

Quelques problèmes mathématiques en hydrodynamique et physique quantique

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Université de Versailles SQY
Laboratoire de Mathématiques, UMR8100

Mémoire

présenté pour l'obtention de

l'Habilitation à Diriger des Recherches

en Mathématiques
par

Vahagn NERSESYAN

Quelques problèmes mathématiques en hydrodynamique et physique quantique

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au vu des rapports de :

Jean-Michel Coron	Université Pierre et Marie Curie
Franco Flandoli	Université de Pise

À mes enfants Aren et Areg,
à mon épouse Anna

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Outline of the content

This manuscript summarises my research activity after my PhD thesis, defended in 2008. It is devoted to the study of some mathematical problems mainly related to the hydrodynamic turbulence and quantum physics. The text is organised in two relatively independent chapters.

In **Chapter 1**, we focus on questions such as the ergodicity and large-time behavior of solutions of randomly forced PDE's. We start our presentation with a series of results obtained with V. Jakšić, C.-A. Pillet, and A. Shirikyan [P9, P12, P14], on the large deviations principle (LDP) and Gallavotti–Cohen type symmetry for a class of parabolic PDE's (such as the Navier–Stokes and Burgers equations) perturbed by a random kick force. With D. Martirosyan [P15], we extend some of the methods developed in these works to the case of the damped nonlinear wave equation driven by a spatially regular white noise. We establish a local version of the LDP. Finally, with S. Kuksin [P8] we consider the stochastic complex Ginzburg–Landau equation in any space dimension. We prove that this equation defines a Markov process in the space of continuous functions, which has a unique stationary measure and is mixing.

Chapter 2 is composed of two unequal parts. In the first (longer) part, we consider the problem of the controllability of a quantum particle by the amplitude of an electric field. The position of the particle is described by a wave function which obeys the bilinear Schrödinger equation. In [P5-P7, P10, P11], with K. Beauchard, H. Nersisyan, and M. Morancey we use variational methods to study the global controllability properties of this equation. We establish approximate controllability, feedback stabilisation, and simultaneous controllability results. The second part of this chapter is concerned with the problem of controllability of the 3D Navier–Stokes system by a finite-dimensional force [P13]. Using some methods from the geometric control theory, we establish approximate controllability of Lagrangian trajectories of the system. We provide some examples of finite-dimensional saturating spaces which ensure the controllability.

These chapters are followed by a short section, where we introduce the notation used in this manuscript.

List of publications

Publication from my Baccalaureat thesis (4$^{\text{th}}$ year diploma in Armenia).

[P1] T. Harutyunyan, V. Nersesyan. A uniqueness theorem in the inverse Sturm–Liouville problem. *J. Contemp. Math. Anal.*, 39(6):27–36, 2004.

Publications from my PhD thesis.

[P2] V. Nersesyan. Polynomial mixing for the complex Ginzburg–Landau equation perturbed by a random force at random times. *J. Evol. Eq.*, 8(1):1–29, 2008.

[P3] V. Nersesyan. Exponential mixing for finite-dimensional approximations of the Schrödinger equation with multiplicative noise. *Dyn. PDE*, 6(2):167–183, 2009.

[P4] V. Nersesyan. Growth of Sobolev norms and controllability of Schrödinger equation. *Comm. Math. Phys.*, 290:371–387, 2009.

Publications presented in this thesis.

[P5] V. Nersesyan. Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications. *Ann. I. H. Poincaré-AN*, 27:901–915, 2010.

[P6] K. Beauchard, V. Nersesyan. Semi-global weak stabilization of bilinear Schrödinger equations. *C. R. Math. Acad. Sci. Paris*, 348(19-20):1073–1078, 2010.

[P7] V. Nersesyan, H. Nersisyan. Global exact controllability in infinite time of Schrödinger equation. *J. Math. Pures et Appl.*, 97(4):295–317, 2012.

[P8] S. Kuksin, V. Nersesyan. Stochastic CGL equations without linear dispersion in any space dimension. *Stoch. PDE: Anal. Comp.*, 1(3): 389–423, 2013.

[P9] V. Jakšić, V. Nersesyan, C.-A. Pillet, A. Shirikyan. Large deviations from a stationary measure for a class of dissipative PDE's with random kicks. To appear in Comm. Pure Appl. Math.

[P10] M. Morancey, V. Nersesyan. Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations. *J. Math. Pures et Appl.*, 103(1):228–254, 2015.

[P11] M. Morancey, V. Nersesyan. Global exact controllability of 1D Schrödinger equations with a polarizability term. *C. R. Math. Acad. Sci. Paris*, 352(5):425–429, 2014.

[P12] V. Jakšić, V. Nersesyan, C.-A. Pillet, A. Shirikyan. Large deviations and Gallavotti–Cohen principle for dissipative PDE's with rough noise. *Comm. Math. Phys.*, 336:131–170, 2015.

[P13] V. Nersesyan. Approximate controllability of Lagrangian trajectories of the 3D Navier–Stokes system by a finite-dimensional force. *Nonlinearity*, 28(3):825–848, 2015.

[P14] V. Jakšić, V. Nersesyan, C.-A. Pillet, A. Shirikyan. Large deviations and mixing for dissipative PDE's with unbounded random kicks. Preprint, 2014.

[P15] D. Martirosyan, V. Nersesyan. Local large deviations principle for occupation measures of the damped nonlinear wave equation perturbed by a white noise. Preprint, 2015.

Chapter 1

LDP and mixing for randomly forced PDE's

1.1 LDP for Navier–Stokes system with random kicks

1.1.1 The model

In this section, we discuss the results obtained in collaboration with V. Jakšić, C.-A. Pillet, and A. Shirikyan in [P9, P14]. In these papers, we study the LDP for a class of dissipative PDE's perturbed by a random kick force. We present the results on the example of the two-dimensional Navier–Stokes (NS) system

$$\partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = h(x) + \eta(t,x), \quad \operatorname{div} u = 0, \quad u\big|_{\partial D} = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in D, \tag{1.2}$$

which describes the motion of an incompressible fluid in a bounded domain $D \subset \mathbb{R}^2$ with a smooth boundary ∂D. Here $\nu > 0$ is the kinematic viscosity, $u = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are unknown velocity field and pressure of the fluid, and $\langle u, \nabla\rangle = u_1\partial_1 + u_2\partial_2$. The functions h and η are, respectively, the deterministic and random components of the external force.

The ergodic properties of this problem have been extensively studied in recent years. In particular, it is known that, if η is a sufficiently non-degenerate random kick force or white noise, then the Markov process associated with (1.1) admits a unique stationary measure μ, which is exponentially mixing. In the case when η is a kick force this result is recalled below in Theorem 1.1.1. We refer the reader to the book [63] for a detailed discussion of this topic, and the papers [16, 58–61, 74, 92] for different results in the case of a kick force and [17, 27, 31, 33, 37, 48, 49, 62, 72, 76, 82, 94] for a white noise. Our goal was to study the large-time behavior of the probabilities of large deviations of trajectories from μ in the case when the random force η is a *regular* function in the space variable x. Before stating the main results of this section, let us introduce some notation and give the exact

conditions on η. The problem is considered in the space of divergence-free vector fields

$$H = \{u \in L^2(D,\mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u,n\rangle = 0 \text{ on } \partial D\},$$

where n stands for the outward unit normal to ∂D. As usual, system (1.1) is projected onto H, in order to eliminate the pressure term and to obtain an evolution equation for the velocity field (e.g., see Section 6 in Chapter 1 of [67])

$$\dot{u} + \nu Lu + B(u) = \Pi h + \Pi\eta(t,x), \tag{1.3}$$

where $L := -\Pi\Delta$ is the Stokes operator, $B(u) := \Pi(\langle u,\nabla\rangle u)$, and Π is the orthogonal projection onto H in L^2 (Leray projector). We assume that $h \in H$ and $\eta(t,x)$ is a random kick force of the form

$$\eta(t,x) = \sum_{k=1}^{+\infty} \delta(t-k)\eta_k(x), \tag{1.4}$$

where δ is the Dirac measure concentrated at zero, and η_k are independent identically distributed (i.i.d.) random variables defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with range in H. Then any trajectory u_t of (1.3), (1.4) is completely determined [1] by its restriction to integer times u_k. If we denote by $S : H \to H$ the time-1 shift along trajectories of $(1.3)_{\eta=0}$, then the sequence $\{u_k\}$ satisfies the equality

$$u_k = S(u_{k-1}) + \eta_k, \quad k \geq 1. \tag{1.5}$$

Let $(u_k,\mathbb{P}_u)$ be the Markov family associated with this dynamical system parametrised by the initial condition $u = u_0 \in H$, and define the corresponding Markov semigroups [2]

$$\begin{aligned} &\mathfrak{P}_k : C_b(H) \to C_b(H), && \mathfrak{P}_k f(u) = \int_H f(v) P_k(u,\mathrm{d}v), \\ &\mathfrak{P}_k^* : \mathscr{P}(H) \to \mathscr{P}(H), && \mathfrak{P}_k^*\sigma(\Gamma) = \int_H P_k(v,\Gamma)\sigma(\mathrm{d}v), \quad k \geq 0, \end{aligned}$$

where $P_k(u,\Gamma) = \mathbb{P}_u\{u_k \in \Gamma\}$ is the transition function. Recall that a measure $\mu \in \mathscr{P}(H)$ is stationary if $\mathfrak{P}_1^*\mu = \mu$. The existence of a stationary measure is a relatively simple question. It is proved by using the classical Bogolyubov–Krylov argument, under the sole condition that $\mathbb{E}\|\eta_1\| < \infty$ (see Section 2.5 in [63]). The uniqueness of a stationary measure and the mixing properties are difficult problems, they are established under some additional assumptions on the kicks $\{\eta_k\}$. We shall distinguish two cases, depending on whether the distribution of η_k has a bounded support or not. We start with the more complicated unbounded case, where we impose the following condition:

[1] The trajectory u_t is normalised to be right-continuous.

[2] $\mathscr{P}(H)$ is the set of probability Borel measures on H, it is endowed with the topology of the weak convergence.

(A) Structure of the noise. *The random variables η_k can be written as*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{1.6}$$

where $\{e_j\}$ is an orthonormal basis in H consisting of the eigenfunctions of the Stokes operator L with eigenvalues $\{\alpha_j\}$, $\{\xi_{jk}\}$ are independent scalar random variables whose laws possess C^1-smooth positive densities ρ_j against the Lebesgue measure verifying $\mathrm{Var}(\rho_j) \le 1$ for all $j \ge 1$, and $b_j \ge 0$ are such that

$$\mathfrak{B} := \sum_{j=1}^{\infty} \alpha_j |b_j|^2 < \infty.$$

Moreover, there is $\delta > 0$ such that

$$\int_H e^{\delta \|u\|^2} \, \ell(\mathrm{d}u) < \infty,$$

where $\ell = \mathscr{D}(\eta_1)$ and $\|\cdot\|$ is the norm in L^2.

Roughly speaking, this condition implies that the kicks $\{\eta_k\}$ are valued in the Sobolev space H^1, they have a finite exponential moment, and $\operatorname{supp} \ell = H$. Moreover, this condition ensures the uniqueness of a stationary measure and the exponential mixing property; the following result is Theorem 3.4.1 in [63].

Theorem 1.1.1. *There is an integer $N = N(\nu, \mathfrak{B}) \ge 1$ such that if*

$$b_j \ne 0 \quad \text{for } j = 1, \dots, N, \tag{1.7}$$

then the Markov chain $(u_k, \mathbb{P}_u)$ admits a unique stationary measure $\mu \in \mathscr{P}(H)$. Moreover, there are constants $C > 0$ and $\alpha > 0$ such that, for any $\sigma \in \mathscr{P}(H)$, we have

$$\|\mathfrak{P}_k^* \sigma - \mu\|_L^* \le C e^{-\alpha k} \left(1 + \int_H \|u\| \sigma(\mathrm{d}u)\right), \quad k \ge 0,$$

where $\|\cdot\|_L^*$ is the dual-Lipschitz metric defined by (2.15).

Under a roughness condition imposed on η, one has an exponential mixing property in the total variation norm (see Theorem 1.2.1). According to [49], if the problem is considered on the torus $\mathbb{T}^2$, the function h is zero, and η is a white noise of the form (1.20), then the mixing property holds, under condition (1.7) with an integer N not depending on ν and $\mathfrak{B}$.

This exponential mixing property has many important consequences, such as the strong law of large numbers (SLLN), the law of the iterated logarithm, and the central limit theorem (see Chapter 4 in [63]). By the SLLN, we have the following limit

$$\mathbb{P}_u \left\{ \frac{1}{k} \sum_{n=1}^{k} f(u_n) \to \langle f, \mu \rangle \right\} = 1 \quad \text{as } k \to \infty$$

for any Hölder-continuous function $f : H \to \mathbb{R}$. Thus, for any open set $O \subset \mathbb{R}$ such that $\langle f, \mu \rangle \notin \overline{O}$, we have

$$\mathbb{P}_u \left\{ \frac{1}{k} \sum_{n=1}^{k} f(u_n) \in O \right\} \to 0, \quad k \to \infty.$$

The large deviations results presented below can be used, in particular, to characterize the rate of this convergence. Indeed, we can derive an asymptotic formula of the following form

$$\mathbb{P}_u \left\{ \frac{1}{k} \sum_{n=1}^{k} f(u_n) \in O \right\} = \exp(-ck + o(k))$$

as $k \to \infty$, where $c = c(f, O) \geq 0$ is a constant that can be expressed in terms of the rate function I of the LDP.

1.1.2 Main results

Recall that a mapping $I : \mathscr{P}(H) \to [0, +\infty]$ is a *good rate function* if the level set $\{I(\sigma) \leq \alpha\}$ is compact for any $\alpha \geq 0$. Given a measure $\sigma \in \mathscr{P}(H)$, we set $\mathbb{P}_\sigma(\Gamma) = \int_H \mathbb{P}_u(\Gamma)\sigma(\mathrm{d}u)$, $\Gamma \in \mathscr{F}$, and introduce the family of *occupation measures*

$$\zeta_k = \frac{1}{k} \sum_{n=1}^{k} \delta_{u_n}, \quad k \geq 1, \tag{1.8}$$

defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P}_\sigma)$. The following theorem is a simplified version of the main result of [P14].

Theorem 1.1.2. *Assume that $b_j \neq 0$ for all $j \geq 1$. Then for any numbers $\delta > 0$ and $M > 0$, the family $\{\zeta_k, k \geq 1\}$ satisfies the LDP on $(\Omega, \mathscr{F}, \mathbb{P}_\sigma)$, uniformly with respect to the initial measure σ in the set*

$$\Lambda = \Lambda(\delta, M) := \left\{ \sigma \in \mathscr{P}(H) : \int_H e^{\delta \|v\|^2} \sigma(\mathrm{d}v) \leq M \right\},$$

and with a good rate function $I : \mathscr{P}(H) \to [0, +\infty]$ not depending on σ. More precisely, the following two bounds hold

Upper bound. *For any closed subset $F \subset \mathscr{P}(H)$, we have*

$$\limsup_{k \to \infty} \frac{1}{k} \log \sup_{\sigma \in \Lambda} \mathbb{P}_\sigma\{\zeta_k \in F\} \leq - \inf_{\sigma \in F} I(\sigma).$$

Lower bound. *For any open subset $G \subset \mathscr{P}(H)$, we have*

$$\liminf_{k \to \infty} \frac{1}{k} \log \inf_{\sigma \in \Lambda} \mathbb{P}_\sigma\{\zeta_k \in G\} \geq - \inf_{\sigma \in G} I(\sigma).$$

Literature review. The LDP is well understood in the case of finite-dimensional diffusions and Markov processes with compact phase space, provided that the randomness is sufficiently non-degenerate and ensures mixing in the total variation norm. This type of results were first obtained by Donsker and Varadhan [30] and later extended by many others (see the books [28, 29, 39] and the references therein). In the context of randomly forced PDE's, the problem of large deviations is mostly considered in the case of vanishing random perturbations and provide estimates for the probabilities of deviations from solutions of the limiting deterministic equations (e.g., see [18, 19, 21, 22, 38, 73, 99–101]). Before our works, the LDP for the occupation measures of randomly forced PDE's has been established by Gourcy [44, 45] in the case of stochastic Burgers and Navier–Stokes equations, based on some abstract results due to Wu [107]. In these papers, the force is assumed to be a rough white noise of the form (1.24) with the following condition on the coefficients:

$$c j^{-\alpha} \le b_j \le C j^{-\frac{1}{2}-\varepsilon}, \quad \frac{1}{2} < \alpha < 1, \ \varepsilon \in \left(0, \alpha - \frac{1}{2}\right]. \tag{1.9}$$

The lower bound in this condition does not allow the coefficients b_j to converge to zero sufficiently fast. So the external force is not regular with respect to the space variable, which is not very natural from the physical point of view. In the case of a perturbation which is a regular random kick force, the LDP is first proved in the papers [P9, P14] for a family of dissipative PDE's with parabolic regularisation. These results were extended in [P15] to the continuous-time setting for the stochastic damped nonlinear wave equation.

Scheme of the proof of Theorem 1.1.2. Let us briefly present the main ideas of the proof the theorem. *Step 1: Reduction.* The proof is based on the following three properties.

Property 1. For any $V \in C_b(H)$ and $\sigma \in \Lambda(\delta, M)$, the following limit exists (called *pressure function*)

$$Q(V) = \lim_{k\to+\infty} \frac{1}{k} \log \mathbb{E}_\sigma \exp\left(\sum_{n=1}^{k} V(u_k) \right), \tag{1.10}$$

it does not depend on the initial measure, and is uniform in $\sigma \in \Lambda(\delta, M)$.

If this property is satisfied, then $Q : C_b(H) \to \mathbb{R}$ is a convex 1-Lipschitz function, and its *Legendre transform* is given by

$$I(\sigma) := \begin{cases} \sup_{V\in C_b(H)} \big(\langle V, \sigma \rangle - Q(V) \big) & \text{for } \sigma \in \mathscr{P}(H), \\ +\infty & \text{for } \sigma \in \mathscr{M}(H) \setminus \mathscr{P}(H). \end{cases}$$

The function $I : \mathscr{M}(H) \to [0, +\infty]$ is convex lower semicontinuous in the weak topology, and Q can be reconstructed by the formula

$$Q(V) = \sup_{\sigma\in\mathscr{P}(H)} \big(\langle V, \sigma \rangle - I(\sigma) \big) \quad \text{for any } V \in C_b(H).$$

We shall say that a measure $\sigma \in \mathscr{P}(H)$ is an *equilibrium state* for $V \in C_b(H)$ if it verifies the following equality

$$Q(V) = \langle V, \sigma \rangle - I(\sigma).$$

Property 2. There is a vector space $\mathscr{V} \subset C_b(H)$ such that, for any compact set $K \subset H$, the family of restrictions to K of the functions in $\mathscr{V}$ is dense in $C(K)$, and for any $V \in \mathscr{V}$ there is a unique equilibrium state $\sigma_V \in \mathscr{P}(H)$.

Property 3. There is a function $\Phi : H \to [0, +\infty]$ with compact level sets $\{u \in H : \Phi(u) \le \alpha\}$ for any $\alpha \ge 0$ such that

$$\mathbb{E}_\sigma \exp\left(\sum_{n=1}^{k} \Phi(u_n) \right) \le Ce^{ck}, \quad \sigma \in \Lambda(\delta, M),\ k \ge 1,$$

for some positive constants C and c.

Properties 1-3 ensure that the conditions of Kifer's criterion are satisfied, which immediately implies the LDP. Here we use a non-compact version of the criterion, which is established in Theorem 3.3 in [P14] (see Theorem 2.1 in [55] for Kifer's original result in the compact case). The main part of the proof of Theorem 1.1.2 is the verification of these three properties.

Step 2: Proof of Properties 1-3. For any $V \in C_b(H)$, we introduce the following *Feynman–Kac semigroup*

$$\mathfrak{P}_k^V f(u) = \mathbb{E}_u \left\{ f(u_k) \exp\left(\sum_{n=1}^{k} V(u_n) \right) \right\}, \quad f \in C_b(H), \tag{1.11}$$

and denote by $\mathscr{V}$ the vector space of functions $V \in C_b(H)$ of the form $V(u) = F(\mathsf{P}_N u)$ for some integer $N \ge 1$ and function $F \in C_b(H_N)$. Clearly, $\mathscr{V}$ satisfies the density condition in Property 2. The proof of Properties 1 and 2 is based on the following multiplicative ergodic theorem, which is also of independent interest.

Theorem 1.1.3. *Under the conditions of Theorem 1.1.2, for any $V \in \mathscr{V}$, the following properties hold*

Existence and uniqueness. *The semigroup $\mathfrak{P}_k^{V*}$ admits a unique eigenvector $\mu_V \in \mathscr{P}(H)$ corresponding to an eigenvalue $\lambda_V > 0$. Moreover, the semigroup $\mathfrak{P}_k^V$ admits a unique eigenvector $h_V \in C_+(H)$ corresponding to λ_V normalised by $\langle h_V, \mu_V \rangle = 1$.*

Convergence. *For any $f \in C_b(H)$, $\sigma \in \mathscr{P}(H)$, and $R > 0$, we have*

$$\lambda_V^{-k} \mathfrak{P}_k^V f \to \langle f, \mu_V \rangle h_V \quad \text{in } C_b(B_H(R)) \cap L^1(H, \mu_V) \text{ as } k \to \infty, \tag{1.12}$$

$$\lambda_V^{-k} \mathfrak{P}_k^{V*} \sigma \to \langle h_V, \sigma \rangle \mu_V \quad \text{in } \mathscr{M}_+(H) \text{ as } k \to \infty, \tag{1.13}$$

where $\mathfrak{P}_k^{V*} : \mathscr{M}_+(H) \to \mathscr{M}_+(H)$ is the dual of $\mathfrak{P}_k^V$.

For any $V \in \mathscr{V}$, the existence of limit (1.10) is stablished by taking $f = \mathbf{1}$ in (1.12). Then the case of an arbitrary $V \in C_b(H)$ is obtained by using a simple approximation argument. To show Property 2, we first prove that any equilibrium state σ_V is a stationary measure for the following Markov semigroup:

$$\mathscr{S}_k^V g := \lambda_V^{-k} h_V^{-1} \mathfrak{P}_k^V (g h_V), \quad g \in C_b(H).$$

We then deduce the uniqueness of stationary measure for $\mathscr{S}_k^V$ from limit (1.13), showing that $\sigma_V(\mathrm{d}v) = h_V(v)\mu_V(\mathrm{d}v)$. Property 3 is verified by $\Phi(u) = \gamma \log(1 + \|u\|_1)$, where $\|\cdot\|_1$ stands for the H^1 Sobolev norm.

Thus Theorem 1.1.3 plays a central role in the proof of the LDP. It is established using an abstract result on large-time asymptotics of generalised Markov semigroups provided in [P14]. The main ingredients are the following four properties: *uniform irreducibility, exponential tightness, growth conditions,* and *uniform Feller property*. The verification of the last property is the most technical part of the proof. It is based on a coupling construction and some Foiaş–Prodi type estimates.

1.1.3 The case of bounded kicks

In the case of bounded kicks, considered in [P9], we take $h = 0$, in order to have a convergence to zero for the solutions of the unperturbed equation (deterministic) equation. In that paper, the sequence $\{\eta_k\}$ is assumed to satisfy the following condition.

(B) Structure of the noise. *The random variables η_k are of the form (1.6), where $b_j \geq 0$ are such that $\sum_{j=1}^{\infty} b_j^2 < \infty$, and ξ_{jk} are independent scalar random variables whose laws possess C^1-smooth densities ρ_j against the Lebesgue measure such that $\rho_j(0) > 0$ and $\operatorname{supp} \rho_j \subset [-1,1]$.*

Under this condition, one has the uniqueness of a stationary measure and an exponential mixing property as in Theorem 1.1.1. Moreover, this condition implies that $\mathscr{K} := \operatorname{supp} \mathscr{D}(\eta_1)$ is contained in a Hilbert cube, hence it is compact in H. Let us introduce the sequence of sets

$$\mathscr{A}_0 = \{0\}, \quad \mathscr{A}_k = S(\mathscr{A}_{k-1}) + \mathscr{K}, \quad k \geq 1,$$

and define the *domain of attainability from zero* $\mathscr{A} := \overline{\cup_{k=1}^{\infty} \mathscr{A}_k}$. The set $\mathscr{A}$ is of particular importance, since the restriction of system (1.5) on $\mathscr{A}$ is irreducible: for any $u, v \in \mathscr{A}$ and $r > 0$, there is an integer $k \geq 1$ such that $P_k(u, B_H(v,r)) > 0$, where $B_H(v,r)$ is the closed ball in H of radius r centred at v, and $P_k(u,\Gamma)$ is the transition function of the chain $(u_k, \mathbb{P}_u)$. $\mathscr{A}$ is compact in H and it is invariant for (1.5), i.e., if $u_0 \in \mathscr{A}$, then $u_k \in \mathscr{A}$ almost surely for any $k \geq 1$. So it carries the unique stationary measure μ (in fact $\operatorname{supp} \mu = \mathscr{A}$ by irreducibility).

The LDP for $(u_k, \mathbb{P}_u)$ on $\mathscr{A}$ is established in [P9]; the result is formulated as Theorem 1.1.3 with $\Lambda := \mathscr{P}(\mathscr{A})$. The proof here is simpler than the one in the case of unbounded kicks, due to the fact that the phase space $\mathscr{A}$ is compact. For example Property 3 in this case is automatically satisfied for any continuous functional Φ, since $\mathscr{P}(\mathscr{A})$ is compact, and we do not need to study the growth properties of the Feynman–Kac semigroup. We establish Property 1 for any initial measure $\sigma \in \mathscr{P}(\mathscr{A})$

and Property 2 with the same space $\mathscr{V}$ as in the unbounded case. Then the LDP on $\mathscr{A}$ is derived from Kifer's result obtained in [55].

1.2 Gallavotti–Cohen principle for Burgers equation

The aim of the paper [P12] is twofold: firstly, to establish the LDP for occupation measures of some randomly forced PDE's perturbed by irregular random kick-forces and, secondly, to derive a Gallavotti–Cohen type symmetry for the rate function corresponding to entropy production. We present the results of this paper in the case of the 1D Burgers equation on the circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\partial_t u - \nu\partial_x^2 u + u\partial_x u = h(x) + \eta(t,x), \quad x \in \mathbb{S}, \tag{1.14}$$

$$u(0,x) = u_0(x), \tag{1.15}$$

where $\nu > 0$ is a parameter, h is a function in the space H of square-integrable functions on $\mathbb{S}$ with zero mean value, and $\eta(t,x)$ is a random process of the form (1.4), where $\{\eta_k\}$ is a sequence of i.i.d. Gaussian random variables in H. We denote by ℓ the law of η_k.

As in the previous section, for any $u_0 \in H$, we denote by $\{u_k\}$ the restriction to integer times of the trajectory of (1.14), (1.15). Then it satisfies (1.5), where $S : H \to H$ denotes the time-1 shift along trajectories of $(1.14)_{\eta=0}$. Let $\zeta_k(u_0)$ be the occupation measure defined by

$$\zeta_k(u_0) = \frac{1}{k}\sum_{n=0}^{k-1} \delta_{\mathbf{u}_n}, \quad \mathbf{u}_n = (u_l, l \geq n),$$

where δ_v is the Dirac mass concentrated at $v = (v_l, l \geq 0)$ in the space of probability measures on $\mathbf{H} = H^{\mathbb{Z}_+}$. Let H^s be the space of functions in the Sobolev space of order s on $\mathbb{S}$ whose mean value is equal to zero. We establish the following result.

Theorem 1.2.1. *Assume that $h \in H^s$ for an integer $s \geq 0$ and ℓ is a centred Gaussian measure on H such that H^{s+1} is continuously embedded into its Cameron–Martin space. Then $\{u_k\}$ has a unique stationary measure $\mu \in \mathscr{P}(H)$, and for any $\sigma \in \mathscr{P}(H)$, we have*

$$\|\mathfrak{P}_k^*\sigma - \mu\|_{var} \leq Ce^{-\alpha k}\left(1 + \int_H \|u\|\sigma(\mathrm{d}u)\right), \quad k \geq 0$$

for some positive constants C and α. Moreover, for any $u_0 \in H$, the measures $\zeta_k(u_0)$ satisfy the LDP in the [3] τ_p-topology with a good rate function not depending on u_0.

[3] Here τ_p denotes the weakest topology on $\mathscr{P}(\mathbf{H})$ with respect to which all the functionals $\mu \mapsto \langle f, \mu\rangle$ with $f \in L^\infty(H^k)$ and any $k \geq 1$ are continuous.

The dissipativity and regularising properties of the Burgers equation imply that $\{u_k\}$ satisfies the *hyper-exponential recurrence* property and admits a *Lyapunov function*. The hypothesis on the Cameron–Martin space ensures the *irreducibility* and *uniform strong Feller* properties for the transition probabilities. Thus the conditions of Wu's criterion are satisfied (see Theorem 2.1 in [107]), which implies both the exponential mixing in the total variation norm and the LDP in the τ_p-topology.

We now turn to the question of the Gallavotti–Cohen fluctuation principle. Given a vector $a \in H$, let us denote by ℓ_a the image of ℓ under the translation in H by the vector a. The regularising property of S implies that, under the hypotheses of Theorem 1.2.1, the shifted measure $\ell_{S(u)}$ is equivalent to ℓ. Thus, the transition kernel of the chain $\{u_k\}$ is given by $P(u,\mathrm{d}v) = \ell_{S(u)}(\mathrm{d}v) = \rho(u,v)\ell(\mathrm{d}v)$, where the density $\rho(u,v)$ is positive for any $u \in H$ and ℓ-almost every $v \in H$. This further implies that, for any $k \geq 1$, the law λ_k of the random variable u_k is equivalent to ℓ, irrespective of the law λ_0 of the initial condition u_0. In particular, the stationary measure μ is equivalent to ℓ. We denote by ρ its density. The entropy of the system at time k is given by the relative entropy of λ_k with respect to the stationary measure μ:

$$S(\lambda_k) = \mathrm{Ent}(\lambda_k|\,\mu) = -\int_H \log\left(\frac{\mathrm{d}\lambda_k}{\mathrm{d}\mu}\right)\mathrm{d}\lambda_k.$$

Then the change of entropy in one time step is given by $\delta S(\lambda) = S(\mathfrak{P}_1^*\lambda) - S(\lambda)$. Let $\boldsymbol{\lambda}$ be the law induced on $\mathbf{H}$ by the initial distribution λ, and define the following function on $\mathbf{H}$:

$$J(\mathbf{u}) = \log\frac{\rho(u_0)\rho(u_0,u_1)}{\rho(u_1)\rho(u_1,u_0)}.$$

It is not difficult to verify that

$$\delta S(\lambda) = \mathrm{Ep}(\lambda) - \int_{\mathbf{H}} J(\mathbf{u})\boldsymbol{\lambda}(\mathrm{d}\mathbf{u}), \tag{1.16}$$

where the $\mathrm{Ep}(\,\cdot\,)$ is such that $\mathrm{Ep}(\lambda) \geq 0$ for all λ in the equivalence class of ℓ. Moreover, $\mathrm{Ep}(\lambda) = 0$ if and only if $\lambda = \mu$ and μ satisfies the detailed balance condition $\ell \otimes \ell$-almost everywhere on $H \times H$:

$$\rho(u)\rho(u,v) = \rho(v)\rho(v,u). \tag{1.17}$$

The validity of equation (1.17) is well known to be necessary and sufficient to ensure the time-reversal invariance of the Markov chain under the stationary law μ. The functional $\mathrm{Ep}(\,\cdot\,)$ is thus a measure of the breakdown of time-reversal invariance, a phenomenon usually connected with the production of entropy. We therefore identify $\mathrm{Ep}(\lambda)$ with the entropy production rate of the system in the state λ. Reading (1.16) as an entropy balance relation, we may consequently interpret the observable J as the entropy dissipated into the environment, i.e., the integral of the outgoing entropy flux over the unit time interval. Note that the vanishing of the entropy flux observable J is equivalent to the detailed balance condition (1.17). We prove that the unique stationary measure μ does not satisfy the detailed balance relation, so that $\mathrm{Ep}(\lambda) > 0$ for all λ. Let us define the sequence of

the random variables $\xi_k(\mathbf{u}) = \frac{1}{k}\sum_{n=0}^{k-1}\sigma(u_n,u_{n+1})$, where $\sigma(u,v) = \log\bigl(\rho(u,v)/\rho(v,u)\bigr)$. Then the entropy balance relation over k time steps has the form

$$\frac{1}{k}(S(\lambda_k) - S(\lambda_0)) = \frac{1}{k}\sum_{n=0}^{k-1}\mathrm{Ep}(\mathfrak{P}_n^*\lambda_0) - \int_{\mathbf{H}}\xi_k(\mathbf{u})\lambda(\mathrm{d}\mathbf{u}) + \frac{1}{k}\int_H \log\rho(u)(\lambda_k(\mathrm{d}u) - \lambda_0(\mathrm{d}u)).$$

The last term on the right hand side of this relation becomes negligible in the large time limit. It vanishes in the stationary regime, where the previous relation becomes

$$\mathrm{Ep}(\mu) = \int_{\mathbf{H}}\xi_k(\mathbf{u})\mu(\mathrm{d}\mathbf{u}) = \int_{H\times H}\rho(u)\rho(u,v)\sigma(u,v)\,\ell(\mathrm{d}u)\ell(\mathrm{d}v).$$

According to this, the mean entropy flux is non-negative. By the law of large numbers, the sequence ξ_k converges μ-a.s. towards $\mathrm{Ep}(\mu)$. The Gallavotti–Cohen fluctuation relation is a statement about the large deviations of ξ_k from this limit. Roughly speaking, it says that

$$\frac{\mu\,(\xi_k \simeq -r)}{\mu\,(\xi_k \simeq +r)} \simeq \mathrm{e}^{-kr} \quad \text{for large } k.$$

The fact that the entropy production rate is non-negative and the definition of the entropy flux observable σ are part of the general theory of non-equilibrium statistical mechanics in the mathematical framework of deterministic and stochastic dynamical systems [36, 40, 41, 51, 64, 68, 88–90]. On the other hand, detailed dynamical questions like strict positivity of the entropy production rate, LDP for the entropy flux, and validity of the Gallavotti–Cohen fluctuation relation can be answered only in the context of concrete models. In some cases, it is possible to relate the observable σ to the fluxes of some physical quantities, typically heat or some other forms of energy. In this respect, we refer the reader to [5] for the discussion of a closely related model. The following theorem is proved in [P12].

Theorem 1.2.2. *In addition to the hypotheses of Theorem 1.2.1, let us assume that $h \in H^{2s+1}$ and the set of normalised eigenvectors of the covariance operator for ℓ coincides with the trigonometric basis in H. Then, for any initial condition $u_0 \in H$, the laws of the random variables ξ_k satisfy the LDP with a good rate function $I : \mathbb{R} \to [0,+\infty]$ not depending on u_0. Moreover, the entropy production rate is strictly positive,*

$$\mathrm{Ep}(\mu) = \int_{\mathbf{H}}\sigma(u_0,u_1)\mu(\mathrm{d}\mathbf{u}) > 0,$$

and the Gallavotti–Cohen fluctuation relation holds:

$$I(-r) = I(r) + r \quad \textit{for } r \in \mathbb{R}.$$

Literature review. There is an enormous literature on mathematical, physical, numerical, and experimental aspects of Gallavotti–Cohen fluctuation relation (some of the references can be found in [51, 88]). The previous mathematically rigorous works closest to ours are [32, 34, 35, 65, 84]. Lebowitz and Spohn [65], building on the previous work by Kurchan [64], have developed a general

theory of Gallavotti–Cohen fluctuation relations for finite-dimensional Markov processes with applications to various models, including diffusion and simple exclusion processes. In [32, 34, 35], the authors consider a finite anharmonic chain coupled to two thermal reservoirs at its ends. Its analysis reduces to a study of suitable finite-dimensional Markov process with degenerate noise. In particular, the local Gallavotti–Cohen fluctuation relation for this model has been established in [84]. To the best of our knowledge, there were no previous mathematically rigorous studies of Gallavotti–Cohen fluctuation relation for nonlinear PDE's driven by a stochastic forcing. On the physical level of rigour, Maes and coworkers [69–71] have examined in depth the fluctuation relation for stochastic dynamics. In a somewhat different spirit, inspired by the thermodynamic formalism of dynamical systems, we should also mention the works of Gaspard [41] and Lecomte *et al.* [66].

In conclusion, let us note that the LDP for the Burgers equation stated in Theorem 1.2.1 is true also for the Navier–Stokes system and the complex Ginzburg–Landau equation, while the Gallavotti–Cohen fluctuation relation of Theorem 1.2.2 remains valid for problems with strong nonlinear dissipation, such as the reaction–diffusion system with superlinear interaction. Moreover, the law of η_k does not need to be Gaussian, and the results we prove are true for a rather general class of decomposable measures.

1.3 LDP for stochastic damped nonlinear wave equation

In [P15], with D. Martirosyan we consider the stochastic damped nonlinear wave (NLW) equation in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary:

$$\partial_t^2 u + \gamma \partial_t u - \Delta u + f(u) = h(x) + \eta(t,x), \quad u|_{\partial D} = 0, \tag{1.18}$$

$$[u(0), \dot{u}(0)] = [u_0, u_1], \tag{1.19}$$

where $\gamma > 0$ is a damping parameter and h is a function in $H_0^1(D)$. The nonlinear term f is assumed to verify some standard dissipativity and growth conditions, which are satisfied for the classical examples $f(u) = \sin u$ and $f(u) = |u|^\rho u - \lambda u$, where $\lambda \in \mathbb{R}$ and $\rho \in (0,2)$, coming from the damped sine–Gordon and Klein–Gordon equations. The force $\eta(t,x)$ is a white noise of the form

$$\eta(t,x) = \partial_t \xi(t,x), \quad \xi(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{1.20}$$

where $\{\beta_j\}$ is a sequence of independent standard Brownian motions defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration [4] $\{\mathscr{F}_t, t \geq 0\}$, and $\{b_j\}$ is a sequence of strictly positive numbers satisfying $\sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty$.

[4] The filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ is assumed to satisfy the usual condition, see Definition 2.29 in [53].

This stochastic NLW equation defines a family of continuous-time Markov processes $(\mathfrak{u}_t, \mathbb{P}_\mathfrak{u})$, $\mathfrak{u}_t = [u_t, \dot{u}_t]$ in the space $\mathscr{H} := H^1_0 \times L^2$, parametrised by $\mathfrak{u} = [u_0, u_1] \in \mathscr{H}$. The exponential ergodicity of this family is established in [72]. The aim of [P15] was to extend the results and the methods of the works [P9, P14] and [44, 45], under more general assumptions on both stochastic and deterministic parts of the equation. The random perturbation in our setting is a spatially regular white noise (we do not assume that the sequence $\{b_j\}$ satisfies a condition similar to (1.9)), and the NLW equation lacks a regularising property. We prove the following level-1 LDP for the solutions of problem (1.18), (1.19).

Theorem 1.3.1. *For any non-constant bounded Hölder-continuous function $V : \mathscr{H} \to \mathbb{R}$, there is a number $\varepsilon = \varepsilon(V) > 0$ and a convex function $I^V : \mathbb{R} \to \mathbb{R}_+$ such that, for any initial condition $\mathfrak{u} \in \mathscr{H}^\mathfrak{s} := H^{\mathfrak{s}+1} \times H^\mathfrak{s}$ and any open subset O of the interval $(\langle V, \mu\rangle - \varepsilon, \langle V, \mu\rangle + \varepsilon)$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\mathfrak{u}\{\zeta_t^V \in O\} = -\inf_{\alpha\in O} I^V(\alpha),$$

where μ is the stationary measure of $(\mathfrak{u}_t, \mathbb{P}_\mathfrak{u})$, $\mathfrak{s} > 0$ is a small number, and $\zeta_t^V = \frac{1}{t}\int_0^t V(\mathfrak{u}_\tau)\mathrm{d}\tau$. Moreover, this limit is uniform with respect to $\mathfrak{u}$ in a bounded set of $\mathscr{H}^\mathfrak{s}$.

In the paper we also establish a more general result of level-2 type with a local lower bound; we do not recall its formulation here. These two theorems are slightly different from the standard Donsker–Varadhan form (e.g., see Theorem 3 in [30]), since we prove that the LDP holds only locally on some part of the phase space.

This theorem is proved using a local version of the Gärtner–Ellis theorem (see Theorem A.5 in [52]). According to that result, the limit in Theorem 1.3.1 will be established if we show that the following limit exists

$$Q(\beta) = \lim_{t\to+\infty} \frac{1}{t} \log \mathbb{E}_\mathfrak{u} \exp\left(\int_0^t \beta V(\mathfrak{u}_\tau)\mathrm{d}\tau\right), \quad |\beta| < \beta_0$$

for some $\beta_0 > 0$, and the limit $Q(\beta)$ is differentiable function on the interval $(-\beta_0, \beta_0)$. We derive both conditions from the following continuous-time version of the multiplicative ergodic theorem (cf. Theorem 1.1.3). Existence of limit follows immediately from limit (1.21) with $\psi = 1$. The differentiability is more delicate, it is derived from (1.22) using some convex analysis technics.

Theorem 1.3.2. *For the stochastic NLW equation the Feynman–Kac semigroup is defined by*

$$\mathfrak{P}_t^V \psi(u) = \mathbb{E}_u \left\{ \psi(u_t) \exp\left(\int_0^t V(u_\tau)\mathrm{d}\tau\right)\right\}, \quad \psi \in C_b(\mathscr{H}).$$

Then there are $\delta > 0$ and $\mathfrak{s} > 0$ such that, for any $V \in \mathscr{V}$ satisfying $\mathrm{Osc}(V) < \delta$, we have

Existence and uniqueness. *The semigroup $\mathfrak{P}_t^{V*}$ admits a unique eigenvector $\mu_V \in \mathscr{P}(\mathscr{H})$ corresponding to an eigenvalue $\lambda_V > 0$, and the semigroup $\mathfrak{P}_t^V$ admits a unique eigenvector $h_V \in C_+(\mathscr{H}^\mathfrak{s})$ corresponding to λ_V normalised by the condition $\langle h_V, \mu_V\rangle = 1$.*

Convergence. *For any $\psi \in C_b(\mathscr{H}^{\mathfrak{s}})$, $\sigma \in \mathscr{P}(\mathscr{H})$, and $R > 0$, we have*

$$\lambda_V^{-t}\mathfrak{P}_t^V \psi \to \langle \psi, \mu_V \rangle h_V \quad \text{in } C_b(B_{\mathscr{H}^{\mathfrak{s}}}(R)) \cap L^1(\mathscr{H}, \mu_V) \text{ as } t \to \infty, \tag{1.21}$$

$$\lambda_V^{-t}\mathfrak{P}_t^{V*}\sigma \to \langle h_V, \sigma \rangle \mu_V \quad \text{in } \mathscr{M}_+(\mathscr{H}) \text{ as } t \to \infty. \tag{1.22}$$

As in the discrete-time case considered in [P14], the proof of this multiplicative ergodic theorem is based on uniform irreducibility, exponential tightness, growth condition, and uniform Feller property. The smoothness of the noise and the lack of a regularising property in the equation result in substantial differences in the techniques used to verify these conditions. While in the case of kick-forced models the first two of them are checked directly, they have a rather non-trivial proof in the case of the stochastic NLW equation, relying on a feedback stabilisation result and some estimates for the Sobolev norms of the solutions. Nonetheless, the most involved and highly technical part of the paper remains the verification of the uniform Feller property. Based on the coupling method, its proof is more intricate here, mainly due to a more complicated Foiaş–Prodi type estimate for the stochastic NLW equation. We get a uniform Feller property only for potentials V that have a sufficiently small oscillation, and this is the main reason why we have a constant $\delta > 0$ in the formulation of Theorem 1.3.2 and why the LDP established in this work is of a local type.

1.4 Mixing for stochastic CGL equations in any space dimension

In this section, we present the results obtained with S. Kuksin in [P8]. We consider the stochastic complex Ginzburg–Landau (CGL) equation

$$\partial_t u - \nu\Delta u + i|u|^2 u = \eta(t,x), \quad \dim x = n, \tag{1.23}$$

where the space dimension n is *arbitrary*, $\nu > 0$ is a given number, and η is a random force. This equation is the Hamiltonian system $\dot{u} + i|u|^2 u = 0$, damped by the viscous term $\nu\Delta u$ and driven by the random force η. So it makes a model for the stochastic Navier–Stokes system, which may be regarded as a damped–driven Euler equation. We consider the problem on the cube $K = [0,\pi]^n$ with the Dirichlet boundary conditions, and the solution is assumed to be odd periodic in x. The force $\eta(t,x)$ is white in time and regular in x:

$$\eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_{d\in\mathbb{N}^n} b_d \beta_d(t) e_d(x), \tag{1.24}$$

where b_d are real numbers such that $B_* := \sum_{d\in\mathbb{N}^n} |b_d| < \infty$, β_d are standard independent complex-valued Brownian motions, and $\{e_d(x), d \in \mathbb{N}^n\}$ is an orthonormal basis in $L^2(K)$ formed by the eigenfunctions of the Dirichlet Laplacian.

We do not impose any restriction on the dimension n, so the global solvability of equation (1.23) cannot be established using the L^2-Sobolev spaces. Moreover, as the best a priori estimates, available for its solutions, turned out to be in terms of the L^∞-norm, the methods developed to treat stochastic PDE's in reflexive Banach spaces (e.g., see [26]) also are not applicable to (1.23). Instead we take the approach of the work [57] which exploits fact that the deterministic equation $(1.23)_{\eta=0}$ implies for the real function $|u(t,x)|$ a parabolic inequality with the maximum principle. The main result of [57] states that if $u_0 \in H^m$, $m > n/2$, and $\sum_d b_d^2 |d|^{2m} < \infty$, then (1.23) has a unique strong solution u in the space H^m. Moreover, for any $T \geq 0$, the random variable $X_T = \sup_{T \leq t \leq T+1} \|u(t)\|_\infty^2$ satisfies the estimates

$$\mathbb{E} X_T^q \leq C_q(B_*, \|u_0\|_\infty) \qquad \forall q \geq 0. \tag{1.25}$$

Analysis of the constants C_q implies that the exponential moments of the variables X_T are finite:

$$\mathbb{E} e^{cX_T} \leq C' = C'(B_*, \|u_0\|_\infty), \tag{1.26}$$

where $c > 0$ depends only on B_*. Let $C_0(K)$ be the space of continuous complex functions on K vanishing at ∂K. Approximating the initial condition $u_0 \in C_0(K)$ and the process ζ by regular sequences u_0^n and ζ^n, and using the fact that the constants in (1.25) and (1.26) depend only on B_* and $\|u_0\|_\infty$, we prove the following result.

Theorem 1.4.1. *Let $B_* < \infty$ and $u_0 \in C_0(K)$. Then problem (1.23) has a unique strong solution $u(t,x)$ which almost surely belongs to the space $C(\mathbb{R}_+, C_0(K)) \cap L^2_{loc}(\mathbb{R}_+, H^1)$. The solutions u define in $C_0(K)$ a Feller Markov process.*

Using the Foiaş–Prodi estimate and the Girsanov theorem, we prove the following properties for the solutions constructed in this theorem

Stability. There is $L > 0$ and sequences $\{T_m \geq 0\}$ and $\{\varepsilon_m > 0\}$, $\varepsilon_m \to 0$ as $m \to \infty$ such that

$$\sup_{t \geq T_m} \|\mathscr{D}(u(t)) - \mathscr{D}(u'(t))\|_L^* \leq \varepsilon_m,$$

provided that $u(0)$ and $u'(0)$ belong to the set

$$G_m = \{u \in C_0(K) : \|u\| \leq 1/m,\ \|u\|_{L_\infty} \leq L\}.$$

Here $\|\cdot\|$ is the L^2-norm on the cube K.

Recurrence. For any $m \geq 1$ and $u_0, u_0' \in C_0(K)$, the hitting time

$$\inf\{t \geq 0 : u(t) \in G_m, u'(t) \in G_m\},$$

where $u(t)$ and $u'(t)$ are two independent solutions of (1.23) such that $u(0) = u_0$ and $u'(0) = u_0'$, is almost surely finite.

These two properties allow us to use Theorem 3.1.3 from [63], which implies the second main result of this work:

Theorem 1.4.2. *There is an integer $N = N(B_*, \nu) \geq 1$ such that if $b_d \neq 0$ for $|d| \leq N$, then the Markov process, constructed in Theorem 1.4.1, has a unique stationary measure μ, and every solution u converges to μ in distribution.*

These results remain true also in the case of smooth bounded domains in $\mathbb{R}^n$ with Dirichlet boundary conditions, under appropriate hypotheses on the eigenfunctions $\{e_d\}$. Theorem 1.4.1 generalises to equations

$$\partial_t u - \nu\Delta u + (i+a)g_r(|u|^2)u = \eta(t,x), \tag{1.27}$$

where $g_r(t)$ is a smooth function, equal to $t^r, r \geq 0$, for $t \geq 1$, and Theorem 1.4.2 generalises to (1.27) with $0 \leq r \leq 1$. Similar results hold for the CGL equations (1.27), where η is a kick force, without the restriction that the nonlinearity is cubic (see [63]). Same is true when η is the derivative of a compound Poisson process (see [P2]). Our technique does not apply to equations (1.27) with complex ν. To prove analogies of Theorems 1.4.1 and 1.4.2 for such equations, strong restrictions should be imposed on n and r. See [48, 82] for equations with $\operatorname{Re}\nu > 0$ and $a > 0$, and [94] for the case $\operatorname{Re}\nu > 0$ and $a = 0$.

Chapter 2

Global controllability properties of some PDE's

2.1 Controllability of bilinear Schrödinger equation

2.1.1 Approximate controllability

In this section, we discuss the problem of controllability of a quantum particle in an m-dimensional space driven by a time-dependent amplitude of an electric field. The position of the particle is described [1] by a wave function $\psi(t,x)$, which obeys the Schrödinger equation. In the first-order approximation and after proper rescaling, the equation takes the following bilinear form

$$i\partial_t \psi = -\Delta\psi + V(x)\psi + u(t)Q(x)\psi, \quad x \in D, \tag{2.1}$$

$$\psi|_{\partial D} = 0, \tag{2.2}$$

$$\psi(0,x) = \psi_0(x), \tag{2.3}$$

where $V(x)$ is the potential in which the system evolves, and $Q(x)$ and $u(t)$ are, respectively, the dipole moment and the amplitude of the field. We assume that $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, and Q and V are given real-valued smooth functions. We consider (2.1)-(2.3) as a control problem in which the state ψ depends in a nonlinear way on the control u. For any $\psi_0 \in L^2$ and $u \in L^1_{loc}(\mathbb{R}_+,\mathbb{R})$, the equation has a unique solution $\psi \in C(\mathbb{R}_+,L^2)$, and under some compatibility and regularity conditions on ψ_0 and u, the solution belongs to more regular Sobolev spaces. Furthermore, the resolving operator $\mathscr{U}_t(\cdot,u) : L^2 \to L^2$ taking ψ_0 to $\psi(t)$ satisfies the relation $\|\mathscr{U}_t(\psi_0,u)\| = \|\psi_0\|$, so we consider the restriction of this system to the unit sphere S in L^2.

Literature review. The problem of controllability of the bilinear Schrödinger equation has been widely studied in the literature; let us briefly recall few results closely related with our papers [P4-P7, P10, P11]. A negative controllability result is proved by Turinici [102] as a corollary of a general

[1] Recall that $\int_E |\psi(t,x)|^2 \mathrm{d}x$ is the probability of finding the particle in the region $E \subset \mathbb{R}^m$ at time t.

result by Ball, Marsden, and Slemrod [6]. It states that the set of attainability $\mathscr{A}(\psi_0, L^2)$ of the system from an initial point $\psi_0 \in S$ with L^2 controls, defined by

$$\mathscr{A}(\psi_0, L^2) := \{\mathscr{U}_t(\psi_0, u) : \text{ for all } u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}) \text{ and } t \geq 0\},$$

has a dense complement in S. In particular, this shows that problem (2.1)-(2.3) is not exactly controllable in S. Beauchard proved in [7] that this negative result is due to the choice of the functional setting. Indeed, using Coron's return method, quasi-static deformations, and Nash–Moser theorem, she established local exact controllability in the space H^7 in the case $m = 1$, $Q(x) = x$, and $V(x) = 0$. Beauchard and Laurent [10] extended this result to the case of the space H^3, under generic conditions on the dipole moment Q; they also simplified considerably the proof. In [8], Beauchard and Coron proved local exact controllability between some neighborhoods of two different eigenstates. All these results are based on the spectral gap property of the Schrödinger operator and they do not admit a straightforward generalisation to the multidimensional case (see Theorem 2.1.7).

Approximate controllability results are available in the multidimensional setting. The first result is due to Chambrion, Mason, Sigalotti, and Boscain [20], which relies on some methods from the geometric control theory. The hypotheses of this result were refined later in [12, 13]. For more details and references on the geometric techniques, we refer the reader to the survey [14]. The Lyapunov method was used by Mirrahimi [77] in the case of a mixed spectrum, and by Beauchard and Mirrahimi [11] in the case $Q(x) = x$ and $V = 0$. Both papers prove approximate stabilisation in L^2; an exact stabilisation property is established in [P4]. In this subsection, we discuss the results of [P5], where global approximate controllability is established in higher Sobolev spaces with generic assumptions on the potential and the dipole moment. We shall discuss the contributions of the papers [P6, P7, P10, P11] in the next subsections.

In the paper [P5] we assume that V and Q satisfy the following condition.

Condition 2.1.1. *The functions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are such that:*

(i) $\langle Qe_{1,V}, e_{j,V}\rangle \neq 0$ *for all* $j \geq 2$,

(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *for all* $j, p, q \geq 1$ *such that* $\{1, j\} \neq \{p, q\}$ *and* $j \neq 1$.

Here $\langle \cdot, \cdot \rangle$ is the scalar product in L^2, and $\{e_{j,V}\}$ is an orthonormal basis in L^2 consisting of eigenfunctions of the Schrödinger operator $-\Delta + V$ corresponding to eigenvalues $\{\lambda_{j,V}\}$. The genericity of this condition with respect to the functions V and Q and the domain D is proved in [75, 83], and [P5]. The following three theorems are established in [P5].

Theorem 2.1.2. *Under Condition 2.1.1, for any $s > 0$, problem (2.1), (2.2) is approximately controllable to $e_{1,V}$ in H^s, i.e., for any $\delta > 0$ and $\psi_0 \in S \cap H^{s+\delta}_{(V)}$, there is a time $T > 0$ and control $u \in L^2((0,T), \mathbb{R})$ such that*

$$\|\mathscr{U}_T(\psi_0, u) - e_{1,V}\|_s < \varepsilon.$$

In many relevant examples, the spectrum of the operator $-\Delta+V$ is degenerate, hence property (ii) in Condition 2.1.1 is not verified. The proof of Theorem 2.1.2 can be adapted to show controllability of the Schrödinger equation in the case of an arbitrary potential V, under stronger assumptions on the dipole moment Q:

Theorem 2.1.3. *Let $V \in C^\infty(\overline{D},\mathbb{R})$ and $s>0$ be arbitrary. Then problem (2.1), (2.2) is approximately controllable to $e_{1,V}$ in H^s generically* [2] *with respect to Q in $C^\infty(\overline{D},\mathbb{R})$.*

In the case $m=1$ and $V=0$, combination of this result with the local exact controllability property obtained by Beauchard and Laurent [7, 10] gives global exact controllability in the space $H^{3+\delta}_{(0)}$, $\delta>0$. More precisely, generically with respect to Q in $C^\infty(\overline{D},\mathbb{R})$, for any $\psi_0,\psi_1 \in S\cap H^{3+\delta}_{(0)}$, there are $T>0$ and $u\in L^2([0,T],\mathbb{R})$ such that $\mathscr{U}_T(\psi_0,u)=\psi_1$.

Problem (2.1), (2.2) is said to be approximately controllable in L^2 if for any $\psi_0,\psi_1\in S$ there are $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that

$$\|\mathscr{U}_T(\psi_0,u)-\psi_1\|<\varepsilon.$$

Combining Theorem 2.1.3 with the time reversibility property of the Schrödinger equation, we obtain approximate controllability in L^2.

Theorem 2.1.4. *Let $V\in C^\infty(\overline{D},\mathbb{R})$ be arbitrary. Then problem (2.1), (2.2) is approximately controllable in L^2 generically with respect to Q in $C^\infty(\overline{D},\mathbb{R})$.*

Scheme of the proof of Theorem 2.1.2. We introduce the following Lyapunov function

$$\mathscr{V}(z)=\mathscr{V}_{s+\delta}(z):=\alpha\|(-\Delta+V)^{\frac{s+\delta}{2}}P_{1,V}z\|^2+1-|\langle z,e_{1,V}\rangle|^2,\ \ z\in S\cap H^{s+\delta}_{(V)},$$

where $\alpha>0$ is a parameter that will be chosen later and $P_{1,V}z:=z-\langle z,e_{1,V}\rangle e_{1,V}$. Then $\mathscr{V}(z)\geq 0$ for $z\in S\cap H^{s+\delta}_{(V)}$ and $\mathscr{V}(z)=0$ if and only if z belongs to the steady state $\mathscr{C}:=\{ce_{1,V},c\in\mathbb{C},|c|=1\}$. In general, we are not able to choose a feedback low $u(\psi)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}(\psi(t))<0$ (in contrast to the case $s+\delta=2$; see next section). However, for any $w\in C^\infty([0,T],\mathbb{R})$, the derivative $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathscr{V}(\mathscr{U}_t(\psi_0,\sigma w))|_{\sigma=0}$ can be calculated explicitly. We show that, for an appropriate choice of the parameter α, there is a time $T>0$ and a control w such that $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathscr{V}(\mathscr{U}_t(\psi_0,\sigma w))|_{\sigma=0}\neq 0$. Hence we can choose σ_0 close to zero such that

$$\mathscr{V}(\mathscr{U}_T(\psi_0,\sigma_0 w))<\mathscr{V}(\mathscr{U}_T(\psi_0,0))=\mathscr{V}(\psi_0).$$

Thus for any ψ_0, we find a time $T>0$ and control u such that $\mathscr{V}(\mathscr{U}_T(\psi_0,u))<\mathscr{V}(\psi_0)$. Using an iteration argument, we conclude that there are sequences T_n and u_n such that $\mathscr{U}_{T_n}(\psi_0,u_n)\to e_{1,V}$ in H^s as $n\to\infty$. Thus any point ψ_0 can be approximately controlled to $e_{1,V}$.

[2] This means that the set of dipole moments Q, for which the approximate controllability property holds, contains a residual set in $C^\infty(\overline{D},\mathbb{R})$. Recall that a set is residual if it is a countable intersection of open dense sets. Here the space $C^\infty(\overline{D},\mathbb{R})$ is endowed with the usual topology given by the countable family of norms $p_n(Q)=\sum_{|\alpha|\leq n}\sup_{x\in D}|\partial^\alpha Q(x)|$.

Applications. The main topics of Chapters 1 and 2, ergodicity and control, are known to be closely related fields. For example, the approximate controllability property can be used to derive some uniqueness results for the randomly forced Schrödinger equation. Indeed, let us consider problem (2.1), (2.2) and assume that u is a random process of the form

$$u(t) = \sum_{k=0}^{+\infty} \mathbb{I}_k(t)\eta_k(t-k), \quad t \geq 0, \tag{2.4}$$

where $\mathbb{I}_k(\cdot)$ is the indicator function of the interval $[k, k+1)$ and η_k are i.i.d. random variables in $L^2((0,1),\mathbb{R})$ satisfying some non-degeneracy conditions. Then the Markov chain associated with this problem admits at most one stationary measure on the sphere S. This can be seen by using the irreducibility of the transition functions (which is a consequence of the approximate controllability established in Theorem 2.1.4) and the uniform Feller property (see Theorem 3.5 in [P5]). Existence of a stationary measure, different from the Dirac mass at zero, is an open problem. In the case of finite-dimensional approximations of the Schrödinger equation, the exponential mixing property is proved in [P3], by combining controllability properties and a measure transformation theorem. It is an interesting task to generalise the methods used in that paper to the infinite-dimensional case, where the existing measure transformation theorems seem to be incompatible with the available controllability results.

Approximate controllability also implies that the trajectories of the randomly forced equation are almost surely non-bounded in Sobolev spaces:

$$\mathbb{P}\{\limsup_{t\to\infty} \|\mathscr{U}_t(\psi_0, u)\|_s = \infty\} = 1$$

for any $s > 0$ and $\psi_0 \in H^s \backslash \{0\}$ (see [P4]). In particular, this result completes in some sense the results [105] and [106] of W.-M. Wang, by showing that the growth of the Sobolev norms is a generic phenomenon.

2.1.2 Stabilisation

In this section, we discuss a feedback stabilisation result established in [P6] with K. Beauchard. The main difficulty of the problem comes from the fact that the usual LaSalle invariance principle cannot be directly applied in the infinite-dimensional setting, due to a lack of some compactness properties for the trajectories for the closed loop system (see [9] for the case of the finite-dimensional approximations of the Schrödinger equation). We develop a version of the LaSalle principle based on a weak convergence approach. Following a strategy proposed in [P4], we introduce the Lyapunov function

$$\mathscr{V}(z) = \mathscr{V}_2(z) := \alpha\|(-\Delta + V)P_{1,V}z\|^2 + 1 - |\langle z, e_{1,V}\rangle|^2, \ \ z \in S \cap H^2_{(0)}.$$

We wish to find a feedback law $u : H^2_{(0)} \to \mathbb{R}$ that stabilises the trajectories of (2.1)-(2.3) to the steady state $\mathscr{C}$. Let $\psi(t) = \mathscr{U}_t(\psi_0, u)$ be the trajectory corresponding to some ψ_0 and u. Then direct

verification shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}(\psi(t)) = 2u\,\mathrm{Im}\Big[\alpha\langle(-\Delta+V)P_{1,V}(Q\psi),(-\Delta+V)P_{1,V}\psi\rangle - \langle Q\psi,e_{1,V}\rangle\langle e_{1,V},\psi\rangle\Big].$$

So defining a feedback law $u(\psi)$ by

$$u(\psi) := -\,\mathrm{Im}\Big[\langle\alpha(-\Delta+V)P_{1,V}(Q\psi),(-\Delta+V)P_{1,V}\psi\rangle - \langle Q\psi,e_{1,V}\rangle\langle e_{1,V},\psi\rangle\Big], \tag{2.5}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}(\psi(t)) = -2u^2(\psi(t)). \tag{2.6}$$

Thus $t\mapsto\mathscr{V}(\psi(t))$ is non-increasing and one may expect that $\psi(t)\to\mathscr{C}$, in some sense, as $t\to\infty$. The local well-posedness of closed loop system (2.1)-(2.3) with feedback law $u=u(\psi)$ is standard, and a finite-time blow-up in $H^2_{(0)}$ is impossible by the construction of u. Hence the system has a global in time solution denoted by $\mathscr{U}_t(\psi_0)$. The following theorem is the main result of [P6].

Theorem 2.1.5. *Under Condition 2.1.1, there is an at most countable set $J\subset\mathbb{R}_+^*$ such that for any $\alpha\notin J$ and $\psi_0\in S\cap H^2_{(0)}$ with $0<\mathscr{V}(\psi_0)<1$ we have $\mathscr{U}_t(\psi_0)\rightharpoonup\mathscr{C}$ in H^2 as $t\to\infty$.*

Let us briefly explain the main ideas of the proof of the weak convergence in this theorem. Let $\psi_\infty\in H^2_{(0)}$ and $T_n\to\infty$ be such that $\mathscr{U}_{T_n}(\psi_0)\rightharpoonup\psi_\infty$ in H^2 as $n\to\infty$. We need to show that $\psi_\infty\in\mathscr{C}$. From (2.6) it follows that the function $t\mapsto u[\mathscr{U}_t(\psi_0)]$ belongs to $L^2(0,+\infty)$. Hence the sequence of functions $(t\in(0,+\infty)\mapsto u[\mathscr{U}_{T_n+t}(\psi_0)])_{n\in\mathbb{N}}$ tends to zero in $L^2(0,+\infty)$ as $n\to\infty$, and there is a subsequence $k_n\to\infty$ such that $u[\mathscr{U}_{T_{k_n}+t}]\to 0$ for $t\in(0,+\infty)$ a.s.. Since the system is autonomous, we have $u[\mathscr{U}_{T_{k_n}+t}(\psi_0)]\to u[\mathscr{U}_t(\psi_\infty)]$. The uniqueness of the limit implies that $u[\mathscr{U}_t(\psi_\infty)]=0$, so $\mathscr{U}_t(\psi_\infty)$ is a solution of the free Schrödinger equation (i.e., with $u=0$). From the assumption $\mathscr{V}(\psi_\infty)\le\mathscr{V}(\psi_0)<1$ and Condition 2.1.1 we deduce that $\psi_\infty\in\mathscr{C}$, provided that α belongs to the complement of some at most countable set $J\subset\mathbb{R}_+^*$.

Finally, let us note that the condition $0<\mathscr{V}(\psi_0)<1$ in Theorem 2.1.5 is not restrictive, since it can be achieved for any $\psi_0\in S\cap H^2_{(0)}$ such that $\psi_0\notin\mathscr{C}$, by choosing $\alpha=\alpha(\|\psi_0\|_2)>0$ sufficiently small.

2.1.3 Simultaneous exact controllability

In [P10], we consider with M. Morancey the problem of controllability of N identical independent 1D quantum particles submitted to the same external field described by the following system of equations

$$\begin{cases} i\partial_t\psi^j = \left(-\partial_{xx}^2+V(x)\right)\psi^j - u(t)Q(x)\psi^j, & (t,x)\in(0,T)\times(0,1),\\ \psi^j(t,0)=\psi^j(t,1)=0, & j\in\{1,\dots,N\},\\ \psi^j(0,x)=\psi_0^j(x). \end{cases} \tag{2.7}$$

The state is the vector of wave functions $\Psi(t) := (\psi^1(t), \dots, \psi^N(t))$ and the control is the real-valued function u. For any initial condition $\Psi_0 := (\psi_0^1, \dots, \psi_0^N)$ in the unit sphere $\mathbf{S} := S^N$, the solution $\Psi(t)$ belongs to $\mathbf{S}$. We say that the vectors $\Psi_0, \Psi_f \in \mathbf{S}$ are unitarily equivalent, if there is a unitary operator $\mathscr{U}$ in L^2 such that $\Psi_f = \mathscr{U}\Psi_0$, i.e. $\psi_f^j = \mathscr{U}\psi_0^j$ for all $j = 1, \dots, N$. Note that $\Psi(t)$ is unitarily equivalent to Ψ_0 for any $t > 0$. The following theorem is the main result of [P11].

Theorem 2.1.6. *For any given $V \in H^4((0,1),\mathbb{R})$, problem (2.7) is globally exactly controllable in $H^4_{(V)}$ generically with respect to Q in $H^4((0,1),\mathbb{R})$. More precisely, there is a residual set $\mathscr{Q}_V$ in $H^4((0,1),\mathbb{R})$ such that, for any $Q \in \mathscr{Q}_V$ and any unitarily equivalent vectors $\Psi_0, \Psi_f \in \mathbf{S} \cap H^4_{(V)}$, there is a time $T > 0$ and a control $u \in L^2((0,T),\mathbb{R})$ such that the solution of (2.7) satisfies $\Psi(T) = \Psi_f$.*

Simultaneous exact controllability of quantum particles has been obtained for a finite-dimensional model in [104] by Turinici and Rabitz. In [13], Boscain, Caponigro, and Sigalotti obtain a sufficient condition for the simultaneous approximate controllability for the infinite-dimensional equation, which allows the presence of infinitely many degeneracies and resonances in the spectrum. Simultaneous exact controllability results are obtained by Morancey [79], in the case $V = 0$ and $N = 2, 3$. He proves local exact controllability around the eigenstates either up to a global phase or a global delay. In the arguments of [79] a central role is played by a construction of a suitable reference trajectory which coincides at the final time (up to global phase and a global delay) with the vector of eigenstates. Extension of this result to the case $N \geq 4$ presents two difficulties: in the trigonometric moment problem, which is solved to construct the reference trajectory, some resonant frequencies appear (e.g., $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$, where $\lambda_k = (k\pi)^2$) and the frequency 0 appears with multiplicity N. So the use of a global phase and a global delay is not sufficient. In [P10], we use another approach. Using a Lyapunov strategy, we prove that any initial condition can be driven arbitrarily close to some position φ which is a finite sum of eigenfunctions. Then, designing a reference trajectory and using a rotation property on a suitable time interval combined with a perturbation argument, we prove local exact controllability in $H^3_{(V)}$ around φ. Finally, combining the fact that the equation is linear with respect to the state and a compactness argument, we obtain global exact controllability under favorable hypotheses on Q.

Let us also note that in Theorem 2.1.6 the potential V is arbitrary and the dipole moment Q is generic, so for the case $N = 1$ this theorem is an improvement of the results of Section 2.1.1.

2.1.4 Other results

Non-controllability result

Assume that $D \subset \mathbb{R}^m, m \geq 1$ is a bounded domain with smooth boundary and $V, Q \in C^\infty(\overline{D},\mathbb{R})$ are arbitrary functions. The set of attainability from $\psi_0 \in S$ with $W^{1,1}_{\text{loc}}$-controls is defined by

$$\mathscr{A}(\psi_0, W^{1,1}) := \{\mathscr{U}_t(\psi_0, u) : \text{ for all } u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+,\mathbb{R}) \text{ and } t \geq 0\ \}.$$

The following non-controllability result is established in [P7] with H. Nersisyan.

Theorem 2.1.7. *For any $s \in (0,m)$, $\psi_0 \in S$, and any ball $B \subset H^s_{(V)}$ such that $B \cap S \neq \varnothing$, we have*

$$\mathscr{A}^c(\psi_0, W^{1,1}) \cap B \cap S \neq \varnothing.$$

This theorem is a generalisation of the results of [6] and [102], in the sense that the non-controllability here is established in more regular Sobolev spaces. Let us emphasize that this result does not exclude exact controllability in $H^s_{(V)}$ with controls form a larger space than $W^{1,1}_{\text{loc}}(\mathbb{R}_+,\mathbb{R})$.

The proof of this theorem is an adaptation of the arguments of Shirikyan [96]. The idea is to use some Hölder type estimates for the solutions and to show that the image by $\mathscr{U}_t$ of a ball in the space of controls $W^{1,1}_{\text{loc}}(\mathbb{R}_+,\mathbb{R})$ has a Kolmogorov ε-entropy strictly less than that of a ball B in the space $H^k_{(V)}$. This shows that the set of attainability $\mathscr{A}(\psi_0, W^{1,1})$ does not contain a ball of $H^k_{(V)}$.

Exact controllability in infinite time

In this subsection, we assume that $m = 1$ and $D = (0,1)$. Let $\mathscr{U}_\infty(\psi_0,u)$ be the $H^3_{(V)}$-weak ω-limit set of the trajectory corresponding to a control $u \in L^2_{loc}(\mathbb{R}_+,\mathbb{R})$ and initial condition $\psi_0 \in H^3_{(V)}$:

$$\mathscr{U}_\infty(\psi_0,u) := \{\psi \in H^3_{(V)} : \mathscr{U}_{t_n}(\psi_0,u) \rightharpoonup \psi \text{ in } H^3_{(V)} \text{ for some } t_n \to +\infty\}.$$

Clearly $\mathscr{U}_\infty(\psi_0,u)$ is non-empty, for example, when the trajectory $\mathscr{U}_t(\psi_0,u)$ is bounded in $H^3_{(V)}$. We assume that the following condition is verified.

Condition 2.1.8. *The functions $V, Q \in C^\infty([0,1],\mathbb{R})$ are such that*

(i) $\inf_{p,j\geq 1} |p^3 j^3 \langle Qe_{p,V}, e_{j,V}\rangle| > 0$,

(ii) $\lambda_{i,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *for all* $i,j,p,q \geq 1$ *such that* $\{i,j\} \neq \{p,q\}$ *and* $i \neq j$.

The following theorem is proved in [P7].

Theorem 2.1.9. *Under Condition 2.1.8, problem (2.1), (2.2) is exactly controllable in infinite time in $S \cap H^3_{(V)}$ in the following sense: for any $\psi_0 \in S \cap H^{3+\sigma}_{(V)}, \sigma \in (0,2]$, and $\psi_1 \in S \cap H^3_{(V)}$, there is a control $u \in L^1(\mathbb{R}_+,\mathbb{R})$ such that $\psi_1 \in \mathscr{U}_\infty(\psi_0,u)$.*

The proof of this theorem is based on an inverse mapping theorem applied to $(\psi_0,u) \mapsto \mathscr{U}_\infty(\psi_0,u)$. The controllability of the linearized system is equivalent to an exponential moment problem. As ψ_0 is arbitrary, the frequencies of the exponentials do not verify the gap condition. However, we are able to prove the controllability of the linearised system, since the problem is considered in an infinite time.

Equation with a polarizability term

With M. Morancey we consider in [P11] the problem of controllability of the Schrödinger equation with a so-called polarizability term

$$\begin{cases} i\partial_t \psi = \left(-\partial_{xx}^2 + V(x)\right)\psi - u(t)Q_1(x)\psi - u(t)^2 Q_2(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases}$$

For finite-dimensional approximations of this system, the problem of controllability is studied by Coron, Grigoriu, Lefter, and Turinici [24, 46, 47, 103]. In particular, they proved stabilisation of the first eigenstate using either discontinuous or time oscillating periodic feedback laws. The strategy based on time oscillating feedback laws was extended to the case of the infinite-dimensional equation by Morancey [78]. Finally, Boussaid, Caponigro, and Chambrion [15] proved global approximate controllability for this system using geometric technics.

Adapting the perturbation method we developed in [P10], together with some arguments from [P5] and [10], we obtain controllability property in the case of an arbitrary potential V and dipole moment Q_1 and with some generic assumptions on the polarizability moment Q_2. The following theorem is the main result of [P11].

Theorem 2.1.10. *For any $V, Q_1 \in H^6((0,1),\mathbb{R})$ the problem is globally exactly controllable in $H^6_{(V)}$ generically with respect to Q_2 in $H^6((0,1),\mathbb{R})$.*

The arguments of the proof of this theorem give also exact controllability property in the case where the term $u(t)^2 Q_2(x)\psi$ is replaced by a higher-degree term $\sum_{j=2}^{m} u^j Q_j \psi$.

2.2 Lagrangian controllability of 3D Navier–Stokes system

2.2.1 The model and the result

In the Eulerian approach, the motion of an incompressible fluid is usually described by the velocity field which is given by the Navier–Stokes (NS) system

$$\partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \tag{2.8}$$

$$u(0,x) = u_0(x). \tag{2.9}$$

We assume that the space variable $x = (x_1, x_2, x_3)$ belongs to the torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$, and the external force f is of the following form

$$f(t,x) = h(t,x) + \eta(t,x),$$

where h is the fixed part of the force (given function) and η is the control.

The well-posedness of the 3D NS system is a famous open problem. Given smooth data (u_0, f), the existence and uniqueness of a smooth solution is known to hold only locally in time. The global existence is established in the case of small data. For large data the global existence holds in the case of a weak solution, but in that case the uniqueness is open.

The flow generated by a sufficiently smooth velocity field u gives the Lagrangian trajectories of the fluid:

$$\dot{x} = u(t,x), \quad x(0) = x_0 \in \mathbb{T}^3. \tag{2.10}$$

Since the fluid is assumed to be incompressible, the mapping $\phi_t^u : x_0 \mapsto x(t)$ belongs to the group $\mathrm{SDiff}(\mathbb{T}^3)$ of orientation and volume preserving diffeomorphisms on $\mathbb{T}^3$ isotopic to the identity. This group is often referred as *configuration space* of the fluid (cf. [4, 54]). Thus for sufficiently smooth data, we have a path $(u(t), \phi_t^u)$, which is defined locally in time. The main issue addressed in the paper [P13] is the approximate controllability of the couple $(u(T), \phi_T^u)$ for any $T > 0$. Projecting system (2.8) onto the space (cf. Section 1.1 for the case of Dirichlet boundary conditions)

$$H := \{u \in L^2(\mathbb{T}^3, \mathbb{R}^3) : \operatorname{div} u = 0, \quad \int_{\mathbb{T}^3} u(x)\mathrm{d}x = 0\},$$

we eliminate the pressure and consider the following evolution equation

$$\dot{u} + \nu L u + B(u) = h(t,x) + \eta(t,x). \tag{2.11}$$

Let E a be subset of H and $H^k_\sigma := H^k(\mathbb{T}^3, \mathbb{R}^3) \cap H$. In this section, we use the following notion of controllability.

Definition 2.2.1. *We shall say that system* (2.11) *is approximately controllable by an E-valued control, if for any $\nu > 0$, $k \geq 3$, $\varepsilon > 0$, $T > 0$, $u_0, u_1 \in H^k_\sigma$, $h \in L^2([0,T], H^{k-1}_\sigma)$, and $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$, there is a control $\eta \in L^2([0,T], E)$ and a solution u of* (2.11), (2.9) *defined for any $t \in [0,T]$ and satisfying*

$$\|u(T) - u_1\|_{H^k(\mathbb{T}^3)} + \|\phi_T^u - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

In [P13] , we establish the following result.

Theorem 2.2.2. *There is a finite-dimensional subspace $E \subset H$ such that* (2.11) *is approximately controllable by an E-valued control.*

This theorem shows that, using a finite-dimensional external force, one can drive the fluid flow arbitrarily close to any configuration. Moreover, near the final position, will have approximately the prescribed velocity. Since ϕ_T^u depends not only on $u(T)$, but on the whole path $u(t), t \in [0,T]$, one needs a *path-controllability property* for the velocity field, in order to prove controllability for ϕ_T^u. This path-controllability is one of the novelties of the paper.

Literature review. The proof of Theorem 2.2.2 is based on some methods introduced by Agrachev and Sarychev in [1, 2] (see also the survey [3]). In that papers they prove approximate controllability for the 2D NS and Euler systems by a finite-dimensional force. This method is then developed and generalised by several authors for various PDE's. Rodrigues proves in [85] controllability for the 2D NS system on a rectangle with the Lions boundary conditions, and in [86, 87] he extends the results to the case of more general Navier boundary conditions and the Hemisphere under the Lions boundary conditions. The controllability for the 3D NS system on the torus is studied in [93, 95] by Shirikyan. He also considers the case of the Burgers equation on the real line in [98] and on an interval with the Dirichlet boundary conditions in [97]. Incompressible and compressible 3D Euler equations are considered by Nersisyan in [80, 81], and the controllability for the 2D defocusing cubic Schrödinger equation is established by Sarychev in [91]. In [96] Shirikyan proves that the Euler equations are not exactly controllable by a finite-dimensional external force.

All the above papers are concerned with the problem of controllability of the velocity field. The controllability of the Lagrangian trajectories of 2D and 3D Euler equations is studied by Glass and Horsin [42, 43], in the case of boundary controls. For given two smooth contractible sets γ_1 and γ_2 of fluid particles which surround the same volume, they construct a control such that the corresponding flow drives γ_1 arbitrarily close to γ_2. In the context of our paper, a similar property can be derived from our main result. Indeed, Krygin shows in [56] that there is a diffeomorphism $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$ such that $\psi(\gamma_1) = \gamma_2$. Thus we can find an E-valued control η such that $\phi_T^u(\gamma_1)$ is arbitrarily close to γ_2, and, moreover, at time T the particles will have approximately the desired velocity. The reader is referred to the book [23] by Coron for an introduction to the theory of control of the NS system by distributed controls and for references on that topic.

Scheme of the proof. Let us give a short description of how the Agrachev–Sarychev method is adapted to establish approximate controllability in the above-defined sense. Let E be defined by (2.13) for some generator $\mathscr{K}$ of $\mathbb{Z}^3$, and let $I(t,x)$ be a smooth isotopy connecting $\psi \in \mathrm{SDiff}(\mathbb{T}^3)$ to the identity. Then $\hat{u}(t,x) := \partial_t I(t, I^{-1}(t,x))$ is a divergence-free vector field such that $\phi_t^{\hat{u}}(x) = I(t,x)$ for all $t \in [0,T]$. In particular, $\phi_T^{\hat{u}} = \psi$. The mapping $u \mapsto \phi_T^u$ is continuous from $L^1([0,T], H^k_\sigma)$ to $C^1(\mathbb{T}^3)$, where $L^1([0,T], H^k_\sigma)$ is endowed with the *relaxation norm*

$$|||u|||_{T,k} := \sup_{t\in[0,T]} \left\| \int_0^t u(s)\mathrm{d}s \right\|_{H^k(\mathbb{T}^3)}.$$

Hence we can choose a smooth vector field u sufficiently close to $\hat{u}$ with respect to this norm, so that

$$u(0) = u_0, \quad u(T) = u_1, \quad \|\phi_T^u - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

Then u is a solution of our system corresponding to a control η_0, which can be explicitly expressed in terms of u and h from equation (2.11). In general, this control η_0 is not E-valued, so we need to approximate u appropriately with solutions corresponding to E-valued controls. To this end, we

define the sets

$$\mathscr{K}_0 := \mathscr{K}, \quad \mathscr{K}_j = \mathscr{K}_{j-1} \cup \{m \pm n : m,n \in \mathscr{K}_{j-1}\}, \quad j \geq 1.$$

As $\mathscr{K}$ is a generator of $\mathbb{Z}^3$, one easily gets that $\cup_{j\geq 1}\mathscr{K}_j = \mathbb{Z}^3$, hence $\cup_{j\geq 1}E(\mathscr{K}_j)$ is dense in H^k_σ. Let P_N be the orthogonal projection onto $E(\mathscr{K}_N)$ in H. Then a perturbative result implies that, for a sufficiently large $N \geq 1$, system (2.11), (2.9) with control $P_N\eta_0$ has a strong solution u_N verifying

$$\|u_N(T) - u_1\|_{H^k(\mathbb{T}^3)} + \|\phi_T^{u_N} - \psi\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

On the other hand, if we consider the following auxiliary system

$$\dot{u} + \nu L(u+\zeta) + B(u+\zeta) = h + \eta \tag{2.12}$$

with two controls ζ and η, then the below two properties hold true

Convexification principle. For any $\varepsilon > 0$ and any solution u_j of (2.11), (2.9) with an $E(\mathscr{K}_j)$-valued control η_1, there are $E(\mathscr{K}_{j-1})$-valued controls ζ and η and a solution $\tilde{u}_{j-1}$ of (2.12), (2.9) such that

$$\|u_j(T) - \tilde{u}_{j-1}(T)\|_{H^k(\mathbb{T}^3)} + |||u_j - \tilde{u}_{j-1}|||_{T,k} < \varepsilon.$$

Extension principle. For any $\varepsilon > 0$ and any solution $\tilde{u}_j$ of (2.12), (2.9) with $E(\mathscr{K}_j)$-valued controls ζ and η, there is an $E(\mathscr{K}_j)$-valued control η_2 and a solution u_j of (2.11), (2.9) such that

$$\|u_j(T) - \tilde{u}_j(T)\|_{H^k(\mathbb{T}^3)} + |||u_j - \tilde{u}_j|||_{T,k} < \varepsilon.$$

These two principles and the above-mentioned continuity property of ϕ_T^u with respect to the relaxation norm imply that, for any solution u_j of (2.11), (2.9) with an $E(\mathscr{K}_j)$-valued control η_1, there is an $E(\mathscr{K}_{j-1})$-valued control η_2 and a solution u_{j-1} of (2.11), (2.9) such that

$$\|u_j(T) - \tilde{u}_{j-1}(T)\|_{H^k(\mathbb{T}^3)} + \|\phi_T^{u_j} - \phi_T^{u_{j-1}}\|_{C^1(\mathbb{T}^3)} < \varepsilon.$$

Combining this with the above-constructed solution u_N, we get the approximate controllability of (2.11) by a control valued in $E(\mathscr{K}) = E$. The proofs of convexification and extension principles are strongly inspired by [93].

2.2.2 Examples of spaces ensuring controllability

We provide in [P13] three types of examples of finite-dimensional spaces which ensure the controllability of the 3D NS system in the sense of Definition 2.2.1.

Spaces associated with generators of $\mathbb{Z}^3$

For any subset $\mathscr{K} \subset \mathbb{Z}^3$, we denote

$$E(\mathscr{K}) := \operatorname{span}\{c_\ell, c_{-\ell}, s_\ell, s_{-\ell} : \ell \in \mathscr{K}\}, \tag{2.13}$$

where $c_0 = s_0 = 0$, and $c_\ell(x) = l(\ell)\cos\langle \ell, x\rangle$, $s_\ell(x) = l(\ell)\sin\langle \ell, x\rangle$ for $\ell \in \mathbb{Z}^3_*$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^3$ and $\{l(\ell), l(-\ell)\}$ is an arbitrary orthonormal basis in

$$\ell^\perp := \{x \in \mathbb{R}^3 : \langle x, \ell\rangle = 0\}.$$

We shall say that $\mathscr{K} \subset \mathbb{Z}^3$ is a *generator* if any $a \in \mathbb{Z}^3$ can be represented as a finite linear combination of elements of $\mathscr{K}$ with integer coefficients.

Proposition 2.2.3. *If $\mathscr{K}$ is a generator, then (2.11) is approximately controllable by an $E(\mathscr{K})$-valued control.*

The simplest example of a generator is

$$\mathscr{K} = \{(1,0,0), (0,1,0), (0,0,1)\},$$

in which case $\dim E(\mathscr{K}) = 12$. The following simple criterion is useful for constructing generators of $\mathbb{Z}^3$ (see Section 3.7 in [50]).

Lemma 2.2.4. *A set $\mathscr{K} \subset \mathbb{Z}^3$ is a generator if and only if the greatest common divisor of the set $\{\det(a,b,c) : a,b,c \in \mathscr{K}\}$ is 1, where $\det(a,b,c)$ is the determinant of the matrix with rows a, b and c.*

Controls with two vanishing components

Let us consider the NS system

$$\partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = h(t,x) + (0,0,1)\eta(t,x), \quad \operatorname{div} u = 0, \tag{2.14}$$

where η is a control taking values in a finite-dimensional space of the form

$$\operatorname{span}\{\cos\langle m, x\rangle, \sin\langle m, x\rangle : m \in \mathscr{K}\},$$

where $\mathscr{K}$ is again a subset of $\mathbb{Z}^3$. Let us rewrite (2.14) in an equivalent form

$$\dot{u} + \nu L u + B(u) = h(t,x) + \tilde{\eta}(t,x),$$

where $\tilde{\eta} := \Pi(e\eta)$ and $e := (0,0,1)$. Then the control $\tilde{\eta}$ takes values in the space

$$\tilde{E}(\mathscr{K}) := \operatorname{span}\{(P_m e)\cos\langle m, x\rangle, (P_m e)\sin\langle m, x\rangle : m \in \mathscr{K}\}.$$

Proposition 2.2.5. *Let*

$$\mathscr{K} := \{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\}.$$

Then (2.11) *is approximately controllable by an* $\tilde{E}(\mathscr{K})$*-valued control.*

This proposition is related to a recent result obtained by Coron and Lissy [25]. They establish local null controllability of the velocity for the 3D NS system controlled by a distributed force having two vanishing components, i.e., the controls are valued in a space of the form $\{(0,0,1)\eta : \eta \in \mathscr{P}\}$, where $\mathscr{P}$ is the space of space-time L^2-functions supported in a given open subset.

6-dimensional example

The space $\tilde{E}(\mathscr{K})$ constructed in Proposition 2.2.5 is 8-dimensional. The following result shows that the 3D NS system can be approximately controlled by η taking values in a 6-dimensional space.

Proposition 2.2.6. *Let us define the following* 6*-dimensional space*

$$\begin{aligned}\hat{E} := \operatorname{span}\{&a\cos\langle(1,0,1),x\rangle, a\sin\langle(1,0,1),x\rangle,\\ &e\cos\langle(0,1,1),x\rangle, e\sin\langle(0,1,1),x\rangle,\\ &b\cos\langle(0,0,1),x\rangle, b\sin\langle(0,0,1),x\rangle\},\end{aligned}$$

where $a := (1,1,1), b := (1,0,0), e := (0,0,1)$*. Then* (2.11) *is approximately controllable by an* $\hat{E}$*-valued control.*

It would be interesting to find a characterisation of finite-dimensional spaces of the following general form ensuring the controllability of the NS system

$$E(\mathscr{K}_c,\mathscr{K}_s,a,b) := \operatorname{span}\{a_m\cos\langle m,x\rangle; b_n\sin\langle n,x\rangle : m\in\mathscr{K}_c, n\in\mathscr{K}_s\},$$

where $\mathscr{K}_c,\mathscr{K}_s \subset \mathbb{Z}^3$, $a = \{a_m\}_{m\in\mathscr{K}_c} \subset \mathbb{R}^3_*$, and $b = \{b_n\}_{n\in\mathscr{K}_s} \subset \mathbb{R}^3_*$. It is clear that both $\mathscr{K}_c$ and $\mathscr{K}_s$ are necessarily generators of $\mathbb{Z}^3$.

Notation

Given a Banach space X, we denote by $B_X(a,R)$ the closed ball in X of radius R centred at a. In the case when $a=0$, we write $B_X(R)$. We always assume that X is endowed with its Borel σ-algebra $\mathscr{B}_X$. For any function $V : X \to \mathbb{R}$, we set $\mathrm{Osc}_X(V) := \sup_X V - \inf_X V$. We use the following spaces:

$L^\infty(X)$ is the space of bounded measurable functions $f : X \to \mathbb{R}$ endowed with the norm $\|f\|_\infty = \sup_{u\in X} |f(u)|$.

$C_b(X)$ is the space of continuous functions $f \in L^\infty(X)$, and $C_+(X)$ is the space of positive continuous functions $f : X \to \mathbb{R}$.

$L_b(X)$ is the space of functions $f \in C_b(X)$ for which the following norm is finite

$$\|f\|_L = \|f\|_\infty + \sup_{u\neq v} \frac{|f(u)-f(v)|}{\|u-v\|}.$$

$\mathscr{M}(X)$ is the vector space of signed Borel measures on X with finite total mass endowed with the topology of the weak convergence. $\mathscr{M}_+(X) \subset \mathscr{M}(X)$ is the cone of non-negative measures.

$\mathscr{P}(X)$ is the set of probability Borel measures on X. For $\mu \in \mathscr{P}(X)$ and $f \in C_b(X)$, we denote $\langle f,\mu\rangle = \int_X f(u)\mu(du)$. If $\mu_1,\mu_2 \in \mathscr{P}(X)$, we set

$$\|\mu_1-\mu_2\|_L^* = \sup_{\|f\|_L\leq 1} |\langle f,\mu_1\rangle - \langle f,\mu_2\rangle|. \tag{2.15}$$

The total variation metric on $\mathscr{P}(X)$ is defined by

$$\|\mu_1-\mu_2\|_{\mathrm{var}} = \frac{1}{2}\sup_{\|f\|_\infty\leq 1} |\langle f,\mu_1\rangle - \langle f,\mu_2\rangle| = \sup_{\Gamma\in\mathscr{B}_X} |\mu_1(\Gamma)-\mu_2(\Gamma)|.$$

For an open set $D \subset \mathbb{R}^m, m \geq 1$, we introduce the following function spaces:

$L^p = L^p(D)$ is the Lebesgue space of measurable functions whose p^{th} power is integrable. In the case $p=2$ the corresponding norm is denoted by $\|\cdot\|$ and the scalar product by $\langle\cdot,\cdot\rangle$.

$H^s(D)$, $s \geq 0$ is the Sobolev space of order s endowed with the norm $\|\cdot\|_s$.

$H^s_{(V)}(D), s \geq 0$ is the domain of definition of the operator $(-\Delta+V)^{s/2}$.

The distribution of a random variable ξ is denoted by $\mathscr{D}(\xi)$.

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Abstract. This thesis is organised in two relatively independent chapters.

Chapter 1 is devoted to the study of some mathematical problems arising in the theory of hydrodynamic turbulence. Our results focus on questions related to the large deviations principle (LDP), Gallavotti–Cohen type symmetry, and ergodicity (existence and uniqueness of a stationary measure and its mixing properties) for a family of randomly forced PDE's. We establish the LDP for parabolic PDE's, such as the Navier–Stokes system or the complex Ginzburg–Landau equation, perturbed by a random kick force. Then we extend this result to the case of the damped nonlinear wave equation driven by a spatially regular white noise, by proving a local LDP. We establish a Gallavotti–Cohen type symmetry for the rate function of an entropy production functional for PDE's with strong nonlinear dissipation, such as the Burgers equation. Finally, we prove a mixing property for the complex Ginzburg–Landau equation with a white-noise perturbation in any space dimension.

In Chapter 2, we first consider the problem of controllability of a quantum particle by the amplitude of an electric field. The position of the particle is described by a wave function which obeys the bilinear Schrödinger equation. We are mainly interested in the global controllability problems of this equation. Using some variational methods, we establish approximate controllability, feedback stabilisation, and simultaneous controllability results. The second part of this chapter is concerned with the problem of controllability of Lagrangian trajectories of the 3D Navier–Stokes system by a finite-dimensional force. We provide some examples of saturating spaces which ensure the approximate controllability of the system.

Keywords: Navier–Stokes system, Ginzburg–Landau equation, Burgers equation, nonlinear wave equation, large deviations principle, Gallavotti–Cohen symmetry, kick force, white noise, coupling method; Schrödinger equation, Lyapunov function, approximate controllability, stabilisation, simultaneous controllability, Lagrangian trajectories.

MSC: 35R60, 37A25 60F10, 60J25, 76D05; 35Q41, 37B25 81Q05, 93B05, 93D15.

Résumé. Ce mémoire est composé de deux chapitres relativement indépendants.

Le chapitre 1 est consacré à l'étude de quelques problèmes mathématiques issus de la théorie de la turbulence en hydrodynamique. Nos résultats portent principalement sur des questions liées au principe de grandes déviations (PGD), relation de Gallavotti–Cohen et ergodicité (existence et unicité d'une mesure stationnaire et ses propriétés de mélange) pour une classe d'EDP perturbées par une force aléatoire. Nous établissons un PGD pour des EDP paraboliques, comme les équations de Navier–Stokes ou de Ginzburg–Landau complexe, perturbées par une force aléatoire discrète en temps. Nous étendons ce résultat au cas de l'équation d'onde non linéaire amortie soumise à une force aléatoire de type bruit blanc en temps et lisse par rapport à la variable spatiale, en prouvant un PGD local. Nous obtenons une relation de type Gallavotti–Cohen pour la fonction de taux d'une fonctionnelle de production d'entropie pour des EDP avec une dissipation non linéaire forte, comme l'équation de Burgers. Enfin, nous prouvons une propriété de mélange pour l'équation complexe de Ginzburg–Landau avec un bruit blanc dans un espace de dimension quelconque.

Dans le chapitre 2, nous considérons d'abord le problème de la contrôlabilité d'une particule quantique par l'amplitude d'un champ électrique. L'état de la particule est décrit par une fonction d'onde qui obéit à l'équation de Schrödinger bilinéaire. En utilisant des méthodes variationnelles, nous obtenons des résultats de contrôlabilité approchée, stabilisation et contrôlabilité simultanée. La deuxième partie de ce chapitre aborde le problème de la contrôlabilité lagrangienne de l'équation de Navier–Stokes $3D$ par une force de dimension finie. Nous donnons des exemples d'espaces qui assurent la contrôlabilité approchée du système.

Mots clés: système de Navier–Stokes, équation de Ginzburg–Landau, équation de Burgers, équation d'onde non linéaire, principe de grandes déviations, relation de Gallavotti–Cohen, bruit blanc, méthode de couplage; équation de Schrödinger, fonction de Lyapunov, contrôlabilité approchée, stabilisation, contrôlabilité simultanée, trajectoires lagrangiennes.