Kupin Theorem to the result (5.1.3) obtained with Hansmann Theorem.

For this we consider a sequence $(\lambda_n)_\mathbb{N} \subset \sigma_d(K)$ which converges to $\lambda \in \sigma_{\text{ess}}(K)$. Without loss of generalities, we assume $d(\lambda_n, \sigma(K_m)) \leq 1$.

The simpler case is $\lambda \in]m; +\infty[$. In this case, (5.1.3) is better than (3.6.2). The case $\lambda = \infty$ is also easy since, in this case, the term in (3.6.2) becomes $\frac{d(\lambda_n, \sigma(K_m))^{p+\tau}}{|\lambda_n|^{d+2\tau}}$ and the term in (5.1.3) becomes $\frac{d(\lambda_n, \sigma(K_m))^p}{|\lambda_n|^{d+\tau}}$. We have $\frac{d(\lambda_n, \sigma(K_m))^\tau}{|\lambda_n|^\tau} \leq 1$. Hence (5.1.3) is better than (3.6.2). The integration method introduced by Demuth, Hansmann, and Katriel is relevant here because, without it, the inequality (3.6.4) give a better estimate at ∞ than (5.3.1).

When $\lambda = m$, the situation is more complicated: if $d(\lambda_n, \sigma(K_m)) \approx |\lambda_n - m|$, that is, $\text{Re}(\lambda_n) \leq m$ or $0 \leq \text{Re}(\lambda_n) - m \leq C|\text{Im}(\lambda_n)|$ with $C > 0$; then (3.6.2) gives a better result than (5.1.3). Else, when $d(\lambda_n, \sigma(K_m)) = |\text{Im}(\lambda)|$ and $\text{Re}(\lambda_n) - m \geq |\text{Im}(\lambda_n)|$, there are two cases. Indeed, the term in (3.6.2) becomes $\frac{|\text{Im}(\lambda)|^{p+\tau}}{|\lambda_n - m|}$ and the term in (5.1.3) becomes $|\text{Im}(\lambda)|^p$. Hence the quotient gives $\frac{|\lambda_n - m|}{|\text{Im}(\lambda)|^\tau} \approx \frac{\text{Re}(\lambda_n) - m}{|\text{Im}(\lambda)|^\tau}$. The two cases are then clear.

5.1.3 Distortion for resolvent

We will apply the Theorem 5.1.1 to the resolvent of the operators (since they are unbounded). To compute the distortion between the spectrum of the operator and the one of its resolvent, we need to know the distortions of the mapping $\lambda \mapsto \frac{1}{\lambda}$ between $\mathbb{C} \setminus [a; +\infty[$ and $\mathbb{C} \setminus [0; \frac{1}{a}]$. This is the content of the next proposition.

Proposition 5.1.5. *Let $a > 0$. The distortion between $\mathbb{C} \setminus [0; \frac{1}{a}]$ and $\mathbb{C} \setminus \mathbb{R}^+$ via the inverse function is given by*

$$d\left(\frac{1}{\lambda}, \left[0; \frac{1}{a}\right]\right) \geq \frac{1}{\sqrt{5}} \cdot \frac{d(\lambda, [a; +\infty[)}{|\lambda| \cdot (a + |\lambda|)}. \quad (5.1.4)$$

Proof. We detail the computation. For this, we distinguish four parts denoted by 1., 2., 3., and 4. (see figure 5.1). We denote $I_z = [0; \frac{1}{a}]$ and $I_\lambda = [a; +\infty[$.

When $\lambda \in 1.$, we have $d(\lambda, I_\lambda) = |\lambda - a|$ and $d(z, I_z) = |z| = \frac{1}{|\lambda|} = \frac{|\lambda - a|}{|\lambda| \cdot |\lambda - a|}$. Since $|\lambda - a| \leq a + |\lambda|$, we deduce that

$$\text{for } \lambda, z \in 1., d(z, I_z) \geq \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

Similarly, when $\lambda \in 2.$, we have $d(\lambda, I_\lambda) = |\text{Im}(\lambda)|$ and $d(z, I_z) = |\text{Im}(z)| = \frac{|\text{Im}(\lambda)|}{|\lambda|^2}$, and since $|\lambda| \leq a + |\lambda|$, we find

$$\text{for } \lambda, z \in 2., d(z, I_z) \geq \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

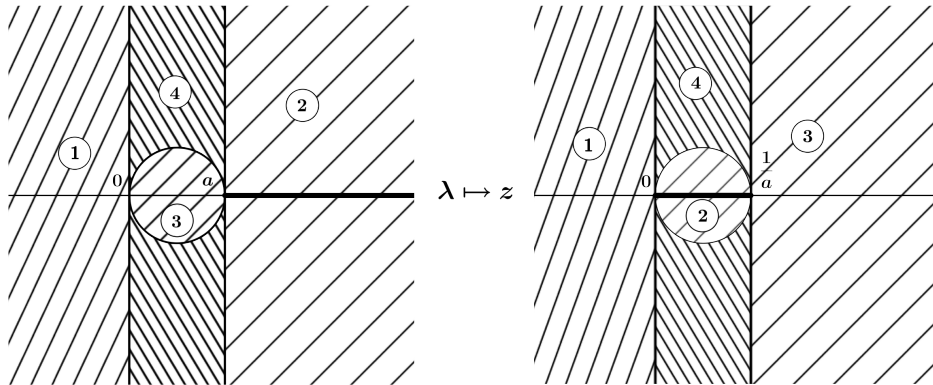


Figure 5.1: The different parts for the inverse map

In the same way, when $\lambda \in 3.$, we have $d(\lambda, I_\lambda) = |\lambda - a|$ et $d(z, I_z) = \left| z - \frac{1}{a} \right| = \left| \frac{a - \lambda}{a\lambda} \right| = \frac{|\lambda - a|}{a|\lambda|}$. But $a \leq a + |\lambda|$ hence

$$\text{for } \lambda, z \in 3., d(z, I_z) \geq \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

For $\lambda \in 4.$, we always have $0 \leq \text{Re}(\lambda) \leq a$. So we always have $d(\lambda, I_\lambda) = |\lambda - a|$ and $d(z, I_z) = |\text{Im}(z)| = \frac{|\text{Im}(\lambda)|}{|\lambda|^2}$. We consider three cases

1. $|\text{Im}(\lambda)| \geq \frac{a}{2}$,
2. $|\text{Im}(\lambda)| \leq \frac{a}{2}$ and $\text{Re}(\lambda) \leq \frac{a}{2}$,
3. $|\text{Im}(\lambda)| \leq \frac{a}{2}$ and $\text{Re}(\lambda) \geq \frac{a}{2}$.

We start with $|\text{Im}(\lambda)| \geq \frac{a}{2}$. Then

$$\begin{aligned} |\lambda - a| &= \sqrt{(\text{Re}(\lambda) - a)^2 + \text{Im}(\lambda)^2} \\ &\leq \sqrt{a^2 + \text{Im}(\lambda)^2} \quad \text{since } 0 \leq \text{Re}(\lambda) \leq a \\ &\leq \sqrt{5}|\text{Im}(\lambda)| \quad \text{since } |\text{Im}(\lambda)| \geq \frac{a}{2}. \end{aligned}$$

Hence, as before,

$$d(z, I_z) \geq \frac{1}{\sqrt{5}} \cdot \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

Case 2. We have $|\operatorname{Im}(\lambda)| \geq \operatorname{Re}(\lambda)$, hence $|\operatorname{Im}(z)| \leq |z| \leq \sqrt{2}|\operatorname{Im}(z)|$ since $|\operatorname{Im}(z)| = \frac{|\operatorname{Re}(\lambda)|}{|\lambda|^2} \leq \frac{|\operatorname{Im}(\lambda)|}{|\lambda|^2} = |\operatorname{Im}(z)|$. Consequently, $d(z, I_z) \geq \frac{1}{\sqrt{2}}|z| = \frac{1}{\sqrt{2}|\lambda|}$. But $\frac{|\lambda - a|}{a + |\lambda|} \leq 1$ so

$$d(z, I_z) \geq \frac{1}{\sqrt{2}} \cdot \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

Case 3. The last case is similar to the previous one, we use this time $|\operatorname{Im}(\lambda)| \geq a - \operatorname{Re}(\lambda) \geq 0$, and we obtain

$$d(z, I_z) \geq \frac{1}{\sqrt{2}} \cdot \frac{d(\lambda, I_\lambda)}{|\lambda|(a + |\lambda|)}.$$

This concludes the proof of the distortion's inequality. \square

5.2 Proof for the fractional Laplacian

We introduce the function $g : \mathbb{C} \setminus [0, +\infty[\rightarrow \mathbb{C} \setminus [-\frac{1}{a}, 0]$ defined by $g(\lambda) = \frac{-1}{a + \lambda}$. The next proposition is the application of Proposition 5.1.5 to H_0 and its resolvent at $-a$: we have $(-a - H_0)^{-1} = g(H_0)$.

Proposition 5.2.1. *The distortion between $\mathbb{C} \setminus [-\frac{1}{a}, 0]$ and $\mathbb{C} \setminus \mathbb{R}^+$ is given by*

$$d\left(g(\lambda), \left[-\frac{1}{a}, 0\right]\right) \geq \frac{1}{2\sqrt{5}} \cdot \frac{d(\lambda, \mathbb{R}^+)}{(a + |\lambda|)^2}.$$

Proof. We compose relation 5.1.4 by the translation $T : \lambda \mapsto \lambda + a$, hence

$$\begin{aligned} d\left(\frac{1}{\lambda + a}, \left[0; \frac{1}{a}\right]\right) &\geq \frac{1}{\sqrt{5}} \cdot \frac{d(\lambda + a, [a; +\infty[)}{|\lambda + a| \cdot (a + |\lambda + a|)} \\ &\geq \frac{1}{2\sqrt{5}} \cdot \frac{d(\lambda, [0; +\infty[)}{|\lambda + a| \cdot (a + |\lambda|)}, \end{aligned}$$

then we bound from below with $|\lambda + a| \leq a + |\lambda|$. \square

The next theorem is a first Lieb-Thirring-type inequality obtained thanks to Theorem 5.1.1.

Theorem 5.2.2. *Let H be the operator defined in (4.1.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > \max\{1; \frac{d}{2s}\}$ with $s > 0$. Then the next inequality holds*

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(a + |\lambda|)^{2p}} \leq K_2 \cdot \frac{a^{\frac{d}{2s} - p}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \quad (5.2.1)$$

where (see Proposition 4.2.5 for the definition of M_1 and N_1)

$$1. K_2 = (2\sqrt{5})^p \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot C_\omega^p \cdot M_1 \text{ if } 0 < s \leq \frac{d}{2},$$

$$2. \text{ or } K_2 = (2\sqrt{5})^p \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot C_\omega^p \cdot N_1 \text{ if } s > \frac{d}{2}.$$

Proof. We put $A = (-a - H_0)^{-1}$ which is normal and $B = (-a - H)^{-1}$ which is bounded for $a > \omega$ so that A and B exist. We know that $B - A = BVA \in \mathbf{S}_p$, hence we can apply Theorem 5.1.1. It implies, for $p \geq 1$,

$$\sum_{\mu \in \sigma_d(B)} d(\mu, \sigma(A))^p \leq \|B - A\|_{\mathbf{S}_p}^p. \quad (5.2.2)$$

Now we bound the right-hand side of the inequality (5.2.2), for $p > \max\{1; \frac{d}{2s}\}$, with the help of Propositions 4.2.4, 4.2.5, and 4.2.7

$$\begin{aligned} \|B - A\|_{\mathbf{S}_p}^p &\leq (2\pi)^{-d} \|(-a - H)^{-1}\|^p \cdot \|V\|_{L^p}^p \cdot \|(-a - |\cdot|^{2s})^{-1}\|_{L^p}^p \\ &\leq K_1 \cdot \frac{a^{\frac{d}{2s}-p}}{|\omega - a|^p} \cdot \|V\|_{L^p}^p, \end{aligned} \quad (5.2.3)$$

where $K_1 = \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot C_\omega^p \cdot M_1$ if $0 < s \leq \frac{d}{2}$ or $K_1 = \frac{n^{p/2} s_{d-1}}{2s(2\pi)^d} \cdot C_\omega^p \cdot N_1$ if $s > \frac{d}{2}$ (see Proposition 4.2.5). Then from Proposition 2.1.13, $\mu = (-a - \lambda)^{-1} = g(\lambda) \in \sigma_d(B)$ if and only if $\lambda \in \sigma_d(H)$, hence

$$\begin{aligned} \sum_{\mu \in \sigma_d(B)} d(\mu, \sigma(A))^p &= \sum_{\{g(\lambda), \lambda \in \sigma_d(H)\}} d(g(\lambda), \sigma(A))^p \\ &\geq \frac{1}{(2\sqrt{5})^p} \cdot \sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(a + |\lambda|)^{2p}}. \end{aligned} \quad (5.2.4)$$

The last inequality result from Proposition 5.2.1. The result is a consequence of (5.2.3) and (5.2.4). \square

The last step to obtain Theorem 5.1.3 is an integration with respect to a the same way as in [DHK13] in the proof of Theorem 5.3.3.

Proof of Theorem 5.1.3. From Theorem 5.2.2 we know that, for all $a > \omega$,

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(a + |\lambda|)^{2p}} \cdot \frac{|\omega - a|^p}{a^{\frac{d}{2s}-p+1+\tau}} \leq C \cdot \frac{1}{a^{1+\tau}} \cdot \|V\|_{L^p}^p,$$

where C does not depend on a . we integrate with respect to a between ω and $+\infty$ the previous inequality :

$$\begin{aligned} \int_\omega^{+\infty} \sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(a + |\lambda|)^{2p}} \cdot \frac{|\omega - a|^p}{a^{\frac{d}{2s}-p+1+\tau}} da &\lesssim \int_\omega^{+\infty} \frac{1}{a^{1+\tau}} da \cdot \|V\|_{L^p}^p \\ \implies \sum_{\lambda \in \sigma_d(H)} d(\lambda, \sigma(H_0))^p \int_\omega^{+\infty} \frac{a^{p-\frac{d}{2s}-1-\tau} |\omega - a|^p}{(a + |\lambda|)^{2p}} da &\lesssim \frac{\|V\|_{L^p}^p}{\tau \omega^\tau}. \end{aligned}$$

In the last inequality the interchanging of the sum and the integral is valid by Fubini Theorem (the functions are positive). We bound from below the left-hand side of the above inequality whether $p - d/2s \leq 1$ or $p - d/2s > 1$ and then make the change of variable $t = \frac{a - \omega}{|\lambda| + \omega}$. Hence, assuming first that $p - d/2s > 1$,

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{a^{p-\frac{d}{2s}-1-\tau}(a-\omega)^p da}{(a+|\lambda|)^{2p}} &= \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{2p}} \int_{\mathbb{R}^+} \frac{[(|\lambda| + \omega)t + \omega]^{p-\frac{d}{2s}-1-\tau} t^p}{[t+1]^{2p}} dt \\ &\geq \frac{(|\lambda| + \omega)^{p-\frac{d}{2s}-1-\tau}}{(|\lambda| + \omega)^{p-1}} \int_0^{+\infty} \frac{t^{2p-\frac{d}{2s}-1-\tau}}{[t+1]^{2p}} dt \\ &\geq \frac{1}{(|\lambda| + \omega)^{\frac{d}{2s}+\tau}} \int_0^{+\infty} \frac{t^{2p-\frac{d}{2s}-1-\tau}}{(t+1)^{2p}} dt. \end{aligned}$$

When $p - d/2s < 1$, we have

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{a^{p-\frac{d}{2s}-1-\tau}(a-\omega)^p da}{(a+|\lambda|)^{2p}} &= \frac{(|\lambda| + \omega)^{p+1}}{(|\lambda| + \omega)^{2p}} \int_{\mathbb{R}^+} \frac{[(|\lambda| + \omega)t + \omega]^{p-\frac{d}{2s}-1-\tau} t^p}{[t+1]^{2p}} dt \\ &\geq \frac{(|\lambda| + \omega)^{p-\frac{d}{2s}-1-\tau}}{(|\lambda| + \omega)^{p-1}} \int_0^{+\infty} \frac{(t+1)^{p-\frac{d}{2s}-1-\tau} t^p}{[t+1]^{2p}} dt \\ &\geq \frac{1}{(|\lambda| + \omega)^{\frac{d}{2s}+\tau}} \int_0^{+\infty} \frac{t^p}{(t+1)^{p+\frac{d}{2s}+1+\tau}} dt. \end{aligned}$$

This concludes the proof. □

5.3 Proof for the Klein-Gordon operator

To prove the Theorem 5.1.4, we start with the following result.

Theorem 5.3.1. *Let $K = K_m + V$ be a perturbation of the Klein-Gordon operator $K_m, m > 0$, defined in (3.6.1) and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then we have the following inequality*

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(K_m))^p}{(a+m+|\lambda|)^{2p}} \leq K_2 \cdot \frac{(a+m)^{d-p}}{(a-\omega)^p} \cdot \|V\|_{L^p}^p, \quad (5.3.1)$$

with $K_2 = (2\sqrt{5})^p \cdot \frac{s_{d-1} n^{\frac{p}{2}}}{(2\pi)^d} \cdot C_{\omega}^p \cdot 2^{\delta} M$ where

1. $\delta = 2d - 3$ and $M = \max\{M_1; M_2\}$ if $d \geq 2$ (see Proposition 3.3.6),
2. or $\delta = 1$ and M is defined in Proposition 3.4.1 if $d = 1$.

Proof. We consider the operators $A = (-a - K_m)^{-1}$ and $B = (-a - K)^{-1}$ for $a > \omega$. From Proposition 2.1.13, we know that $\sigma((-a - K_m)^{-1}) = [-\frac{1}{a+m}; 0]$. We put $\lambda_1 = \lambda + a$, then from Proposition 5.1.5 we have

$$\begin{aligned} d\left(\frac{1}{\lambda_1}, \left[0; \frac{1}{a+m}\right]\right) &\geq \frac{1}{\sqrt{5}} \cdot \frac{d(\lambda_1; [a+m; +\infty])}{|\lambda_1|(a+m+|\lambda_1|)} \\ &\geq \frac{1}{\sqrt{5}} \cdot \frac{d(\lambda; [m; +\infty])}{|\lambda+a|(a+m+|\lambda+a|)} \\ &\geq \frac{1}{2\sqrt{5}} \cdot \frac{d(\lambda; [m; +\infty])}{(a+m+|\lambda|)^2}. \end{aligned}$$

Using the results of Proposition 3.3.6 (or 3.4.1 when $d = 1$)

$$\begin{aligned} \|B - A\|_{\mathfrak{S}_p}^p &\leq \|B\|^p \|VA\|_{\mathfrak{S}_p}^p \\ &\leq (2\pi)^{-d} \|(-a - K)^{-1}\|^p \|V\|_{L^p}^p \|(-a - \sqrt{|x|^2 + m^2})^{-1}\|_{L^p}^p \\ &\leq K_1 \cdot \frac{C_\omega^p}{(a-\omega)^p} \cdot \|V\|_{L^p}^p \cdot 2^{\delta'} (a+m)^{d-p} \\ &\leq K_2 \cdot \frac{(a+m)^{d-p}}{(a-\omega)^p} \cdot \|V\|_{L^p}^p, \end{aligned}$$

with, if $d \geq 2$, $\delta' = d - 1$, $K_1 = \frac{s_{d-1} n^{\frac{p}{2}}}{(2\pi)^d} \cdot 2^{d-2} M$ with $M = \max\{M_1; M_2\}$ (see Proposition 3.3.6), or, if $d = 1$, $\delta' = 1$, $K_1 = \frac{s_0 n^{\frac{p}{2}}}{(2\pi)^d} \cdot M$, with M defined in Proposition 3.4.1; then $K_2 = K_1 C_\omega^p 2^{\delta'}$.

We apply now Theorem 5.1.1 to A and B , and we obtain inequality (5.3.1).

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(K_m))^p}{(a+m+|\lambda|)^{2p}} \leq \frac{(2\sqrt{5})^p K_2}{(a-\omega)^p (a+m)^{p-d}} \cdot \|V\|_{L^p}^p,$$

with K_2 described above. □

Proof of Theorem 5.1.4. We now integrate the inequality (5.3.1) with respect to a between ω and $+\infty$ to find (5.1.3). Recall that $\omega \geq 1$.

$$\sum_{\lambda \in \sigma_d(H)} d(\lambda, \sigma(K_m))^p \int_{\omega}^{+\infty} \frac{(a-\omega)^p (a+m)^{p-d-1-\tau}}{(a+m+|\lambda|)^{2p}} da \lesssim \int_{\omega}^{+\infty} \frac{da}{(a+m)^{1+\tau}} \cdot \|V\|_{L^p}^p.$$

In the integral in the left-hand side of the inequality, we make the change of variable

$$t = \frac{a - \omega}{|\lambda| + \omega + m},$$

$$\begin{aligned} & \int_{\omega}^{+\infty} \frac{(a - \omega)^p (a + m)^{p-d}}{(a + m + |\lambda|)^{2p} (a + m)^{1+\tau}} da = \\ &= \int_0^{+\infty} \frac{(|\lambda| + \omega + m)^{p+1} t^p [(|\lambda| + \omega + m)t + \omega + m]^{p-d}}{[(|\lambda| + \omega + m)(t + 1)]^{2p} [(|\lambda| + \omega + m)t + \omega + m]^{1+\tau}} dt \\ &\geq \frac{(|\lambda| + \omega + m)^{2p-d+1}}{(|\lambda| + \omega + m)^{2p}} \int_0^{+\infty} \frac{t^{2p-d}}{(t + 1)^{2p} [(|\lambda| + \omega + m)(t + 1)]^{1+\tau}} dt \\ &\geq \frac{1}{(|\lambda| + \omega + m)^{d+\tau}} \int_0^{+\infty} \frac{t^{2p-d}}{(t + 1)^{2p+1+\tau}} dt. \end{aligned}$$

In the penultimate inequality we use $\omega + m > 0$ and $|\lambda| > 0$.

This ends the proof. □

6 Appendix

Table of Contents

6.1 Integration of the Lieb-Thirring-type inequality in the case of massless Dirac operator	79
6.2 Glossary of the notation	82

6.1 Integration of the Lieb-Thirring-type inequality in the case of massless Dirac operator

One of the reasons we cannot “integrate” the Lieb-Thirring-type inequalities in Theorem 3.2.1, as for half-bounded operators, is the absence of the parameter b in the behaviour at infinity of the distortion.

We realize lately that this parameter is present in the case of the massless Dirac operator. That is why we present in this appendix the proof of the following improvement of Theorem 3.2.3.

Theorem 6.1.1 (case $m = 0$). *Let D be the Dirac operator defined in (3.1.3) with $m = 0$ and $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$, $p > d$. Then, for $0 < \tau \leq p - d$, we have*

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(1 + |\lambda|)^{d+2\tau}} \leq C \cdot \frac{K_1}{I} \cdot \frac{\omega^{d+\tau} C_\omega^p}{\tau} \cdot \|V\|_{L^p}^p, \quad (6.1.1)$$

where

1. C depends on p, d , and τ ,
2. $K_1 = \Gamma_p 2^{2p-1} K$, K is defined in Proposition 3.5.2,
3. $I = \int_{\mathbb{R}^+} \frac{t^{p+(p-d-1)_+} dt}{(t+1)^{2(p+\tau)-(p-d-1)_+}}$,
4. and C_ω is defined in Proposition 6.1.3.

To prove the result, we show another distortion inequality in Proposition 3.5.1. We keep the same functions and we work in \mathbb{C}^+ .

Proposition 6.1.2 (distortion between \mathbb{C}^+ and \mathbb{D}). *Let $b > 0$, the distortion between \mathbb{C}^+ and \mathbb{D} are*

$$\frac{b}{2} \cdot \frac{d(z, \mathbb{T})}{|z-1|^2} \leq d(\lambda, \sigma(D_0)) \leq 4b \cdot \frac{d(z, \mathbb{T})}{|z-1|^2},$$

and

$$b \cdot \frac{d(\lambda, \sigma(D_0))}{(b + |\lambda|)^2} \leq d(z, \mathbb{T}) \leq 16b \cdot \frac{d(\lambda, \sigma(D_0))}{(b + |\lambda|)^2}.$$

Proof. The proof is the same as the proof of Proposition 3.5.1 except that we use $\frac{1}{2}(|\lambda| + b)^2 \leq |\lambda + ib|^2 \leq (|\lambda| + b)^2$. \square

We know from Proposition 3.5.2 the bound of the norm-resolvent of D_0 . The new fact is the following bound of the norm-resolvent of the perturbed operator D :

Proposition 6.1.3. *There is $\omega \geq 1$ depending on d, n, p , and V such that $i\omega \in \rho(D)$ and, for all $b \geq \omega$,*

$$\|(ib - D)^{-1}\| \leq \frac{C_\omega}{|b - \omega|}.$$

Proof. Indeed, looking back at the proof of Proposition 4.2.7, we use that

$$\|V(ib - D_0)^{-1}\|_{\mathbf{S}_p}^p \leq K \|V\|_{L^p}^p \frac{b^{d-1}}{b^{p-1}}$$

tends to 0 when b tends to $+\infty$. \square

Proof of Theorem 6.1.1. We go back to the inequality (3.5.1) which becomes, since $p > d$,

$$\sum_{g(z)=0} d(z, \mathbb{T})^{p+\tau} \leq C \cdot K_1 \cdot \frac{C_\omega^p b^d}{|b - \omega|^p} \cdot \|V\|_{L^p}^p,$$

But, with the new expression of the distortion, we have $d(z, \mathbb{T}) \geq b \cdot \frac{d(\lambda, \sigma(D_0))}{(b + |\lambda|)^2}$, hence

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(b + |\lambda|)^{2(p+\tau)}} \lesssim \frac{b^{d-p-\tau}}{|b - \omega|^p} \cdot \|V\|_{L^p}^p.$$

We integrate this inequality with respect to b between ω and $+\infty$ in the following way

$$\sum_{\lambda \in \sigma_d(D)} d(\lambda, \sigma(D_0))^{p+\tau} \int_\omega^{+\infty} \frac{|b - \omega|^p b^{p-d-1}}{(b + |\lambda|)^{2(p+\tau)}} db \lesssim \|V\|_{L^p}^p \int_\omega^{+\infty} \frac{db}{b^{1+\tau}}. \quad (6.1.2)$$

The interchanging of the sum and the integral is valid by Fubini Theorem. The integral in the right-hand side member equals $\frac{1}{\tau \omega^\tau}$. In the left-hand side, the bound depends on the

sign of $p - d - 1$. If $p - d - 1 \geq 0$ then, with the change of variable $t = \frac{b - \omega}{\omega + |\lambda|}$, we have

$$\begin{aligned} \int_\omega^{+\infty} \frac{|b - \omega|^p b^{p-d-1}}{(b + |\lambda|)^{2(p+\tau)}} db &= \frac{(\omega + |\lambda|)^{p+1}}{(\omega + |\lambda|)^{2(p+\tau)}} \int_{\mathbb{R}_+} \frac{t^p ((\omega + |\lambda|)t + \omega)^{p-d-1}}{(t+1)^{2(p+\tau)}} dt \\ &\geq \frac{(\omega + |\lambda|)^{p-d-1}}{(\omega + |\lambda|)^{p-1+2\tau}} \int_{\mathbb{R}_+} \frac{t^{2p-d-1}}{(t+1)^{2(p+\tau)}} dt \\ &\geq \frac{1}{(\omega + |\lambda|)^{d+2\tau}} \int_{\mathbb{R}_+} \frac{t^{2p-d-1}}{(t+1)^{2(p+\tau)}} dt. \end{aligned}$$

If $p - d - 1 < 0$, we bound from above b by $b + |\lambda|$, and then with the same change of variable, we find

$$\begin{aligned} \int_{\omega}^{+\infty} \frac{|b - \omega|^p b^{p-d-1}}{(b + |\lambda|)^{2(p+\tau)}} db &\geq \int_{\omega}^{+\infty} \frac{|b - \omega|^p}{(b + |\lambda|)^{p+d+1+2\tau}} db \\ &\geq \frac{(\omega + |\lambda|)^{p+1}}{(\omega + |\lambda|)^{p+d+1+2\tau}} \int_{\mathbb{R}^+} \frac{t^p dt}{(t + 1)^{2(p+\tau)}} \\ &\geq \frac{1}{(\omega + |\lambda|)^{d+2\tau}} \int_{\mathbb{R}^+} \frac{t^p dt}{(t + 1)^{p+d+1+2\tau}}. \end{aligned}$$

Hence, going back to (6.1.2), we obtain

$$\sum_{\lambda \in \sigma_d(D)} \frac{d(\lambda, \sigma(D_0))^{p+\tau}}{(\omega + |\lambda|)^{d+2\tau}} db \leq C \cdot \frac{K_1}{I} \cdot \frac{C_{\omega}^p}{\tau \omega^{\tau}} \cdot \|V\|_{L^p}^p,$$

with C , K_1 , and C_{ω} described above, and $I = \int_{\mathbb{R}^+} \frac{t^{2p-d-1}}{(t + 1)^{2(p+\tau)}} dt$ if $p - d - 1 \geq 0$, or

$$I = \int_{\mathbb{R}^+} \frac{t^p dt}{(t + 1)^{p+d+1+2\tau}} \text{ if } p - d - 1 < 0. \quad \square$$

6.2 Glossary of the notation

- $L^2(\mathbb{R}^d, \mathbb{K}^n) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{K}^n \text{ measurable, } \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty \right\}$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . This is a complex Hilbert space.
- C_c^∞ denote the space of infinitely differentiable functions with compact support. See Sections 3.1, 3.6.1, and 4.1.
- \mathbb{C}^+ (resp. \mathbb{C}^-) denote the half-plane with positive (resp. negative) imaginary part. See Section 3.5.1.
- $\mathbb{D} = \{z : |z| < 1\}$ is the open unit disc in \mathbb{C} .
- $d(z; A)$ denote the distance between z and A . See Section 2.3.3.
- \mathcal{H}^1 denote the Sobolev space of order 1. See Sections 3.1 and 3.6.1.
- $s_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$ is the total mass of the invariant measure on the unit sphere of dimension $d - 1$. If $d = 1$, it is convenient in this work to put $s_0 = 2$. See Propositions 3.3.6, 3.5.2, and 4.2.5.
- \mathbf{S}_p : Schatten-von Neumann classes. See Section 2.2. In particular \mathbf{S}_∞ denote the space of compact operators.
- $\mathbb{T} = \{z : |z| = 1\}$ is the unit circle in \mathbb{C} .
- $W^{2,2s}$ denote the fractional Sobolev space. See Section 4.1.
- $Z_b = \frac{1 + |z_b|}{1 - |z_b|}$, constant appearing in bounds of conformal mapping; see remark 3.3.1.
- Γ_p has two definitions : except for Chapter 5, it is a constant in the bound of the regularized perturbation determinant. See Formula (2.2.1).
In Chapter 5, this is a constant appearing in Theorem 5.1.2.
- $\|\cdot\|_F$ denote the Frobenius norm. See Section 2.4.
- $f \approx g$ and $f \lesssim g$. Comparison's inequalities between two nonnegative functions. See Section 2.3.2.
- \hat{f} and \check{g} denote respectively the image by the Fourier transform of f and the image by the inverse Fourier transform of g .

Bibliography

- [AAD01] A. A. Abramov, A. Aslanyan, and E. B. Davies. Bounds on complex eigenvalues and resonances. *J. Phys. A, Math. Gen.*, 34(1):57–72, 2001. 3
- [BE11] A. A. Balinsky and W. D. Evans. *Spectral analysis of relativistic operators*. Imperial College Press, London, 2011. 2, 43
- [BGK09] A. Borichev, L. Golinskii, and S. Kupin. A Blaschke-type condition and its application to complex Jacobi matrices. *Bull. Lond. Math. Soc.*, 41(1):117–123, 2009. 3, 4, 16
- [BO08] V. Bruneau and E. M. Ouhabaz. Lieb-Thirring estimates for non-self-adjoint Schrödinger operators. *J. Math. Phys.*, 49(9):093504, 10, 2008. 3, 4
- [CLN96] C. Cancelier, P. Lévy-Bruhl, and J. Nourrigat. Remarks on the spectrum of Dirac operators. *Acta Appl. Math.*, 45(3):349–364, 1996. 3, 4, 24
- [CLT14] J.C. Cuenin, A. Laptev, and C. Tretter. Eigenvalue estimates for non-selfadjoint Dirac operators on the real line. *Ann. Henri Poincaré*, 15(4):707–736, 2014. 4, 8
- [Dav07] E. B. Davies. *Linear operators and their spectra*. Cambridge: Cambridge University Press, 2007. 11
- [DH13] M. Demuth and F. Hanauska. On the distribution of the discrete spectrum of nuclearly perturbed operators in Banach space. arxiv, preprint 1309.4594., 2013. 8
- [DHK09] M. Demuth, M. Hansmann, and G. Katriel. On the discrete spectrum of non-selfadjoint operators. *J. Funct. Anal.*, 257(9):2742–2759, 2009. 3, 4, 14, 47
- [DHK13] M. Demuth, M. Hansmann, and G. Katriel. Eigenvalues of non-selfadjoint operators: a comparison of two approaches. In *Mathematical physics, spectral theory and stochastic analysis*, pages 107–163. Basel: Birkhäuser/Springer, 2013. 4, 47, 48, 70, 75
- [Dub14] C. Dubuisson. On quantitative bounds on eigenvalues of a complex perturbation of a Dirac operator. *Integral Equations Oper. Theory*, 78(2):249–269, 2014. 44, 46, 47, 52, 53
- [Dub15] C. Dubuisson. Notes on Lieb-Thirring type inequality for a complex perturbations of a fractional Schrödinger operator. accepted in *Journal of Mathematical Physics, Analysis, Geometry*, 2015. 52

- [EE89] D.E. Edmunds and W.D. Evans. *Spectral theory and differential operators*. Oxford: Clarendon Press, paperback ed. edition, 1989. 11, 12
- [FG09] S. Favorov and L. Golinskii. A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators. In *Linear and complex analysis. Dedicated to V. P. Havin on the occasion of his 75th birthday*, pages 37–47. Providence, RI: American Mathematical Society (AMS), 2009. 17
- [FLLS06] R. L. Frank, A. Laptev, E. H. Lieb, and R. Seiringer. Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials. *Lett. Math. Phys.*, 77(3):309–316, 2006. 3, 4
- [FLS08] R. L. Frank, E. H. Lieb, and R. Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Am. Math. Soc.*, 21(4):925–950, 2008. 3, 4, 6, 45, 55
- [Fra11] R. L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.*, 43(4):745–750, 2011. 4
- [FS11] R. L. Frank and B. Simon. Critical Lieb-Thirring bounds in gaps and the generalized Nevai conjecture for finite gap Jacobi matrices. *Duke Math. J.*, 157(3):461–493, 2011. 4, 24
- [GGK00] I.C. Gohberg, S. Goldberg, and N. Krupnik. *Traces and determinants of linear operators*. Basel: Birkhäuser, 2000. 14
- [GK69] I.C. Gohberg and M.G. Krein. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969. 13, 14, 19
- [GK07] L. Golinskii and S. Kupin. Lieb-Thirring bounds for complex Jacobi matrices. *Lett. Math. Phys.*, 82(1):79–90, 2007. 3, 4
- [GK12] L. Golinskii and S. Kupin. A Blaschke-type condition for analytic functions on finitely connected domains. Applications to complex perturbations of a finite-band selfadjoint operator. *J. Math. Anal. Appl.*, 389(2):705–712, 2012. 4
- [GK13] L. Golinskii and S. Kupin. On discrete spectrum of complex perturbations of finite band Schrödinger operators. In *Recent trends in analysis. Proceedings of the conference in honor of Nikolai Nikolski on the occasion of his 70th birthday, Bordeaux, France, August 31 – September 2, 2011*, pages 113–121. Bucharest: The Theta Foundation, 2013. 4, 8, 70
- [Han10] M. Hansmann. *On the discrete spectrum of linear operators in Hilbert spaces*. PhD thesis, TU Clausthal, 2010. 14, 32, 46, 59, 60

- [Han11] M. Hansmann. An eigenvalue estimate and its application to non-selfadjoint Jacobi and Schrödinger operators. *Lett. Math. Phys.*, 98(1):79–95, 2011. 4, 5, 69, 70
- [Han13] M. Hansmann. Variation of discrete spectra for non-selfadjoint perturbations of selfadjoint operators. *Integral Equations Oper. Theory*, 76(2):163–178, 2013. 4, 69, 70
- [HK11] M. Hansmann and G. Katriel. Inequalities for the eigenvalues of non-selfadjoint Jacobi operators. *Complex Anal. Oper. Theory*, 5(1):197–218, 2011. 4, 17
- [HS02] D. Hundertmark and B. Simon. Lieb-Thirring inequalities for Jacobi matrices. *J. Approx. Theory*, 118(1):106–130, 2002. 3
- [LL01] E. H. Lieb and M. Loss. *Analysis*. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2001. 2
- [LS09a] A. Laptev and O. Safronov. Eigenvalue estimates for Schrödinger operators with complex potentials. *Commun. Math. Phys.*, 292(1):29–54, 2009. 4
- [LS09b] E. H. Lieb and R. Seiringer. *The stability of matter in quantum mechanics*. Cambridge: Cambridge University Press, 2009. 2, 4, 43, 45, 53
- [LT75] E. H. Lieb and W. E. Thirring. Bounds for the kinetic energy of fermions which proves the stability of matter. *Phys. Rev. Lett.*, 35:687–689, 1975. Errata *ibid.*, 1116 (1975). 2
- [LT76] E. H. Lieb and W. E. Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Stud. math. Phys., Essays Honor Valentine Bargmann*, 269-303 (1976)., 1976. 2
- [LW00] A. Laptev and T. Weidl. Recent results on Lieb-Thirring inequalities. In *Journées “Équations aux dérivées partielles”, La Chapelle sur Erdre, Nantes, France, 5 au 9 juin 2000. Exposés Nos. I-XX*, page ex. Nantes: Université de Nantes, 2000. 2
- [Obo98] E. Obolashvili. *Partial differential equations in Clifford analysis.*, volume 96 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman, Harlow, 1998. 22
- [Pom92] C. Pommerenke. *Boundary behaviour of conformal maps*. Berlin: Springer-Verlag, 1992. 18
- [RS78] M. Reed and B. Simon. *Methods of modern mathematical physics. IV: Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. 12
- [RS80] M. Reed and B. Simon. *Methods of modern mathematical physics. I: Functional analysis*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. 10, 11, 12

- [Sam14] D. Sambou. Lieb-Thirring type inequalities for non-self-adjoint perturbations of magnetic Schrödinger operators. *J. Funct. Anal.*, 266(8):5016–5044, 2014. 4, 70
- [Sch12] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*. Dordrecht: Springer, 2012. 10
- [Sei10] R. Seiringer. Inequalities for Schrödinger operators and applications to the stability of matter problem. In *Entropy and the quantum. Arizona school of analysis with applications, March 16–20, 2009, University of Arizona, Tucson, AZ, USA.*, pages 53–72. Providence, RI: American Mathematical Society (AMS), 2010. 43
- [Sim77] B. Simon. Notes on infinite determinants of Hilbert space operators. *Adv. Math.*, 24:244–273, 1977. 13, 14
- [Sim05] B. Simon. *Trace ideals and their applications*. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2005. 13, 18, 19
- [Tha91] B. Thaller. *The Dirac equation*. Berlin: Springer-Verlag, 1991. 4, 21, 22, 43

Index

- conformal map, 4, 17, 26, 27, 40, 46, 56
- determinant
 - regularized, **13**
 - regularized perturbation, 4, 12, **14**, 14–16
- distortion, 17, 27, 28, 56
- distortion
 - inverse function, 72
- eigenvalue, 1, 3, 4, 11, 13, 16
- inequality
 - Birman-Solomyak, 12, **18**, 30, 56, 59
 - classical Lieb-Thirring, 2
 - Lieb-Thirring-type, 2–7, 23, 25, 44, 45, 54, 55, 70, 71, 74, 76, 79
- multiplicity
 - algebraic, 1, 10, **11**, 11, 14, 16
 - geometric, 10, **11**, 11
- operator
 - adjoint, 9, **10**
 - bounded, 5, **9**
 - compact, **10**, 10, 12
 - Dirac, 1, 3–5, 21, **22**, 23, 25, 43, 79
 - essentially self-adjoint, 3, **10**, 10
 - fractional Laplacian, 1, 3, 4, 6, 7, **53**
 - fractional Schrödinger, 6, 7, 45, **53**, 54, 55, 70, 74
 - Jacobi matrices, 3–5
 - Klein-Gordon, 1, 4, 6, 7, **43**, 44, 45, 71, 76
 - Laplacian, 1
 - multiplicative, 1
 - perturbed, **1**, 3, 22, 53
 - relativistic Schrödinger, 4, 43
 - Schrödinger, **1**, 2–5
 - self-adjoint, 1, **10**, 10–12
 - unbounded, 9
 - fractional Laplacian, 45
- potential, 1, 3, 4
- relatively compact perturbation, **12**, 12
- relatively Schatten-von Neumann, **12**, 22, 53
- resolvent, 5, **11**
- resolvent
 - norm, 4, 5
 - set, **11**
- s-number
 - see singular value, 13
- singular value, 13
- space
 - Schatten-von Neumann, **13**
- spectrum
 - compact operator, 10
 - definition, 11
 - discrete, 1, 2, **11**, 12, 14, 17, 45
 - essential, 1, 11, **12**
- Theorem
 - Borichev-Golinskii-Kupin, 4, **16**, 17, 35, 39, 43, 47, 51, 61, 64
 - Hansmann, 4, 7, 17, **69**, 75, 77
 - Weyl, **12**, 22, 43, 54