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The ” Science of Numbers ” in action in Richard Dedekind’s works: between mathematical explorations and foundational investigations

Emmylou Haffner

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Présentée et soutenue par
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8 Décembre 2014

The “Science of Numbers” in action
in Richard Dedekind’s works:
between mathematical explorations
and foundational investigations

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“Good morning” said Bilbo, and he meant it. The sun was shining, and the grass was very green. But Gandalf looked at him from under long bushy eyebrows that stuck out further than the brim of his shady hat.

“What do you mean?” he said. “Do you wish me a good morning, or mean that it is a good morning whether I want it or not; or that you feel good this morning; or that it is a morning to be good on?”

— J.R.R. Tolkien, *The Hobbit*.

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Introduction

*On dit qu'un Allemand a fait un
livre sur un zeste de citron.*

Jean-Jacques Rousseau

In 1868, Alfred Clebsch left Giessen for Göttingen. Discussing the person who would take his position in Giessen, he wrote to his former colleagues that “it [would] not be easy to find somebody suitable who has, in the same time, accomplished something from a scientific point of view (among which I can, unfortunately, not count Dedekind, for example).”¹ It took Dedekind, who was already 37 years old, several more years to publish any major works. It wasn’t until 1880, that he started to gain the recognition of his peers with a nomination at the Berlin Akademie der Wissenschaften. By the next century, Emmy Noether was repeatedly stating that “Everything is already in Dedekind”, as if in Dedekind’s writings was a most powerful seed, the whole of modern algebra in waiting.

Richard Dedekind (1831-1916)

In 1855, after Gauss’s death, Johann Peter Gustav Lejeune-Dirichlet left Berlin for Göttingen to take the chair of mathematics. For Richard Dedekind, who defended his doctoral dissertation on Eulerian integrals in 1852 and his *Habilitation* in 1854 both under C. F. Gauss, meeting Dirichlet turned him into “a new man”. Dedekind became close to Dirichlet as a student and as a friend and this was to be, along with his friendship with Bernhard Riemann, one of the most significant events in Dedekind’s early mathematical life.

Dedekind was chosen to edit Dirichlet’s lectures on number theory, the *Vorlesungen über Zahlentheorie* ([Lejeune-Dirichlet, 1863]), after Dirichlet’s death in 1859. In them, Dedekind tried to stay “as faithful as possible” to Dirichlet’s lectures on number theory in Göttingen, between 1855 and 1858 and which, according to Dedekind, “contributed so extraordinarily to the circulation of the newest and the subtlest parts of mathematics.” The *Vorlesungen*, which had four editions in 1863, 1871, 1879 and 1894, were seen

¹Quoted in [Dugac, 1976b], 134.

as one of Dedekind's most important contributions to mathematics by his contemporaries – as were his other editorial works. To the publishers's request, Dedekind included supplements presenting additional developments. In 1871 and in the later editions, Dedekind took this opportunity to publish his own original work, the Xth and XIth Supplements, in an attempt to reach a larger audience² and as a way to mark his desire to work in the continuity of Dirichlet's mathematics.

Dedekind's Xth Supplement, *Über die Komposition der binären quadratischen Formen*, exposes developments on the theory of binary quadratic forms. It was initially excluded to avoid moving too far away from Dirichlet's own works. An important part of the Xth Supplement is the investigation of the laws of divisibility of algebraic integers, generalizing Kummer's works on the factorization of certain complex numbers such as cyclotomic integers. For this purpose, Dedekind introduces for the first time the notions of algebraic integer and of field, module, and ideal. The Xth Supplement is often said to be the birth place of modern algebra, a game changing work and, thus, one of the most important works of contemporary mathematics. In 1917, Edmund Landau wrote that by founding the theory of algebraic fields, Dedekind brought “the light in the darkness, the order in chaos, and the methods, which he created here, have had a fruitful influence on other, very different, problems of number theory and algebra.”³ Bourbaki considers it a “masterpiece” in which the theory of modules and ideals is “entirely created” and “brilliantly exposed” ([Bourbaki, 1984], 130). It is presented by H. Edwards as the “birthplace of the modern set-theoretic approach to the foundations of mathematics” ([Edwards, 1983]) and by Dugac as “one of the sources of today mathematics” ([Dugac, 1976b], 29, who, like Edwards, underlines the role of the Supplement in set theory).

Dedekind came back several times to his algebraic number theory,⁴ and rewrote it anew each time, presenting each new version as an improvement upon the previous one, modifying the “presentation” or the “foundation” of the theory. Three different versions were published: in 1871, in 1876/77 (and a similar version in 1879) and in 1894. Except for the 1876/77 which was published in French in the *Bulletin des Sciences Astronomiques et Mathématiques*, all versions were published as supplements of the successive versions of Dirichlet's *Vorlesungen*.⁵ The theory of algebraic numbers is arguably

²Letter to Lipschitz, 29 April 1876: “I thought that the inclusion of this investigation in Dirichlet's *Zahlentheorie* would be the best way to attract a wider circle of mathematicians to the field. . .” (transl. in [Stillwell, 1996], 45)

³[Landau, 1917], quoted and translated in [Dugac, 1976b], 29.

⁴When using the phrase “algebraic number theory”, I mean “the theory of algebraic numbers”, rather than “the algebraic theory of numbers”.

⁵In the dissertation, references to these works will be the following: [Dedekind, 1871] will be referred to with page numbers referring to Avigad's translation ([Avigad, 2004]). The last paragraphs were however not translated and will be in reference to the 1871 edition of Dirichlet's *Vorlesungen*. [Dedekind, 1876-1877] will be referred to with page num-

Dedekind's most pivotal work, if only because it introduces the concepts of field, module and ideal, which will be used in a majority of Dedekind's other mathematical works. Besides, Dedekind's mathematical papers are often related to the theory of algebraic numbers, for they investigate questions and problems which arose from the study of algebraic integers and connected questions deepening aspects of, for example, ideal theory (e.g., class of ideals).

Dedekind was not only involved in publishing Dirichlet's *Vorlesungen über Zahlentheorie*, he also participated to the publication of Gauss's collected works, and was in charge, with Heinrich Weber (1842-1913), of the publication of Riemann's collected works. The edition of Riemann's works, which involved the tedious process of clearing up Riemann's manuscripts, was the beginning of a long friendship with Weber, as attested by their rich correspondence. One of the most significant side-effect of their collaboration and of their friendship is the co-writing of an article on Riemannian function theory, published in 1882, *Theorie der algebraischen Funktionen einer Veränderlichen*⁶ which is often presented as a milestone in the development of algebraic geometry. In this paper, Dedekind and Weber transfer to algebraic functions of one complex variable, the methods elaborated by Dedekind for the theory of algebraic integers. By doing so, they give themselves the means to develop a purely "algebraico-arithmetic" theory of algebraic functions. Yet, they do not propose a more abstract theory of fields and/or ideals of which function and number theories would be instances, nor do they state that they treat functions 'like' numbers or that they apply number theory to function theory. For Dieudonné, Dedekind and Weber brought together "very remote" domains of mathematics with a "remarkable originality" which can only be compared "in all the history of algebraic geometry" to Riemann's ([Dieudonné, 1974], 61). For him, "this conception of algebraic geometry is for us the clearest and simplest one, due to our familiarity with abstract algebra" ([Dieudonné, 1972], 838).

In parallel to these mathematical works, Dedekind was deeply interested, throughout his career, in foundations of mathematics and in particular in providing a rigorous definition for the systems of numbers, thereby securing the foundations of arithmetic. This constitutes two of his most well-known works: *Stetigkeit und irrationale Zahlen* in 1872 and *Was sind und was sollen die Zahlen?* in 1888. In *Stetigkeit und irrationale Zahlen*, Dedekind defines the irrational numbers by means of cuts. In *Was sind und was sollen die Zahlen?*, he gives a "logical" definition of the natural numbers.⁷

bers referring to Stillwell's translation ([Stillwell, 1996]); [Dedekind, 1879] and [Dedekind, 1894a] will be referred to with page numbers referring to the 1879 and 1894 editions of Dirichlet's *Vorlesungen* respectively.

⁶References to this work will be to Stillwell's translation ([Stillwell, 2012]). I will refer to this article as *Algebraische Funktionen* for brevity.

⁷For these works, page references will be to Ewald's translation [Ewald, 2005]. I will

The interest in the foundations of mathematics was relatively common at the time. However, the approach adopted by Dedekind differed sensibly from the common practice by its use of set-theoretic notions. In addition, it defines the whole domain of natural numbers as a ‘structure’, with a set of necessary and sufficient conditions. Dedekind adopts what we can recognize as being an axiomatic approach. He is thus considered on one hand as a precursor of Hilbert and on the other hand as one of the fathers of set-theory. Notably, H. Benis Sinaceur suggests that the presence of “structures” in Dedekind’s foundational works is hidden by his use of “concept” or “general concept” for what we call structures:

[In Dedekind’s *Zahlen* and *Stetigkeit*], the point is precisely that of bringing out the *structural* properties of positive integers and real numbers respectively. Simply, Dedekind calls “concept” or “general concept” what we name “structures”, a turn of phrase of which Hilbert, Emmy Noether, B. L. van der Waerden and other members of the Göttingen school will inherit, while Bourbaki will give the pride of place to “structures”.⁸ ([Sinaceur and Dedekind, 2008], 108)

Dedekind has been granted a place of choice in the genealogy with which mathematicians represent the shaping of mathematical knowledge, the accumulation of discoveries leading inexorably to the present day state of mathematics. Dedekind introduced several of the most important concepts for the development of modern algebra to the point that he is often said to have influenced our practice and knowledge of mathematics more than any other mathematician of the second half of the 19th century.⁹ After the reception of his works remained dormant for about twenty years, Dedekind’s works eventually changed the face of algebraic number theory – especially through their adoption by David Hilbert in his *Zahlbericht* ([Hilbert, 1897]).

The previous pages show how Dedekind has been (and still is, sometimes) perceived in the history of mathematics and by present-day mathematicians. Dedekind’s works also often play a key role in certain discussions in contemporary philosophy of mathematics.

refer to [Dedekind, 1872] as *Stetigkeit* and to [Dedekind, 1888] as *Zahlen*.

⁸ “[Dans *Zahlen* et dans *Stetigkeit*], il s’agit bien de dégager les propriétés structurelles des entiers et des réels respectivement. Tout simplement, Dedekind appelle ‘concept’ ou ‘concept général’ ce que nous nommons ‘structure’, trait d’expression dont hériteront Hilbert, Emmy Noether, B. L. van der Waerden et d’autres membres de l’école de Göttingen, tandis que Bourbaki fera la part belle aux ‘structures’.”

⁹It is, notably the case of [Dugac, 1976b], [Bourbaki, 1984], but also [Avigad, 2006] more recently.

0.1 *State of the art and our expectations about Dedekind*

0.1 State of the art and our expectations about Dedekind

0.1.1 Philosophical readings of Dedekind's works

For the last ten to fifteen years, philosophers of mathematics have shown a great interest in Dedekind's mathematics, and it is among philosophers that one finds the most vivid scholarship on Dedekind.

It is well known that Dedekind's mathematics is deeply shaped by epistemological questions and values. Because of this and of the decisive role played by Dedekind in shaping today's mathematical knowledge, it seems rather natural that philosophers find food for thought in his works. In particular, Dedekind's particular tendency to adopt strong philosophical positions as directing principles for his mathematical works has made him a very good subject for philosophers of mathematical practice (especially for [Avigad, 2006], and Tappenden's inquiries about fruitfulness and definition). On the other hand, as an historian, it is imperative to take Dedekind's strong epistemological concerns into consideration, to account for his practice and his mathematics. And whatever the assessment of Dedekind's ideas might be, the role played by his epistemological concerns is never denied. H. M. Edwards, for example, wrote that

[Dedekind's] insistence on philosophical principles was responsible for many of his important innovations. However, in the case of ideal theory, this insistence seems to have impelled him to abandon his first, very simple generalization of Kummer's theory, and to replace it with two later formulations which were more in accord with his philosophical principles but which required much more space and technique to reach the same objective. ([Edwards, 1980], 349)

The importance given to epistemological values constitutes one of the most significant aspects of Dedekind's approach, insofar as he was rarely completely content with the treatment of certain mathematical theories, whether they had been written by his contemporaries, or (albeit far less often) by himself. I will put forward the idea that the key point here is how Dedekind prized, understood, and worked with certain epistemological values.

In fact, Dedekind's works have held the philosophers's attention for a long time: Husserl, Cassirer, and Cavaillès took a deep interest in Dedekind's works. Recently, interest in Dedekind's works have grown among philosophers, stemming from interest in the concept of number ([Belna, 1996]) and in foundational issues. In particular, Dedekind has been considered in relation to philosophical positions, such as structuralism (e.g., by Erich Reck ([Reck, 2003], [Reck, 2009]) who tries to identify the structuralism peculiar

to Dedekind's works¹⁰) or logicism. Many considerations of Dedekind as a logicist are connected to the always renewed interest of philosophers for Frege.¹¹ Alternate propositions of a Dedekindian logicism have been made in Ferreirós's works and by Michael Detlefsen ([Detlefsen, 2012]). Hourya Benis Sinaceur proposed a different view and argues against a logicist reading of Dedekind (in [Sinaceur and Dedekind, 2008] and more extensively in [Sinaceur, Forthcoming]). Finally, Dedekind's impact on or participation in the early development of set theory ([Ferreirós, 2008]) and of the axiomatic method ([Sieg and Schlimm, 2005], [Sieg, 2009], [Schlimm, 2011]) also attracted considerable attention.

These works, insightful and epistemologically subtle, adopt an approach with two characteristic features. Firstly, they tend to only consider a small part of Dedekind's mathematics. In particular, if one follows these works, Dedekind's mathematics seems to have two facets, his mathematical works and his foundational works, while in the same time being thought to present a striking and profound unity, particularly sensible as regards the methodology. If philosophers do consider Dedekind's mathematical works, it is usually for a very particular point, often isolated from the remaining parts of the corpus. Notably, philosophical works take a particular interest in Dedekind and the infinite. While this point is deeply interesting for philosophy of mathematics, for the history of mathematics, and even for the history of philosophy of mathematics, Dedekind himself did not seem to consider it as a particularly problematic issue. In general, philosophical works tend to detach the foundational works from the mathematical works. Some publications do escape this tendency. They are those devoted to larger questions such as set theory (in particular, Ferreirós's works), epistemological values in mathematical practice such as Tappenden's papers or again Avigad's inquiry about "methodology" ([Avigad, 2006], which I will consider in more details in the dissertation).

The second feature of recent publications on Dedekind is a tendency to project expectations on Dedekind's works, which had a strong impact on the philosophical interpretations of Dedekind's ideas. This is an important point, for Dedekind, given the importance of epistemological considerations for his writings. It prompted many attempts to characterize Dedekind's ideas by means of categories that were only elaborated after his own writings – or at best by Frege – and to try to pinpoint which was the ontological position underlying Dedekind's mathematics. Commentators built their interpretations on *Zahlen* (sometimes adding *Stetigkeit* and extracts from the correspondence). The two main positions mentioned above thus emerged:

¹⁰See also [Yap, 2009] on Dedekind and structuralism.

¹¹These discussions however fall out of the scope of this dissertation. For a comparison of Dedekind's and Frege's ideas on logic and whether they should be treated in the same way, one can refer to [Sinaceur, Forthcoming].

0.1 State of the art and our expectations about Dedekind

Dedekind is a logicist, and Dedekind is a structuralist.

Dedekind's statement that "arithmetic is a part of logic" led to consider him, often, as one of the founding fathers of logicism along with Frege, despite their very opposed ideas on mathematics and logic. This led to many comparisons of their approaches and discussions of Dedekind's logicism. More finely-shaded descriptions of a Dedekindian logicism have been proposed by Ferreirós ([Ferreirós, Unpublished]) and Detlefsen ([Detlefsen, 2012]). By "logicism", they both mean something different. Ferreirós defends the idea that logicism has been developed in two phases, an "early, triumphant phase from 1872 to the shocking discovery of the contradictions in the years 1897–1903" and Russellian logicism. For him, Dedekind is a major reference of the first phase of logicism, and adopts "the doctrinal thesis that the basic principles of mathematics can be derived from logical principles." For Ferreirós, Frege and Dedekind develop their approaches on the basis of the same fundamental idea: to subsume arithmetic under the concepts of logic. However, Dedekind conceived logic as a theory of inference – a conception of logic significantly different from Frege's. In [Sinaceur, Forthcoming], H. Benis Sinaceur gives a thorough analysis of the fundamental differences between Dedekind's and Frege's conceptions of logic and mathematics to argue that there is no logicism in Dedekind. Sinaceur's argument is based on a detailed and subtle analysis of Dedekind's conception of logic which suggests convincingly that the standards of logicism are not applicable to Dedekind. Detlefsen's take on a Dedekindian logicism is significantly different from Ferreirós (and even from the idea of a Dedekindian logicism against which Sinaceur argues), since from his viewpoint, Dedekind's logicism is rooted in his demand that "everything that can be proven ought not to be believed without a proof". Dedekind's logicism, according to Detlefsen, is related to the pursuit of rigor all the way to unprovable premisses, the laws of pure thought, which for Dedekind are laws of logic. Detlefsen's ideas will be the starting point of a reflection on rigor, in the last part of this dissertation, hence I will not develop it more here.

Another possibility which has been discussed at length is Dedekind's structuralism. Again, there are many ways to understand Dedekind's structuralism, but the common denominator is a philosophical, foundational understanding of "structuralism". Structuralists, such as Hellman and Shapiro, consider Dedekind as the first structuralist about objects of mathematics. Reck has rightly shown ([Reck, 2003]) that Dedekind's ideas correspond exactly neither to Shapiro's *ante rem* structuralism, nor to Hellmann's modal structuralism, nor to "set-theoretic structuralism" (the position that assumes the existence of sets and considers objects of mathematics as set-theoretical structures). In fact, if one follows Reck, the only sort of structuralism that Dedekind's *Zahlen* seems to satisfy is what Reck calls "methodological structuralism". The "methodological structuralism" is described by Reck as a "general, largely conceptual approach [such as that of modern

abstract algebra] (as opposed to more computational and particularist approaches). It is typically tied to presenting mathematics in a formal axiomatic way” ([Reck, 2003], 371). From these observations, Reck concludes that one needs to define a new version of structuralism, peculiar to Dedekind, which he calls “logical structuralism”. “Logical” refers to Dedekind’s specific process of abstraction. The “methodological structuralism” identified by Reck has been recognized in Dedekind’s works by several commentators.¹² In particular, for Sieg and Schlimm, Dedekind’s structuralism is linked to his axiomatic approach, which for them must be understood in connection to Hilbert’s and Bernays’s conception of axiomatics (a point strongly defended by Sieg in [Sieg, 2009]¹³).

To read Dedekind as a structuralist because he introduces some of the mathematical concepts which will prove capable of supporting a structuralist interpretation seems to me to be tendencious.¹⁴ It is judging Dedekind’s works against later philosophical standards. Dedekind never stated that mathematics is about studying structures, and his mathematical works rather goes in the opposite direction. In fact, one may offer an alternative explanation of the same phenomena by placing them in the wider context of Dedekind’s mathematics¹⁵.

0.1.2 The historians’s way(s)

I have considered, in the previous pages, the philosophical interpretations of Dedekind’s works and suggested that the considerations of specific parts of his works as well as expectations about the conceptions underlying his works might have created certain biases. The projection of certain expectations can also happen for mathematical concepts. A typical example would be to read Dedekind from a set theoretical standpoint and thus to want to recognize the concept of *Abbildung*, which Dedekind developed in the 1870s, and actually used in the 1880s, in the concept of “substitution” which Dedekind first uses in his lectures on Galois theory in 1865-58. If Dedekind does develop a rather general notion of “substitution”, his use of the Galoisian terminology is evidently following Galois’s use. Yet, with a reading Dedekind’s mathematics under the light of set theory, the use of “substitution” is seen by

¹²We saw that for Sinaceur, Dedekind simply uses the word “concept” for what we would call “structure” ([Sinaceur and Dedekind, 2008], 108).

¹³On Dedekind and Hilbert, see also [Klev, 2011].

¹⁴Corry has invited us to read Dedekind’s algebraic works without the teleology of structural algebra. It seems likewise important to read Dedekind’s number concept without anachronistically using today’s philosophical ideas.

¹⁵In particular, the fact that Dedekind defines at once the entire domain of natural numbers by its structure can be related to Dedekind’s demand that the definitions be uniform, that is, valid for all cases in a determinate framework. Thus, for example, real numbers should not be defined alternatively as roots, logarithms, etc. Rather one should be able to provide a definition of all real numbers in one move.

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Ferreirós as an “obscure” terminology.

In the same line of ideas, retrospective readings identified algebraic structures in Dedekind’s works for a long time, a point that was efficiently opposed by Corry in [Corry, 2004b]. Corry argued that Dedekind was not a (mathematical) “structuralist”, for his mathematical works are not investigating structures themselves but rather using (some) algebraic structures as tools. Corry has highlighted that while Dedekind does possess the essential concepts of (would be) structural algebra, in particular fields and ideals, the roles played by these concepts and the status of each concept are very different from modern algebra.

Indeed, Dedekind does not consider these ‘structural’ concepts as instances of a more general concept of structures. Dedekind does not define ‘structures’ in an abstract general way (i.e., as a non-empty set with abstract operations) but always for numbers or functions, and he does not define the ‘structures’ in relation to each other. Rather, he defines and works with ‘structures’ as if they were “independent mathematical entities”. While in a structural approach, there is a certain unity in the idea of structure, Dedekind treats each concept differently and gives them different roles. Thus, by a close attention to “mathematical knowledge and practice”, Corry highlighted that Dedekind’s approach does not display the overall homogeneity of structuralist mathematics. In the course of this dissertation, I will emphasize this point and investigate in more detail the development of ideal (and module) theory, so as to highlight how the approach adopted by Dedekind answers certain requirements of rigor.

In the last decades, historians of mathematics discarded a practice of historiography that had been dominant until the 1980s and focused on the grand narrative of the shaping of present day knowledge. The type of “Whig history of science”¹⁶ and/or of a Bourbakist idea of the development of mathematics was rejected, while other issues came into focus: larger selections of actors with notably ‘peripheral’ actors; educational, institutional and sociological issues; interest for journals, non-academic practices of mathematics, applied vs. pure mathematics... One of the main themes of historical researches about 19th and 20th century mathematics has been the understanding of the (many) sources of our modernity and the desire to exhibit the many complexities of the development of ‘modern’ mathematics. Dedekind became mainly interesting as one of the marks of a turning point, a junction between classical and modern mathematics. For example, he is the first step in Corry’s history of mathematical structuralism (even if he is not identified as a structuralist). Dedekind is also considered as one of the very first important steps in the development of modern mathe-

¹⁶That is, is the tendency to judge mathematical works in terms of the modern standards. See [Shapin and Schaffer, 1989].

matics. In particular, he plays an important role in paving the way to the rise of modernism in mathematics in Jeremy Gray's inquiries on the subject ([Gray, 1992], [Gray, 2008b], [Gray, 2008a]). In this renewed historiography, Dedekind's mathematics was approached so as to shed light, by contrast, on the more 'traditional' practice of mathematics in the 19th century, an entry door to parts of the history of mathematics that had been left out in previous historical researches (notably Leopold Kronecker's works). A point of convergence of many of these works is a depiction of the mathematical developments between 1850 and 1880 as the opposition between the "algorithmic" Berlin school of mathematics and the "conceptual approach" in mathematics (adopted by Dedekind and which took over the conceptions of mathematics in the 20th century).

Considering the large amount and the great quality of studies about or involving Dedekind's works, one can wonder what there is to add. I will suggest that elements have been overlooked and that some obscure parts are left to investigate further. I will propose, on the bases of these works, which have greatly informed my own research, to explore these points.

The first element to consider is, of course, that of the inner unity of Dedekind's mathematics which I have mentioned above. Indeed, while philosophers have been considering mainly foundational works, historians of mathematics tend to explore Dedekind's mathematical works (algebraic number theory, algebraic function theory) and show little interest for foundational works. In fact, it seems that neither philosophers nor historians have tried to understand Dedekind's corpus as a united whole. The tendency of commentators to consider that Dedekind's overall works have a noticeable unity (to the point that it is systematically mentioned), but at the same time to treat both aspects of his works separately has always seemed puzzling to me. The approach I would like to propose is a global approach of Dedekind's writings between 1854 and 1888, so as to point what can be seen by looking at Dedekind's corpus as a whole rather than in separate pieces.

A second point appeared to me as requiring further thought. As I just mentioned, historiography has emphasized the opposition between the algorithmic and the conceptual approach. The "conceptual approach" is identified as a tendency in the second half of the 19th century to try to substitute concepts to computations in mathematics. It is, for example, described by Ferreirós in the following way:

[The mathematicians working in the conceptual approach] consistently attempted to frame mathematical theories within the most general appropriate setting, in such a way that "outer forms of representation" were avoided, new basic objects were chosen, and a definition of the characteristic "inner" properties of these objects (i.e., a fundamental concept) was placed at the very be-

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ginning of the theory.¹⁷ ([Ferreirós, 2008], 31)

The commonly accepted idea that Dedekind’s mathematics is “conceptual”, in which “conceptual” can essentially be understood as “non algorithmic”, does not yield a description as precise as one might wish. I believe much more can be made to characterize in which ways and how Dedekind actually developed his “conceptual” mathematics. In the first chapter of this dissertation, I will question this category of analysis for a thorough account of Dedekind’s approach. I will thus propose to look beyond the mere statement of “conceptuality” so as to provide a clear account of Dedekind’s elaboration of new concepts and rewriting of theories.

The last point concerns the insertion of Dedekind in his historical context. Although he brought key innovations and changes in mathematics, Dedekind was claiming to walk in the footsteps of such central figures of mathematics as Gauss, Dirichlet or Riemann. He did not consider his works as marking a strong divergence with common approaches in mathematical research but rather as developing certain of their ideas and principles. In addition to Dedekind’s explicit desire to continue some of his predecessors’s works, he also developed his works on the basis of ideas that were, in fact, rather common. Notably, Ferreirós explained that Dedekind’s conception of logic was rather traditional ([Ferreirós, 2008], 119-124 in particular). Another particularly important point is Dedekind’s conception of numbers. The core idea of *Zahlen* is that numbers are creations of the mind, flowing directly from the pure laws of thought. As Jahnke and Otte have underlined, in the 19th century, “[t]here was the widespread idea that number theory was the purest expression of the laws ruling our thoughts” ([Jahnke and Otte, 1981], 30). An account of Dedekind’s mathematics that would propose an historical approach for his foundational works should acknowledge the importance of these points.

By looking at Dedekind’s texts with the question “What kind of structuralist was Dedekind?” or “How can we extend our idea of logicism so that Dedekind can fit in?”, one projects the 21th century philosopher’s questions onto Dedekind – an epistemological reconstruction comparable to the mathematical reconstructions that were decried by historians. Where philosophy could fruitfully be used as a tool for the exegesis, the approach proposed by the authors mentioned above is an extrapolation of some isolated points of *Zahlen*. I would like to explore the idea that a close analysis of the role and status of arithmetic in Dedekind’s works highlights the fact that his foundational writings and his mathematical researches are intimately linked.

¹⁷In this respect, I will suggest that while Dedekind does state the fundamental concepts he will use in his works at the beginning of his writings, many of his mathematical works are, in fact, entirely dedicated to providing a suitable definition to the concepts studied (e.g., real numbers, natural numbers, the Riemann surface), but this is not the main point, here.

Let me now expose more precisely the objectives of this dissertation, in relation with the hypothesis that Dedekind's his foundational writings and the methods developed in his mathematical works are deeply connected.

0.2 Aims of this study

0.2.1 Some methodological considerations

0.2.1.1 Retrospections

In 1998, Ken Saito presented his arguments against mathematical reconstructions and in favor of textual studies for Greek mathematics as going against the tendency previously exhibited to “rationalize” the arguments of Ancient mathematics.¹⁸ To this approach, he opposed an approach relying on texts solely and approaching them in a different way, with an “attitude based on reasonable doubts as to the validity of logical conjectures”:

For example, apparently redundant or roundabout passages call for more attention, because these might reveal some of the ancients' particular thoughts of which modern minds are unaware. ([Saito, 1998], 135)

Despite ample discussion among historians, these issues are still very pregnant in the history of mathematics. For example, in [Chemla, Chorlay, and Rabouin, Forthcoming], a collective book to be published in which practices of generality are studied across time, space and disciplines, David Rabouin proposes a reconsideration of generality in Euclid's *Elements*, suggesting that modern interpretations of generality in ancient texts were biased by wrong expectations as to what generality should be:

a large part of the “defaults” found in the *Elements* could be seen as linked to our expectations about the way “generality” is supposed to hold in science. This teaches us two lessons: first, we have criteria for “generality” which do not necessarily coincide with what was at stake in ancient Greek mathematics; second, the kind of generality which is to be found in these mathematics may be not less, but indeed more subtle than the one which we project on the texts today. ([Rabouin, Forthcoming])

¹⁸“Previous scholars (say, from Tannery and Zeuthen to van der Waerden) were, I believe, confident in the power of something like universal reason, and took it for granted that a careful mathematico-logical reasoning was able to restore the essence of ancient mathematics. Today scholars are more skeptical: the type of reasoning that once played an essential role tends to be regarded as a mere rationalising conjecture. They are even convinced that the modern mind will always err when it tries, without the guide of ancient texts, to think as the ancients did.” ([Saito, 1998], 134)

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The projection of our expectations and categories on historical texts is far from only affecting the history of Ancient mathematics. Closer to us, Paolo Mancosu, in a 2009 paper ([Mancosu, 2009]), considers the “measure” of infinite sets. He presents recent mathematical developments which provide an alternative to Cantor’s theory of infinite, thereby opening the possibility of a philosophical reflection on the supposed “inevitability” of Cantor’s theory of infinite.¹⁹ In this last part of his study, Mancosu criticizes a “Whig” approach to the history of mathematics and the tendency to consider that there exists only one “right way” to proceed for the development of, for example, the theory of infinite, as if mathematics and reason were progressing with an absolute necessity towards *the* truth as it is now known. This presents the risk to judge previous authors against Cantor’s works or even to “assimilate [them] to the later Cantorian accomplishments”. To do so leads to “a completely anachronistic reading of many of the medieval and later contributions (this was certainly the case with Duhem’s interpretation of Gregory of Rimini and Maier’s interpretation of Albert of Saxony; recent scholarship has been more cautious (Murdoch, Dewender, etc.))” ([Mancosu, 2009], 626). Besides, this often implies that one would miss “the complexities of the contrasting intuitions that have shaped the attempts to cope with such a recalcitrant object” (*ibid.*, 636).²⁰

In [Chemla, Chorlay, and Rabouin, Forthcoming] mentioned above, two papers raise similar historiographical points, of the kind which underlie my own inquiry. The first one by Frédéric Jaëck ([Jaëck, Forthcoming]) considers the introduction of the concept of Banach space and the writings by Banach in which former historians have identified the introduction of the concept of Banach space. Jaëck stresses that historians might have been mistaken in recognizing this object in Banach’s 1922 *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, for, he argues, the axioms stated here, and later used for the definition of Banach spaces, do not play the role of defining a new mathematical object. The concept of Banach spaces only appears in a later work, published in 1932, which suggests that the development of Banach spaces was made in at least two steps. This conclusion brings to the fore the important issue of the historian’s assessment of which degree of generality is given to certain axioms, definitions, theorems, since, Jaëck tells us, a core difference between the 1922 and 1932 papers is the generality of the axioms. Without the close attention for the interpretation of the texts, here, one is tempted to anachronistically see, in Banach’s statement of the axioms in 1922, the first emergence of the

¹⁹The idea of Cantor’s theory of infinite being “inevitable” was an argument advanced by Gödel, in [Gödel, 1990].

²⁰Note that the two authors I just mentioned are philosophers, who warn us against the dangers of retrospective readings in philosophy. The battle against retrospective readings should not be seen as a battle between philosophers and historians, of course.

concept itself, while in fact, at that point, the status and meaning of the axioms do not produce a new concept.

Similar caution can be important in reading Dedekind. A simple example is the introduction, in his algebraic number theory a concept formally similar to that of ring appears, which is named *Ordnung*, but which is not given the fundamental status it has in modern algebra. It is the most striking case, but as Corry underlined, Dedekind’s ‘algebraic’ concepts are considerably different from the concepts of modern algebra, by the role and status they have in the theory.

The second paper, by Frédéric Brechenmacher ([Brechenmacher, Forthcoming]), studies different conceptions of generality in the 1870s and suggests (not unlike Rabouin’s point) that our expectations of what “generality” meant for 19th century algebraists kept the variety of conceptions of generality out of our sight. Brechenmacher emphasizes the importance to be cautious when identifying epistemological values in mathematical texts. Studying the controversy between Camille Jordan and Leopold Kronecker in 1874, he explains that, in fact, this quarrel “sheds light on two conflicting perspectives on ‘generality.’”²¹ More interestingly, it seems that Kronecker’s own historiography of general reasonings and what constitutes a “truly general approach” was taken up by Thomas Hawkins. Hawkins called this particular generality the “generic reasoning in algebra” and identified it as playing a key role in the history of algebra (since Viète). But by adopting this particular view of generality and its history, historians might have missed other conceptions of “generality”, whose various meanings were “an organization of knowledge of the type that was used before the emergence of object-oriented disciplines” (*ibid.*). Historiographical considerations about generality were thus altered by our own expectations of what a “general” approach or reasoning was supposed to be.

The point, here, is not to provide a definition of the different kinds of generality and trace their respective history – neither for Brechenmacher nor for me. Rather, it is to emphasize the idea that our readings of the texts is affected by our preconceived ideas about the epistemological values themselves and how they might have played a role in this history of mathematics. It becomes important to be able to identify the actors’s categories, if one is to recognize the variety of conceptions (and even understand the benefits of satisfying such epistemological values). The same issue appears, quite strikingly, in Dedekind’s and Kronecker’s assessment of the “abstraction” or, a point that I will consider more deeply, of the “arithmeticity” of an approach.

²¹ “[W]hile Jordan criticized the lack of generality of Kronecker’s invariant computations because they did not reduce pairs of forms to their simplest expression, Kronecker considered Jordan’s canonical form as a ‘formal notion’ with no ‘objective meaning’ which therefore failed to achieve true generality. What made one generality true was exactly what made the other generality false”.

0.2 Aims of this study

Any such reflection also ought to avoid the opposite excess, that is, suggesting that the history of the particular matter considered should be completely rethought or even rewritten – and it is, of course, not what I am suggesting in emphasizing these methodological issues. Indeed, the point is not about Dedekind’s place in the history of mathematics: it is wrong neither to give Dedekind pride of place in the development of modern algebra and of modern set theory, nor to recognize in his approaches the premisses of axiomatic definitions. However, it is important to be able to provide a non-anachronistic account of his practice of mathematics and of the key epistemological values which guided them. It seems indeed doubtful that Dedekind’s ideas and conceptions would be similar to ours – a simple example being the co-existence in his definition of the domains of numbers of a genetic approach and axiomatic definitions (see [Sieg and Schlimm, 2005]). In particular, *his* approach should not be assessed or explained in terms of our own categories. However, in the case of our reading of Dedekind, the projection / reconstruction is made less at the level of the concepts themselves, than at an epistemological level. Indeed, our expectations of Dedekind’s conceptions are distorted by a feeling of proximity resulting from the preponderance in our mathematics of concepts grown *from* Dedekind’s ideas. Not only did he father some of the core concepts of modern algebra, but certain parts of his writings appear so very familiar to modern day readers that they need next to no actual “reconstruction” to fit modern standards. However, Dedekind’s approach, while based on concepts still in use today, is far from identical to modern algebra or algebraic number theory, for it was shaped by specific epistemological values and principles, as well as by the set up of specific strategies to satisfy these values.

Because of the proximity of certain of his works with our mathematics, the historiographical treatment of Dedekind’s mathematics slipped through the net of many of the historiographical criticisms mentioned above. However, one should not forget that Dedekind’s ideas were reinterpreted, filtered so to speak, by mathematicians such as Emmy Noether, and are not, in themselves, so close to present day knowledge and practice. Indeed, as Corry emphasized, the concepts invented by Dedekind and to some extent the approach he advocated for (against Kronecker’s) presents strong differences with our modern concepts. It should not be overlooked that the actual adoption of Dedekind’s concepts only occurred after Hilbert’s *Zahlbericht* (1897) and that Emmy Noether was greatly responsible in developing both Dedekind’s ideas and his notoriety. In this dissertation, I would like to expose the epistemological justification underlying Dedekind’s mathematics so as to support the idea that his approach is, in fact, very different from ours.

0.2.1.2 Arithmetical reading and rewriting of Riemannian function theory?

As I have mentioned above, I would like to adopt a global approach of Dedekind's corpus. As an historian interested in philosophy, and in an attempt to propose a comprehensive account of Dedekind's mathematics, I would like to be able to understand how the more philosophical aspects of Dedekind's works are integrated into his mathematics and mathematical practices.

The conjunction of a global approach and of a rigorous historical approach take seriously philosophical questions to guide the historical inquiry incited me to enter into the Dedekindian corpus using a different road than the ones followed before. I would like to start the investigation with a paper that is less studied and which comes less evidently in mind as a first step in an analysis of Dedekind's works, namely the paper co-written with Heinrich Weber and published in 1882, *Theorie der algebraischen Funktionen einer Veränderlichen*.²² By doing so, I hope to suggest a way to observe Dedekind's mathematics from a different perspective. I believe that this allows us to bring to the fore the importance of questions that were not systematically investigated in previous studies and to propose an account that sheds light to certain characteristics of Dedekind's mathematics. By entering through the side door, interesting questions present themselves allowing us to see *how* Dedekind developed his mathematical innovations and in what consisted in practice his "conceptual approach".

Algebraische Funktionen offers a rewriting of the bases of Riemannian function theory, by transferring to functions some of Dedekind's methods from algebraic number theory. Dedekind and Weber thus propose what we identify as an "algebraico-arithmetic" definition of the Riemann surface, together with the reformulation of several basic concepts of Riemannian function theory, such as the genus, and with new proofs of important theorems, such as the Riemann-Roch theorem. This transfer sheds a brighter light on systematic features of Dedekind's practice. Here like in algebraic number theory, the treatment aims at a more uniform and more rigorous treatments of (the foundation of) certain theories. Interestingly enough, in both cases, Dedekind achieves this goal through the use of the same concepts and methods. This treatment yields an arithmetical reading of the theory and provides an arithmetical treatment of its foundations.

The meaning that a reader gives to a mathematical theorem depends on her historical and sociological background. Catherine Goldstein highlighted in [Goldstein, 1995] the fact that mathematical results cannot be taken as universal and fixed, neither in time nor in space. This idea was pursued by

²²Hence the title of my first part, which marks the entry into Dedekind's mathematics through a different door than the entry points usually taken – be it ideal theory or the definition of natural numbers.

0.2 Aims of this study

Caroline Ehrhardt²³ who considers Galois’s notion of group and highlights the strong differences of the diverse receptions (in her words, the “reelaborations” of Galois’s memoir).²⁴ Goldstein’s and Ehrhardt’s analyses bring to the fore how the understandings and interpretations of mathematical results, concepts and theories are historically, sociologically and epistemologically correlated. This last point, not present in Goldstein’s and Ehrhardt’s works, will be one of the pillars of my argument. In particular, as a starting point for the dissertation, I will follow ideas highlighted in Karine Chemla’s works on epistemological values in mathematical cultures,²⁵ and suggest that certain epistemological values, significant in Dedekind’s conception of mathematics, produced a certain reading of Riemann’s theory of algebraic surfaces. This reading, I will argue, appears to be deeply arithmetical. From this, follow several questions that I will investigate throughout the dissertation: what are the nature and the role of arithmetic in Dedekind’s mathematics? Which peculiar questions is the resort to arithmetic supposed to answer? Why is arithmetic considered as the answer to these questions?

The re-definition of the Riemann surface given in 1882 by Dedekind and Weber is essentially a reinterpretation of Riemann’s ideas, a *rewriting* of the fundamental concepts of algebraic function theory. The idea of “rewritings” was developed in a paper by Norbert Schappacher ([Schappacher, 2010]), in which he explains that “rewritings on all scales make up the very fabric of mathematical activity through the centuries.” The first step of his study are arithmetical rewritings of the notion of points by Dedekind (real numbers as points of the straight line) and Dedekind and Weber (points of the Riemann surface). The aim of comparisons between the original concept and its rewriting is to “describe explicit transformations of epistemic objects and techniques”²⁶ ([Schappacher, 2010], 3260). Dedekind engages into “rewritings” at many points in his works, proposing a new definition of the Riemann surface, “new clothes” for Kummer’s ideal numbers, and even rewriting his own works so that they agree more closely to his epistemological requirements.²⁷

²³[Ehrhardt, 2012], [Ehrhardt, Forthcoming].

²⁴I will consider this point briefly in Chapter 4.

²⁵See, for example, [Chemla, 2003], [Chemla, 2009], in which she emphasizes how epistemological values such as generality “are subject to specific practices, according to the culture in which they are valued” ([Chemla, 2009], 148). It is important, she writes, that these values be “identified and described” (*ibid*) for a profitable study of the mathematical practices of the actors.

²⁶“Epistemic objects and techniques” are a reference to [Epple, 1999], who himself was referring to Rheinberger’s works.

²⁷Another “rewriting” of Riemannian function theory is proposed by Heinrich Weber in his *Lehrbuch der Algebra* (1895-96) in which the Dedekind-Weber approach is mixed with Kronecker’s.

0.2.2 The argument of the thesis. The corpus and its limits

The 1882 paper, while recognized as a major work (of Dedekind and Weber, and in the history of mathematics), is often mentioned rather quickly as yet another example of Dedekind's ("conceptual") method.²⁸ It is thus treated as a side work, often taken away from the context of Dedekind's works. I rather will propose to consider it as an integral (and important) part of Dedekind's corpus. Indeed, this paper combines several important characteristics to be taken into more consideration when studying Dedekind's mathematics. It is precisely the transfer from one domain to the other, the change of framework, that will allow us to capture in a different way Dedekind's methods. In particular, the arithmetical form given to the theory appears considerably more striking when one looks closely at the overall strategy of redefinition of the Riemann surface proposed in the paper.

The arithmetical rewriting in this non-arithmetical context suggests a specific role played by arithmetic. I will highlight, in Dedekind's works, the preeminence of arithmetical methods, that is, methods built on elementary arithmetical notions, by bringing out their importance in *Algebraische Funktionen* and following this thread in a selection of Dedekind's works. This will bring to the fore the importance to pay a closer attention to the characterization of arithmetic in Dedekind's works and to question its role, its status. I will suggest that Dedekind deploys a strategy of arithmetization and that this strategy is tightly linked to epistemological values and strong convictions about the 'right' approach in mathematics. I will suggest that this strategy does not only account for Dedekind's methods in algebraic number and function theory, it also is reflected in Dedekind's foundational researches. That is, I shall argue that to some extent *Zahlen*, published in 1888, bears the mark of mathematical works, and notably the 1882 function theory. To be clear, by this I do not mean such a bold statement as a complete transformation of Dedekind's ideas on numbers. Rather, I mean that the definition of the number concept is intrinsically linked to the uses of arithmetic in Dedekind's mathematical practice, that arithmetic and the number concept were affected by the mathematical practice and should be understood as embedded in a more global perspective of Dedekind's works. This thesis relies on the conviction that the practice of mathematics is a key factor in the shaping and reshaping the objects of mathematics and the mathematician's conceptions.

²⁸It is notably the case of [Corry, 2004b] or [Dugac, 1976b]. Exceptions to the rule are [Geyer, 1981], [Strobl, 1982], [Houzel, 2002] and [Schappacher, 2010].

0.2 Aims of this study

0.2.2.1 Issues addressed and plan adopted

To reach this goal and avoid an anachronistic exegesis, my analysis will focus on understanding Dedekind's strategy by unfolding the characteristic elements of practice that can be identified in a selection of texts.²⁹ The novelty of the work I am proposing is not in the texts selected, but rather in the project of inquiring into the relationships between the texts which, I will argue, reveal important characteristics of Dedekind's conception of arithmetic. In fact, it rather seemed more relevant to investigate major and well known texts, so as to highlight the significance of the argument – other less significant texts being used for precisions when needed.

The dissertation will be organized in four parts, each containing two chapters. The strategy I propose will thus be the following: in a first part, I will expose in detail *Theorie der algebraischen Funktionen einer Veränderlichen*, after necessary preliminaries about elements of context. The paragraphs concerned with the historical and mathematical contexts aim not only at setting up the scenery, but also at outlining in which ways Dedekind did not see himself as diverging from his predecessors's ideas. In studying the 1882 paper, I will highlight its essential features. This shall allow me to have solid elements on which to rely to show that the treatment of the (grounds of the) Riemann surface provided is thoroughly “arithmetical”. This suggests a series of questions about arithmetic, its status and its characterization in Dedekind's mathematics.

From there, the three following parts will address these questions. For this, my general plan of action consists in unraveling the steps of the developments of Dedekind's arithmetic, from his *Habilitationsvortrag* to *Was sind und was sollen die Zahlen?*.

In the second Part, I will propose to return to Dedekind's earlier works. This will allow us to see in Chapter 3 that in 1854, Dedekind's *Habilitationsvortrag*, titled *Über die Einführung neuer Funktionen in der Mathematik*, presented a characterization of arithmetic as essentially related to domains of numbers and elementary operations. It appears that in this context the kind of divisibility questions which will be essential in his next works and were part of typical number theoretical issues remain absent. A couple of years later, Dedekind produced his first works in number theory, which I will present in the following Chapter: his lectures on Galois theory in 1856-58 and his first published article on number theory in 1857, *Abriß einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus*. In these works, ideas about arithmetic start to move away from simple considerations of domains of numbers to a closer attention to divisibility. In particular, emerges the definition of arithmetic operations to

²⁹As a side-effect of this approach, the study of *Algebraische Funktionen* will concentrate on Dedekind's role in the paper. I hope to provide a more thorough consideration of Weber's part in future works.

reinterpret relations of inclusion for concepts such as groups.

This peculiar definition of arithmetic operations is used in the two other texts I will consider in the third Part, namely the 1871 and 1876-77 (and 1879) versions of algebraic number theory, respectively titled *Über die Composition der binären quadratische Formen* and *Über die Theorie der ganzen algebraischen Zahlen*. A full chapter will be devoted to each version of Dedekind's famous algebraic number theory. I will propose to study the increasing importance of an arithmetical approach for developing the theory, a point that should be made clear by a comparison of the changes between the 1871 and the 1877 versions of the theory. The development of a certain arithmetical strategy indeed culminates in the 1877 version of ideal theory, which is later transferred to algebraic functions, resulting in the 1882 paper with Weber.

The last Part of the dissertation will be more reflexive. In Chapter 7, I will take a step back in time and review the historiography on the issues, in the 19th century, of extending concepts of number and of the so-called "arithmetization of mathematics". In particular, I will consider *Stetigkeit und irrationale Zahlen* and some of Dedekind's manuscripts. Rather than a mere survey of an already well researched subject, I aim at exploring the idea that "arithmetization" in the first sense of the term, i.e., to give an arithmetical form to a definition or an argument, should be taken into account in the consideration of Dedekind's "arithmetization". I will thus reconsider the conclusions drawn in Part III, to put forward the hypothesis of a certain inner coherence of Dedekind's works. I will suggest that an identification of the underlying epistemological motives for Dedekind's methods can provide a consistent explanation of his recurrent use of arithmetical operations.

One of the core stakes of this dissertation is to offer an interpretation of what "arithmetic" and its objects are for Dedekind and to understand the role attributed to arithmetic operations. As a consequence, it is essential for us to consider these questions in relation to Dedekind's foundational researches. In particular, the issues addressed by Dedekind's foundational works and the methods employed to do so, are important to exhibit a potential relation between the foundation of arithmetic and its place in Dedekind's mathematics. This will lead me to propose an interpretation of the concept of number and its definition for Dedekind, in the last Chapter. I will highlight how *Was sind und was sollen die Zahlen?* offers an answer to the demand of a definition of the natural numbers, and does so in a way able to account for the previous developments of arithmetic. I will conclude with a suggestion regarding how we could link the definition given to previous and later works of Dedekind.

0.2 Aims of this study

0.2.2.2 About the corpus

In the thesis, I have decided to expose the texts with a great amount of details so as to identify the most noticeable elements of practice. I will do so for two reasons. The first reason is that my goal is to draw conclusions from the texts themselves: the study of *Algebraische Funktionen* suggests questions, whose answers ought to be found in Dedekind's writings, in the form of claims made by Dedekind and which need to be interpreted in their context. By this detailed study, I wish to identify recurrent practices and strategies, without downplaying evolutions.

My second reason for staying close to the texts is inspired by the hope of identifying clues in the texts likely to allow me to draw information to understand *how* the elaboration of Dedekind's writings was made. Hence, I wish to involve as little modern mathematical formulations as possible. I will in particular expose Dedekind and Weber's theory of algebraic functions, as well as that of Dedekind's lectures on Galois theory, the first two versions of his algebraic number theory, and his definition of several domains of numbers. While I have no pretension to exhaustiveness, I do hope to provide a representative account of Dedekind's practice of mathematics.

The approach I adopt, which could be seen as micro-historical, has obvious limits. This work relies on a close attention to details. While I do not intend to treat Dedekind independently from the larger history (hence the paragraphs of contextualisation), this does entail that my focus is rather narrow. Consequently, larger timescales and more sociological issues had to be kept out of the dissertation,³⁰ and the space devoted to comparing Dedekind's approaches to alternative ones had to be limited. More significantly, the choice of a tight focus also imposed limits to the attention given to the reception of Dedekind's works. These later works are remarkable by the way in which they combine Dedekind's approach with Kronecker's, often replacing Dedekind's arguments with Kroneckerian ideas. I am conscious of these limits and have tried to avoid producing a work that would have been completely airtight to these untreated matters.

Two points, in my defense, should however be underlined. The first is the academic isolation in which Dedekind put himself and the somewhat delayed reception of his works, which make a more sociologically oriented study of Dedekind's works less prominent than in other cases.³¹ The second concerns

³⁰Notably, Schappacher suggests, in [Schappacher, 2010], to consider the 1882 paper as part of larger interrogations about organization of mathematics into disciplines or subdomains. I will not pursue this line of thought.

³¹Of course, it would be of great interest to understand more clearly the sociological and mathematical underpinnings of this point. They are already indicated on one hand by the institutional importance of Kronecker, and on the other hand by Dedekind's uses of actual infinities. Dedekind saw the use of infinite sets as an extrapolation of Gauss's notion of class but they were hardly accepted by his contemporaries. This, however, is a

the reception of Dedekind's works. Dedekind was deeply convinced that his approach was the most appropriate one, and was rarely open to suggestions (especially on a methodological level). The few mathematicians who adopted his methods did not, however, perpetuate this idea. Rather than acknowledging that Dedekind was right (against Kronecker, in particular), they recognized that some of his ideas and some of his concepts (in particular the concept of field, and his approach for group theory) were fruitful and that the divergences between Dedekind's and Kronecker's approaches were far from necessary. I hope that my dissertation will allow us to understand more clearly the specificities of Dedekind's thought, and his own arguments for the superiority of his approach.

completely different issue.

Part I

Through the side-door

Introduction to Part I

Dedekind is most well-known for his number theoretical works, in which were introduced the concepts of field and ideal, which would become some of the most important concepts of today's mathematics. The “groundbreaking” new notions introduced in number theory were to be extended, generalized to more abstract concepts by mathematicians in the early 20th century. For Dedekind, those notions were tightly related to the context of the study in which they appeared and were used for the purpose of investigating the properties of mathematical objects (for example, algebraic numbers). The elaboration of Dedekind's ideas was led by the desire to set up a uniform approach, that is to be able to treat, in one move, entire classes of objects. The new notions introduced by Dedekind were thus never considered in and for themselves and out of the precise theoretical framework in which they were to be used, yet they were defined without reference to the individual nature of the objects concerned. This peculiar point allowed the transfer of concepts and methods developed in number theory, to function theory, in a 1882 paper co-written with Heinrich Weber, *Theorie der algebraischen Funktionen einer Veränderlichen*.

In the midst of what historians tend to call the development of a “conceptual approach” in mathematics led by Riemann and Dedekind from the late 1850s on, Dedekind and Weber wrote a paper whose goal was to redefine the basic notions of Riemann's function theory and to set up a solid ground for the investigation of complex functions and Riemann surfaces. But one should not be mistaken, Dedekind and Weber's paper is not a shining proof of the effectiveness of the “conceptual approach”. Initially, it is a criticism of previous treatments of Riemann's ideas and an attempt to go back to Riemann's true methodological and epistemological principles hitherto overlooked by his successors. It highlights that the “conceptual approach” was, in fact, very sparsely adopted – and even when adopted, as it was the case for Dedekind, the response from the mathematical community was not an overly enthusiastic one.

For Dedekind and Weber, ever since Riemann's works in function theory, mathematicians had failed to follow some of the key precepts of his elaboration of the concept of Riemann surface, namely, to treat entire classes of functions without distinguishing between special cases, without relying on individual or explicit expressions, and without taking computations as a ground of the theory when they should be its results. To be able to correct these flaws, Dedekind and Weber adopt an approach opposite to the usual approach one. They start by considering fields of algebraic functions, in which they will be able to transfer and adapt Dedekind's number theoretical methods. Their goal is to thereby provide a more rigorous and more uniform treatment of Riemann surfaces.

A singular paper in the Dedekindian corpus by its subject, the work on algebraic functions co-written with Weber exhibits nonetheless typical characteristics of Dedekind's approach in mathematics. It presents itself as an interesting entry door into Dedekind's way of working in mathematics, which I will propose to take. Indeed, it allows to see Dedekind's corpus under a different light and to enter in it with different questions. In particular, I hope to be able to deepen our understanding of "arithmetic" in Dedekind's approach.

To start on this path, I will divide my inquiry in two chapters. First, I will introduce contextual elements. In particular, I will start by considering questions of historiography, in which I will highlight that to consider Dedekind's contribution to mathematics as a part of a "conceptual approach" might be less fruitful than other approaches. I will also briefly expose in this chapter historical and mathematical elements about Riemann's function theory that will be important for our understanding of Dedekind and Weber's paper, as well as some interesting aspects of Weber's multifaceted career.

In a second chapter, I will give a detailed presentation of the content of Dedekind and Weber's theory of algebraic functions, which will serve as an essential basis for my argument. This paper has been largely, in later years, acknowledged as a milestone of algebraic geometry. However, it is long, complicated and based on ideas which were already found difficult to apprehend in Dedekind's number theory, and its immediate reception is almost non-existent. My aim, here, will be to consider the 1882 paper as a part of Dedekind's corpus, rather than a piece of cutting-edge mathematics. With all the mathematical elements at hand, I will call attention to the "arithmetical" nature of the methods developed by Dedekind and Weber. The idea will be to provide a thorough presentation of the said paper, which will allow to show the importance of arithmetic – rather than to exhibit the arithmetical parts and let the other elements silent. The comprehensive treatment of Dedekind's mathematics I would like to propose in the next chapters of the dissertation will grow from the elements highlighted in reading *Algebraische Funktionen*. To avoid a characterization of Dedekind's approach as merely "conceptual" implies to be able to characterize it in some other way. With the analysis of the 1882 paper, I would like to suggest that to look at Dedekind's works from the viewpoint of the role he attributed to arithmetic within mathematics yields most interesting insights about his works.

Chapter 1

Elements of contextualisation

It is commonly accepted that, in the late 1850s in Göttingen, Dirichlet, Riemann and Dedekind, formed the heart of a “conceptual approach” to mathematics. The importance of this approach is widely recognized, in particular for setting the deep transformations in mathematics which happened at the end of the 19th century. It is also usually acknowledged that this approach was, for several decades, restricted to a small isolated group of mathematicians. Most readings of Dedekind’s works are so deeply embedded in this idea, so tainted by this approach, that a critical reflection is inevitable. But the reference to a “conceptual approach” is, in itself, relatively nebulous. I will expose the outlines of the “conceptual approach” to point out reasons for which this reconstruction might not be the most appropriate category to work with for an account of Dedekind’s mathematics. As a way of accounting for Dedekind’s ideas, as it was used for example by Edwards, Avigad, or Ehrhardt, it appears to leave room for further inquiries.

A first way in which the “conceptual approach” can be understood is as referring to a gradual rejection of purely algorithmic approaches, which has its roots in algebraic equation theory in the 18th century but reveals to be as wide as it is slowly growing: from Lagrange to Dedekind, via Dirichlet and Gauss but also Galois, Abel, Jacobi or Eisenstein – in that case, Dedekind’s works are merely an accentuation of this tendency. A second way to understand to “conceptual approach” is to make of Dedekind (and Riemann) a far-ahead of his time precursor of Hilbert’s and E. Noether’s works, an idea always dangerous and slightly contradictory with Dedekind’s insistence on his belonging to the tradition of Gauss’s and Dirichlet’s number theory.

Following this, the next paragraphs will expose some elements of Riemann’s function theory and Weber’s works, which will be pertinent for the study of *Algebraische Funktionen*.

1.1 Remarks on the “conceptual approach”

Königsberg, in which Jacobi taught from 1829 to 1842, was the first place to develop a mathematical school, which Klein described as being

a scientifically-oriented neohumanism, which regards as its aim the inexorably strict cultivation of pure science, and in search of that aim establishes a specialized higher culture, with a splendor never seen before, through a concentrated effort of all its powers.¹ (Quoted and translated in [Ferreirós, 2008], 8)

The ideas developed in Königsberg and in Berlin, notably by Jacobi and Kummer, were later brought to Göttingen by Dirichlet.

After Gauss’s death in 1855, Dirichlet accepted the chair of mathematics in Göttingen. He brought with him deep transformations in the teaching of mathematics, notably by introducing the teaching system in use in Berlin or Königsberg universities, in which research and teaching were mixed together – a system which later became more largely spread in Germany. Dedekind, who became a *Privatdozent* in Göttingen in 1854, reports that Dirichlet, when he came to Göttingen, “opened up a new era for mathematical studies” by giving brilliant in-depth lectures at the verge of research. Dedekind was strongly impressed and deeply influenced by Dirichlet, about whom he wrote

[T]hrough his teaching, as well as frequent conversations (...), he turned me into a new man. In this way he had an enlivening influence on his many students. (Quoted and translated in [Ferreirós, 2008], 26.)

Importantly enough, Dirichlet seems to have had a decisive influence on Dedekind’s turn towards number theory. As Dedekind’s letters to his family tell us, Wilhelm Weber’s seminar of mathematical physics was, until Dirichlet’s arrival, the one that attracted Dedekind’s attention the most.²

The mathematical activity in Göttingen became then articulated mainly around Dirichlet, Riemann and Dedekind. It is often said that these three mathematicians, all followers of Gauss, formed the first core of the so-called Göttingen conceptual tradition in mathematics, whose founding and most important principle was to avoid long computations and to rather use concepts than explicit forms of representation (*Darstellungsformen*). This principle is one of the core precepts of leading Riemann’s approach in function theory but was overlooked by Riemann’s immediate successors. In 1882, a

¹ “*der naturwissenschaftlich gerichtete Neuhumanismus, der in der unerbittlich strengen Pflege der reinen Wissenschaft sein Ziel sieht und durch einseitige Anspannung aller Kräfte auf dies Ziel hin eine spezialfachliche Hochkultur von zuvor nicht gekannter Blüte erreicht.*”

² Dedekind’s *Inauguraldissertation* in 1852 was on Eulerian integrals, and his *Habilitationschrift* was on transformations of rectangular coordinate systems.

1.1 Remarks on the “conceptual approach”

core motivation for the arithmetical rewriting of Riemann’s function theory by Dedekind and Weber was to provide an account of it which would meet the principles which had led Riemann’s approach.

Dedekind and Riemann studied together at Göttingen and became close friends. Dedekind, who was teaching probability theory, geometry and algebra, also followed Dirichlet’s and Riemann’s classes. He highly valued Riemann’s works and explicitly stated close links between their mathematics.

Let us consider briefly the context in which the idea of “conceptual mathematics” is presented, in particular since the historiography underlines key changes in mathematics, in this period. I will propose a brief general overview so as to highlight how Dedekind’s contribution is integrated in the larger picture.

Many recent works in the history of 19th century mathematics³ have been referring to the 1850s as the start (or even a golden age) of the “conceptual approach” which was (supposedly) central in Göttingen, especially as opposed to Berlin tradition of algorithmic and constructive mathematics. The words “tradition” and “school” should be avoided (as they are by Ferreirós, Ehrhardt, Stein, etc.). None of the actors was able (for diverse reasons) to start a school or assemble enough followers to build up a tradition. The vagueness inherent to the word “approach” allows to convey the idea that they shared certain basic premisses on the methods one should adopt for doing ‘good’ mathematics and assure its development. In the same time, it does not engage any strong sociological meaning. In fact, it seems that the “conceptual approach” can either be understood as a tendency, beginning long before Dirichlet, to move away from approaches relying strictly on computations in mathematics, or as a movement which might have its roots in the 1850s Göttingen mathematicians but only really became tangible with Hilbert’s and Emmy Noether’s works.

In the following, I will question the consistency of this reconstruction, by suggesting that the net of influences between mathematicians is more complex than it first appears, and that the ideas involved in the “conceptual approach” should not be considered as peculiar to Dirichlet, Riemann and Dedekind. This approach is usually associated to a larger transformation in mathematics. I will thus consider the broader context in which historiography locates the key changes in mathematics. In particular, I will bring to the fore the presentation of Dedekind in this larger context, to better point out the insufficiencies of this approach.

³See for example, but this list is not exhaustive, [Edwards, 1992], [Edwards, 1983], [Ferreirós, 2008], [Ferreirós, 2007], [Ferreirós, 2006], [Laugwitz, 2009], [Stein, 1988], as well as [Boniface, 2004], [Boniface, 2002], [Ehrhardt, 2012], [Ehrhardt, Forthcoming], and philosophers of mathematics among which [Avigad, 2006], [Reck, 2012], [Reck, 2009], [Tappenden, 2005a], [Tappenden, 2005b], . . .

Howard Stein ([Stein, 1988]) considers that the change underwent by mathematics, at that time, were

a transformation so profound that it is not too much to call it a second birth of the subject – its first birth having occurred among the ancient Greeks. ([Stein, 1988], 238)

For Detlef Laugwitz, the 1850s in Göttingen were a “golden age” and played a key role in this general change. It was “an extraordinary time (...) that prepared to a considerable extent the mathematics of the 20th century” ([Laugwitz, 2009], 24)...even if it lasted a very short time: Gauss wasn't active anymore in the 1850s, Dirichlet died after only four years in Göttingen, Dedekind left for Zurich in 1858. But it still was the time and place where Riemann wrote his “most important works” and Dedekind made his first steps in number theory with Dirichlet. Riemann is usually placed at the very center of the radical changes happening during the second half of the 19th century and Dedekind is considered as a “true conceptual thinker”. According to Laugwitz, the “conceptual approach” is a continuation of certain aspects in the thoughts of Gauss and Dirichlet by Riemann and Dedekind. For the latter, “the view of mathematics as a form of thinking in concepts (*Denken in Begriffen*) is second nature”. In fact, Dedekind and Riemann seem to have been the only two mathematicians, at that precise moment, to be aware that a “gentle revolution” was in march.

For Jeremy Gray, the changes in mathematics that happened in the 19th century is essentially an ontological revolution, for

although the objects of study remained superficially the same, the way they were defined, analysed theoretically, and thought about intuitively was entirely transformed. This new framework was incompatible with older ones, and the transition to it was much greater than scientists are accustomed to. ([Gray, 1992], 245)

Riemann and Dedekind are two key figures for this “revolution”. In Gray's view, many different parts of mathematics, such as algebraic number theory or projective geometry, went through such important changes that “central features as number and line were left dramatically altered in status and meaning” (*ibid.*). Riemann and Dedekind are considered to be two of the most influential mathematicians in bringing about these changes (Riemann in geometry, Dedekind in number theory), bringing about new mathematics perceived as “modern”. However, Gray seems to locate the actual passage to “modernist” mathematics around 1890: Riemannian geometry “provided the ideology for a revolutionary change in geometrical ideas” ([Gray, 2008b],

1.1 Remarks on the “conceptual approach”

151) and Dedekind laid the foundations for Hilbert’s *Zahlbericht*. Dedekind’s approach, according to Gray, is

recognizably modernist in its emphasis on the need for internal definitions of objects, tailored to meet internally set goals of the theory. The concept of integer is stretched to include new integers, and the concept of prime is likewise stretched. (*ibid.*, 150)

The importance of Riemann for the development of new conceptions and concepts in mathematics during the second half of the 19th century is indisputable. For M. A. Sinaceur, mathematics was marked in its “style and orientation” by Riemann’s “genius” ([Sinaceur, 1990]). Riemann is the instigator of a “new mathematical spirit, by his way of taking head-on a great number of problems, as well as to provoke radical questions” (*ibid.*, 222). The desire for a certain “conceptual clarity” is central in both Riemann’s and Dedekind’s works, and is guided by the idea that

concepts, in mathematics, are used for knowing when computations are necessary and when they are not. This way of looking at things is Dirichlet’s. It excessively impregnates Riemann’s methods: (...) to avoid, as much as possible, the long computations and to substitute them with ideas.⁴ (*ibid.*, 223)

This description, philosophically heavy but mathematically vague, is typical of statements about the “conceptual approach”. It is true that Dedekind was a strong advocate for substituting concepts to computations. However he stayed a rather lonely voice until the end of the century. He stated repeatedly that mathematicians working in function theory failed to follow Riemann’s precepts. H. Weber (in particular with his textbooks) revealed to be instrumental in passing these ideas to the next generation of mathematicians (Hilbert, E. Noether, Van der Waerden, ...) who fully developed them.

Often identified by commentators with Riemann’s ideas, the new “conceptual” approach can be summed up as being non-algorithmic, which “implies a new mode of thought: mathematical objects are no longer given primarily as formulas but rather as carriers of conceptual properties” ([Laugwitz, 2009], 34). For Laugwitz, this (supposedly) new possibility had already surfaced in previous works:

⁴ “*les concepts servent, en mathématiques, à savoir où il faut calculer, où il ne faut pas le faire. Cette façon de voir est celle de Dirichlet. Elle imprègne la méthode de Riemann jusqu’à l’excès : serrer de près les concepts étudiés ; y adhérer ; les présenter dépouillés de tout ‘bruit’ avec les seules indications indispensables ; enfin, éviter, autant que faire se peut les longs calculs et leur substituer les idées.*”

Leibniz had both possibilities in mind, but the algorithmic approach triumphed beginning with Euler. Then came the day of the conceptual approach. Bolzano and Cauchy (1817/1821) casually announce the notion of continuity of functions, and Dirichlet (1829) deduces the representability by Fourier series for a class of functions defined by concepts. Riemann defines a complex function by its property of differentiability rather than by an analytic expression, which he regards as a secondary matter. In algebra, Dedekind moves from expressions made up of terms to the concepts of ring, field, and ideal. (*ibid.*, 34)

Changes of this type happened in analysis, geometry, number theory... The emblematic example of the ideas carried by the conceptual approach is taken to be Riemann's approach to function theory, by contrast to Weierstrass's. Weierstrass starts from the definition of an analytic function of a complex variable, centered on a certain z_0 as a function that can be developed into

$$f(z) = \sum a_i(z - z_0)^i.$$

Riemann, on the other hand, gives a definition of the function which makes no reference whatsoever to an explicit representation – in series or otherwise. To characterize functions, Riemann argues that one only needs to know a certain number of basic properties about the values of a function, their singularities. For example, an elliptic function is fully characterized by its zeros, its poles, and the orders of the zeros and the poles. The functions are determined up to a constant, which is a crucial point of Riemann's approach. Indeed, Riemann is the first to propose to consider the functions not one by one, but in collections, the *class* of functions i.e., the totality of functions $F(\theta, z)$ which can be rationally expressed in terms of θ and z . Riemann then talks about classes of polynomial equations equivalent by rational transformations in two variables to designate what we call “surface”.⁵

The idea of a Riemann surface of a holomorphic function makes possible a kind of investigation that is largely independent of specific expressions and provides a basis for new (topological) concept formations and methods of proof. ([Laugwitz, 2009], 35)

Classes of functions are infinite systems treated as totalities. The consideration of infinite systems is a crucial element of the new methods developed by both Riemann and Dedekind. Dedekind considered systems of infinitely many numbers, such as fields, modules and ideals. In 1876, he explained in a letter to Lipschitz that his theory of ideals, in which infinite sets of numbers are considered as “wholes”, was directly inspired by Gauss's notion of class, in the *Disquisitiones Arithmeticae*:

⁵On Riemann and set theory, see [Ferreirós, 2008].

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Just as we can conceive of a collection of infinitely many functions, which even if they are still dependent on variables, as one whole, in the same way as when we collect all equivalent forms in a form-class, denote this again by a single letter, and submit it to composition, with the same right I allow myself to conceive of a system \mathfrak{a} of infinitely many, completely determined numbers in [the system of integers] \mathfrak{o} , which satisfies two extremely simple conditions I. and II., as one whole, and name it an ideal.⁶ ([Lipschitz and al. 1986], 62-63, transl. slightly altered in [Ferreirós, 2008], 110)

In this letter to Lipschitz, Dedekind refers to both Riemann and Gauss as an inspiration for his introduction of ideals as sets. Let us examine more closely the potential sources for this new approach.

1.1.1 Before Riemann

If Dedekind is also referring to Riemann in the letter to Lipschitz, it seems that the common inspiration of a Göttingen conceptual tradition could be seen as being Gauss himself. Indeed, as soon as in the *Disquisitiones Arithmeticae* in 1801, Gauss expresses the idea central to the “conceptual approach”, when commenting of the following theorem:

The product of all numbers less than a given prime number, when increased by unity is divisible by this prime number. ([Gauss, 1801], §76, 50)

Gauss remarks that neither Waring, who published it, nor Wilson, to whom it was attributed, could find a proof for the theorem:

Waring confessed that the demonstration seemed more difficult because no *notation* can be devised through which one could express a prime number. But in our opinion truths of this kind ought to be drawn out of notions [*notionibus*, translated in German by *Begriffen*] and not out of notations. (*ibid.*, 50)

Gauss’s statement invites to be careful when choosing modes of proofs. He emphasizes that the notation itself is not an adequate basis when investigating mathematical truths. In fact, there is a part of arbitrariness in the

⁶“So gut, wie man einen Inbegriff von unendlich vielen Functionen, die sogar noch von Variablen abhängen, als ein Ganzes auffasst, wie man z.B. alle äquivalenten Formen zu einer Formen-Classe vereinigt, diese wieder mit einem einfachen Buchstaben bezeichnet und einer Composition unterwirft, mit demselben Rechte daß ich ein System \mathfrak{a} von unendlich vielen, aber vollständig bestimmten Zahlen in \mathfrak{o} , welches zwei höchst einfachen Bedingungen I. und II. genügt, als ein Ganzes auffassen und ein Ideal nennen.”

choice of a notation which shouldn't be a prelude to any mathematical reasoning – a point which will be often emphasized later by Dedekind. From the *impetus* of Gauss's works, then, is thus developed the so-called new “conceptual approach”, whose core characteristic is the desire to base proofs or even theories on notions rather than *Darstellungsformen* or computations.

Dedekind himself, almost a century after the publication of Gauss's *Disquisitiones*, commented on the passage just quoted by saying:

In these last words lies, if they are taken in the most general sense, the statement of a great scientific thought: the decision for the internal in contrast to the external. This contrast also recurs in mathematics in almost all areas; it suffices to think of function theory, and Riemann's definition of functions through internal characteristic qualities, from which the external forms of representation flow with necessity.⁷ ([Dedekind, 1895b], 54–55, partially translated in [Tappenden, 2005b], 8-9)

Gray underlines that the idea expressed here by Dedekind, tightly linked to Gauss's and Riemann's approaches, does not suggest that

objects cannot be studied via their representations, but (...) that one must be vigilant to ensure that one establishes properties of the objects themselves and not the properties of merely this or that representation, and to this end it was best to avoid explicit representations whenever possible. ([Gray, 2008a], 670)

If one takes these ideas to be the core of the “conceptual approach”, then in fact mathematicians outside of the small group of 1850s Göttingen mathematicians had expressed similar views. Notably, in 1843, Gotthold Eisenstein, a student of Dirichlet in Berlin, attests to the fact that in the actors's view, these precepts were embraced by a wider circle:

The essential principle of the latest mathematic school, established by Gauss, Jacobi and Dirichlet, is in contrast to the past, such that it would enclose a problem as a whole by a brilliant method – with one leading idea – and would show the final result by a single strike in an elegant way, while previous methods tried to succeed by tedious, complicated calculations and deductions (as still used in Gauss's *Disquisitiones*). ([Eisenstein, 1843], 8)

⁷ “In diesen letzten Worten liegt, wenn sie im allgemeinsten Sinne genommen werden, der Ausspruch eines großen wissenschaftlichen Gedankens, die Entscheidung für das Innerliche im Gegensatz zu dem Äußerlichen. Dieser Gegensatz wiederholt sich auch in der Mathematik auffast allen Gebieten; man denke nur an die Funktionentheorie, an Riemanns Definition der Funktionen durch innerliche charakteristische Eigenschaften, aus welchen die äußerlichen Darstellungsformen mit Notwendigkeit entspringen.”

1.1 Remarks on the “conceptual approach”

Eisenstein’s vision of what he perceived as a “new school” explicitly mentions avoiding “tedious, complicated calculations.” However, this statement should be taken with a grain of salt, for it is more directed towards the simplicity and elegance of mathematical reasonings. Moreover, for the three mathematicians mentioned here are all great calculators, and this by their own standards (and so was Eisenstein!). In 1852, in Jacobi’s obituary, Dirichlet himself points out this new tendency, and carefully underlines that not only are computations as good as the new approach, but Jacobi distinguished himself in both:

If it is the more and more prominent trend in the new analysis to put thoughts in place of computations, there are nevertheless certain domains in which computation keeps its rights. Jacobi who had so considerably promoted this trend, also contributed in these branches by his technical mastery.⁸ ([Lejeune-Dirichlet, 1852], 245)

Jacobi’s mathematics would hardly be considered “conceptual” by later commentators, and he is never mentioned, for example, by Dedekind. Note that Dirichlet mentions this “conceptual approach” in relation with the ongoing efficiency of the computational approach. Moreover, he does not explicitly consider himself as a part of the new approach, nor does he particularly praise this new way of working. He clearly sees advantages in both approaches.

According to [Klein, 1979], Dirichlet had a “single goal, which he strove for with his whole being, [which] was a clear insight into the ideal coherence of mathematical thought, a goal which led him to renounce external influence and success” ([Klein, 1979], 90). A great calculator himself, Dirichlet is also considered to be at the origin of some of the distinct traits of the conceptual approach. He introduced non-constructive methods in some of his number-theoretic works, giving proofs of existence without providing means of construction – an approach he was the first to use, long before Dedekind or Hilbert’s *Nullstellensatz*. This was to be a core point of Dedekind’s methods, and a point of dissension with ‘non-conceptual’ mathematicians like Kronecker.

Also often taken as a reference by later commentators for a “conceptual” approach, is the notion of arbitrary function proposed in Dirichlet’s “*Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire*” ([Lejeune-Dirichlet, 1829]) and its German version “*Über die Darstellung ganz willkürlicher Functionen durch Sinus- und Cosinusreihen*” ([Lejeune-Dirichlet, 1837]). Here, Dirichlet’s aim is to pursue Fourier’s works

⁸“Wenn es die immer mehr hervortretende Tendenz der neueren Analysis ist, Gedanken an die Stelle der Rechnung zu setzen, so giebt es doch gewisse Gebiete in denen die Rechnung ihr Recht behält. Jacobi, der jene Tendenz so wesentlich gefördert hat, leistete vermöge seiner Meisterschaft in der Technik auch in diesen Gebieten Bewundernswürdiges.”

on the representation of arbitrary functions in trigonometric series. In the *Théorie analytique de la chaleur*, Fourier refers to a notion of function as not being “subjected to a consistent law” but rather representing the succession of arbitrary ordinates ([Fourier, 1822], 12). Dirichlet considers an “entirely arbitrary function” f , i.e., a function for which no explicit representation is given: no formula and no “rule of formation” by the arithmetic operations. Dirichlet explicitly follows Fourier’s approach. According to Dirichlet, Fourier “introduced the way of expressing [in trigonometric series] the said arbitrary functions” ([Lejeune-Dirichlet, 1829], 117). So, Dirichlet is, here, using Fourier’s own terminology and not departing from his notion of function.⁹ Dirichlet seems to be less inaugurating a new concept of function than giving a more rigorous treatment of the notion of arbitrary function, to which he finally gives a more precise definition in the 1837 paper:

We mean by a and b two constant values and by x a variable quantity which must gradually take all the values between a and b . Now, to every x corresponds a single, finite y , that is in such a way that while x continuously runs through the interval between a and b , $y = f(x)$ likewise gradually changes. Then, y is called a continuous function of x for this interval. It is at the same time not necessary that y in this entire interval be dependent of x with always the same law, so one does not need once to think of a dependency expressible by the mathematical operations.¹⁰ ([Lejeune-Dirichlet, 1837], 135)

For the arbitrary function¹¹ f studied, only the necessary properties are considered: f takes finite values and is continuous in a certain interval $[0; h]$ with $0 \leq h \leq \frac{\pi}{2}$. This allows Dirichlet to give a general proof of the convergence of the trigonometric series he is studying. As underlined by Fourier himself in 1822, the definition of the notion of function allows for the arbitrary functions to be “submitted to the ordinary procedures of analysis” (*op. cit.*, 12). Dirichlet, in fact, essentially proposes, here, a more detailed and rigorous definition for Fourier’s concept of function. Dirichlet’s concept of function will be the basis of the very general concept of function used by Riemann.

⁹For more details, see [Youschkevitch, 1976] and [Chorlay, Forthcoming].

¹⁰“Man denke sich unter a und b zwei feste Werthe und unter x ein veränderliche Grösse, welche nach und nach alle zwischen a und b liegenden Werthe annehmen soll. Entspricht nun jedem x ein einziges, endliches y und zwar so, dass, während x das Intervall von a bis b stetig durchläuft, $y = f(x)$ sich ebenfalls allmählich verändert. So heisst y eine stetige oder continuirliche Function von x für dieses Intervall. Es ist dabei gar nicht nöthig, dass y in diesen ganzen Intervall noch demselben Gesetze von x abhängig sei, ja man braucht nicht einmal an eine durch mathematische Operationen ausdrückbare Abhängigkeit zu denken.”

¹¹The functions studied by Dirichlet are assumed to be continuous.

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Riemann studied with Dirichlet in Berlin and in Göttingen. Like Dedekind, he was profoundly influenced by Dirichlet’s works and ideas, from what he called the “Dirichlet principle” to his concept of function:

If one designates by z a variable magnitude which can take gradually all the possible real values, then when to each value corresponds a single value of the indeterminate magnitude w , one calls w a function of z . And if, while z continuously runs through all the values between two fixed values, w changes continuously as well, then this function is called continuous [*stetig oder continuirlich*] in this interval.¹² ([Riemann, 1851], 3)

1.1.2 Riemann’s and Dedekind’s mathematics

I will present, in this section, the outlines of what is commonly considered as the “conceptual approach”, from its very roots in Dirichlet’s works to its archetypal expression in Riemann’s function theory and Dedekind’s ideal theory. This will allow me, in the next paragraph, to point out the insufficiencies of this reading alone.

Dedekind and Riemann had a close relationship during their time together in Göttingen, which can be seen in the letters Dedekind sent to his family and to Riemann’s widow (see [Scharlau, 1981a], [Dugac, 1976b]). Dedekind expressed his admiration for Riemann’s works at numerous occasions and was profoundly influenced by his approach. Riemann’s impact on Dedekind was essentially methodological and epistemological, and acted as an inspiration for his own works, as Dedekind explained in a letter to Lipschitz:

My efforts in number theory have been directed towards basing the research not on arbitrary representations or expressions but on simple fundamental concepts and thereby – although the comparison may sound a bit arrogant – to achieve in number theory something analogous to what Riemann achieved in function theory.¹³ (Letter to Lipschitz, 6 June 1876, in [Dedekind, 1932] III,

¹²*Denkt man sich unter z eine veränderliche Grosse, welche nach und nach alle möglichen reellen Werthe annehmen kann; so wird, wenn jedem ihrer Werthe ein einziger Werth der unbestimmten Grosse w entspricht, w eine Function von z genannt; und wenn, während z alle zwischen zwei festen Werthen gelegenen Werthe stetig durchläuft, w ebenfalls stetig sich ändert, so heisst diese Function innerhalb dieses Intervalls stetig oder continuirlich.*

¹³*“Mein Streben in der Zahlentheorie geht dahin, die Forschung nicht auf zufällige Darstellungsformen oder Ausdrücke sondern auf einfache Grundbegriffe zu stützen und hierdurch – wenn diese Vergleichung auch vielleicht anmaßend klingen mag – auf diesem Gebiete etwas Ähnliches zu erreichen, wie Riemann auf dem Gebiete der Functionentheorie.”*

468-468, transl. slightly altered in [Edwards, 1983], 11.)

He also explicitly drew this link in the 1877 version of algebraic number theory:

[S]uch a theory, based on calculation, would not yet offer, in my opinion, the highest degree of perfection. It is preferable, as in the modern theory of functions, to try to infer the demonstrations immediately from the fundamental characteristic concepts rather than from calculations, and to construct the theory in a way that it is, on the contrary, able to predict the results of the calculation.¹⁴ ([Dedekind, 1876-1877], 102)

Riemann's influence on Dedekind was, thus, precisely what has been described as the adoption of the conceptual approach. Concretely, for both Riemann and Dedekind – despite the differences of their fields of study – this can be summed up by the set up of a “higher level”, the recourse to a collections of functions or numbers to avoid, or hide away, the computations. According to Laugwitz, then:

The view of mathematics as the study of sets with structures, a view canonized by Bourbaki in the middle of the 20th century, can be regarded as a consistent continuation of the Dedekind-Emmy Noether-van der Waerden line in algebra and the Riemann-Hausdorff-Frechet line in set-theoretic topology. ([Laugwitz, 2009], 35)

Nowadays, the principles of the conceptual approach, exposed above, appear as “red-letters clichés”, as Tappenden suggests ([Tappenden, 2008a], 277). However, in the 1870s, Dedekind was not very successful in getting mathematicians to follow Riemann's lead (especially not in number theory). Only at the end of the century were these principles a significantly important part of the way in which function theory was taken up to be explicitly emphasized. For example, they are presented in Stahl's textbook on elliptic functions:

... the peculiarities of Riemann's treatment lie first in the abundant use of geometrical presentations, which bring out in a flexible way the essential properties of the elliptic functions and at the same time immediately throw light on the fundamental values and the true relations of the functions and integrals which

¹⁴“Une telle théorie, fondée sur le calcul, n'offrirait pas encore, ce me semble, le plus haut degré de perfection ; il est préférable, comme dans la théorie moderne des fonctions, de chercher à tirer les démonstrations, non plus du calcul, mais immédiatement des concepts fondamentaux caractéristiques et d'édifier la théorie de manière qu'elle soit, au contraire, en état de prédire les résultats du calcul.”

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are also particularly important for applications. Second, in the synthetic treatment of analytic problems which builds up the expression for the functions and integrals solely on the basis of their characteristic properties and nearly without computing from the given element and thereby guarantees a multifaceted view into the nature of the problem and the variety of its solutions. ([Stahl, 1899], p. iii. Quoted and translated in [Tappenden, 2008b], 276-277)

1.1.2.1 Riemann’s conceptual mathematics

Riemann, in his 1851 doctoral dissertation on the foundation of the theory of complex functions, proposed a completely new approach to function theory. In a transparent allusion to Weierstrass’s works, he explained how his work brought to the fore the importance of not defining functions by explicit expressions:

The previous methods of treating such functions always set down as a definition an expression for the function, whereby its value was given for every value of its argument; our investigation shows that, as a result of the general character of a function of a variable complex quantity, in such a definition some of the data are a consequence of the remaining ones, namely, the proportion of data has been reduced to those indispensable for the determination.¹⁵ ([Riemann, 1851], translated in [Laugwitz, 2009], 101).

Riemann emphasized the importance of defining the functions based on intrinsic properties (e.g., nature and location of the singularities). The idea defended by Riemann in 1851 was, thus, that

[a] theory of these functions [algebraic, circular or exponential, elliptical and Abelian] on the basis provided here would determine the presentation of a function (i.e., its value for every value of its argument) independently of its mode of determination by operations on quantities [i.e., analytical expressions], because one would add to the general concept of a function of a variable complex quantity just the attributes necessary for the determination of the function, and only then would one go over to the different

¹⁵ “Die bisherigen Methoden, diese Funktion zu behandeln, legten stets als Definition einen Ausdruck der Function zu Grunde, wodurch ihr Werth für jeden Werth ihres Arguments gegeben wurde; durch unsere Untersuchung ist gezeigt, dass, in Folge des allgemeinen Charakters einer Function einer veränderlichen complexen Grösse, in einer Definition dieser Art ein Theil der Bestimmungsstücke eine Folge der übrigen ist, und zwar ist der Umfang der Bestimmungsstücke auf die zur Bestimmung nothwendigen zurückgeführt worden.”

expressions the function is fit for.¹⁶ (*ibid.*, translated in [Laugwitz, 2009], 101-102)

Riemann was then arguing for the conjunction of analytical (singularities and conditions on the imaginary and real parts of the functions) and geometrical (the surface) considerations, so as to define the functions with only the necessary and sufficient informations, avoiding the redundancies found in purely analytical treatments. The kind of functions Riemann is interested in, such as Abelian functions, imply to consider algebraic functions, which are often multi-valued functions (e.g., the square-root function). In order to avoid a purely analytic treatment of such functions, Riemann proposes to consider surfaces multiply covering (parts of) the complex plane – what we now call Riemann surfaces. The importance of a geometric representation (the surface) was a core point of his approach. Scholz ([Scholz, 1992]) showed that Riemann’s celebrated notion of manifold (*Mannigfaltigkeit*) was elaborated between 1851 and 1854, when he presented these ideas in his *Habilitationsvortrag*. The idea of manifolds grew from a desire to find an acceptable conceptualization for Riemann surfaces, introduced in 1851.

According to Ferreirós, Riemann’s ideas grew more abstract with Riemann’s attempts to find a satisfying foundation to his approach. Riemann first related his approach to “*Analysis Situs*”. He went from continuous manifolds defined “in reference to a ‘variable object’ that admits of different ‘forms of determination,’ i.e., that can be in different states” ([Ferreirós, 2008], 58) to the acceptance of an abstract approach to geometry. Then, rather than declaring that function theory, and more generally the theory of magnitudes, depend on geometry, Riemann introduced the concept of manifold, independent of intuition. For Ferreirós, this move “made possible an abstract derivation of geometry” and was in fact “actually indispensable for an abstract theory of magnitudes.” The definition of a manifold, given in 1854 is the following:

Notions of quantity are possible only where there already exists a general concept which allows different realizations. Depending on whether or not a continuous transition of instances can be found between any two of them, these realizations form either a continuous or a discrete manifold; individual instances in the first case are called points and in the latter case elements of the manifold.¹⁷ ([Riemann, 1854], 273, transl. in [McCleary, 1994], 314)

¹⁶ “*Eine Theorie dieser Functionen auf den hier gelieferten Grundlagen würde die Gestaltung der Function (d.h. ihren Werth für jeden Werth ihres Arguments) unabhängig von einer Bestimmungsweise derselben durch Grössenoperationen festlegen, indem zu den allgemeinen Begriffe einer Function einer veränderlichen complexen Grösse nur die zur Bestimmung der Function nothwendigen Merkmale hinzugefügt würden, und dann erst zu den verschiedenen Ausdrücken deren die Function fähig ist übergehen.*”

¹⁷ “*Grössenbegriffe sind nur da möglich, wo sich ein allgemeiner Begriff vorfindet, der*

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Riemann investigates the concept of manifold in a very general way, linking it to numbers through the concept of discrete manifolds:

Distinct portions of a manifold, distinguished by a mark or by a boundary, are called quanta. Their quantitative comparison is effected in the case of discrete quantities by counting, in the case of continuous quantities by measurement.¹⁸ (*ibid.*, 274, transl. in [McCleary, 1994], 314)

What interested Riemann the most, however, was continuous manifolds which form the ground for geometrical and (what we would call) topological inquiries:

In a concept whose instances form a continuous manifold, if one passes from one instance to another in a well-determined way, the instances through which one has passed form a simply extended manifold, whose essential characteristic (*Kennzeichen*) is that from a point in it a continuous movement is possible in only two directions, forwards and backwards. If one now imagines that this manifold passes to another, completely different one, and once again in a well-determined way, that is, so that every point passes to a well-determined point of the other, then the instances form similarly a doubly extended manifold.¹⁹ (*ibid.*, 275, transl. in [McCleary, 1994], 315)

Riemann’s *Habilitationsvortrag* is dedicated to the study of continuous manifolds from the second paragraph to the end. He considers mostly questions about n dimensionality and the parametrization of n dimensional manifolds.

The demand that the theory of functions, or the theory of magnitudes in general, be based on characteristic properties (here, singularities and boundary conditions) embodied by the Riemann surface or the concept of manifold is the core of Riemann’s “conceptual approach”, and the main

verschiedene Bestimmungsweisen zulässt. Je nachdem unter diesen Bestimmungsweisen von einer zu einer andern ein stetiger Übergang stattfindet oder nicht, bilden sie eine stetige oder discrete Mannigfaltigkeit; die einzelnen Bestimmungsweisen heissen im erstern Falle Punkte, im letztern Elemente dieser Mannigfaltigkeit.”

¹⁸ “Bestimmte, durch ein Merkmal oder eine Grenze unterschiedene Theile einer Mannigfaltigkeit heissen Quanta. Ihre Vergleichung der Quantität nach geschieht bei den discreten Grössen durch Zählung, bei den stetigen durch Messung.”

¹⁹ “Geht man bei einem Begriffe dessen Bestimmungsweisen eine stetige Mannigfaltigkeit bilden, von einer Bestimmungsweise auf eine bestimmte Art zu einer andern über, so bilden die durchlaufenen Bestimmungsweisen eine einfach ausgedehnte Mannigfaltigkeit, deren wesentliches Kennzeichen ist, dass in ihr von einem Punkte nur nach zwei Seiten, vorwärts oder rückwärts, ein stetiger Fortgang möglich ist. Denkt man sich nun, dass diese Mannigfaltigkeit wieder in eine andere, völlig verschiedene, übergeht, und zwar wieder auf bestimmte Art, d. h. so, dass jeder Punkt in einen bestimmte Punkt der andern übergeht, so bilden sämtliche so erhaltene Bestimmungsweisen eine zweifach ausgedehnte Mannigfaltigkeit.”

idea to which Dedekind is referring when he wishes to walk in Riemann's footprints.

The elaboration of new concepts and methods so as to satisfy this requirement involves the consideration of actual infinities, such as the continuous manifold as an infinite collection of points. According to Ferreirós, Riemann did accept the actual infinite, a point suggested by his definition of manifold. He considered that infinity was a “well determined” notion, albeit not directly definable.²⁰ The infinitary aspect of Dedekind's works seems to have (at least some) roots in Riemann's works. Indeed, Dedekind takes (like Cantor) “*Mannigfaltigkeit*” to be a synonym for his “*System*”. It is, in particular, the case for Dedekind's concept of ideal,²¹ which plays an essential role in his number theory. Dedekind's ideal theory can be seen as a typical example of the “conceptual approach”, however I will suggest that it demands to be nuanced.

1.1.2.2 Dedekind's ideal theory

Dedekind's approach is marked by the joined influences of Riemann who left a deep epistemological imprint; and Dirichlet who was decisive for his interest for number theory and the importance attached to rigor. Dedekind's most important work in number theory, the theory of algebraic integers, in which is introduced the concept of ideal, is a great example of his methods. It is also a good example of the limits of an analysis of his innovations as essentially related to a “conceptual” approach. I will come back to this point in greater detail below. First, let us outline elements of the definition of ideals to support this claim.

First published in 1871 as a part of the Xth Supplement to Dirichlet's *Vorlesungen über Zahlentheorie*, which Dedekind edited, the theory of algebraic integers is famous for being the birthplace of field and ideal theories. In this work, among other things, Dedekind successfully extends to any algebraic integer Kummer's theory of ideal numbers. Ideal numbers are introduced to “save” the unicity of the decomposition in primes which fails for the decomposition of certain complex prime numbers (see Sec. 5.1.2). Ideals are defined by the conditions of divisibility they satisfy and only by these conditions of divisibility – which implies that they are not defined as existing objects. For example, consider the system of numbers of the form $a + b\sqrt{-5}$, with a and b integers. In this system, one can write the number

²⁰It is well-known that Riemann's notion of *Mannigfaltigkeit* inspired Cantor's set theory.

²¹Note however that Dedekind's methodological references for the introduction of ideals is not solely Riemann, it is also Gauss (see p. 44) which seems to be related to the fact that being infinite is not the only remarkable property of ideals.

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9 in two essentially different prime decompositions:

$$9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

The ideal factors, here, are β_1 and β_2 such that $\beta_1^2 = 2 + \sqrt{-5}$, $\beta_2^2 = 2 - \sqrt{-5}$, and $\beta_1\beta_2 = 3$. Then, one has the prime decomposition $9 = \beta_1^2\beta_2^2$, which is unique. Kummer’s approach, however, can’t be generalized as such to any algebraic numbers, and both Dedekind and Kronecker engaged in attempts to provide a valid generalization.

Dedekind introduces the concept of ideal specifically for this generalization. Ideals initially consist in the collection of all numbers divisible by a certain (existing or ideal) factor p .²² This way, Dedekind avoids the introduction of new (ideal) entities, and he can also avoid the divisibility tests used by Kummer, which are the kind of long, tedious computations that should not (according to him) be taken as the basis of concepts. This new concept, the ideal, is undeniably an archetypical example of the so-called conceptual approach: in an attempt to avoid relying on *Darstellungsformen* and get rid of computations, one introduces a new concept which is expected to focus on “characteristic properties” – here, a divisibility condition. Dedekind demands that the definitions allow a better understanding of the nature of the concept investigated. *Darstellungsformen* and computations tend to hide some important properties, and there is a certain arbitrariness involved in the choice of a certain notation, equation or indeterminate variable, that he wishes to avoid.

Dedekind’s ideals are infinite sets. Their introduction was motivated by his desire to be able to give a definition “exact and covering all the ideal numbers that must be introduced in a particular numerical domain σ ” ([Dedekind, 1876-1877], transl. slightly altered, 57). This requirement leads to the following reasoning. What Kummer defined, in his works on cyclotomic numbers, is the divisibility of ideal numbers which amounts to the following:

If a number α has a certain property A , consisting always in that α satisfies one or more congruences, he says that α is divisible by a determinate ideal number corresponding to the property A .²³
(*ibid.*, transl. slightly altered, 57)

Dedekind considers that, in principle, Kummer’s introduction of ideal numbers is perfectly rigorous, but not only does it rely too heavily on a computational approach, it “may [also] lead to hasty conclusions and incomplete proofs. And in fact this danger is not always completely avoided” (*ibid.*, 57).

²²See p. 338.

²³“Si un nombre α possède une certaine propriété A , consistant toujours en ce que α satisfait à une ou plusieurs congruences, il dit que α est divisible par un nombre idéal déterminé, correspondant à la propriété A .”

Dedekind detects two additional issues with the approach adopted by Kummer. First, it does not provide a general definition of ideal numbers. Kummer's approach, while efficient for cyclotomic integers, can't be generalized for all algebraic integers and Dedekind is interested in definitions and methods valid for entire classes of objects. Secondly, Kummer does not give a definition of the multiplication of ideal numbers. This is particularly problematic for Dedekind for the definition of arithmetical elements.

Dedekind's ideal theory has a long genesis. Dedekind explains that he first considered the possibility to extend Kummer's approach using higher congruences. This did not yield results of a generality meeting Dedekind's expectations :

In my researches, the goal of which has been to arrive at a definitive answer to [the] question [of the divisibility laws for algebraic integers], I began by building on the theory of higher order congruences, since I had already previously noticed that the application of the latter theory could shorten Kummer's researches considerably. However, while this method led me very close to the goal my efforts aimed at achieving, I have not been able to surmount, by this route, certain apparent exceptions to the laws holding in other cases. I achieved the general theory, without exceptions, that I first published in the place mentioned above, only when I entirely abandoned the old, more formal, approach...²⁴
(*ibid.*, 57)

In fact, Dedekind explains in a 1878 paper, that he detected several shortcomings in the approach using higher congruences, which are essentially related to the fact that this method requires to base

[the] investigation of a domain of algebraic integers initially (...) on the consideration of a determinate number and the equation corresponding to it, which is treated as a congruence; and that the definition of ideal numbers (or rather, of divisibility by ideal numbers) so obtained, as a result of this determinate chosen form of representation, does not allow one to distinguish, from the outset, the character of *invariance* which is attached to these

²⁴ "Dans mes recherches, qui avaient pour but d'amener la question [des lois de divisibilité de entiers algébriques] à une solution définitive, j'ai commencé par m'appuyer sur la théorie des congruences d'ordre supérieur, parce que j'avais déjà précédemment remarqué que par l'application de cette théorie les recherches de Kummer pouvaient être considérablement abrégées, mais, bien que ce moyen conduisit jusqu'à un point très-voisin du but de mes efforts, je n'ai pu toutefois réussir par cette voie à soumettre certaines exceptions apparentes aux lois constatées pour les autres cas. Je ne suis parvenu à la théorie générale et sans exceptions, que j'ai publiée pour la première fois au lieu indiqué plus haut, qu'après avoir entièrement abandonné l'ancienne marche plus formelle..."

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concepts.²⁵ ([Dedekind, 1878], in [Dedekind, 1932] I, 202, my translation, original emphasis)

An additional reason for Dedekind to discard the first approach was that, in his view, it was hiding the fact that it is possible to define concepts that do not depend on the equation initially chosen. With the concept of ideals adopted later, however, it is possible to define a notion of ideal (and the associated notion of divisibility) which holds for *any* field of algebraic numbers. And it is possible to do so with no reference whatsoever to a specific number or equation. The concepts are consequently independent of any reference to an individual representation of the numbers or the (ideal) factors – they can be defined *a priori* without any reference to these individual properties of (ideal or existing) numbers. For Avigad, avoiding references to the “way [the objects] are represented” in the definition of ideals allows to understand

why calculations with and properties of the objects [i.e., ideals] do not depend on these choices of representations. This is why Dedekind, in the passage above, complains that his first attempt at a theory of ideal divisors did not allow one to recognize the invariance the concepts in fact have from the outset. ([Avigad, 2006], 173).

Long complicated computations are avoided, hidden by the use of ideals: computations were linked to the way ideal numbers were defined and studied and ideals are defined without reference to ideal numbers. Indeed, Dedekind notices that the systems “of numbers divisible by a factor p ” satisfy properties of closure: if \mathfrak{a} is an ideal of a field Ω of algebraic numbers and \mathfrak{o} the system of integers of the field, then

- I. The sum and difference of two numbers of \mathfrak{a} are again numbers of \mathfrak{a} .
- II. The product of any number in \mathfrak{a} by any number in \mathfrak{o} is again a number of \mathfrak{a} .

It is possible to see that these conditions are also sufficient conditions, that is, any system satisfying these conditions is a “system of numbers divisible by a factor p ”. This incites Dedekind to take these properties as a definition for the concept:

A fact of the highest importance, the truth of which I was able to prove rigorously only after numerous vain attempts, and after surmounting the greatest difficulties, is that, conversely, each

²⁵“(. . .) die Untersuchung eines Gebietes von ganzen algebraischen Zahlen sich zunächst auf die Betrachtung einer bestimmten Zahl und der ihr entsprechenden Gleichung gründet, welche als Kongruenz aufgefasst wird, und dass die so erhaltenen Definitionen der idealen Zahlen (oder vielmehr der Teilbarkeit durch die idealen Zahlen) zufolge dieser bestimmt gewählten Darstellungsform nicht von vornherein den Charakter der Invarianz erkennen lassen, welcher in Wahrheit diesen Begriffen zukommt.”

system enjoying properties I and II is also an ideal. That is, \mathfrak{a} is the set of all numbers α of the domain \mathfrak{o} divisible by a determinate existing number or by an ideal number indispensable for the completion of the theory. (...) This observation naturally led me to base the whole theory of numbers in the domain \mathfrak{o} on this simple definition, entirely freed from any obscurity and from the admission of ideal numbers.²⁶ ([Dedekind, 1876-1877], transl. modified, 59-60)

Dedekind's and Riemann's approaches are coming from the same efforts to meet new epistemological requisites linked to a desire to distance the mathematical practice from heavy computational apparatuses. But despite their being "conceptual thinkers", Dedekind and Riemann worked in very different ways. According to Dedekind in his biography of Riemann ([Riemann, 1876]), Riemann had "[a] brilliant power of thought and [an] anticipatory imagination [which] led him frequently to take very great steps that others could not follow so easily". And indeed, as P. Ullrich explains

[t]he account [of function theory by Riemann in 1851] is awe-inspiring but cryptic in the extreme; the topology was left entirely intuitive, and the scope of the approach uncertain, especially relative to the properties of functions that were then known. Exegeses of the theory would occupy mathematicians for generations, with an important stimulus coming in the 1890s from Henri Poincaré and in 1913 from Hermann Weyl (1885–1955). (p. Ullrich, in [Grattan-Guinness, 2005], 453)

Dedekind, by contrast, confessed, in a letter to Cantor, from August 29 1899, his "*Treppen-Verstand*" (stepped-understanding, letter reproduced in [Dugac, 1976b], 261), emphasizing his need to unfold carefully and explicitly each step of the theory.²⁷ This apparently superficial difference is an important point to understand some elements of Dedekind's approach. Dedekind strove to make his algebraic number theory as simple and easy to understand as possible, even going so far as rewriting his own works to ease the

²⁶ "[U]n fait de la plus haute importance, et dont je n'ai pu démontrer rigoureusement la vérité qu'à la suite de nombreux et vains efforts et après avoir surmonté de grandes difficultés, c'est que, réciproquement, tout système \mathfrak{a} qui jouit des propriétés I et II, est aussi un idéal, c'est-à-dire que \mathfrak{a} forme l'ensemble de tous les nombres α du domaine \mathfrak{p} qui sont divisibles par un nombre existant déterminé, ou par un nombre idéal, indispensable pour compléter la théorie. (...) Cette constatation m'a conduit naturellement à fonder toute la théorie des nombres du domaine \mathfrak{o} sur cette définition simple, entièrement délivrée de toute obscurité et de l'admission des nombres idéaux."

²⁷The analytical researches on Riemann's *Inauguraldissertation* which were used by Weber for [Weber and Riemann, 1900-1901] and later published by M.A. Sinaceur, testify of Dedekind's careful unfolding of the mathematical content.

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reader’s way in them. As noted by Ullrich, the tools to develop Riemann’s theory were not fully or clearly developed in Riemann’s writings. Topological concepts were of course not developed in any precise way and neither were the methods of proof. Dedekind, on the contrary, introduced methods of proof with his theory of ideals – which were to be later extended but appeared self-sufficient for Dedekind’s purpose. Rather than merely a ‘conceptual’ approach, it is the possibility to provide what can be identified as *arithmetical* methods for the theory of algebraic numbers that is decisive for Dedekind’s preference for the concept of ideal over other possible generalizations of Kummer’s ideal factors – a point on which I will come back later in this dissertation.

Despite the attention given by Dedekind to the elaboration of (what he considered to be) simple definitions and methods of proof, the reception of his works stayed cold and almost inexistent until the 1880s (and by then, not all receptions of his works were enthusiastic ones.²⁸). The reception of Riemann’s works was as well difficult at first, but the initial difficulties encountered by the readers with Riemann’s mathematically imprecise dissertation started to be dissolved by his later works, whose depth and originality were quickly recognized. As Bottazzini and Gray explain, Riemann’s 1851 dissertation did not help the “diffusion of Riemann’s message” especially insofar as

for any who took the trouble to consult the paper, its many novelties worked against its immediate acceptance. So too did its vagueness at certain crucial points and its often murky language. The novelties included the emphasis on such topological concepts as cuts and connectivity and on the central role of harmonic functions, but most of all on the way a complex function was to be specified and hence thought about. ([Bottazzini and Gray, 2013], 277)

An actual reception of Riemann’s ideas happened only by the end of the 1850s, with the lectures given in Göttingen from 1855-56 to 1861-62, which adopted a more conventional approach. More important is his 1857 paper on Abelian functions:

The depth of originality in Riemann’s remarkable paper [on Abelian functions] can be measured by the fact that on reading it no less an authority than Weierstrass withdrew a paper of his own on the same subject, preferring to wait until he had assimilated what Riemann had to say. (*ibid.*, 286)

²⁸See notably 5.7.

The interpretation of Dedekind as a conceptual thinker, while pertinent, focuses the analysis on Dedekind's first steps in the set up of new concepts and methods meeting the requirement that computations and *Darstellungsformen* be avoided or be the result of the theory. It focuses the exegete's attention on the elaboration of new concepts. However, the interpretations following this approach have not examined closely how these concepts can be associated to new methods of proof, how they are used in practice and articulated with other parts of the theory.

Modern readings of Dedekind's and Riemann's works are deeply marked by the idea of them as conceptual thinkers. While I will not dwell upon Riemann's case, I will suggest that to analyze Dedekind's works only in these terms might alter our understanding of some important aspects of the developments of his works. To refer to the approach as "conceptual" is only the beginning of the analysis. One should go further and describe the actual practice to which the adjective corresponds. I will argue that, in Dedekind's case, a close analysis of his mathematical texts, to understand how the new concepts are effectively used in the mathematical practice, brings to the fore additional elements and shows how his conceptual approach is in fact impossible to dissociate from his conception of arithmetic.²⁹

1.1.3 Diffuse ideas in space and time?

The undeniable vagueness in the description of the mathematical practices attached to the "conceptual tradition" is echoed by the fact that this approach is also not very well delimited, neither in space nor in time. The opposition often made between the Berlin computational approach and the Göttingen conceptual one is clearly irrelevant in the case of Eisenstein or Dirichlet. Moreover, the web of influences between mathematicians is very complex and leads to close relationships between "conceptual" and "computational" mathematicians.

We saw that it is often proposed, both by the historians and by the actors, that Gauss is, in some ways, the pioneer of the conceptual approach. The timespan between his *Disquisitiones Arithmeticae*, published in 1801, and Dirichlet's or Eisenstein's statements about a new "conceptual" approach is rather long, especially insofar as Gauss's *Disquisitiones* met immediately a considerable success in the mathematical community (see [Goldstein and Schappacher, 2007a] and in this dissertation 1.2.1). Statements about the growing importance of avoiding the resort to computations can be found in several other mathematicians's works. This can be detected in

²⁹In addition, by pushing the investigation further and considering these elements as embodied in a more global perspective of his works, one can be led to question the role, status and understanding of arithmetic and numbers for Dedekind, which I will do in the last third of this dissertation.

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Abel’s works on algebraic equations, in which he explains that (computational) attempts to find formulae solving equations are not the appropriate approach anymore, rather one should ask whether it is possible to solve the equations:

[I]f unfortunately the resolution was impossible, one could look for it for an eternity without finding it. To infallibly obtain something on that matter, one has to take another road. One has to give to the problem a form such that it would always be possible to solve it, which can always be done for any problem. Instead of asking for a relation for which we don’t know if it exists or not, it must be asked if such a relation is indeed possible.³⁰ ([Abel, 1839b], 217)

This idea can also be detected in Abel’s works on the so-called Abelian functions, as Jacobi perceived it himself:

To be sure, mathematics has the property that one can come to discovery through calculation, for if you make a mistake at the start, you recognise it so to speak by calculating a falsehood. Since one calculates with letters, that show how they have been obtained, the result itself shows the shortest way that one has to take. This path, through calculation to discovery, is completely impossible for the Abelian transcendental functions, for if one departs even so little from the true path one can find no result at all because of the vast complexity of the calculation. It seems therefore that the whole of mathematics must be raised to a higher level³¹ for the direction of this research. ([Jacobi, 1836-1837], quoted and translated in [Gray, 1992], 237)

“Higher level” is also the phrase used by Dedekind to refer to his introduction of ideals (although there is no evidence that he took it from Jacobi).

The idea that computations will not be sufficient to solve problems of algebraic equation theory can also be found in Galois’s works in 1831 (who, at that time, hadn’t read Abel’s works):

I believe that the simplifications produced by the elegance of calculations (intellectual simplifications, of course; there are no

³⁰ “[S]i par malheur la résolution était impossible, on aurait pu la chercher une éternité, sans la trouver. Pour parvenir infailiblement à quelque chose dans cette matière, il faut donc prendre une autre route. On doit donner au problème une forme telle qu’il soit toujours possible de le résoudre, ce qu’on peut toujours faire d’un problème quelconque. Au lieu de demander une relation dont on ne sait pas si elle existe ou non, il faut demander si une telle relation est en effet possible.”

³¹For Gray, Riemann is a perfect example of working at a “higher level”.

material ones) have their limits; I believe that the time will come when the algebraic transformations foreseen by the speculations of the Analysts will find neither the time nor the place for their realisation; so much so that one will have to be content with having foreseen them.³² ([Galois, 1831], transl. slightly altered in [Galois, 2011], 253)

The fundamental idea of the “conceptual approach” is strongly reminiscent of Galois’s words, in 1831:

Jump with both feet over calculations, put operations into groups, classify them according to their difficulty and not according to their form; that is, according to me, the mission of future geometers, that is the path that I have entered in this work.³³ (*ibid.*, transl. slightly altered, 253)

Such statements can also be found in the testament letter to Chevalier written in 1832. When Galois submitted his memoir to the Académie des Sciences, in 1831, this standpoint put him at odds with his contemporaries, and his memoir was rejected.³⁴ It was published by Liouville in 1846 and only started to attract attention two decades later.³⁵

Galois also suggests that the long and tedious computations that mathematicians now have to jump over are a recent phenomenon. In fact, Galois states, it is Euler’s inheritance. The increasing difficulty of the computations after Euler’s works awoke in the mathematicians a desire to design new methods to shorten these computations and make it so the mind can grasp a larger set of operations. But for Galois, the elegance that was supposed to help mathematicians out of too long computations is not sufficient. It has become necessary to introduce new methods avoiding computations.

Long algebraic calculations were at first hardly necessary for progress in Mathematics; the very simple theorems hardly gained from being translated into the language of analysis. It is only since Euler that this briefer language has become indispensable to the new extensions which this great geometer has given to science. Since Euler calculations have become more and more necessary but more and more difficult, at least insofar as they

³² “Or je crois que les simplifications produites par l’élégance des calculs, (simplifications intellectuelles, s’entend; de matérielles il n’y en a pas) ont leurs limites; je crois que le moment arrive[ra] où les transformations algébriques prévues par les spéculations des Analystes ne trouveront plus ni le tems ni la place de se produire; à tel point qu’il faudra se contenter de les avoir prévues.”

³³ “Sauter à pieds joints sur les calculs, grouper les opérations, les classer suivant leurs difficultés et non suivant leurs formes ; telle est, suivant moi, la mission des géomètres futurs; telle est la voie où je suis entré dans cet ouvrage.”

³⁴ See [Ehrhardt, 2011b] and [Ehrhardt, 2012].

³⁵ see Sec. 4.1.1

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are applied to more advanced objects of science. Since the beginning of this century algorithmics had attained such a degree of complication that any progress had become impossible by these means, unless one used the elegance geometers have learnt to give to their computations, and by means of which the mind promptly and with a single glance grasps a large number of operations. It is clear that such praised elegance, and so properly claimed, has no other goal.³⁶ (transl. modified in [Galois, 2011], 251-253)

Abel and Galois both read Gauss’s *Disquisitiones*, but they were also continuing works on the resolution of algebraic equations developed by Lagrange. Lagrange himself was trying to find *a priori* reasons for which a method of resolution could effectively work or fail. Indeed, Lagrange considered that by 1770, when he wrote his *Mémoire sur la résolution algébrique des équations*, the theory of equations was “not more advanced than in Cardano’s times”.³⁷ So as to allow effective developments, Lagrange proposed to investigate the “general principles” of the known algebraic methods and to highlight “*a priori* why these methods succeed for the third and fourth degrees and fail for higher degrees.” For this, he introduced the use of permutations and substitutions, by which he

[shows] what the necessary conditions are in order that the equation in question can admit the resolution, by supposing only the resolution of equations of a lower degree than the proposed equation, and [he gives] on that occasion the true principles and, so to speak, the metaphysics of the resolution of the equations of third and fourth degrees.³⁸ ([Lagrange, 1770-1771], 357)

The kind of questions and methodological problems found in the new /

³⁶ “*Les longs calculs algébriques ont d’abord été peu nécessaires au progrès des Mathématiques, les théorèmes fort simples gagnaient à peine à être traduits dans la langue de l’analyse. Ce n’est guère que depuis Euler que cette langue plus brève est devenue indispensable à la nouvelle extension que ce grand géomètre a donnée à la science. Depuis Euler les calculs sont devenus de plus en plus nécessaires, mais de plus en plus difficiles à mesure qu’ils s’appliquaient à des objets de science plus avancés. Dès le commencement de ce siècle, l’algorithme avait atteint un degré de complication tel que tout progrès était devenu impossible par ce moyen, sans l’élégance que les géomètres modernes ont su imprimer à leurs recherches, et au moyen de laquelle l’esprit saisit promptement et d’un seul coup un grand nombre d’opérations. Il est évident que l’élégance si vantée et à si juste titre, n’a pas d’autre but.*”

³⁷ For more on Lagrange and Galois, see Sec. 4.1.1.

³⁸ “*Je ferai voir ensuite quelles sont les conditions nécessaires pour que l’équation dont il s’agit puisse admettre la résolution en supposant uniquement celle des équations des degrés inférieurs à celui de l’équation proposée; et je donnerai à cette occasion les vrais principes et, pour ainsi dire, la métaphysique de la résolution des équations du troisième et du quatrième degré.*”

conceptual approach can thus be found in works written a century before Dedekind's ideal theory. These works appear to turn towards different ways of handling problems because, in fact, computations were becoming too complicated to be fruitful.

Thus, we see that actors acknowledged – and sometimes promoted – modifications in the practices. Actors point out changes that seem to appear at two levels. First, the questions asked change, albeit gradually, in particular in algebra in which the central question of the theory of equations slowly drifts from resolution to resolubility.³⁹ This point becomes particularly central with Abel's proof of the impossibility to solve the general quintic by radicals, and with Galois's memoir and its late publication by Liouville. As emphasized above, Lagrange was already interested in *a priori* understanding the success of certain methods of resolution. In fact, in Lagrange's works, it is the desire to provide a general (algebraic) method for solving equations that prompts him to consider this kind of questions. This is the level in the change actors acknowledge: practices change in the way a problem is approached. Mathematicians express the desire to treat, in one move, an entire class of problems, for example to be able to solve the cyclotomic equation of degree p for any prime number p . The demand to “put ideas in place of computations” appears as a side effect of the conjunction of these two ideas – a side-effect whose importance later increases drastically.

It seems likely that it is due to change in mathematical practice that in the 1840s, Galois's works became acceptable for some mathematicians, and were in particular praised by Liouville who published Galois's memoir in 1846 (see [Ehrhardt, 2011b] and 4.1.1 in this dissertation). Liouville, who was one of the leading mathematicians in France at the time, was a close friend and collaborator of Dirichlet's.⁴⁰ At that time, Dirichlet was in Berlin, where he played a great role in the mathematical formation of many mathematicians such as Lipschitz, Kronecker, or Eisenstein.

Eisenstein, as we saw, wrote the first significant account of this new “conceptual approach”. He was close friend with Riemann during the latter's stay in Berlin. Nevertheless, Eisenstein was a “master of formal manipulation” ([Bottazzini and Gray, 2013], 279). In fact, according to Dedekind's biography of Riemann, Eisenstein and Riemann had strongly opposed ideas on what should be taken as grounding principles of function theory:

³⁹Note that this kind of change is not restricted to algebra. For example, Bolyai's development of non-Euclidean geometry is motivated by his failed attempts to prove Euclid parallel postulate, which led to him to change his questions about geometry and the fifth postulate, and try to create a new geometry.

⁴⁰Liouville to Dirichlet, 5 February 1840: “*Il n'y a pas de géomètre que j'ai estimé et aimé autant que vous, avant même de vous rencontrer, et depuis que je vous connais, je n'ai assurément pas changer d'avis.*”

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Riemann later explained that he [and Eisenstein] had discussed the introduction of complex quantities into the theory of functions but had been of markedly different opinions about the principles that should be taken as basic. Eisenstein stood by the formal calculus, while [Riemann] himself had seen in the partial differential equations the essential definition of a function of a complex variable. (Dedekind, in [Riemann, 1876], 576, quoted in [Bottazzini and Gray, 2013], 279).

The key element of what Eisenstein calls “the new school’s” methods is the manner in which mathematical problems (and theories) are approached and solved. The “old methods”, explains Eisenstein, well exemplified by Euler’s works, would proceed step by step, laboriously going from one intermediary theorem to another so as to reach a certain goal, whose sight one may very well have lost during the intermediary steps. This method, according to Eisenstein, does not allow to “see a theory’s true character, its actual inner mechanism”. The new method adopts an entirely different viewpoint from the first contact with the problem, and tries to approach it by presenting a “*formula* which, right from the beginning, contains the full scope of truths of a whole complex” ([Eisenstein, 1843], my emphasis). In fact, it seems that the difference between the old and the new approaches is less a matter of technical and computational length, for Eisenstein, than it is a change in the conception of what should be the scope of a proof. The demand of the new method, then, is to treat problems by embracing a more general perspective so as to consider the problem from a standpoint which allows for a better grasp of “its actual mechanism”. This can be achieved if one does not proceed by “tedious” progressive steps and keeps sight of the result aimed for. Eisenstein gives two examples of this new method:

Jacobi built the whole idea of inversion of elliptic integrals on the single idea that there exists every power of x except the term $\frac{1}{x}$ in the differential of a two way power series [i.e., Laurent series]; Gauss built his magnificent idea of circle division, about which men had, since the time of Euclid despaired, on a peculiar order of integer numbers following the exponents of their congruent powers. ([Eisenstein, 1843], 8)

Note the reference to Jacobi’s use of power series, which doesn’t seem to be in harmony with Riemann’s ideas.

The network of relationships in the context of which a conceptual approach took shape is usually described as going from Gauss to Riemann and Dedekind through Dirichlet, when they all were in Göttingen together. The “new approach” described by Eisenstein summons up some of the same first actors, even though Eisenstein studied and made his short (but productive) career in Berlin. Recall that Berlin, in the historiography describing the development of a “conceptual approach” is the place in which a

non-conceptual, computational tradition for mathematics was favored, as opposed to the Göttingen's conceptual approach.

In addition, Dirichlet himself also had a profound importance for the “Berlin school”. Indeed, Dirichlet taught at Berlin university for almost thirty years and had a strong influence on Kronecker, one of the greatest representatives of the algorithmic approach. The reception of Dirichlet's ideas is very different for Kronecker and for Dedekind. Of course, it is not the case that Dedekind and Kronecker each deliberately ignored parts of Dirichlet's mathematics.

For example, in relation with a strongly “conceptual” element of Dirichlet's practice, namely the adoption of non-constructive proofs, Kronecker merely saw it as a temporary, fixable flaw. It is underlined by Kronecker in his *Vorlesungen über Zahlentheorie*, in which he proudly announces to his students that he was able, in 1885, to provide a constructive proof and present the perfected result to them:

It was reserved to the great number theorist Gustav Lejeune-Dirichlet to conduct a completely rigorous proof of Legendre's claim. However, he once more made use of a treatment completely deviating from the Euclidean one, and was therefore not able to be up to this classical model, and to design his proof in such a way that he could identify an interval in which a new prime number of the arithmetical sequence must always be located. In 1885, I have then managed to give Dirichlet's proof its completion, and in this lecture, I will explain it to you for the first time in its more exact version.⁴¹ ([Kronecker, 1901], I, 11.)

From Kronecker's viewpoint, it seems that the non-constructive aspect of Dirichlet's result was a temporary insufficiency of the proof, rather than a methodological stance. And this comes from a student of Dirichlet's, who calls him a “great number theorist”. Kronecker, despite his strong opposition to non-constructive proofs, claimed to be influenced by Dirichlet. Note that Kronecker was in charge of publishing Dirichlet's mathematical works after his death, while Dedekind published Dirichlet's *Vorlesungen über Zahlentheorie*.

⁴¹ “Es war dem großen Zahlentheoretiker Gustav Lejeune-Dirichlet vorbehalten, einen völlig strengen Beweis jener Legendre'schen Behauptung zu führen. Aber er wiederum bediente sich eines von dem Euklidischen ganz abweichenden Verfahrens und vermochte daher nicht, dessen klassisches Vorbild zu erreichen und seinen Beweis so auszubilden, daß er ein Intervall erkennen läßt, in dem notwendig immer eine neue Primzahl der arithmetischen Reihe liegen muß. Im Jahre 1885 ist es mir dann gelungen, dem Beweise Dirichlets jene Vollendung zu geben, und in diesen Vorlesungen werde ich ihn zum ersten Male in seiner exakteren Fassung auseinandersetzen.”

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Dirichlet as well as Gauss, even if they are considered as founding fathers of the conceptual approach, were not always prone to avoid calculations and both deeply influenced many different mathematicians. In particular, Dedekind and Kronecker walked in Gauss’s and Dirichlet’s footprints. The difference between two receptions of Gauss is sometimes subtle. For example, both Kronecker and Dedekind invoke Gauss for the consideration of classes of numbers or functions: Kronecker (followed Gauss and) always worked with representatives of the class so as to avoid any use of actual infinities, whereas Dedekind worked directly with the class itself as “one whole”, uninhibitedly manipulating infinite sets (and sometimes, infinite sets whose elements are themselves infinite).

The conceptual approach is thus merely a ‘tendency’ slowly developing in mathematics. It was, by then, far from dominant neither globally in German mathematics, nor locally in each of the (supposed) early representatives of the new approach. If there is, the actors tell us, a new ‘tendency’ in mathematics which starts to be noticeable in 1840-50, according to Eisenstein’s testimony, it stays rather diffuse, sporadic, and far from purely “conceptual”.

1.1.3.1 Was Riemann’s mathematics really conceptual?

In fact, the same kind of observations can apply to the “conceptual” reading of Riemann’s works as well. To get rid of computations at the foundation of the theory, in Riemann’s case, is intrinsically linked to changes in “the type of questions one could ask about a complex function, away from the computational and towards the conceptual” ([Bottazzini and Gray, 2013], 279). However, this should not be taken as implying that all computations disappeared from Riemann’s works.

Neuenschwander ([Neuenschwander, 1981]) showed that Riemann was strongly influenced by Cauchy’s works in complex function theory. He further highlighted the importance of Gauss’s works for the development of Riemann’s ideas. He also stressed the fact that the common interpretation about how Cauchy, Weierstrass and Riemann were opposed and did not influence each other was seriously challenged by a closer attention to their earlier works.

In particular, Neuenschwander emphasized the fact that the relationship between Riemann and Weierstrass was not that of a deep opposition, as is often suggested.⁴² He highlighted mutual influences between Riemann and Weierstrass. It seems to suggest that the opposition between “conceptual” and “computational” is probably more superficial than it appears.

⁴²Not only did Riemann stayed in Berlin, in which he completed his mathematical formation, but Neuenschwander gives several evidences that Riemann’s and Weierstrass’s personal relationship must have been relatively good (see [Neuenschwander, 1981], 95).

Neuenschwander explains that after the publication of several of Weierstrass's works on Abelian functions in 1856 in Crelle's Journal, Riemann published his own work in 1857, explaining his position towards Weierstrass's works:

Jacobi's inversion problem, which is settled here, has already been solved for the hyperelliptic integrals in several ways through the persistent works of Weierstrass crowned by such a beautiful success, of which a survey has been communicated in vol. 47 of the *Journ. für Math.* (page 289). Until now, however, the actual working out has been published (vol. 52, page 285 of the *Journ. für Math.*) only for part of these works, namely the part that was outlined in §§1 and 2 of the quoted paper and in the first half of §3, concerning elliptic functions for the most part. Only after the presentation of these results and methods shall we be able to tell to what extent the later parts of these works presentation agree with the article presented here not only in the results but also in the methods leading to these.⁴³ ([Riemann, 1857], quoted and translated in [Neuenschwander, 1981], transl. slightly altered, 94)

And in fact, when Dirichlet praised Riemann's works on function theory, in letters to Kummer, he essentially underlined that Riemann had developed a theory whose basis was "even more comprehensive than the researches of Weierstrass" (*ibid.*, 94).

After the publication of Riemann's paper, Weierstrass decided to withdraw his own work, and later explained that he wanted to understand Riemann's approach and the "entirely different" foundation of the theory, whose "algebraic" investigations were difficult for him to fully understand. Once "this difficulty was overcome, it seemed to [him] that a thoroughgoing overhaul of [his] paper was necessary." (*ibid.*, 95).

According to Neuenschwander,

[i]t would be entirely conceivable that the general systematic construction of the Weierstrassian function theory, achieved around 1860 (...) could have been inspired by, among other things, the

⁴³ "Das hier erledigte Jacobi'sche Umkehrungsproblem ist für die hyperelliptischen Integrale schon auf mehreren Wegen durch die beharrlichen mit so schönem Erfolge gekrönten Arbeiten von Weierstrass gelöst worden, von denen eine Uebersicht im 47. Bande des *Journ. für Mathm.* (S. 289) mitgetheilt worden ist. Es ist jedoch bis jetzt nur von dem Theile dieser Arbeiten, welcher in den §§1 und 2 und der ersten die elliptischen Functionen betreffenden Hälfte des §3 der angeführten Abhandlung skizzirt wird, die wirkliche Ausführung veröffentlicht (Bd. 52, S. 285 d. *Journ. f. Math.*); in wie weit zwischen den späteren Theilen dieser Arbeiten und meinen hier dargestellten eine Uebereinstimmung nicht bloss in Resultaten, sondern auch in den zu ihnen führenden Methoden stattfindet, wird grossentheils erst die versprochene ausführliche Darstellung derselben ergeben können.

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works of Riemann pertaining to the same set of ideas. While Riemann freely employed the methods of others and also took these up in his lectures, (...) by contrast, Weierstrass seems to have admitted them into his theory usually only after a systematic reworking. (*ibid.*, 95)

The idea that Weierstrass studied and assimilated Riemann’s works up to the point where he thoroughly reformulated them and integrated them in his own works, without acknowledging Riemann’s insights, is also defended by [Bottazzini and Gray, 2013] (see later in this dissertation p. 101).

However, after Riemann’s death, Weierstrass became much more critical of his methods, openly attacking him in letters to many different mathematicians. He also published proofs that some of Riemann’s results were false.⁴⁴ The academic position of Weierstrass in Berlin, and the fact that Riemann’s theory still suffered from lack of rigorous and precise treatments of certain fundamental points made the reception of Riemann’s works complicated – albeit not impossible.

In addition to the artificial opposition between Weierstrass and Riemann, H. Edwards has argued that Riemann did not embrace entirely the conceptual approach more than Dirichlet did. For Edwards, Riemann’s mathematics had a strongly algorithmic component:

Surely a mathematician of Riemann’s greatness would want to simplify and organize his formulas in the clearest possible way, but to say that Riemann would insist that *Darstellungsformen* should always be results, not tools, of the theory is, I believe, a serious misrepresentation. (...) He was, rather, a virtuoso of *Darstellungsformen*. ([Edwards, 2010], 63)

With four examples taken from Riemann’s mathematical works, Edwards shows how Riemann, while he may have been “primarily interested in grand general abstract concepts,” at several occasions “did not venture into these higher realms without doing a lot of serious computation to lay the groundwork for his flights” (*ibid.*, 64). From Edwards’s viewpoint, Riemann’s great technical abilities and his leaning towards algorithmic mathematics can also be found in Riemann surfaces, the archetypal example of conceptual mathematics.

To describe a polynomial by its roots or a rational function by its zeros and poles often serves very concrete algorithmic purposes. Such a method also gives insight into the number of arbitrary parameters in an algebraic function of a certain type – which is the number evaluated by the Riemann-Roch theorem. (*ibid.*, 67)

⁴⁴See [Neuenschwander, 1981], 96-97.

The legend surrounding Riemann's purely conceptual approach is due, according to Edwards, to Dedekind and later to Klein – a point which I will develop in the next section.⁴⁵

It should be underlined that Dedekind's testimonies, on which accounts of Riemann's conceptual ideals are often heavily relying, albeit stemming from recollections of conversations with Riemann himself, were written two decades after Riemann's death. As we will see, Riemann only had five students in Göttingen: Abbe, Roch, Hattendorff, Hankel, and Prym. But among them, Roch, Hankel and Hattendorff died between 1866 and 1882, and Abbe left mathematics. While the reception and responses to Riemann's function theory was rather considerable, albeit difficult on some points (see Bottazzini and Gray, 2013, 311-339), the spreading of Riemann's ideas about a "conceptual" approach in mathematics was far less significant. Dedekind seems to have been the only true "Riemannian" – something he appeared to be highly aware of. Incidentally, Dedekind's methods in number theory, which appear to be very faithful to the principles (that he himself states) of what we now see as "conceptual mathematics", did not find an immediate reception either, for they were often considered too abstract.

In fact, Klein, during his time in Göttingen, seems to have been the first to state some of the core ideas of Riemann's function theory – the intimate relationship between function and surface, the desire to treat entire classes of functions. . . – be acknowledged and adopted. This was continued by Hilbert, and later Weyl.⁴⁶

Klein gave, in 1894, a discourse about Riemann's influence on modern mathematics ([Klein, 1894]), in Vienna, which was reproduced as a foreword to the French translation of Riemann's collected works. In this address, he told that Riemann's 1857 theory of Abelian functions is

the most brilliant of all the marvelous productions of his genius.
Indeed, the results in it are obtained by means which are not

⁴⁵This was also defended by Carl Siegel, in his publication of the Riemann-Siegel formula (1932):

The legend according to which Riemann found his mathematical results through grand general ideas without requiring the formal tools of analysis, is not as widely believed today as it was during Felix Klein's lifetime. Just how strong Riemann's analytic technique was is especially clearly shown by the derivation and transformation of his asymptotic series for $\zeta(s)$. (Quoted and translated in [Edwards, 2010], 67)

Note however that Siegel was very hostile to the evolution of 20th century mathematics towards more abstract and "conceptual" mathematics.

⁴⁶Note, however, that this only means that they acknowledged the importance of these methodological components (which had hitherto been overlooked) for Riemann's theory. It does not imply that they adopted Dedekind's approach itself. The agreement around this specific methodological point does not necessarily entail an agreement about the actual treatment of Riemann's function theory and/or the understanding of what constitutes its essence.

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tiresome, with the help of immediate thoughts based on the geometrical methods [related to Riemann surfaces].⁴⁷ ([Klein, 1894], xxiii)

Klein worked on giving Riemann his good reputation back, and this translated partly into setting up an important and very influent school in Göttingen. More will be mentioned in the next paragraph about the reception of Riemann’s works, but as Bottazzini and Gray explain, “neither Berlin nor Clebsch’s followers could talk up their connection to Riemann”, and Klein’s ideas which were taken up by Hilbert and his school and became clearly oriented towards a “conceptual” reading of Riemann, became the dominant ideas of Riemann’s works in the 20th century.

1.1.3.2 A case of nostrification?

The tradition established by Klein in Göttingen is often considered as a natural continuation (or a renewal) of the great years of Dirichlet, Dedekind and Riemann in Göttingen. I will suggest that Hilbert and Klein take up Dedekind’s reading of Dirichlet and Riemann, and in their interpretation, they put forward a certain “tradition” of conceptual mathematics in Göttingen.⁴⁸

Firstly, even if we were to accept that, in the 1850s, the new (conceptual) tendency finds a place to grow and develop, with Dirichlet’s arrival in Göttingen and his impact on Dedekind and Riemann, it should be underlined that the group of researchers in Göttingen, working in the context this conceptual approach or in mathematics in general, was very small at that moment. Indeed, it only consisted in Dirichlet, Riemann and Dedekind. Moreover, neither Dirichlet nor Riemann and Dedekind “succeeded in establishing a viable mathematical school at Göttingen” ([Rowe, 1989]). The actual impact of a tradition they would have tried to develop is virtually

⁴⁷ “[L]a plus brillante de toutes les merveilleuses productions de son génie. En effet, les résultats y sont obtenus par des moyens qui ne sont pas pénibles, à l’aide des réflexions immédiates basées sur les méthodes géométriques”.

⁴⁸For more on the idea of “nostrification”, see [Corry, 2004a], p. 221, p. 496 and in particular §9.2. An explanation by Corry is : “There are no available hints as to the origins of the term or who was the first to use it, but over the first three decades of the twentieth century, it was widely understood, among German mathematicians at least, that “nostrification” encapsulated the peculiar style of creating and developing scientific ideas in Göttingen (...). The Göttingen atmosphere implied a constant discussion and adoption of new ideas, techniques, and problems that had originally been created or suggested by others, either at home or outside. Whenever these ideas appeared to be fruitful and relevant to current concerns of the local community, they were immediately absorbed into the common scientific patrimony.”

The idea is also used by Petri and Schappacher in Schappacher and Petri, 2007 in reference to the Klein and the “nostrification” of the movement of arithmetization.

nonexistent.⁴⁹ The Georg-August Universität in Göttingen was famous for the great minds who taught there, in particular Gauss, Dirichlet, Riemann, but for a long time (in fact, before Klein's arrival), it was not as influential as its fame suggests, and was not the place of choice for students to study mathematics. In the second half of the 19th century, the Berlin university was, Rowe explains, "clearly the dominant place for German mathematics," closely followed by Königsberg, and a development of the conceptual approach is considerably challenged by the absence of students:

Few of [Dirichlet's, Riemann's and Dedekind's] students were ever exposed to this new mathematics, and fewer still were capable of understanding the work and communicating it to others. ([Rowe, 1989], 187)

Secondly, if both Dedekind and Riemann had a tremendous impact on the mathematical ideas, concepts and practices of the end of 19th century and later, their immediate impact was limited – especially Dedekind's. To be clear, certain circumstances did not help the forming of a school: Dirichlet and Riemann died young and so did Riemann's few students. Dedekind did not start publishing significant works before the 1870s, and taught in Braunschweig at the *Technische Hochschule* all his career, putting himself in a certain academic isolation. Even when he started publishing, the reactions to Dedekind's works in number theory were not particularly positive, and the preeminence of Kronecker's works can certainly not be attributed solely to circumstances. Dedekind's advocacy for conceptual methods was, in fact, not really widespread. Dedekind complained of being very little read, in a letter to Lipschitz. Before 1880, his algebraic concepts were barely used by anybody but himself. His methods were considered too abstract and too difficult by his contemporaries. Kronecker wrote, when reporting for Dedekind's nomination at the Berlin Akademie der Wissenschaften in 1880, that Dedekind's choices of methods and terminology made it difficult to judge his works – he did, however, recommend Dedekind's election to the Akademie. Dedekind and Kronecker worked on the same topics in significantly different ways. Kronecker's prominent position in German mathematics, at the time, may have been an obstacle to the reception of Dedekind's works – problems of rivalry and priority as regards algebraic number and function theories even arose between Dedekind and Kronecker.

In 1888, Dedekind's methods still did not reach many mathematicians. *Zahlen* was coldly received in Berlin. By 1893, the two mathematicians who worked closest to Dedekind or using ideas coming from Dedekind's works were H. Weber and Frobenius who both had reservations about the abstraction of his approach, and both worked closely to Berlin mathematicians as

⁴⁹And it was certainly not Clebsch and his students who were going to continue the "conceptual" approach, which they considered too imprecise. see Sec. 1.3.2.

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well. Notably, Frobenius, who refers to Dedekind as “our admired friend and master”, wrote to Weber in December 1893:

I hope you often walk on the paths of Dedekind, but avoid the too abstract corners, which he now likes so much to visit. His newest edition contains so many beauties, §173 is highly ingenious, but his permutations are too disembodied, and it is also unnecessary to push abstraction so far.⁵⁰ (Repr. in [Dugac, 1976b], 269, transl. in [Sieg and Schlimm, 2005], 121)

In the 1890s, the rise of ‘modernism’ marks the beginning of a stronger and more positive reception of Dedekind’s works, as are his interactions with other mathematicians. Weber turns to algebra and, with Frobenius, gradually adopt more “Dedekindian” approaches, thereby making his methods more available. The first important use of Dedekind’s (and Weber’s) ideas is to be found in Hilbert’s *Nullstellensatz* (1888). Schappacher, in [Schappacher, 2010], tells us that in articles written in German, “the [*Jahrbuch über die Fortschritte der Mathematik*] database lists 69 papers between the first one in 1882 (an article by Dedekind) and 1914 which have the word *Körper* (in the algebraic sense) or the word *Zahlkörper* already in the title. The little industry really took off around 1900” (*ibid.*, 3269). Considering more broadly the papers using also the words “ideal”, the number of papers published on these topics started to grow in the 1890s with works by Weber, Meyer, Fricke, Hurwitz, Hensel and Hilbert. A ‘growth’ relative to the very small (if not null) number, since there were up to six papers on ideal theory in 1894, and two of them were by Dedekind himself.⁵¹ Only with the publications of Weber’s *Lehrbuch der Algebra* and Hilbert’s *Zahlbericht*, did Dedekindian number theory start to receive a wider acknowledgement.

If it started to be more widely adopted among mathematicians of later generations, especially after Hilbert’s *Zahlbericht* in 1897, the “conceptual approach” was far from a typical way of doing mathematics. Dedekind’s ideas are discussed and used by mathematicians such as Hilbert and Hurwitz – sometimes leading to unpleasant quarrels of priority or to Dedekind being unhappy with the treatment of ideal theory (in particular in [Dedekind, 1895b]).⁵² Indeed, Hurwitz, like Hilbert, developed an approach in which Dedekind’s and Kronecker’s ideas are mixed together – not unlike some parts of Weber’s *Lehrbuch*, written around the same time. They “cared little about

⁵⁰ “Hoffentlich gehen Sie vielfach die Wege von Dedekind, vermeiden aber die gar zu abstrakten Winkel, die er jetzt so gern aufsucht. Seine neueste Auflage enthält so viele Schönheiten, der §173 ist hochgenial, aber seine Permutationen sind zu körperlos, und es ist doch auch unnötig, die Abstraktion so weit zu treiben.”

⁵¹ These elements come from a keyword research in the *Jahrbuch über die Fortschritte der Mathematik* and on *Zentralblatt*. Galois theory was a slightly more popular topic, but Dedekind himself published very little on the subject and until 1894 excluded most considerations about Galois theory from his algebraic number theory.

⁵² see Sec. 5.7 and [Dugac, 1976b].

the methodological preferences of either Dedekind or Kronecker in this area of research” ([Schappacher, 2005], 703). In fact, they found “both of them horrible (*scheußlich*)” (Hilbert, quoted and translated in [Corry, 2004a], 21).

It seems, in fact, that the idea of a “conceptual approach” has its source with Dedekind’s reconstruction of Riemann’s and Dirichlet’s “conceptual approaches”, but that Dedekind seems to have been, at that time, the only one to truly embrace what he presents as Riemann’s methodological principles. His view was however later picked up by Minkowski, Klein and Hilbert. From their own statements, the view of a “conceptual approach” with Riemann and Dirichlet became largely accepted.

Hilbert⁵³ presented in famous *Zahlbericht* as following attempts to “avoid Kummer’s elaborate computational machinery” to realize “Riemann’s principle” and have proofs be “completed not by calculations but purely by ideas”⁵⁴ ([Hilbert, 1897], viii)

Minkowski, in 1905, for the celebration in Göttingen of Dirichlet’s centenary, gave a talk in which, while underlining that strangely enough Dirichlet never founded an actual school despite his “tremendous influence on many mathematicians who received from him the strongest impulse of their scientific aspiration,”⁵⁵ he nevertheless presents Dirichlet as the father of the so-called conceptual tradition in mathematics. In particular, he mentions Riemann:

What mathematician could fail to understand that the luminous path of Riemann, this gigantic meteor in the mathematical heaven, took its starting-point in the constellation of Dirichlet. It may be that the sharp sword that Riemann named the Dirichlet principle was first brandished by William Thomson’s young arm, but the modern era in the history of mathematics dates from the other Dirichlet principle, to conquer the problems with a minimum of blind calculation, a maximum of clear-seeing thoughts.⁵⁶ ([Minkowski, 1911] II, 460-61, partially translated

⁵³Hilbert learnt number theory with H. Weber, who used Dirichlet’s *Vorlesungen* and many of Dedekind’s ideas.

⁵⁴“*Ich habe versucht, den grossen rechnerischen Apparat von Kummer zu vermeiden, auch hier der Grundsatz von Riemann verwirklicht würde, demzufolge man die Beweise nicht durch Rechnung, sondern lediglich durch Gedanken zwingen soll.*”

⁵⁵“*So außerordentlich anregend Dirichlet als Lehrer gewirkt hat, eine besondere mathematische Schule hat er nicht gegründet. Aber viele, die sich hernach auf individuell sehr verschiedene Wege zerstreuten, verdankten ihm die stärksten Impulse ihres wissenschaftlichen Strebens.*”

⁵⁶“*Welcher Mathematiker hätte kein Verständnis dafür, daß die leuchtende Bahn Riemanns, dieses riesigen Meteors am mathematischen Himmel, vom Sternbilde Dirichlets ihren Ausgangspunkt nahm. Mag auch das von Riemann Dirichletsches Prinzip benannte scharfe Schwert zuerst von William Thomsons jungem Arm geschwungen sein, von dem anderen Dirichletschen Prinzipie, mit einem Minimum an blinder Rechnung, einem*

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in [Stein, 1988], p. 241.)

If Dedekind did read Dirichlet and Riemann as conceptual mathematicians, he was not able to develop that heritage into a “conceptual tradition” himself, and his works were merely referred to as “too abstract”. In fact, the “conceptual approach” has been seen by several commentators as beginning later in reference to Hilbert’s and E. Noether’s works. For Dieudonné and Carrell (quoted in [Boniface, 2004], 53), Hilbert’s *Nullstellensatz* was the “first paper in ‘modern algebra’ by its conceptual approach and methods.” David Rowe explains how “Hilbert’s abstract, conceptual approach” for the *Nullstellensatz* and in the *Zahlbericht*, which was “largely inspired by Dedekind”, was at odds with “the spirit of Kronecker’s work and, for that matter, to the entire classical tradition of number theory” ([Rowe, 1989], 199). The conceptual approach is also considered as typical of Emmy Noether’s approach⁵⁷, who was described by Alexandrov as

the creator of a new direction in algebra and the leading, the most consistent and prominent representative of a certain general mathematical doctrine – all that which is characterized by the words “begriffliche Mathematik”. (...) It was she who taught us to think in terms of simple and general algebraic concepts – homeomorphic mappings, groups and rings with operators, ideals – and not in terms of cumbersome algebraic computations; and thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation which was not at all suited for the accustomed approach of the classical algebraists. (in [Dick, 1970], 156-158)

A viewpoint seconded by Van der Waerden, according to whom Emmy Noether’s “maxim” was

Any relationships between numbers, functions and operations only become transparent, generally applicable, and fully productive after they have been isolated from their particular objects and been formulated as universally valid concepts. (*ibid.*, 101)

The picture of a “Göttingen tradition” preceding Klein’s arrival at Göttingen, which can be found in some accounts by Hilbert or Minkowski, reflects

Maximum an sehenden Gedanken die Probleme zu zwingen, datiert die Neuzeit in der Geschichte der Mathematik.”

⁵⁷Even if, as Hermann Weyl underlines Noether’s “mathematical orbit set out” from “a formalist like Gordan”. Gordan was, according to his friend Max Noether, an “*Algorhythmiker*” whose “strength rested on the invention and calculative execution of formal processes” (in [Dick, 1970], 120-121).

rather strongly their own view. It seems that their interpretation of Dirichlet's and Riemann's thoughts – maybe even, to a certain extent of Dedekind's – may have been biased. As suggested by L. Corry ([Corry, 2004a])⁵⁸, if one wants “to characterize the peculiar Göttingen scientific culture” in Hilbert's and Klein's times, a key idea is that of “nostrification”, which broadly corresponds to a tendency, particularly strong in Hilbert's works, to “reinterpret other people's thoughts, so as to make them fit [their] own current picture of the domain in question”. The desire, strongly expressed by Klein, to create a new kind of scientific institution might have led to the reconstruction of a history, an inheritance, which selected and overemphasized some isolated ideas.

Note that as far as Dedekind is concerned, considering the emphasis put by Emmy Noether on the essential importance of Dedekind's works for modern algebra, it is easy to overestimate how close his works actually were to modern algebra, as well as their immediate impact. It seems that the emphasis of a “conceptual approach” developed in Göttingen under the *impetus* of Dirichlet's works could be yet another part of the “nostrification”. The clear-cut tradition depicted in Klein's or Minkowski's writings appears as a story told with only isolated elements of (their reading of) Dirichlet's and Riemann's practices.

We saw that Eisenstein's enthusiasm for a “new school” and Dedekind's own statements, are the most detailed accounts available, for the 19th century. A closer look at the mathematicians of the “new” approach suggested that the so-called new “conceptual approach” was rather following a tendency towards moving away from 18th century complicated computational machinery, in attempts to bypass the growing difficulties coming with the increasing complexity of computations. The “conceptual approach”, during the second half of the 19th century, thus, corresponds to the slow development of the emergence of new mathematical practice shaped by specific epistemological requisites, which would reveal to be the fertile soil on which works such as Emmy Noether's could grow.

If Riemann's and Dedekind's approaches are exhibiting clear signs of being “conceptual”, such a description of their works remains superficial. I suggest that one can characterize in greater detail not only the mathematical content (or at least not exclusively), but also the account of the underlying conceptions guiding the development of Dedekind's works. What I will propose, in the following chapters, is to analyse Dedekind's texts so as to uncover the methods at play to fulfill Dedekind's “conceptual” demands:

⁵⁸Some of D. Rowe's works also abound in this sense. In [Schappacher and Petri, 2007], it is the “arithmetization of analysis” which is presented as being “nostrified” by Göttingen after Klein's talk (see Sec. 7.1.1). What I am proposing, here, follows the same line of ideas, although I will not consider in any detail the idea of a nostrification of the movement of arithmetization.

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avoiding computations, providing uniform definitions and proofs, answering a certain ideal of rigor... For example, I have mentioned that Dedekind's invention of ideals is a stereotypical example of the "conceptual approach" but not how Dedekind did prove theorems for algebraic integers, which is, after all, the essential point of his work.

The definition of ideals consists essentially in an extension of the notion of divisibility in such a way that methods of proof can be elaborated in a purely arithmetical way. This point plays a central role in Dedekind's ideal theory and, as I will show, has important ramifications. I will base my argumentation on close readings of Dedekind's texts, to exhibit specific elements of practice which complement the understanding in terms of "conceptual mathematics" and will lead me to concentrate on questions related to Dedekind's conception and use of arithmetic.

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Michael Potter explains in the introduction of his book *Reason's Nearest Kin. Philosophies of Arithmetic from Kant to Carnap* that:

In popular parlance the word 'arithmetic' means the study of calculations involving numbers of all sorts, natural, integral, rational, real, and complex. Mathematicians, on the other hand, often use the word to mean only the study of the properties of the natural (i.e., counting) numbers 0, 1, 2, etc. ([Potter, 2002], 1)

This view is largely embraced by philosophers of mathematics. However, the term took various meanings in different historical contexts. It will be useful for my purpose to examine them.

This section will present parts of the mathematical context in which Dedekind developed his number theoretical works. It will also allow to underline that "arithmetic", for 19th century mathematicians, such as Gauss and later Dedekind, does not designate the properties of natural numbers as a the sequence but number theory in a relatively large sense.⁵⁹ In fact, interpretations can go as far as José Ferreirós, who considers that "arithmetic" was a generic word for *all* pure mathematics.⁶⁰ According to him, "arithmetic" for Dedekind "is understood (...) in a broad sense that embraces algebra and analysis".⁶¹ I will suggest that arithmetic had a rather

⁵⁹On a side note, this is still the case in French nowadays.

⁶⁰See [Ferreirós, 2007].

⁶¹In the introduction to *Was sind und was sollen die Zahlen?*, Dedekind writes that he is considering "arithmetic (algebra, analysis) as merely a part of logic", which does not keep him from giving a perfectly determinate definition of arithmetic later in the book: the laws which derive from the definition of numbers. I will consider the status of arithmetic in more detail in Chapter 8.

clear meaning for Dedekind.

C. F. Gauss used to repeat,⁶² according to several witnesses, a motto inspired by a sentence which Plutarch attributed to Plato⁶³:

ὁ θεὸς ἀριθμητίζει

God arithmetizes.

This quote, immortalized in an etching of Gauss and W. Weber by A. Weger appearing in Zöllner's *Wissenschaftliche Abhandlungen* (1878, vol. 2, part I), is linked to Gauss's conviction that arithmetic was at "the top" of the edifice of science. As Ferreirós underlines in the beginning of his paper titled "The Rise of Pure Mathematics as Arithmetic with Gauss" ([Ferreirós, 2007]), the theological assertion behind the motto is somewhat "less interesting than the way in which his adaptation of Plato's words reflects changing perceptions of mathematical knowledge" (*ibid.*, 237). Indeed, one can observe, in the 19th century, the development of many 'arithmetical' approaches to mathematics. As Hilbert put it, the mathematics of this century developed "under the sign of number".

From Gauss's *Disquisitiones Arithmeticae* in 1801 to the emphasis put on a "movement of arithmetization" with Klein in 1895, the meaning of the term "arithmetic" slowly moved from the theory of integers to being associated, rather loosely, to another conception of numbers and ideals of rigor and certainty, or even *a priori* knowledge. Weierstrass, who famously insisted on giving rigorous definitions and "sufficiently detailed exposition of the basic notions and the arithmetical operations" (*ibid.*, 238), demanded, for example, that the "definition of complex magnitudes" be "purely arithmetical" (*ibid.*, 239) – as did Dedekind. Arithmetic was opposed to geometry, which is the core of claims such as Weierstrass's:

Furthermore we shall give a purely arithmetical definition of complex magnitudes. The geometrical representation of the complex magnitudes is regarded by many mathematicians not as an explanation, but only as a sensorial representation, while the arithmetical representation is a real explanation of the complex

⁶²And he repeated until his last days, as reported by his physician Dr. Wilhelm Baum in a letter to A. von Humboldt: "The last days of his life were often very painful owing to the aggravated complaint of dropsy, which the hypertrophy of his heart produced – but still he always maintained his freedom and greatness of spirit, the strongest conviction of his personal permanence, the firmest hope in the still deeper intelligent insight into the number relationships, which God places in matter and which he would perhaps be able to recognize in the intensive magnitudes, for he used to say: ὁ θεὸς ἀριθμητίζει." (Quoted in [Ferreirós, 2008], 236).

⁶³The version attributed to Plato is "ὁ θεὸς ἀεὶ γεωμετρεῖ", i.e., "God geometrizes eternally".

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magnitudes. In analysis we need a purely arithmetical foundation, which was already given by Gauss. Although the geometrical representation of the complex magnitudes constitutes an essential means for investigating them, we cannot employ it, for analysis must be kept clean of geometry.⁶⁴ (From Weierstrass's lecture of 1874 summer semester, translated and quoted in [Freirós, 2007], 239)

Opposition to geometry and association to rigor do not suffice, of course, to precisely characterize what arithmetic consisted of. More precisely, the epistemological values attributed to arithmetic do not help us circumscribe the domain of research.

1.2.1 Gauss's *Disquisitiones Arithmeticae*

Gauss, in the preface of his celebrated *Disquisitiones Arithmeticae*, describes the scope of his investigations in the following way:

The Inquiries which this volume will investigate pertain to that part of Mathematics which concerns itself in particular with integers, sometimes fractions, but from which irrational [surds] numbers are always excluded. (...) Just as we include under the heading of [Diophantine] Analysis all researches concerning the general affections of quantities, the considerations of integers and fractions, insofar as they are expression by means of integers, constitute the proper object of *Arithmetica*. ([Gauss, 1801], transl. modified Clarke and Waterhouse, xviii)

Gauss mentions an ordinary meaning of arithmetic, which he immediately challenges, a distinction which should allow for a more finely-shaded description of 'arithmetic':

One ordinarily calls Arithmetic only the art of forming numbers and calculating [with] them, i.e., the art of expressing numbers by suitable symbols (for example by a decimal representation), and of carrying out arithmetic operations. It sometimes includes other points, some which do not pertain to arithmetic, like the

⁶⁴ "Wir werden ferner eine rein arithmetische Definition der complexen Grössen geben. Die geometrische Darstellung der complexen Grössen wird von vielen Mathematikern nicht als eine Erklärung, sondern nur als eine Versinnlichung betrachtet, während die arithmetische Darstellung die complexen Grössen wirklich erklärt. Wir bedürfen jedoch für die Analysis eine rein arithmetische Begründung, die schon Gauss gegeben hat. Obgleich die geometrische Repräsentation der complexen Grössen ein wesentliches Hülfsmittel zur Untersuchung derselben ist, können wir sie hier nicht anwenden, da die Analysis von der Geometrie rein erhalten werden muss."

theory of logarithms, and others which are not peculiar to integral numbers but are common to all quantities. (*ibid.*, transl. modified, xviii)

The important distinction made, then, is between the part of arithmetic just described, which Gauss calls “elementary arithmetic”, and the “general investigations on the particular properties of integers” which are part of “higher arithmetic” (“*Arithmeticae sublimiori*”, translated in German by “*höhere Arithmetik*” and usually called “*arithmétique transcendante*” in France).

Gauss argued for the consideration of imaginary quantities as actual numbers and called for their introduction in “higher arithmetic”, expanding its limits and drawing the outlines of a “general arithmetic” including Gaussian integers:

Accordingly, one soon realizes that in this rich domain of higher arithmetic, one can only penetrate through completely new roads (...) for this end, a specific expansion of the entire field of higher arithmetic is essentially necessary.⁶⁵ ([Gauss, 1831], transl. modified in [Ewald, 2005], 308)

What does appear rather clearly is that Gauss seems to consider that arithmetic is limited to study of discrete domains, continuity falling in the scope of analysis (sometimes called “*Größenlehre*”).⁶⁶ Ferreirós suggests that Gauss “placed severe restrictions as to the subject matter and methods of number theory, namely that one studies discrete domains (from our standpoint, rings of integers) by characteristically number-theoretical means.” ([Ferreirós, 2008], 262)

Higher arithmetic covers the study of congruences, first defined in Gauss’s *Disquisitiones*, and the notions and objects deriving from it, such as binary quadratic forms or algebraic numbers. For example, in a paper entitled “Simplification of the theory of binary quadratic forms with positive determinant” (“*Vereinfachung der Theorie der binären quadratischen Formen von positiver Determinante*”), read at the Berlin Akademie der Wissenschaften, on 13 July 1854 ([Lejeune-Dirichlet, 1889-1897], II, 141-158), Dirichlet explains:

The wider scope the domain of higher arithmetic has gained, since Gauss’s epoch-making work, and the works which followed it, the more it appears desirable that the access to this beautiful

⁶⁵ “Man erkennt demnach bald, dass man in dieses reiche Gebiet der höhern Arithmetik nur auf ganz neuen Wegen eindringen kann (...) dazu eine eigenthümliche Erweiterung des ganzen Feldes der höhern Arithmetik wesentlich erforderlich ist.”

⁶⁶That point will be very different in Dedekind’s works, who considers the linear continuum formed by real numbers to be part of arithmetic.

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branch of Analysis be as facilitated as possible by the simplification of its elementary parts.⁶⁷ (*ibid.*, 141, my translation)

Gauss's *Disquisitiones Arithmeticae* marks the birth of number theory as a discipline in its own right in Germany (see [Goldstein and Schappacher, 2007a]). The book can be divided in three distinct parts. The first part defines the notion of congruence⁶⁸ and studies the various properties of congruences of first and second degree. Gauss gives new proofs of Fermat's theorem, Wilson's theorem and the law of quadratic reciprocity. In the second part, which takes half of the book's length, quadratic forms $ax^2 + bxy + cy^2$ (with a , b and c integers) are introduced. While relying on works by Fermat, Euler, Lagrange and Legendre, Gauss proposes a new way of studying quadratic forms, considering only the triplet of the coefficients of the form (a, b, c) and using the new notions of discriminant and equivalence of forms. The third part is dedicated to solving cyclotomic equations, that is, equations of the form $x^n - 1 = 0$ with n a prime number. Gauss explains in the introduction that

The theory of the division of the circle, or of regular polygons, treated in Section VII, *of itself* does not pertain to Arithmetic, but its *principles* can only be drawn upon Higher Arithmetic. ([Gauss, 1801], transl. modified, xx, emphasis in the original)

This Section VII, is probably the best known part of the book, especially because Gauss gives the conditions to construct regular polygons with ruler and compass. Gauss underlines an "intimate connection" between Section VII and the arithmetical inquiries of the book. In addition to the technical relationship between this section and preceding ones, Goldstein and Schappacher ([Goldstein and Schappacher, 2007a]) suggest that there are certain specificities in "the systemic architecture of the treatise" (*ibid.*, 16) that explain the "intimate connection" between the division of the circle and the arithmetical considerations developed in the next sections of the book. Indeed, as they explain, an eighth chapter about the theory of higher congruences was planned by Gauss. This way, the *Disquisitiones Arithmeticae* would thus have "come full circle in several respects: beginning with ordinary congruences and ending with higher congruences; encountering various periodic structures along the way (...); and proving quadratic reciprocity four separate times in the process" (*ibid.*, 16).

⁶⁷"Je grosser der Umfang ist, welchen die höhere Arithmetik durch das Epoche machende Werk von Gauss und andere spätere Arbeiten gewonnen hat, um so wünschenswerther erscheint es, dass der Zugang zu diesem schönen Zweige der Analysis durch Vereinfachung des elementaren Theiles desselben so viel als möglich erleichtert werde."

⁶⁸"If a number a divides the difference of the numbers b and c , b and c are said to be congruent relative to a , if not they are noncongruent." ([Gauss, 1801], 1)

As is well known, Gauss's book had a considerable impact on the development of number theory and more largely mathematics, in the 19th century. Poincot, in 1807, writes:

The doctrine of numbers, in spite of [the works of previous mathematicians] has remained, so to speak, immobile, as if it were to stay for ever the touchstone of their powers and the measure of their intellectual penetration. This is why a treatise as profound and as novel as his *Arithmetical Investigations* heralds M. Gauss as one of the best mathematical minds in Europe.⁶⁹ (in *Gazette nationale ou Le Moniteur universel* 80 (1807), 312, quoted and translated in [Goldstein and Schappacher, 2007a], 3)

Lagrange, in a letter to Gauss on May 31st 1804, displays his admiration of his work:

Your *Disquisitiones* have put you at once among the first mathematicians, and I consider the last section as one which contains the most beautiful analytic discovery made in a long time.⁷⁰ ([Lagrange, 1867-1892], XIV, 299, quoted and translated in [Goldstein and Schappacher, 2007a], 19)

A few years earlier, the *Disquisitiones Arithmeticae* were mentioned at the Académie des Sciences as a geometrical discovery:

Citizen Legendre communicates a geometrical discovery, made in Germany by M. Charles Frédéric Bruce [sic], from Brunswick, and published by him in his work entitled *Disquisitiones arithmeticae*, Leipsik, 1801.⁷¹ (Quoted and translated *ibid.*, 19)

By then, as underlined by Goldstein and Schappacher, Section VII was the part of the treatise which first retained the mathematicians' attention, leading them to consider Gauss's work as analytical or geometric. In any case, the delineation of "disciplines" in the *Disquisitiones* as well as in its reception, appears to be less clear than Gauss suggested.

⁶⁹"La doctrine des nombres malgré leurs travaux [antérieurs] est restée, pour ainsi dire, immobile ; comme pour être dans tous les tems, l'épreuve de leurs forces et la mesure de la pénétration de leur esprit. C'est pourquoi Monsieur Gauss, par un ouvrage aussi profond et aussi neuf que ses Recherches arithmétiques s'annonce certainement comme une des meilleures têtes mathématiques de l'Europe."

⁷⁰"Vos Disquisitiones vous ont mis tout de suite au rang des premiers géomètres et je regarde la dernière section comme contenant la plus belle découverte analytique qui ait été faite depuis longtemps."

⁷¹"Le Citoyen Legendre communique une découverte géométrique, faite en Allemagne par M. Charles Frédéric Bruce, de Brunswick, et publiée par lui dans son ouvrage intitulé *Disquisitiones arithmeticae*, Leipsik, 1801." (Procès verbaux de l'Académie des sciences, registre 114, vol. II, séance du 6 pluviôse an 10 (26 janvier 1802), p. 457)

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1.2.2 After Gauss

Number theory developed essentially in Germany, in a group of mathematicians who followed Gauss's lead and adopted the meaning of "arithmetic" and "higher arithmetic" that he proposed. However, as Goldstein and Schappacher underline in [Goldstein and Schappacher, 2007a], the lines between disciplines become rather blurry, in the works of the second generation of mathematicians who continued the researches undertaken in Gauss's treatise. To account for it, Goldstein and Schappacher propose to consider what they call the "Arithmetic Algebraic Analysis" ([Goldstein and Schappacher, 2007a], 26) and present as a new domain of research which embedded the different approaches to the *Disquisitiones Arithmeticae* and number theory, a domain of research which "knits together reciprocity laws, infinite series with arithmetical interpretations, elliptic functions and algebraic equations" ([Goldstein and Schappacher, 2007a], 52). This field of research was very active until the 1860s, and both Dedekind and Kronecker were in contact with it during the first years of their mathematical career. By the time Dedekind started to publish important works, the Arithmetic Algebraic Analysis had mostly died off. While in the works of mathematicians such as Dirichlet or Jacobi, "higher arithmetic" meets other domains of research, such as analysis via infinite series or algebraic equations, the next generation of mathematicians insisted on the importance of finding the 'right' methods, leading to disputes such as the one evoked in the introduction between Jordan and Kronecker, or of course between Dedekind and Kronecker.

I will concentrate, in the following, in the characterization of arithmetic for the mathematicians working in Arithmetic Algebraic Analysis, so as to clarify which idea of arithmetic Dedekind inherited.⁷²

In 1836-1837, Jacobi wrote:

Number Theory in its present state consists of two big chapters, one of which may be called the theory of the solution of pure equations, the other the theory of quadratic forms. Here I will deal mainly with the first part whose discovery we owe to Gauss.⁷³ (Quoted and transl. in [Goldstein and Schappacher, 2007a], 28)

Equations and forms appear as a sort of mathematical and conceptual middle ground, between higher arithmetic, i.e., the congruences, and analytic notions such as infinite series. Infinite series are particularly important if

⁷²For a broader view on these questions, one can refer to [Goldstein, Schappacher, and Schwermer, 2007].

⁷³"Die Zahlentheorie auf ihrem jetzigen Standpunkte zerfällt in zwei große Kapitel, von denen das eine als die Theorie der Auflösung der reinen Gleichungen, das andere als die Theorie der quadratischen Formen bezeichnet werden kann. Ich werde hier hauptsächlich von dem ersten Theile handeln, dessen Erfindung wir Gauß verdanken." (*Theorie der Zahlen*, first course, p.5, Lecture notes by J.G. Rosenhain, copied by hand.)

one wants to use functions of a real variable and tools from Fourier analysis in “higher arithmetic.” Goldstein and Schappacher bring to the fore the fact that although a domain of research in itself, number theory is developed in close relation with other areas of mathematics, at that moment. It was, of course, also the case for Dirichlet who imported in number theory tools from Fourier analysis. Commenting on his own works in which he used infinitesimal analysis to number theory, in 1839/40, Dirichlet wrote:

The Memoir in question is about the theory of *quadratic forms*, a theory which, prepared by some propositions by Fermat and the ingenious researches by Euler, and definitely founded by Lagrange, later received noticeable developments by the works of Legendre and mostly those of M. Gauss, who devoted to it the largest part of his “*Disquisitiones Arithmeticae*”, such that it constituted today one of the main branches of the science of numbers.⁷⁴ (“Recherches sur les formes quadratiques à coefficients et à indéterminées complexes”, in [Lejeune-Dirichlet, 1889-1897] I, 535, my translation)

Quadratic forms with integral coefficients play a crucial role in the 19th century number theory. We will see, in Chapter 5, that Dedekind’s introduction of ideals is made in a work dedicated to binary quadratic forms. In number theory, quadratic forms are essentially characterized by the concept of determinant, a positive or negative integer which allows to distinguish classes of forms for which certain arithmetical properties can be given. Interested by the dependency between the determinant and the number of corresponding distinct forms, Dirichlet is led to the distinction between positive and negative determinants:

We identify that the expression of the number of forms corresponding to any given determinant presents two very different cases depending on whether the determinant is a negative or a positive number. In the first case, the expression of the law in question has a purely arithmetical character, while for a positive determinant, it is of a more composite nature, mixed in a way, since, in addition to the arithmetic elements on which it depends, it contains others which present themselves in certain

⁷⁴ “*Le Mémoire dont il s’agit se rapporte à la théorie des formes quadratiques, théorie qui, préparée par quelques énoncés de Fermat et par les ingénieuses recherches d’Euler et définitivement fondée par Lagrange, a reçu plus tard de notables accroissements par les travaux de Legendre, et surtout par ceux de M. Gauss qui y a consacré la plus grande partie de ses ‘Disquisitiones Arithmeticae’, en sorte qu’elle constitue aujourd’hui l’une des branches principales de la science des nombres.*” The work referred to is “Recherches sur diverses applications de l’Analyse infinitésimale à la Théorie des nombres”, published in *Journal für die reine und angewandte Mathematik*, vol 19, 324-369; vol 21, 1-12 and 134-155, in 1839 and 1840.

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auxiliary equations present in the theory of binomial equations, pertaining thus to Algebra.⁷⁵ (*ibid.*, 536, my translation)

Here, one can see that arithmetic is more a characteristic of the approach, than a theory – a point that will be important for our understanding of Dedekind’s “arithmetical” methods.

Dirichlet explains that, following the “natural” desire to expand the results already obtained, one is led to try to “solve by [these] means other analogous questions but of a higher order” (*ibid.*, 536). One can either consider forms of a higher degree, or quadratic forms with complex integers as coefficients, which is what Dirichlet proposes to consider. The use of Gaussian integers allows to obtain laws for biquadratic residues which are of “an extreme simplicity and elegance” and “perfectly analogous” to what was known for quadratic residues (*ibid.* 537). But there is more to this “generalized number theory” than mere simplifications, it opens “a new field to arithmetical speculations”. Dirichlet’s goal, here, is to “import, in the thus generalized theory of numbers, the previously considered question” about quadratic forms. To consider a wider concept of integer, here the Gaussian integers, allows to investigate objects of number theory, here quadratic forms, in a more general framework, and it opens new possibilities of “arithmetical speculations.”

Provided that the concept of integral number be extended correctly, as was done by Gauss with complex integers, then “arithmetic” is, for Dirichlet, what “is related” to integers. Dirichlet favors efficiency of methods to any purity ideal:

While the preceding theorems [about determinant and number of classes of forms] do not contain any element which is not related to integral numbers, it seems difficult to establish them with purely arithmetical considerations, whereas the mixed method we just used and which is based partly on the use of quantities varying by insensible degrees, led us to them by the most natural way, and so to speak, effortlessly.⁷⁶ (*ibid.*, 618, my translation)

⁷⁵ “On reconnaît que l’expression du nombre des formes qui répondent à un déterminant quelconque présente deux cas très distincts suivant que ce déterminant est un nombre négatif ou positif. Dans le premier cas, l’expression de la loi dont il s’agit a un caractère purement arithmétique, tandis que pour un déterminant positif, elle est d’une nature plus composée et en quelque sorte mixte, puisque, outre les éléments arithmétiques dont elle dépend, elle en renferme d’autres qui ont leur origine dans certaines équations auxiliaires qui se présentent dans la théorie des équations binômes et appartiennent par conséquent à l’Algèbre.”

⁷⁶ “Quoique les théorèmes précédents ne contiennent aucun élément qui ne soit relatif aux nombres entiers, il paraît difficile de les établir par des considérations purement arithmétiques, tandis que la méthode mixte dont nous venons de faire usage, et qui est fondée en partie sur l’emploi de quantités variant par degrés insensibles, nous y a conduit de la manière la plus naturelle et, pour ainsi dire, sans effort.”

Not only were quadratic and biquadratic forms used in the theory of numbers, so did cyclotomic equations, modular equations and the division equations of elliptic functions (which originated in the study of algebraico-arithmetical properties of the integrals). All these theories were considered as pertaining both to the theory of algebraic equations and to arithmetic, as [Goldstein and Schappacher, 2007a] makes clear. Arithmetic then developed in an intricate relation with these theories, widening considerably while at the same time being still considered as the study of properties related to integral numbers. Eventually, in the works of Kronecker and Dedekind, a new more general concept of number is introduced: the algebraic numbers defined by means of a polynomial equation. Note that Kronecker, whose works rely essentially on polynomial and modular equations, qualifies his own works as being arithmetical. At the same time, Dedekind denies him this quality, finding his approach too formal to be arithmetical.

During the 19th century, the distinction between “arithmetic” and “higher arithmetic” faded – probably (or at least partly) because, with the tremendous developments of number theory, congruences soon were considered as elementary as were basic properties of integers. Divisibility properties appeared to be the first properties whose study impulsed new development in the theory of numbers, notably with Gauss’s congruences. Divisibility was, then, at the core of most of the inquiries. In 1876/77, Dedekind goes as far as saying that arithmetic is in some way founded on divisibility:

The theory of divisibility of numbers, which serves as a foundation of arithmetic, was already established in its essentials by Euclid. At any rate, the major theorem that each composite integral number can always be represented as a product of only prime numbers is an immediate consequence of the theorem, proved by Euclid, that a product of two numbers is not divisible by a prime number unless the prime divides at least one of the factors.⁷⁷ ([Dedekind, 1876-1877], transl. modified, 53)

It seems that “foundation”, here, should be taken in a practical sense: divisibility is at the foundation of arithmetic as an area of research insofar as it is its study which motivates its developments.

⁷⁷ “La théorie de la divisibilité des nombres, qui sert de fondement à l’arithmologie, a déjà été établie par Euclide dans ce qu’elle a d’essentiel ; du moins, le théorème capital que tout nombre entier composé peut toujours se mettre, et cela d’une seule manière, sous la forme d’un produit de nombres tous premiers, est une conséquence immédiate de ce théorème démontré par Euclide, qu’un produit de deux nombres ne peut être divisible par un nombre premier que si celui-ci divise au moins l’un des facteurs.”

1.3 Riemannian function theory

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We saw, in the first section, the epistemological context in which Dedekind's and Riemann's works grew. I will propose to consider in greater details the mathematical context in which Riemann worked.⁷⁸ I will then proceed to look into Riemann's 1851 *Inauguraldissertation* and his 1857 paper on Abelian functions, so as to expose the core ideas that will be studied by Dedekind and Weber in 1882. The account will be partial for its goal is to present the concepts and results which Dedekind and Weber are going to work on, in their 1882 paper, in which they do not develop any topological notion. This section will close with a brief (and, again, very partial) look at the first responses to Riemann's works. Indeed, Dedekind and Weber's paper is greatly motivated by their dissatisfaction with the reception of Riemann's works, between Riemann's paper and the late 1870s.

Before considering Riemann's works, let me say a few words about algebraic functions and the beginning of complex analysis.⁷⁹

Algebraic functions⁸⁰ are functions which are solutions of a polynomial equation: f is an algebraic function of x_1, x_2, x_3, \dots (x_i are real or complex variables) if it satisfies a polynomial equation with polynomial coefficients $F(f, x_1, x_2, x_3, \dots) = 0$ with

$$F(f, x_1, x_2, x_3, \dots) = a_0 f^n + a_1 f^{n-1} + \dots + a_{n-1} f + a_n$$

in which the coefficients a_0, a_1, \dots, a_n are polynomials in x_1, x_2, x_3, \dots with rational coefficients and without common divisors. We will only consider algebraic functions of one variable, that is, a function f of z satisfying the polynomial equation $F(f, z) = 0$, with

$$F(f, z) = a_0 f^n + a_1 f^{n-1} + \dots + a_{n-1} f + a_n$$

(a_0, a_1, \dots, a_n are polynomials in z with rational coefficients and no common divisors).

Algebraic functions are mainly of interest to mathematicians in the 19th century because of their integrals, namely for the study of integrals of the form

$$F(w) = \int_{z_0}^w f(s, z) dz$$

⁷⁸For Dedekind's number theoretical works, the next chapters will give all the necessary details.

⁷⁹On the concept of function, whose extension is an essential point for Riemannian function theory, one can refer to [Chorlay, Forthcoming], [Lützen, 2003b], [Lützen, 2003a], [Youschkevitch, 1976]. For more detailed accounts on the episodes in the history of analysis mentioned here, one can refer to [Bottazzini, 1986], and [Bottazzini and Gray, 2013].

⁸⁰The reader can refer to [Houzel, 2002] and [Bottazzini and Gray, 2013] for more details. I rely on their works in this section.

where s satisfies a polynomial equation $P(s, z) = 0$. While these integrals are not too difficult to handle for polynomials of degree 1 or 2 because the integration by elementary functions (rational functions, logarithms, arcsin, arctan) is possible, this strategy stops working for polynomials of degree 3 and higher. This leads to the introduction of new functions, whose study constitutes one of the central themes of 19th century Analysis.

The interest for integral of algebraic functions originated in the study of elliptic functions. The root of these researches was Wallis's attempt to find the length of an arc of the ellipse. The arc-element of an ellipse of half axes a and b is given by

$$ds = \sqrt{(dx^2 + dy^2)} = \sqrt{\frac{a^2 - ex^2}{a^2 - x^2}} dx$$

with $e = 1 - \frac{b^2}{a^2}$. But this is not the differential of an "elementary function", and the "rectification" raises problems. Mathematicians such as Newton, Euler, MacLaurin or Lambert attempted to solve the problem only to have to be satisfied with power or trigonometric series.

Leibniz and the Bernoulli brothers encountered the same kind of problems studying the "paracentric isochrone", for which Jakob and Johann Bernoulli used the lemniscate whose arc-element is

$$ds = \frac{dx}{\sqrt{1 - x^4}}$$

The general theory of elliptic integrals is a particular case of integration of algebraic functions, and deals with integrals of the form

$$E(z) = \int_{z_0}^z \frac{ds}{\sqrt{s^2}}$$

with s^2 polynomial of degree 3 or 4 (for example, in the case of the lemniscate, $s^2 = 1 - x^4$). Fagnano, Euler, Lagrange, Legendre, among others, worked on these questions. Notably Legendre established that elliptic integrals (for a real variable) can all be reduced to three canonical forms:

$$\text{Integrals of the first kind: } \int \frac{dx}{(1 - x^2)(1 - k^2x^2)}$$

$$\text{Integrals of the second kind: } \int \sqrt{\frac{1 - k^2x^2}{1 - x^2}} dx$$

$$\text{Integrals of the third kind: } \int \frac{dx}{(1 - nx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

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Riemann will continue on the same line of idea and reformulate Legendre's classification in his own terms – and so will Dedekind and Weber in 1882.

Cauchy's works were decisive for complex analysis (see [Bottazzini and Gray, 2013], Chapter 2). In his *Cours d'Analyse*, Cauchy insisted, and he was the first to do so, on the fact that real and complex variables (“imaginary quantities”) are to be treated differently and that a function of a complex variable is a completely different function:

When the constants or variables contained in a given function, having been considered real are later supposed to be imaginary, the notation that was used to express the function cannot be retained in the calculation except by virtue of new conventions able to determine the sense of this notation under the new hypothesis.⁸¹ ([Cauchy, 1821], 159)

In several memoirs on the integral calculus, Cauchy studied integrals on complex paths, which he defined analogously to real ones. Cauchy's theory of complex paths will be crucial, notably through Puiseux's works (see [Bottazzini and Gray, 2013], 192-198). Before Riemann, Puiseux developed some of Cauchy's ideas in complex analysis. He stressed the fact that the elliptic integral does not depend solely on the boundaries but on the whole path of integration. Puiseux's goal was to study the behavior of algebraic functions, and for this, he proposed a geometrical setting to investigate algebraic functions and their integrals (see [Bottazzini and Gray, 2013], 192). Puiseux used Cauchy's results in complex analysis to provide what Bottazzini and Gray consider to be “a coherent theory of algebraic functions and their integrals that provided a geometric setting which prefigured Riemann's work” (*ibid.*, 198).⁸²

Riemann knew well Cauchy's works on about integrals on complex paths⁸³ and he did know Puiseux's works. It has been argued (notably by Neuenschwander) that Riemann's 1851 dissertation was “an excellent geometric commentary and at the same time what appears to be a profound summary of Puiseux's work” ([Bottazzini and Gray, 2013], 267). Bottazzini and Gray, however, disagree on that point. For them, as for many commentators, Riemann's theory goes well beyond Puiseux's memoir. Besides, while Puiseux's geometrical constructions were merely tools, the Riemann surface is more

⁸¹ “Lorsque les constantes ou variables comprises dans une fonction donnée, après avoir été considérées comme réelles, sont ensuite supposées imaginaires, la notation à l'aide de laquelle on exprimait la fonction dont il s'agit ne peut être conservée dans le calcul qu'en vertu de conventions nouvelles propres à fixer le sens de cette notation dans la dernière hypothèse.”

⁸² [Bottazzini and Gray, 2013] Chapter 3 provides a thorough account of Cauchy's works on complex analysis which I will not develop here.

⁸³ See [Neuenschwander, 1981].

than a tool or an image. Riemann surfaces, as it has been underlined by Weyl,

are not merely a device for visualizing the many-valuedness of analytic functions, but rather an indispensable essential component of the theory; not a supplement, more or less artificially distilled from the functions, but their native land, the only oil in which the functions grow and thrive. ([Weyl, 2013], vii)

Finally, and more importantly, in the following section we will see that Riemann's work does not aim at a study of the behavior of complex algebraic functions, it has a wider, more general goal, as Bottazzini and Gray suggest:

he was searching for a general method for embracing and treating large classes of functions in a unified manner and precisely this search led him to his concept of a "Riemann" surface. ([Bottazzini and Gray, 2013], 267)

1.3.1 Outlines of Riemann's function theory

Riemann exposed his function theory in two main works. First, he presented the core ideas in his doctoral dissertation, "*Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*", defended in 1851, in Göttingen. Six years after his doctoral dissertation and three years after his *Habilitation*, Riemann published his function theory in a somewhat reworked form, with an application to Abelian functions.⁸⁴ In the following, I will expose the essential ideas of Riemann's theory using both works.

Riemann's theory of functions draws an intimate link between analytical and geometrical conceptions of functions. The concept of surface allows to understand the multi-valuedness of algebraic functions by giving it a geometrical meaning. Rather than defining a function in the complex (Argand-Cauchy) plane, Riemann proposes to consider a surface with 'piled' sheets, each corresponding to a branch of the multi-valued function. Surfaces and functions are indissociable, for Riemann's theory. Indeed, it is the surface which allows to determine the values of the functions living on the surface without resorting to explicit representations "by operations on quantities", such as power series. Let me now give some elements of Riemann surfaces as presented in [Riemann, 1851] and [Riemann, 1857], so as to have a better idea of the works Dedekind and Weber were developing. In addition,

⁸⁴Some of the further developments about Riemann surfaces given in this paper are important for Dedekind and Weber's rewriting of Riemann's works, in particular what we now call Riemann's inequality. I do not claim to have anything new to write about Riemann and will not go into much details.

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Dedekind and Weber claim to be more faithful to the principles of Riemann than their predecessors. We will see that while they do follow closely the demand to avoid computations and to provide a uniform approach for classes of functions, they do erase the geometrical component of Riemann's concept of a Riemann surface.⁸⁵

1.3.1.1 Riemann surfaces

There was a significant widening of the function concept in the 19th century, with the adoption of the concept in which the analytical expression does not play a definitional role anymore. This move is usually attributed to Dirichlet in 1829 and 1837, as we saw earlier. The viewpoint adopted by Dirichlet, thus, is that of an "arbitrary function", a concept of function completely detached from any consideration of explicit, individual, expression (*Darstellungsformen*) of the functions. Functions (which, here, are always continuous) appear as a process of attributing finite values to an indeterminate variable quantity according to certain (undefined) rules.⁸⁶

Riemann's doctoral dissertation begins with the distinction between "function" and "expression" which arose from the desire to consider "arbitrary" functions. Riemann's definition of function is of a great generality and close to Dirichlet's:

If one designates by z a variable magnitude which can take gradually all the possible real values, then when to each value corresponds a unique value of the indeterminate magnitude w , one calls w a function of z . And if, while z continuously runs through all the values between two fixed values, w changes continuously as well, then this function is called continuous [*stetig oder continuirlich*] in this interval.⁸⁷ ([Riemann, 1851], 3)

⁸⁵My presentation will, here again, rely on [Bottazzini and Gray, 2013], as well as on the first Chapter of [Chorlay, 2007].

⁸⁶"We mean by a and b two constant values and by x a variable quantity which must gradually take all the values between a and b . Now, for every x corresponds a single, finite y , and such that while x continuously runs through the interval between a and b , $y = f(x)$ likewise gradually changes. Then, y is called a continuous function of x for this interval. It is at the same time not necessary that y in this entire interval be dependent of x with always the same law, so one does not need once to think of a dependency expressible by the mathematical operations. ([Lejeune-Dirichlet, 1837], in [Lejeune-Dirichlet, 1889-1897] I, 135)

⁸⁷"Denkt man sich unter z eine veränderliche Grosse, welche nach und nach alle möglichen reellen Werthe annehmen kann; so wird, wenn jedem ihrer Werthe ein einziger Werth der unbestimmten Grosse w entspricht, w eine Function von z genannt; und wenn, während z alle zwischen zwei festen Werthen gelegenen Werthe stetig durchläuft, w ebenfalls stetig sich ändert, so heisst diese Function innerhalb dieses Intervalls stetig oder continuirlich."

Riemann explicitly states that it is “indifferent” to consider the dependence between z and w to be arbitrary or given by a certain determinate calculation. He also clearly states that he considers the appropriate definition of function to *not* be “an expression that yields its value for each value of the argument”.

A complex function, for Riemann, is defined as a mapping of the complex variable $z = x + iy$ into the complex variable $w = u + iv$.

Our study shows that, because of the general nature of a function of a complex variable, a part of the determination through a definition of this kind yields the rest. Indeed, we reduce this part of the determination to that which is necessary for complete determination of the function.⁸⁸ (*ibid.*, 38, translated in [Bottazzini and Gray, 2013], 273-274)

Riemann uses a definition of continuity which is rather imprecise, but remaining notes studied by Neuenschwander tell us that he drew it from the notion of continuity given by Cauchy and used by Dirichlet. A complex function satisfies what we now call the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

From the Cauchy-Riemann equations follows that the functions are harmonic.⁸⁹ The Cauchy-Riemann equations are necessary for the functions to be holomorphic (i.e., differentiable) but not, as Riemann seemed to have assumed, sufficient. Note that Riemann does not use the word “holomorphic” or even “differentiable”. He only talks about a “function of z ” with $z = x + iy$, as opposed to the complex function conceived as a function of two real variables (see [Chorlay, 2007], Chapter 1). Riemann later takes the Cauchy-Riemann equations as a definitional property of complex functions.

Riemann brings to the fore the idea of a function as a correlation between two variables. To study complex algebraic functions (which are often many-valued), Riemann introduces a new notion, that of a surface spread over the complex plane. The idea is to think of the complex plane, the “ (x, y) -plane”, as being covered by a surface spread over it “(or lying at an infinitesimal distance above it) for as far as the function is defined”. As he explains in his doctoral dissertation, this choice is motivated by the possibility to consider a multi-sheeted surface:

⁸⁸ “*durch unsere Untersuchung ist gezeigt, dass, in Folge des allgemeinen Charakters einer Function einer veränderlichen complexen Grosse, in einer Definition dieser Art ein Theil der Bestimmungsstücke eine Folge der übrigen ist, und zwar ist der Umfang der Bestimmungsstücke auf die zur Bestimmung nothwendigen zurückgeführt worden.*”

⁸⁹I.e., they satisfy $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

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We leave open the possibility that the locus of the point O can stretch out over the same part of the plane several times; in such case though, we suppose that the parts of the surface lying on each other are not linked up alongside a line, so that it does not happen that the surface is folded or that it is split in the superposed parts.⁹⁰ ([Riemann, 1851], 7)

This, for Riemann, enables to understand and grasp correctly the said correlation and give a geometrical representation of multi-valued functions.

The core idea of the surface is an intrinsic relation between the function and the surface itself, which embodies the functional correspondence. The key element is what Riemann calls the branch points (or ramification points or winding points, “*Windungspunkte*”), which are multiple points on the surface, where two or more sheets are joined. In 1857, Riemann explains that

For a more comfortable designation of these relations, the different continuations of a function for the same part of the z -plane will be named the *branches* (*Zweige*) of this function, and a point, on which a branch of a function is continued into another one will be called a branch-point (*Verzweigungsstelle*) of this function. Where no ramification occurs, the function is said to be *single-valued* or *monodrom*.⁹¹ ([Riemann, 1857], 83)

To explain how the branches are joined in the neighborhood of a branch point σ , Riemann proposes to consider a mobile point which goes along the boundary of the surface (multi-sheeted) T , then

[t]he mobile point around σ comes back after m circuits in the same patch of surface, and is limited to m of the superposed patches of surface which assemble in one unique point via σ . We call this point a branch point of $m - 1$ -th order of the surface T .⁹² (*ibid.*, 8)

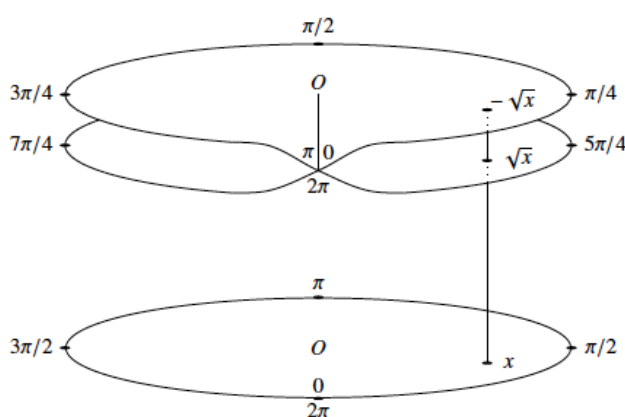
⁹⁰ “Wir lassen die Möglichkeit offen, dass der Ort des Punktes O über denselben Theile der Ebene sich mehrfach erstrecke, setzen jedoch für einen solchen Fall voraus, dass die auf einander liegenden Flächentheile nicht längs einer Linie zusammenhängen, so dass eine Umfaltung der Fläche, oder eine Spaltung in auf einander liegende Theile nicht vorkommt.”

⁹¹ “Zur bequemeren Bezeichnung dieser Verhältnisse sollen die verschiedenen Fortsetzungen einer Function für denselben Theil der z -Ebene *Zweige* dieser Function genannt werden und ein Punkt, um welchen sich ein *Zweig* einer Function in einen andern fortsetzt eine *Verzweigungsstelle* dieser Function; wo keine *Verzweigung* stattfindet, heisst die Function einädrig oder *monodrom*.”

⁹² “Der um σ sich bewegendende Punkt kommt alsdann nach je m Umläufen in denselben Flächentheil zurück und ist auf m der auf einander liegenden Flächentheile eingeschränkt, welche sich über σ zu einem einzigen Punkte vereinigen. Wir nennen diesen Punkt einen *Windungspunkt* $m - 1$ ster Ordnung der Fläche T .”

The mobile point describes a path (*Umlauf*) around the ramification point σ and permutes the sheets of the surfaces. After a certain number of such “circuits”, the function goes back to its initial value. When this happens, the sheets are ‘permuted’, the “topmost sheet passes back through the others to join the lowest” ([Riemann, 1857], quoted and translated in [Koch, 1991], 123). If the permutation in question is a cycle of order m , then the branch point is said to be of order $m - 1$.

For example, for the square root function, the Riemann surface allows to figure and understand clearly the multiplicity of the function and the branch point is 0, and is of order 1.⁹³



The Riemann surface for the square root function.

Thus, for a complete characterization of the surface, or even a class of surfaces, Riemann considers the “boundary” and the ramification of T :

When the position and the sense of the boundary of T and the position of its ramification points are given, then T is either completely determined, or limited to a finite number of different forms. This last point comes from the fact that these characteristic elements can correspond to different superposed parts of surface.⁹⁴ ([Riemann, 1851], 8)

The essential point, here, is that the surface T determines completely a function (or a class of functions). The surface is not merely a tool to represent the multi-valued functions, the functional correspondence of the complex variables determines the inner nature of the surface and conversely:

⁹³I borrow this image to [Stillwell, 2010], 303.

⁹⁴“Wenn die Lage und der Sinn der Begrenzung von T und die Lage ihrer Windungspunkte gegeben ist, so ist T entweder vollkommen bestimmt oder doch auf eine endliche Anzahl verschiedener Gestalten beschränkt; Letzteres, in so fern sich diese Bestimmungsstücke auf verschiedene der auf einander liegenden Flächentheile beziehen können.”

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A variable magnitude which for each point of the surface T generally, that is without excluding the exception of some singular lines or points, takes a determinate value varying continuously with the position of this point, can obviously be considered as a function of x, y wherever, in the following, we will speak about a function of x, y , we will define the concept in this way.⁹⁵ (*ibid.*, 9)

The surface allows to represent the ramification of a multi-valued function and the intimate links between one and the other are made obvious. In fact, the multi-sheeted surfaces allow to represent and understand the multi-valued functions in a way that reconquer a certain form of uniformity: the multiplicity of the values is replaced by the multiplicity of the sheets (for each branch).

Hence, the surface allows to provide the determination of the values of functions (for each value of the argument) independently of a representation or determination by operations on quantities. Once this determination obtained, the explicit (analytic) expression of the function can be looked for:

A theory of these functions based on the principles introduced here would fix the form of the function (that is, its value for every value of the argument) independently of a means of determining the function via operations on quantities. We determine the function by appending to the general idea of a function of a complex variable just those features that are necessary for the determination. Only at this stage do we pass to the various representations that the function permits.⁹⁶ (*ibid.*, 38-39, translated in [Bottazzini and Gray, 2013], 273-274)

Riemann is thus clearly putting into action the precept according to which one should ground the definition of and researches about a concept on essential characteristics, rather than *Darstellungsformen*. And in particular, an algebraic function will be characterized by its surface (rather than by an explicit expression such as a power series).

⁹⁵ "Eine veränderliche Grösse, die für jeden Punkt der Fläche T , allgemein zu reden; d. h. ohne eine Ausnahme in einzelnen Linien und Punkten auszuschliessen, Einen bestimmten mit der Lage desselben stetig sich ändernden Werth annimmt, kann offenbar als eine Function von x, y angesehen werden, und überall, wo in der Folge von Functionen von x, y die Rede sein wird; werden wir den Begriff derselben auf diese Art festlegen."

⁹⁶ "Eine Theorie dieser Functionen auf den hier gelieferten Grundlagen würde die Gestaltung der Function (d. h. ihren Werth für jeden Werth ihres Arguments) unabhängig von einer Bestimmungsweise der selben durch Grossoperationen festlegen, indem zu dem allgemeinen Begriffe einer Function einer veränderlichen complexen Grosse nur die zur Bestimmung der Function nothwendigen Merkmale hinzugefügt würden, und dann erst zu den verschiedenen Ausdrücken deren die Function fähig ist übergehen."

Moreover, rather than providing a characterization for *one* function, Riemann wants (and succeeds) to characterize classes of functions possessing the same essential properties. Indeed, as underlined by Bottazzini and Gray, Riemann's aim was not just to study multi-valued functions but "he was searching for a general method for embracing and treating large classes of functions in a unified manner and precisely this search led him to his concept of a 'Riemann' surface." ([Bottazzini and Gray, 2013], 267). For algebraic functions, then, as Bottazzini and Gray explain:

If the domain of the variable z covers the entire infinite plane simply or multiply and the function is allowed to be discontinuous only at a finite number of isolated points (where it becomes infinite to finite degree, i.e., it has poles, as we would say today), then the function is algebraic, and reciprocally every algebraic function satisfies these conditions. (*ibid.*, 271)

Riemann observes that these results are valid on *any* surface and that the variable needs not to run through the whole complex plane. That is, a complex function could be defined on any two-dimensional domain, which is a crucial move for Riemann's function theory.

For Riemann, the gains allowed by the concept of surface is considerable, compared to an analytic and formal definition and study of functions, for example:

to show the equality of two expressions for the same function, one formerly needed to transform one into the other: that is, show that the expressions coincide for every value of the variable. Now it suffices to prove the expressions coincide in a much more restricted domain.⁹⁷ (*ibid.*, 38, quoted and transl. in [Bottazzini and Gray, 2013], 273-275)

Finally, one should note that Riemann's theory relies essentially on a principle used by Gauss and Dirichlet in potential theory, which Riemann called the "Dirichlet principle". Basically, the idea is that of the existence of a function minimizing a surface integral (see [Bottazzini and Gray, 2013], 271-275). Riemann uses it in the proofs of existence and unicity that form the foundation of his function theory. But the "Dirichlet principle" was not commonly accepted by the mathematical community and "widely regarded as of heuristic value only and (...) this part of the paper was to occupy mathematicians for the next 70 years." ([Bottazzini and Gray, 2013], 271).

⁹⁷ "Um z. B. die Gleichheit zweier Ausdrücke derselben Function zu beweisen, musste man sonst den einen in den andern transformiren, d. h. zeigen, dass beide für jeden Werth der veränderlichen Grösse übereinstimmen; jetzt genügt der Nachweis ihrer Uebereinstimmung in einem weit geringern Umfange."

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While essential for Riemann’s theory, the “Dirichlet principle” is not considered at all by Dedekind and Weber, I will thus not develop it further, here. One can refer to [Bottazzini and Gray, 2013], 271-275.

1.3.1.2 A note on Riemann’s inequality

An important notion introduced by Riemann is that of boundary cuts (or cross-cuts, *Querschnitte*) and the connectivity of the surface related to it. Boundary cuts correspond to the idea of cutting or breaking a surface into simpler patches. The boundary cuts themselves are simple curves, not intersecting themselves, which connect two points on the boundary of the surface. A surface is said to be connected if any two points can be connected by a continuous curve, and if consequently any boundary cut would split the surface in two. A simply connected surface is a connected surface which, roughly speaking, does not have any ‘hole’⁹⁸. Such a surface is equivalent (one-to-one) to the plane disc.

Riemann, using boundary cuts, transforms a surface into a simply connected surface on which the functions admit a unique integral. This allows him to define the “order of connectivity” (*Zusammenhangszahl*): if by the system of n_1 boundary cuts, the surface is split into m_1 patches, and if by the system of n_2 boundary cuts, it is split into m_2 patches, then $n_1 - m_1 = n_2 - m_2$. In general, $n - m$ is a constant, the “order of connectivity”. This number is independent of the way in which the boundary cuts have been made, and is consequently a characteristic property of the surface ([Bottazzini and Gray, 2013], 263).

Riemann claimed that for the surface of an algebraic function of equation $F(x, y) = 0$, the order of connectivity is of the form $2p - 1$ (this integer p will be called the genus by Clebsch).⁹⁹

This integer p is notably involved in the first important result of the 1857 paper on Abelian function, which we now call the Riemann inequality, Riemann’s part of the Riemann-Roch theorem.

To prove Riemann’s inequality, Riemann starts with an irreducible equation in s of degree n , whose coefficient are polynomials in z of degree m . This equation is associated to a n -sheeted surface T spread over the complex plane. This surface is “without boundary”, which means that it can be considered as a surface whose boundary is “rejected to infinity” (infinitely far away) or as a closed surface. Moreover

⁹⁸That is, a connected surface of genus 0.

⁹⁹The genus of the surface in topology corresponds to the (maximum) number of non-intersecting closed curves that can be drawn on the surface without rendering it disconnected. In other words, it is the number of holes in the surface.

any rational function of s and z is obviously also a single-valued function of the locus the surface T and therefore possesses the same mode of ramification as the function s , and we will see later that the converse is also true.¹⁰⁰ ([Riemann, 1857], 95-96)

The integration of such a rational function gives multi-valued functions which differ only by constants. Riemann proposes, then, to study the classes of “equivalently branching algebraic functions and their integrals” by studying their discontinuities using the Dirichlet principle, rather than using “expressions of these functions” (see [Bottazzini and Gray, 2013], 288).

Riemann gives indications on the singularities and how to use coordinates to define the order of vanishing of a function. At the points corresponding to the zeros of the function, it is said to be “infinitely small”. It is the case, for example of $(z - a)^{1/\mu}$ when $z = a$ or of $\frac{1}{z}^{1/\mu}$ for $z = \infty$. At the points corresponding to poles, the function is said to become infinitely great. Riemann, here, only considers simple zeros (“infinitely small of the first order”) and proposes, when the function is infinitely small (or infinitely great) of order ν , to consider it as being infinitely small (or infinitely great) of first order at ν coinciding points. Dedekind and Weber will use a similar terminology.

Considering a surface T , Riemann turns it into a simply connected surface T' by $2p$ boundary cuts. Then, following Legendre’s classification of integrals (see p. 88), Riemann establishes three kinds of integrals:

- The integrals of first kind are the (multi-valued) functions w which are everywhere finite and continuous. There are at most p linearly independent such functions. They are the result of the integration of functions which are everywhere holomorphic.
- The integrals of second kind are the (multi-valued) functions w which become infinite on only one single point ϵ of T . They are the result of the integration of rational functions without simple infinities.
- The integrals of third kind are the (multi-valued) functions w which have two (opposite) logarithmic infinities on two points ϵ_1, ϵ_2 of T . They are the result of the integration of functions with simple infinities which have opposite residues. ([Bottazzini and Gray, 2013], 288-289)

Following this, Riemann goes on to prove what is called nowadays the Riemann inequality, which G. Roch was able to complete into an equality

¹⁰⁰ “Jede rationale Function von s und z ist offenbar ebenfalls eine einwerthige Function des Orts in der Fläche T und besitzt also dieselbe Verzweigungsart wie die Function s , und es wird sich unten ergeben, dass auch das Umgekehrte gilt.”

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in 1864 (but for which he only gave a partial proof). First, Riemann gives the “general expression of a function which becomes infinitely great of first order at m points $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ of the surface T ,” which is

$$s = \beta_1 t_1 + \beta_2 t_2 + \dots + \beta_m t_m + \alpha_1 \omega_1 + \alpha_2 \omega_2 + \dots + \alpha_p \omega_p + k$$

where the t_i are arbitrary functions, and k , α_i and β_i are constants. This function s has $2p$ moduli of periodicity dependent on the $m + p$ constant α_i and β_i .

For $m \geq p + 1$, $2p$ of the α_i and β_i can be determined as linear functions of the other ones in such a way that the moduli of periodicity vanish. Thus, one obtains a function s depending linearly on $m - p + 1$ constants (which amounts to it being the sum of $m - p$ functions which have each $p + 1$ infinities). This result is called nowadays Riemann’s inequality. Riemann – and Roch as well – referred to it as the number of arbitrary constants contained in the function s (that is, the number of parameters of which the function is dependent).¹⁰¹

Before moving on to a few considerations about the reception of Riemann’s works, let me mention one last (for us) important result of Riemann’s 1857 paper, namely the fact that the functions with a determinate number of singularities can be written as the quotient of two polynomial functions in s and z (which will later lead to birational geometry). Consider s' the functions “ramified like” the surface T and m' the number of their simply infinite branch point, then

if it is possible, as we will show later, to form rational expressions in s and z which become infinite of first order for m' randomly given pairs s and z , satisfying the equation $F = 0$ and which are linear functions of $m' - p + 1$ arbitrary constant, then any function s' can be represented by these expressions.¹⁰² (*ibid.*, 107)

The functions s' can be expressed, then, as rational functions on the surface. This result is one the bases for later readings of Riemann’s theory which will propose to transfer the concepts in an algebraico-geometric framework.

The second half of Riemann’s 1857 paper is dedicated to the study of Abelian integrals, which Riemann calls Abelian functions. The study of

¹⁰¹This result is a lower bound for what we would call the dimension of linear spaces of functions having m singular points on T ([Bottazzini and Gray, 2013], 289).

¹⁰²“Lassen sich also, wie jetzt gezeigt werden soll, rationale Ausdrücke von s und z bilden, die für m' beliebig gegebene, der Gleichung $F = 0$ genügende Werthenpaare von s und z unendlich von der ersten Ordnung werden und lineare Functionen von $m' - p + 1$ willkürlichen Constanten sind, so kann durch diese Ausdrücke jede Function s' dargestellt werden.”

Abelian functions is linked to what we would call topological considerations, which Dedekind and Weber refer to as “investigation of continuity and the related questions”. This is out of the scope of my inquiry, since Dedekind and Weber did not include these questions in their paper – in fact, they didn’t have the tools for it.¹⁰³

1.3.2 Receptions and developments of Riemann’s works

Riemann’s inaugural dissertation was only printed as a separate thesis and never as a research paper (e.g. in Crelle’s Journal), which didn’t help an early adoption of Riemann’s ideas. More significantly, the ideas proposed here are mathematically relatively vague “at certain crucial points and its often murky language”, which opposed an immediate acceptance of the many novelties of the work, according to Bottazzini and Gray ([Bottazzini and Gray, 2013], 277). If the response to Riemann’s works took a few years, and has often been presented as sparse, Bottazzini and Gray draw up a list of works following Riemann which, from the 1860s to the beginning of the 1880s, counts as much as 24 items, from Clebsch and Roch to Christoffel, Kraus and Dedekind and Weber.¹⁰⁴ This suggests, for Bottazzini and Gray, in the light of the young age at which Riemann died and the few students he had, that the reception of Riemann’s works was rather important. In fact, if some of his ideas were difficult to develop due to their novelty and to a certain “mathematical vagueness”, his works were recognized immediately as valuable by the mathematical community – suffice it to mention that Weierstrass himself proposed his nomination at the Berlin Akademie der Wissenschaften in 1859.

The density and richness of Riemann’s works in complex function theory leads to a multifaceted reception of his works. Indeed, many aspects were to be considered problematic and/or fruitful, and several different readings and adaptations of his ideas emerged during the following decades. Not only were Riemann’s works vast and fruitful, each point considered will raise problems. The amount of problems to investigate and of possible directions that could be taken after Riemann’s “visionary presentation” of his function theory is very wide, as was already alluded to above and is well illustrated by the variety of authors studied in works such as [Chorlay, 2007], [Houzel, 2002] or [Bottazzini and Gray, 2013].

My intention in outlining the reception of Riemann’s ideas is to give an

¹⁰³For more details, one can see [Chorlay, 2007] Part I, [Bottazzini and Gray, 2013] which provides a large amount of historical informations about complex analysis, [Houzel, 2002] for the point of view of the history of algebraic geometry, and [Scholz, 1999] for the history of topology.

¹⁰⁴Their list concerns the responses to all but three of Riemann’s works (the 1855 paper on Bessel’s equation, the 1859 paper on the prime number theorem and the 1860 paper about shock waves).

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idea of what dissatisfied Dedekind and Weber in the treatment of the theory of algebraic functions on a Riemann surface. Indeed, it is clear from the introduction of Dedekind and Weber's paper, that from their view, Riemann's precepts – to base the investigation on characteristic properties rather than computations, to consider classes of functions rather than individual functions, etc. – were not followed by the mathematicians who continued his works. Dedekind and Weber do not give any name, of course, and it is difficult to point exactly with which works they are most dissatisfied. I will present two different possibilities. It should be borne in mind that Dedekind and Weber are also opposed to any use of geometrical intuition. I will concentrate on aspects related to the uniformity of the approach and the desire to avoid external representations, in relation with a more precise characterization of the “conceptual approach”.

A first example is Weierstrass's reception of and response to Riemann's works. Strongly opposed to Riemann's “conceptual” approach, Weierstrass did recognize the value of Riemann's works despite disapproving their patent lack of rigor. Not only did Weierstrass propose Riemann to the Berlin Akademie der Wissenschaften in 1859, but as I mentioned, after the publication of Riemann's paper on Abelian functions, he withdrew one of his own papers on the same subject, to be sure that he had fully understood everything Riemann wrote. In addition, Weierstrass strongly encouraged his students to work on certain aspects of Riemann's works “and re-derive Riemann's results by better (i.e., Weierstrassian) methods” ([Bottazzini and Gray, 2013], 11). Weierstrass (and to a certain extent, his students) imported some of Riemann's ideas into the Berlin algorithmic tradition, inserting them in his own mathematical and methodological framework, adapting them to his own viewpoint and methods to the point that “the large audiences drawn to Berlin would not necessarily know when he was entering Riemannian territory” (*ibid.*, 322). The ‘conceptual’ Riemannian function theory entered the Berlin community through the rework of his ideas by Weierstrass. In fact, according to Bottazzini and Gray, “within Germany the response was increasingly along lines inimical to the deepest tendencies in Riemann's thought.” The increasing tendency of Weierstrass's function theory to stress the resort to power series led to a “fully developed algebraic approach to complex analysis [which] obliterates Riemann's – at the price, a later generation came to feel, of making it impossible to think for oneself” (*ibid.*, 322).

Another reception of Riemann's theory is the algebraic approach adopted by Rudolf Clebsch, who gave the concept of “genus” (the number designated as p by Riemann) its name. Clebsch was a student of Jacobi's in Königsberg and succeeded Riemann in Göttingen, in 1866. Upon arriving in Göttingen, Clebsch started to study Riemann's works and developed an approach that

can be designated as algebraic geometry.

Clebsch's taking up of Riemann's ideas led to great developments in algebraic geometry, after the works of mathematicians such as Plücker, Steiner, Hesse, or Salmon. Clebsch and his students, notably Max Noether, aimed at a translation and use of Riemann's ideas into projective algebraic geometry. This did not please Prym, the only one left of Riemann's students after Roch's death, who wrote that "the attempt to base function theory on algebra is completely useless" (quoted in [Bottazzini and Gray, 2013], 317).

Clebsch, along with Noether, Brill and Gordan, considered the generality inherent to Riemann's "conceptual" approach to be misleading to say the least:

At such a level of generality the idea of a function, incomprehensible and evaporating, no longer delivers reliable conclusions. In order to delineate precisely the domain of validity of the theorems under consideration, people have recently abandoned Riemann's path completely.¹⁰⁵ ([Brill and Noether, 1892], 265, transl. in [Bottazzini and Gray, 2013], 322)

They preferred to approach the problems through the equations themselves, and they rewrote every result and every proof in Riemann's paper in what they thought was an appropriate form. Less than a question of appropriation (they gave credit to Riemann for all his results) as it seems to have been the case with Weierstrass, their reformulation of Riemann's ideas was motivated by the idea that algebra held the key to an innovative and rigorous approach to the study of (plane) curves. But the focus on algebraic equations leads to formulate a theory of plane curves attached to definite equations. In this way, it is in the equations that the characteristic properties, such as genus, singularities, etc., are to be determined. This point is one that Riemannian mathematics would try to avoid.¹⁰⁶

Clebsch wrote, with Paul Gordan, a book on Abelian functions ([Clebsch and Gordan, 1866]) whose aim was to be so thorough that it could be seen as definitive. They presented Riemann's approach to function theory as containing difficulties which for them "are mostly responsible for bewildering the researcher". The generality looked for by Riemann was, for them, far from being a benefit of the approach – precisely the point on which Dedekind

¹⁰⁵ *"In solcher Allgemeinheit lässt der Functionsbegriff, unfassbar und sich verflüchtigend, controlirbare Schlüsse nicht mehr zu. Um den Gültigkeitsbereich der aufgestellten Sätze genau zu umgrenzen, hat man neuerdings den von Riemann betretenen Weg ganz verlassen."*

¹⁰⁶ Of course, the algebraic approach did yield interesting results and technical improvements, such as the definition of the genus by Clebsch. However, Clebsch, who first accepted Riemann's definition of the genus, later completely moved away from considerations about boundary cuts. The definition of genus was thus given exclusively in terms of the degree of the defining equation of the curve and the number of its double points and cusps. See [Bottazzini and Gray, 2013], 336.

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and Weber will strongly disagree.

1.3.2.1 A note about the Riemann-Roch theorem

The Riemann inequality, which was given earlier, exhibits a lower bound for the “number of arbitrary constants contained in a function s which only becomes infinite of the first order for m points of the surface T and stays continuous otherwise”. This result was refined by Riemann’s student, Gustav Roch,¹⁰⁷ who proved the equality

$$\mu = m - p + q + 1$$

where q is the number of linearly independent Abelian differentials of the first kind, with zeroes at the m given points.

Roch stated that

If a function s' is infinitely great of the first order at m points, and there can be, on these m points, q vanishing functions

$$\frac{\varphi(s, z)}{\frac{\partial F}{\partial s}}$$

between which there exist no linear equations with constant coefficients, then the function s' contains $m - p + q + 1$ arbitrary constants.¹⁰⁸ ([Roch, 1865], 375)

in which $F(s, z)$ defines the algebraic function s associated to the surface T studied; s' is an algebraic function of s rational in z and s , which has the same mode of ramification as s and can be written as the quotient of two functions φ .

Roch, who died in 1866 at age 26, was Riemann’s student in Göttingen, got his doctorate degree in Leipzig and his *Habilitation* in Halle, and he was one of the few people to accept and use the Dirichlet principle (see [Bottazzini and Gray, 2013], 315-317). The use of the Dirichlet principle in his proof of the Riemann-Roch theorem made his demonstration incomplete and his paper difficult for his contemporaries. Several attempts to prove the Riemann-Roch theorem were made during the following years, but Dedekind

¹⁰⁷For more details on the Riemann-Roch theorem, one can refer to the historical studies mentioned above, and to [Bottazzini and Gray, 2013], 337-339, [Gray, 1987], [Gray, 1997], [Gray, 1998].

¹⁰⁸“Wird eine Function s' in m Punkten unendlich gross erster Ordnung und können in diesen m Punkten q Functionen $\frac{\varphi(s, z)}{\frac{\partial F}{\partial s}}$ verschwinden, zwischen denen keine lineare Gleichung mit constanten Coefficienten besteht, so enthält s' die Zahl $m - p + q + 1$ willkürlicher Constanten.”

and Weber were the first to be able to provide an actually complete and rigorous proof (see Sec. 2.5.6).

Clebsch, in his book with Gordan, attempted an algebraic reformulation of the result but only for a very restricted class of functions. After Clebsch's premature death, Brill and Noether continued the algebraization of Riemann's function theory. From their viewpoint, Clebsch and Gordan did not go far enough in the algebraization, since

For them algebra was the source of rigour, and moreover, in Brill's opinion Riemann's work on the Riemann-Roch Theorem was in a form foreign to geometry. This was a sound, critical response, but the price was high: the very definition of genus became entangled with the nature of the singular points a plane curve might have, and the invariance of genus under birational transformations now had to be proved. Clebsch and Gordan had given such a geometric proof by means of a subtle elimination process, which Brill and others wanted to simplify.¹⁰⁹ ([Gray, 1998], 814)

Brill and Noether, who gave the theorem its name in [Brill and Noether, 1874], gave a reformulation of the theorem in terms of families of plane curves. They completely moved away from complex function theory to embrace an algebraico-geometric viewpoint.

An important follower of Riemann, who has not yet been mentioned, is Heinrich Weber. Particularly well-known for his *Lehrbuch der Algebra*, Weber worked in many areas of mathematics and was one of the very few mathematicians of his time to understand and acknowledge the value of Riemann's *Inauguraldissertation*. He was also, a few years later, the co-editor with Dedekind, of Riemann's *Gesammelte Werke*. Following their collaboration on the edition of Riemann's *Gesammelte Werke*, Dedekind and Weber wrote together a paper which reformulates the bases of Riemannian function theory (for algebraic functions of one complex variable), which will be the focus of my second chapter and will play an important role in investigating the role of arithmetic in Dedekind's practice. Considering Weber's role in the writing of this paper, but also as one of Dedekind's closest friends,¹¹⁰ I will propose, in the next section, to consider closer some aspects of Weber's works.

¹⁰⁹For the details of the difficulties encountered by Brill and Noether, see [Gray, 1998].

¹¹⁰Most of their long and rich correspondence is still unpublished. A forthcoming book by Katrin Scheel (Braunschweig TU) will consider their relationships through the edition of their letters (at least, those which are still available) .

1.4 Heinrich Weber (1842-1913)

1.4 Heinrich Weber (1842-1913)

Heinrich Weber was a very versatile mathematician. He is well-known for his works in algebra (Galois theory and group theory, as well as the first elements of what is now called class field theory¹¹¹). His textbook *Lehrbuch der Algebra* (first published in 1895-96) was one of the most influential of the late 19th and early 20th centuries (E. Noether, van der Waerden, etc. learned algebra with the *Lehrbuch*¹¹²). But he was also a recognized specialist of Riemannian function theory, mathematical physics, number theory and wrote elementary mathematics textbooks. Able to understand thoroughly and deeply new ideas in many areas of mathematics, Weber was very active and respected throughout his career, but with a tendency to stay in the shadow of other mathematicians or institutions.¹¹³

Associated with many mathematicians, from Kronecker to Frobenius to Dedekind or again Hilbert, Weber studied and taught in many different places. From 1860 to 1863, he studied at Heidelberg, where he obtained his doctorate under Otto Hesse. Hesse had been Jacobi's student and according to Cremona was a very "elegant" algebraist which probably had an important influence on Weber.¹¹⁴ While he was a student, Weber went to Leipzig in 1861-62, where he followed classes by Möbius and Scheibner. After his doctorate, he went to Königsberg and fully took advantage of the rich scientific activities around the mathematics and physics seminars (*mathematisch-physikalisches Seminar*) founded by Jacobi and Neumann, and studied with Franz Neumann and Richelot (a student of Jacobi as well). In 1866, Weber went back to Heidelberg and obtained his Habilitation.¹¹⁵ From 1869 to 1870, Weber was *Privatdozent* and *ausserordentlicher Professor* in Heidelberg. In 1870, Weber took Prym's position at the ETH Zürich (where Dedekind taught for a few years a decade earlier). During these years, he worked with Dedekind on the publication of Riemann's works.

¹¹¹For example, see [Kiernan, 1971], [Frei, 1989], [Hawkins, 1978], [Hawkins, 2013], and [Avigad and Morris, 2014].

¹¹²See [Corry, 2004b] and [Corry, 2005]

¹¹³Little has been published about Weber. [Schappacher and Volkert, 2000] gives a general survey of Weber's works, with a focus on his time in Strasbourg. Voss's obituary of Weber ([Voss, 1914]) also provides many informations on Weber (although it states the wrong date of birth, 5 May 1842, while Weber was in fact born the 5th of March). Weber does appear as a secondary character in many studies, most notably in the history of algebra in the works mentioned above, and more general studies, such as [Waerden, 1985], [Corry, 2004b]. Weber also plays an important role in the history of the Kronecker-Weber theorem ([Schappacher, 1998]). Finally, he also appears in studies about history of universities in Germany, such as [Scharlau, 1990]. In [Bourbaki, 1984], however, Weber is mentioned exclusively in reference to *Algebraische Funktionen*.

¹¹⁴[Voss, 1914]. Most of the informations below provide from Voss's obituary, which I have completed with some details. For the Göttingen part of Weber's career and his relationship with Hilbert, [Rowe, 1989] and [Rowe, 2003] provide useful information.

¹¹⁵His *Habilitationsschrift* was entitled "*Zur Theorie der regulären Lösungen partieller Differentialgleichungen erster Ordnung*". The subject was proposed to him by Richelot.

In 1875, Weber left Zürich for Königsberg, where he succeeded to Richelot. In Königsberg, Weber was Hilbert's and Minkowski's teacher, who followed his lectures on number theory and function theory. Weber also seems to have been the first mathematician to have a tangible influence on Hilbert. His lectures on number theory were more sympathetic to Dedekind's works in number theory than any other course on this topic, by this time. According to C. Reid, Hilbert "made his first acquaintance with the theory of invariants, the most fashionable mathematical theory of the day" in Weber's lectures and the only set of notes to which Hilbert came back and worked over are those from Weber's lectures on number theory.

In 1883, Weber went to Berlin to teach at the newly founded Technische Hochschule Charlottenburg. Although he stayed barely a year in Berlin and despite his strong affinities with the "conceptual" approach of Dedekind, he wrote Kronecker's obituary in 1893. In fact, Weber was a natural intermediary between the Berlin tradition and Dedekind's ideas. He was able to size up the benefits and drawbacks of each approach and to take the best of both. Weber was instrumental in spreading Dedekind's ideas and methods, and his ability to balance conceptual and non-conceptual approaches might have played an important role in the possibility to communicate efficiently these new ideas.

In 1884, Weber went to Marburg, where he stayed until 1892. In 1892, he was given Schwarz's chair in Göttingen, which was already under Klein's reign. Although Klein had expressed the wish to have Hurwitz or Hilbert come to Göttingen, in order to establish a young dynamic school, the professorship was given to Weber. David Rowe explains that despite Weber's brilliance,

at age fifty he was well past his prime, and, more crucial still, his personality and research style reflected the neohumanistic values of the older generation. He lacked the dynamism Klein was looking for in a new colleague, and although they got along well, Klein was hardly disappointed when Weber accepted a call to Strasbourg in 1895. ([Rowe, 1989], 197)

The chair went to Hilbert, and Weber stayed in Strasbourg until his death in 1913.¹¹⁶

1.4.1 Weber as a Riemannian mathematician

In 1870, Weber was one of the very few mathematicians to take interest in (and the defense of) Riemann's *Inauguraldissertation*. He tried to prove the Dirichlet principle in an article which he starts by explaining:

¹¹⁶About Weber's time in Strasbourg, see [Schappacher and Volkert, 2000].

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The Dirichlet principle, whose application by Riemann in the theory of Abelian functions has such great results, and from whose further developments one can promise oneself still some success, has lately been contested repeatedly in terms of its generality and rigor. The doubts are directed primarily against the liability of the application of the calculus of variations to a function whose properties are a priori completely unknown.¹¹⁷ ([Weber, 1870], 29)

It is interesting to notice that Weber was able to point out, before Weierstrass's publication of his own criticisms, that the core of the problem with the Dirichlet principle was the assumption of arbitrary functions. But such a criticism is powerful and very wide, and as Bottazzini and Gray underline, it would "render invalid any use of the calculus of variations, which is aimed at finding which unknown function maximises or minimises a given integral" ([Bottazzini and Gray, 2013], 278). It appears, then, that the crux of the matter is that the Dirichlet principle only 'shows' that the function exists, but does not provide concrete information on the function (and certainly not an explicit construction of it).

Weber explains that he does not want to try to prove the Dirichlet principle on the basis of "essentially new foundations" but rather wishes to "complete, at some vulnerable points, the proof that was more suggested than carried out by Riemann"¹¹⁸ ([Weber, 1870], 29). For this, Weber starts by making two restrictive hypotheses for his proof: that the boundary has a well-defined and finite curvature everywhere, and that the function considered is continuous and could be extended into a continuous function u which is "nowhere discontinuous or infinite" within S and satisfies that the integral

$$\Omega(u) = \iint \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 dx dy$$

is finite. Weber then proceeds to give an iteration process which can lead to a suitable (harmonic) extension of the function considered. By doing this, he achieves the proof of the validity of the Dirichlet principle under the very restrictive initial hypotheses. However, while the process given by Weber is valid in principle, it is not useful in practice. Even if it were seen to be sufficient to secure the Dirichlet principle, Bottazzini and Gray underline

¹¹⁷ "Das Dirichletsche Princip, dessen Anwendung durch Riemann in der Theorie der Abelschen Functionen zu so grossen Resultaten geführt hat, und von dessen weiterer Entwicklung man sich noch manche Erfolge versprechen darf, ist hinsichtlich seiner Allgemeinheit und Strenge neuerdings mehrfach angefochten worden. Die Zweifel richten sich hauptsächlich gegen die Befugniss der Anwendung der Variationsrechnung auf eine Function von a priori völlig unbekanntem Eigenschaften."

¹¹⁸ "Es ist indessen nicht meine Meinung, den Beweis des Dirichletschen Princips auf wesentlich neue Grundlagen zu bauen, sondern nur den von Riemann mehr angedeuteten als durchgeführten Beweis in einigen angreifbaren Punkten zu ergänzen."

that the proof is made under “vague and unduly restrictive” hypotheses ([Bottazzini and Gray, 2013] 278), which makes the proof unsatisfying.¹¹⁹ Weber’s paper went unnoticed by his contemporaries.

Weber, in his 1870 paper, is tackling a prevailing problem, and does so with the desire to provide a rigorous ground to Riemann’s theory without departing from Riemann’s initial proposal.

It is not the place, here, to give a detailed account of Weber’s works (neither on function theory nor in the many other areas he investigated). However, it seems important to underline that in many of his papers, he adopted an approach which did not present the kind of strong methodological and/or mathematical statements against Riemann of some his contemporaries, such as Clebsch or Weierstrass.

One can see in his 1873 paper “*Zur Theorie der Transformation algebraischer Functionen*”, that he wants to adopt an approach following Riemann’s desire not to base the investigations on particular representations of the functions and to adopt a general viewpoint – i.e., a viewpoint valid for *any* algebraic function. Briefly, the idea of this short paper is to investigate the properties of the Riemann surface of an algebraic function when subjected to rational transformations. Let s be an algebraic function of z satisfying the equation $F(s, z) = 0$ of degree n in s and m in z and associated with a surface T . Let z_1 be such that

$$z_1 = \varphi(s, z) \quad (1)$$

in which φ is a rational function and z_1 becomes infinitely great of the first order in n_1 points of the surface T – a new surface T_1 is thus defined spread over the z_1 plane. And let

$$s_1 = \psi(s, z) \quad (2)$$

in which ψ is a rational function and s_1 becomes infinitely great of the first order in m_1 points of the surface T , and is uniquely determined on T_1 . So, one obtains the equation $F_1(s_1, z_1) = 0$ of degree n_1 in s_1 and m_1 in z_1 , in which F_1 is a power of a function $\Psi(s_1, z_1)$ of degree ν in s_1 and μ in z_1 which cannot be decomposed into rational functions:

$$F_1(s_1, z_1) = \Psi(s_1, z_1)^\lambda$$

with $n_1 = \lambda\nu$, $m_1 = \lambda\mu$. Here, one can see that the substitutions (1) and (2) allow to transform $F(s, z)$ (of degree n in s and m in z) into $\Psi(s, z)$ (of degree ν in s and μ in z). From there, Weber investigates properties such as the genus of the new surface. Weber does not consider a particular transformation at any point of his paper, nor does he give explicit expressions of

¹¹⁹For more on the Dirichlet principle, see [Bottazzini and Gray, 2013], whose presentation of Weber’s works I followed here.

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the functions.

By 1874, when the edition of Riemann's collected works needed the help of a mathematician well versed into Riemann's works, Weber was warmly recommended to Dedekind.¹²⁰

1.4.2 Weber and algebra

Upon meeting Dedekind, Weber took interest in algebra and number theory and, in his research papers, adopted Dedekind's methods. By 1876, he was, according to Dedekind, the only one who had read his 1871 theory of algebraic integers (in the letter to Lipschitz from 29 April 1876, transl. in [Stillwell, 1996], 45).

Weber was particularly interested in group theory and worked closely with Frobenius as well as Dedekind.¹²¹ The proximity with Dedekind's ideas is particularly significant in the way in which he introduces his 1893 "*Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie*":

In the following, an attempt is made to found Galois' theory of algebraic equations in a way which will include as far as possible all cases in which this theory has been used. Thus, we present it here as an immediate consequence of the group concept extended to the field concept, as a formal law completely without reference to any numerical meaning of the elements used. This foundation, as a consequence, is thus also completely independent of the fundamental theorem of algebra on the existence of roots. The theory under this interpretation appears, it is true, as a pure formalism, which only gains meaning and life by the substitution of the individual elements with numeric values. In contrast, this method of procedure is applicable to every conceivable case in which the assumed hypotheses hold, reaching over to function theory on the one hand, to number theory on the other hand.¹²² ([Weber, 1893], 521)

¹²⁰Weber also did a great amount of work on Bessel functions, Abelian functions, elliptic functions, and the theta function. Still on the Riemannian side of his works, Weber also did important contributions to mathematical physics and published, in 1900, a two-volume textbook *Die partiellen Differentialgleichungen der mathematischen Physik nach Riemann's Vorlesungen* which would become, for decades the unique reference for Riemann's works in mathematical physics.

¹²¹Note that Weber only started to publishing about groups and curves, or elliptic functions and number theory after the writing of the 1882 paper with Dedekind.

¹²²"*Im Folgenden ist der Versuch gemacht, die Galois'sche Theorie der algebraischen Gleichungen in einer Weise zu begründen, die soweit möglich alle Fälle umfasst, in denen diese Theorie angewandt worden ist. Sie ergibt sich hier als eine unmittelbare Konsequenz des zum Körpergriff erweiterten Gruppenbegriffs, als ein formales Gesetz ganz ohne Rücksicht auf die Zahlenbedeutung der verwendeten Elemente. Diese Begründung ist hiernach also auch ganz unabhängig von dem Fundamentalsatz der Algebra über die Wurzelexistenz.*

Weber, here and in several of his algebraic and number-theoretic works, adopts a Dedekindian approach, with many references to the supplements of Dirichlet's *Vorlesungen über Zahlentheorie* written by Dedekind. Weber also uses a very Dedekindian terminology such as “systems of things” to define groups in the most general way possible:¹²³

A system \mathfrak{G} of things (elements) of any type, in finite or infinite number, becomes a *group* when the following conditions are fulfilled.

- 1) A rule is given according to which from a first and a second element of the system is deduced a completely determined third element of the same system. (...)
- 2) The associative law is assumed. (...)
- 3) It is supposed that when there is $AB = AB'$ or $AB = A'B$, then necessary there is $B = B'$ or $A = A'$. (...)
- 4) When from three elements A, B, C , two arbitrary are given in \mathfrak{G} , then one can always determine the third one in such a way that one has

$$AB = C.^{124}$$

(*ibid.*, 522)

When he considers the concept of field, Weber presents it as an extension of the concept of group, and goes back to the arithmetical roots found in Dedekind, since the composition laws, which were completely indefinite in the definition of a group, are called “addition” and “multiplication” for fields. Weber, in the 1893 paper, is interested in divisibility properties and introduces the notion of “form field” (*Formenkörper*): the field formed by the rational functions in several variables with coefficients in a given field Ω (see [Kiernan, 1971], 136-137).

Die Theorie erscheint bei dieser Fassung freilich als ein reiner Formalismus, der durch Belegung der einzelnen Elemente mit Zahlwerten erst Inhalt und Leben gewinnt. Dagegen ist diese Form auf alle denkbaren Fälle, in denen die gemachten Voraussetzungen zutreffen, anwendbar, die einerseits in die Functionentheorie andererseits in die Zahlentheorie hinübergreifen.”

¹²³Weber is, in fact, more formal than Dedekind is.

¹²⁴“Ein System \mathfrak{G} von Dingen (Elementen) irgend welcher Art in endlicher oder unendlicher Anzahl wird zur Gruppe, wenn folgende Voraussetzungen erfüllt sind.

1) Es ist eine Vorschrift gegeben, nach der aus einem ersten und einem zweiten Element des Systems ein ganz bestimmtes drittes Element desselben Systems abgeleitet wird, (...) 2) das associative Gesetz vorausgesetzt wird, (...) 3) Es wird vorausgesetzt dass, wenn $AB = AB'$ oder $AB = A'B$ ist, nothwendig $B = B'$ oder $A = A'$ sein muss. (...) 4) Wenn von den drei Elementen A, B, C zwei beliebig aus \mathfrak{G} genommen werden, so kann man das dritte immer auf eine Weise so bestimmen dass $AB = C$ ist.”

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In fact, here, Weber shows how he is able to combine both Dedekind's and Kronecker's approaches. Indeed, his investigations are heavily dependent on Kronecker's own researches on the divisibility of such fields (although, of course, Kronecker did not use this terminology), while elaborating a framework with a strong Dedekindian flavor. In addition, Weber explained that this paper was greatly inspired by Dedekind's manuscripts on Galois theory from his lectures in Göttingen in 1856-58 (see Sec. 4.2.2).

Weber's work in group theory is also known for his contributions to the development of the notion of "character", in [Weber, 1882] (see [Hawkins, 2013] and [Avigad and Morris, 2014]), in the framework of arbitrary (in the sense of his definition of group quoted above) finite Abelian groups. In his works on number fields and cyclic fields, Weber considers questions linked to the famous Kronecker-Weber theorem and its many failed proofs (see [Schappacher, 1998]). Weber also shows a clear interest for arithmetic on elliptic functions and problems pertaining to the complex multiplication, with several articles and chapters in the *Lehrbuch* dedicated to it. The influence of Kronecker is often tangibly obvious, yet supplemented by the use of Dedekindian algebraic concepts, as is particularly noticeable in Weber's *Lehrbuch der Algebra*.

1.4.3 *Lehrbuch der Algebra*

Weber explains, in the introduction to the *Lehrbuch*¹²⁵ that

It was [his] intention to write a textbook which, without assuming too much previous knowledge, would introduce the reader to the modern algebra as well as to lead him into the higher and more difficult parts in which the interest in the subject first become really lively. Thus, the required tools, the elementary ones as well as the higher ones, should result from the development itself, so that the presentation is made as independent from other textbooks as possible.¹²⁶ ([Weber, 1895-96], v)

For this reason, the book starts with an introduction in which the different systems of numbers are discussed in detail, from the natural numbers

¹²⁵On Weber's *Lehrbuch der Algebra*, see [Corry, 2004b] and [Corry, 2005]. While it is widely acknowledged as one of the most important textbook in algebra in the early 20th century, very little has been written on Weber's *Lehrbuch*.

¹²⁶"Es war meine Absicht, ein Lehrbuch zu geben, das, ohne viel Vorkenntnisse vorauszusetzen, den Leser in die moderne Algebra einführen und auch zu den höheren und schwierigeren Partien hinführen sollte, in denen das Interesse an dem Gegenstande erst recht lebendig wird. Dabei sollten die erforderlichen Hilfsmittel, die elementaren sowohl als die höheren, aus dem Gange der Entwicklung selbst abgeleitet werden, um die Darstellung von anderen Lehrbüchern möglichst unabhängig zumachen."

(which are considered as given) and elementary arithmetic (from basic properties like associativity to the unique factorization theorem) to the complex numbers. The presentation of natural numbers is followed by the concept of set, “*Mannigfaltigkeit oder Menge*”, a “system of objects or elements of any type” (a “naïve” set-theoretic viewpoint). Weber defines properties of order, the notion of discrete and dense (*dicht*) sets. Following Dedekind’s remarks in his 1872 *Stetigkeit und irrationale Zahlen*, Weber distinguishes between density and continuity, which gives him the ground to introduce rational numbers (a dense but discontinuous set) and real numbers (a dense and continuous set). Real numbers are introduced using Dedekind cuts, and following Dedekind’s 1872 definition almost word for word.

As regards the idea of “set”, Weber’s vocabulary and the order in which he introduces the ideas seem to relate him more closely to Riemann than to Dedekind, who placed the idea of “System” at the very foundation of mathematics, before even natural numbers.¹²⁷ Weber also introduces the notion of measurable sets (*messbare Menge*). The introductory parts end by explaining that literal calculus is the “most important” tool in algebra. In this literal calculus, one should distinguish between variables (letters that can be replaced by a continuous succession of numerical values) and unknowns which “impose conditions to the numbers” which one can put in place of the letters: it is the forming of equations, whose resolution is the essential task of algebra.

For Corry, the *Lehrbuch* starting with this long preliminary suggests that, despite the great level of abstraction that can be encountered, for example, in the treatment of groups, and the acknowledgement that abstract / arbitrary groups are an interesting and promising object of study, the systems of numbers hold nevertheless a deeply fundamental place in algebra ([Corry, 2004b], 38).¹²⁸ In fact, according to Corry in [Corry, 2005], there is little difference between a textbook such as Serret’s or Jordan’s and the “picture of algebra” in the *Lehrbuch* – despite the additional results provided in Weber’s *Lehrbuch*. Indeed, “in spite of including a great deal of material that would eventually be incorporated as the basis of van der Waerden’s presentation, it does not envisage the kind of fundamental change in conception that *Moderne Algebra* intended to imply” ([Corry, 2005], 691).

The new concepts (of group, ideal, field) were introduced because they appeared to be efficient in proving new results, shortening proofs and highlighting the possibility of new developments in certain theories (e.g., of quadratic forms or complex functions). In addition, they allowed to do so avoiding tedious computations. A good example of this is the resolution of polynomial equations and the role of the theory of groups in solving these

¹²⁷see Sec. 8.2.

¹²⁸And in fact, it was also the case for Dedekind.

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questions, as Weber himself underlines:

One of the oldest questions which the new algebra has preferentially developed is that of the so-called algebraic roots of equations, by which one means the representation of the solutions of an equation through a series of radicals, or their calculation through a series of root-extractions. The clearest light is shed on this question using the theory of groups.¹²⁹ (*ibid.*, 644, transl. slightly altered in [Corry, 2004b], 40)

In the introduction of the *Lehrbuch*, group theory is said to “spread light everywhere”. Fields and groups are only introduced in Chapter XIII of the third book in the first volume of the *Lehrbuch*, after about 450 pages on the resolution of equations by more traditional methods (symmetric functions, for example). Weber’s approach in Galois theory relies on the concepts of field (of numbers)¹³⁰ and adjunction, in a fashion very similar to Dedekind’s 1856-58 lectures and 1894 algebraic number theory. The *abstract* notion of group is only introduced in the second volume of the *Lehrbuch*.

The words “Gruppe” and “Körper” appear respectively approximately 660, 2180 and 370 times, and 370, 850 and 410 times in each volume of the second edition of the *Lehrbuch*. The concept of ideal, on the other hand, is completely absent from the first volume and appears only about 260 and 370 times in the two other volumes. Weber, thus, if he adopted Dedekind’s field and in great part his understanding of the concept of group, does not appear to have endorsed Dedekind’s view about ideals being a new grounding concept. In particular, in Weber’s treatment of the theory of algebraic functions (*Lehrbuch* III) follows the overall idea of the paper co-written with Dedekind, and indeed some parts are almost exactly similar, but it does not make any use of the concept of ideal, which as we will see was essential for *Algebraische Funktionen*.

It seems important to underline that Weber made a clear difference between the mathematics of research and that of teaching, being careful of not taking the abstract turn in algebra in his textbooks that can be seen in his papers, for example, on Galois theory. This is reflected, in his *Lehrbuch der Algebra*, first published in 1895-96. The said dissimilarities seem to be related to Weber’s sensitivity to the difference between the audiences of a textbook and a research paper. Kronecker and Dedekind are referred to the

¹²⁹ “Eine der ältesten Fragen, an der sich vorzugsweise die neuere Algebra entwickelt hat, ist die nach der sogenannten algebraischen Auflösung der Gleichungen, worunter man eine Darstellung der Wurzeln einer Gleichung durch eine Reihe von Radicalen, oder die Berechnung durch eine endliche Kette von Wurzelziehungen versteht. Auf diese Frage fällt von der Gruppentheorie das hellste Licht.”

¹³⁰ Like Dedekind, Weber only considers fields of numbers and introduces, here again, the divisibility notion between algebraic concepts to describe the inclusion relationships.

same amount of times in total in the three volumes of *Lehrbuch der Algebra*, but Kronecker appears twice as often as Dedekind in the first volume.¹³¹

Weber's *Lehrbuch*, was one of the most popular textbooks in Algebra in Germany in late 19th and early 20th century and was only replaced by van der Waerden's *Modern Algebra* in 1930. The *Lehrbuch* dominated the teaching of Algebra in Germany and the first volume was translated into French in 1898. A lot of the terminology and ideas introduced by Weber were used by later algebraists – and in fact, many can still be found in today's textbooks (see [Corry, 2004b], 33-43). According to Schappacher and Volkert ([Schappacher and Volkert, 2000]), it is precisely because Weber found a certain balance between the traditional (formal and algorithmic) methods in algebra and the new more abstract and “conceptual” methods, “bearing the mark of the transitory period”, that the *Lehrbuch* was so rich and so successful for such a long time.

Leo Corry (*ibid.*) sees the ‘balance’ in question in the consideration of algebra as the science of (polynomial) equations together with the introduction of the new “abstract” concepts such as field or groups to play what Corry called “secondary” roles. That is, they are new concepts that can shed light and allow new developments on the main subject matter of algebra: the theory of equations. This conception of algebra and the concepts of field, ideal, and group is similar to Dedekind's.¹³² Weber's originality, in finding a ‘balance’ between Dedekind's and Kronecker's approaches rather consists in introducing Kronecker's methods into the new framework established by Dedekind, and, for example, replace Dedekind's ideals by the “*Functionale*”, for the investigation of (arbitrary) fields of algebraic functions.

Corry showed clearly, in [Corry, 2004b], that Dedekind had not adopted the “modern” conception of algebra as being a theory of structures, and neither did Weber. For Dedekind, as for Weber, the concept of group and the concepts introduced in Dedekind's theory of algebraic numbers were new concepts to investigate questions pertaining to number theory and the “algebraic kinship of numbers”. The concepts proved, later, to be transferable (or translatable) in other theories, such as algebraic function theory – an approach which is fully taken advantage of in *Algebraische Funktionen*.

¹³¹Many thanks to Cédric Vergnerie who communicated me this information collected for his own PhD dissertation.

¹³²Indeed, for him properties of fields express properties of equations to which they are related: “The properties of an equation will (...) correspond to properties of the corresponding field. The relations between two equations are replaced by kinship between the fields.” (see p. 284)

1.5 A note about Dedekind's theory of algebraic numbers

Dedekind, who was very close to Dirichlet as a student and as a friend, was put in charge, after Dirichlet's death, in 1859, of publishing Dirichlet's lectures on number theory, the *Vorlesungen über Zahlentheorie*, based on the notes he took during Dirichlet's lectures, and on manuscripts left by Dirichlet. Dedekind followed Dirichlet's lectures from 1855 to 1858 and had long discussions with him that allowed him to "get to know and understand the grounds of his method of exposition." He tried, as he explains, in the preface to the 1863 edition of the *Vorlesungen*, to give a retranscription "as faithful as possible" of Dirichlet's lectures which "contributed so extraordinarily to the circulation of the newest and the subtlest parts of mathematics."

Dirichlet's *Vorlesungen* had four editions: in 1863, 1871, 1879 and 1894. From the first edition, the publishers asked Dedekind to add original supplements, which completed the main text, presenting additional proofs or results omitted by Dirichlet, such as his theorem on prime numbers in arithmetic progressions and his "pigeonhole" solution to Pell's equation. Dirichlet's *Vorlesungen* is an exceptionally thorough and clear general survey of 19th century number theory, from the most basic results on divisibility of rational integers, to results at the frontier of research. It presents fundamental and classic results from Fermat, Euler, Legendre, Jacobi, and Gauss. It also provides a complete and clear treatment of Dirichlet's class number formula for quadratic forms, and exposes, in some Supplements, more recent developments on the subject matter. Most notably, Dedekind used this opportunity, in 1871 and the subsequent versions, to expose his main work in number theory: the theory of algebraic integers.

The second edition, published in 1871, is essentially the same as the first one, except for the Xth Supplement, *Über die Komposition der binären quadratischen Formen*, which was initially excluded to avoid moving too far away from Dirichlet's own works. Dirichlet did investigate binary quadratic forms, albeit "only the first fundamental theorem", but did not work on the factorization of complex or algebraic numbers. What encouraged the publication, in addition of the publishers's pressure, was that

because, despite the immense interest and the increasing importance of this theory, still no attempt to clear away the biggest difficulties, which rise for the beginner at his entry [in the theory], and because the remaining parts of this work are all perfectly suited to ease such an effort.¹³³ (Preface to the second edition of Dirichlet's *Vorlesungen*, repr. in [Dedekind, 1932], III, 399, my translation)

¹³³ "weil trotz des ungemeinen Interesses und der steigenden Wichtigkeit dieser Theorie noch immer kein Versuch gemacht ist, die großen Schwierigkeiten hinwegzuräumen, welche beim Eindringen in dieselbe sich dem Anfänger entgegenstellen, und weil die übrigen Abschnitte des Werkes ganz vorzüglich geeignet sind, einen solchen Versuch zu erleichtern."

Dedekind's famous Xth Supplement is the birthplace of field and ideal theories. In this work, among other things, he succeeds in proving the existence of a unique factorization in primes for algebraic integers. As I have mentioned in the introduction, this Supplement, and to a certain extent its later rewritings, are considered as one of the most important number theoretical writings of the second half of 19th century. However, as it is rightly as often underlined, Dedekind's presentation differs significantly from the usual style of 1870s mathematics. The "modernity" of his approach, that is, his use of actual infinities – or, to put it in more modern terms, his adoption of infinitary set-theoretic methods and the abstraction attached to it – appear to have made it very difficult for his contemporaries to read and understand, or at least appreciate, his theory. In fact, it seems that it was Dedekind's endeavors to develop methods following the principles of the so-called "conceptual" approach, which made the reception of his works difficult.

Aware of, and annoyed by, the lack of reception of his works, Dedekind considered that the lack of immediate reception of his theory of algebraic integers was due to the presentation of his work, which therefore needed a complete rewriting. He thus decided to write his theory a second time, so as to make it clearer. He wanted his work to be easier to read, easier to apprehend for the readers. He wrote to Lipschitz, who had relayed an invitation to publish his theory in the *Bulletin des Sciences Mathématiques et Astronomiques*:

[F]or a few years I had more or less given up hope that my presentation and conception of a general theory of ideals would at the present day interest anybody but me. With the exception of Professor H. Weber (...) you are the first (...) to express interest in the subject (...) [L]ittle by little I have become convinced that the presentation itself is to blame for the failure (...) I can only suppose that the presentation deterred readers through excessive brevity and terseness, and since autumn I have been spending my free time, obtained by resigning my three year directorship of the local polytechnic, working out a more detailed presentation of the theory of ideals, and I have come so far as to obtain a somewhat improved form of the essential foundation (the content of §163).¹³⁴ (Translation modified of Dedekind's letter to Lipschitz, in [Stillwell, 1996], 45, original in [Lipschitz and

¹³⁴ "[I]ch [hatte] seit einigen Jahren so ziemlich die Hoffnung aufgegeben (...), dass meine Darstellung und Auffassung einer allgemeinen Theorie der Ideale in jetziger Zeit noch irgend Jemand ausser mir interessiren würde. Mit Ausnahme des Prof. H. Weber (...) sind Sie der erste, der (...) ein Interesse an dem Gegenstande äussert (...) allein ich habe mich nach und nach davon überzeugt, dass die Darstellung selbst wohl die Schuld an dem Misslingen (...) trägt. Ich muss vermuthen, dass die Darstellung durch übertriebene

1.5 A note about Dedekind's theory of algebraic numbers

al. 1986], 48-49.)

The new version of the theory is, according to Dedekind, essentially the same in a reworked form and with an improved foundation, i.e., an improved theory of ideals. A closer look at the new version shows that the theory is developed with considerably more attention to the possibility to develop ideal theory in a way that would be perfectly analogous to rational number theory.

The strategy introduced in the 1871 version, and pushed much further in the 1877 one, consists in transferring the study of divisibility from algebraic integers to ideals. For this, Dedekind defines a relation of divisibility between ideals: an ideal \mathfrak{a} is said to be divisible by an ideal \mathfrak{b} if $\mathfrak{a} \subset \mathfrak{b}$. One can show that the divisibility between ideals is equivalent to the divisibility of numbers and then study the laws of divisibility of ideals. To show that the ideals obey the same divisibility laws as rational numbers is, then, easier than for algebraic integers. The central concepts of divisibility of rational numbers, such as GCD and LCM, are redefined for ideals. The theory is re-modeled in 1877 so that it follows the exact lines of rational number theory, giving for ideals theorems such as Euclid's lemma or the existence of a unique decomposition in primes.

Dedekind's algebraic number theory will be published two more times as a Supplement of Dirichlet's *Vorlesungen*, once in 1879 in the same form as in 1877, and once in 1894 in another completed reworked version, taking a more 'abstract' turn and involving a great amount of Galois theory (almost absent of the previous versions).¹³⁵

In 1882, field and ideal theories are transferred to a different framework and used in the theory of algebraic functions of one complex variable, in a common work with Heinrich Weber, "*Theorie der algebraischen Funktionen einer Veränderlichen*", which was published in 1882 in the *Journal für die reine und angewandte Mathematik*. As I have mentioned earlier, this paper was written by the two mathematicians after their collaboration in the edition of Riemann's *Gesammelte Werke*. In this paper, they propose a new treatment of the Riemannian theory of algebraic functions, on the ground of ideas and methods imported from Dedekind's algebraic number theory. This article, birthplace of modern algebraic geometry in many aspects, was elaborated through letters during the years 1877 to 1879, written properly by Weber, submitted in 1880 and published in 1882.

Kürze und Gedrängtheit die Leser abgeschreckt hat, und ich habe daher seit dem Herbst meine freie Zeit, die ich durch Niederlegung meines dreijährigen Directorats des hiesigen Polytechnikums gewonnen habe, dazu benutzt, eine ausführlichere Darstellung der Theorie der Ideale auszuarbeiten, mit welcher ich auch so weit gekommen bin, dass die eigentliche Grundlage (der Inhalt des §163) in einer etwas verbesserten Form gewonnen ist.."

¹³⁵While Dedekind entirely reworked his theory of algebraic numbers for each re-edition of Dirichlet's *Vorlesungen*, the other supplements were not modified in the same way.

Chapter 2

Dedekind and Weber's *Theorie der algebraischen Funktionen einer Veränderlichen*

Dedekind and Weber's 1882 *Theorie der algebraischen Funktionen einer Veränderlichen* proposes an arithmetico-algebraic treatment of Riemann's function theory, with a new definition of the concept of Riemann surface. Unsatisfied with previous treatments of Riemann's theory, their aim is to be able to provide a more general treatment of the theory. Here "general" means "uniform", that is, a presentation of the theory which would not need to make restrictive initial hypotheses, nor to distinguish between cases.

In previous investigations on this topic, as a rule certain restrictive assumptions about the singularities of the functions under consideration have been made and the so-called exceptional cases have either been mentioned casually as limit cases, or even left aside entirely.¹ ([Dedekind and Weber, 1882], transl. modified, 41)

Via the set up of the algebraico-arithmetic concepts of field, module and ideal, Dedekind and Weber propose a presentation of the theory valid for any algebraic function of one complex variable.

In addition, they wish to avoid any use of geometric intuition, especially if it is taken as a reason to admit the truth of certain theorems about functions, such as their developability. Ultimately, the idea is to be able to provide a treatment of the bases of Riemannian function theory on which the mathematicians would be able to rely and which would allow a more

¹"Bei den bisherigen Untersuchungen über diesen Gegenstand werden in der Regel gewisse beschränkende Voraussetzungen über die Singularitäten der betrachteten Funktionen gemacht, und die sogenannten Ausnahmefälle entweder als Grenzfälle beiläufig erwähnt oder auch ganz beiseite gesetzt."

2.1 Returning to Riemann's epistemological ideals?

rigorous, simpler and more general development of the theory.

Few extensive commentaries have been made on this paper ([Geyer, 1981], [Strobl, 1982], and to a lesser extent, the introductory notes in [Stillwell, 2012]). Most of them are integrated into larger inquiries about history of mathematics ([Dieudonné, 1974], [Laugwitz, 2009], [Houzel, 2002], [Schapacher, 2010], one can also refer to [Couche, 2006]). I will avoid, as much as possible, to restate Dedekind and Weber's ideas in modern mathematics. As a consequence, in my own commentary very little reference will be made to the mentioned commentaries, which unfold the mathematical content of the paper in modern terms. However, it should be emphasized that I have greatly benefited from their explanations in my reading. The study of Dedekind and Weber's article will be the center of this chapter.²

My aim in this chapter is twofold. Firstly, I would like to offer an account of the 1882 paper with a fair amount of details, so as to enable a better understanding of the approach adopted by Dedekind and Weber. Dedekind and Weber wrote their paper as a two stores building, and in order to understand their rewriting of the Riemann surface, it is necessary to go through the study of function fields. As I have stated earlier, my aim is to exhibit the importance of arithmetic for Dedekind's approach. However, I do not wish to give a partial account of the paper that would present only what can be seen as arithmetical. Rather, I hope to convince my reader that arithmetic is, indeed, important.³

Secondly, I will analyse how *Algebraische Funktionen* is embedded in the Dedekindian corpus and which benefits can be derived from exploring its connections with Dedekind's contributions in other domains. This question will play an essential role in my dissertation. Indeed, the strategy used by Dedekind and Weber, in 1882, involves the set up of an arsenal relying deeply on the definition of concepts rooted in arithmetic. This strategy was already used by Dedekind in several of his other mathematical works, and the transfer of his works to another framework allows to shed a different light on his ideas, providing additional tools understand certain aspects of Dedekind's mathematics.

2.1 Returning to Riemann's epistemological ideals?

As I have mentioned, the introduction of Dedekind and Weber's paper develops the same kind of arguments against post-Riemann Riemannian function theory that Dedekind brought to the fore in earlier writings: function theo-

²I will rely on Stillwell's translation of the 1882 paper, [Stillwell, 2012]. All references will be to [Stillwell, 2012]. Occasionally, modifications have been made to Stillwell's translation to stay closer to the original text. I will mention them explicitly.

³Lesser details on the texts will be given in the next chapters.

rists did not respect Riemann's methodological precepts. Indeed, Dedekind's praise of Riemann's approach in function theory was often accompanied by criticisms towards the treatments subsequently given to Riemann's theory – which Roger Godement calls “le ‘verbiage’ considérablement moins génial de certains de ses successeurs,” in [Godement, 2001].

In the letter to Lipschitz quoted earlier, in which Dedekind explains that what he attempted in his number theoretical works is, in some way, analogous to what Riemann accomplished in function theory, he adds a “passing remark” that he “cannot suppress”, namely that:

the Riemann's principles are not being put into application in a consistent way by most writers – for example, even in the newest work on elliptic functions. Almost always they disfigure the simple theory by unnecessarily bringing in forms of representation which should in fact be results, not tools, of the theory.⁴ (Letter to Lipschitz, 6 June 1876, in [Dedekind, 1932] III, 467-468, transl. slightly altered in in [Edwards, 1983], 11)

Dedekind and Weber present their work as returning to Riemann's true intentions by avoiding to rely on explicit expressions or computations. By proposing approaches to the theory attached to definite equations or to explicit representations in series, the previous authors had to impose initial restrictions on the functions treated (e.g., on their singularities). This is the second point to which Dedekind and Weber are opposed. They wish to offer a theory of functions which treats, in one move, entire classes of functions. That is, to meet the level of generality established by Riemann's treatment of entire classes of functions in previous treatments of the theory, be they algebraic or geometric or purely analytical.

The theory, for Dedekind and Weber, must be grounded on concepts that do not use computations or *Darstellungsformen* in any essential way, that is, as grounds for definitions and deductions. It does not need to be contradictory with the fact that Riemann could have come to a deep knowledge of the nature of algebraic functions through computations and *Darstellungsformen*, as Edwards suggested. The fact that Riemann's ideas were developed relying on a great mastery of *Darstellungsformen* does not change the fact that Riemann formulated the theory itself in such a way that *Darstellungsformen* do not form its ground, nor play any significant role in its development. Dedekind and Weber's aim, here, is to supplement the work accomplished by Riemann with a “precise” definition of the Riemann surface, so that further works on the subject can adopt the “general”

⁴“(...) die Riemannschen Principien von den meisten Schriftstellern, z.B. auch in den neuesten Werken über elliptische Functionen, nach meiner Ansicht nicht in consequenter Weise zur Anwendung gebracht werden; fast immer wird die einfache Theorie verunziert durch unnöthige Einmischung der Darstellungsformen, welche doch eigentlich nur Resultat, nicht Hilfsmittel der Theorie sein sollten.”

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approach without it being “confusing”, as it was according to Clebsch.

2.1.1 Criticisms and the need of a sound basis for Riemann's theory

Dedekind and Weber explain that the treatments of function theory before their work had a tendency to unduly accept the validity of certain unproven theorems at the basis of their researches.

Likewise, certain fundamental theorems on continuity or developability have been admitted, the evidence of which based on some geometric intuition of one sort of another.⁵ (*ibid.*, transl. modified 41)

This statement is strongly reminiscent of Dedekind's reflections on the definition of real numbers, in 1872 (see Sec. 7.2). That point was noted by Schappacher in [Schappacher, 2010]. Schappacher links the definition of a point of a Riemann surface to the definition of real numbers using cuts as points of the linear continuum and explains that “both [definitions] try to conceptualize the intuition of what a concrete point does for you.” As he underlines, Dedekind is adopting the same approach, proposing the same conceptual move, in these two works. I will come back on this, in relation to the methods used for both definitions.

Facing what they perceived as conspicuous lack of rigor and recurring restrictive assumptions that can be found in treatments of Riemannian function theory, Dedekind and Weber wish to reformulate the fundamental concepts of the theory, so as to solve these problems. The authors explicitly state so, in the introduction of the paper:

The purpose of the investigations communicated in what follows is to found the theory of algebraic functions of one variable, one of the most important results of Riemann's creation, from a standpoint which would be simple, and at the same time rigorous and completely general.⁶ (*ibid.*, transl. modified, 41)

The three epistemological values mentioned here, simplicity, rigor and generality (as uniformity), are regularly invoked by Dedekind, so much so that they appear to play an essential role in his works. One can observe, that these requirements, often combined with each other, play important roles in Dedekind's shaping of new concepts and new methods. Dedekind seems to explicitly design his methods to fulfill simultaneously these requirements,

⁵“Ebenso werden gewisse Grundsätze über die Stetigkeit und Entwickelbarkeit zugelassen, deren Evidenz sich auf geometrische Anschauung verschiedener Art stützt.”

⁶“Die im nachstehenden mitgetheilten Untersuchungen verfolgen den Zweck, die Theorie der algebraischen Funktionen einer Veränderlichen, welche eines der Hauptergebnisse der Riemannschen Schöpfung ist, von einem einfachen und zugleich strengen und völlig allgemeinen Gesichtspunkt aus zu begründen. ”

to provide a uniform and rigorous theory that does not “disfigure the simplicity” of the theory. Conversely, simplicity (which contains the demand that no external tools be introduced in the theory) and uniformity appear as essential components of a rigorous approach. I will come back on the question of “rigor” later in this dissertation, once we will have more elements to support the analysis.

Dedekind and Weber propose to fulfill their aim to provide a “more reliable basis for the fundamental notions, as well as for a general and exceptionless treatment of the theory” by

[taking as a basis] a generalization of the theory of rational functions of one variable, in particular of the theorem according to which every polynomial⁷ of one variable admits a decomposition into linear factors.⁸ (*ibid.*, transl. modified, 41)

In the case of surfaces of genus 0, such as the sphere, the approach has already been considered and is “simple and well known”. Dedekind and Weber’s aim is to treat the general case, transl. to provide a treatment valid for any genus p . This general case is related to the surfaces of genus 0 “in the same way that general algebraic numbers are related to rational numbers” (*ibid.*). To treat the general case, they will thus transfer and adapt to the functions of one complex variable, the number theoretical methods “derived from Kummer’s creation of ideal numbers”, which proved to be “most successful” in Dedekind’s own works in number theory.

To fulfill Dedekind and Wever’s goal of rewriting the bases of Riemannian function theory, one important task presents itself as essential: to supply the mathematicians with

a completely precise and rigorous definition of the “point of a Riemann surface” that can also serve as a basis for the investigation of continuity and related questions.⁹ (*ibid.*, 42-43)

For this, they deploy a vast amount of technical preliminaries, but do not provide any new results. Rather, they introduce in function theory new concepts so as to give new definitions of Riemannian notions such as the genus,

⁷Dedekind and Weber, as it was common at the time, use “integral rational function” (*ganze rationale Funktion*) to designate polynomials. To avoid confusion with the integral functions defined later in the text as the integers of the field, I will use the “polynomial”.

⁸“Eine sichere Basis für die Grundvorstellungen sowie für eine allgemeine und ausnahmslose Behandlung der Theorie läßt sich gewinnen, wenn man von einer Verallgemeinerung der Theorie der rationalen Funktionen einer Veränderlichen, insbesondere des Satzes, daß jede ganze rationale Funktion einer Veränderlichen sich in lineare Faktoren zerlegen läßt, ausgeht.”

⁹“(…) eine vollkommen präzise und strenge Definition des ‘Punktes der Riemannschen Fläche’ (...), welche auch als Basis für die Untersuchung der Stetigkeit und der damit zusammenhängenden Fragen dienen kann.”

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and, on this basis, to provide new proofs of results such as the Riemann-Roch theorem, all of which is done with the expected level of rigor and generality. Both of those things are acknowledged by Dedekind and Weber. In their work, the Riemann surface remains the core of Riemann function theory, despite the long and winding road they had to take to arrive at a definition for it. They explain, in fact, that for the time being, providing a rigorous foundation to the theory is too difficult to avoid preliminaries and detours:

Admittedly, all these results can be obtained from Riemann's theory with a much more limited investment of means, and as special cases of a much more extensive general presentation. However, it is known that a rigorous foundation of this theory still presents certain difficulties, and until one has managed to completely overcome these difficulties, it may very well be that the path we have taken, or at least one related to it, is in fact the only one leading to this goal for algebraic function theory with satisfactory rigor and generality.¹⁰ (*ibid.*, transl. modified, 42)

To be able to provide uniform and rigorous definitions of the basic, grounding notions of the (already existing) theory seems essential to ensure its (again, rigorous) development. For this, their strategy is to take full advantage of the well-known similarities between the arithmetical behaviors of numbers and of functions. That is, they define a notion of field of algebraic functions:

In an analogy with number theory, a *field of algebraic functions* is understood to be a system of such functions with the property that application of the four fundamental operations of arithmetic (*Spezies*) to the functions of this system always leads to functions of the same system.¹¹ (*ibid.*, transl. slightly altered, 42)

In such a field, any function can be chosen as “independent variable and the rest as dependent on it”. This is a specific “mode of representation”, but insofar as any function of the field can be taken as “independent variable”

¹⁰“*Freilich ergeben sich alle diese Resultate durch einen weit geringeren Aufwand von Mitteln und als Spezialfälle einer vielumfassenden Allgemeinheit aus Riemanns Theorie; allein es ist bekannt, daß diese Theorie bezüglich einer strengen Begründung noch gewisse Schwierigkeiten bietet, und bis es gelungen ist, diese Schwierigkeiten vollständig zu überwinden, dürfte der von uns betretene Weg oder wenigstens ein verwandter, wohl der einzige sein, der für die Theorie der algebraischen Funktionen mit befriedigender Strenge und Allgemeinheit zum Ziele führt.*”

¹¹“*Versteht man, analog der Zahlentheorie, unter einem Körper algebraischer Funktionen ein System solcher Funktionen von der Beschaffenheit, daß die Anwendung der vier Spezies auf Funktionen des Systems immer zu Funktionen desselben Systems führt.*”

and the totality of the functions remain unaltered by the choice, this does not constitute a problem for the generality of their approach. A notion of *integral algebraic function* can thus be defined in very similar way as the algebraic integers were defined:

<p>A number θ is an <i>algebraic integer</i> if it satisfies an equation</p> $\theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n = 0$ <p>of finite degree n and whose coefficients a_1, a_2, \dots, a_n are rational integers.</p>	<p>A function ω of the field Ω is called an <i>integral algebraic function</i> of z if in the equation of lowest degree satisfied by ω</p> $\varphi(\omega) = \omega^e + b_1\omega^{e-1} + \dots + b_{e-1}\omega + b_e = 0$ <p>the coefficients b_1, b_2, \dots, b_n are all <i>polynomials</i> of z.</p>
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From there, a reasoning similar to the one used for the introduction of ideals leads to the understanding that Dedekind's ideal theory can be carried over to functions. Since the arithmetic operations can easily be defined for functions, it is easy to see that

among these integral algebraic functions, groups of functions may now be distinguished again, to which are attached the characteristic properties of polynomials sharing a common divisor.¹² (*ibid.*, transl. slightly altered, 42)

Such a system of functions corresponds to the concept of ideal, as initially thought of by Dedekind: the system of numbers divisible by an (existing or ideal) factor. Hence, in the same way that Dedekind transferred the study of the divisibility of algebraic integers to systems of numbers (i.e., ideals), they will study the divisibility of functions by means of systems of functions, i.e., ideals of functions.

This divisor does not "exist in the general case". However it is possible to consider the "system of functions divisible by the said divisor" rather than the divisor itself, just as Dedekind did with ideal factors and the definition of ideals. Then, if the theorems about the rational functions

are linked not to the divisor itself but to the system of functions divisible by the said divisor, then they can be completely transferred to the general algebraic functions.¹³ (*ibid.*, transl. slightly altered, 42)

¹²"Unter diesen ganzen Funktionen lassen sich nun wieder Gruppen von Funktionen aussondern, welchen die charakteristischen Merkmale solcher ganzen rationalen Funktionen zukommen, die einen gemeinschaftlichen Teiler haben."

¹³"Ein solcher Teiler existiert zwar im allgemeinen Falle nicht, wenn man aber die bezüglichen Sätze über rationale Funktionen nicht an den Teiler selbst, sondern an das System der durch denselben teilbaren Funktionen knüpft, so gestatten sie eine vollkommene Übertragung auf die allgemeinen algebraischen Funktionen."

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The transfer of the methods elaborated by Dedekind for algebraic numbers constitutes the first step of Dedekind and Weber's strategy. They can, from there, set up the algebraic arsenal which they will need in the paper. Indeed, with an appropriate definition of arithmetical operations between ideals, it becomes possible to "calculate with ideals using the same rules as for rational functions", to define notions such as "prime ideal" and to prove the validity of divisibility theorems such as the existence of a unique factorization in primes for ideals of algebraic functions. On this basis, they can proceed to the second step: the definition by purely arithmetico-algebraic means of the Riemann surface associated with the field, and of the basic notions of a Riemann surface (e.g., the genus).

It is the possibility to provide the first elements of a theory of Riemann surfaces on the sole basis of the arithmetical 'structure' of a field of functions, that allows for a more general and more rigorous treatment of the theory. Indeed, the algebraic concepts are defined and used in such a way that the nature of their elements is completely indifferent,¹⁴ and all "restrictive assumptions about the singularities of the functions" are avoided. The theory can thus be truly "exceptionless".

The purely algebraico-arithmetic investigations elaborated by Dedekind and Weber allow to develop a theory of functions completely freed from any assumption about continuity, a notion which is itself neither evident nor intuitive. Continuity ought to be mathematically defined. In addition, it is not an intrinsic property of the functions, or the Riemann surface or even the geometric space. Indeed, Dedekind argues that there is no reason to consider continuity as an intrinsic property of space, as early as 1872 when he defines the real numbers and, by that, the linear continuum. Dedekind's insistent statement about the fact that continuity is not a necessary character of space is based on his proof that the propositions of Euclidean geometry hold when one takes algebraic integers as coordinates. In 1872, it is one of Dedekind's main reasons to give an arithmetical definition of continuity, for continuity is in no way a self-evident or immediate idea. Dedekind's explanation about why continuity is neither a necessary condition, nor a consequence of Euclidean geometry appears in *Stetigkeit*, and is developed in some details in the letters to Lipschitz. The considerations on Euclidean geometry made with algebraic numbers as coordinates can be found first in a commentary to Paul du Bois-Reymond's book on algebraic function theory, and in the first introduction to *Zahlen* (see Sec. 7.2). Considerations about "continuity and related questions" for Riemann surfaces, essential in particular for integration theory, ought to be developed from a precise and solid definition of the notion of point. Dedekind and Weber were aware of the

¹⁴As we will see, not only is the definition of fields and ideals relying solely on properties of closure, but the development of ideal theory is made by considering ideals as wholes, which erases completely the individual nature of the functions as elements of the ideal from all considerations.

fact that since all considerations about “continuity and related questions” are excluded from their article, the piece of Riemann’s theory of functions that they are able to rewrite is relatively limited. In particular, it does not comprise anything regarding integration and Abelian functions. On the other hand, their treatment implies that they have shown that a great part of the theory was independent of such considerations about continuity.

2.1.2 Dedekind and epistemological requirements

What is now seen as “red-letter clichés” of the “conceptual approach” appear generously in Dedekind’s mathematics and have been at least as generously commented upon. For example, H. Edwards, who opens his 1980 article on the genesis of ideal theory by explaining that Dedekind’s “strong philosophical principles” were shaped simultaneously with his reflections on ideal theory and were responsible for the rewritings of his algebraic integer theory ([Edwards, 1980], 321). Edwards, later in his paper, states that Dedekind’s “insistence on philosophical principles was responsible for many of his important innovations” (*ibid.*, 349). Indeed, many of Dedekind’s works are motivated by a desire to provide a better foundation to theories: it is obviously the case of his foundational works but also appears in number theory and, in what occupies us here, in function theory.¹⁵

His attention to the foundation of theories led Dedekind to formulate certain rules that a definition should meet to be acceptable. When explaining the requirements that a definition of real numbers should meet, he explains that there must be no intrusion of “foreign” elements, imprecise notions, or intuition¹⁶:

As first demand, I reckon that arithmetic must be kept free from intermixture with foreign elements, and for this reason I reject the definition of number as the ratio of two quantities of the same kind. On the contrary, the definition or creation of irrational number ought to be based on phenomena one can already observe clearly in the domain R of rational numbers.¹⁷ ([Dedekind, 1876-

¹⁵In number theory, the concepts of field and ideal are presented as new grounding concepts, see Chapters 5 and 6. Their role in Dedekind’s attempts to prove the general validity of the theorems of rational arithmetic is essential. Note that there is, in Dedekind’s works, no claim that his approach is the only one susceptible to lead to the right results, on the contrary.

¹⁶“I regard such invocation of geometric intuition (*Anschauung*) in a first presentation of the differential calculus as exceedingly useful from a pedagogic standpoint (...) but no one will deny that this form of introduction into the differential calculus can make no claim to being scientific.” ([Dedekind, 1872], 768)

¹⁷“*Comme première exigence, je reconnais que l’Arithmétique doit être maintenue exempte de tout mélange d’éléments étrangers, et pour cette raison je rejette la définition d’après laquelle le nombre serait le rapport de deux grandeurs de même espèce; au contraire, la définition ou la création du nombre irrationnel doit être fondée uniquement sur*

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1877], transl. slightly altered, 57, footnote)

The definition must satisfy a criterion of uniformity related to a definite framework, that is, one must give one definition for all the elements falling under the concept, and no distinction of several cases, no exception or special treatment. For example, for irrational numbers:

Secondly, one should demand that all real irrational numbers be possibly engendered simultaneously by a common definition, and not successively as roots of equations, as logarithms, etc.¹⁸ (*ibid.*, 57-58, footnote, transl. slightly altered)

When defining new numbers, the definition ought to provide the possibility to define operations:

Thirdly, the definition should be of a kind which also permits a perfectly clear definition of the calculations (addition, etc.) one will have to carry out with the new numbers.¹⁹ (*ibid.*, 58, footnote, transl. slightly altered)

More generally, the definition should allow to develop methods of proof for the new concepts and the theorems to be proved.

Dedekind also adds, in several works, that the definition must be done using “earlier notions”, i.e., concepts and methods already well-defined (e.g., the definition of real numbers should be done with only means pertaining to the rational numbers). It is in particular the case of the extensions of the domains of number:

[T]he step-by-step extension of the number-concept is subsequently to be carried out (...) always by a reduction to earlier concepts.²⁰ ([Dedekind, 1888], 792)

Finally, if the definition is an extension, it must be valid for the restricted concept as well (e.g., all rational integers are algebraic integers).

In addition, one should not develop a theory on the basis of notions which rely on computation, notation or explicit representation. The conjunction of such demands are guiding Dedekind's efforts to find the most appropriate definition for the concept(s) grounding a theory. For example, in the case of the theory of ideals:

des phénomènes que l'on puisse déjà constater clairement dans le domaine R.”

¹⁸ “*En second lieu, on devra exiger que tous les nombres réels irrationnels puissent être engendrés à la fois par une commune définition, et non successivement comme racines des équations, comme logarithmes, etc.*”

¹⁹ “*La définition devra, en troisième lieu, être de nature à permettre aussi une définition parfaitement claire des calculs (addition, etc.) que l'on aura à faire sur les nouveaux nombres.*”

²⁰ “*In welcher Art später die schrittweise Erweiterung des Zahlbegriffes (...) stets durch Zurückführung auf die früheren Begriffe herzustellen ist*”

One notices, in fact, that the proofs of the most important propositions depend upon the representation of ideals by the expression $[ma, m(b + \theta)]$ and on the effective realisation of multiplication, that is, on a calculus which coincides with the composition of binary quadratic forms given by Gauss. If we want to treat fields Ω of arbitrary degree in the same way, then we shall run into great difficulties, perhaps insurmountable ones. Even if it were not the case, such a theory, based on calculation would still not be of the highest degree of perfection.²¹ ([Dedekind, 1876-1877], transl. slightly altered, 102)

These ideas are particularly important, from Dedekind's viewpoint, insofar as a theory based on notions lacking the previously described characteristics would fail to provide the appropriate treatment of the subject investigated. It could also lead to "insurmountable difficulties" and/or exhibit lack of rigor.

In fact, Dedekind affirms that if the concepts grounding a theory are appropriately defined, one should be able to give more appropriate demonstrations – e.g., the definition of real numbers and the proof of a proposition such as $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$. Moreover, this should ease the way toward further developments hitherto out-of-reach.

The latter point was particularly important for algebraic number theory, in which the concept of ideal is said to be a new ground for the "theory of (...) all the integers of the field" investigated (see Sec. 5.4):

We ground the theory of the numbers of \mathfrak{o} , i.e., all the integers of the field Ω , on the following new concept. A system \mathfrak{a} of infinitely many numbers contained in \mathfrak{o} will be called an ideal if it satisfies the following pair of conditions:

I. The sum and difference of any two numbers in \mathfrak{a} are again numbers in \mathfrak{a} .

II. Each product of a number in \mathfrak{a} and a number in \mathfrak{o} is again a number in \mathfrak{a} .²² ([Dedekind, 1871], 38)

²¹ "On peut remarquer, en effet, que les démonstrations des propositions les plus importantes se sont appuyées sur la représentation des idéaux par l'expression $[ma, m(b + \theta)]$ et sur la réalisation effective de la multiplication, c'est-à-dire sur un calcul qui coïncide avec la composition des formes quadratiques binaires, enseignée par Gauss. Si l'on voulait traiter de la même manière tous les corps Ω de degré quelconque, on se heurterait à de grandes difficultés, peut-être insurmontables. Mais, lors même qu'il n'en serait pas ainsi, une telle théorie, fondée sur le calcul, n'offrirait pas encore, ce me semble, le plus haut degré de perfection."

²² "Wir gründen die Theorie der in \mathfrak{o} enthaltenen Zahlen, d. h. aller ganzen Zahlen des Körpers Ω , auf den folgenden neuen Begriff. Ein System \mathfrak{a} von unendlich vielen in \mathfrak{o} enthaltenen Zahlen soll ein Ideal heissen, wenn es den beiden Bedingungen genügt:

2.1 Returning to Riemann's epistemological ideals?

Ideals are grounding concepts because all propositions and theorems are deduced from the concept of ideal and the rules of computation designed for it.²³ Ideals are defined in such a way that they are a simple yet powerful concept. Moreover, as Dedekind always takes the time to underline and illustrate, their efficiency extends to other parts of number theory, such as the theory of binary quadratic forms.

The problems faced in Riemannian function theory are the same kind of problems as the ones Dedekind saw in Kummer's ideal numbers and the ones which made him give up higher congruences for his developments of Kummer's works: lack of uniformity in the treatments of the theory, investigations too attached to a determinate representation, lack of rigor, etc. As I have mentioned above, Dedekind and Weber wish to be able to give definitions and proofs for algebraic functions without restrictions made *a priori* on the singularities of functions and to provide a sound, rigorous ground to the theory, that mathematicians will be able to exploit. This is a desire that comes back several times in Dedekind's writings, before and after *Algebraische Funktionen*, and notably in 1888 when he defines the natural numbers (see Sec. 8.2).

In the case of Riemann's function theory, it is particularly crucial to be able to provide such a definition of the Riemann surface and other basic notions of the theory, inasmuch as Riemann did not give a precise, rigorous treatment of his ideas, and his successors were not able to adequately correct this flaw.

While they were working on the edition of Riemann's works, in 1874, Dedekind wrote to Weber that

I am not the profound expert on Riemann's work that you take me to be. I certainly know those works, and I *believe* in them, but I do not master them, and I will not master them until having overcome in my way, with the rigor that is customary in number theory, a whole series of obscurities.²⁴ (Letter to Weber, November 1874. Quoted and translated in [Ferreirós, 2008], 78)

As I have mentioned earlier, Riemann was described by Dedekind as having a "brilliant power of thought and [an] anticipatory imagination [which] led

I. Die Summe und die Differenz je zweier Zahlen in \mathfrak{a} sind wieder Zahlen in \mathfrak{a} . II. Jedes Product aus einer Zahl in \mathfrak{a} und einer Zahl in \mathfrak{o} ist wieder eine Zahl in \mathfrak{a} ."

²³The definition of ideals will be considered in more details in Sec. 5.4 and 6.1, once we will have more elements from Dedekind's writings on the subject.

²⁴"Namentlich bin ich nicht der gründliche Kenner der Riemann'schen Werke, für den Sie mich halten. Ich kenne zwar diese Werke und glaube an sie, aber ich beherrsche sie nicht, und ich werde sie nicht eher beherrschen, als bis ich eine ganze Reihe von Dunkelheiten mir auf meine Weise und mit der in der Zahlentheorie üblichen Strenge überwunden haben werde."

him frequently to take very great steps that others could not follow so easily” (quoted and translated in [Ferreirós, 2008], 78). When asked for details, examples or clarifications, Dedekind tells us that Riemann was “puzzled” and actually seemed to have a strong dislike for detailed explanations, leaving his readers with ambiguous and relatively vague statements. On the other hand, I mentioned that Dedekind described himself as slow to understand, needing to master each and every step of a theory. He spoke of his “*Treppen-Verstand*” in a letter to Cantor in 1899. For Dedekind, who learned the importance of rigor with Dirichlet, it was paramount to be in possession of a clear knowledge and understanding of the basic concepts of a theory and of the different stages of demonstrations, which he took to be typified by number theory.

2.1.3 Weber’s role?

The demand that a more general and more rigorous version of Riemann’s theory be given is shared by Weber, whose works on elliptic functions were directed towards providing an algebraic treatment of Riemann’s ideas. The 1882 paper shows obvious signs of Dedekind’s approach, adopting some of his methods and ideas with a literal translation from number to function theory, but Weber’s role should probably not be underestimated. Indeed, it is well known that Weber was the one who wrote up the paper properly after the exchange of letters with Dedekind in 1878-79, and sent it to Crelle’s Journal. In small details, such as the use of the word “*Spezies*”, common in Weber’s writings and never used by Dedekind, one can discern Weber’s touch. The few letters available also suggest that he played an important role in the elaboration of the theory.

As of yet, the letters in which Weber and Dedekind elaborated together their rewriting of Riemann’s theory have only been very partially published, but they show that Weber played a more than active role in setting up the new version of the theory.²⁵ In March 1879, while they were working on their paper, Weber wrote to Dedekind:

We must wait and see if the whole thing is going to lead to something new. What we have so far is not fundamentally new. Anyway, it is a very elegant and neat presentation for known propositions, and in this respect, it satisfies an esthetic requirement. What I expect from this in the first place is in fact a more rigorous, or at least general foundation for Riemann’s theory.²⁶
(Quoted in [Strobl, 1982], 232)

²⁵The forthcoming book by K. Scheel mentioned earlier shall provide us with more information on that matter. Unfortunately I have not been able to access the letters and was not be able to consider this aspect in any systematic way.

²⁶“*Ob die ganze Sache zu Etwas Neuem führen wird, müssen wir abwarten. Was wir bis jetzt haben ist im Grunde nicht neu. Immerhin ist es eine sehr elegante und hübsche Aus-*

2.1 Returning to Riemann's epistemological ideals?

The demands expressed by Dedekind are thus shared by Weber, whose experience in Riemannian mathematics was marked, as we saw (p. 106), by an attempt to prove the infamous Dirichlet principle.

A few months after this letter, Weber wrote to Dedekind, on December 18th 1879, that he considered the concept of point as “the main difficulty”, which could be overcome by relating it to prime ideals:

The main difficulty is, in my view, to be found still only in the definition of the “point of the R[iemann] surface”, and lately, I have made various attempts on it, which have led me to the conviction that the best could be to return to your original foundation of the theory of ideals, where then as long as possible, we are not talking about points, but only about prime ideals. From there, it should not be then too difficult to introduce the concept of “point” in a completely satisfactory way. Then, the theory of ideals would enter in its own right, and the path I have taken presents itself as a *Petitio Principii*.²⁷ (Quoted in [Strobl, 1982], 242)

But the observations made on p. 113 show that the presentation of algebraic function theory given by Weber himself in his *Lehrbuch*, gave up the use of ideals. It is not that Weber was against the use of Dedekind's algebraic concepts, though, since the notion of field is plainly exploited and ideals are used in number theory and for the study of forms. In fact, Weber's presentation of algebraic function theory is partly similar (even exactly similar at some points) to that of 1882, but rather than resorting to ideal theory, he reformulates all these paragraphs using *Functionale*. The main ideas underlying the 1882 paper are, however, kept (e.g., the definition of the point), and the theory proposed by Weber intertwines the approach adopted in 1882 with a focus on functions, proposing a study of the divisibility of the field by means of *Functionale* and relating points to “*Primfunctionale*” rather than prime ideals.

druckweise für bekannte Sätze und genügt insofern einem ästhetischen Bedürfnis. Was ich zunächst davon hoffe ist übrigens eine strenge oder wenigstens allgemeinere Begründung der Riemannschen Theorie.”

²⁷ *“Die hauptsächlichste Schwierigkeit erblicke ich nur noch in der Definition des ‘Punktes der R. Fläche’ und ich habe in der letzten Zeit mancherlei darauf bezügliche Versuche gemacht, die mich zu der Überzeugung geführt haben, dass es das Beste sein dürfte, zu Deiner ursprünglichen Begründung der Idealtheorie zurückzukehren, wo dann möglichst lange von den Punkten gar nicht die Rede ist, sondern nur Primidealen. Von da aus würde es dann wohl nicht schwer sein den Begriff ‘Punkt’ in ganz befriedigender Weise einzuführen. Dann würde die Idealtheorie in ihr volles Recht eintreten und der von mir eingeschlagene Weg stellt sich dar als eine Petitio Principii.”*

2.2 Fields of algebraic functions of one complex variable

The first and essential move to set up the “rigorous, or at least general foundation for Riemann’s theory” will be to introduce the notion of a field of functions. While Riemann, as well as many mathematicians, was well aware that rational and polynomial functions reproduce themselves by the four fundamental operations of arithmetic (+, −, ×, ÷), the choice to consider this very property as the ground of the theory is an important departure from the usual approaches to the theory.

2.2.1 Algebraic functions, integral algebraic functions

For the definition of algebraic functions, Dedekind and Weber define an algebraic function of an independent variable z in the following way:²⁸

A variable (*Variable*) θ is called an algebraic function of an independent variable (*abhängigen Veränderlichen*) z if it satisfies an irreducible equation

$$(1) \quad F(\theta, z) = 0.$$

Here F denotes an expression of the form

$$F(\theta, z) = a_0\theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n$$

where the coefficients a_0, a_1, \dots, a_n are polynomials in z without common divisor.²⁹ ([Dedekind and Weber, 1882], transl. slightly altered 45)

They further establish that θ does not satisfy any equation of lower degree in θ as well as in z , and that

²⁸According to Bourbaki, in [Bourbaki, 1984], “pas plus que leurs prédécesseurs, Kronecker ni Dedekind et Weber ne définissent en réalité la notion de “fonction algébrique” d’une ou plusieurs variables complexes. On ne peut en effet définir correctement une “fonction algébrique” d’une variable complexe (au sens de l’Analyse) qu’une fois définie la surface de Riemann correspondante, et c’est précisément la définition de la surface de Riemann (par des moyens purement algébriques) qui est le but poursuivi par Dedekind et Weber, Ce cercle vicieux apparent disparaît bien entendu quand on définit un corps de “fonctions algébriques” comme une extension algébrique abstraite d’un corps de fractions rationnelles : en fait, c’est uniquement de cette définition que se servent Dedekind et Weber, ce qui légitime pleinement leurs résultats.”

²⁹“Eine Variable θ heißt eine algebraische Funktion einer unabhängigen Veränderlichen z , wenn dieselbe einer irreduktibeln algebraischen Gleichung $F(\theta, z) = 0$ genügt. F bedeutet hierin einen Ausdruck von der Form $F(\theta, z) = a_0\theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n$ worin die Koeffizienten a_0, a_1, \dots, a_n , ganze rationale Funktionen von z ohne gemeinschaftlichen Teiler sind.”

2.2 Fields of algebraic functions of one complex variable

The system $\Phi(\theta, z)$ of all rational functions of θ and z has the property that its members reproduce themselves through the elementary arithmetic operations of addition, subtraction, multiplication, division, and this system is therefore called a *field Ω of algebraic functions of degree n* .³⁰ (*ibid.*, 46)

In modern terms, Dedekind and Weber further establish that $\mathbb{C}(z)[\theta]$ is a field, which is a finite algebraic extension of $\mathbb{C}(z)$, the field of rational functions of one variable z with coefficients in \mathbb{C} . An algebraic function in Ω must satisfy an irreducible equation (1) with coefficients (without common divisors) in $\mathbb{C}[\theta]$. The field generated by the algebraic function θ is thus $\mathbb{C}(z)[X]/(F(z, \theta))$.

The striking similarities of behavior between numbers and functions (which we would call structural similarities) are shed light on by taking the field Ω as the starting point of the theory. The concept of field, here, acts as a framework in which the theory is going to be developed.

Dedekind and Weber consider the function f obtained by the division of all the coefficients of F by a_0 the coefficient of θ^n :

$$f(\theta, z) = \theta^n + b_1\theta^{n-1} + \dots + b_{n-1}\theta + a_n.$$

From there, they define what they call an “algebraic division” of a function φ by the function f , using the Euclidean division between polynomials θ whose coefficients are rational functions of z . This allows to show that

Each function ζ in the field Ω is uniquely expressible in the form

$$\zeta = x_0 + x_1\theta + \dots + x_{n-1}\theta^{n-1}$$

where the coefficients x_0, x_1, \dots, x_{n-1} are rational functions of z . Conversely, each function of this form obviously belongs to the field Ω .³¹ (*ibid.*, 46)

With a good understanding of the functions of Ω , one has to identify the integers of the field (i.e., define the ring of integers). Indeed, the idea of the transfer of methods from number theory is to define for functions counterparts of the notions used in number theory, in order to be able to unfold

³⁰“Das System aller rationalen Funktionen von θ und z , $\Phi(\theta, z)$, hat die Eigenschaft, daß seine Individuen sich durch die elementaren Rechenoperationen, Addition, Subtraktion, Multiplikation und Division reproduzieren, und dies System wird daher als ein Körper algebraischer Funktionen Ω vom Grade n bezeichnet.”

³¹“Jede Funktion ζ des Körpers Ω läßt sich auf eine einzige Weise in die Form setzen: $\zeta = x_0 + x_1\theta + \dots + x_{n-1}\theta^{n-1}$ worin die Koeffizienten x_0, x_1, \dots, x_{n-1} rationale Funktionen von z sind. Umgekehrt gehört jede Funktion dieser Form selbstverständlich dem Körper Ω an.”

the ideal theory in function fields. In number theory, Dedekind defines algebraic numbers as numbers which are solution of a polynomial equation whose coefficients are rational numbers, and algebraic integers as algebraic numbers which are solution of monic polynomial equation whose coefficients are rational integers. In this context, he can introduce the notion of ideal and use it to investigate the divisibility laws of the integers of the field.

While Dedekind and Weber did mention Dedekind's algebraic integer theory and its relation to the 1882 paper, they also emphasized that they did not expect from their reader to be acquainted with them and consequently did not want to presuppose that any of the definitions and theorems they were about to give were already known. As a consequence, they give all definitions and proofs in extenso: every result, although already proved in 1877, is proved again. The theories will appear to be extremely similar but not fully identical, and some points are simpler to prove for functions – although, as Dieudonné mentions, Dedekind and Weber do not seem to clearly know why and, in any case, do not attempt to explain it. The fact that module and ideal theories are entirely developed for the theory of functions without relying on their development and validity in number theory emphasizes the idea that Dedekind and Weber are proceeding to a “translation” of the methods, implying no presupposition of the validity of the methods in a different (albeit somewhat analogous) framework.

For the transfer to fields of algebraic functions, there is the following correspondence in given fields:

rational integers	→	polynomials
rational numbers	→	rational functions
algebraic numbers	→	algebraic functions
algebraic integers	→	integral algebraic functions

Integral algebraic functions are defined by:

A function ω of the field Ω is called an *integral* function of z if in the equation of lowest degree satisfied by ω

$$\varphi(\omega) = \omega^e + b_1\omega^{e-1} + \dots + b_{e-1}\omega + b_e = 0$$

the coefficients b_1, b_2, \dots, b_n are all *polynomial functions* of z .³² (*ibid.*, transl. slightly altered, 52)

The collection of all integral algebraic functions in Ω is designated by \mathfrak{o} .

Interesting properties about the integral algebraic functions are properties related to the *system* of such functions, such as:

³²“Eine Funktion ω des Körpers Ω soll eine ganze Funktion von z heißen, wenn in der Gleichung niedrigsten Grades, welcher dieselbe genügt: $\varphi(\omega) = \omega^e + b_1\omega^{e-1} + \dots + b_{e-1}\omega + b_e = 0$ die Koeffizienten $s b_1, b_2, \dots, b_n$ ganze rationale Funktionen von z sind.”

2.2 Fields of algebraic functions of one complex variable

A rational function of z belongs to the system \mathfrak{o} if and only if it is a polynomial function of z .

Any function η in Ω can be transformed into a function of the system \mathfrak{o} by multiplication by a non zero polynomial of z . (...)

A function ω in the field Ω that satisfies any equation of the form

$$\psi(\omega) = \omega^m + c_1\omega^{m-1} + \dots + c_{m-1}\omega + c_m = 0$$

where the coefficients c_1, \dots, c_m are polynomial functions of z , is an *integral algebraic function*.³³ (*ibid.*, 52)

An important point is the arithmetical behavior of integral algebraic functions *as elements of the system* \mathfrak{o} . The system \mathfrak{o} is proved to be closed under addition, subtraction and multiplication (§3, 5).³⁴ The definition of the arithmetical operations follows the known rules of manipulation of rational functions. Divisibility in \mathfrak{o} requires a specific (albeit familiar) definition:

An integral algebraic function ω is said to be divisible by another integral algebraic function ω' if there exists a third integral algebraic function ω'' satisfying the condition

$$\omega = \omega'\omega''.$$

The first properties of the divisibility of integral algebraic functions are similar to divisibility of numbers, e.g., transitivity of divisibility or if $\omega_1, \omega_2, \dots$ are divisible by ω and $\omega'_1, \omega'_2, \dots$ are any functions in \mathfrak{o} , then $\omega'_1\omega_1 + \omega'_2\omega_2 + \dots$ is divisible by ω .

The complete study of the divisibility laws governing fields of algebraic functions will be done by resorting to ideal theory. Before the introduction of modules and ideals, Dedekind and Weber consider the notions of norm of functions, trace and discriminant in the field Ω .

2.2.1.1 Norms, discriminant, trace

In *Algebraische Funktionen*, Dedekind and Weber develop notions pertaining to what we would call “linear algebra” for the field Ω : norm of a function, determinant, trace and discriminant.

³³ “Eine rationale Funktion von z gehört dann und nur dann zu dem System \mathfrak{o} , wenn sie eine ganze rationale Funktion von z ist.

Jede Funktion in Ω kann durch Multiplikation mit einer von Null verschiedenen ganzen rationalen Funktion von z in eine Funktion des Systems \mathfrak{o} verwandelt werden. (...)

Eine Funktion ω des Körpers Ω , welche irgend einer Gleichung von der Form genügt $\psi(\omega) = \omega^m + c_1\omega^{m-1} + \dots + c_{m-1}\omega + c_m = 0$ in welcher die Koeffizienten c_1, \dots, c_m ganze rationale Funktionen von z sind, ist eine ganze Funktion.”

³⁴Dedekind and Weber do not introduce anything similar to a ring, in this paper.

The “analogy” with number theory likely played an important role in the set up of these notions, which are most useful in the proofs. Indeed, for algebraic numbers, norms play an essential role in the study of divisibility: insofar as the norm of an algebraic number is a rational number, using norms allows to reduce the problems to the study of rational numbers. This strategy was used by Gauss and Kummer (see Sec. 5.1.2) and was well-known. In the case of algebraic functions, the norm will be a rational function, which continues further the correspondence stated earlier. Determinants and discriminants were usual tools for the theory of forms, and can be found in works from Gauss, Sylvester, Gordan, Kronecker. . . The notion of discriminant is also particularly important because of its relations with the ramification of the field, which we will see later. Dedekind’s and Weber’s innovations thus stem from prevalent methods.

Note that “trace” was not a very widespread *word* and seems to have been introduced by Dedekind in algebraic number theory. It was usually solely referred to as a “coefficient of the characteristic equation” ([Hawkins, 2008], 500). As such, it was well-known, and in 1895, Dedekind referred to “norms, discriminants and traces” as “common” ([Dedekind, 1895a], in [Dedekind, 1932] II, 59). According to Hawkins, it was not such a common expression, and Frobenius, “following the lead of Dedekind”, was the one to begin to popularize it in his works on matrices. Weber also uses it in his *Lehrbuch*. The notions and proofs given here, about the basis of the field and the notions of trace, discriminant, etc., played an important role in the development of linear algebra as a discipline.

A “basis” of the field is defined, a strategy used generously in all of Dedekind’s and Weber’s works. Here, in modern terms, the field Ω studied by Dedekind and Weber is considered as a vector space over $\mathbb{C}(z)$.³⁵ We saw in the above paragraph that any function ζ in Ω can be uniquely written as

$$\zeta = x_0 + x_1\theta + \dots + x_{n-1}\theta^{n-1}$$

(where x_i are rational functions of z) and conversely any function of this form belongs to Ω .

Dedekind and Weber chose n functions in all the functions of Ω such that:

$$\begin{aligned} \eta_1 &= x_0^{(1)} + x_1^{(1)}\theta + \dots + x_{n-1}^{(1)}\theta^{n-1} \\ \eta_2 &= x_0^{(2)} + x_1^{(2)}\theta + \dots + x_{n-1}^{(2)}\theta^{n-1} \\ &\dots \\ \eta_n &= x_0^{(n)} + x_1^{(n)}\theta + \dots + x_{n-1}^{(n)}\theta^{n-1} \end{aligned}$$

and such that the determinant $\sum \pm x_0^{(1)} x_1^{(2)} \dots x_{n-1}^{(n)} \neq 0$. They show that every function of Ω can be written as a linear combination of η_i with rational

³⁵[Stillwell, 2012]

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functions y_i as coefficients:

$$\zeta = y_1\eta_1 + y_2\eta_2 + \dots + y_n\eta_n$$

This system $\eta_1, \eta_2, \dots, \eta_n$ is called a *basis* of the field Ω ([Dedekind and Weber, 1882], 47). In particular, the remark above is equivalent to saying the functions $1, \theta, \theta^2, \dots, \theta^{n-1}$ are a basis of Ω .

To define the norm of a ζ function of Ω , Dedekind and Weber consider $\eta_1, \eta_2, \dots, \eta_n$ a basis of Ω . Any function ζ of Ω gives rise to the following decompositions:

$$\begin{aligned}\zeta\eta_1 &= y_{1,1}\eta_1 + y_{1,2}\eta_2 + \dots + y_{1,n}\eta_n \\ \zeta\eta_2 &= y_{2,1}\eta_1 + y_{2,2}\eta_2 + \dots + y_{2,n}\eta_n \\ &\dots \\ \zeta\eta_n &= y_{n,1}\eta_1 + y_{n,2}\eta_2 + \dots + y_{n,n}\eta_n\end{aligned}$$

Dedekind and Weber derive from the fact that the η_i form a basis that the determinant of the system must vanish:

$$\begin{vmatrix} y_{1,1} - \zeta & y_{1,2} & \dots & y_{1,n} \\ y_{2,1} & y_{2,2} - \zeta & \dots & y_{2,n} \\ \dots & \dots & \dots & \dots \\ y_{n,1} & y_{n,2} & \dots & y_{n,n} - \zeta \end{vmatrix} = 0.$$

This determinant has the “form” (*Gestalt*) of a polynomial ϕ in ζ of degree n in ζ whose coefficients are rational functions of z (completely determined by ζ). It is independent of the choice of the basis. If one arranges the functions according to the powers of ζ , some of its coefficients are particularly important “for what follows” and will, thus, be “given special names”. The norm of ζ , $N(\zeta)$, is the coefficient b_n up to the sign:

$$(-1)^n b_n = \begin{vmatrix} y_{1,1} & y_{1,2} & \dots & y_{1,n} \\ y_{2,1} & y_{2,2} & \dots & y_{2,n} \\ \dots & \dots & \dots & \dots \\ y_{n,1} & y_{n,2} & \dots & y_{n,n} \end{vmatrix}$$

The norm of ζ satisfies common properties of the notion of norm, in particular, $N(\zeta) = 0$ if and only if $\zeta = 0$.

The trace $S(\zeta)$ ³⁶ is

$$S(\zeta) = -b_1 = y_{1,1} + y_{2,2} + \dots + y_{n,n}.$$

The trace of ζ satisfies the properties:

³⁶Stillwell replaced Dedekind and Weber’s notation S for the trace (*Spur*) by the nowadays common Tr . I will keep the original notation.

- $S(0) = 0, S(1) = n$
- For a rational function $x, S(x\zeta) = xS(\zeta)$
- $S(\zeta + \zeta') = S(\zeta) + S(\zeta')$

These previous considerations allow to prove that

Every function ζ in Ω satisfies an n th degree equation $\varphi(\zeta) = 0$ whose coefficients are rationally dependent from z . When this equation is irreducible, the functions $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$ form a basis of Ω .³⁷ (*ibid.*, transl. slightly altered, 49)

Lastly, for the definition of the discriminant, Dedekind and Weber consider an arbitrary system $\eta_1, \eta_2, \dots, \eta_n$ of functions of Ω . The discriminant of this system is a special kind of determinant:

$$(12) \quad \Delta(\eta_1, \dots, \eta_n) = \begin{vmatrix} S(\eta_1, \eta_1) & S(\eta_1, \eta_2) & \dots & S(\eta_1, \eta_n) \\ S(\eta_2, \eta_1) & S(\eta_2, \eta_2) & \dots & S(\eta_2, \eta_n) \\ \dots & \dots & \dots & \dots \\ S(\eta_n, \eta_1) & S(\eta_n, \eta_2) & \dots & S(\eta_n, \eta_n) \end{vmatrix}$$

The discriminant is different from zero if and only if the system $\eta_1, \eta_2, \dots, \eta_n$ is a basis of Ω , so it can be used to “test” whether a certain system is a basis of the field or not. The “fundamental theorem on discriminants” gives the following relation: Consider a system of n functions

$$\eta'_k = x_{1,k}\eta_1 + x_{2,k}\eta_2 + \dots + x_{n,k}\eta_n$$

and let X be the determinant of the functions $x_{i,k}$, i.e.,

$$X = \sum \pm x_{1,1}x_{2,2} \dots x_{n,n}$$

then

$$\Delta(\eta'_1, \dots, \eta'_n) = X^2 \Delta(\eta_1, \dots, \eta_n)$$

I will not develop further the considerations on these notions, for they are mostly formal and, besides the essential properties stated above, they will not be essential to understand the rest of the theory.

2.3 Ideal theory and the laws of divisibility in the field Ω

The word “analogy” was used earlier to designate the “transfer” (*Übertragung*) of methods from number theory to function theory.³⁸ This word

³⁷ “Es hat sich aus dieser Betrachtung ergeben, daß jede Funktion ζ in Ω einer Gleichung n ten Grades, $\varphi(\zeta) = 0$ genügt, deren Koeffizienten rational von z abhängen. Wenn diese Gleichung irreduktibel ist, so bilden die Funktionen $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$ eine Basis von Ω .”

³⁸ On analogies in terms of “axiomatic characterization” and how they can be used to transfer results or methods from one domain to another, see also [Schlimm, 2008].

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should be taken as meaning a middle ground between “seeing a resemblance” and “exploiting a correspondence” between algebraic numbers and algebraic functions. In particular, it does not mean the weaker version of the word “analogy” in which the correspondence works in only one way.³⁹

On the other hand, it should be clear that the correspondence between numbers and functions is essential for the sole reason that it allows to transfer and set up, in the context of function theory, the conceptual arsenal used by Dedekind in number theory, using the fact that the concepts were defined solely by conditions of closure by operations. So, it must be stressed that in the previous considerations, as well as in what follows, Dedekind and Weber do not identify algebraic functions with algebraic numbers, they do not take functions ‘as if’ they were numbers. Ideals play the central role in both the theory of numbers and the theory of functions. The role attributed to the ideals is crucial as a way to investigate properties of the integers of the field, and this appears to be very clear to the authors.⁴⁰

Finally, the “analogy” should not be taken as suggesting that Dedekind and Weber are investigating ‘structural similarities’ between fields of functions and fields of numbers. It is not their aim at all – and in addition, fields, ideals or modules are not thought of as abstract concepts, in the sense of (our) modern algebra. If these concepts have an ‘abstract’ definition (which Weber calls “arbitrary” ([Weber, 1893]) and Dedekind calls “invariant” ([Edwards, Neumann, and Purkert, 1982]), that is, a definition did not refer to the individual nature of their elements, never is a general theory of these concepts developed by Dedekind himself.⁴¹

The basis, the foundation of Dedekind’s algebra is the concept of field (whose relation to equations has been evoked earlier), and only the operations defined and the closure by the operations are necessary to apply his methods. Ideals, like fields, are defined as being systems of elements united by the fact that they satisfy certain properties of closure of arithmetic operations, and without reference to the individual nature of the elements. With the appropriate notion of integer of the field and the definition of divisibility, it is possible to transfer the algebraic tools.

³⁹Note that Dedekind is fully aware of such a distinction between “analogy” and “correspondence”, as he mentions something similar in *Stetigkeit*:

This analogy between rational numbers and the points of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point O and a definite unit of length for the measurement of segments.

In particular, Dedekind will transfer back to number theory some of the notions developed here in a paper published in 1882, *Über die Diskriminanten endlicher Körper*. See p. 374.

⁴⁰Later, Weber’s use of a middle ground between Kronecker’s approach (published in 1881), in the *Lehrbuch*, highlights that what he considered as essential is to be able to link the definition of the Riemann surface to the divisibility of the field.

⁴¹It will be developed in the works of Steinitz, E. Noether, etc. On these matters, see [Corry, 2004b].

The main idea behind the translation of algebraic number methods to the theory of algebraic functions is to study divisibility in a field of algebraic functions relying on the introduction of the concept of integral functions, just as Dedekind did with algebraic numbers. The development of ideal theory is very much similar to what is done in 1877 – it is sometimes the same word for word, with “function” in place of “number” – but it is *not strictly* identical. Dedekind and Weber underline the differences when they appear. It is not my aim, here, to compare the two theories, and I will not mention every difference.⁴² In fact, if Dedekind and Weber do mention the differences in footnotes, their aim is not to try to expose what the theory of algebraic numbers and the theory of algebraic functions have in common. They do not explain the differences and do not try to extract a common core for the two theories.

In addition, the two theories are fundamentally different in their aims, which, I believe, makes a comparison slightly out of phase with the initial goals of the theories. In algebraic number theory, ideal theory is essential, and if ideal theory can indeed have many applications, the understanding of how the divisibility laws hold uniformly and generally is the goal of Dedekind’s research. In 1882, the investigation of divisibility laws, i.e., the part of the paper similar to algebraic number theory, holds a significantly smaller place, and the goal of the research, ultimately, is to improve on Riemann’s function theory. Insofar as Dedekind and Weber are not proposing an algebraic (in the modern abstract sense) investigation of fields, it seems that this difference is important enough to not dwell on the comparison of ideals of numbers and ideals of functions. Rather, it would be interesting to understand how Dedekind developed an approach for algebraic numbers that can be transferred to algebraic functions of one variable.

Since the notion of field of functions, as Dedekind and Weber write in the introduction, “coincides completely with that of Riemann class of algebraic functions”, they first study in depth the properties of the field, and then build the concept of Riemann surface on this basis. Dedekind and Weber will, in the second part of the paper, exhibit the said “coincidence” by establishing a one-to-one correspondence between the field and the surface.

2.3.1 Modules

Before presenting the theory of ideals of the field Ω ,⁴³ Dedekind and Weber consider the “auxiliary” theory of modules. In earlier works, modules

⁴²For such a comparison, see [Strobl, 1982].

⁴³In fact, of course, it is the theory of ideals of the ring of integers of Ω . Dedekind, however, does not talk about rings, and tends to use indifferently “ideals of \mathfrak{o} ”, the system of integers of the field, or “ideals of the field”. Following him, I will often make the misuse of language of talking about “ideals in the field Ω ”.

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were presented by Dedekind as an “independent theory”, general and only interesting for its applications.

it seems appropriate to begin with a small number of very simple theorems [about modules theory], even though their interest lies mainly in their applications. (...) The researches in this first chapter have been expounded in a special form suited to our goal, but it is clear that they do not cease to be true when the Greek letters denote not only numbers, but any objects of study. ([Dedekind, 1876-1877], 62 and 92)

Module theory is, as soon as it is introduced, given as a theory potentially independent from the theory in which it is used. Modules will be particularly interesting because they allow for the definition of a congruence relation for functions.⁴⁴

A module is defined by the following closure conditions:

A system of functions (in Ω) is called a module if its functions reproduce themselves by addition, subtraction and multiplication by a polynomial function of z .⁴⁵ (*ibid.*, transl. modified, 55)

A particular kind of module will be used copiously: modules with a finite basis, which are called “finite modules” by Dedekind and Weber:⁴⁶

One designates by $\alpha_1, \alpha_2, \dots, \alpha_m$ any m given functions and by x_1, x_2, \dots, x_m arbitrary polynomials in z . Then, the collection of all functions of the form

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

is a module. Such a module is called a *finite module* and is denoted by

$$\mathfrak{a} = [\alpha_1, \alpha_2, \dots, \alpha_m]$$

The system of functions $\alpha_1, \alpha_2, \dots, \alpha_m$ is called the *basis* of this module.⁴⁷ (*ibid.*, transl. modified, 55)

⁴⁴Weber, in his own version of algebraic function theory, in the *Lehrbuch*, did not use module theory at all. In fact, the concept of module doesn’t appear as such in Weber’s *Lehrbuch*.

⁴⁵“*Ein Funktionensystem (in Ω) heißt ein Modul, wenn sich die Funktionen desselben durch Addition, Subtraktion und durch Multiplikation mit ganzen rationalen Funktionen von z reproduzieren.*”

⁴⁶Although they do not correspond to what we would call a “finite module” but rather to “finitely generated modules”, I will adopt a literal translation, unlike Stillwell.

⁴⁷“*Bezeichnet man mit $\alpha_1, \alpha_2, \dots, \alpha_m$ irgend m gegebene Funktionen, mit x_1, x_2, \dots, x_m willkürliche ganze rationale Funktionen von z , so bildet der Inbegriff aller Funktionen von der Form $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$ einen Modul. Ein solcher soll ein endlicher Modul gent und mit $\mathfrak{a} = [\alpha_1, \alpha_2, \dots, \alpha_m]$ bezeichnet werden. Das Funktionensystem $\alpha_1, \alpha_2, \dots, \alpha_m$ heißt die Basis dieses Moduls.*”

Note that the concept of basis used here corresponds only to a spanning property. The linear independence, which Dedekind and Weber call “rational independence” is an additional property of the basis of finite modules but not a necessary one.

A very important part of the paper is the definition of arithmetical operations, here for modules, but also for ideals – and later, again, on the surface. The idea used here is the one that constitutes the core of Dedekind’s algebraic number theory and allowed him to prove the general validity of the propositions and theorems of arithmetic. A new notion of divisibility is defined for modules:

A module \mathfrak{a} is said to be divisible by a module \mathfrak{b} , or \mathfrak{b} is said to be a divisor (*Teiler (Divisor)*) of \mathfrak{a} , or \mathfrak{a} a multiple (*Vielfaches (Multiplum)*) of \mathfrak{b} (\mathfrak{b} taken up in \mathfrak{a} [\mathfrak{b} geht in \mathfrak{a} auf]), if all function in \mathfrak{a} also belongs to \mathfrak{b} . We call \mathfrak{b} a proper divisor of \mathfrak{a} if \mathfrak{a} is divisible by \mathfrak{b} but not identical to \mathfrak{b} .⁴⁸ (*ibid.*, 57)

The definition of divisibility for modules here is a different way of expressing the relation of inclusion: to divide is to contain. Thus, the divisor “contains a greater number of functions” than the module it divides. The definition inverts the ‘size’ of the divisor and the multiple, which can seem “intuitively opposite” to divisibility of numbers, as Dedekind and Weber note. However, once accepted this definition of divisibility, which as underlined by Stillwell corresponds to the idea that the multiples of \mathfrak{a} are included in the multiples of \mathfrak{b} , the study of divisibility properties will be developed following the lines similar to rational number divisibility. Insofar as, as we will see, it can easily be seen that “every ideal is itself a module”, Dedekind and Weber will only refer to the definitions and propositions given for modules, for the basis of the divisibility of ideals, which is why I will take the time to give a fair amount of details, here.

Divisibility for modules satisfies the transitivity property. Just like functions are not taken to be numbers for the transfer of methods, the definition of arithmetic operations between objects that are not numbers (ideals, modules, ...) does not imply that they should be considered as being ‘like numbers’. In particular there is no proposition or theorem considered to be evident because of its validity for numbers.

Dedekind and Weber rely on the reformulation of inclusion of modules as a divisibility relation to put into play arithmetical notions and ideas in their study of the properties of the field. Dedekind and Weber develop this new

⁴⁸ “Ein Modul \mathfrak{a} heißt durch einen Modul \mathfrak{b} teilbar, oder \mathfrak{b} ein Teiler (Divisor) von \mathfrak{a} , \mathfrak{a} ein Vielfaches (Multiplum) von \mathfrak{b} (\mathfrak{b} geht in \mathfrak{a} auf), wenn jede Funktion in \mathfrak{a} zugleich in \mathfrak{b} enthalten ist. \mathfrak{b} soll ein echter Teiler von \mathfrak{a} heißen, wenn \mathfrak{a} durch \mathfrak{b} teilbar, aber nicht mit \mathfrak{b} identisch ist.”

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terminology, in such a way that a new level of arithmetic is established, in which they can transfer the study of the divisibility laws of the field Ω from integral algebraic functions to modules and ideals.⁴⁹ For this, they develop the module counterparts of familiar arithmetical notions:

The collection \mathfrak{m} of all the functions belonging to both modules \mathfrak{a} and \mathfrak{b} , unless it consists only of the “zero” function, is a module (according to the general definition) called the *least common multiple of \mathfrak{a} and \mathfrak{b}* .⁵⁰ (*ibid.*, 57)

This choice of terminology, the least common multiple, to designate the intersection of two modules, is justified by the fact that any module which is a multiple of both \mathfrak{a} and \mathfrak{b} (i.e., included in both \mathfrak{a} and \mathfrak{b}) is also a multiple of \mathfrak{m} (i.e., contained in their intersection). The LCM of an arbitrary number of modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$, is thus the collection of all the functions belonging to all these modules. It is possible to form this latter LCM since one can always take any two modules \mathfrak{a} and \mathfrak{b} , find their LCM and use it in the place of the two modules \mathfrak{a} and \mathfrak{b} , to find the LCM of \mathfrak{a} and \mathfrak{b} and a third module, and so on.

Likewise, a notion of greatest common divisor of modules is given:

If α is an arbitrary function in \mathfrak{a} , β an arbitrary function in \mathfrak{b} , then the collection of all the functions of the form $\alpha + \beta$ forms a module \mathfrak{d} called the *greatest common divisor of \mathfrak{a} and \mathfrak{b}* .⁵¹ (*ibid.*, 57)

The justification for the arithmetical terminology goes along the same lines: the GCD is defined in such a way that it is “a divisor of both \mathfrak{a} and \mathfrak{b} ” (i.e., $\mathfrak{a} \subset \mathfrak{d}$ and $\mathfrak{b} \subset \mathfrak{d}$). And if a module \mathfrak{d}' divides “both \mathfrak{a} and \mathfrak{b} , then the functions α [of \mathfrak{a}] and β [of \mathfrak{b}]” belong to \mathfrak{d}' , hence “the functions $\alpha + \beta$ belong to \mathfrak{d}' ”, which means that $\mathfrak{d} \subset \mathfrak{d}'$ i.e., \mathfrak{d}' divides \mathfrak{d} .

Once the notion of divisibility of modules is established, Dedekind and Weber give the definition of the product and the quotient of two modules. They appear to be setting up an arithmetical framework for the study of modules, so that they can handle the theory (of modules and of ideals) with methods of proof that are similar to those in elementary number theory.

⁴⁹As we will see, the arithmetic of ideals is relatively simple, inasmuch as it is similar to that of rational integers.

⁵⁰“Der Inbegriff \mathfrak{m} aller derjenigen Funktionen, welche zugleich in zwei Moduln $\mathfrak{a}, \mathfrak{b}$, enthalten sind, bildet, falls er nicht aus der einzigen Funktion ‘Null’ besteht, einen Modul (nach der allgemeinen Definition), welcher das kleinste gemeinschaftliche Vielfache von \mathfrak{a} und \mathfrak{b} heißt.”

⁵¹“Ist α eine beliebige Funktion in \mathfrak{a} , β eine beliebige Funktion in \mathfrak{b} , so bildet der Inbegriff aller Funktionen von der Form $\alpha + \beta$ einen Modul, welcher der größte gemeinschaftliche Teiler der beiden Moduln \mathfrak{a} und \mathfrak{b} .”

The reformulation of inclusion as a divisibility relation is not an *ad hoc* introduction of some familiar denomination, the inquiry takes an overall arithmetical form, and the theorems and their proofs are given under an arithmetical form.

The multiplication of two modules is defined in the following way:

If \mathfrak{a} , \mathfrak{b} are two modules, and if α , β are any two functions in \mathfrak{a} and \mathfrak{b} respectively, then by the product

$$\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a} = \mathfrak{c}$$

we understand the collection of all products of a function α by a function β , and all the sums of such products, and thus of all the functions that can be represented by the expression

$$\gamma = \sum \alpha\beta.^{52}$$

(*ibid.*, transl. modified, 58)

It is easy to prove that this system is always a module. Dedekind and Weber stress that fact that it is a finite module if \mathfrak{a} and \mathfrak{b} are finite modules. The definition can be extended to any number of modules, and satisfies the “fundamental theorem of the multiplication”, that of the permutability of factors (*Vertauschbarkeit der Faktoren*). And they further define a notion of m th power of a module \mathfrak{a} , by taking m times all the factors equal to \mathfrak{a} .

[W]e denote it by \mathfrak{a}^m and we have

$$\mathfrak{a}^{m+m'} = \mathfrak{a}^m \mathfrak{a}^{m'}.^{53}$$

(*ibid.*, 58, transl. slightly altered)

Note that, in general, as Dedekind and Weber emphasize, the “product $\mathfrak{a}\mathfrak{b}$ is *not* divisible by \mathfrak{a} ”, but we have the property that “if \mathfrak{a}_1 divides \mathfrak{a} and \mathfrak{b}_1 divides \mathfrak{b} , then $\mathfrak{a}_1\mathfrak{b}_1$ divides $\mathfrak{a}\mathfrak{b}$.”

The quotient of two modules \mathfrak{a} and \mathfrak{b} , denoted by $\frac{\mathfrak{b}}{\mathfrak{a}}$, is defined as:

the collection of all those functions γ with the property that $\gamma\mathfrak{a}$ is divisible by \mathfrak{b} . This quotient, unless it consists of the single function “zero” is itself a module \mathfrak{c} , as is immediately clear from the definition. The product $\frac{\mathfrak{b}}{\mathfrak{a}}\mathfrak{a}$ is always divisible by \mathfrak{b} though

⁵² “Sind \mathfrak{a} , \mathfrak{b} zwei Moduln, α , β sämtliche Funktionen in \mathfrak{a} , resp. in \mathfrak{b} , so verstehen wir unter dem Produkt $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a} = \mathfrak{c}$ den Inbegriff aller Produkte einer Funktion α und einer Funktion β und aller Summen solcher Produkte, also sämtlicher Funktionen, welche durch das Zeichen $\gamma = \sum \alpha\beta$ bezeichnet werden können.”

⁵³ “So wird dasselbe mit \mathfrak{a}^m bezeichnet, und es ist $\mathfrak{a}^{m+m'} = \mathfrak{a}^m \mathfrak{a}^{m'}$ ”

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not always equal to \mathfrak{b} .⁵⁴ (*ibid.*, 58)

Modules, as their name suggests, allow to define a notion of congruence for functions, which makes them very important for developing an ‘arithmetical’ theory. Two functions α, β are said to be congruent modulo \mathfrak{a} (or “according to \mathfrak{a} ”) when the difference $\alpha - \beta$ is contained in \mathfrak{a} . This notion of congruence satisfies the usual properties of congruence as defined by Gauss, such as the transitivity of the relation, multiplication by a third function, addition of congruences, etc.

Pursuing the transfer of number-theoretical notions to the higher level of modules, Dedekind and Weber next define a notion of norm for modules (and later for ideals). For this, they first consider what they call a “family” (*Schaar*) of functions.⁵⁵ A family of functions noted $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is the collection formed by the functions of the form

$$c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m$$

where the λ_i are given functions in Ω and the c_i are arbitrary constants. The functions $\lambda_1, \lambda_2, \dots, \lambda_m$ form a “basis” (in Dedekind and Weber’s sense) of the family. Here again, the functions are said to be linearly irreducible or linearly independent “if an equation (identity) of the form

$$c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m = 0$$

holds only when the constant coefficients c_1, c_2, \dots, c_m all vanish.” Such a basis is called an irreducible basis by Dedekind and Weber. They state that since any family has an irreducible basis, they will use the word “basis” in reference to “irreducible basis” from this point on. The number of functions in the irreducible basis is always the same, here m , and is called the dimension of the family. A family with dimension m is called an m -tuple by Dedekind and Weber.

The notion of linear independence can be extended into a congruence version of it:

The functions $\lambda_1, \lambda_2, \dots, \lambda_m$ are called *linearly independent according to the module \mathfrak{a}* if a congruence of the form

$$c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m \equiv 0 \pmod{\mathfrak{a}}$$

⁵⁴ “Unter dem Quotienten $\frac{\mathfrak{b}}{\mathfrak{a}}$ zweier Moduln $\mathfrak{a}, \mathfrak{b}$ soll der Inbegriff aller derjenigen Funktionen γ , verstanden werden, welche die Eigenschaft haben, daß $\gamma\mathfrak{a}$ durch \mathfrak{b} teilbar ist. Dieser Quotient ist, falls er nicht aus der einzigen Funktion ‘Null’ besteht, ein Modul \mathfrak{c} , was sofort aus der Definition erhellt. Das Produkt ist jederzeit durch \mathfrak{b} teilbar, wenn auch nicht immer gleich \mathfrak{b} .”

⁵⁵In modern terms, this corresponds to a vector space, which is the translation adopted by Stillwell. I will, however, not follow him on that point and rather use the word “family”. Note that “*Schaar*” is translated in French by “*faisceau*” in [Müller, 1900].

holds only when the constants c_1, c_2, \dots, c_m vanish.⁵⁶ (*ibid.*, transl. slightly altered, 60)

This allows to define the notion of “complete system of remainders” of a module \mathfrak{a} according to another module \mathfrak{b} . For this, Dedekind and Weber suppose, first, that there exists in \mathfrak{b} only a finite number of functions linearly independent modulo \mathfrak{a} . Then, they take such a system with the greatest number of functions possible $\lambda_1, \lambda_2, \dots, \lambda_m$ in \mathfrak{b} linearly independent according to \mathfrak{a} , then any function β in \mathfrak{b} can be uniquely written as

$$\beta \equiv c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m \pmod{\mathfrak{a}}$$

where the coefficients c_1, c_2, \dots, c_m are constants. The family $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is called “a complete system of remainders of \mathfrak{b} modulo \mathfrak{a} ,” whose basis is by definition $\lambda_1, \lambda_2, \dots, \lambda_m$. It can be written as

$$\mathfrak{b} \equiv (\lambda_1, \lambda_2, \dots, \lambda_m) \pmod{\mathfrak{a}}$$

So, the family $(\lambda_1, \lambda_2, \dots, \lambda_m)$ consists of the congruence classes of \mathfrak{b} modulo \mathfrak{a} .

The norm of a module according to another module is a function dependent on only these two modules. To define it, Dedekind and Weber consider $(\lambda_1, \lambda_2, \dots, \lambda_m)$ a complete system of remainders of \mathfrak{b} modulo \mathfrak{a} . Since the module the module $z\mathfrak{b}$ is contained in the module \mathfrak{b} , there exists m^2 complex numbers $c_{h,k}$ such that:

$$\left. \begin{array}{l} z\lambda_1 \equiv c_{1,1}\lambda_1 + c_{2,1}\lambda_2 + \dots + c_{m,1}\lambda_m \\ z\lambda_2 \equiv c_{1,2}\lambda_1 + c_{2,2}\lambda_2 + \dots + c_{m,2}\lambda_m \\ \dots \\ z\lambda_m \equiv c_{1,m}\lambda_1 + c_{2,m}\lambda_2 + \dots + c_{m,m}\lambda_m \end{array} \right\} \pmod{\mathfrak{a}}$$

The resolution of the system allows to see that the multiplication by the determinant of the system will transform any function λ_i in \mathfrak{b} into a function in \mathfrak{a} via the multiplication by

$$(\mathfrak{b}, \mathfrak{a}) = (-1)^n \begin{vmatrix} c_{1,1} - z & c_{2,1} & \dots & c_{m,1} \\ c_{1,2} & c_{2,2} - z & \dots & c_{m,2} \\ \dots & \dots & \dots & \dots \\ c_{1,m} & c_{2,m} & \dots & c_{m,m} - z \end{vmatrix}$$

The function $(\mathfrak{b}, \mathfrak{a})$ “is independent of the choice of the basis $\lambda_1, \lambda_2, \dots, \lambda_m$ and hence dependent only on the two modules $\mathfrak{a}, \mathfrak{b}$ ” (*ibid.*, 61) and is called

⁵⁶ “Die Funktionen $\lambda_1, \lambda_2, \dots, \lambda_m$ heißen linear unabhängig in bezug auf den Modul \mathfrak{a} , wenn eine Kongruenz von der Form

$$c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m \equiv 0 \pmod{\mathfrak{a}}$$

für keine anderen als verschwindende konstante Koeffizienten c_1, c_2, \dots, c_m besteht.”

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“the norm of \mathfrak{a} relative to \mathfrak{b} ”. In particular, if \mathfrak{b} is divisible by \mathfrak{a} , then by convention $(\mathfrak{b}, \mathfrak{a}) = 1$. If \mathfrak{b} contains “more than a finite number of functions (...) linearly independent modulo \mathfrak{a} ”, then Dedekind and Weber set $(\mathfrak{b}, \mathfrak{a}) = 0$ (*ibid.*, 61).

This norm satisfies:

- If \mathfrak{m} is the LCM and \mathfrak{d} the GCD of \mathfrak{a} and \mathfrak{b} , then $(\mathfrak{b}, \mathfrak{a}) = (\mathfrak{b}, \mathfrak{m}) = (\mathfrak{d}, \mathfrak{a})$.
- If \mathfrak{a} is divisible by \mathfrak{b} which is itself divisible by \mathfrak{c} , then $(\mathfrak{c}, \mathfrak{a}) = (\mathfrak{c}, \mathfrak{b})(\mathfrak{b}, \mathfrak{a})$ – the multiplicative property of the relative norm.

2.3.2 Ideals

In the field Ω , an ideal is a system of integral algebraic functions (i.e., contained in \mathfrak{o}) defined by the following necessary and sufficient conditions:

A system \mathfrak{a} of integral functions of z in the field Ω is called an *ideal* if it satisfies the two following conditions:

- I. The sum and difference of every pair of functions of \mathfrak{a} are again functions of \mathfrak{a} .
- II. The product of any function in \mathfrak{a} by any function in \mathfrak{o} is again a function of \mathfrak{a} .⁵⁷ (*ibid.*, transl. slightly altered, 66)

Note that Dedekind and Weber do not define a notion of ring. \mathfrak{o} is only defined as containing all the integers of the field Ω (and proved to satisfy certain closure properties).

Dedekind and Weber explain that since “every ideal is at the same time a module, all the terms and notations defined for modules can be applied to ideals”.⁵⁸ In particular, the notions of divisibility and product can be carried over to ideals. Since every ideal is contained in \mathfrak{o} (which is of course itself an ideal), each ideal is divisible by \mathfrak{o} which plays the role of the unit for the arithmetic of ideals.

A particular kind of ideal will be particularly useful, the principal ideals, defined by:

If μ is any non-zero function in \mathfrak{o} , the module $\mathfrak{o}\mu$ (the system of all the integral functions divisible by μ) is an ideal. Such an

⁵⁷“Ein System \mathfrak{a} von ganzen Funktionen von z im Körper Ω heißt ein Ideal, wenn es die beiden folgenden Bedingungen erfüllt: I. Summe und Differenz je zweier Funktionen in \mathfrak{a} ergeben wieder eine Funktion in \mathfrak{a} . II. Das Produkt einer jeden Funktion in \mathfrak{a} mit einer jeden Funktion \mathfrak{o} ist wieder eine Funktion in \mathfrak{a} .”

⁵⁸“Jedes Ideal ist also zugleich ein Modul und alle für die Moduln erklärten Begriffe und Bezeichnungen können auf die Ideale angewandt werden.”

ideal is called a *principal ideal*.⁵⁹ (*ibid.*, transl. slightly altered, 66)

A principal ideal $\mathfrak{o}\mu$ can be written as a finite module, using the basis of \mathfrak{o} : if $\omega_1, \omega_2, \dots, \omega_n$ is a basis of \mathfrak{o} then $\mathfrak{o}\mu = [\omega_1\mu, \omega_2\mu, \dots, \omega_n\mu]$.

The notion of relative norm given for modules can be used for ideals as well. The norm of \mathfrak{a} relative to \mathfrak{o} , $(\mathfrak{o}, \mathfrak{a})$, is a polynomial in z . It is called the norm of the ideal \mathfrak{a} and denoted by $N(\mathfrak{a})$. The degree of this polynomial is called the degree of the ideal \mathfrak{a} . It is easy to see that for any function $\alpha \in \mathfrak{a}$, $N(\alpha) = \text{const.}N(\mathfrak{a})(\mathfrak{o}, \mathfrak{o}\alpha)$ “ i.e., the norm of any function in \mathfrak{a} is divisible by the norm of \mathfrak{a} ” (*ibid.*, 67).

A closer look is given to the properties of multiplication and division of ideals, which will form the ground of the theory. The notion of divisibility for ideals is similar to that given for modules: An ideal \mathfrak{a} is said to be divisible by an ideal \mathfrak{b} if every function of \mathfrak{a} is also contained in \mathfrak{b} . In addition, Dedekind and Weber detail the particular cases of principal ideals and of the divisibility between functions and ideals:

A principal ideal $\mathfrak{o}\mu$ is divisible by a principal ideal $\mathfrak{o}\nu$ if and only if the integral function μ is divisible by the integral function ν .

A function α in \mathfrak{o} is said to be *divisible* by the ideal \mathfrak{a} if the principal ideal $\mathfrak{o}\alpha$ is divisible by \mathfrak{a} or, which means the same thing, if α is a function of \mathfrak{a} .⁶⁰ (*ibid.*, transl. slightly altered, 68)

It can easily be seen that the divisibility for principal ideals is equivalent to the divisibility for the integral functions generating them. The idea for the study of the divisibility of the functions of the field, then, is to study the laws of divisibility governing the ideals and show that they are the same as the laws governing the polynomials. Having shown that, Dedekind and Weber will have thus shown that the laws of divisibility governing the (integral) algebraic functions are the same as the laws governing the polynomials (which are the same as those governing the rational integers).⁶¹

⁵⁹“[W]enn μ eine beliebige von Null verschiedene Funktion von \mathfrak{o} bedeutet, der Modul $\mathfrak{o}\mu$ (das System aller durch μ teilbaren ganzen Funktionen) [ist] ein Ideal. Ein solches Ideal soll ein Hauptideal genannt werden.”

⁶⁰“Ein Hauptideal $\mathfrak{o}\mu$ ist dann und nur dann teilbar durch ein Hauptideal $\mathfrak{o}\nu$, wenn die ganze Funktion μ teilbar ist durch die ganze Funktion ν . (...) Eine Funktion α in \mathfrak{o} soll durch das Ideal \mathfrak{a} teilbar heißen, wenn das Hauptideal $\mathfrak{o}\alpha$ durch \mathfrak{a} teilbar, oder, was dasselbe sagt, wenn α eine Funktion in \mathfrak{a} ist.”

⁶¹Note that Dedekind’s (and Weber’s) method works so well because it is developed in a Dedekindian ring, in which every non-zero prime ideal is a maximal ideal. In higher dimensions, it ceases to be valid.

2.3 Ideal theory and the laws of divisibility in the field Ω

Additional definitions and properties for the divisibility of ideals are given, presenting a striking similarity with rational arithmetic. Dedekind and Weber carry over the notion of LCM and GCD from modules (see p. 143) to ideals, and define a notion of relative primality: two ideals are said to be relatively prime when their GCD is \mathfrak{o} . Dedekind and Weber state that for two modules \mathfrak{a} and \mathfrak{b} to be relatively prime, it is necessary and sufficient that “there exist a function α in \mathfrak{a} and a function β in \mathfrak{b} such that $\alpha + \beta = 1$ ” (*ibid.*, 68).

They also introduce the notion of prime ideal:

An ideal \mathfrak{p} different from \mathfrak{o} is called a *prime ideal* if no other ideal than \mathfrak{p} et \mathfrak{o} divides \mathfrak{p} .⁶² (*ibid.*, transl. slightly altered, 68)

The first elements about the divisibility of ideals are given using norms of ideals. Recall that, in this context, norms are useful because they allow to reduce the problems to questions about rational objects. They obtain, for example:

- If \mathfrak{b} divides \mathfrak{a} , then $N(\mathfrak{b})$ divides $N(\mathfrak{a})$.
- If \mathfrak{b} divides \mathfrak{a} , and $N(\mathfrak{b}) = N(\mathfrak{a})$, then $\mathfrak{b} = \mathfrak{a}$.

While norms do play a role in the proofs, Dedekind and Weber focus on properties pertaining to relations of divisibility for ideals themselves. The theorems and proofs about the laws of divisibility of ideals are only expressed in terms of ideals, the properties of norms such as the last two given above play decisive roles in the proofs. The propositions and theorems given about the divisibility of ideals appear as familiar theorems about the divisibility properties of rational numbers, such as the one already given for modules: if \mathfrak{a}_1 divides \mathfrak{a} and \mathfrak{b}_1 divides \mathfrak{b} , then $\mathfrak{a}_1\mathfrak{b}_1$ divides $\mathfrak{a}\mathfrak{b}$. One also has:

Each ideal \mathfrak{a} different from \mathfrak{o} is divisible by a prime ideal \mathfrak{p} .

If \mathfrak{a} is an arbitrary ideal and \mathfrak{p} a prime ideal, then either \mathfrak{a} is divisible by \mathfrak{p} or \mathfrak{a} is relatively prime to \mathfrak{p} .

If \mathfrak{a} is relatively prime to \mathfrak{b} and to \mathfrak{c} , then \mathfrak{a} is also relatively prime to \mathfrak{bc} .

If \mathfrak{a} is relatively prime to \mathfrak{c} and $\mathfrak{a}\mathfrak{b}$ is divisible by \mathfrak{c} , then \mathfrak{b} is divisible by \mathfrak{c} .⁶³ (*ibid.*, 69-70)

⁶²“Ein von \mathfrak{o} verschiedenes Ideal \mathfrak{p} heißt ein Primideal, wenn kein anderes Ideal außer \mathfrak{p} und \mathfrak{o} in \mathfrak{p} aufgeht.”

⁶³“Jedes von \mathfrak{o} verschiedene Ideal \mathfrak{a} ist durch ein Primideal \mathfrak{p} teilbar. Ist \mathfrak{a} ein beliebiges ideal, \mathfrak{p} ein Primideal, then entweder \mathfrak{a} durch \mathfrak{p} teilbar oder \mathfrak{a} relativ prim zu \mathfrak{p} . Ist \mathfrak{a} a relativ prim zu \mathfrak{b} und zu \mathfrak{c} , so ist \mathfrak{a} auch relativ prim zu \mathfrak{bc} . Ist \mathfrak{a} relativ prim zu \mathfrak{c} und $\mathfrak{a}\mathfrak{b}$ durch \mathfrak{c} teilbar, so ist \mathfrak{b} durch \mathfrak{c} teilbar.”

However, these propositions are not sufficient for Dedekind and Weber's purpose. Indeed, as they explain

All these theorems, which mostly follow immediately from the definition of ideal, do not suffice to prove the complete analogy between the laws of divisibility of ideals and those of polynomial functions.⁶⁴ (*ibid.*, transl. slightly altered, 70)

The following propositions are going to completely unfold and explain the divisibility laws governing ideals, and bring to the fore the fact that these laws are the same as the ones governing polynomials.

The key for the following properties, which are labelled "Laws of divisibility of ideals" is to show that the notion of divisibility defined as an inclusion relationship is equivalent to the arithmetical notion of divisibility, that is:

If \mathfrak{c} is an ideal divisible by [i.e., contained in] an ideal \mathfrak{a} , then there exists *one and only one* ideal \mathfrak{b} that satisfies the condition

$$\mathfrak{a}\mathfrak{b} = \mathfrak{c}.$$

It is called the *quotient of \mathfrak{c} by \mathfrak{a}* .⁶⁵ (*ibid.*, 73)

Note that there is clear distinction made between the quotient and the notion of divisibility, here. With the proof of this theorem, it becomes clear that the notion of divisibility given for ideals can indeed be considered as an extension of the arithmetical notion.

This equivalence between the 'set-theoretic' notion of divisibility and the arithmetical one was, in algebraic number theory, a difficult property to prove. For algebraic functions, it is much easier, a point which Dedekind and Weber do notice in a footnote without explaining it. It is, in fact, related to peculiar properties of polynomials, in particular the possibility to factorize them into linear factors ([Stillwell, 2012], 70).

As it should be clear, the theory of ideals so far has been developed focused on divisibility properties and using essentially (1) the notions of norm, basis, etc. and (2) the new arithmetical operations defined for ideals. Norms are quickly reduced to properties (such as the ones p. 147) that do not necessitate manipulation of determinants once proven. Ideal theory is thus developed with no other means than these arithmetical notions and following a path very similar to rational number theory. Dedekind and Weber are able not only to give analogous theorems, but also demonstrations mirroring the

⁶⁴ "Alle diese Sätze, die sich meist unmittelbar aus der Definition der Ideale ergaben, reichen nicht aus, um die vollständige Analogie zu beweisen, die zwischen den Gesetzen der Teilbarkeit der Ideale und denen der ganzen rationalen Funktionen herrscht."

⁶⁵ "Ist ein Ideal \mathfrak{c} teilbar durch ein Ideal \mathfrak{a} , so gibt es ein und nur ein Ideal \mathfrak{b} , welches der Bedingung $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$ genügt, welches der Quotient von \mathfrak{c} durch \mathfrak{a} heißt."

2.3 Ideal theory and the laws of divisibility in the field Ω

proofs given in rational number theory. They can prove non-trivial results by means of only methods extending rational arithmetic. Indeed, everything is done by means of divisibility of ideals, the statements of propositions and theorems *and* their proofs.

If the perfect correspondence between divisibility laws of ideals and of polynomials is the first goal of ideal theory, it seems to be in great part because it will assure the foundation of the theory for further developments. In addition, since it is proven that divisibility of ideals behaves in the same way as rational divisibility, it gives the possibility to use these concepts following the lines of rational arithmetic.⁶⁶

A central result for the divisibility of ideals, here as in number theory, is the proof that every ideal has a unique factorization in primes:

Each ideal different from \mathfrak{o} is either a prime ideal or else uniquely expressible as a the product of prime (and only prime) ideals.⁶⁷
(*ibid.*, 73)

The proofs of the theorems, such as the unique factorization theorem, are modeled on the same tactics as in algebraic number theory in which Dedekind, very carefully, proves the theorems by using only operations of elementary arithmetic between ideals, giving demonstrations strikingly resembling the usual ones in rational number theory. The fact that the methods of proofs themselves can be of arithmetical form seems to play an important role in the simplicity attributed to the concept of ideal and in the possibility to get rid of complicated computations. By moving the study of the properties of algebraic functions up to the level of *systems* of such functions and setting up a new arithmetical framework for these systems considered as objects, Dedekind and Weber are replacing the long tedious computations by very simple ones.

This last point is particularly clear in the proof of the unique factorization theorem in primes for ideals, which goes as follows. Let \mathfrak{a} be a non-prime ideal different from \mathfrak{o} . Then there exists certain prime ideal \mathfrak{p}_1 dividing \mathfrak{a} . Hence, one has $\mathfrak{a} = \mathfrak{p}_1\mathfrak{a}_1$ with \mathfrak{a}_1 different from \mathfrak{a} . Hence, the degree of $N(\mathfrak{a}_1)$ is lower than the degree of $N(\mathfrak{a})$. If \mathfrak{a}_1 is not prime and is different from \mathfrak{o} (a property that can be verified with its norm), the procedure is repeated on \mathfrak{a}_1 and one obtains

$$\mathfrak{a}_1 = \mathfrak{p}_2\mathfrak{a}_2$$

and thus

$$\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{a}_2.$$

⁶⁶Already, in 1877, Dedekind explained that for number theory the “certainty that these general laws [of divisibility] exist eases to the highest degree the discovery and proof of special phenomena in a given field Ω ”.

⁶⁷“*Jedes von \mathfrak{o} verschiedene Ideal ist entweder ein Primideal, oder es lässt sich, and nur auf eine Weise, als Produkt von lauter Primidealen darstellen.*”

The successive repetition of this procedure leads, after a finite number of such decompositions, to an ideal $\mathfrak{a}_{r-1} = \mathfrak{p}_r \mathfrak{a}_r$, with $N(\mathfrak{a}_r) = 1$ i.e., $\mathfrak{a}_r = \mathfrak{o}$. Consequently, one has

$$\mathfrak{a} = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_r.$$

To prove the uniqueness of this decomposition, suppose that there exists another one, for example that

$$\mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_r = \mathfrak{q}_1 \mathfrak{q}_2 \dots \mathfrak{q}_s$$

(with \mathfrak{q}_i primes). Hence, there exists at least one ideal in these prime ideals

$$\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$$

which can be divided by \mathfrak{q}_1 and consequently is equal to \mathfrak{q}_1 . Let's suppose it is \mathfrak{p}_1 , then there is

$$\mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 \dots \mathfrak{p}_r = \mathfrak{p}_1 \mathfrak{q}_2 \mathfrak{q}_3 \dots \mathfrak{q}_s.$$

It is possible to eliminate \mathfrak{p}_1 on both sides and to have:

$$\mathfrak{p}_2 \mathfrak{p}_3 \dots \mathfrak{p}_r = \mathfrak{q}_2 \mathfrak{q}_3 \dots \mathfrak{q}_s.$$

In the same way, one obtains $\mathfrak{p}_2 = \mathfrak{q}_2$ and so on. Finally, all identical ideals can be assembled to form powers so as to have

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \dots \mathfrak{p}_r^{e_r}.$$

The greatest common divisor and the least common multiple of two ideals can be given using their prime decompositions, in the exact same way it is usually done in rational number theory.

The proof of the existence of a unique factorization in primes allows to prove the multiplicative property of the norm of ideals:

$$N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b}).$$

It also allows to give a further characterization of a prime ideal: an ideal is prime if and only if it is an ideal of first degree (i.e., its norm is a linear polynomial). This last point “distinguishes in an essential way” the theory of algebraic functions from the theory of algebraic numbers, in which a prime ideal is *not necessarily* of the first degree.⁶⁸

⁶⁸The difference lies in the notion of norm of an ideal. In algebraic number theory, the degree of an ideal is defined in the following way: the norm of an ideal \mathfrak{a} is a rational integer, the (always finite) number of incongruence classes in \mathfrak{o} modulo \mathfrak{a} (i.e., the number of integers in \mathfrak{o} which are not congruent modulo \mathfrak{a}). In the case of a prime ideal \mathfrak{p} , its norm is a rational integer divisible by \mathfrak{p} and which put together with all the rational numbers divisible by \mathfrak{p} forms a module $[p]$ in which p is the smallest rational number greater than 0 divisible by \mathfrak{p} . This p is necessarily indecomposable (or it wouldn't be the smallest number divisible by \mathfrak{p}) and it cannot be 1, because in that case $\mathfrak{p} = \mathfrak{o}$. Since $\mathfrak{o}p$ is divisible by \mathfrak{p} , $N(\mathfrak{o}p) = p^n$ is divisible by the norm of \mathfrak{p} which is thus of the form p^f and not necessarily of the first degree.

2.3 Ideal theory and the laws of divisibility in the field Ω

As mentioned earlier, there is a complete equivalence between the laws of divisibility of ideals and the laws of divisibility of functions. So, with all the theorems proved about the divisibility of ideals it is tantamount to having proven the validity of these divisibility properties for algebraic functions. This entails that Dedekind and Weber have proven that the laws of divisibility of integral algebraic function agree completely with the laws of divisibility of rational numbers.

Overall, it is only in the beginning of the second part of the paper that the reasons for all these investigation of the properties of the field Ω appear more clearly. Indeed, the study of the divisibility laws is not an aim in and of itself, but the basis on which to construct a more rigorous and uniform treatment of Riemannian function theory. The set up of the theory of ideals and their divisibility leads to more algebraic investigations: the first part of the 1882 paper continues for four more sections, in which Dedekind and Weber pursue their analysis of the properties of the field.

An essential new concept for the definition of the Riemann surface is the ramification ideal (*Verzweigungsideal*). This notion will, as its name indicates, allow Dedekind and Weber to describe the ramification of the field, so as to define the notion of ramification of a Riemann surface.

The ramification ideal is introduced as a purely algebraic notion without mention of its relation with the ramification of the (to be defined) Riemann surface. The aim of Dedekind and Weber's investigation is really unclear, at that point.⁶⁹ The ramification ideal is defined as an ideal \mathfrak{z} such that

$$\mathfrak{z} = \prod \mathfrak{p}^{e-1}$$

in which the product is “taken over all the prime ideals \mathfrak{p} for which a power higher than the first, namely the e th, divides their norm” (*ibid.*, 83). One of the first and most remarkable properties of the ramification ideal is that its norm is the discriminant of the field: $N(\mathfrak{z}) = D$.⁷⁰

Dedekind and Weber, even if they are not telling their readers so, are setting up the framework and the toolbox which will serve to establish what they hope to be a more general and more rigorous treatment of Riemann's function theory. We will see that most of the considerations about the

⁶⁹See [Stillwell, 1996], 75-85, for more details on the technical preliminaries for the definition of the ramification ideal.

⁷⁰For more details, see Stillwell, 1996, 75-85. Dedekind will, in a paper published in 1882, transfer the ramification ideal (with a different name) back to number theory. The back transfer is allowed by the generality and the “formal” character of the investigations conducted here, as it was the case for the transfer from number theory to function theory. I will consider it briefly on p. 374, but the link between Dedekind and Weber's first introduction of the ramification ideal, the notion to number theory by Dedekind and today's notion of ramification in algebraic number theory should be investigated further than what I am proposing in this dissertation – a work I hope to accomplish in the future.

arithmetical structure of the field Ω and the definition of special ideals will have a counterpart in the definition of the surface.

2.4 Further investigations on function fields

There are two points left, for Dedekind and Weber, before moving on to the definition of the Riemann surface. The first one is the consideration of “fractional functions of z ”, i.e., functions of Ω written as quotient of integral algebraic functions. Such an expression is always possible, just as it is always possible to write a rational number as a quotient of two integers. The second point is the proof of the birational invariance of their results.

2.4.1 Fractional functions

The study of “fractional functions” is a way to enlarge the investigation of the properties of the field outside of the system of integral algebraic functions. One can consider any arbitrary function in the field and study them using ideal theory.

It was proved in the beginning of the paper that it is always possible to write a function of Ω as a quotient of two integral algebraic functions, and this in infinitely many different ways. Dedekind and Weber thus explain that with μ and ν in \mathfrak{o} two integral algebraic functions of z such that for a function η in Ω

$$\eta = \frac{\nu}{\mu}$$

(μ can be a polynomial), then it is possible to consider the two principal ideals $\mathfrak{o}\nu$ and $\mathfrak{o}\mu$ and their GCD \mathfrak{m} . Thus, there exists two relatively prime ideals \mathfrak{a} and \mathfrak{b} such that

$$\mathfrak{o}\mu = \mathfrak{a}\mathfrak{m}, \quad \mathfrak{o}\nu = \mathfrak{b}\mathfrak{m}$$

Hence, laws about the multiplication and division of ideals give

$$\mathfrak{b}\mu = \mathfrak{a} \text{ or } \mathfrak{a}\nu = \mathfrak{b}$$

(*ibid.*, 86). If now α is an integral function of z such that $\alpha\eta = \beta$ is again an integral function, then

$$\alpha\nu = \beta\mu \Rightarrow \alpha\mathfrak{o}\nu = \beta\mathfrak{o}\mu \Rightarrow \alpha\mathfrak{b} = \beta\mathfrak{a}$$

Since \mathfrak{a} and \mathfrak{b} are relatively prime, it is necessary that α be divisible by \mathfrak{a} and β divisible by \mathfrak{b} . Weber and Dedekind thus prove that \mathfrak{b} is the LCM of $\mathfrak{o}\eta$ and \mathfrak{o} , and \mathfrak{a} is the LCM of $\frac{\mathfrak{o}}{\eta}$ and \mathfrak{o} .

2.4 Further investigations on function fields

Dedekind and Weber then consider two ideals \mathfrak{a} and \mathfrak{b} (not necessarily relatively prime), such that $\mathfrak{a}\eta = \mathfrak{b}$. Then, it is easy to see, by simple manipulation of products of ideals, that

$$\sigma\eta = \frac{\mathfrak{b}}{\mathfrak{a}}$$

Then, in the case of \mathfrak{a} and \mathfrak{b} relatively prime, Dedekind and Weber call \mathfrak{b} the *upper ideal* (*Oberideal*) and \mathfrak{a} is the *lower ideal* (*Unterial*) of the function η .

Without using anything else than the arithmetic of ideals, Dedekind and Weber deduce easily that the norm of η is the quotient of the norms of \mathfrak{a} and \mathfrak{b} up to a constant. One can also exhibit laws about the product and addition of such functions:

If η and η' are two functions in Ω and if

$$\mathfrak{a}\eta = \mathfrak{b}, \quad \mathfrak{a}'\eta' = \mathfrak{b}'$$

(...) but regardless of whether $\mathfrak{a}, \mathfrak{b}$ or $\mathfrak{a}', \mathfrak{b}'$ are relatively prime, then

$$\mathfrak{a}\mathfrak{a}'\eta\eta' = \mathfrak{b}\mathfrak{b}'$$

The equations

$$\sigma\eta = \frac{\mathfrak{b}}{\mathfrak{a}} \quad \text{and} \quad \sigma\eta' = \frac{\mathfrak{b}'}{\mathfrak{a}'}$$

therefore yield the equations

$$\sigma\eta\eta' = \frac{\mathfrak{b}\mathfrak{b}'}{\mathfrak{a}\mathfrak{a}'} \quad ; \quad \sigma\frac{1}{\eta} = \frac{\mathfrak{a}}{\mathfrak{b}} \quad ; \quad \sigma\frac{\eta}{\eta'} = \frac{\mathfrak{b}\mathfrak{a}'}{\mathfrak{a}\mathfrak{b}'}$$

(*ibid.*, 88)⁷¹

Likewise, for the addition, one obtains that $\mathfrak{a}(\eta \pm \eta') = \mathfrak{b}''$ and thus

$$\sigma(\eta \pm \eta') = \frac{\mathfrak{b}''}{\mathfrak{a}}.$$

The notions of upper ideal and lower ideal will be essential for a complete definition of the Riemann surface. Indeed, they will be used to define the zeros and the poles, and for the definition of differentials.

⁷¹ "Sind η and η' zwei Funktionen in Ω und ist (...) $\mathfrak{a}\eta = \mathfrak{b}$, $\mathfrak{a}'\eta' = \mathfrak{b}'$, gleichviel ob $\mathfrak{a}, \mathfrak{b}$; $\mathfrak{a}', \mathfrak{b}'$ relativ prim sind oder nicht, so folgt $\mathfrak{a}\mathfrak{a}'\eta\eta' = \mathfrak{b}\mathfrak{b}'$. Es folgen also aus $\sigma\eta = \frac{\mathfrak{b}}{\mathfrak{a}}$ und $\sigma\eta' = \frac{\mathfrak{b}'}{\mathfrak{a}'}$ die Gleichungen $\sigma\eta\eta' = \frac{\mathfrak{b}\mathfrak{b}'}{\mathfrak{a}\mathfrak{a}'}$, $\sigma\frac{1}{\eta} = \frac{\mathfrak{a}}{\mathfrak{b}}$, $\sigma\frac{\eta}{\eta'} = \frac{\mathfrak{b}\mathfrak{a}'}{\mathfrak{a}\mathfrak{b}'}$."

2.4.2 Invariance by rational transformations

The last paragraph of the first part of Dedekind and Weber's paper is important for their generality requirement and state that the results obtained for the study of the field Ω are invariant by "rational transformations of functions in the field Ω ". The idea, here, is to chose another independent variable z_1 in place of z and an algebraic function θ_1 in place of θ . It has been proved that there exist an "irreducible algebraic equation" between z and z_1 of degree e in z and e_1 in z_1 with $n = ef$ and $n_1 = e_1f$. Then

not only can z_1, θ_1 be expressed rationally in terms of z, θ , but conversely, z, θ can be expressed rationally in terms of z, θ .⁷²
(*ibid.*, 91)

The path leading to this conclusion implies several pages of manipulation of systems of equations, the details of which I will not give here.

The birational invariance result entails that the "arbitrary nonconstant function z " chosen as "independent variable" can be "any (nonconstant) function" in Ω . A result similar was obtained by Dedekind, in algebraic number theory, where he chooses to associate the field to a certain equation.

Note that while this assures that the choice of the variable does not mar the generality of the approach, especially since the "totality of the functions in the field Ω remains completely unaltered", the concepts developed (basis, norm, discriminant, integral algebraic function, module and ideal) are dependent on the choice of the variable (*ibid.*, 92). Some of these concepts, namely the basis, norm, trace and discriminant, stay identical if and only if z and z_1 are linearly dependent of one another, that is $z_1 = \frac{az+b}{cz+d}$ ($ad - bd \neq 0$) ([Stillwell, 1996], 89).

The study of the arithmetical and algebraic behavior of the functions in Ω was preliminary work, in order for Dedekind and Weber to be able to define the Riemann surface. The said preliminary work was done with only concepts built on arithmetical notions, and even, as we saw with ideals, designed so that their development can follow lines very close to rational number theory.

As Dedekind and Weber mentioned in the introduction of their paper, the notion of a field of algebraic functions "coincides completely" with that of a "class of algebraic functions" used by Riemann. The aim of the second part of the paper is thus to exhibit the equivalence between the field and the Riemann surface. This would make it possible to use, for developments of the theory of Riemann surfaces, results obtained in algebraic investigations on functions. Indeed, the correspondence between surface and field implies that the singularities of the surface can be expressed in terms of (divisibility

⁷² "Es lassen sich also sowohl z_1, θ_1 rational durch z, θ , als auch umgekehrt z, θ rational durch z, θ darstellen."

2.5 The point and the Riemann surface

of) ideals. For example, to describe the ramification of the surface, Dedekind and Weber will propose a definition that relates it to the ramification ideals.

2.5 The point and the Riemann surface

Dedekind and Weber qualify the first part of their paper as “formal”:

The previous considerations on the functions in the field Ω were of purely formal nature. (...) The numerical values of these functions did not come under consideration at any moment. (...) But now that we have carried the formal part of the investigation this far, the pressing question is *to what extent is it possible to assign particular numerical values to the functions in Ω so that all rational relations (identities) existing between these functions become correct numerical equalities?*⁷³ (*ibid.*, transl. slightly altered, 94)

It will be “useful”, for the definition of Riemann surfaces, to consider “ ∞ ” as “one definite (constant) number”. The rules of computation with arithmetic operations can be extended to the domain of numbers to which ∞ has been added, as long as no “indeterminacy” of the form $\infty \pm \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ is encountered. If such indeterminacy were to appear, Dedekind and Weber state that the equation just does not have a truth value anymore.

The formal aspect of the study of the field Ω is a requirement for the assurance of a general ground of Riemann’s theory. The next step is, thus, to attribute numerical values (in $\mathbb{C} \cup \{\infty\}$) to the functions, so as to satisfy if the “rational relations” investigated in the first part are preserved in the numerical domain. This statement appears to be a conservativity requirement for the relations between functions on the so-called Riemann sphere. “Formal” refers the fact that all results are inferred from the equation and the manipulation of the four fundamental operations of arithmetic ($+$, $-$, \times , \div).⁷⁴ Weber, who had the same use of the term “formal”, wrote, in his 1893 paper on Galois theory, that the only way for the “pure formalism” to “[gain] meaning and life” is by “the substitution of the individual

⁷³“Die bisherigen Betrachtungen über die Funktionen des Körpers Ω , waren rein formaler Natur. (...) Die numerischen Werte dieser Funktionen kamen nirgends in Betracht. (...) Nachdem nun aber der formale Teil der Untersuchung soweit geführt ist, drängt sich die Frage auf, in welchem Umfange es möglich ist, den Funktionen in Ω solche bestimmten Zahlenwerte beizulegen, daß alle zwischen diesen Funktionen bestehenden rationalen Relationen (Identitäten) in richtige Zahlengleichungen übergehen.”

⁷⁴Stillwell translate “Spezies” by “algebraic operations”. While it is understandable if one is to look at the 1882 paper as a part of algebraic geometry, I believe it is misleading. Uses of the term “Spezies” were essentially made to refer to the four fundamental operations of arithmetic, as a synonym for “Grundoperationen der Arithmetik” (see [Müller, 1900]). It is, for example, the term used by Gauss in [Gauss, 1929] to designate the operations of arithmetic.

elements with number values”, in reference to the passage from a formal treatment to the attribution of numerical values.

2.5.1 Points of the Riemann surface

The attribution of numerical values is the core idea for the definition of the point of a Riemann surface. Dedekind and Weber define the point of the surface as a correspondence from the field of functions into the field of numerical constants:

If *all* the individual elements $\alpha, \beta, \gamma, \dots$ of the field Ω are replaced by *definite* numerical values $\alpha_0, \beta_0, \gamma_0, \dots$ in such a way that:

(I) $\alpha_0 = \alpha$ if α is constant, and in general,

(II) $(\alpha + \beta)_0 = \alpha_0 + \beta_0$

(III) $(\alpha - \beta)_0 = \alpha_0 - \beta_0$

(IV) $(\alpha\beta)_0 = \alpha_0\beta_0$

(V) $(\frac{\alpha}{\beta})_0 = \frac{\alpha_0}{\beta_0}$.

Then, to such a conjunction of definite values must be assigned a point $\mathfrak{P} (\dots)$ and we say that $\alpha = \alpha_0$ at \mathfrak{P} or that α has the value α_0 at \mathfrak{P} . Two points are called different if and only if there is a function α in Ω which has a different values for each of these two points.⁷⁵ (*ibid.*, transl. modified, 94)

The evaluation of the functions,⁷⁶ as defined here by Dedekind and Weber is, thus, a field morphism. The mapping from Ω to the “numerical constants” is defined exactly (albeit without alluding to it) as field morphisms were

⁷⁵“Wenn alle Individuen $\alpha, \beta, \gamma, \dots$ des Körpers Ω durch bestimmte Zahlwerte $\alpha_0, \beta_0, \gamma_0, \dots$ so ersetzt werden, daß

(I) $\alpha_0 = \alpha$ falls α konstant is, und allgemein:

(II) $(\alpha + \beta)_0 = \alpha_0 + \beta_0$

(III) $(\alpha - \beta)_0 = \alpha_0 - \beta_0$

(IV) $(\alpha\beta)_0 = \alpha_0\beta_0$

(V) $(\frac{\alpha}{\beta})_0 = \frac{\alpha_0}{\beta_0}$

wird, so soll einem solchen Zusammentreffen bestimmter Werte ein Punkt \mathfrak{P} zugeordnet werden (...) und wir sagen, in \mathfrak{P} sei $\alpha = \alpha_0$, oder α habe in \mathfrak{P} den Wert α_0 . Zwei Punkte heißen stets und nur dann verschieden, wenn eine Funktion α in Ω existiert, die in beiden Punkten verschiedene Werte hat.”

⁷⁶Dedekind and Weber are considering an application for a function ω of Ω taken as variable, such that $\omega \mapsto \omega(x)$, with $\omega(x)$ a constant in $\mathbb{C} \cup \{\infty\}$. This move, Bourbaki tells us, has become such a common thing to do in mathematics, that we do not notice anymore how original it is ([Bourbaki, 1984], 134).

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defined by Dedekind in his algebraic number theory. Dedekind called field morphisms “permutations”. When a mapping is defined from a certain field Ω onto another system of numbers Ω' , it is said to be a permutation if:

a determined number of Ω

$$\alpha, \beta, \alpha + \beta, \alpha - \beta, \alpha\beta, \frac{\alpha}{\beta}$$

is changed into a corresponding number

$$\alpha', \beta', (\alpha + \beta)', (\alpha - \beta)', (\alpha\beta)', \left(\frac{\alpha}{\beta}\right)'$$

in such a way that the conditions

$$(1) (\alpha + \beta)' = \alpha' + \beta'$$

$$(2) (\alpha\beta)' = \alpha'\beta'$$

are satisfied and the substitute numbers α', β', \dots are not all zero. (...) [T]he set Ω' of the latter numbers forms a new field, and (...) the permutation also satisfies the following conditions:

$$(3) (\alpha - \beta)' = \alpha' - \beta'$$

$$(4) (\alpha/\beta)' = \alpha'/\beta'.$$
⁷⁷

([Dedekind, 1876-1877], transl. modified, 108-109)

These two fields are called “conjugate fields”. In [Dedekind, 1879], Dedekind explains that these conditions are providing an answer to whether it is possible “to represent the numbers ω of the field Ω by the numbers ω' , in such a way that all the rational relations existing between the numbers ω are completely transferred to the image ω' ”⁷⁸ ([Dedekind, 1879], 470, my translation). In 1882, the definition of the point allows to show the conservation of the rational relations between functions when a numerical value is attributed to it. That is, that all rational relations between functions correspond to numerical equalities. It seems to yield the assurance of the consistency of their “formal” results. Indeed, the morphism appears to be giving content

⁷⁷ [N]ous entendons par une permutation de Ω une substitution par laquelle un nombre déterminé contenu dans Ω , $\alpha, \beta, \alpha + \beta, \alpha - \beta, \alpha\beta, \frac{\alpha}{\beta}$ se change en un nombre déterminé correspondant $\alpha', \beta', (\alpha + \beta)', (\alpha - \beta)', (\alpha\beta)', \left(\frac{\alpha}{\beta}\right)'$. Et cela de telle manière que les deux conditions (1) $(\alpha + \beta)' = \alpha' + \beta'$ (2) $(\alpha\beta)' = \alpha'\beta'$ soient remplies, et que les nombres substitués α', β', \dots ne s'annulent pas tous. Nous allons faire voir que l'ensemble Ω' de ces derniers nombres forme un nouveau corps, et que la permutation satisfait aussi aux deux conditions suivantes : (3) $(\alpha - \beta)' = \alpha' - \beta'$ (4) $(\alpha/\beta)' = \alpha'/\beta'$ ”

⁷⁸ “ob es möglich ist, die Zahlen ω des Körpers Ω in der Weise durch Zahlen ω' abzubilden, dass alle zwischen den Zahlen ω bestehenden rationalen Beziehungen sich vollständig auf die Bilder ω' übertragen”

to the formal investigations, the equalities translated in numbers granting a guarantee of the coherence of the relationships between functions.

Dedekind and Weber suggest the possibility to attribute to the point a geometrical meaning, to represent it “sensibly” as “somehow located in space”. However, they immediately add that it is not necessary and in fact “does not make comprehension easier”. Any implicit geometrical meaning of the word point should, thus, be taken out of Dedekind and Weber’s concept of point, which designates only “the coexistence of values” ([Dedekind and Weber, 1882], 94).

The definition of a “point” is an “invariant concept” of the field, it does not depend on the choice of a variable z , but only on the functions of Ω . The definition has to be completed by further elaborations, so as to “deduce the existence as well as the scope” of the concept of point. For this, Dedekind and Weber establish the existence of a one-to-one correspondence between points and prime ideals.

This is made in several steps. Firstly, they note that there is always a variable z with a finite value in \mathfrak{P} . Indeed, if z_0 were infinite, then $(\frac{1}{z})_0$ would be finite; and any integral function ω of such variable z as a finite value in \mathfrak{P} as well. Secondly, for any variable z with a finite value in \mathfrak{P} , they consider the collection of all the integral functions π of z which have the value 0 in \mathfrak{P} . This collection is a prime ideal \mathfrak{p} . The point \mathfrak{P} is said to “generate” the prime ideal \mathfrak{p} . Moreover, if ω has the value ω_0 at the point \mathfrak{P} , this translates into a congruence between the functions modulo the ideal \mathfrak{p} : $\omega \equiv \omega_0 \pmod{\mathfrak{p}}$. In a third time, they show that the correspondence is injective:

The same prime ideal \mathfrak{p} cannot be generated by two different points.⁷⁹ (*ibid.*, 95)

Finally, they show that this correspondence is biunivoque, that is, for a variable z and a prime ideal \mathfrak{p} in z , there always exists one, and according to the previous point, only one, point \mathfrak{P} which generates this prime ideal. This point is called the “null point” (*Nullpunkt*) of the ideal \mathfrak{p} (*ibid.*, 95). Thus, for any finite variable z , there is a one-to-one correspondence between the points and the prime ideals of z in Ω .

To prove the last result, Dedekind and Weber use the fact, proven earlier, that any function η of the field can be written as $c\rho^m + \eta_1\rho^{m+1}$, in which ρ is a function whose upper ideal is divisible by a prime ideal \mathfrak{p} but not by \mathfrak{p}^2 , m an integer, c a function and η_1 a function. They design a way to attribute values to η on this ground. With the point \mathfrak{P} obtained in this way, they explain that a function whose upper ideal is divisible by \mathfrak{p} (in particular, all the functions contained in \mathfrak{p}) is always attributed the value 0 at \mathfrak{P} by this

⁷⁹ “Dasselbe Primideal \mathfrak{p} kann nicht durch zwei verschiedene Punkte erzeugt werden.”

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peculiar attribution of values (i.e., the point \mathfrak{P} generates the ideal \mathfrak{p}). The functions whose lower ideal is divisible by \mathfrak{p} are always attributed the value ∞ at the point \mathfrak{P} . They are the only functions which take the value ∞ at \mathfrak{P} .

One can start to see how the upper and lower ideals will allow to express the zeros and poles of the functions. These considerations allow to see more clearly that there is a relationship between the values taken by the functions and the fact that they are or are not integral algebraic functions of a variable z :

an *integral algebraic* function of z is infinite at no point where z has a finite value and, since a fractional function of z certainly has one prime ideal in its lower ideal, hence it must be infinite at at least one point where z is finite, thus also conversely each function, which is infinite at no point for which z has a finite value, is an integral algebraic function of z .⁸⁰ (*ibid.*, transl. modified, 96)

Finally, the characterization of the concept of point is completed by an indication on how to obtain “all the existing points \mathfrak{P} exactly once”. Dedekind and Weber explain that it should be done in the following way. First, one has to choose (*ergreifen*) a variable z in Ω . To obtain the points at which z is finite, one forms all the prime ideals in z and constructs the null point of each ideal: these are all the points at which z “remains finite”. To obtain the points at which z becomes infinite, one starts by considering a point \mathfrak{P}' different from all the points just studied (i.e., not a point where z remains finite), and takes the variable $z' = \frac{1}{z}$ which has the value 0 in \mathfrak{P}' . Conversely, all the points at which z' is finite are not one of the points \mathfrak{P} first described. Thus, if one considers the ideal \mathfrak{p}' prime in z' generated by \mathfrak{P}' , that is, the prime ideal constituted by all the integral functions of z' which vanish at \mathfrak{P}' , then z' is contained in \mathfrak{p}' (“ \mathfrak{p}' (...) divides z' ”). Conversely, if the null point of a prime ideal in z' divides z' , then it is a point at which z' takes the value 0 and $z = \infty$. These “complementary points” (*Ergänzungspunkte*) \mathfrak{P}' are in finite number. If one collects (once) all the points \mathfrak{P} first described and the complementary points \mathfrak{P}' , the totality of the points is obtained, which form the “Riemann surface”. Formulated with ideals, one has:

With this finite number of complementary points corresponding to the different \mathfrak{p}' , and those previously derived from the prime

⁸⁰“(...) eine ganze Funktion von z in keinem Punkte, in welchem z einen endlichen Werte hat, unendlich ist, und, da eine gebrochene Funktion von z im Unterideal gewiß ein Primideal enthält, also mindestens in einem Punkt, in welchem z endlich ist, unendlich sein muß, so ist auch umgekehrt jede Funktion, die in keinem Punkte, in welchem z einen endlichen Wert hat, unendlich ist, eine ganze Funktion von z .”

ideals \mathfrak{p} in z , the totality of all points \mathfrak{P} is exhausted, whose collection constitutes the *Riemann surface* T .⁸¹ (*ibid.*, transl. modified, 96)

While Dedekind and Weber do call the collection of all points the “Riemann surface”, what is defined here is a “simple totality” (as they will state a few pages later) in which each point appears only once. It does not describe the multiplicity and ramification of the surface, and the surface as such is not fully described.

2.5.2 Order numbers of the points

So as to characterize the singularities of the functions, Dedekind and Weber introduce the notion of “order number” (*Ordnungszahl*). The order number is attributed to the functions π of Ω which vanish in \mathfrak{P} in the following way. First consider ϱ which is one of the functions π “vanishing at \mathfrak{P} ”:

Such a function ϱ has order number 1 or is said to be *infinitely small of first order* or again 0^1 at \mathfrak{P} when all quotients $\frac{\pi}{\varrho}$ remain finite at \mathfrak{P} .⁸² (*ibid.*, 97)

This corresponds to the zeros of the function ϱ . The order of the other functions can be given according to ϱ by considering quotients like the ones used to define the order number of ϱ . Thus, a function ϱ' has the same order number as ϱ , if and only if $\frac{\varrho'}{\varrho}$ is neither 0 nor ∞ at \mathfrak{P} .

The order of a function π is thus defined as the positive integer r such that $\frac{\pi}{\varrho^r}$ is neither 0 nor ∞ at \mathfrak{P} . Such a number r always exists for any function π . The same is valid for the quotient $\frac{\pi}{\varrho^r}$ and thus

π receives the order r or is said to be infinitely small of order r at \mathfrak{P} . We also shall say that π is 0^r at \mathfrak{P} or that π is 0 at \mathfrak{P}^r .⁸³ (*ibid.*, 97)

One notices that Dedekind and Weber are using Riemann’s vocabulary to designate the zeros of the function. They also adopt the convention that if a zero is of order r , then it is the same as having the function vanishing at r superposed points. But their adopting Riemann’s “infinitely small of order r ” does not mean that their approach involves uses of infinitesimals such

⁸¹ “Mit diesen in endlicher Anzahl vorhandenen, den verschiedenen \mathfrak{p}' , entsprechenden Ergänzungspunkten und den vorher aus den Primidealen \mathfrak{p} in z abgeleiteten ist die Gesamtheit aller Punkte erschöpft, deren Inbegriff die Riemannsche Fläche T bildet.”

⁸² “Eine solche Funktion ϱ hat die Ordnungszahl 1, oder heißt unendlich klein in der ersten Ordnung oder 0^1 in \mathfrak{P} , wenn alle Quotienten $\frac{\pi}{\varrho}$ in \mathfrak{P} endlich bleiben.”

⁸³ “ π erhält die Ordnungszahl r oder heißt unendlich klein in der Ordnung r im Punkte \mathfrak{P} . Wir werden auch sagen, π ist 0^r in \mathfrak{P} oder π is 0 in \mathfrak{P}^r .”

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as one can find in Riemann's works. The use of a Riemannian terminology simply appears to bring to the fore the fact, already mentioned several times, that they are rewriting the concepts of Riemann's theory, rather than proposing something completely new or merely applying number theory to function theory.

The existence of ρ and r is proved by Dedekind and Weber using the representation of π as a quotient of two relatively prime ideals: the functions whose upper ideal is divisible by \mathfrak{p} (the prime ideal in z generated by \mathfrak{P}) but not by \mathfrak{p}^2 are of order 1. The order of the other functions π will be "the highest power of \mathfrak{p} dividing the upper ideal".

For a value η_0 different from 0, Dedekind and Weber say that a function η takes the value η_0 at \mathfrak{P} r -tuply (or at r coinciding points or at \mathfrak{P}^r) when $\eta - \eta_0$ is infinitely small of order r at \mathfrak{P} .

It is possible that $\eta_0 = \infty$, which then describes what we would call a pole. Then, Dedekind and Weber say that " η takes the value ∞ r -tuply" (or at r coinciding points or at \mathfrak{P}^r) or is ∞^r at \mathfrak{P} , if $\frac{1}{\eta}$ vanishes at \mathfrak{P}^r . In this last case, the order number of η is $-r$.

At any point where the function's value is neither 0 nor ∞ , its order number is 0.

This way, Dedekind and Weber attribute an order number to each function of Ω at a certain point \mathfrak{P} ,⁸⁴ and only the constants 0 and ∞ do not have an order.

2.5.3 Polygons

So as to described the singularities appropriately, one has to consider complexes of points, called "polygons", and their products. Indeed, polygons provide a tool for the definition and study of the singularities of the Riemann surface⁸⁵ and will be of great use in the rest of the paper.

We give the name *polygons* to complexes of points, which may contain the same point more than once and denote them by \mathfrak{A} , \mathfrak{B} , \mathfrak{C} ,

We also let \mathfrak{AB} denote the polygon obtained from the points of the polygons \mathfrak{A} and \mathfrak{B} put together, in such a way that a point \mathfrak{P} that appears r -tuply in \mathfrak{A} and s -tuply in \mathfrak{B} , appears $r + s$ -tuply in \mathfrak{AB} .⁸⁶ (*ibid.*, transl. modified, 98)

⁸⁴This corresponds to what is nowadays called a discrete valuation.

⁸⁵A polygon corresponds to what is today called a positive divisor, see [Dieudonné, 1974] or [Stillwell, 2012] note 39, page 98.

⁸⁶"Komplexe von Punkten, welche denselben Punkt auch mehrmals enthalten können, nennen wir Polygone and bezeichnen dieselben mit \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , Es bedeute ferner \mathfrak{AB} das aus den Punkten von \mathfrak{A} und von \mathfrak{B} zusammengesetzte Polygon in der Weise, daß, wenn ein Punkt \mathfrak{P} r -mal in \mathfrak{A} , s -mal in \mathfrak{B} auftritt, er $(r + s)$ -mal in \mathfrak{AB} vorkommt."

The product of two polygons is thus, roughly speaking, the ‘putting together’ of two polygons, the multiplicity of the points of each polygons being added – indeed, it was shown that the order number of a product of functions is the addition of the order number of each function.

Hence, from the definition of the product of polygons, one can deduce the “meaning” (*Bedeutung*) of the power of a point \mathfrak{P}^r , which was used in the definition of the order number: it is a point that appears n times in the polygon, and thus is repeated n times on itself. One can also deduce the meaning of

$$\mathfrak{A} = \mathfrak{P}^r \mathfrak{P}_1^{r_1} \mathfrak{P}_2^{r_2} \dots$$

It is possible, therefore, to decompose a polygon into the product of all its points, each with its own power – the number of times it appears in the polygon – which is basically a decomposition into prime elements. Dedekind and Weber call the number of points in a polygon its order, and a polygon of order n is called an n -gon.

This allows Dedekind and Weber to study the *laws of divisibility of polygons*, established by the one-to-one correspondence between (composite) ideals and polygons:

the laws of divisibility of polygons agree completely with the laws of divisibility of integers and of ideals. Points play the role of prime factors.⁸⁷ (*ibid.*, transl. slightly altered, 98)

If one wants to have a unit, for this new notion of divisibility, one has to admit the polygon containing 0 points, denoted by \mathfrak{D} (called “*Nulleck*”).

These developments lead Dedekind and Weber to define the GCD and the LCM of two or more polygons:

The *greatest common divisor* of two polygons \mathfrak{A} and \mathfrak{B} is the polygon that contains each point the *least* number of times it occurs in \mathfrak{A} and in \mathfrak{B} . If it is [the empty polygon] then \mathfrak{A} , \mathfrak{B} are called *relatively prime*.

The *least common multiple* of two polygons \mathfrak{A} and \mathfrak{B} is the polygon that contains each point the *greatest* number of times it occurs \mathfrak{A} and in \mathfrak{B} . If \mathfrak{A} , \mathfrak{B} are relatively prime, then $\mathfrak{A}\mathfrak{B}$ is their least common multiple.⁸⁸ (*ibid.*, 99)

⁸⁷“[D]ie Gesetze der Teilbarkeit der Polygone in vollkommener Übereinstimmung mit denen der Teilbarkeit der ganzen Zahlen und der Ideale. Die Rolle der Primfaktoren übernehmen dabei die Punkte.”

⁸⁸“Der größte gemeinschaftliche Teiler zweier Polygone \mathfrak{A} , \mathfrak{B} ist dasjenige Polygon, welches jeden Punkt so oft enthält, als er in \mathfrak{A} und \mathfrak{B} mindestens vorkommt. Ist dies \mathfrak{D} , so heißen \mathfrak{A} , \mathfrak{B} relativ prim. Das kleinste gemeinschaftliche Vielfache von \mathfrak{A} und \mathfrak{B} ist dasjenige Polygon, welches jeden Punkt so oft enthält, als er in \mathfrak{A} und \mathfrak{B} höchstens vorkommt. Sind \mathfrak{A} , \mathfrak{B} relativ prim, so ist $\mathfrak{A}\mathfrak{B}$ ihr kleinstes gemeinschaftliches Vielfache.”

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Note that this corresponds exactly to the definition of the GCD and the LCM by powers of prime numbers, in number theory – or for ideals.

The correspondence between functions (vanishing at some fixed points) and ideals is not yet complete, for one needs to prove that polygons correspond to non-prime ideals. Dedekind and Weber explain that it can easily be proved that there is always a function z in Ω that never becomes infinite at any points of an arbitrary polygon $\mathfrak{A} = \mathfrak{P}^r \mathfrak{P}_1^{r_1} \mathfrak{P}_2^{r_2} \dots$. Indeed, if z were to become infinite, one only needs to take a constant c such that $z - c \neq 0$ and consider $\frac{1}{z-c}$ which is finite at every points of the polygon. Consequently, taking z as a variable, one can form the collection of the integral algebraic functions that vanish at the points of the polygon \mathfrak{A} (without forgetting to count their multiplicities). This collection of points forms the ideal $\mathfrak{a} = \mathfrak{p}^r \mathfrak{p}_1^{r_1} \mathfrak{p}_2^{r_2} \dots$, which is said to be generated by the polygon \mathfrak{A} . Dedekind and Weber call this polygon \mathfrak{A} the “*Nullpolygon*” of \mathfrak{a} .

So, from the decomposition of a polygon into the product of its points, one deduces that polygons are in correspondence with the composite (non prime) ideals.

$$\mathfrak{A} = \mathfrak{P}^r \mathfrak{P}_1^{r_1} \mathfrak{P}_2^{r_2} \dots$$

corresponds to the ideal

$$\mathfrak{a} = \mathfrak{p}^r \mathfrak{p}_1^{r_1} \mathfrak{p}_2^{r_2} \dots$$

whose prime factors correspond to the points of the polygon.

As a prime ideal is said to be generated by a point, a composite ideal is generated by a polygon. The GCD (resp. the LCM) of two ideals is generated by the GCD (resp. the LCM) of the polygons which generate the said two ideals. The null-gon \mathfrak{D} generates the ideal \mathfrak{o} .

This way, Dedekind and Weber write, is established “the complete coincidence between ideals and the collection of integral functions equal to zero at the same fixed points” (*ibid.*, 99).

There is a connection between the order of the field Ω relative to a variable z chosen, and the values of this function. Indeed, Dedekind and Weber prove that for any variable z in Ω , if Ω of degree n relative to z , then z takes each particular value c at exactly n points. This can be seen by considering the principal ideal $\mathfrak{o}(z - c)$ and its prime decomposition $\mathfrak{o}(z - c) = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \dots$ with $e_1 + e_2 + \dots = n$. From what has been previously shown, if $\mathfrak{P}_1, \mathfrak{P}_2, \dots$ are the null-points of the prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots$, then z takes the value c at \mathfrak{P}_i as many times as the prime ideal generated by \mathfrak{P}_i appears in the prime decomposition of the ideal $\mathfrak{o}(z - c)$. This is valid for each of the points, which means that z takes the value c at each n points of the polygon. It is also true for $c = \infty$. Thus, the number n is the number of points at which the variable z takes the same value c , and it is called the order (*Ordnung*) of z . The *Ordnung* of z is always a positive integer and

always equals to the order of the field relative to z . Constants, and only constants, have order 0.

2.5.4 The Riemann surface

With the multiplicity of the points adequately characterized, and with the polygons describing the singularities of the functions, Dedekind and Weber can define the Riemann surface with its structure and not as a mere collection of points. They call the Riemann surface with ramification the “absolute Riemann surface”.

The Riemann surface is defined as the product of all polygons “moving through” (*sich bewegen*), i.e., taking successively all the values in $\mathbb{C} \cup \{\infty\}$. As they proved just before, for any numerical value c , there is a polygon \mathfrak{A} with n points (not necessarily distincts) for which the variable z takes the value c . These n points are said to be “conjugate” for z . This entails that z can take continuously all the values in $\mathbb{C} \cup \{\infty\}$. This is what Dedekind and Weber designate as the polygon \mathfrak{A} “moving through” all the possible values “and indeed in such a way that all its points change simultaneously.” This allows for the discrete characterization used so far to adequately describe the surface associated to the field Ω .

In this way, one obtains all the existing points, including several times the points for which $z - z_0$ or $\frac{1}{z}$ vanishes for an order higher than 1 (which always happens only a finite number of times).

Hence, if one takes the product of all these polygons, one obtains:

$$\prod \mathfrak{A} = T\mathfrak{Z}_z$$

where T is the “simple totality” formed by collecting *all* the points *once*, and \mathfrak{Z}_z is “a particular finite polygon called the ramification or winding polygon of T in z ” (*ibid.*, 100). The Riemann surface is an invariant concept of the field.

The ramification polygon \mathfrak{Z}_z is a *finite* polygon whose points describe the ramification of the Riemann surface. These points are called winding points or ramification points (*Verzweigungs- oder Windungspunkt*) of T in z , and each point is said to be “of order s if it appears s -tuply in \mathfrak{Z}_z .” (*ibid.*, 100). If $z - z_0$ or $\frac{1}{z}$ is infinitely small of degree e , then $s = e - 1$. The order of the ramification polygon is called “the ramification or winding number w_z .” Moreover, as its name suggests:

The points of the ramification polygon at which z has finite value together generate the *ramification ideal* in z .⁸⁹ (*ibid.*, 100)

⁸⁹ “Diejenigen Punkte des Verzweigungspolygons, in welchen z einen endlichen Wert hat, erzeugen zusammen das Verzweigungsideal in z .”

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The correspondence between surface and field implies that the singularities of the surface can be expressed in terms of (divisibility of) ideals: the ramification of the surface is thus translated into what we would now call the ramification in the field Ω , and which is characterized for Dedekind and Weber by the “ramification ideal” \mathfrak{z} .

It can be shown that the ramification polygon and the ramification number are left unchanged by a linear transformation of z into $z' = \frac{c+dz}{a+bz}$ (a, b, c, d constants and $ad - bc \neq 0$), which is linked to the fact that norms, traces and discriminants are not affected by such transformations.

Moreover, the ramification number is equal to the difference between the degree of the discriminant of Ω relative to z , $\Delta_z(\Omega)$, and the degree of the discriminant of Ω relative to $z' = \frac{1}{z}$, $\Delta'_z(\Omega)$.

Finally, further down the paper, Dedekind and Weber can also show that the ramification number is necessarily an even number (*ibid.*, 113).

Hence, as one can see, the correspondence between surface and field implies that the singularities of the surface can be expressed in terms of (divisibility of) ideals. The polygons allow to describe the singularities, and in particular the ramification of the surface is translated into the ramification in the field Ω , characterized by the “ramification ideal” \mathfrak{z} .

The definition of the “‘absolute’ Riemann surface” given by Dedekind and Weber has no geometrical component and its link with the Riemannian idea of a Riemann surface seems difficult to grasp. Dedekind and Weber mention that to “pass from this definition of the ‘absolute’ Riemann surface to the well-known Riemannian conception”, it suffices to “think the surface spread over a z -plane which is then covered n -tuply everywhere except at the ramification points” (*ibid.*, 100). But they don’t explore further this possibility, which appeals at least to a geometrical representation – if not to spatial intuition. The Riemann surface defined by Dedekind and Weber is rather detached from any spatial consideration. By relying exclusively on rational relations between functions, Dedekind and Weber develop their theory in such a way that topological notions are not *needed* in the definition but should eventually follow from the notions introduced.

2.5.5 More investigations about polygons

So far, only polygons with a positive power have been considered. This implies that only positive order numbers have been taken into account, since the poles have a negative order number. Following their arithmetical approach and the correspondence between polygons and ideals, Dedekind and Weber propose a representation of the functions of Ω as quotients of polygons. The denominator of the quotient will serve for the description of negative order numbers.

Why is this important? We saw that functions in Ω are completely characterized by their order numbers: if η in Ω takes the value 0 with multiplicity

r at \mathfrak{P} , its order number is r ; if η takes the value ∞ with multiplicity r at \mathfrak{P}' , its order number is $-r$; if η takes any other (constant) value, its order number is 0. There is only a finite number of points at which η has an order number different from 0. In addition, “the sum of all these order numbers is 0, and hence the sum of positive order numbers equals the sum of negative order numbers, and indeed it equals the order of the function η ” (*ibid.*, 103). In addition

If the order numbers of a function η are known for each point \mathfrak{P} , then the function η is determined up to a constant factor. Because if η' has everywhere the same order number as η then $\frac{\eta}{\eta'}$ has order number zero everywhere and hence is a constant.⁹⁰ (*ibid.*, 103)

So, if one were to construct a polygon \mathfrak{A} with each point at which η has a positive order number, and a polygon \mathfrak{B} with each point at which η has a negative order number, then these two polygons are of order the order of η and characterize the function completely up to a constant. But the definition adopted so far does not include polygons whose points have a negative power. Rather than to propose the definition of an ‘arbitrary’ polygon with arbitrary (positive or negative) powers, Dedekind and Weber “symbolically set” the function η as quotient of two polygons \mathfrak{A} and \mathfrak{B}

$$\eta = \frac{\mathfrak{A}}{\mathfrak{B}}.$$

In accordance with the names used for quotients of ideals, \mathfrak{A} is called the upper polygon and \mathfrak{B} the lower polygon. This allows Dedekind and Weber to introduce negative powers – those of the lower polygon – without adopting an approach which would not agree with the arithmetic of ideals as developed in the first part. In fact, by adopting the presentation of negative powers as the powers of the denominator of a quotient, they lean even more towards an arithmetical treatment of the theory.

The multiplication and division of two functions η and η' show that quotients of polygons obey the following (familiar looking) rules:

$$\text{if } \eta = \frac{\mathfrak{A}}{\mathfrak{B}} \text{ and } \eta' = \frac{\mathfrak{A}'}{\mathfrak{B}'} \text{ then } \eta\eta' = \frac{\mathfrak{A}\mathfrak{A}'}{\mathfrak{B}\mathfrak{B}'} \text{ and } \frac{\eta}{\eta'} = \frac{\mathfrak{A}\mathfrak{B}'}{\mathfrak{B}\mathfrak{A}'}.$$

Thus, one can see that the function η' is an integral algebraic function of η (taken as variable) if and only if “each point in the lower polygon of η' also appears in the lower polygon of η ” (*ibid.*, 104). Indeed, the characterization by means of order numbers given by Dedekind and Weber show that a

⁹⁰ “Sind die Ordnungszahlen einer Funktion η für jeden Punkt \mathfrak{P} bekannt, so ist damit die Funktion η bis auf einen konstanten Faktor bestimmt; denn hat η' überall dieselbe Ordnungszahl wie η so hat überall die Ordnungszahl Null und ist also eine Konstante.”

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function of a certain variable z is an integral algebraic function of z if and only if it does not have infinite values at the points at which z takes a finite value. If $\mathfrak{B}' \subset \mathfrak{B}$, then η' is infinite only at points where η is also infinite (the points of \mathfrak{B}).

From the representation of functions as quotients of polygons, Dedekind and Weber define an equivalence relation between two polygons $\mathfrak{A}, \mathfrak{A}'$ with the same number of points:

Two polygons \mathfrak{A} and \mathfrak{A}' with the same number of points are called *equivalents* if there is a function η in Ω with the notation

$$\eta = \frac{\mathfrak{A}}{\mathfrak{A}'}.^{91}$$

(*ibid.*, 105)

This relation (not introduced as an ‘equivalence relation’ by modern standards) is transitive and allows to define a notion of equivalence class for polygons:

All the polygons $\mathfrak{A}', \mathfrak{A}'', \dots$ equivalent to a given polygon \mathfrak{A} constitute a *polygon class* A . (...) Each polygon belongs to one and only one class.⁹² (*ibid.*, 105)

All the polygons in a polygon class have the same order, which is thus called the order of the class. It is possible to have polygon classes constituted by only one polygon, this polygon is called “isolated” polygon. The equivalence is preserved by the multiplication by an arbitrary polygon, i.e., if \mathfrak{A} equivalent to \mathfrak{A}' , then $\mathfrak{M}\mathfrak{A}$ equivalent to $\mathfrak{M}\mathfrak{A}'$ for any \mathfrak{M} .

Dedekind and Weber then introduce the notions of product and of divisibility for the polygon classes. Since \mathfrak{A} equivalent to \mathfrak{A}' and \mathfrak{B} equivalent to \mathfrak{B}' implies $\mathfrak{A}\mathfrak{B}$ equivalent to $\mathfrak{A}'\mathfrak{B}'$, they can consider the class of the polygons $\mathfrak{A}\mathfrak{B}$. The class C which contains $\mathfrak{A}\mathfrak{B}$ contains all the products of a polygon of A by a polygon of B . It is thus called the product of A and B and $C = AB = BA$.

In the same order of ideas, if a polygon \mathfrak{A} in A divides a polygon \mathfrak{C} in C , one has $\mathfrak{A}\mathfrak{B} = \mathfrak{C}$ and for any \mathfrak{A}' in A and \mathfrak{C}' in C , then $\mathfrak{A}'\mathfrak{B} = \mathfrak{C}'$. Thus, one can say that C is divisible by A .

The notion of “*Schaar*” can be extended to polygons as well, by considering s equivalent polygons $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$ and an arbitrary polygon \mathfrak{A} in

⁹¹ “Zwei Polygone \mathfrak{A} und \mathfrak{A}' von gleichviel Punkten heißen äquivalent, wenn eine Funktion in Ω existiert welche die Bezeichnung hat $\eta = \frac{\mathfrak{A}}{\mathfrak{A}'}$.”

⁹² “Alle mit einem gegebenen Polygon \mathfrak{A} äquivalenten Polygone $\mathfrak{A}', \mathfrak{A}'', \dots$ bilden eine Polygonklasse A . (...) kommt jedes beliebige Polygon in einer und nur in einer Klasse vor.”

the same polygon class A . Thus, s functions can be formed by taking the quotients of each $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$ by \mathfrak{A} :

$$\eta_1 = \frac{\mathfrak{A}_1}{\mathfrak{A}}, \quad \eta_2 = \frac{\mathfrak{A}_2}{\mathfrak{A}}, \quad \dots, \quad \eta_s = \frac{\mathfrak{A}_s}{\mathfrak{A}}$$

If one considers these functions as forming a family, then at a certain point \mathfrak{P} , any function of the form

$$\eta = c_1\eta_1 + \dots + c_s\eta_s$$

where the c_i are constants, can be shown to be such that

$$\eta = \frac{\mathfrak{A}'}{\mathfrak{A}}$$

with \mathfrak{A}' in A . All the polygons \mathfrak{A}' corresponding to the different constants c_i depend only on the polygons $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$, which can thus legitimately be called a “family of polygons” with basis $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$ and denoted by $(\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s)$. The linear dependency / independency of $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$ is likewise defined.

A notion of *divisor of the family* is then defined, which corresponds to the (possible) greatest common divisor \mathfrak{M} of all the $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_s$. Moreover, if one takes in a family S (of dimension s) all the polygons that are divisible by an arbitrary r -polygon \mathfrak{A} , these polygons form a family of dimension at least $s - r$ (*ibid.*, 108). One can thus lower the dimension of a family of polygons by choosing the appropriate divisor \mathfrak{A} .

The dimension of the family formed by all the polygons in a same class, which is always finite, is called the dimension of the class. Conditions of divisibility are again given: if the class C contains exactly s linearly independent polygons divisible by a polygon \mathfrak{A} in a class A , then C is divisible by A and the number s depends only of A and C and is denoted by (A, C) . The dimension of A will consequently be denoted by (O, A) , where O is the class of \mathfrak{D} , the polygon null.⁹³ In particular, if $C = AB$, $(A, C) = (A, AB) = (O, B)$ and in general $(O, B) \geq (O, AB) - a$, where a is the order of A .

One defines a “proper (*eigentlich*) class” as a family whose polygons are all relatively primes. In the opposite case, the class is said to be “improper” of divisor \mathfrak{M} . This notion will be important for the proof of the Riemann-Roch theorem.

2.5.6 On the Riemann-Roch theorem

The Riemann-Roch theorem has been described as the most striking example of the intimacy of the relationships between topological notions and

⁹³One can note that the notation is analogous to that used for the relative norm of modules and ideals.

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properties of functions,⁹⁴ which seems to be lost in Dedekind and Weber's version of it. We saw that in Riemann's and Roch's version of it, it gives the number of arbitrary constant contained in a function (i.e., the number of parameters of which the function depends) which becomes infinite at a certain number m of given points, in relation with its genus p .

In Dedekind and Weber's rewriting of Riemann's theory, the Riemann-Roch theorem becomes a relation between the order and the dimension of a polygon class, or between a class and its "supplementary" class. The proof of the Riemann-Roch theorem, given by Dedekind and Weber only involves manipulations of (quotients of) polygons and classes of polygons.⁹⁵

The genus p is, following Riemann, given by

$$p = \frac{1}{2}w - n + 1$$

in which w is the sum of the ramification numbers and n is the degree as earlier. The genus is independent of the choice of variables.

If two functions α, β , of order n, m respectively, generate all the functions of the field Ω , then Dedekind and Weber show, with manipulations of quotients of polygons, that for r the number of double points:

$$p = (n - 1)(m - 1) - r$$

For the proof of the Riemann-Roch theorem, Dedekind and Weber introduce purely algebraic notions of differential quotients and differentials. Dedekind and Weber's surface, defined by purely arithmetico-algebraic means, lacks the possibility to set up the appropriate structure for an integration theory. From their viewpoint, continuity should not be taken as a given property of surfaces (or, in fact, of space in general) nor as a necessary condition for the definition of the surface. Dedekind and Weber thus introduce notions related to integration theory by purely algebraic means, without any recourse to "continuity".

Dedekind and Weber define differentials of different "kinds" according to the classification given by Riemann (following Legendre, see p. 88). The proof of the Riemann-Roch theorem will be done in several times, according to the different "kinds" of differentials.

Differentials of the first kind are "the simplest differentials in Ω ", that

⁹⁴See the Introduction of [Stillwell, 2012], as well as [Bourbaki, 1984] and the articles by Jeremy Gray mentioned on p. 103.

Dedekind and Weber also prove Abel's theorem, in a form stated by Stillwell to be "hardly recognizable", since the genus does not even appear in the theorem. I will not develop the proof of Abel's theorem, and will concentrate on the Riemann-Roch theorem.

⁹⁵For more details, see Stillwell, 1996, 110-133.

is, differentials of first kind are those without poles.⁹⁶ Differentials can be written as quotients of polygons, and the upper polygon \mathfrak{W} of differentials of first kind dw is of order $2p - 2$. It is called “fundamental polygon of w ” (*ibid.*, 123). Dedekind and Weber call this polygon a “complete polygon of first kind”, and its divisors “polygons of first kind.”

Two polygons \mathfrak{A} and \mathfrak{B} are said to be “supplementary polygons of each other” if $\mathfrak{A}\mathfrak{B} = \mathfrak{W}$. Any polygon that is not a divisor of a complete polygon of first kind, and in particular a polygon whose order is greater than $2p - 2$, is said to be a “polygon of the second kind”.

All complete polygons of the first kind form a polygon class W .⁹⁷
(*ibid.*, 124)

Dedekind and Weber show that $(O, W) = p$.

A supplementary class is defined following the same approach that Dedekind and Weber adopted for most of the paper. They extend the notions given above to classes of polygons: the equivalence between polygons preserves the property of a polygon to be of first kind, hence a class of polygons which contains polygons of first kind is called “polygon class of the first kind”. The other classes are called “polygon classes of second kind.”

If \mathfrak{A} and \mathfrak{B} are supplementary, then their respective polygon classes are such that $AB = W$ and consequently called “supplementary classes”. The notion can also be related to functions, which will be called of the first or second kind according to the kind of their lower polygon.

Dedekind and Weber prove that for an arbitrary class of the first kind A , and q the number of independent polygons \mathfrak{W} divisible by a polygon \mathfrak{A} in A , then

$$q = (A, W) = (O, B)$$

that is, the number of independent polygons that can be divided by a polygon in A is equal to the dimension of the supplementary class of A . Since $(O, W) = p$, and since $(A, W) = 0$ for any class A of the second kind, then “each class of order $\leq p - 1$ is of the first kind” (*ibid.*, 127).

The Riemann-Roch theorem is then stated as follows.

If A, B are supplementary classes of first kind, at least one of which is proper, and if a, b are their orders, so that

$$a + b = 2p - 2$$

⁹⁶Riemann’s restatement of Legendre’s classification, given on p. 98 gives the conditions on the singularities of functions for them to correspond to the first, second or third kind of integrals established by Legendre. Dedekind and Weber follow this idea.

⁹⁷“bilden alle vollständigen Polygone erster Gattung eine Polygonklasse W .”

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then

$$(O, A) - \frac{1}{2}a = (O, B) - \frac{1}{2}b.^{98}$$

(*ibid.*, 129)

If the case $(A, W) = 0$ is not excluded, then the Riemann-Roch theorem reads:

If A is a proper class of order n , then its dimension is

$$(O, A) = n - p + 1 + (A, W).^{99}$$

(*ibid.*, 129)

in which one can recognize the Riemann-Roch theorem in its original form. Indeed, for a variable z , (O, A) is the number of linearly independent polygons in A of order n ,¹⁰⁰ p is the genus and $(A, W) = q$ is the number of linearly complete polygons of first kind, i.e., upper polygons of differentials of the first kind.

As we can see, Dedekind and Weber do not only reformulate the theory so as to have a definition of the Riemann surface answering certain epistemological requisites. Indeed, their theory gives them the means to offer what is considered by historians as the first rigorous proof of the Riemann-Roch theorem. This proof illustrates well how Dedekind and Weber are able to obtain and prove important results using solely the arithmetico-algebraic arsenal developed in their paper.

After this proof, Dedekind and Weber's paper goes on for several paragraphs, in which they shed further light on differentials of second and third kind, using the Riemann-Roch theorem and relating them to the singularities of the surface. They also define the residues of an algebraic differential. They are able to prove, with the conceptual apparatus developed in the previous paragraphs, the theorem stating that the sum of the residues of an algebraic function is zero.

Dedekind and Weber are able, as they wished, to treat entire classes of functions at once, without having to distinguish between special cases or make restrictive assumptions about their singularities, and without having to rely on *Darstellungsformen*. Moreover, Dedekind and Weber's treatment

⁹⁸ "Sind A, B Ergänzungsklassen erster Gattung, von denen wenigstens die eine eine eigentliche ist, und a, b ihre Ordnungen, also $a + b = 2p - 2$, so ist $(O, A) - \frac{1}{2}a = (O, B) - \frac{1}{2}b$."

⁹⁹ "Ist A eine eigentliche Klasse von der Ordnung n , so ihre Dimension $(O, A) = n - p + 1 + (A, W)$."

¹⁰⁰The order of A is the order of the polygons contained in A , which is itself the number of points of the polygons.

does keep the idea that the surface defined and the functions of Ω are intimately related and indispensable to each other (with, notably, the intricate relation between the definition of the points and the values of the functions, in particular the study of their singularities).

However, overall, any geometrical consideration, any recourse to “*Analysis situs*”, disappear from Dedekind and Weber’s treatment. Indeed, notions which were considered by Riemann as intrinsically related to the surface and pertaining to *Analysis situs* were all defined by Dedekind and Weber by arithmetico-algebraic means involving polygons (and through them, ideals). The properties of the surface and their intrinsic relationships with functions are emptied of any geometrical meaning, essential for Riemann but deemed unnecessary (if not useless) by Dedekind and Weber.

While Riemann’s Riemann surfaces allowed to acquire a deep understanding of multi-valued functions via a thoroughly geometrical concept, Dedekind and Weber’s rewriting erases the geometrical core of Riemann’s works. It seems that the fact that they completely give up any geometrical meaning in Riemann’s theory makes their approach not exactly as faithful to Riemann’s initial ideas as Dedekind and Weber claimed it to be.

2.5.6.1 Reception of the 1882 paper

Dedekind and Weber’s paper arose very little interest, for some time. While Hilbert did use some of their ideas for his *Nullstellensatz*, the adoption of the ideal theory as a basis for a treatment of Riemann function theory did not stir up much enthusiasm. As I have mentioned, Weber himself did not use ideals in algebraic function theory, in the third volume of his *Lehrbuch*.

Brill and Noether, in 1894 ([Brill and Noether, 1892]), openly exclude Dedekind and Weber’s “theory relying on number theory” (as well as Kronecker’s) from their report on algebraic geometry.¹⁰¹

A first significant response can be found in Hensel and Landsberg’s 1902 book ([Hensel and Landsberg, 1902], dedicated to Dedekind), which changes the basis of the theory from ideal theory to function theory, so as to be able to penetrate deeper into Riemann’s theory, in particular as regards Abelian functions.¹⁰² They do, however, introduce ideal theory later in the book, in the part about the Riemann-Roch theorem. But Hensel, as Geyer explains, replaces many of the “conceptual conclusions” of Dedekind and Weber by computations using expansions in series and products, losing what for Geyer constitutes the “conceptual elegance, the formal beauty and the flavorful brevity” of the paper ([Geyer, 1981], 115). According to Laugwitz,

¹⁰¹An additional reason for them do so, they explain, is the fact that Klein’s lectures on Riemann surfaces in 1892 did give a detailed account of their work ([Brill and Noether, 1892], v) – which does not mean, that Klein adopted their approach.

¹⁰²See [Laugwitz, 2009], 160, and [Geyer, 1981].

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Hensel and Landsberg’s book was “found (...) easier to read than the work of Dedekind and Weber,” however, the standpoint adopted as regards the foundation of the theory “represents a watering down of Dedekind’s idea in that it largely replaces the ideal-theoretic by the function-theoretic.” Hence, even in the works of mathematicians who felt they owed so much to Dedekind’s works that they dedicated their book to him, one can find what Laugwitz considers to be

one more instance of a common phenomenon, which is that many people are likely to reject an ever-so-beautiful algebraic theory in favor of a (possibly more complicated) analytic treatment. ([Laugwitz, 2009], 160)

2.6 On the “arithmetical” rewriting

In the following section, I will argue that Dedekind and Weber’s rewriting of Riemannian function theory is an *arithmetical* rewriting. For this, I will try a first clarification of how “arithmetical” should be understood from a Dedekindian viewpoint. These paragraphs set up a series of statements and questions about Dedekind’s mathematics and will highlight a thread to follow. Hence, these paragraphs constitute a first analysis and the comments and statements made will be tested and investigated more deeply in the next chapters of the dissertation.

2.6.1 An arithmetical definition of the Riemann surface?

Let me sum up quickly what we just saw. In their 1882 paper, Dedekind and Weber start “analogously to number theory” by defining fields, modules and ideals of functions. The first essential step of the theory is to thoroughly study the *laws of divisibility of ideals*, which correspond to the laws of divisibility of polynomial functions. This is done by defining arithmetical operations for ideals in such a way that it becomes possible to “calculate” with ideals:

With a proper definition of multiplication, it is possible to calculate with these ideals using exactly the same rules as for rational functions.¹⁰³ ([Dedekind and Weber, 1882], transl. slightly altered, 42)

Once this is done, additional notions in ideal theory are defined such as the ramification ideal, which are developed on the same ground, namely using *products, decomposition in primes, and divisibility* of ideals.

¹⁰³ “Mit diesen Idealen läßt sich nach gehöriger Erklärung der Multiplikation ganz nach denselben Regeln rechnen, wie mit rationalen Funktionen.”

Following the unfolding of the theory of ideals, the surface is defined by its points: points are defined by a mapping from the elements of the field Ω into the field of numerical values in such a way that the image of the arithmetical operations (of functions) is consistent with the arithmetical operations for the numerical values. There is a one-to-one correspondence between points and prime ideals. Then, because points do not allow to have more than a “simple totality” which does not provide an appropriate description of the singularities of the functions, complexes of points, called polygons, are defined, for which it can be shown that they correspond to a unique non-prime ideal. The one-to-one correspondence between ideals and polygons allows to carry divisibility properties considerations over to polygons.¹⁰⁴ Hence, in general, *the laws of arithmetic of ideals can be carried over to polygons*. This yields a *decomposition into primes for polygons*, and notions such as the GCD and the LCM of polygons, which will be used in the rest of the theory. For any numerical value c , there exists a n -gon \mathfrak{A} for which the variable z takes the value c . To obtain all the points, including multiple points, one makes c “run through” $\mathbb{C} \cup \{\infty\}$, which yields a product of polygons composed of the simple totality of all points and the ramification polygon describing the ramification of the surface.

The ramification number, the genus, the differentials are defined in terms of polygons, divisibility relations between polygons, quotients of polygons and equivalence classes relation between polygons. Notions of divisor, multiplication, and the like are likewise carried over to classes of polygons, differential quotients, families of differentials, etc.

Emmy Noether, in her editor’s note, presents *Algebraische Funktionen* as the creation of the arithmetical theory of functions. And in fact, Dedekind and Weber’s theory was received by their contemporaries as an application of number theory to function theory, seeing in it a certain “arithmetical” aspect.

However, the treatment of the theory of functions, given by Dedekind and Weber, is not presented by them as a mere “application” of number theoretical methods. Rather, they are proposing a thorough rewriting of Riemann’s ideas so as to instigate, in Riemannian function theory, the rigor and “generality” they consider essential for its development. In fact, the inherently arithmetical nature of Dedekind and Weber’s rewriting of Riemann’s theory, I will argue, is brought about by the fundamental use of the rational operations to develop the core ideas and to propose a brand new ground of Riemann’s theory. Indeed, as we saw, if we interpret arithmetical in this way, the approach used not only for ideals (where it could be seen as natural, given the number theoretical roots of ideals), but for

¹⁰⁴This argument was already used by Dedekind for the fact that studying the divisibility of ideals is tantamount to studying the divisibility of algebraic numbers.

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polygons and notions related to the surface, is indeed arithmetical. Thus, in this sense, the attempt to provide a definition of a Riemann surface that would be considered more reliable and more rigorous, consists essentially in an “arithmetical” rewriting of Riemann’s theory of functions.

To be clear, by “arithmetic”, I do not mean the properties of (the sequence of) natural numbers themselves, but rather what Dedekind and many 19th century referred to as the “science of numbers”: the properties, theorems and laws following from the definition of the four elementary operations. In particular, the reference to the development of an “arithmetical” theory should not be understood as suggesting that Dedekind is involved in a reduction of the objects of mathematics to integral numbers. The pre-eminence of arithmetical methods in Dedekind’s works is not the kind of arithmetical reductionism that one can find in Kronecker’s mathematics, a point that I will try to make clear in the rest of this dissertation.¹⁰⁵ The key reason for it not being reductionism is that for Dedekind, the invention of new concepts is an essential part of the development of mathematics and, in particular, the extension of the concept of number appears as crucial for the development of arithmetic. Dedekind’s efforts are directed towards arithmetical relationships in the domain he is studying.

Furthermore, as we saw in *Algebraische Funktionen* and will see in more details in the following chapters, Dedekind defines arithmetical relationships for various objects, from groups to polygons. While this could suggest that the operations should be understood as algebraic operations, rather than arithmetic¹⁰⁶, it is important to note that Dedekind systematically proceeds to study the *laws of divisibility*. And, from Dedekind’s viewpoint, divisibility is the first and fundamental concern of *arithmetic*:

The theory of divisibility of numbers, which serves as a foundation of arithmetic, was already established in its essentials by Euclid. ([Dedekind, 1876-1877], transl. slightly altered, 53)

The idea that the study of divisibility plays a central role in the development of number theory was rather common in the 19th century as we saw in Sec. 1.2.

If arithmetic is mostly concerned with divisibility, then Dedekind and Weber’s theory is, indeed, deeply arithmetical, since the operations used in

¹⁰⁵Kronecker, as is well known, demanded to develop methods and algorithms based on the sole use of natural numbers which were considered to be the only actual existing numbers, defining negative and rational numbers as magnitudes stemming from congruences but without giving them the status of *numbers*. I will come back to the difference(s) between Dedekind’s and Kronecker’s ideas about arithmetic on several occasions in this dissertation.

¹⁰⁶In the sense of algebra as being the science of calculation with ‘symbols’ and arithmetic being the science of numbers.

their paper are still intrinsically related to divisibility. However, they do seem to be completely devoid of any relation to *numbers*, with which arithmetic is usually related – in Dedekind’s statements as well. I will propose in the following paragraphs to consider the first, most immediate elements at our disposal to clarify Dedekind’s conception of arithmetic, so as to set the outlines for my investigations.

2.6.2 Arithmetic as the science of numbers, from Dedekind’s viewpoint?

Dedekind characterizes several times arithmetic as “the science of numbers” (*Stetigkeit* 1872, 771; *Zahlen* 1888, 791-792, 795, 809; the 1894 version of algebraic number theory in [Dedekind, 1932] III, 24; . . .). Or again, in this rather radical quote from his *Nachlass*, which can be dated from after 1888:

Of all the auxiliary means (*Hilfsmitteln*), that the human mind has until now created to ease its life, i.e., the work in which the thinking consists, none is as far-reaching and as inextricably connected to its innermost Nature, as the concept of number. Arithmetic, whose sole subject matter is this concept, is already by now a science of immeasurable extension, and there is no doubt that no limits are set to its further developments. Equally limitless is the area of its applications, since every thinking man, even when he is not distinctly aware of it, is a number-man, an arithmetician.¹⁰⁷ (*Zum Zahlbegriff*, in [Dugac, 1976b], 315, my translation)

In *Stetigkeit*, arithmetic is characterized as the science of numbers to underline that one should not introduce “foreign notions into arithmetic itself, the *science of numbers*” (*ibid.*, my emphasis). In *Zahlen*, Dedekind makes that point again, and explains that the process of extending the domains of numbers, the “step-by-step extension of the number-concept”, can only be made perfectly clear “through the science of numbers” (*ibid.*, 792), that is by “arithmetical” means – a good example of this is, of course, the definition of irrational numbers.¹⁰⁸

¹⁰⁷“Von allen Hilfsmitteln, welche der menschliche Geist zur Erleichterung seines Lebens, d.h. der Arbeit, in welcher das Denken besteht, bis jetzt erschaffen hat, ist keines so folgenreich und so untrennbar mit seiner innersten Natur verbunden, wie der Begriff der Zahl. Die Arithmetik, deren einziger Gegenstand dieser Begriff ist, ist schon jetzt eine Wissenschaft von unermesslicher Ausdehnung und es ist keinem Zweifel unterworfen, dass ihrer ferneren Entwicklung gar keine Schranken gesetzt sind; ebenso unermesslich ist das Feld ihrer Anwendung, weil jeder denkende Mensch, auch wenn er dies nicht deutlich fühlt, ein Zahlenmensch, ein Arithmetiker ist.”

¹⁰⁸see Sec. 7.2.

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In *Zahlen*, in which natural numbers and arithmetical operations are defined, Dedekind qualifies arithmetic as the “Science of numbers” at several points in the essay. For example in the Preface of the book:

Upon this unique and moreover absolutely indispensable foundation (...) the whole *science of numbers* must, in my opinion, be established (...) I hope that the following pages, as an attempt to establish the *science of numbers* upon a more uniform foundation, will find a generous welcome (...) ¹⁰⁹ ([Dedekind, 1888], transl. slightly altered, 791-792, my emphasis)

The “foundation” of the “science of numbers” is explicitly the aim of Dedekind’s 1888 book. I will give a more detailed analysis of the way in which Dedekind defines the system of natural numbers and gives a rigorous foundation to the “science of numbers” in Sec. 8.2. For now, I only want to make clear that his concern is the *science* of numbers, as he explains, for example, in clarifications about some of the keystones of the book:

[N]either [Bolzano nor Cantor] made the attempt to use this property for the definition of the infinite¹¹⁰ and *upon this foundation to establish with rigorous logic the science of numbers.* (*ibid.*, 792, my emphasis)

Already in the third edition of [Dirichlet’s *Vorlesungen* (1879, remark p. 470) it is stated that no thought of any kind is possible without the capacity of the mind to compare a thing a with a thing a' , or to relate a to a' or to allow an a' to correspond to a – and that the entire science of numbers also rests on this capacity. (*ibid.*, 834)

Once the natural numbers defined (by a series of conditions equivalent to Peano axioms), Dedekind gives a clear statement of what arithmetic is firstly composed of:

[t]he relations or laws which are derived entirely from the sole conditions [defining the system of natural numbers] (...) form the first object of the *science of numbers* or *arithmetic*.¹¹¹ (*ibid.*, transl. slightly altered, 809)

¹⁰⁹“Auf dieser einzigen, auch sonst ganz unentbehrlichen Grundlage muß nach meiner Ansicht, (...) die gesamte Wissenschaft der Zahlen errichtet werden. (...) mögen die folgenden Blätter als ein Versuch, die Wissenschaft der Zahlen auf einheitlicher Grundlage zu errichten, wohlwollende Aufnahme finden...”

¹¹⁰Dedekind is referring to: “64. Definition. A system S is said to be infinite when it is similar to a proper part of itself.” ([Dedekind, 1888], 806)

¹¹¹“Die Beziehungen oder Gesetze, welche ganz allein aus den Bedingungen $\alpha, \beta, \gamma, \delta$ in 71 (...) bilden den nächsten Gegenstand der Wissenschaft von den Zahlen oder der Arithmetik.”

Following the idea that arithmetic is the science of numbers, it is important to clarify what is meant by “science” and to which extent it may be different from “theory”. In the Chapter 3, we will see that Dedekind considers “science” as an activity of the human mind. This shall help to understand why it is an important part of the characterization of arithmetic. In particular, it shall clarify how it allows for an idea of arithmetic as something different than a rigid theory, and such for example the relation between arithmetic for the natural numbers and arithmetic for the real or complex numbers.

Let me underline now that the fact that arithmetic is an “activity” whose objects are numbers implies a tension in what is indeed essential for arithmetic: the numbers or the arithmetical operations? In fact, one cannot be thought of without the other. Indeed, the properties and laws of natural numbers, which are, as we saw, the first object that the science of numbers studies, are given by the operations. In the same time, in the 1870s and later, the operations of arithmetic are defined for Dedekind by the objects on which they act. The initial definition of operations is made for the system of natural numbers, which gives a fundamental place to natural numbers. And through the extensions of the number concept, operations will likewise be extended – and so is arithmetic.

2.6.3 What is not arithmetical for Dedekind

For a finer idea of what is characteristic of an theory based on arithmetic, for Dedekind, it should be useful to have a look at what he considers as *not* arithmetical. Such statements can be found in the comments Dedekind wrote about Kronecker’s *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* (1881), the so-called “Bunte Bemerkungen”.¹¹² Dedekind criticized Kronecker’s use of “arithmetical” to qualify his own work:

Neither from this introduction nor from the essay itself is it clear why exactly this theory should be called an *arithmetical* one. Under this name, one should suspect that the consideration of the realm of numbers (the absolute constants) would form the main foundation, it is by no means the case. I would very much like to rather call this theory *formal*, because it predominantly is based on the “auxiliary methods of indefinite coefficients” (p. 47, 48, 69) and the “association of forms (formed by these coefficients or auxiliary variables u, u', u'')” (§15 and §22, p. 93-96).¹¹³ ([Edwards, Neumann, and Purkert, 1982], 54)

¹¹²Dedekind’s “Bunte Bemerkungen” were never published by Dedekind, although according to H. Edwards, he seemed to intend to do so. Edwards, Neumann and Purkert edited and published the “Bunte Bemerkungen” in 1982 ([Edwards, Neumann, and Purkert, 1982]). References will be to their paper.

¹¹³“Weder aus dieser Einleitung noch aus der Abhandlung selbst wird es deutlich, weshalb diese Theorie gerade eine arithmetische genannt wird. Nach diesem Namen sollte man

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In Dedekind’s view, then, for Kronecker’s work to be an *arithmetical* theory, it should be built upon investigations of the natural numbers (“*der absoluten Constanten*”). Instead, Kronecker’s paper makes an extensive use of indeterminate variables and polynomials. This statement by Dedekind offers a rather narrow view about what should be taken to be arithmetic(al). It seems to suggest that Dedekind considered that only what concerns or uses the natural numbers is truly arithmetical. This is not the case, for it would imply that the extensions of the domain of natural numbers are not actually within the scope of arithmetic, or that the only true numbers are the positive integers. This is clearly not what Dedekind argued for – on the contrary, he even was the first, and for some time the only one, to talk about irrational *numbers* and not irrational magnitudes:

[N]ot all cuts are produced by rational numbers. (...) Now, whenever we have a cut (A_1, A_2) produced by no rational number, we *create* a new number, an *irrational* number α .¹¹⁴ ([Dedekind, 1872], transl. slightly altered, 773, original emphasis)

The elements obtained from the extension of the system of natural numbers to negative numbers, and from there the successive extensions to rational, real, algebraic and complex numbers, are unambiguously still numbers:

And this requirement [that the indirect operations be unconditionally executable] leads to the necessity to create new classes of numbers, since with the original sequence of positive integers the requirement cannot be satisfied. Thus one obtains the negative, rational, irrational, and finally also the so-called imaginary numbers.¹¹⁵ ([Dedekind, 1854a], 757)

Kronecker, on the other hand, distinguished between natural numbers, the only true numbers, and magnitudes.¹¹⁶ This distinction, however, is not proper to Kronecker. What can be called a number varies from one author

vermuthen, daß die Betrachtung des Reiches der Zahlen (der absoluten Constanten) die hauptsächlichste Grundlage bilden würde, was keineswegs der Fall ist. Viel eher möchte ich diese Theorie eine formale nennen, weil sie vorwiegend auf dem ‘methodischen Hilfsmittel der unbestimmten Coefficienten’ (S. 47, 48, 69) und der ‘Association der (aus diesen Coefficienten oder Hilfsvariablen $u, u', u'' \dots$ gebildeten) Formen’ (§15 und §22, S. 93-96) beruht.

¹¹⁴“[N]icht alle Schnitte durch rationale Zahlen hervorgebracht werden (...). Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so erschaffen wir eine neue, eine irrationale Zahl α .”

¹¹⁵“Die Forderung der unbedingten Ausführbarkeit der indirekten, umgekehrten Operationen (...) führt auf die Notwendigkeit, neue Klassen von Zahlen zu schaffen, da mit der ursprünglichen Reihe der absoluten ganzen Zahlen dieser Forderung kein Genüge geleistet werden kann. So erhält man die negativen, gebrochenen, irrationalen und endlich auch die sog. imaginären Zahlen.”

¹¹⁶On Kronecker’s conception of arithmetic, see see [Boniface, 2005] and in this dissertation 7.1.3.

to another – some consider only natural integers as numbers, other include rationals – but, except for Dedekind, all the major actors of the so-called arithmetization of analysis from Méray to Weierstrass to Cantor and Kronecker, talk about irrational magnitudes (see 7.1.2).

In Dedekind’s case, the objects defined (created) after the extensions are still numbers. Hence, their study is still arithmetic, the science of numbers. A characterization of arithmetic in Dedekind’s works should, then, take this point into account: arithmetic is the science of numbers, a concept whose scope and meaning is not fixed, and the focus of arithmetic widens according to the modifications inflected to the number concept. By contrast, Kronecker’s arithmetic only admits natural integers as being effectively *numbers*. To define the other magnitudes (i.e., negative, rational, algebraic numbers), Kronecker uses congruences and polynomials with positive integers as coefficients. Polynomials, for Kronecker, as a series of operations on positive integers, become constitutive of an arithmetical inquiry.

2.6.4 From the science of numbers to arithmetic of polygons?

In the 1882 paper, the existence of a one-to-one correspondence (between ideals and polygons, notably) allowed for the transfer of arithmetical properties. For the divisibility properties of ideals themselves, the correspondence between divisibility of ideals corresponds to divisibility of algebraic numbers (or functions) is invoked as a justification for transferring the study of divisibility laws to the level of ideals. Yet, the properties of divisibility are to be *proved*, they cannot be assumed because of the said correspondence, since the laws of divisibility of algebraic numbers (or functions) are not known.

How did arithmetic go from being the “science of numbers” to being usable in algebraic function theory? Is there a characterization of arithmetic for Dedekind that would allow for it to be both the “science of numbers” and a set of operations apparently general or abstract enough that it can be carried over to concepts like ideals and polygons?

A first hypothesis on Dedekind’s conception of arithmetic, which also sheds light on Dedekind’s statement about the non-arithmeticity of Kronecker’s approach, could be that natural numbers are the *first* object of arithmetic. For a theory or a method to be arithmetical, it should use notions grounded in the natural numbers. A conceptual analysis should allow to trace the origins of the concept back to natural numbers – it doesn’t seem to be the case of Kronecker’s indeterminate coefficients and variables, for Dedekind. In particular, a constant of Dedekind’s thought is the demand “that arithmetic shall develop out of itself” ([Dedekind, 1872], 771), that is, without the introduction of “foreign elements” such as geometrical methods or indeterminate variables, or vague notions such as magnitudes, so as to preserve the “identity of nature” of numbers.

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On the other hand, if the “first object” of arithmetic is the integral numbers, Dedekind also underlines that one should not forget that “the concept of integral number, in this century, has known an enlargement, through which number theory discovered new paths”¹¹⁷ ([Dedekind, 1879], 434, my translation). Arithmetic, as the science of numbers, is enlarged consequently. Yet, there can be at least two directions for the extension of the number concept and arithmetic: the extensions of the number domains and the extension of the concept of integer. Although not always cleanly separated, these two sorts of extension are nevertheless treated in different ways. In this dissertation, I will be interested in the practices involved in Dedekind’s explorations of the extensions of numbers and of arithmetic.

Dedekind showed particular interest in identifying what can provide effective and fruitful notions for mathematics. For example, in 1854 already, new classes of numbers are created because “the definitions we thereby obtain [in the natural numbers] for these fundamental operations [of addition and multiplication] no longer suffice for the further development of arithmetic” ([Dedekind, 1854a], 757). In later number theoretical works, Dedekind seems to identify divisibility as the source of potentially fruitful concepts and even, more generally, the study of divisibility is a powerful driving force in the development of number theory. In particular, the demand that the laws of divisibility be unconditionally valid is the strongest motive behind the invention of ideal theory.

In the words of Dedekind, the theory of divisibility of numbers is at the basis of arithmetic, as I have alluded to earlier in the paragraph – a point which highlights the importance of operations and their particular status in the development of the “science of numbers”. We will see that the importance attached to divisibility is not as prominent in Dedekind’s earliest works, and in particular in his account of the development of arithmetic in his 1854 *Habilitationsvortrag*. The weight given to divisibility seems to increase with the experience acquired by Dedekind in working in number theory. Indeed, number theory appears to be stimulated by the study of (problems induced by) divisibility.¹¹⁸ It is possible that a significant part of the explanation as to why divisibility takes over other properties of numbers as being the most important ones, or at least interesting ones, is of a merely practical or pragmatic aspect: as the history of mathematics tends to show, divisibility leads to the kind of “circumstances”, as Dedekind puts it, in which important difficulties appear. Overcoming these difficulties, Dedekind tells us, is a “singular stimulus for mathematicians” ([Dedekind, 1852], 1) and leads to important innovations.

¹¹⁷“*Der Begriff der ganzen Zahl hat in diesem Jahrhundert eine Erweiterung erfahren, durch welche der Zahlentheorie wesentlich neue Bahnen eröffnet sind.*”

¹¹⁸see Sec. 5.1.3.

2.6.5 Arithmetical methodology in Dedekind's mathematics, a first view

In 1882, Dedekind and Weber go from arithmetic of functions to arithmetic of ideals to arithmetic of (complexes of) points. This is done very explicitly, and can be detected in the notations chosen. In particular, the multiplicative notation adopted for the polygons (which has been replaced, in the modern expositions, by an additive notation) enables the development of a theory reproducing the ideas from the arithmetic of ideals – and through that, rational arithmetic. At each “level” defined, Dedekind and Weber carry over their arithmetical considerations and the notions defined, from the divisibility to norms, order and equivalence classes which are handled just like congruence classes. Ultimately, this allows to provide the first rigorous proof of the Riemann-Roch theorem.

As a way to close this chapter and make the transition to the next parts of the dissertation, I will propose to consider the main clues for the investigation of arithmetic and methods built on arithmetic, in Dedekind's mathematics. I will point out the main points at which our attention should be directed, to underline the overall importance of the question of arithmetic for Dedekind's works, as well as to clarify the motivation for further developments of the dissertation.

2.6.5.1 Higher levels of arithmetic?

The idea of different “levels” of study is expressed by Dedekind himself, in the presentation of the Xth Supplement of the 1871 edition of Dirichlet's *Vorlesungen*. In the announcement of the second edition of Dirichlet's *Vorlesungen über Zahlentheorie*, Dedekind presents his new algebraic concepts as setting up a “higher level” which allows to shed a “new light” on number theory, the subject of the book:

In the following paragraphs, I tried to introduce the reader to a higher level at which algebra and number theory join each other in the most intimate manner.¹¹⁹ (Anzeige of Dirichlet's *Vorlesungen* 2nd edition 1871, repr. in [Dedekind, 1932] III, 399, my translation)

He went further, in the Xth Supplement itself, in which he explains that he was pursuing Kummer's researches on ideal numbers which had provided a

¹¹⁹“In den nun noch folgenden Paragraphen habe ich versucht, den Leser in ein höheres Gebiet einzuführen, in welchem Algebra und Zahlentheorie sich auf das Innigste miteinander verbinden.”

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“deeper understanding of the true nature of algebraic numbers”, and that in trying to “introduce the reader to these new ideas”, he was

setting up for ourselves a somewhat higher standpoint and, from there, begin to introduce a new concept that seems to be well-suited to serve as a foundation for higher algebra and the related parts of number theory.¹²⁰ ([Dedekind, 1871], transl. slightly altered, 12)

In both quotes, the concept alluded to is the concept of field of algebraic numbers. In practice, working at this “higher level” consists essentially in working at the level of (infinite) collections of objects. Numbers and later functions, as individuals, do not have any role to play in the theory.

The fields on which Dedekind is working are field extensions, in which one can identify a ring (the polynomials in z in *Algebraische Funktionen*) and integers (the system \mathfrak{o}). In these domains, one can form modules and ideals, for which new arithmetical operations can be defined. Dedekind indeed defines arithmetic operations at the higher level, e.g., between ideals: for two ideals \mathfrak{a} and \mathfrak{b} , \mathfrak{a} divides \mathfrak{b} means that \mathfrak{a} contains \mathfrak{b} , or again that $\mathfrak{b} \subseteq \mathfrak{a}$ (we will see that such a notion of divisibility is also defined for groups, modules and even fields).

In 1871, the divisibility is first introduced between a number and an ideal:

If α is contained in [the ideal] \mathfrak{a} , we will say that α is *divisible by* \mathfrak{a} , and that \mathfrak{a} *divides* α , since by this manner of expression we gain in facility.¹²¹ (*ibid.*, transl. slightly altered, 38)

It is this notion that leads to the divisibility between ideals, in 1871. The arithmetical terminology used to refer to a characterization of inclusions is, here, explicitly introduced because the investigations will be eased by it. Why should a new arithmetical operation make the development of a theory easier? Let me outline some elements of answer.

To be able to reduce everything to considerations relating to rational operations has a clear advantage, that of familiarity. But to take familiarity as the reason for the new arithmetic to make things easier seems a weak argument in the face of the systematicity with which Dedekind uses this strategy – and to some extent, the 1882 paper would contradict such statement. In the case of ideals in algebraic number theory, the familiarity and the easiness could be seen as related, for all considerations about *arithmetic*

¹²⁰“Indem wir versuchen, den Leser in diese neuen Ideen einzuführen, stellen wir uns auf einen etwas höheren Standpunkt und beginnen damit, einen Begriff einzuführen, welcher wohl geeignet scheint, als Grundlage für die höhere Algebra und die mit ihr zusammenhängenden Theile der Zahlentheorie zu dienen.”

¹²¹“Ist α in \mathfrak{a} enthalten, so sagen wir, α sei theilbar durch \mathfrak{a} , \mathfrak{a} gehe in α auf, weil die Ausdrucksweise hierdurch an Leichtigkeit gewinnt.”

of algebraic integers, which is notoriously difficult, are reduced to arithmetic of ideals, which is revealed to behave just like arithmetic of rational integers. And, in fact, the set up of an arithmetical framework for ideals is not, in itself, surprising, insofar as Dedekind is engaged in a purely number theoretical work whose aim is the proof of the validity of divisibility properties for algebraic integers.

A second clear benefit for the use of rational operations, in ideal theory, is that long computations tied to representations are reduced to simpler ones (see Sec. 6.1).¹²² Could this be the only reason for the arithmetical rewriting of Riemannian function theory? Or could there be something special about arithmetic, for Dedekind, that stimulates the desire to rewrite completely Riemann's ideas by arithmetical means? Could there be a shift in the conception of arithmetic brought about by the discovery of the efficiency of the same concepts and operations in the context of algebraic functions? Indeed, the argument of familiarity and/or easiness is already less obvious in the case of algebraic functions, and becomes even less so for the arithmetic of complexes of points of a Riemann surface. On what ground does the use of arithmetical operations rely, in this case? Is it just about following the one-to-one correspondence between points and ideals? Does this actually make things easier? In particular, how is it suppose to ease the understanding and treatment of Riemann surfaces, if all considerations about algebraic functions are replaced by arithmetical operations on ideals, polygons, classes of polygons, and so on?

2.6.5.2 Why an arithmetical treatment for Riemann's theory?

One could consider the arithmetical development observable in Dedekind and Weber's work as a mere side-effect of the transfer of ideal theory, which is itself developed in such a way that it can be fully similar to rational number theory. And these endeavors to develop ideal theory in algebraic number theory might as well be related to the demand that, in the extended domain of algebraic numbers, the methods of proof be an extension of the methods of the restricted domain. Jeremy Avigad suggests that it is related to

Dedekind's insistence that definitions and methods of proof used in an extended domain should parallel the definitions and methods of proof that have been effective in more restricted domains. ([Avigad, 2006], 171)

¹²²In elementary arithmetic, the introduction of new operations, described at length by Dedekind, is motivated by two main desires: to shorten the computations (e.g., the addition of a with itself b times is replaced by one multiplication $a.b$).

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However, Dedekind does not explicitly express this demand. Rather he stresses the importance, when extending *systems of numbers* and *number theoretical* notions, to develop arithmetical methods of proof. Dedekind does not give a more general statement, and the demand appears to be more specific than Avigad is suggesting: when investigating the number concept, be it for extending the systems of numbers or investigating the divisibility laws of algebraic integers, one should develop arithmetical methods – a point linked to the demand that no “foreign elements” be introduced in the theory.

What would then be the argument for the use of arithmetical methods in algebraic function theory? Is it because algebraic functions are themselves part of arithmetic? Aren’t algebraic functions of one complex variable more susceptible to be considered as some kind of extension of algebraic numbers rather than as arithmetical elements *per se*? Yes, indeed, and I do not intend to argue that algebraic functions themselves were considered as arithmetical. However, the idea pointed out by Dedekind and Weber is that of an analogy rather than an extension – a point that they would not have omitted, if they were considering their work as being, in some way, an extension of algebraic number theory. The idea that polynomial, rational and algebraic functions “reproduce themselves” through the arithmetical operations and could be studied with methods analogous to (or inspired by) number theory was well-known. The use of a thoroughly arithmetical approach, in algebraic function theory, can neither be justified by a kind of requirement of purity of methods, nor by the demand for parallel definitions and methods of proof in extended domains. In fact, it should be underlined that the arithmetical investigations are essentially developed for *ideals* (and related notions). Indeed, we will see that ideals are presented by Dedekind as answering demands inherent to the definition of arithmetical elements, such as the possibility to provide a definition of the arithmetical operations (see Sec. 6.1.3 for more details about this point). The theory of ideals developed in 1876-77 exhibits clearly how ideals can be developed arithmetically, that is, following the exact lines of rational number theory. Insofar as the definition and the development of ideal theory are made without reference to the individual nature of the elements, their arithmetical nature does not depend on the fact that they are or are not used in *number theory*.

Arithmetic, in *Algebraische Funktionen*, seems to hold a central place in the attempts to provide a reliable, more rigorous basis of function theory. The role of arithmetic in such attempts is already alluded to by Dedekind, when he explains to Weber that he would only be sure to fully understand Riemann’s works if he could approach them in arithmetical way.¹²³ For

¹²³“I am not the profound expert on Riemann’s work that you take me to be. I certainly know those works, and I *believe* in them, but I do not master them, and I will not master them until having overcome in my way, with the rigor that is customary in number theory, a whole series of obscurities.” See p. 129.

Dedekind, the “rigor customary in number theory” is essential to throw light on Riemann’s theory. The presentation of the 1882 paper suggests that arithmetical operations were involved in the design or building of new concepts (from ideals to polygon classes) and to develop what Dedekind (and Weber) considers to be a more rigorous and more general theory of Riemann surface.

Riemann surfaces are not the first concept involving continuity that Dedekind chooses to treat “arithmetically”. Indeed, already in 1872, he defined the linear continuum formed by the real numbers by arithmetical means with the concept of cuts (see Sec. 7.2). The reasons for an arithmetical treatment of the linear continuum are related to the considerations about extensions of numbers evoked above, and to a demand that intuition be eliminated from mathematical definitions and proofs. The arithmetical treatment of Riemann’s theory of functions contains the demand not to rely on an intuitive geometrical representation of Riemann’s surfaces, as Schappacher explains:

So, if you ban intuition but still want to define a point (...) on an algebraic Riemann surface, a point will be something where you can evaluate (sometimes getting the value ∞ ...) rational functions living on the Riemann surface; and you know or postulate that rational functions ought to be sufficiently plentiful to separate points. So, if you can’t see the Riemann surface, but still have its field of rational functions, define a point arithmetically as an evaluation homomorphism (including possible values ∞) on rational functions which leaves constant functions invariant. ([Schappacher, 2010], 3262)

Schappacher links the definition of a point of a Riemann surface to the definition of real numbers using cuts as points of the linear continuum and explains that “both [definitions] try to conceptualize the intuition of what a concrete point does for you.” As he underlines, Dedekind is adopting the same approach, proposing the same conceptual move, in these two works.

I would like to suggest that the strategy of resorting to arithmetic is, in fact, present throughout all of Dedekind’s works, and gives us important informations about his practice of mathematics, his conception of arithmetic, and about his ideal of rigor. By resorting to arithmetic as he does, Dedekind seems to be adopting what could be called a strategy of arithmetization consisting in the elaboration of arithmetical notions (of notions built by arithmetical means) so as to *define* given concepts, such as points of the Riemann surface (or of the linear continuum). I will try to exhibit how the role of arithmetic in Dedekind’s mathematics, which has been (more or less) acknowledged but never dug into very deeply, tells us more about how he

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developed his very innovative mathematics.

2.6.5.3 The “simplest principles of arithmetic”

Another possibility to consider, so as to obtain a better understanding of Dedekind’s conception of arithmetic are his works on the theory of algebraic integers. These works are motivated by the demand for the laws of divisibility to be unconditionally valid, which led to tremendous developments. Dedekind reworked, remodeled and developed several times his theory. Nevertheless, the purpose of the paper was each time the proof of the general validity of the divisibility laws. Moreover, even though the refinements make for significant changes in presentation and content,¹²⁴ the basis of the theory remains the same (i.e., field and ideal theories).

And indeed, a more comprehensive investigation of Dedekind’s works will allow to see that Dedekind’s approach is often underpinned by the wish to define concepts only using what he refers to as the “simplest principles of arithmetic”:

The conviction imposed itself to me that the study of the algebraic kinship of numbers is founded the most adequately on a concept immediately related to the simplest arithmetical principles.¹²⁵ ([Lejeune-Dirichlet, 1871], Anzeige, repr. in [Dedekind, 1932] III, 399, my translation)

The concept to which Dedekind is referring, here, is the concept of field, which he defines by (and only by) conditions of closure by the four arithmetic operations. It appears more and more clearly, from 1871 to 1879, in Dedekind’s algebraic number theory that the “simplest arithmetical principles” are a reference to the four fundamental operations and elementary arithmetic – which covers broadly operations, laws of divisibility and congruences.

The “simplest arithmetical principles” are effectively used in Dedekind’s algebraic works at two levels: firstly, the new algebraic concepts of field, module and ideals are defined using uniquely the property of closure by arithmetic operations; secondly arithmetical relationships are defined and divisibility laws are studied as a basis for the theory of the said algebraic concepts, in particular ideals and modules.¹²⁶ The “conviction which imposes itself” in 1871 transforms into a precept in the following works, as I will suggest.

¹²⁴See Chapter 5, 6.2.

¹²⁵“[D]rängte sich mir die Überzeugung auf, dass das Studium der algebraischen Verwandtschaft der Zahlen am zweckmässigsten auf einen Begriff gegründet wird, welcher unmittelbar an die einfachsten arithmetischen Prinzipien anknüpft.”

¹²⁶see Sec. 5, 6.2.

In fact, one can find, in Dedekind's writings, different considerations about simplicity linked with arithmetic. In particular, arithmetic is presented as being "the simplest science" in *Zahlen*. The superlative, here, seems to relate to the elementariness of arithmetic and the place held by the science of numbers in a conceptual hierarchy of science(s), according to Dedekind. Some mathematicians, such as Lipschitz, considered that geometry is first, that it is in geometry that mathematicians found their basic concepts and the intuitions leading to more developments of the rest of mathematics (e.g., geometry provides the foundation for irrational numbers). Dedekind, like Kronecker, Gauss or Weierstrass, on the contrary, considers that arithmetic is first. As we will see, arithmetic, for Dedekind, is the science whose first objects "flow directly from the laws of thought" and on this account is the 'first science', and it is also the *simplest*:

(...) the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. ([Dedekind, 1888], 790)

In this case, "simplest" implies that arithmetic is considered as the first theory in the order of logical construction of mathematics. But arithmetic being the "simplest science" does not mean that it is simple in an absolute sense: it is not immediate nor given, and its truths are not unprovable. In fact, while arithmetic is the simplest activity of the mathematician, and its objects and operations are the first layer of mathematics, it is constructed upon more fundamental concepts, systems and mappings, "without which no thinking at all is possible" (see Sec. 8.2).

Where should the gain in simplicity allowed by the use of a divisibility relationship in ideals be coming from? As I mentioned, it is not, for Dedekind, a systematic reduction of concepts to natural numbers. Yet, Dedekind considers numbers (and consequently arithmetic) to be one of the most powerful and "inextricably connected to [the human mind's] innermost Nature" of the auxiliary means created by the human mind to facilitate its work (in *Zum Zahlbegriff*, in [Dugac, 1976b]).

In *Algebraische Funktionen*, the study of divisibility laws for algebraic functions with the definition of arithmetical operations for modules and ideals and its extension to concepts building the Riemann surface (points, polygons) constitute the core elements of the strategy adopted to give a treatment of Riemann's theory of functions that is both rigorous and uniform. In addition, *Algebraische Funktionen* extends the scope of the methods and tools transferred from algebraic number theory, giving them a greater generality. Arithmetic, here, seemed to guide the set up of the tools susceptible to provide a satisfactory answer to the demand to reformulate the theory. Following the model of Dedekind's ideal theory, Dedekind and Weber developed their theory using only elementary arithmetic operations, and while

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the theorems proved are far from elementary ones, they were able to prove them using methods of proofs built on the “simplest principles of arithmetic”.

A particularly important point is, thus, that arithmetic exhibits certain specificities which allow for its application in different frameworks. The 1882 paper suggests that arithmetic plays a role in the way Dedekind chooses to develop his works, and in the introduction of the new concepts that have become so important in today’s mathematics. Resorting to arithmetical methods appears to provide, in many of Dedekind’s works, a mean to develop the theory in a more rigorous and more uniform way. While arithmetic as a tool for rigor is, at least since the 19th century, a common statement, the links tied between arithmetic, rigor and uniformity in Dedekind’s works do raise questions about the possibilities of uses of arithmetic for Dedekind. Indeed, if the “arithmetization” in the case of real numbers can be justified by rigor and ‘purity’ requirements, the “arithmetization”, i.e., the arithmetical construction of Riemann surfaces, at first sight, does not answer the same requirement of ‘purity’. And insofar as the theory is developed by resorting, almost exclusively, to notions relating to arithmetical operations, rendering the 1882 paper thoroughly arithmetic despite its non-arithmetical subject, the nature of arithmetic itself comes into question.

What kinds of epistemological requirements is arithmetic fulfilling for Dedekind? In which way(s) arithmetic can and is indeed used for this purpose by Dedekind? Following these questions, it becomes important to understand why Dedekind attributes this possibility to arithmetic. In particular, this brings up a question raised earlier in the paragraph: how does one go from the “science of numbers” to arithmetic of polygons, in which operations are used as epistemic tools to build up new definitions?¹²⁷

2.6.5.4 More questions by way of conclusion

To unfold and understand more deeply the specificities and whether Dedekind’s ideas changed throughout his works, the assumption that the practice influenced Dedekind’s conception of mathematics is important. I will try to answer the questions about the specificities of arithmetic and its changes by arguing in this direction, in the rest of this dissertation, through an investigation of the role played by arithmetic in several of the most important of Dedekind’s mathematical works and the reflections he developed in his foundational works. I will inquire into the possibility to answer questions about the specificities of arithmetic in their mathematical context, via the study of his mathematical texts.

It seems, indeed, that to be able to explain the role of arithmetic in the set up of more rigorous and uniform definitions and proofs, it is important

¹²⁷I will consider these points in more detail in Chapters 7 and 8.

to go through the details of the mathematical texts. In investigating this, arises an important concern: if the rational operations are initially defined as operations on numbers, and extended up to arithmetic of ideals and arithmetic of polygons, are the fundamental ideas about arithmetic modulated by this extension? If so, how? But then again, pushing the questioning further, how do the uses of arithmetical operations to build up new concepts relate to natural numbers, the first object of arithmetic? If the conception of arithmetic is indeed subjected to changes through practice, if it is extended slowly through its uses in the mathematical works, does this have an impact on the concept of number itself?

Part II

Early years

Introduction to Part II

Dedekind's career spans over more than fifty years, and if his ideas were refined over the years, many of them were born in his earlier works. I suggested, in the previous pages, that not only did arithmetic play crucial role in Dedekind's mathematics, it also was subjected to a certain evolution by this very role and through the uses made of arithmetical operations in mathematical works.

In order to argue for this idea, I will follow a broadly chronological path, and go back from 1882 to 1854, starting from Dedekind's doctoral work and *Habilitationsvortrag*. This shall enable us to get an idea of the conception of arithmetic with which Dedekind started. The 1854 *Habilitationsvortrag* is an interesting and important work for two main reasons. First, it allows us to see what Dedekind was thinking about arithmetic and mathematics more generally *before* starting to work in number theory. Indeed, his first works were on Eulerian integrals, and his interest in number theory was stimulated by Dirichlet's arrival in Göttingen, in 1855. Secondly, the *Habilitationsvortrag* provides certain general epistemological ideas about Dedekind's conception of science and the development of mathematics which are valuable because they are rarely expressed so explicitly. In particular, Dedekind's conception of science is interesting in relation to the assumption that the ideas about arithmetic evolved with Dedekind's experience in mathematics, since he considers that science is an *activity* of the human understanding, subjected to more or less drastic changes according to the needs of the development of knowledge.

Dedekind's interest in number theory arises with Dirichlet's arrival in Göttingen, after Gauss's death in 1855. Arithmetic, in his *Habilitationsvortrag* played the role of a model to explain the processes of developments of mathematics more generally. It did not hold a special place, apart from being the first, simplest part of mathematics. In his subsequent works, starting with a course on Galois theory in 1856-58, Dedekind's approach to algebra and number theory seems to have a noticeable impact on his conception of arithmetic, as I will try to show. Indeed, to study certain algebraic concepts such as groups, Dedekind proposes to define arithmetical relationships for these concepts, in such a way that the theories are modeled on rational arithmetic. I will propose to consider how Dedekind first approaches these questions.

Chapter 3

Dedekind's *Habilitationsvortrag*, in 1854

Dedekind's *Inauguraldissertation, Über die Elemente der Theorie der Eulerischen Integrale*¹, which he presented, under Gauss's supervision, in 1852 to obtain his doctorate at the university of Göttingen, has been described as a good work but which did not show the great innovative mathematician Dedekind was.² Nevertheless, in the first paragraph, one can detect some of the leading threads of Dedekind's conception of mathematics and directing principles of his mathematics. Indeed, this very early work opens with considerations on the introduction of new concepts in mathematics as means to bypass difficulties or even impossibilities, and enable more developments.

It is known, that the execution of the indirect operations in Analysis most often come across many more important difficulties than that of the direct ones; but exactly these seemingly unfortunate circumstances have always exerted the most beneficial influence on the development of mathematics. Not only the overcoming of these difficulties, when it is possible, always presents a singular stimulus for the mathematician, but also precisely the cases in which this was not possible in the previously introduced concepts and auxiliary means were not possible have always inaugurated completely new fields in further development of mathematics. Thus, for example, the operations of subtraction, division and root extraction have led to the concepts of negative, fractional and imaginary numbers each of has so amazingly extended the domain of mathematics.³ ([Dedekind, 1852], 1-2, my

¹Reproduced in [Dedekind, 1932] I, 1-26. Translations of this text are mine.

²Notably, Landau called it a "useful and independent contribution, but which doesn't reveal the future great scholar." ([Landau, 1917])

³"*Es ist bekannt, daß die Ausführung der indirekten Operationen in der Analysis meist auf viel bedeutendere Schwierigkeiten stößt als die der direkten; aber gerade dieser schein-*

translation.)

For the first time, and in a still relatively imprecise, if not naive manner, Dedekind refers to ideas and concerns about the nature of the development of mathematics which will appear recurrently in his researches, sometimes even being their driving force. This way of introducing his very first work reveals that were already present, in a very young Dedekind, questions and concerns that stayed at the core of his reflections and even revealed to be essential throughout his works. A more careful and thorough treatment of these questions can be found in his 1854 *Habilitationsvortrag*, an epistemological inquiry in the development of arithmetic and mathematics. Moreover, the issue of how mathematics develops and the ideas sketched in 1852 appear to be important since this concern might account for two cornerstones of Dedekind's conception of mathematics: his advocacy for the introduction and creation of new concepts in mathematics, and the idea that mathematics develops through a repeated process of generalization, which becomes significant in his later works.

Dedekind's approach to mathematics is often underlaid by attempts of developing notions in such a way that they allow to treat wider classes of objects. In 1852, the concern for inverse operations and the possibility to perform them, rooted in his work on integral calculus, leads him to reflect on the possibility to extend operations. Then, the "difficulties" and "unfortunate circumstances" which appear using the indirect operations result from the demand to apply the said operations without restrictions, that is, to make their applicability as wide as possible.

This view is expressed often and clearly by Dedekind, from the 1854 *Habilitationsvortrag* to *Zahlen*.⁴ Moreover, Dedekind identifies similar forms of extension, *mutatis mutandis*, in other areas of mathematics. As Dedekind suggests, in the opening remarks of his *Inauguraldissertation*, the same kind of phenomenon occurs for integral calculus:

By contrast, the kind of inverse computation that the integral calculus constitutes in its entirety succeeds, only in relatively few cases, in achieving the same [as differential calculus]. In most cases, one has hitherto not succeeded, or it may also be completely impossible, to represent the integral of given functions with the help of precisely similar functions. But precisely this

bar unglückliche Umstand hat auf die Entwicklung der Mathematik stets den günstigsten Einfluß ausgeübt. Nicht nur, daß die Besiegung dieser Schwierigkeiten, wenn sie möglich, immer einen eigentümlichen Reiz für den Mathematiker darbietet, sondern auch gerade die Fälle, in welchen dies mit den früher eingeführten Begriffen und Hilfsmitteln nicht möglich war, haben immer der weiteren Ausbildung der Mathematik ganz neue Felder eröffnet; so führen z. B. die Operationen der Subtraktion, Division und Wurzelausziehung auf die Begriffe der negativen, gebrochenen und imaginären Zahlen, von denen jeder das Gebiet der Mathematik so außerordentlich erweitert hat."

⁴*Zahlen*, 792; *Stetigkeit*, §3. A more detailed analysis will be given in Sec. 7.2.

circumstance led to a considerable extension of the concept of function, in that one has attached new names and descriptions to such non-representable integrals and in this has introduced them in the sphere of the previous functions.⁵ (*ibid.*, 1-2, my translation.)

These remarks are again taken up and clarified in his 1854 *Habilitationsvortrag*, in which he explains that even though demanding the “general applicability of integration” cannot be done with the “same justification” (since it is not a question of arithmetical results), one can notice “a similar course of development” when looking at the birth and developments of Eulerian integrals or elliptic functions⁶ ([Dedekind, 1854a], 761).

Dedekind, in his *Habilitationsvortrag*, states that developments of mathematics are greatly led by “concepts [developing] from those which relate only to a restricted domain into more general ones”⁷ (*ibid.*, 760). So as to overcome the difficulties, the mathematician has to *define* new concepts. Already in his *Inauguraldissertation*, Dedekind explained that certain new functions introduced to develop integral calculus, such as Eulerian integrals, became of the greatest importance and are studied by reduction to previously known functions. For him, this allows to understand better what is at stake and what could be expected. It also provides means to ease the calculations of the said new functions by reduction to already known ideas.⁸ This idea is used again in the *Habilitationsvortrag* (see Sec. 3.2.3): if one considers the Γ -function, i.e., a definite integral for positive values of x such that

$$\Gamma(x) = \int_0^{\infty} t^{(x-1)} e^{(-t)} dt$$

⁵“Dagegen ist es der umgekehrten Rechnungsart, welche in ihrer Gesamtheit die Integralrechnung bildet, nur in verhältnismäßig wenigen Fällen gelungen, dasselbe zu leisten; in den meisten ist es bisher nicht gelungen, oder vielleicht auch ganz unmöglich, die Integrale gegebener Funktionen mit Hilfe eben solcher darzustellen. Aber gerade dieser Umstand hat zu einer beträchtlichen Erweiterung des Begriffs der Funktion geführt, indem man solchen nicht darstellbaren Integralen neue Namen und Bezeichnungen beigelegt, und sie dadurch in den Kreis der früheren Funktionen eingeführt hat.”

⁶see Sec. 3.2.3

⁷“Diese Beispiele werden genügen, um die Eigentümlichkeit des Fortschritts von Begriffen, die sich nur auf ein beschränktes Gebiet beziehen, zu allgemeinem in der Mathematik nachzuweisen.”

⁸“Bei der Entwicklung der Theorie solcher Integralfunktionen sind nun namentlich die Fälle von der größten Wichtigkeit, in denen sie sich auf die bisher allein gebräuchlichen Funktionen zurückführen lassen, indem dadurch ihr Verlauf deutlicher hervortritt, und auch oft Mittel an die Hand gegeben werden, ihre Berechnung zu erleichtern. Die Zusammenstellung dieser Fälle für die Eulerschen Integrale, mit besonderer Rücksicht auf die dabei anzuwendende Methode, ist der Hauptzweck der folgenden Abhandlung.” ([Dedekind, 1852], 2)

and the B -function defined by

$$B(x+1, y+1) = \int_0^1 t^x (1-t)^y dt,$$

then one can show that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Computations with the B -function are, then, reduced to computations with the already well-known Γ -function. In addition, although the idea that such a reduction allows to shorten computations is not clearly stated, here, it seems to underlie the desire to replace the integrals in the B -function by elementary computations on the Γ -function itself, which somewhat hide the integrals.

Such a reduction to earlier and already well-known notions, subtly combined with a defense of innovation in mathematics, is a pillar of Dedekind's mathematics.⁹

In 1854, Dedekind presented his *Habilitation* in Göttingen under the supervision of C. F. Gauss, so as to qualify as a university lecturer and obtain a position as a *Privat-dozent* at Göttingen university. His *Habilitationsschrift* is entitled “*Über die Transformationsformeln für rechtwinklige Koordinatensysteme*” ([Dedekind, 1854b])¹⁰ and, according to Pierre Dugac, “does not present any special mathematical interest” ([Dugac, 1976b], 19). In order to complete his examination, Dedekind delivered a lecture before C.F. Gauss, Karl Hoeck, Georg Waitz and Wilhelm Weber, on 30 June 1854, “*Über die Einführung neuer Funktionen in der Mathematik*”,¹¹ in which he proposes a (tentatively systematic) description of the development of mathematics as a process of generalization of definitions and laws.

This short epistemological text attempts to give a uniform presentation of the principle guiding how new concepts are shaped and how operations are extended (notably leading to changes of the number concept). It tries to explain the way in which mathematics is driven by innovations introduced

⁹In the introduction to the first edition of *Zahlen*, Dedekind famously advocated for the introduction of new concepts as being particularly fruitful: “(...) the greatest and most fruitful advances in mathematics and other sciences have invariably been made by the creation and introduction of new concepts” ([Dedekind, 1888], 792), in which he also refers to his *Habilitationsvortrag*. I will develop this point further in the dissertation, in particular in Sec. 7.3 and in Sec. 8.4.

¹⁰A short paper published in Crelle's journal is extracted from it: “*Ein Satz aus der Theorie der dreiachsigen Koordinatensysteme*”, in [Dedekind, 1932] I, 32-35.

¹¹Reproduced in [Dedekind, 1932] III, 428-438; English translation “On the introduction of new functions in mathematics” by W. Ewald, in [Ewald, 2005], 754-762; French translation “Sur l'introduction de nouvelles fonctions en mathématiques” by H. Benis Sinaceur, in [Sinaceur and Dedekind, 2008], 220-233. All page numbers will refer to Ewald's translation.

so as to bypass problems and impossibilities in a certain framework. While Dedekind's ideas evolved and were refined, one can discern the outlines of what is often seen as a program which Dedekind pursued throughout his works. Ewald and Sinaceur, each presenting their own translation of Dedekind's *Habilitationsvortrag*, both underlined that the observations he developed constituted a research program that Dedekind carefully followed:

This early paper can be read as the announcement of a mathematical research programme that embraced his work both in foundations and in algebra. His foundational writings of 1872 and 1888 deepen the analysis he gives here. ([Ewald, 2005], 755)

This epistemological text (...) is indispensable to understand the work program (research program) Dedekind set himself the task of developing and which he patiently realized in a work essentially dedicated to numbers.¹² ([Sinaceur and Dedekind, 2008], 219, my translation)

Dedekind never stated explicitly that his *Habilitationsvortrag* was indeed a program, however he did develop some of the ideas presented in it. One can detect, in this text, directing principles of his overall approach. As was underlined by [Sieg and Schlimm, 2005], in the *Habilitationsvortrag*,

Dedekind's observations reveal general aspects of his intellectual approach as well as special features of his understanding of the classical number systems. ([Sieg and Schlimm, 2005], 125)

It should be underlined that "On the introduction of new functions in mathematics" is an early work. This, on one hand, should prompt us to not put too much significance in the conceptions drawn in it. The *Habilitationsvortrag* was written before any of Dedekind's major works, and while a certain continuity or at least a certain permanence is clearly noticeable, changes did appear later – especially insofar as practice of mathematics may have affected Dedekind's conceptions.

In the light of the notion of science presented in the *Habilitationsvortrag*, it seems that the evolution of some of the basic conceptions of numbers and operations can be regarded as an integral part of this conception of mathematics. Dedekind considers that the mathematician invents and develops himself the concepts of his science: to make progresses in mathematics and prove theorems, it can be necessary to introduce new concepts, abandon old ones, impose changes in conceptions and practices. The ideas on the

¹²"Ce texte à caractère épistémologique (...) est indispensable pour comprendre le programme de travail que Dedekind s'est proposé et qu'il a patiemment réalisé dans une œuvre essentiellement consacrée aux nombres."

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nature of mathematics presented in Dedekind's *Habilitationsvortrag*, then, seem to be conveying the idea that evolutions and changes are inherent to the mathematician's practice.

In any case, Dedekind himself referred to his *Habilitationsvortrag*, more than thirty years later, when arguing for the usefulness (if not necessity) of the introduction of new concepts in mathematics in *Zahlen*, a point he maintained consistently all along his mathematical career.

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3.1.1 Dedekind's idea of science

The *Habilitationsvortrag* provides us with the lineaments of some of Dedekind's philosophical ideas and allows for a clearer view on some of the epistemological principles that guided and supported Dedekind's works. These principles can usually only be drawn from scattered remarks and based on the analysis of a practice openly and deeply attentive to methodological and epistemological issues. In the case of Dedekind, epistemological and methodological considerations, which I have outlined in Sec. 2.1.2, play a significant role in shaping his practice of mathematics, so much so that it seems that an understanding of Dedekind's ideas can't dispense with taking them into account.

Dedekind's understanding of the process of development of mathematics relies on a conception of science as representing "the course of human knowledge up to" the truth ([Dedekind, 1854a], 756). The "chief goal" of science can be regarded as being "the endeavour to fathom the *truth*" (*ibid.*, 755, emphasis in the original). If the pursuit of the truth is an activity of the human understanding, truth itself does not however depend on us, it is objective and "wholly external to us": it is never "our arbitrary creation" but rather "a necessity independent of our activity" (*ibid.*). Science, in consequence, is not the truth itself according to Dedekind, but the human scaffolding of reasonings which attempts to understand, grasp or deepen our understanding of the objective truth. Inasmuch as the goals of any science are objective truths, the results are declared "to be invariable, to be unchangeable" (*ibid.*), but science is bound to human understanding. This entails two important aspects of it.

Firstly, there may be multiple, if not innumerably many different ways of trying to fathom or ground the truth. As Dedekind puts it, science, whose object is fixed ("immutable"), is itself "infinitely manifold" (*ibid.*). Thus, the "object of a science" can be investigated in many different ways, as well as *conceived* in diverse manners. This "finds its expression in the different forms, the different systems, in which one seeks to embed those conceptions" (*ibid.*, 756). This is also the case for mathematics: despite

its peculiar status of science providing universal truths, mathematics is still produced by the human mind. The multiplicity of possible practices and conceptions in mathematics can be given numerous illustrations, from the well-known many different ways to discover and define, for example, the negative numbers or the real numbers, to the possibility to give several different proofs for the same theorem.

Secondly, the possibility to reach these goals is subjected to the limits of human understanding. As a human activity, science is of course subjected to the “arbitrariness” of the “work of man” and “to all the imperfections of his mental powers” (*ibid.*). The discovery or knowledge of truth is subjected to the limited and finite character of human understanding. The human mind only grasps the truth progressively, thus making the development of knowledge dependent on “long chain[s] of inferences” (*ibid.*, 756), which for Dedekind constitute the essence of science.

In fact, for a man gifted with an unbounded understanding, for whom the final conclusions, which we attain through a long chain of inferences, would be immediately evident truths, there would exist no more science; and this would be so even if he stood in exactly the same relation to the objects of science as we do.¹³ (*ibid.*, transl. slightly altered, 755-756)

If there were a man with unlimited understanding and immediate knowledge of truth, then, science as an activity would cease to exist for him, although the independence of truth would leave him in the “same relation to the objects of science as we do”. Science, for humans with finite understanding, appears to be indispensable if one is to explore the realm of rationality. Reciprocally, there isn't any science without the human “endeavour[s] to fathom the truth” – and again mathematics is not an exception to the rule. There is no mathematics without mathematicians doing it, for there is no chain of inferences without the mathematician's endeavours to “fathom the truth” and extend scientific knowledge.

As regards mathematical concepts, Dedekind explicitly states that the mathematician *creates* the concepts he studies, he is creator and actor of his science. Mathematical objects, such as numbers, are not pre-existing entities. The idea that numbers are creations of the mind is not a new idea.¹⁴ Berkeley stated that number is “entirely the creature of the mind” ([Berkeley, 1710], Part I, §12). Closer to Dedekind, Gauss wrote to Bessel in 1830:

¹³“Für einen mit unbegrenztem Verstande begabten Menschen, dem die letzten von uns durch eine lange Kette von Schlüssen erhaltenen Konsequenzen unmittelbar evidente Wahrheiten wären, würde eigentlich keine Wissenschaft mehr existieren, wenn er auch den Objekten derselben genau ebenso gegenüberstände, wie wir es tun.”

¹⁴Note that Dedekind holds a particularly radical view on the matter, insofar as he considered any mathematical concept to be a creation of the mind, geometrical as well as arithmetical or analytical notions.

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We must admit with humility that, while number is merely a product of our minds, space has a reality outside of our minds, and that we cannot completely prescribe its laws *a priori*. (in [Ewald, 2005], 949)

Many mathematicians disagreed, of course, with such a statement, either holding a Platonist view on the existence of mathematical objects, as did for example Cantor or Frege; or a more empiricist one, such as Charles Hermite, who, in a letter to Mittag-Leffler from 13 April 1883, stated:

The successive generalizations of operations which are born in the arithmetical mechanism, which from addition lead us to functions, are not, I believe, creations of our mind but a product of the intellectual activity applied to realities which exist outside and independently of ourselves and of our understanding, as the world of physical and natural sciences exists outside of physicists and naturalists.¹⁵ (Letter from 13 April 1883, quoted in [Goldstein, 2011])

In his *Habilitationsvortrag*, Dedekind's first illustration of the development of science is in mineralogy, rather than mathematics. This simple example, borrowed from a descriptive classificatory science, allows for a good grasp of the ideas argued for. When setting up a classification of some objects, such as minerals, the "investigator of nature" chooses which concept he considers as "suited to distinguishing and classification" ([Dedekind, 1854a], 756). In certain cases, as in mineralogy¹⁶ when Dedekind writes, there can be different or even contradictory systems which can't be brought "into complete harmony with each other", nevertheless,

[e]ach of these systems has a good reason for itself, for science itself shows that similar bodies group themselves together most naturally in these ways.¹⁷ (*ibid.*, 756)

¹⁵ "Les généralisations successives d'opérations qui ont leur naissance dans le mécanisme arithmétique, qui de l'addition nous conduisent aux fonctions, ne sont pas à mon avis, des créations de notre esprit, mais un fruit de l'activité intellectuelle, appliqué à des réalités qui existent en dehors et indépendamment de nous et de notre intelligence, comme le monde des sciences physiques et naturelles existe en dehors des physiciens et des naturalistes."

¹⁶ Auguste Bravais's works on the symmetry of crystals are a particularly good illustration of the second classification. Benoît Timmermans, in [Timmermans, 2012], gives a remarkably clear presentation of Bravais's works (pp. 171-196). Timmermans's overall aim in this book is to explore the possible links between the development of group theory from Galois to Weyl, and that of philosophy, crystallography or chemistry from Eschenmayer's *Naturphilosophie* to quantas.

¹⁷ "Jedes dieser Systeme hat ein großes Recht für sich, weil die Wissenschaft weiterhin selbst lehrt, daß die ähnlichen Körper sich so am natürlichsten zusammengruppieren."

Eventually, says Dedekind, one of the criteria will prove more fruitful, more efficient, and scientists will be able to settle the conflict. Moreover, even if the state of knowledge made for conflicting classifications, it is certain that “it will occur to no mineralogist to advance, say, *differences of colour* as the characteristic features and to prefer a classification resting on them”¹⁸ (*ibid.*, 756, emphasis original). Indeed, experience taught mineralogists that color “is not of such great significance for the true nature of bodies” (*ibid.*, 756, emphasis original) or again that it doesn't provide us with any *certainty* regarding the nature of bodies – unlike the criteria mentioned earlier. The choice of a criterion to organize a certain system, that is, “[t]he introduction of such a concept as a motive for the arrangement of the system”, constitutes, according to Dedekind, “an hypothesis one puts to the inner nature of a science”¹⁹ (*ibid.*, 756, transl. slightly modified). There is a part of randomness involved in the postulate that a certain property is a characteristic one, and only “the further development” of the science can verify the actual efficiency of the said hypothesis. Indeed, once these concepts are introduced, they “appear as definitions in the science” and are tools for the scientist to “express the recognizable general truths that emerge from the infinite manifold of the particular”²⁰ (*ibid.*, 756, transl. slightly altered). Changes in concepts appear when, in turn, these general truths influence the formation of new definitions, and the said efficiency of the concepts is to be determined:

So, it may well happen that the concepts, introduced for whatever motive, because they were initially conceived either too narrowly or too broadly, will require modification so their effectiveness, their implications, can be extended to a larger domain. The greatest art of the systematizer lies in this turning and manipulation of definitions for the sake of the discovered laws or truths in which they play a role.²¹ (*ibid.*, 756)

These ideas will reveal important in Dedekind's later texts, for the role of definitions, of grounding concepts and the deductive structure of theories

¹⁸“*es wird keinem Mineralogen einfallen, etwa die Farbenverschiedenheiten als die charakteristischsten Merkmale hervorzuheben und eine hierauf beruhende Einteilung allen andern vorzusetzen.*”

¹⁹“*Die Einführung eines solchen Begriffs, als eines Motivs für die Gestaltung des Systems, ist gewissermassen eine Hypothese, welche man an die innere Natur der Wissenschaft stellt.*”

²⁰“*[Diese Begriffe] als Definitionen in die Wissenschaft eintreten, und mit deren Hilfe er imstande ist, die aus der unendlichen Mannigfaltigkeit des Einzelnen erkennbaren allgemeinen Wahrheiten auszusprechen.*”

²¹“*So zeigt sich wohl, daß die aus irgendeinem Motive eingeführten Begriffe, weil sie anfangs zu beschränkt oder zu weit gefaßt waren, einer Abänderung bedürfen, um ihre Wirksamkeit, ihre Tragweite auf ein größeres Gebiet erstrecken zu können. Dieses Drehen und Wenden der Definitionen, den aufgefundenen Gesetzen oder Wahrheiten zuliebe, in denen sie eine Rolle spielen, bildet die größte Kunst des Systematikers.*”

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play a crucial role in his reflections.

As Dedekind himself mentioned, experience teaches us to choose and/or abandon some concepts in favor of more effective, fruitful and often more general (i.e., with a larger scope of validity or applicability) ones. The development of science consists in pushing further the limits of our understanding by extending knowledge (which can be observed in the historical progress of all sciences) and one of its main aims is to obtain a deeper and wider knowledge of the objective truths. From this viewpoint, then, while the truth is immutable and independent from us, our conceptions, insofar as they are intimately linked to the state of science, are (susceptible to be) constantly changing and evolving.

The same principles as these general ones prevail for mathematics, however the generation of concepts is strictly regulated by the inner necessity peculiar to mathematics. The extensions of definitions “no longer allow scope for arbitrariness; on the contrary, they follow with compelling necessity from earlier restricted definitions” (*ibid.*, 757). Dedekind’s ideas are sharper and more detailed when it comes to mathematics, as we will see in the following, and in particular he describes how he considers that the changes in concepts are made.

Before continuing and looking into Dedekind’s ideas about the development of mathematics, I would like to mention the striking resemblance between his ideas about science and Lotze’s. In *Mikrokosmos*, Lotze gives a characterization of science which seems quite close to the statement made by Dedekind at the beginning of his *Habilitationsvortrag*:

What is science ? Not truth itself, for this existed always, and did not need to be produced by human effort. So that science means simply knowledge of the truth; but this knowledge has become so vast that it can no longer be comprehended in the knowledge possessed by any individual. Such is the strange life of science now-a-days; it exists, but for any individual it means only the possibility of investigating and learning to know each of its parts. ([Lotze, 1856], II, 152)

Lotze was a dominant philosopher in Germany at the time, even though he fell from grace relatively fast, and we know that Dedekind followed Lotze’s lectures on the history of philosophy.²² I will not develop this point, because it would take me too far, but it has been suggested²³ that Lotze’s influence on Dedekind might have been much deeper than thought.

²²Incidentally, the quote given above is in the chapter about the “meaning of history”.

²³[Gandon, 2006] notably for published works alluding to this question. One can find such considerations in [Sieg and Morris, 2014] (unpublished).

3.2 Development of mathematics according to the *Habilitationsvortrag*

The core of Dedekind's *Habilitationsvortrag* is an epistemological reflection on mathematics, in which he explains how, from his viewpoint, the development of mathematics answers a "principle of induction" ([Dedekind, 1854a], 757), a principle according to which mathematicians invent, generate a series of concepts starting from the narrow framework of the natural numbers ("*absolute ganze Zahlen*").

The main idea argued for by Dedekind, here, is that mathematics develops following a sort of generalization process. The main example developed is arithmetic, which is "based upon the formation of ordinal and cardinal numbers" (*ibid.*, 757). Its development or enlargement is done through introduction of new operations (or in the title, new functions²⁴), by successively generating new domains of numbers in order to assure the general applicability of operations. The process has two steps: the demand that operations be unconditionally applicable leads to the creation of new numbers, and the requirement to preserve the validity of the laws formerly defined guides the definition of the new numbers.

As Ferreirós underlines, Dedekind's approach is very reminiscent of Ohm's:

Ohm (...) attempted to give a consistent (i.e., unitary, systematic) presentation of arithmetic, algebra and analysis on the only basis of natural numbers. (...) [He] introduced the inverse operations (...) and observed that they cannot always be carried out in the limited domain of the natural numbers. The requirement that it ought to be possible to realize them motivates an extension of the number system. In order to be rigorous, this extension must be accompanied by new definitions of equality and of the basic operations that can be applied to the new numbers; the process is guided by the requirement that the fundamental equations that were valid for natural numbers should be preserved. ([Ferreirós, 2008], 121)

Ohm's writings were highly influential among Gymnasium teachers until the mid-19th century. Moritz Stern, who was a professor in Göttingen from 1848 to 1884 where he directed the teaching of mathematics with G. Ulrich, published in 1860 a textbook ([Stern, 1860]) adopting an approach very close to Ohm's. It is likely, as suggested by Ferreirós, that Dedekind was familiar

²⁴"This lecture is not about the introduction of a determinate class of new functions into mathematics (although one could perhaps interpret the title in that way); rather, it is about the general manner in which, in the progressive development of this science, new functions, or, as one can equally well say, new operations, are added to the chain of previous ones." (*ibid.*, 755)

3.2 Development of mathematics according to the Habilitationsvortrag

with Ohm's approach, at least through Stern's teaching. The family resemblance between Ohm's and Dedekind's works should not be emphasized too strongly, for Ohm's also considered the natural numbers to be given and "the basic operations on them were established by taking into account their intuitive meaning" (*ibid.*, 121). Moreover, for Ohm, only the natural numbers were actual numbers, the extensions were mere symbols, as they were for the previous authors mentioned above. In fact, for most authors, only integers were considered *numbers*, fractional and irrational numbers were magnitudes – even for Gauss, who advocated for an arithmetic generalized to complex *integers*. As is often underlined, Dedekind was the first to talk about irrational numbers rather than quantities or magnitudes (see Sec. 7.2).

Considerations similar to Ohm's can be found in the works of Hermann Grassmann or Hermann Hankel. In *Lehrbuch der Arithmetik*, Grassmann proposes a systematic treatment of arithmetic based on mathematical induction, leading to what we would call a recursive definition of the rational operations and proofs of their properties ([Grassmann, 1861], 1-10, 17-28, 73-78). Hankel wrote, in 1867, *Vorlesungen über der complexen Zahlen und ihre Functionen*, that

It is obvious that when $b > c$, there does not exist any number x of the sequence 1, 2, 3, ... which can be solution of our problem: the subtraction is then *impossible*. Nothing prevents us, however, from considering in this case the difference $(c - b)$ as a *sign* which would solve our problem and with which it would be correct to operate as if it were a number of the sequence 1, 2, 3, ...²⁵
([Hankel, 1867], 5, my translation)

Hankel's ideas are explicitly guided by Peacock's principle of permanence of forms (published in 1830). As with Ohm's works, the similarities between Dedekind's and Hankel's conceptions are only partial ones. In particular, the formalism adopted by Hankel, Grassmann and Ohm is opposed by Dedekind. Nevertheless, the desire to give a systematic presentation of arithmetic based on the statement that the general applicability of the operations requires extensions of the number domains, proceeds from the same basic (methodological) principles.

Note that while these ideas were relatively widespread, many mathematicians did not adopt them. Kronecker, for example, strongly criticizes Hankel's method and calls it an "evaporation of arithmetical concepts" ([Kronecker, 1887], 56).

²⁵ "Es liegt auf der Hand, dass wenn $b > c$ ist, keine Zahl x in der Reihe 1, 2, 3, ... gibt welche die betreffende Aufgabe löst: die Substraction is dann unmöglich. Nichts hindert uns jedoch, dass wir in diesen Falle die Differenz $(c - b)$ als ein Zeichen ansehen, welches die Aufgabe löst und mit welchem genau so zu operieren ist, als wenn es eine numerische Zahl aus der Reihe 1, 2, 3, ..., wäre."

3.2.1 The particular nature of mathematics

Dedekind proposes, in his *Habilitationsvortrag*, a systematic way of defining the successive generalizations of the concept of number, instead of *ad hoc* introductions. Dedekind's approach, here, is a back and forth movement between laws and definitions of numbers: once the (restricted) system is defined, the laws themselves become a source for the definition of the new numbers. It is important to underline that the fact that mathematics is "the most certain of all the sciences" ([Dedekind, 1854a], 756) does not entail that the said development is made any differently than other sciences:

In mathematics too, the definitions necessarily appear at the outset in a restricted form, and their generalization emerges only in the course of further development.²⁶ (*ibid.*, 757)

Mathematics is nevertheless different from other sciences in the sense that its very nature does not allow any arbitrariness in the introduction of new concepts or the extensions of definitions:

[T]hese extensions of definitions no longer allow scope for arbitrariness; on the contrary, they follow with compelling necessity from the earlier restricted definitions.²⁷ (*ibid.*, 757)

Insofar as mathematics, as a science, is a human activity, the introduction of new concepts becomes not only desirable but a need for mathematics to actually progress – a point Dedekind consistently advocated. However, the prohibition of any randomness imposes strong limitations on how the generalization ought to be done.

The core idea is, as Sinaceur explains in her introductory note to the *Habilitationsvortrag*, that "[t]he truth is not our arbitrary creation, but we create the *concepts* with which we approach it" ([Sinaceur and Dedekind, 2008], 219). That is, concepts are the tools forged by the human mind to attain the truth. More precisely, concepts are, for Dedekind, the tools to unfold these series of inferences to fathom the truth. From Dedekind's viewpoint, then, the work of the mathematician appears to be to invent new concepts – an invention controlled by the inner necessity of mathematics – which would allow them to progress towards a better understanding or a clearer view of the (objective) truth. Progress, in mathematics, if understood as in Dedekind's *Habilitationsvortrag* as a possibility to extend the scope of validity of concepts and laws, proceeds from a deep and careful analysis of the restricted concept – a domain of numbers, for example – to identify the conditions for the laws (logically) derived from the concept

²⁶"Auch [der Mathematik] Definitionen treten anfangs notwendig in beschränkter Form auf, und erst durch weitere Entwicklung ergibt sich die Verallgemeinerung derselben."

²⁷"[D]iese Erweiterungen der Definitionen lassen der Willkür keinen Raum mehr, sondern sie folgen mit zwingender Notwendigkeit aus den frühern beschränkten."

3.2 Development of mathematics according to the Habilitationsvortrag

to become perfectly general and unconditionally valid.²⁸ Those conditions are *necessary* conditions, for the laws to generalize become the source of the extended definition. To quote Sinaceur again, “from a first concept to its progressive extensions, the passage is strictly regulated by internal exigences” and the passage from a “first” concept to its extension is not only an historical progress, it is “a link of necessity: the conquest of a larger and more complex rationality” (*ibid.*, 219).

The development of mathematics, for Dedekind, appears to be a *chain* of inventions of concepts. As noted in the opening remarks of Dedekind’s *Inauguraldissertation*, the best recourse of the mathematician, when put in an impossible situation, is to introduce new concepts, new tools, to overcome the difficulty. Dedekind considers that very often, these new concepts inaugurate entire new fields in mathematics proving the approach to be doubly fruitful – in this, Dedekind’s mathematical works match remarkably well his epistemological ideas, if one thinks of the context of introduction of Dedekind’s concepts of fields and ideals and their tremendous posterity.

The demand to generalize concepts stated by Dedekind, here, appears to be linked with fruitfulness concerns. It is not required for the laws to be general for the sake of following a certain ideal of generality, but because it conveys the hope that the mathematician will be able to *know* more about the concept(s) he is investigating and to *do* more with them.²⁹ Indeed, working with more restricted or narrower concepts seems to confine the moves allowed to the mathematician, or even to close doors:

[T]he definitions we thereby obtain for these fundamental operations [of addition, multiplication and exponentiation] no longer suffice for the further development of arithmetic, and that is, because it assumes that the numbers with which it teaches us to operate are restricted to a very narrow domain.³⁰ ([Dedekind, 1854a], 757)

The crux of the matter, in Dedekind’s *Habilitationsvortrag*, is to identify which are the conditions allowing for the definition of more general domains of numbers – a domain of numbers in which the subtraction or the division are unrestrictedly applicable, for example. Dedekind, hence, is not considering the problem of the *definition* of more general domains of numbers,

²⁸In later works, and in particular in *Zahlen*, Dedekind’s attention is turned towards identifying the foundations, the concepts *from which* natural numbers are derived. In that case, it is again a deep and careful analysis of the concept that indicates the answer.

²⁹It seems that, for Dedekind, if the answer for a tentative of extension is that the concept ought to be replaced by a new one, this still entails a gain of knowledge.

³⁰“[D]ie so gegebenen Definitionen dieser Grundoperationen genügen der weitern Entwicklung der Arithmetik nicht mehr, und zwar aus dem Grunde, weil sie die Zahlen, mit denen sie operieren lehrt, auf ein sehr kleines Gebiet beschränkt annimmt.”

but of finding a reliable and rigorous method to extend the validity of the laws. This states the limits of the *Habilitationsvortrag*: Dedekind alludes to creating new domains of numbers which assure the general validity of characteristic laws so that the operations are correctly extended, but he does not explain how the said new domains of numbers are to be created or defined. Later works, in particular *Stetigkeit* and manuscripts that can be found in his *Nachlass* (see Sec. 7.2), consider these questions.

3.2.2 Development of arithmetic

In 1854, arithmetic, in Dedekind's attempt to give a systematic account of its development, presents itself as a hierarchy of concepts, each invented on the ground of the preceding one. The basis of elementary arithmetic, Dedekind writes in 1854, is "the formation of ordinal and cardinal numbers" (*ibid.*, 757) and the "first and simplest operation of arithmetic" is the successor operation – "the successive progress from one member of the sequence of positive integers to the next" (*ibid.*). The *act* of counting is, thus, the most elementary act of mathematics insofar as arithmetic is the "simplest science", the first step of the building of mathematics. Dedekind did not discard this idea later on. Indeed, in *Zahlen*, the basis for the definition of the natural numbers is, albeit in a much more abstract presentation, the successor operation establishing the order for the structure of the simply infinite system(s).

The *act of counting* is the basis upon which the arithmetic operations are to be defined. The addition is the collection "into a single act the multiply-repeated performance of this elementary operation"; the multiplication is formed from addition "in a similar manner, and from multiplication that of exponentiation" (*ibid.*, 757). Still in the natural numbers, subtraction, division and root-extraction are defined as inverse operations of addition, multiplication and exponentiation respectively. However, as one knows well, the definition of the inverse operations in the domain of positive integers only enables to perform the said operations in a very restricted way: inverse operations are not unconditionally applicable in \mathbb{N} and the possibilities of performing and developing arithmetic are rather limited, for the initial definitions of the fundamental operations defined "are restricted to a very narrow domain." Dedekind explains, then, that at that point of development of arithmetic, the inverse operations require extended definitions so as to be generally applicable, that is:

[t]he requirement of arithmetic to produce, by means of each of these [inverse] operations, each time anew, the entire existing domain of numbers, or in other words: the requirement for the unconditional executability of the indirect, inverse operations, subtraction, division, and the like, makes it necessary to cre-

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ate new classes of numbers, since with the original sequence of positive integers, the requirement cannot be satisfied.³¹ (*ibid.*, transl. modified, 757)

These extensions should be carried on following the principle stated by Dedekind as directing the development of mathematics:

Laws which emerge from the initial definitions and which are characteristic for the concepts that they designate are to be considered as of general validity. Then these laws conversely become the source of the generalized definitions if one asks: How must the general definition be conceived in order that the discovered characteristic laws be constantly satisfied?³² (*ibid.*, transl. slightly altered, 757)

Dedekind's aim, in this part of the *Habilitationsvortrag*, is to provide a systematic description of the extension of arithmetic and to propose a general principle for the definition or creation of new systems of numbers. Arithmetic, as we will see later, seems to provide a good basic example for his purpose. His presentation shows the part of his approach pertaining to the so-called "genetic method", which plays an important role in his foundational researches.³³ To adopt the "genetic" approach³⁴ consists essentially in extending a system of numbers into a new one, for example to define the real numbers as an extension of the rational numbers.³⁵ The extension is made by adding the irrationals to the system of rational numbers, and by extending operations and laws as well. It is the basic idea found in Ohm's works³⁶ mentioned earlier. In Dedekind's case, it should be underlined that the genetic method is a consequence of more general ideas, in particular what he called his "principle of induction".³⁷

³¹"Die Forderung der Arithmetik nämlich, durch jede dieser Operationen das gesamte vorhandene Zahlgebiet jedesmal von neuem zu erzeugen, oder mit andern Worten: die Forderung der unbedingten Ausführbarkeit der indirekten, umgekehrten Operationen, der Subtraktion, Division usw., führt auf die Notwendigkeit, neue Klassen von Zahlen zu schaffen, da mit der ursprünglichen Reihe der absoluten ganzen Zahlen dieser Forderung kein Genüge geleistet werden kann."

³²"Gesetze, welche aus den anfänglichen Definitionen hervorgehen und charakteristisch für die durch sie bezeichneten Begriffe sind, als allgemeingültig anzusehen; dann werden umgekehrt diese Gesetze die Quelle der verallgemeinerten Definitionen, wenn man fragt: Wie muß die allgemeine Definition gefaßt werden, damit dem gefundenen charakteristischen Gesetze stets Genüge geschieht?"

³³I will come back to these questions later.

³⁴[Hilbert, 1900] coined the term "genetic approach" and provided the description.

³⁵In particular, it is opposed to axiomatic definitions made such that each domain of numbers is defined by its own set of axioms, independently from the other domains.

³⁶*Versuch eines vollkommen consequenten Systems der Mathematik*, the first volume of [Ohm, 1822-1853] was published in 1822, and Ohm kept publishing additional volumes until 1852.

³⁷As underlined by [Sieg and Schlimm, 2005], in his later works, Dedekind's approach

At the first steps of the development of arithmetic, the necessity of introducing new numbers seems clear, if one considers the limits set to the applicability of subtraction: with the definition of subtraction in the positive integers, it is impossible to subtract b from a when $a < b$. Following the principle enunciated earlier, one states the general validity of the characteristic law, for example for all a and b , there exists a c such that $a - b = c$, and *creates* new domains of numbers for the laws to be always satisfied:

[T]he requirement for the unconditional applicability of the indirect, inverse operations, subtraction, division, and the like, makes it necessary to create new classes of numbers, since with the original sequence of positive integers, the requirement cannot be satisfied. Thus one obtains the negative, rational, irrational, and finally also the so-called imaginary numbers.³⁸ (*Habilitationsvortrag.*, transl. slightly altered, 757)

It is essential – necessary, even – that once the number-concept has been extended, one defines again the arithmetic operations, “for until now their effectiveness was only determined for the sequence of positive integers” and they are now to be valid for the new more general domain of numbers as well. The re-definition of the operations ought to be done at every extension. In his *Habilitationsvortrag*, Dedekind develops a couple of examples, such as the definition of multiplication for negative numbers (see below), so as to highlight why and how the re-definition is to be done. In his later works, whenever he introduces a new concept of number (e.g., irrational or algebraic numbers), Dedekind systematically re-defines arithmetic operations.

The extension of the definition does not involve any arbitrariness whatsoever, provided one follows the principle:

[O]ne declares the laws which the operations obeyed in their restricted conception (*Auffassung*) to be valid in general, and from this conversely, one derives the meaning of the operations for the new numbers domain.³⁹ (*ibid.*, 757)

The application of this general principle protects the mathematician from finding definition by a “lucky guess, a happy coincidence” (*ibid.*, 758). As underlined by Sinaceur, “the analysis of a mathematical situation reveals

is both axiomatic and genetic, in the sense that the systems are defined as successive extensions, yet each definition is expressed as a set of necessary and sufficient conditions.

³⁸ “[Diese] Forderung führt auf die Notwendigkeit, neue Klassen von Zahlen zu schaffen, da mit der ursprünglichen Reihe der absoluten ganzen Zahlen dieser Forderung kein Genüge geleistet werden kann. So erhält man die negativen, gebrochenen, irrationalen und endlich auch die sog. imaginären Zahlen.”

³⁹ “(...) die Gesetze, welchen die Operationen in ihrer beschränkten Auffassung gehorchten, für allgemeingültig erklärt, und daraus umgekehrt die Bedeutung der Operationen für die neuen Zahlengebiete ableitet.”

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the *necessary conditions* for the laws, valid in this situation, to become (...) generally and unconditionally valid” ([Sinaceur and Dedekind, 2008], 219). Once these conditions identified, the concept should be extended in such a way that they are, indeed, verified. In 1854, the issue of actually *defining* the extended concepts is not addressed.

Sieg and Schlimm note that there is a “subtle circularity” ([Sieg and Schlimm, 2005], 127) in Dedekind’s principle:

the new numbers are generated by the unrestricted inverse of a restricted operation, which is then extended to this generated broader domain! (*ibid.*, 127)

Dedekind’s use of only internal resources, namely operations whose nature is inherently linked to numbers themselves, and the back and forth movement between restricted and unrestricted operations does create a “subtle circularity”. However, Sieg and Schlimm’s remark seems to suppose that the unrestrictedly valid operation is responsible for the creation of new numbers. Here, it is not clear whether Dedekind does consider that the unrestricted inverse operations is defining the new numbers, or if the requirement for unconditional applicability *highlights the need for new numbers*, then created by the human mind to meet the requirement. This last solution is explicitly the position adopted by Dedekind in later works, especially in *Stetigkeit*.⁴⁰

3.2.2.1 The examples of the extension of multiplication and exponentiation

The definition of multiplication for the negative numbers provides a good example to understand the process advocated for by Dedekind.

In the domain of positive integers, the multiplication $m.n$ is defined as the collection in one single act of the m (the multiplier) repetitions of the addition of a number n (to itself) (the multiplicand). Once the domain of numbers has been extended to the negative numbers, $m.n$ should be, according to the initial definition, the collection in one single act of the m repetitions of the addition of a *negative or positive* number n . But then, the multiplier m is necessarily a positive integer, for it would not make any sense to repeat something a negative number of times. Thus, “a special definition is (...) needed in order to admit negative multipliers as well, and thereby to liberate the operation from the initial constraint” ([Dedekind, 1854a], 758). Such a definition, affirms Dedekind, “involves an *a priori* complete arbitrariness” (*ibid.*, 758). Should one then rely on sheer luck for identifying the appropriate new definition? This would certainly not be true

⁴⁰An objection similar to Sieg and Schlimm’s was made by Russell ([Russell, 1903], 71), and later Dugac ([Dugac, 1976b], 43) about cuts and the creation of irrational numbers. In the case of *Stetigkeit*, cuts are presented as tools bringing out the necessity of irrational numbers which are *then* created by the human mind. see Sec. 7.2.3.

to the scientific method, and in fact, writes Dedekind, rather than counting on a “happy coincidence”, one applies “a general principle”:

We must investigate to which laws the product is submitted if the multiplier undergoes in succession the same alterations which led to the production of the sequence of negative integers out of the sequence of positive integers. For this it suffices if we determine the alteration which the product undergoes if one makes the simplest numerical operation with the multiplier, namely, allowing it to go over into the next-following number.⁴¹ (*ibid.*, transl. slightly altered, 758)

From $(m+1)n = mn+m$, one can derive the “addition theorem”, namely the law for the multiplication of a number by a sum: $(m+m')n = mn+m'n$. We know that the extended domain of numbers has been obtained by considering the operation $(m - m'')$ to be valid for *any* m, m'' . So, from the “addition theorem”, one can derive a “subtraction theorem”: $(m - m'')n = mn - m''n$, which is valid even if $m - m'' < 0$ i.e., if $mn < m''n$. Then, it suffices to “declare this law to be valid in general” (*ibid.*, 758), and

one obtains the definition of multiplication with negative multipliers; and it is then of course no accident that the general law which multiplication obeys is exactly the same for both cases.⁴² (*ibid.*, transl. Ewlad slightly altered, 758)

Sieg and Schlimm underline that the “subtle circularity” detected is still present in Dedekind's definition of multiplication for negative numbers (or of exponentiation for rational numbers, which is done in the same way), for Dedekind “appeals to the very character of [the] generation [of the new numbers]” ([Sieg and Schlimm, 2005], 127). The definition of multiplication explained here gives a clear example of the “intricate dependency” (*ibid.*), since to obtain “the definition of multiplication with negative multipliers”, one uses the law which served for the creation of new numbers.

In spite of Dedekind's protestation — that the definition of the extended operation “involves an *a priori* complete arbitrariness” — he appeals to the very character of that generation. (*ibid.*, 127)

⁴¹“Man muß untersuchen, welchen Gesetzen das Produkt unterworfen ist, wenn der Multiplikator sukzessive dieselben Veränderungen erleidet, durch welche überhaupt aus der absoluten ganzen Zahlenreihe die der negativen erzeugt wurde. Dazu genügt allein schon die Bestimmung der Veränderung, welche das Produkt erleidet, wenn man mit dem Multiplikator die einfachste Zahlenoperation vornimmt, nämlich ihn in die nächstfolgende Zahl übergehen läßt.”

⁴²“erhält man hieraus die Definition der Multiplikation mit negativen Multiplikatoren; und es ist dann natürlich kein Zufall mehr, daß das allgemeine Gesetz, dem die Multiplikation gehorcht, für beide Fälle genau dasselbe ist.”

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There is, indeed, an “intricate dependency” in Dedekind’s method here, but Sieg and Schlimm seem to take Dedekind’s mention of the possible “*a priori* complete arbitrariness” as applying to his own definition of multiplication. It doesn’t however seem to be the case. Dedekind appeals to his “general principle” of considering the characteristic law to be unconditionally valid *so as to avoid all arbitrariness* – this very principle is acknowledged to be the source of the necessity of the extension of definition ([Dedekind, 1854a], 757).

After the definition of multiplication for negative integers, Dedekind continues with the definition of exponentiation in the domain of rational numbers, since powers have only been defined for exponents which are positive integers. A new definition has to be given for exponents which are not positive integers, but again “instead of choosing the definition arbitrarily one must rather investigate how one has to frame it so that the laws following out of the original definition will turn out to be generally valid” (*ibid.*, 758). Following a sort of principle of permanence, the method employed is the same:

[O]ne must ask about the alterations which the power undergoes when one subjects the exponent to the operations of addition, subtraction, multiplication, and division, while leaving the altered exponent a positive integer. Once the laws prevailing here are known, they yield in turn the generalized definition if one requires that these laws set the standard for the character of exponentiation in general.⁴³ (*ibid.*, 758)

The extension to null and negative exponents is fairly easy. The multiplication of exponents defined as an exponentiation ($(a^{mn}) = (a^m)^n$) is more complicated, because the division of powers requires to be able to perform the inverse operation of exponentiation (n -th root extraction), which is not always possible in the rational numbers. Thus, one is led to create new classes of numbers, the irrational numbers and the imaginary numbers, in order to satisfy the requirement of the unrestricted applicability of exponentiation. Note that only algebraic irrational and complex numbers are defined this way, which doesn’t seem to bother Dedekind – if he noticed, since the definition of transcendental numbers is not addressed.

Dedekind mentions that with the creation of the irrational numbers “the concept of limit appears at the same time”, which calls for two remarks. As Pierre Dugac suggests ([Dugac, 1976b], 20), this mention of the concept of

⁴³“Man muß daher (...) nach der Veränderungen fragen, welche die Potenz erleidet, wenn man den Exponenten den Operationen der Addition, Subtraktion, Multiplikation, Division unterwirft, solange der so veränderte Exponent immer noch eine absolute ganze Zahl bleibt. Sind die hierin herrschenden Gesetze erkannt, so liefern sie umgekehrt die verallgemeinerten Definitionen, wenn man die Forderung stellt, daß diese Gesetze maßgebend für den Charakter der Potenzierung überhaupt sein sollen.”

limit made in passing gives us an insight of Dedekind's conception of limit in 1854. However, and this is linked to the second point, Dedekind's conception of irrational numbers and limits will change in the next two decades. Indeed, it is important to notice that, here, Dedekind considers the introduction of the concepts of irrational numbers and of limit as being simultaneous. In *Stetigkeit*, the concept of limit is defined *after* the definition of the irrational numbers (which are, in fact, defined precisely to provide a rigorous ground to Analysis). Limits are *derived from* the definition of the domain of irrational numbers, which, as underlined by Sinaceur ([Sinaceur and Dedekind, 2008], 229), entails that irrational numbers are *logically* anterior to limits – from Dedekind's viewpoint at least.⁴⁴ This remark is not for the sole purpose to note that Dedekind's ideas evolved – sometimes very significantly – compared to his *Habilitation*, but to underline the role played by the (logical) ordering of the concepts. The identification of the conceptual structure and of the order of reasoning appears to be a crucial step in finding the 'right' definition of a concept: to give a definition of real numbers which will serve as a basis for investigations of continuity or a definition of the Riemann surface which will help the research on their topological properties, or again to give a definition of the sequence of positive integers in terms of more general and logically anterior notions so as to be able to deduce the arithmetical properties.

In 1854, Dedekind states that the progress made by creating these last new domains of numbers “are so immense that it is difficult to decide which of the many different paths which open before us we should first pursue” ([Dedekind, 1854a], 759). To pursue arithmetic with these new domains of numbers, the definitions of the operations ought to be extended once again. However, it might happen that the extensions of operations cannot be carried over with preservation of the familiar laws (e.g., for hypercomplex numbers). And indeed, Dedekind considers that “at least with the appearance of imaginary numbers” (*ibid.*, 759), a systematic development of arithmetic becomes difficult to carry out. Dedekind seems openly dissatisfied with the definitions of irrational and imaginary numbers proposed so far:

However, one might well hope that by persistently applying the principle, by not permitting ourselves any arbitrariness but rather always to be led on by the discovered laws, a truly solid edifice of arithmetic will be attained. Everybody knows that until now, an unobjectionable theory of the imaginary numbers, not to mention those newly invented by Hamilton, does not exist, or at any rate has not been published yet.⁴⁵ (*ibid.*, 759)

⁴⁴It is notably not the case for Cantor, who defines the irrational numbers as limits of Cauchy sequences.

⁴⁵“*Indessen ist wohl zu hoffen, daß man durch beharrliche Anwendung des Grundsatzes,*

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As we know, Dedekind proposes what he considers to be a satisfying definition of irrational numbers in 1872. He never, however, wrote about complex numbers and it seems likely that he adopted Hamilton's approach defining complex numbers as pairs of real numbers. Ferreirós mentions that, in 1857, Dedekind read Hamilton's *Lectures on Quaternions*, in the introduction of which Hamilton presents his definition of complex numbers as ordered pairs of real numbers ([Ferreirós, 2008], 220). Ferreirós concludes that this example of the genetic method of definition of new numbers (which Dedekind favored) certainly satisfied Dedekind. And indeed, he did use pairs of numbers to define new numbers, in one of his attempts to define the domain of negative integers from the natural numbers.⁴⁶

3.2.2.2 Some elements about Gauss's idea of arithmetic

Dedekind strongly underlined that Gauss approved of the reflections developed in his *Habilitationsvortrag*. If one considers more closely Gauss's writings, his positions seems, however, to be slightly different. To contrast his ideas with Dedekind's, I will consider mainly two texts, *Zur Metaphysik der Mathematik*⁴⁷ written around 1800, and "*Theoria residuorum biquadraticorum. Comm. II*"⁴⁸.

In *Zur Metaphysik der Mathematik*, quantities are presented, following traditional ideas, as conglomerates or repetitions of "a known quantity (the unit), or an aliquot part of the unit in order to obtain a quantity equal to it" ([Gauss, 1929], 295). Numbers are, then, the *expression* of a quantity, rather than elements of a sequence. Gauss's presentation is for any kind of magnitude, not for natural numbers to be later extended.

Relations between quantities are, writes Gauss, the "proper object of mathematics" – indeed, the arithmetical operations are considered as expressing relations – and in particular, "one quantity in itself" do not constitute an object of study. Quantities are given an "arithmetical representation", which corresponds to the indication of "the way and manner in which, from the idea of an immediately given quantity, one can achieve the idea of the sought quantity." Numbers are used to "show how many times one must

sich auch hier keine Willkürlichkeit zu erlauben, sondern immer durch die gefundenen Gesetze selbst sich weiterleiten zu lassen, zu einem wirklich festen Gebäude der Arithmetik gelangen wird. Bis jetzt ist bekanntlich eine vorwurfsfreie Theorie der imaginären, geschweige denn der neuerdings von Hamilton erdachten Zahlen entweder nicht vorhanden, oder doch wenigstens noch nicht publiziert.

⁴⁶See [Sieg and Schlimm, 2005] and [Schlimm, 2000].

⁴⁷This short text was found in Gauss's *Nachlaß* and published in 1929 in [Gauss, 1863-1874, 1900-1917] XII, 57-61. It is translated in [Ewald, 2005] and references will be to the translation.

⁴⁸In Gauss's *Werke* II, translated in [Ewald, 2005]. I will refer to this text as "Notice" and references will be to Ewald's translation.

imagine the immediately given quantity reiterated to obtain an idea of the sought quantity." Fractions are obtained when one imagines the reiteration of a part of the "given quantity".

Elementary arithmetic is a first and essential way to study relations of quantities, since

we have to make ourselves familiar with the most important of these relations, and especially with those that, on account of their simplicity, can be regarded as the elements of the others – although in fact even here the first (addition and subtraction) underlie the others (multiplication and division). (*ibid.*, 295)

There are essentially two sorts of relations between quantities. The first is that "between the wholes and the parts" (which is the "simplest" one), which generates addition and subtraction:

addition, shows how to find the whole from the parts; the second m[ode of] c[alculation], subtraction, shows how, given the whole and a part, one finds the other. With respect to addition the parts are called the quantities summed and the whole the sum or the aggregate; with respect to subtraction the whole is called the major or minuend, the known part the minor, and the sought part the difference or the remainder. (*ibid.*, 295)

The second relation is that between the simple and the multiple, and is the origin of multiplication and division:

Multiplication shows how to find [the multiple] from [the simple] and [the number which indicates what sort of a multiple it is, i.e., the multiplier]; division, how to find [the multiplier] from the first two: with respect to multiplication, the simple is called the multiplicand, the number that determines the sort of multiplicity, the multiplier, both the factors, and the multiple the product. With respect to division the simple is called the divisor, the number that determines the sort of multiplicity the quotient, and the multiple the dividend. (*ibid.*, 295)

Operations are not themselves the relations to be studied: the relations between quantities lead to the definition of the elementary operations (*Spezies*). Operations seem to be expressing the relations, and may be a tool to study the said relations – tools for mathematics, then. Note that the definition of operations in terms of relations is given for magnitudes in general. For addition and subtraction, Gauss states that the most important principle, taken as an axiom, is "that the parts, if they are united in any order, and if none is omitted, are equal to the whole" (*ibid.*, 295). Relations between the whole and the parts determine the "mode[s] (species) of calculation",

3.2 Development of mathematics according to the *Habilitationsvortrag*

i.e., the operations. Operations are, so to speak, reconstituted from the relations.

Gauss goes on to give the usual properties of multiplication (commutativity, associativity, distributivity), and calls them “principal truths of multiplication” (*ibid.*, 295-296) and he underlines the similarity (reciprocity) between multiplication and division.

While it can be easily noted, in Gauss’s vocabulary, that subtraction and division are the inverse operations of addition and multiplication, it is not the property chosen to *define* them, and, in fact, it seems to be a property of lesser importance. More importantly, the whole edifice of arithmetic is not organized as it is in Dedekind’s *Habilitationsvortrag* – or any of the earlier works mentioned, such as Ohm’s. Moreover, insofar as the operations are given, here, for a general notion of magnitude, considerations about the extension of domains of numbers does not seem relevant. It would, however, be considered later by Gauss as a central driving force of the development of mathematics.

Ferreirós suggests that Gauss proceeded in the following way:

One would begin with utmost generality, introducing quite generally relations, magnitudes and operations, and developing the theory of the universal properties of all magnitudes. Such results would not belong to arithmetic, but to the pure theory of magnitudes; they apply to the natural numbers because they are a particular case, namely the theory of discrete magnitudes. ([Ferreirós, 2007], 261)

Indeed, it is important to underline that Gauss had a very general concept of magnitude, in which he distinguished between “extensive” and “intensive”:

Mathematics has for its object all extensive quantities (those of which parts can be thought); intensive quantities (all non-extensive quantities) only to the extent that they depend on the extensive. To the first sort of quantities belong: space or the geometric quantities (which include lines, surfaces, bodies, and angles), time, number; to the latter: speed, density, hardness, height and depth of tones, strength of tones and of light, probability, etc. (*op. cit.*, 294)

As is well known, Gauss further distinguished between geometry and arithmetic, stating that geometry should be considered along with mechanics, as an experimental science. As regards geometry, the empirical intuition plays an important role to determine, for example, the validity of the parallel postulate, insofar as he considered logical principles to be insufficient in themselves. Logical principles in geometry “only bloom sterile flowers, if the fructifying, living intuition of the object does not act everywhere”⁴⁹

⁴⁹Dedekind does not follow Gauss on this matter.

(Gauss 1900). Arithmetic, on the other hand, “stands purely *a priori*”. The representations (*Darstellung*) of magnitudes shall, then, be obtained in different ways depending on their nature. The geometrical magnitudes are obtained by “immediate intuition (an immediate representation) (...), [by] geometrical representation or construction” (*op. cit.*, 294). The arithmetical representation is obtained when “from an immediately given quantity, one can achieve the idea of the sought quantity”, which is done “by means of numbers”⁵⁰.

Zur Metaphysik was written very early, and is, of course, not Gauss's only writing on this matter. Thirty years later, during which he produced an extraordinary body of work (notably in number theory), Gauss advocated for the introduction of the so-called Gaussian integers in “higher arithmetic”. In these reflections, his arguments were closer to the ones developed by Dedekind. The Gaussian integers, i.e., numbers of the form $a + ib$ with a and b rational integers and $i^2 = -1$ and called “complex integers” by Gauss, were conceived as a more general concept of number which allowed for better developments of the theory of biquadratic residues:

For the true grounding of the theory of biquadratic residues, one must extend the field of higher arithmetic⁵¹, which has hitherto been confined to the real integers, into the imaginary integers, and must concede to the latter the same legitimacy as the former. As soon as one has seen this, that theory appears in an entirely new light, and its results acquire a startling simplicity.

But before the theory of biquadratic residues can be developed in this widened domain of numbers, the doctrines of higher arithmetic, which hitherto have been explored only for real numbers, must be extended as well. ([Gauss, 1831], 308-309)

Gauss underlines that “complex quantities are not opposed to the real, but contain them as a special case where $b = 0$ ”. Moreover, “[j]ust as in the arithmetic of real numbers” there are two units for real numbers $(1, -1)$, one should consider four units for the complex numbers $(1, -1, i, -i)$. This is not a new idea for Gauss, who seems to have drawn it from his practice of number theory. Already, two decades earlier, in a letter to Bessel, Gauss wrote that “in the realm of magnitudes, the imaginary numbers $a + b\sqrt{-1} = a + bi$ have

⁵⁰Arithmetical and geometrical representations are not necessarily mutually exclusive: “To represent geometrically quantities that have arithmetical relations – as was so common among the ancients – is no longer so common today; otherwise one would have to regard this as a part of geometry. On the contrary, one applies the arithmetical manner of representation extremely frequently to quantities in geometric relation, for example in trigonometry...” (*ibid.*, 294)

⁵¹As regards the distinction between arithmetic and higher arithmetic, see Sec. 1.2.1.

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to be considered as having the same rights as the real ones” (8 December 1811, quoted and translated in [Boniface, 2007], 325).

In the 1831 “Notice”, Gauss explains how arithmetic has extended its scope to become the “general arithmetic” proper to “modern times”:

[Our general arithmetic] started from the concept of absolute integers, and has gradually extended its territory; the fractions have been added to the integers, the irrationals to the rationals, the negatives to the positives, and the imaginaries to the reals. (*op. cit.*, 312)

But Gauss’s aim is not to describe the process of generation of new numbers, it is mainly to show that complex integers are as legitimate as objects of arithmetic as any real number. Real numbers are all commonly accepted now but provoked “a few scruples”: for example, negative roots of equations were called “false roots” but the “reality of the negative numbers is sufficiently justified because they have an adequate substrate in innumerable other cases.” The idea of a “general arithmetic” does suggest that Gauss’s views changed since *Zur Metaphysik*. It suggests that there is a generation of different sorts of numbers, although it is not by extending domains so as to satisfy the general applicability of operations. For Gauss, it is necessary to have a relation, to which one can refer, to be able to express clearly what a number, such as -3 actually is – a point crucially different from Dedekind’s viewpoint. Indeed, Gauss still considered mathematics to study relationships between (abstract) objects:

Positive and negative numbers can find an application only where that which is counted has an opposite, so that the thought of them as united is to be equated with annihilation. Precisely regarded, this condition occurs only where what is counted is not substances (objects thinkable in themselves) but relations between any two objects. It is then supposed that these objects are ordered into a sequence in a determinate way... (*ibid.*, 312)

In the case of imaginary numbers, such an ordering cannot be given. Gauss proposes, then, to consider the ordering of “sequences of sequences”, one given by the unit 1, and the other by the unit i :

The mathematician abstracts totally from the nature of the objects and the content of their relations; he is concerned solely with the counting and comparison of the relations among themselves: just as he regards the relations designated by $+1$ and -1 in themselves, as similar, so is he entitled to regard all four elements $+1$, -1 , $+i$, $-i$ as similar. (*ibid.*, 312)

So as to bring “these relationships (...) to intuition”, Gauss introduces the famous geometrical representation of complex integers. Such a representation is argued for on the basis of an analogy with the “infinite straight line”

used to “represent the whole domain of real quantities” (Letter to Bessel, 8 December 1811, *op. cit.*, 326). More importantly, Gauss gives the first elements of the computations on complex numbers and underlines that he will answer the question “why the relations between things that form a manifold of more than two dimensions cannot supply yet another type of number that is, admissible in higher arithmetic”⁵² showing that he is fully aware that the geometrical representation does not justify the complex numbers – it eases our understanding of them.

3.2.3 In the case of definitions in less elementary parts of mathematics

Dedekind's considerations about the development of mathematics are not restricted to elementary arithmetic. Indeed, he argues that it is possible to find examples of such developments in more developed, less elementary parts of mathematics. In his 1852 *Inauguraldissertation*, the observations on the development of mathematics were introductory remarks on the formation of Eulerian integrals, the topic of his dissertation. The extension of definition for Eulerian integrals presents peculiar difficulties, because a modification of the definition is required, which wasn't the case in the examples considered so far by Dedekind. In fact, it is easy to see that it is not always possible to apply a systematic treatment such as the one proposed by Dedekind for arithmetic, in ‘complicated’ theories like Eulerian integrals or even trigonometric functions, as acknowledged by Dedekind himself. Nevertheless, the main idea, namely that “the definitions necessarily appear at the outset in a restricted form, and their generalization (*Verallgemeinerung*) emerges only in the course of further development” ([Dedekind, 1854a], 757), seems to be widely valid from Dedekind's viewpoint and can be articulated with the idea of a progressive extension of arithmetic.

In his *Habilitationsvortrag*, Dedekind proposes two examples: the trigonometric functions and the integral calculus. Those additional examples are discussed to support Dedekind's argument, by showing to what extent the ideas presented may indeed be relevant not solely to elementary mathematics. They display very clearly the outlines for what Dedekind seems to consider as being the most efficient and most rigorous way to proceed when shaping a theory. To be clear, Dedekind seems to propose a way of re-ordering the theory following his “principle” so that one gets a more rigorous and general presentation of the theory. We will see, in the following, that Dedekind presents mathematics as proceeding from successive (attempts of) generalization. While this is a rather widespread idea among 19th and 20th century mathematicians, it should be acknowledged that it is not (always) an historically accurate description of the development of mathematics. Indeed, even Dedekind's simplest example, the negative numbers, had a more

⁵²Gauss never published such works.

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complicated and multifold birth – notably, until the 19th century, they were more often considered as *debts* than as new quantities introduced to allow the general applicability of subtraction. It is not completely clear, neither here nor in later writings, whether Dedekind considered that the process he is describing is objectively what happens in any mathematical enterprise, or simply as a better method allowing to rearrange a theory following the right logical order. It does, however, seem more likely that Dedekind is prescribing an appropriate way of developing mathematics, and provides illustrations to sustain his point.

Dedekind's first example is, for him, a striking illustration of the gain of simplicity and clarity made possible by using his principle. When defining the sine and the cosine, one defines them for acute angles and while they can be conceived more generally “the style and manner strikes [Dedekind] as thoroughly arbitrary” ([Dedekind, 1854a], 759). Dedekind, drawing from various textbooks, gives an example of an attempt to generalize the definition and prove fundamental theorems valid in all cases. But this example, from Dedekind's viewpoint, appears to suffer from several imperfections. He explains that the generalization itself is led in such a way that it happens that the geometrical construction carried out to prove a theorem, such as the addition formula,⁵³ is only carried out “for the case in which all three angles are acute”, after which

one either considers it going without saying and not worth the trouble to prove that this theorem is valid for all cases, no matter what value one may give to the angle, or, if this gap is felt, such a proof turns out to be extraordinarily complicated.⁵⁴ (*ibid.*, 760)

Besides, Dedekind detects the arbitrariness mentioned above again in later developments of this theory.

To be sure, it always remains interesting to see that, even with the definitions that were formerly assumed, the theorem coincidentally fits all cases. I say *coincidentally*, because one cannot call it anything else.⁵⁵ (*ibid.*, 760)

This situation should be regarded as problematic, at least from Dedekind's viewpoint. Even more so inasmuch as it is possible, Dedekind states, “with

⁵³For a , b and $a+b$ acute angles, $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.

⁵⁴“entweder hält man es für selbstverständlich und nicht für der Mühe wert, zu beweisen, daß dieser Satz für alle Fälle gültig ist, welche Werte man auch den Winkeln beilegen mag, oder wenn dieser Mangel gefühlt wird, so fällt ein solcher Nachweis außerordentlich umständlich aus.”

⁵⁵“Freilich bleibt es immer interessant zu sehen, daß bei den einmal angenommenen Definitionen der Satz zufällig auf alle Fälle paßt. Ich sage zufällig, denn anders kann man es nicht nennen.”

little trouble [to] discover a consistent and natural path to the same results" (*ibid.*, 760). Indeed, it suffices to apply Dedekind's "general principle":

One really needs only the definitions of the sine and cosine of acute angles; and if the aforementioned addition theorem is proved for these, then it yields, in the simplest way, when it is elevated to a general law, the extended definitions with compelling necessity.⁵⁶ (*ibid.*, transl. slightly altered, 760)

Then, one takes one of the two angles to be equal to a right angle "while the other remains arbitrarily acute" and the definitions for the obtuse angles are obtained. It is possible to obtain "the meaning of the sine and cosine of angles up to the size of four right angles" by giving the value of two right angles to one of the angles and allow "the other to run through the domain of concave angles" (*ibid.*, transl. slightly altered, 760).

[A]nd if one continues so, one attains easily to the general definition for positive angles, and also in a similar manner, via subtraction, to that for negative angles.⁵⁷ (*ibid.*, 760)

Here again, the principle proposed by Dedekind indicates how to obtain a generalized definition from an initially restricted one, and exhibits "the peculiarity of the way in which, in mathematics, concepts develop from those which relate only to a restricted domain into more general ones" (*ibid.*, 760). More importantly, even, the "law teaches us also here, how the concept should be conceived, for it to be the most efficient",⁵⁸ the law indicates how to appropriately extend the concept. It is, then, the analysis of the 'situation' established by the consideration of the unrestricted validity of the characteristic properties which teaches us how the concept is to be understood and its definition reshaped, so that it can acquire its widest scope – in the same way as the identification of the conditions for the extensions of the domains of numbers. The idea that mathematics is an *activity* of the human thought is illustrated, here, very clearly.

All the examples given until then may seem somewhat *ad hoc*, insofar as they all allow "the original definition for the restricted domain [to remain]

⁵⁶"Man bedarf wirklich nur der Definitionen von Sinus und Kosinus spitzer Winkel; und wenn für diese das genannte Additionstheorem nachgewiesen ist, so liefert dasselbe, wenn es zu einem allgemeinen Gesetz erhoben wird, auf dem einfachsten Wege die erweiterten Definitionen mit zwingender Notwendigkeit."

⁵⁷"[U]nd wenn man so fortführt erhebt man sich leicht zur allgemeinen Definition für die positiven und auch auf analoge Weise durch Subtraktion zu der für negative Winkel."

⁵⁸"Das Gesetz lehrt auch hier, wie man den Begriff fassen soll, auf daß er am wirksamsten werde." (*ibid.*, in [Dedekind, 1932] III, 436. This sentence is not in Ewald's translation.)

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unchanged”⁵⁹ (*ibid.*, transl. slightly modified, 760), which is not always possible. Dedekind is well aware of this and acknowledges that in some parts of “higher mathematics”, such as integral calculus, it happens that the initial definitions “must be utterly abandoned in order to make room for others” (*ibid.*, 760). For example, this is what happens with the definition of integration as the inverse operation of differentiation. Insofar as this is a formal approach, and not an arithmetical or computational one, it is not possible to “demand the general applicability of integration”, for it would mean to find “a form (...) which, when subjected to the mechanical operation of differentiation, returns the given form” which seems to only lead to unmanageable complications (*ibid.*, 761). However, if one “[properly explores] the relationship between differential and integral”, there is a possibility to find a connection “fully independent of their formal expression”⁶⁰ (*ibid.*, 761) and which involves the concept of limit. Hence,

[w]hen one gives [the integrale] this meaning, namely, to be the boundary-value of a sum of an ever-increasing number of ever-smaller parts, as a general definition, the operation of integration appears in complete self-sufficiency and independence of the differential calculus, without however losing the earlier connection with it.⁶¹ (*ibid.*, 761)

Eulerian integrals and elliptic functions are an example of the fruitfulness of such a conception of integration. While the extension of the definition, here, is significantly different from what was done for arithmetic operations, “it is interesting to see that, as in the examples from elementary arithmetic which we mentioned earlier, so with these newly introduced functions a similar course of development has made itself visible”⁶² (*ibid.*, 761). Indeed, for the Eulerian integrals, one defines the “so-called Γ -functions”, definite integrals on the domain of positive real numbers x :

$$\Gamma(\mu) = \int_0^{\infty} x^{(\mu-1)} e^{(-x)} dx$$

“or at any rate, the effectiveness of this function ceases for every negative value of the argument, since the function then becomes infinitely great”

⁵⁹“(...) blieb aber die ursprüngliche Definition für das beschränkte Gebiet unangetastet stehen.”

⁶⁰“(...) bei der eigentlichen Erforschung der Beziehungen zwischen Differential und Integral stellt sich ein von diesem formellen Ausdruck völlig befreiter Zusammenhang heraus.”

⁶¹“Dann aber, wenn man ihr diese Bedeutung, nämlich den Grenzwert einer Summe einer immer größer werdenden Anzahl immer kleinerer Teile zu bilden, als allgemeine Definition unterlegt, tritt die Operation des Integrierens in vollkommener Selbständigkeit und Unabhängigkeit von der Differentialrechnung auf, ohne indessen den frühern Zusammenhang mit dieser zu verlieren.”

⁶²“(...) interessant ist es zu sehen, daß auch bei diesen neu eingeführten Funktionen ein ähnlicher Entwicklungsgang sich bemerkbar gemacht hat wie in den oben angeführten Beispielen aus der Elementararithmetik.”

(*ibid.*, 761). And we have $\Gamma(n) = (n - 1)!$, for any positive integer n . Gauss introduced the Π -function which is identical to Γ for all positive values and “which also remains a variable function for all other values (even though at certain places it becomes infinite)” (*ibid.*). The Π -function was first introduced by Gauss and is defined, for $s > -1$ by:

$$\Pi(s) = \int_0^{\infty} x^s e^{-x} dx$$

which, for any positive integer n , is such that $\Pi(n) = n!$. Hence, $\Pi(n - 1) = \Gamma(n)$ and one can show⁶³ that

$$\Pi(s) = \prod_{n=1}^{\infty} \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^s}{1 + \frac{s}{n}}.$$

Edwards explains that this infinite product, the definition chosen by Dedekind, allows to show that $\Pi(x)$ “is an analytic function of the complex variable s , which has simple poles at $s = -1, -2, -3, \dots$. It has no zeros” ([Edwards, 2001], 8). Consequently,

[s]ince one found that this function Γ is identical for all positive values of its argument with a function Π , which is defined as an infinite product and which also remains a variable function for all other values (even though at certain places it becomes infinite) one naturally abandons the earlier definition and transforms it into a theorem, and instead defines the function Π as the main function.⁶⁴ ([Dedekind, 1854a], 761)

Pierre Dugac, in his short presentation of Dedekind's *Habilitationsvortrag* underlines how Dedekind, already in 1854, had a profound understanding of the “principles of mathematics” and “their development” ([Dugac, 1976b], 19).⁶⁵ The guidelines described by Dedekind suggest that mathematics mainly develops following a process of extensions of the conditions of validity and applicability of its methods, concepts and even theories. A great part of the work of the mathematician, from Dedekind's viewpoint, appears to be to investigate if and how it is possible to enlarge the scope of validity of

⁶³See [Edwards, 2001], 8.

⁶⁴“Seitdem man aber gefunden hat, daß diese Funktion Γ für alle positiven Werte ihres Arguments mit einer Funktion Π , welche als unendliches Produkt definiert wird und für alle andern Werte gleichfalls eine veränderliche, wenn auch an bestimmten Stellen unendlich werdende Funktion bleibt, identisch ist, wird man natürlich die frühere Definition verlassen und in ein Theorem verwandeln, und statt dessen die Funktion Π als die Hauptfunktion definieren.”

⁶⁵This suggests a view of mathematics as progressing towards ‘our’ truth, in accordance to Dugac's conception of mathematics close to Bourbaki's.

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a certain concept or theory: what are the properties and laws which should be considered as characteristic, so as to obtain a concept which can be used more widely? What can be preserved of the properties of the restricted concept in the extension? With the complex numbers, and even more so with the hypercomplex numbers, the principle of permanence brought to the fore in the previous developments of the *Habilitationsvortrag* should be questioned.⁶⁶ Yet, although he did acknowledge that a systematic development of the extensions of the number domains started to raise difficulties with these extensions, Dedekind did not address these issues either here or in later works.

The conception of the process of development of mathematics as successive attempts of generalization has become relatively common, notably after Hilbert's famous "Mathematical problems".⁶⁷ However, Hilbert suggests that generalization is not the sole move for developing mathematics, it has to be joined to a process of "specialization", that is, solving "problems simpler and easier than the one in hand" may play an even bigger role than generalization. In Dedekind's works, the attempt to develop a theory from a more general standpoint appears as driving force in his approach. It isn't completely clear whether Dedekind considered that mathematics develop exclusively from a generalization process, overlooking "specialization", or if he considered generalization to be the most fruitful and most interesting part.

In addition to stating that mathematics is a human activity, Dedekind also insists on the "inner necessity" of science. For him, in particular in mathematics, this "inner necessity" is guiding the scientific development:

It is not only an historical fact, but also a fact that rests on an inner necessity, that the further development of each science always reacts creatively on the system through which one tries to conceive the organism of the science. ([Dedekind, 1854a], 756)

Sinaceur underlines that the image outlined by Dedekind can be found later in Cantor's, Hilbert's and Cavaillès's writings.⁶⁸ The conceptual analysis

⁶⁶Indeed, with the hypercomplex numbers, the extension of the operations do not allow to preserve all their properties, and one has to work without certain of their properties, such as the commutativity of multiplication for the quaternions. See [Detlefsen, 2005].

⁶⁷"If we do not succeed in solving a mathematical problem, the reason frequently consists in our failure to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems. After finding this standpoint, not only is this problem frequently more accessible to our investigation, but at the same time we come into possession of a method which is applicable also to related problems." ([Hilbert, 1900], transl. in [Ewald, 2005], 1101)

⁶⁸According to Sinaceur, Dedekind's *Habilitationsvortrag* shows the first elements of a "school of modern philosophy" that developed in "parallel to the development of axiomatics" and which developed "the conception of an objective and rational history of

brought to the fore by Dedekind's approach, here, has been described by C. Ehrhardt as "the research of foundations, the rise of logic and the development of the idea of structure" ([Ehrhardt, 2011a], 551), as part of the "conceptual approach." As we saw, according to some accounts, the demand for a more uniform and general approach to problems and definitions appears as a methodological guideline of many developments in the 19th century mathematics. These are the principles identified by Eisenstein as constituting the core of a new approach to mathematics developed by Gauss, Dirichlet and Jacobi. The demand that a problem, or a definition, be treated in one move – for example, to have one definition for *all* real numbers, rather than as roots, logarithms, infinite series, etc. – was not one made by earlier authors. In fact, concerns about definition really started to become central with Hankel's works. Besides, as regards the extension of arithmetic, our short detour by Gauss's ideas tends to show that these ideas were not exactly shared by all of Dedekind's contemporaries. Another example is Cauchy who, in his first *Cours d'Analyse*, presents the positive and negative quantities as representing respectively "increase or decrease of a fixed magnitude" and the signs + and – placed in front of numbers to be mere adjectives. Moreover, complex numbers were, for Cauchy purely symbolic.⁶⁹

A strong opponent to the approach proposed in Dedekind's *Habilitationsvortrag* is, of course, Leopold Kronecker. From Kronecker's viewpoint, "definitions, in themselves, constitute an impossibility" ([Kronecker, 1887]), insofar as every definition involves concepts, each of which is in need of a definition. . . and so on. Kronecker considers mathematics as being a natural science, hence supposed to "describe the phenomena simply and completely." Mathematics, then, should be based on phenomena observed, almost empirically, rather than on definitions.⁷⁰

To ask such questions as whether Dedekind was or wasn't right is not the point, here. However, I should point out that the ideas proposed by Dedekind in his *Habilitationsvortrag* are very consistent with his own prac-

mathematics", which is particularly well represented Caillaud's works. ([Sinaceur and Dedekind, 2008], 220). Indeed, Caillaud read and was strongly impressed by Dedekind's *Habilitationsvortrag* while he was working on his PhD dissertations and developed ideas that were underlying Dedekind's text. I will not, however, develop the relation between Caillaud's and Dedekind's ideas. For more on Caillaud's philosophy of mathematics, one can refer to [Sinaceur, 1994].

⁶⁹ "En analyse, on appelle expression symbolique ou symbole toute combinaison de signes algébriques qui ne signifie rien par elle-même., (...) Parmi les expressions ou équations symboliques dont la considération est de quelque importance en analyse, on doit surtout distinguer celles que l'on a nommées imaginaires. (...) [Les] expressions symboliques [du type $\text{cosa} + \sqrt{-1}\text{ sina}$] ne peuvent s'interpréter d'après les conventions généralement établies, et ne représentent rien de réel. On les a nommées pour cette raison expressions imaginaires." ([Cauchy, 1821])

⁷⁰ For Kronecker's conceptions of mathematics and numbers, see [Boniface, 2005] and in this dissertation 7.1.3.

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tice of mathematics. As mentioned earlier, the *Habilitationsvortrag* is better considered as providing methodological and epistemological guidelines for (his) mathematics.

Dedekind's observations imply a strong methodological claim. Dedekind seems to be proposing that one should try to follow conjointly two principles. The first is the importance of introducing new concepts to develop mathematics. The second is the quest for a greater generality as a motive for mathematical research. The generality looked for by Dedekind seems to be, so to speak, a localized generality, insofar as Dedekind is trying to give definitions which are valid for all cases possible in a specific framework. This, for Dedekind, will appear to be an important component of a developing concepts and methods in mathematics which allow to bypass difficulties or impossibilities, resolve problems, and even ease certain aspects of the work of the mathematicians.

In the course of their work, the mathematicians are confronted with two possible situations. The generalization might be possible: one should, then, introduce new concepts which generalize the old ones and allow to develop more general methods of definition and proof – and preferably parallel to the restricted theory. It is what happens, for example, when one extends the systems of numbers up to the real numbers. However, such an approach can rarely be systematically applied, for difficulties soon appear – e.g., with complex numbers in the extension of the number concept. On the other hand, the generalization might not be possible, in which case, the mathematician should look for new concepts and methods to overcome the impasse. In this situation, the mathematician could be led to develop a new theory, bringing to the fore evolutions or changes in the questions or the objects to be studied – as for example with Eulerian integrals mentioned above. The invention of new concepts can thus serve the purpose of generalization, and the mathematician may face a situation in which proving that the extended notion does satisfy a certain number of essential properties requires to introduce new tools. It is the case with Dedekind's works on algebraic numbers whose main aim is to prove the general validity of the fundamental theorem of arithmetic.⁷¹ For this purpose, Dedekind proposes a new, wider notion of integer, the algebraic integers which are “the most general concept of integer”, and develops a whole arsenal of new tools, notably constituted of ideals, so as to establish that the unique factorization in primes holds for

⁷¹Note that as far as the extension of the number concept is concerned, there seems to be two distinct possibilities: one can extend the domains of number from the natural numbers up to complex and hypercomplex numbers, or one can extend the concept of *integer*. In both cases, the properties investigated are distinct, and in particular, in the second case, which leads to algebraic integers, the “most general concept of integer” according to Dedekind, the core properties to investigate are properties such as the fundamental theorem of arithmetic. In considering the divisibility of integral numbers as the essential properties to investigate in arithmetic, Dedekind seems to be following Gauss's lead. See Sec. 1.2.

any integer.

From the *Habilitationsvortrag* – and as I hope to make clear in Dedekind's subsequent works as well – one understands two important things. Firstly, the introduction of new notions to extend definitions is an essential move for the mathematician. In Dedekind's case, this idea is closely linked to that of science as a activity of the human understanding which seems to run through his mathematics. And secondly, arithmetic holds a peculiar place in mathematics, as it is more fundamental than any other parts of mathematics. However, this does not mean that arithmetic is exempt from being defined (contrary to what mathematicians such as Kronecker would assess) or extended and subjected to innovations. Indeed, arithmetic is a part the human activity that constitutes science in general, and mathematics in particular. The links between invention of new notions and arithmetic is of particular interest to understand the role and status of arithmetic in Dedekind's approach, not only for the new concepts of numbers introduced, but also – and maybe even more so – for the role arithmetic will play in the invention of new notions.

As it is already alluded to in 1854 and will become pregnant in later foundational works, numbers are very explicitly considered to be productions of the mind (*Stetigkeit*, 771, *Zahlen*, 791, 809, *Zum Zahlbegriff* . . .), following a view advocated notably by Gauss. More significantly, they are the first and the most powerful tool produced by the mind to work with. Once the arithmetic operations are introduced, they offer “an inexhaustible wealth of remarkable laws” ([Dedekind, 1872], 768).

Inasmuch as mathematics is an activity of the human understanding, whose finiteness entails that our discovery of truth and conquest of rationality only happen progressively and require to build up our own tools, then significant changes are susceptible to appear in its development. The introduction of new concepts as a driving force of the progress of mathematics is a key element of Dedekind's conception of mathematics. It is clear in his practice of mathematics: Dedekind introduced the concepts of field, ideal, cut, or chain as new concepts on which to base his and further works. He also developed the general bases of certain theories referred to as auxiliary theories, such as module theory, and the theories of systems and mappings. And, as is well known, he openly defended innovation in mathematics, with an explicit reference to his *Habilitationsvortrag*:

[T]he greatest and most fruitful advances in mathematics and other sciences have primarily been made by the creation and introduction of new concepts, rendered necessary by the frequent recurrence of complex phenomena which could be mastered by the old notions only with difficulty. On this subject, I had to give a lecture before the philosophical faculty in the summer

3.2 Development of mathematics according to the Habilitationsvortrag

of 1854 on the occasion of my Habilitation as *Privat-Dozent* in Göttingen. The purpose of this lecture met with the approval of Gauss; but this is not the place to go into further detail.⁷² (*Zahlen*, transl. slightly altered, 792-793.)

In the course of my analysis of Dedekind's works, in the next chapters, I will highlight how his particular view of conceptual innovation is linked to arithmetic. The introduction of new concepts concerns (the development of) arithmetic as well as all mathematics. Besides, arithmetic, as the science of numbers, can provide tools to enhance the development of mathematics. Indeed, as mentioned in the quote from *Zum Zahlbegriff*: numbers are invented to ease the work of the mind, thus playing this role in science, and more particularly in mathematics. Arithmetic is thus susceptible to change according to the needs of mathematical practice. From there, it is still essential to give mathematical characteristics which would allow to identify uses of arithmetic in Dedekind's works.

⁷²“[D]ie größten und fruchtbarsten Fortschritte in der Mathematik und anderen Wissenschaften sind vorzugsweise durch die Schöpfung and Einführung neuer Begriffe gemacht, nachdem die häufige Wiederkehr zusammengesetzter Erscheinungen, welche von den alten Begriffen nur mühselig beherrscht werden, dazu gedrängt hat. Über diesen Gegenstand habe ich im Sommer 1854 bei Gelegenheit meiner Habilitation als Privatdozent zu Göttingen einen Vortrag vor der philosophischen Fakultät zu halten gehabt, dessen Absicht auch von Gauß gebilligt wurde; doch ist hier nicht der Ort, näher darauf einzugehen.”

Chapter 4

Dedekind's first works in number theory

In the 1854 *Habilitationsvortrag*, the extension of the number concept and arithmetic appear essentially as the successive widening of the domains of numbers, without implying any particularity of divisibility, whereas in 1876/77, divisibility is presented as the foundation of arithmetic. In fact, I will suggest in the next three chapters that divisibility, in Dedekind's mathematical works, is subjected to a slow extension from a property of natural integers to the possibility of studying divisibility properties of "polygons" on a Riemann surface.

Following Dirichlet's suggestion in 1856, Dedekind studied Galois's works, which he immediately considered in relation to number theory. In his lectures on Galois theory, given in Göttingen in 1856-58, Dedekind first defines arithmetical relationships between sets and highlights a certain analogy with rational number theory. In 1857, Dedekind published his first paper on number theory, investigating the possibility to develop an approach to higher congruences which would be "rigorously tied to an analogy with elementary number theory."

I will propose to consider first Galois's works, and will stay very close to Galois's text, so as to have a clear idea of what Dedekind was working with. For the same reason, I will briefly expose shortly Cayley's 1854 article on groups, which likely inspired some aspects of Dedekind's interpretation of the notion of groups. I will then expose the content of Dedekind's two first inquiries in number theory: a manuscript containing Dedekind's lectures on "higher algebra"; and his first published paper in number theory, *Abriß einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus* in which he proposes to develop a completely arithmetical approach for higher congruences. Dedekind's reading of Galois's works on groups, in 1856-58 has been presented, by Caroline Ehrhardt, as deeply influenced by the "conceptual approach" in Göttingen. Without coming back to these

4.1 Galois's Galois theory

questions, I would like to exhibit how, in Dedekind's first works in algebra and number theory, one can see the beginnings of an arithmetical approach which will dominate Dedekind's works.

4.1 Galois's Galois theory

The methodological principles underlining the changes in mathematical practices in the 19th century, which can be detected in certain of Gauss's works and gradually take more importance in the 19th century mathematics in Germany, are very close to Galois's idea that one should "jump over" the computations that were becoming too wearisome for mathematicians's (and mathematics's) sake. However, as I have underlined, it seems rather unlikely that Galois's memoir would be the origin for all the statements about "conceptual" methods in mathematics, since Eisenstein's account dates from the year Liouville rediscovered Galois's works.

Let me first give the outlines of the roots of Galois's works on the resolvability of equations.¹

Algebraic works, in the 19th century, were meant to exhibit general methods to solve polynomial equations, which was considered as acceptably done only with an explicit formula allowing to compute the said solutions. The problem could be tackled in two ways: the algebraic resolution, in which one would provide a general solution for any (literal) equation of a certain degree (such as Cardano's formulae for degree 3 polynomials), or the numerical resolution, in which one would find the numerical (often approximate) values of the roots of the equation. When adopting the second possibility, mathematicians tried to provide methods which could work in as many cases as possible and which usually implied the use of arguments and methods from Analysis, such as the continuity of polynomial functions. Yet, from the beginning of the 19th century, the theory of equations appeared to be encountering its own limits. As I have mentioned in the first chapter of this dissertation, the limits of the theory of equation was acknowledged by actors such as Lagrange ([Lagrange, 1770-1771], [Lagrange, 1808]). The difficulties to find effective methods to resolve equations of degree higher than 3 led, in particular with Lagrange, to look for ways to identify the *a priori* effectiveness of methods of resolution.

In the 1830s, the theory of equations, in which most of algebra consisted, was in an impasse. Niels Henrik Abel proved in 1826 ([Abel, 1826]) that it was impossible to solve by radicals the general quintic.² The first ground

¹For more on the roots of Galois's works, see [Ehrhardt, 2011b], [Ehrhardt, 2012]. See also [Sinaceur, 1991], Part I, in particular Chapter IV on Lagrange and Fourier on the resolution of equations.

²Abel first published his proof in 1824, in French, at his own expense. The paper was

of Abel's work was to change the questions asked. He underlines an idea which becomes essential to the development of modern mathematics: the importance of determining conditions for the solvability of problems. With the great difficulties faced in the resolution of general equations, Abel, in a later work about algebraic resolution of equations ([Abel, 1839b]), observes that it might be impossible to obtain a solution following the path taken so far. Indeed, it can only work if the equations are actually solvable by radicals, which has never been proved to be possible:

[I]f unfortunately the resolution was impossible, one could look for it for an eternity without finding it. To infallibly obtain something on that matter, one has to take another road. One has to give to the problem a form such that it would always be possible to solve it, which can always be done for any problem. Instead of asking for a relation for which we don't know if it exists or not, it must be asked if such a relation is indeed possible.³ ([Abel, 1839b], 217)

The question addressed by Abel, then, is: "Is it possible to solve equations of higher than the fourth degree in general?" ([Abel, 1839a], 65). Abel seems to be in the continuity of Lagrange. But Abel takes a more radical turn than Lagrange did: he gives up the previous approach, which consisted in trying different methods without any assurance of their success. Instead, Abel develops more general and theoretical considerations, and proposes to work with "the general form of algebraic functions" (*ibid.*, 65).

Abel's works on algebraic equations, for the first time, turn the question around. Abel proposes that the possibility of resolution should be taken as being the problem to investigate. Slowly, then, the question to answer starts to change and becomes that of the solvability *per se*. It was also, albeit independently from Abel's works, the case of Galois's 1831 memoir, in which he announces a criterion to determine whether (any) algebraic equations are or are not solvable by radicals. Turning his back to computations and explicit formulae, he solely gives the answer to the yes or no question "Is it possible to solve a given equation?" without providing an effective procedure to actually produce the roots.

republished in longer, more developed form by Crelle, in 1826 ([Abel, 1826]) and translated in French in Férussac's journal. See [Sørensen, 2010].

³ "[S]i par malheur la résolution était impossible, on aurait pu la chercher une éternité, sans la trouver. Pour parvenir infailliblement à quelque chose dans cette matière, il faut donc prendre une autre route. On doit donner au problème une forme telle qu'il soit toujours possible de le résoudre, ce qu'on peut toujours faire d'un problème quelconque. Au lieu de demander une relation dont on ne sait pas si elle existe ou non, il faut demander si une telle relation est en effet possible."

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4.1.1 Galois's 1831 "*Mémoire sur les conditions de résolubilité des équations par radicaux*"

My presentation of Galois's memoir will stay close to the text of his memoir. I have greatly benefited from Caroline Ehrhardt's works cited above and of her very clear explanation of Galois's memoir in [Ehrhardt, 2008].⁴

Admitted at the École Préparatoire (later to be renamed École Normale Supérieure) in 1829, Galois proposes his memoir to the Académie des Sciences in January 1831. Galois's aim, in this memoir, is to give "a general condition satisfied by every equation that is solvable by radicals, and which conversely ensures their solubility"⁵ ([Galois, 1831], transl. in [Galois, 2011], 107⁶). Galois's memoir is composed of three distinct parts: the "Principles" in which he gives definitions and "a sequence of lemmas all of which are known" (*ibid.*, 109), five "propositions" leading to the necessary and sufficient condition for the resolubility of equations, and an "application to irreducible equations of prime degree". The two essential notions introduced in the "Principles" are adjunct quantities and substitutions.

Adjunct quantities are defined in the following way:

When we agree to thus regard certain quantities as known, we shall say that we adjoin them to the equation which it is to be solved. We shall say that these quantities are adjoined to the equation.⁷ (*ibid.*, transl. slightly altered, 109)

The properties of equations can be altered depending on the quantities adjoined, as Galois underlines, "[f]or example, the adjunction of a quantity can render an irreducible equation reducible," making the notion of adjunction central to the theory of equations.⁸ These properties of equations and adjoined quantities were, albeit not rigorously defined or given a systematic treatment, well-known to algebraists.

For example, in Lacroix's textbook *Éléments d'algèbre*, one of the most used algebra textbook in France in the 19th century, negative quantities are introduced as quantities revealed by algebra. The first move, when faced with equations having negative solutions, was to consider them as debts or to rectify the initial problem in order to get a positive solution – this was the

⁴One can also refer to the introduction of [Galois, 2011], and to [Ehrhardt, 2007], [Verriest, 1934], and [Dahan-Dalmedico et al., 1982].

⁵"une condition générale à laquelle satisfait toute équation soluble par radicaux, et qui réciproquement assure leur résolubilité"

⁶References will be to [Galois, 2011].

⁷"Lorsque nous conviendrons de regarder ainsi comme connues de certaines quantités, nous dirons que nous les adjoignons à l'équation qu'il s'agit de résoudre. Nous dirons que ces quantités sont adjointes à l'équation."

⁸For example, the equation $x^2 = 2$ is irreducible in the rational numbers and its roots cannot be written using the coefficients of the equation. However, if one adjoins the quantity $\sqrt{2}$ defined by $(\sqrt{2})^2 = 2$, then this equation can be written $(x - \sqrt{2})(x + \sqrt{2}) = 0$ and its roots can be written as a combination of the coefficients and of the quantity $\sqrt{2}$.

strategy adopted by many algebraists since the 16th century. Algebra allows to understand where the negative quantities come from: they are solving certain problems, they satisfy certain equations. Insofar as any problem whose solutions are negative can be algebraically manipulated to give a positive solution, negative quantities, Lacroix's textbook, are characterized in the following way:

[W]e commonly say that the values affected by the sign $-$, and which are called *negative solutions*, solve, in a sense opposed to the terms [of the problem], the questions they meet. It follows that we can consider as forming, properly speaking, only one question, that whose problem statements are linked together in such a way that the solutions satisfying one statement can by only a change of sign, satisfy another.⁹ ([Lacroix, 1799], 91, my translation)

Negative quantities appear to be necessary in order to be able to solve all equations of first degree, one should be able to manipulate them as easily as positive quantities,

[s]ince the negative quantities solve, in a certain way, the problems which generate them, it is appropriate to examine closer the use of this quantities and, first, to assure ourselves of the way in which we should carry out the indicated operations.¹⁰ (*ibid.*, 91)

Negative quantities are, then, joined to positive quantities: they are developed so as to satisfy the same arithmetical rules and are comprised in the algebraic computations, even though they are still considered to be impossible solutions. The same idea is used for complex or imaginary quantities (which are not quantities, for Lacroix, but mere symbols). Galois, in his memoir, uses a well-known idea, then. It appears, however, that adjunction has a different role and status for Galois: to place it at the beginning of the theory implies that it holds an essential importance in the further developments of the theory. The adjunction of quantities is not, for Galois, a move made in the course of solving equations, but a basic notion of the theory. As Galois mentions, adjunction is intrinsically linked to the nature of equations, insofar as it modifies an essential characteristic of the equation:

⁹ “[O]n dit communément que les valeurs affectées du signe $-$ et qu'on appelle solutions négatives, résolvent, dans un sens opposé à son énoncé, les questions où elles se rencontrent. Il suit de là qu'on peut regarder comme ne formant, à proprement parler, qu'une seule question, celles dont les énoncés sont liés entre eux de manière que les solutions qui satisfont à l'un des énoncés peuvent par un simple changement de signe, satisfaire à l'autre.”

¹⁰ “Puisque les quantités négatives résolvent, dans un certain sens, les problèmes qui leur donnent naissance, il est à propos d'examiner de plus près l'usage de ces quantités, et d'abord, de s'assurer de la manière dont il convient d'effectuer les opérations indiquées à leur égard.”

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its reducibility.

The second basic notion of Galois's memoir are substitutions. A permutation of n letters is an ordered list of those letters, for example (a, b, c) and (c, a, b) are permutations of 3 letters. Substitutions are defined as "the passage from one permutation to another." Permutations were introduced by Lagrange and further studied by Cauchy, whose definition Galois is using.¹¹ However, Galois does not use Cauchy's notation. As underlined by Ehrhardt ([Ehrhardt, 2008], 4), by the time Galois was writing, these notions were only sketched by Lagrange and Cauchy, and Galois was the only mathematician, with Abel, to use them. Only in 1844 would Cauchy give a more thorough treatment of these ideas. Galois, who does not master completely these new tools, often uses the words "permutation" and "substitution" one for another, which can confuse his reader. The notion of group is first introduced here as "group of permutations":

When we will want to group some substitutions, we will make them originate in one and the same permutation. As the concern is always with questions where the original disposition of the letters has no influence, in the groups we will consider, one must have the same substitutions, whichever permutation it is from which one starts. Therefore, if in such a group, we have the substitutions S and T , we are sure to have the substitution ST .¹² ([Galois, 1831], transl. slightly altered, 115)

The notion of "group", then, is not introduced as a set closed by a binary operation by Galois. It is better characterized, here, as a set organized in such a way that it can be written in a table. Indeed, although the above quote refers to the group of the equation as understood in the modern sense,

¹¹Cauchy's definition reads: "*Si, au-dessous de la permutation relative à [une certaine fonction] K , on écrit une autre permutation formée avec les indices $1, 2, 3, \dots, n$, et que l'on remplace successivement dans la fonction K chacun des indices qui composent la permutation supérieure par l'indice correspondant de la permutation inférieure, on aura une nouvelle valeur de K (...). Pour déduire deux de ces valeurs l'une de l'autre, il suffira de former les permutations relatives à ces deux valeurs et de substituer aux indices de la première permutation les indices correspondants pris dans la seconde. Pour indiquer cette substitution, j'écrirai les deux permutations entre parenthèses en plaçant la première au-dessus de la seconde; ainsi, par exemple, la substitution*

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 3 & 1 \end{pmatrix}''$$

([Cauchy, 1815], 67)

¹²"*Quand nous voudrions grouper des substitutions nous les ferons toutes provenir d'une même permutation. Comme il s'agit toujours de questions où la disposition primitive des lettres n'influe en rien, dans les groupes que nous considérerons, on devra avoir les mêmes substitutions quelle que soit la permutation d'où l'on sera parti. Donc si dans un pareil groupe on a les substitutions S et T , on est sûr d'avoir la substitution ST .*"

what Galois calls the “group of permutations”, for the roots of $\frac{x^n-1}{x-1}$ is the following matrix describing the permutations of the roots and which is not a group in the modern sense of the term:

$$\begin{array}{l}abcd\dots k \\ bcd\dots ka \\ \dots\dots\dots \\ kabc\dots i\end{array}$$

Galois explains, in the letter to Auguste Chevalier, that the aim of his work, in the theory of equations, is to identify “the circumstances under which equations [are] solvable by radicals”, and by this research, he was led “to deepen this theory and to describe all possible transformations on an equation even in case it is not solvable by radicals” (Letter to Auguste Chevalier, 29 May 1832, transl. in [Galois, 2011], 85).

The first proposition of Galois’s memoir introduces the group of the equation:

Let an equation be given of which the m roots are a, b, c, \dots . There will always be a group of permutations of the letters a, b, c, \dots which will enjoy the following property:

1. That every function of the roots invariant under the substitutions of this group will be rationally known;
2. Conversely, that every function of the roots that is rationally determinable will be invariant under the substitutions.¹³ (*op. cit.*, 113-115)

Galois’s proof for this theorem is to exhibit the group in question and to satisfy the two statements made in the theorem. Note that, once again, Galois’s handling of the concept of group is very different from the modern uses, and in particular his demonstration does not aim to prove that the “group” in question satisfies the closure properties of the modern definition of a group. The two next theorems explain the link between groups and adjunct quantities: to adjoin a root to a given equation leads to the partition of the initial group into several smaller groups,¹⁴ which “enjoy the remarkable property that one will pass from one to another by operating on all the permutations of the first with one and the same substitution of letters” (*ibid.*, 119).

¹³ “Soit une équation donnée dont a, b, c, \dots sont les m racines. Il y aura toujours un groupe de permutations des lettres a, b, c, \dots qui jouira de la propriété suivante: 1) Que toute fonction des racines invariable par les substitutions de ce groupe soit rationnellement connue; 2) réciproquement, que toute fonction des racines déterminable rationnellement soit invariable par les substitutions.”

¹⁴These smaller groups are, of course, subgroups of the group of the equation.

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Galois develops further this point by showing more precisely the dependence between the group and the adjunct quantities and explaining how the number of elements of the group decreases with each adjunction.

If one adjoins to an equation the numerical value of a certain function of its roots, the group of the equation will be reduced in such a way as to not have any other permutations than those under which this function is invariant.¹⁵ (*ibid.*, 121)

Galois can finally give the necessary and sufficient condition for an equation to be solvable by radicals. Galois adopts a very peculiar approach, since he does not state the result as such, but presents it as an algorithm, so as to get to the result:

[T]o solve an equation, it is necessary to reduce its group successively to the point where it does not contain more than a single permutation.¹⁶ (*ibid.*, 121)

To be clear, Galois is not proving the result with an algorithm, but he presents it under a what Ehrhardt calls a “narrative” form. By doing so, he unfolds the links between adjunction and size of the group. One can understand, by following each steps of the reasoning deployed by Galois, why the result is the one obtained – and even, to some extent, why some equations can be solved by radicals, and some cannot. But Galois does not explain how one should proceed to answer the question of the resolubility of a particular given equation. He does, however, give a reformulation of his result in the language of the theory of equations, replacing it in the more traditional practices of algebra by explaining that it is necessary and sufficient with if any two roots of the equations are known to be able “deduce rationally” the other roots from them.¹⁷

The importance of the notion of group in later developments of algebra and algebraic number theory leads, as underlined by Ehrhardt, to “interpret” Galois’s memoir as centered on the notion of group rather than equations and to overemphasize its importance and its modernity (see for example [Verriest, 1934]). Groups do not take the central stage in Galois’s works, and his notion of group has, in fact, little in common with the modern notion of group. It is even significantly different from the notion of group

¹⁵ “Si l’on adjoint à une équation la valeur numérique d’une certaine fonction de ses racines, le groupe de l’équation s’abaissera de manière à n’avoir plus d’autres permutations que celles par lesquelles cette fonction est invariable.”

¹⁶ “[P]our résoudre une équation, il faut successivement abaisser son groupe jusqu’à ne contenir plus qu’une seule permutation.”

¹⁷ For example, for the equation of third degree $x^3 + px^2 + qx + r = 0$, we know that for x_1, x_2 and x_3 its roots, we have $x_1 + x_2 + x_3 = -p$; $x_1x_2 + x_1x_3 + x_2x_3 = q$; $x_1x_2x_3 = -r$ hence, each root can be expressed rationally by means of the two other ones.

which will be developed by the first mathematicians to pursue his works, Cayley and Dedekind. To be clear, the idea of a group as a system closed by multiplication does appear in Galois's memoir. Nonetheless, this is not the characterization that Galois retains as a definitional property for groups. Galois explains what a group is, and proves certain results, using a tabular presentation, so that it is possible to visualize the object and the internal organization of its elements – a strategy already used, for example, by Lagrange. Besides, Galois does not consider a general notion of group. In his works, only groups of substitutions appear, although it seems that Galois had formed the idea of a group as a mathematical object in itself. This can be found in the letter to Chevalier, in which Galois uses a single letter G to designate the group, which leads to a very different way of using the notion itself:

when a group G contains another H , the group G can be partitioned into groups each of which is obtained by operating on the permutations of H with one and the same substitution, so that $G = H + HS + HS' + \dots$ ¹⁸ ([Galois, 2011], 85)

This last point is important insofar as to consider the group as an object in itself, rather than a mere collection of objects, dismisses the need to attach a definite meaning to the elements of the group – which is the approach Dedekind will adopt in 1856. In the proposition given by Galois, the notion of group seems to be detached from the equation for which it was defined, at least inasmuch as results can be given independently of the said equation. However, it should be underlined that the decomposition here corresponds to that given by the adjunction. Indeed, it is a reformulation of the theorems proved in the memoir, and it is also immediately linked to the equation:

It is easy to see that when the group of an equation is not susceptible of any proper decomposition one may transform the equation at will, and the groups of the transformed equations will always have the same number of permutations.¹⁹ (*ibid.*, 85)

In the successive readings of Galois's memoir during the 19th century, the notion of group is reinterpreted and reshaped. In particular, Dedekind defines the notion of group as a set of elements satisfying a property of closure, and as being part of an auxiliary theory susceptible to be applied to other areas of mathematics (and indeed, he immediately uses Galois's ideas in number theory). Dedekind considers groups of permutations because they

¹⁸ “quand un groupe G en contient un autre H le groupe G peut se partager en groupes que l'on obtient chacun en opérant sur les permutations de H une même substitution, en sorte $G = H + HS + HS' + \dots$ ”

¹⁹ “Il est aisé de voir que quand le groupe d'une équation n'est susceptible d'aucune décomposition propre, on aura beau transformer cette équation, les groupes des équations transformées auront toujours le même nombre de permutations.”

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are the ones useful in the theory of equations, but underlines explicitly that it is for the purpose of this particular work, and that groups (in a more general way) can be used in many other theories – a statement and he will repeatedly make in later works.

Galois's memoir is concise to the point of having been considered incomplete and unintelligible by his contemporaries. Galois's manuscripts show that he shortened his memoir on purpose, masking computations that he did not consider as essential for the overall consistency and understanding of his work. In fact, he explains, in a preface which wasn't published until 1962 (in [Galois, 1962]), that computations are evolving into details on which the mind does not have time to stop and that it becomes necessary, with the developments of mathematics, to be able to work without computations:

But then, I believe that the simplifications produced by elegance of calculations (intellectual simplifications, of course; there are no material ones) have their limits; I believe that the time will come when the algebraic transformations foreseen by the speculations of analysts will find neither the time nor the place for their realisation; at which point one will have to be content with having foreseen them.²⁰ (*ibid.*, transl. slightly altered, 253)

These statements were far from unusual for Galois, as we saw in the beginning of this dissertation. But in 1831, when Galois submitted his memoir to the Académie des Sciences, it was considered unreadable and was rejected.

In 1831 when Galois submitted his memoir to the Académie des Sciences, it was rejected because it was considered unclear, if not incomprehensible:

We have made every effort to understand Mr Galois' proof. His reasoning is neither clear enough nor well enough developed for us to have been able to judge its exactness, and we are in no position to give an idea of it in this report. (...) It is often the case that the different parts of a theory, by mutually clarifying each other, are easier to grasp together than in isolation. One may therefore wait until the author will have published his work in its entirety before forming a final opinion; but given the present state of the part that he has submitted to the Academy, we cannot propose to you that you give it your approval.²¹ (Lacroix and

²⁰ "Or je crois que les simplifications produites par l'élégance des calculs, (simplifications intellectuelles, s'entend; de matérielles il n'y en a pas) ont leurs limites; je crois que le moment arrivera où les transformations algébriques prévues par les spéculations des Analystes ne trouveront plus ni le tems ni la place de se produire; à tel point qu'il faudra se contenter de les avoir prévues."

²¹ "Nous avons fait tous nos efforts pour comprendre la démonstration de M. Galois. Ses

Poisson's report on Galois's paper, in *Procès verbaux des séances de l'Académie*, vol. IX, transl. in [Galois, 2011], 148-149)

As Ehrhardt highlighted in [Ehrhardt, 2012], the memoir's unclarity is not the sole reason for its refusal and the lack of reception of Galois's works for the next two decades. In fact, certain aspects Galois's memoir do not meet the norms and standards governing mathematical practice in 1830. Indeed, the theory of equations, and more generally algebra was still considered as a problem-solving part of mathematics: the theory of equations was not seen as an independent theory, but a theory built for applications. Hence, algebra was supposed to provide effective methods allowing to obtain explicit results, priority was given to being able to provide numerical solutions of equations. Even Lagrange, who was the most attached to an algebraic ideal, despite his methodological ideals, turned towards numerical solutions. Mathematicians were expecting to find different results in the memoir, as Lacroix and Poisson's report states clearly:

It must be noted however that it does not contain, as the title of the memoir promised, the condition for solvability of equations by radicals; for, even accepting Mr Galois' proposition as true, one is hardly further forward in knowing whether or not an equation of prime degree is solvable by radicals, because it would first be necessary to convince oneself whether the equation is irreducible, and then whether any one of its roots may be expressed as a rational function of two others.²² (*op. cit.*, 148)

The "condition for solvability" would be considered by Lacroix and Poisson as being valid if and only if it were provided with an explicit formula. It should have "an external character", which can be verified "by inspection of the coefficients of a given equation" or using the resolvent method. From their viewpoint, Galois is not answering the question – in fact, he is not asking the right question, insofar as a theoretical possibility to solve the equation appears to be far less interesting than an actual, numerical resolution.

raisonnements ne sont ni assez clairs, ni assez développés pour que nous ayons pu juger de leur exactitude et nous ne serions pas en état d'en donner une idée dans ce Rapport. (...) Souvent il arrive que les différentes parties d'une théorie, en s'éclairant mutuellement sont plus faciles à saisir dans leur ensemble qu'isolément. On peut donc attendre que l'auteur ait publié en entier son travail pour se former une opinion définitive ; mais dans l'état où est maintenant la partie qu'il a soumise à l'Académie, nous ne pouvons pas vous proposer d'y donner votre approbation."

²² "Toutefois on doit remarquer qu'il ne renferme pas, comme le titre du Mémoire le promettait, la condition de résolubilité des équations par radicaux ; car en admettant comme vraie la proposition de M. Galois, on n'en serait guère plus avancé pour savoir si une équation donnée dont le degré est un nombre premier est résolue ou non par radicaux, puisqu'il faudrait d'abord s'assurer si cette équation est irréductible, et ensuite si l'une de ces racines peut s'exprimer en fonction rationnelle des deux autres."

4.2 Dedekind's Galois theory

Joseph Liouville announces, on September 4th 1843 at the Académie des Sciences, his intention to publish a commentary of Galois's memoir. Ehrhardt's analysis of the social and mathematical context in which Galois's memoir was published shows that the publication of Galois's memoir is embedded in the evolution of mathematical practices and how the incomprehension faced by Galois was linked to the context and conditions in which his work was read.²³ The evolutions of mathematics and the changes in practices which start to appear around 1850 open the door to a better understanding and possible uses, developments and re-writings of Galois's ideas.²⁴ It was, in any case, a crucial element for the circulation of Galois's ideas that Liouville's reception of Galois resulted in the publication of Galois's memoir. Indeed, a publication gives the memoir an actual material existence allowing for a wider circulation of his ideas, notably in England and in Germany, where Galois's memoir is integrated into different mathematical landscapes: Cayley is interested in a formal notion of group as a fundamental algebraic concept, whereas Dedekind and Kronecker focus on the notion of adjunction and import Galois's ideas in number theory.

4.2 Dedekind's Galois theory

The historical elements presented above are important for us because Dedekind did read French algebraic works. Not only did he read Galois's memoir, he also mentions Serret's textbook in [Dedekind, 1857]. Moreover, Dirichlet was in correspondence with Liouville, and stayed close to the French mathematical community, after his stay in Paris between 1822 and 1826. Dirichlet, who taught in Berlin between 1826 and 1855, suggested the study of Galois's papers to Kronecker. A few years later, in 1856, Dirichlet suggested the same thing to Dedekind.

For Ehrhardt, Dedekind's reception and interpretation of Galois's notion of group²⁵ is a typical example of the tendency to “substitute thinking

²³“*L'intelligibilité d'un texte n'est pas une donnée intrinsèque : quand Poisson et Lacroix écrivent que le mémoire de Galois n'est ni utile ni compréhensible, ils le font selon des critères fondés sur l'état de l'algèbre à leur époque, ou encore sur le type de méthodes qu'ils privilégient et auxquels ils sont habitués...*” ([Ehrhardt, 2012], 28)

²⁴See [Ehrhardt, 2011b], [Ehrhardt, 2012]. In addition, Jenny Boucard's PhD thesis ([Boucard, 2011]) shows that readings of Galois, such as Jordan's in the 1860s, are made in the aftermath of a theory called “theory of order” by Louis Poincaré, a general theory growing from Poincaré's works on algebra and number theory, interested in “the situation of things without any consideration of magnitude” ([Boucard, 2011], 237) and introduced as being to algebra what Gauss's higher arithmetic was to arithmetic.

²⁵I will, here, only consider Dedekind's lectures on Galois theory in 1856-58 and will not consider Dedekind's other works on groups. This is in particular because the aim of this chapter is to highlight evolutions in Dedekind's methods and Dedekind's works on groups (notably with the notion of group character) developed later and flourished in the 1890s

for computations”, brought to the fore in Dirichlet’s obituary of Jacobi. Ehrhardt pays close attention to the sociological context in which Galois’s ideas were received and developed and how it influenced the understanding of his works, and for her, it was Dedekind’s immersion in the so-called “conceptual approach” that led to his peculiar treatment of Galois’s ideas. Even if it was “not necessarily dominant in mathematics at that time”, Dedekind might have been inspired by the “conceptual approach”.²⁶ More than the “conceptual” approach, it seems that Dedekind’s handling of Galois’s ideas is grounded in his interest for number theory awoken by Dirichlet’s arrival. Indeed, an important aspect of Dedekind’s treatment of Galois theory is its insertion into number theory – a point Dedekind has in common with Kronecker. Number theory was one of the most important areas of interest for mathematicians in Germany, during the 19th century, since Gauss’s *Disquisitiones Arithmeticae*.²⁷

One of the most important specificities of Dedekind’s treatment of groups is to consider groups as “wholes”, as objects in their own right, a method he will use systematically for concepts such as groups, ideals, and modules. In a famous letter from 10 June 1876, Dedekind clarifies a certain number of difficulties encountered by Lipschitz which, for a large part, are related to Dedekind’s (method of) definitions. He explains that the use of infinite systems is to be related to Gauss’s approach (see the quotation p. 44).

Dedekind’s work on Galois theory is important because it is his first known work in algebra and number theory, and because Dedekind proposes here, for the first time, the use of arithmetical operations to designate set-theoretic relationships, in particular inclusion.

There is, in Dedekind’s *Nachlass*, a copy of Galois’s works, attesting that Dedekind read it. In addition, in his paper “*Bemerkungen zu einer Aufgabe der Wahrscheinlichkeitsrechnung*” (published in Crelle’s journal in 1855, and reproduced in [Dedekind, 1932] I, 36-39), Dedekind quotes a paper written by Boole and published in the same issue of the *Philosophical Magazine* that contains Cayley’s first paper on groups. It is, then, more than likely that

and of which Frobenius said that they “successfully brighten the African darkness of the theory of groups” (see [Hawkins, 1978] and [Hawkins, 2013]).

²⁶“Indeed, because he still considered himself a student when he started to work on Galois’s memoir, Dedekind attended Dirichlet’s lecture about number theory as well as Riemann’s seminar. (...) In a way, then, the mathematical practice that Dedekind used while teaching Galois’s ideas reproduced a specific way to make mathematics that must have seemed familiar, natural and legitimate to him. This was owed to the specific social and cultural environment in which he did his work, but also to the research training he had received from his two mentors. Just like Cayley’s, Dedekind’s interpretation of Galois’s work bore the mark of the place where and the time when it was made, as well as those of their author’s own trajectory.” ([Ehrhardt, Forthcoming])

²⁷See [Goldstein and Schappacher, 2007a].

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Dedekind read Cayley's paper, having had the journal in his hands. In his lecture, Dedekind will select elements of Galois's and Cayley's works which are fully integrated in his own practice, and propose a treatment of Galois theory as a part of higher algebra, using a notion of group closest to Cayley's and linking the theory to number theory.

4.2.1 Cayley's Galois theory

In 1854, Arthur Cayley, student and later professor in Cambridge, published a series of papers entitled “*On the theory of groups as depending on the symbolical equation $\theta^n = 1$* ” ([Cayley, 1854]), in which he presents the notion of group, which he attributes to Galois. Cayley's papers are the first to develop a theoretical viewpoint on the notion of group introduced by Galois. However, there are strong differences between Galois's and Cayley's notions of group, related to the mathematical and cultural context in which they both worked, which entail noticeable differences in their mathematical education and their conception of mathematics.

In the 19th century, mathematics in Cambridge was dominated by a certain practice of algebra, the so-called Cambridge tradition, which started around 1810. The core idea of the approach in algebra developed in Cambridge is that concrete objects, such as numbers, should not be considered to be the main objects in mathematics. This status should be given rather to the symbolical operations that can be defined between the said concrete objects. Algebra, in Cambridge, is the study of the laws of symbolical calculus, and not the theory of equations, as it was in France and in Germany.²⁸

Cayley's treatment of the notion of group follows the main ideas and guidelines of the Cambridge tradition. From Galois, he only takes the notion of group. But even there, they share neither the definition, nor the way of writing or representing a group. Even the uses of the notion of group differ. While Cayley explicitly refers to Galois's memoir, he does not work in the same framework, nor does he use the same symbolism.

Cayley considers a “symbol of operation” θ (more generally, the Greek letters are used to designate operations), which acts on a system of quantities

²⁸See [Ehrhardt, 2012] for more on the context in which Cayley received Galois's works. The “tradition” in Cambridge is, for example, explained by Augustus de Morgan (1806-1871) as following this idea:

When we wish to give the idea of symbolical algebra, (...) we ask, firstly, what symbols shall be used (without any reference to meaning); next, what shall be the laws under which such symbols are to be operated upon; the deduction of all subsequent consequences is again an application of common logic. Lastly, we explain the meanings which must be attached to the symbols, in order that they may have prototypes of which the assigned laws of operation are true. ([Morgan, 1840], 133-134)

" (x, y, \dots) such that:

$$\theta(x, y, \dots) = (x', y', \dots)$$

where (x', y', \dots) are any functions whatever of (x, y, \dots) " ([Cayley, 1854], 123). First, the basic rules of the symbolic calculus are given:

- It is not necessary for the systems (x, y, \dots) and (x', y', \dots) to have the same number of elements.
- It is not necessary (but it can be done) to attach any meaning to operations such as $\theta \pm \psi$, nor to the symbol 0, and nor, consequently, to equations of the type $\theta = 0$ or $\theta \pm \psi = 0$.
- The symbol 1 "denote[s] an operation which (either generally or in regard to the particular operand) leaves the operand unaltered" (*ibid.*).
- The equation $\theta = \psi$ means that θ and ψ are equivalent operations.
- The composite operation, denoted $\theta\psi$ is, in general, not commutative ("convertible"); it is however associative: " $\theta\psi\chi$, $\theta\psi\chi\omega$, etc., have a definite signification independent of the particular mode of compounding the symbols" (*ibid.*). In consequence, the "distributive law has no application to the symbols $\theta\psi$ " (*ibid.*).

Particular cases can be considered, for example if the transformation of (x, y, \dots) into (x', y', \dots) is a permutation of the quantities, then θ denotes a permutation.

The notion of group is defined in the following way:

A set of symbols

$$1, \alpha, \beta, \dots$$

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*. (*ibid.*, 124)

In a footnote, Cayley mentions that the "idea of a group as applied to permutations or substitutions" was introduced by Galois and that it "may be considered as marking an epoch in the progress of the theory of algebraical equations" (*ibid.*). It is the only mention of Galois by Cayley.

Having defined the notion of group, Cayley explains that if symbols are multiplied by each other, "the effect is simply to reproduce the group", and proposes a representation in a table:

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	1	α	β	...
1	1	α	β	...
α	α	α^2	$\alpha\beta$	
β	β	$\alpha\beta$	β^2	
	...			

With this representation, each line and each column will contain all the elements of the group. This allows to see that any product of any number of elements of the group is still in the group. In particular, if one considers the symbolic equation $\theta^n = 1$, it can be shown that the elements of the group $1, \alpha, \beta, \dots$ of cardinality n satisfy the equation “so that the group may be considered as representing a system of roots of this symbolic binomial equation” (*ibid.*, 125). Indeed, Gauss showed that if n is a prime number, then the solutions of the equation will be of the form:

$$1, \alpha, \alpha^2, \dots, \alpha^{n-1}, (\alpha^n = 1)$$

The roots then form a cyclic group. If n is not prime, the group can be of that form, but not necessarily. Cayley proposes to consider the case for $n = 4$ which allows to understand how the group for the symbolic equation $\theta^n = 1$ can be different from the group of the algebraic equation $x^n = 1$. For that, he examines the various possibilities of definitions for the 4 elements of the group $1, \alpha, \beta, \gamma$. Cayley gives, for each possibility, the representation of the group in a table, for example. The first group considered for the equation $\theta^4 = 1$, which is equivalent to $1, \alpha, \alpha^2, \alpha^3, (\alpha^4 = 1)$ is represented in this square:

	1	α	β	γ
1	1	α	β	γ
α	α	β	γ	1
β	β	γ	1	α
γ	γ	1	α	β

This representation of groups in tables is specific to Cayley and reminiscent of Hamilton's works on quaternions or de Morgan's formal logic, and it will not be used by Dedekind.

Cayley provides other examples of the possible uses of the notion of groups, such as elliptic functions or the equation $\theta^6 = 1$ and seems to be trying to draw links between very different areas of mathematics, by showing that they function by obeying the same underlying laws – the laws for groups. This approach is frequent in Cayley's works, and he seems to see, in the notion of group, a new objects susceptible to answer this kind of questions. However, Cayley did not develop very much his researches on that topic, and, in fact, did not use groups in other works, not even in papers on the theory of algebraic equations, even though, as we saw, he explicitly

draw the link between groups and algebraic equations.²⁹

Cayley seems to be considering the notion of group as a sort of generic concept, a concept which could allow for a unification of a certain number of problems under the same general principles. For Cayley, this is embedded in attempts to find the “true reasons” hidden by the specificities of problem. Dedekind adopts an approach to algebraic concepts (groups, but also modules and ideals) which considers them as “independent”, thus concepts forming auxiliary theories which can be applied to specific situations: typically, groups, defined as systems of elements satisfying a property of closure, and in which the individual nature of the elements does not enter in consideration in any way. These auxiliary theories, to a certain extent, give the possibility to develop a unified method for a certain class of problems. While Dedekind does not adopt a symbolical viewpoint, one should not discard the possibility that his reading of Cayley's works on groups had an impact on the conception of his algebraic concepts, insofar as Cayley worked with arbitrary elements. However, Cayley's investigations on groups are essentially based on the determination of the elements of the group considered, which are identified by the examination of the constraints following from the initial definition, as was done for the equation $\theta^4 = 1$. While this approach allows, because of its perfectly symbolical nature, a general description of certain finite groups, it seems difficult to generalize it, that is to develop it for groups with any number of elements.

Dedekind adopts Cayley's viewpoint as regards the indifference towards the individual nature of the elements of the groups, however his intention is to develop an approach as general as possible for the theory of algebraic equations and number theory. In particular, the theory of groups should be able to give propositions and develop methods using groups without relying in any essential way on knowing specifically the number of elements of the group. For this, the standpoint adopted by Dedekind will be to consider the group as an object in itself, to push the indifference towards the nature of the elements up to considering the system, as a whole, as the object of the investigation in its own right.

4.2.2 Dedekind's 1856-58 “*Eine Vorlesung über Algebra*”

In 1853, Leopold Kronecker published *Über die algebraisch auflösbaren Gleichungen* ([Kronecker, 1853]), in which he refers to Galois's memoir on the resolubility of polynomial equations, thus being the very first mathematician in Germany to show interest in Galois's works. He was, however, very critical of both Galois's and Abel's works on the resolubility of equations.

²⁹From this viewpoint, his approach is strikingly different from Dedekind's who used his algebraic concepts as systematically as he could.

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From his viewpoint, Galois's memoir was "hiding the true nature of solvable equations". Kronecker's aim was to find a way to determine all the solvable equations and give an explicit expression of their roots – an approach, then, opposed to Galois's. Kronecker makes almost no use of the notion of adjunction and does not use the notion of groups at all. Kronecker considered the appropriate approach to be the one that would provide effective solutions to the equations, and his reference to Galois is essentially to point out that Galois chose the wrong approach to the problem.

A few years later, Dedekind, in 1856-58, became the first to teach Galois theory in Germany, during the winter semesters of 1856-57 and 1857-58, when he was *Privatdozent* in Göttingen. Only two students followed Dedekind's course, Paul Bachmann and Hans Zincke. The only account of this class is a manuscript written by Dedekind himself, and entitled "*Eine Vorlesung über Algebra*". The manuscript was reproduced by W. Scharlau (in [Scharlau, 1981a], 59-100)³⁰ and is not dated. As Scharlau explains, one can see that it is an *a posteriori* elaboration on the content of the lectures given during the winter semesters of 1856-57 and 1857-58. Dedekind's use of the word "*rationale Gebiet*" to designate the field of numbers indicates that the manuscript was written before the 1871 Xth Supplement.³¹

The paper is divided in four sections: §I Elements of the theory of substitutions; §II On the interdependency of rational functions of indeterminate variables; §III On the algebraic kinship between numbers; §IV About the resolution of equations by radicals.³²

In the first paragraph, Dedekind gives the bases of the theories of substitutions and groups. Substitutions are defined as a "process through which certain elements a, b, c, \dots are turned into other a', b', c', \dots or a replaced by them." Dedekind indicates that, here, only substitutions for which the initial and final sets of elements are the same are considered, which suggests that he had in mind a more general notion closer to mappings.³³ In this case, only substitutions of one set into itself are of interest. Dedekind represents

³⁰Scharlau added titles to each articles, in Dedekind's paper, using modern terminology (e.g., normal subgroup, splitting field, ...). I will not be using these titles. Translations of this text are mine. References will be to Scharlau's edition.

³¹This point is mentioned by Dedekind himself, in the 1894 reedition of Dirichlet's *Vorlesungen*, referring to the notion of field, he wrote; "At first, in my Gottingen lectures (1857 to 1858), I had used the name "rational domain" (*rationales Gebiet*) for this same concept."

³²This paper itself has been little studied. In addition to Ehrhardt's works, one can find comments in [Corry, 2004b] and in [Scharlau, 1981b]. See also [Sieg and Schlimm, 2014].

³³Here, to put it in modern terms, a substitution is an isomorphism of a set into itself, notion which Dedekind will often use in his works in number theory. However, it seems doubtful that Dedekind's permutations, here, are the general concept of mapping as he developed it later, but with another name. See p. 366. See also [Sieg and Schlimm, 2014].

them as Cauchy did:

$$\begin{pmatrix} a & b & c & \dots \\ a' & b' & c' & \dots \end{pmatrix}.$$

This substitution can also be denoted by the single letter θ . The product of two substitutions is the composition of these two substitutions:

$$\theta\theta' = \begin{pmatrix} a & b & c & \dots \\ a' & b' & c' & \dots \end{pmatrix} \begin{pmatrix} a' & b' & c' & \dots \\ a'' & b'' & c'' & \dots \end{pmatrix} = \begin{pmatrix} a & b & c & \dots \\ a'' & b'' & c'' & \dots \end{pmatrix}$$

This allows to prove that the product for any substitution satisfies associativity and that if $\varphi = \theta$ and $\varphi' = \theta'$, then $\varphi\varphi' = \theta\theta'$. This has the important consequence that the number of substitutions is finite. Dedekind exposes his viewpoint on the considerations developed here:

The following investigations are based only on the two fundamental results proven above [associativity, right- and left-cancelability] and on the assumption that the number of substitutions is finite: The results are therefore valid for any finite domain of elements, things, concepts $\theta, \theta', \theta'', \dots$ admitting an arbitrarily defined composition $\theta\theta'$, for any two given elements θ, θ' , such that $\theta\theta'$ is itself a member of the domain, and such that this composition satisfies the laws expressed in the two main results. In many parts of mathematics, and especially in the theory of numbers and in algebra, one often finds examples of this theory; the same methods of proof are valid here as there.³⁴ (*ibid.*, 63, transl. in [Corry, 2004b], 78, transl. slightly altered)

The generality of Dedekind's investigations, here, is manifest. Although he always restricts his results to substitutions, the above statement seems to express his awareness that the nature of the elements of the group is indifferent, as it has been noticed by Ferreirós ([Ferreirós, 2008]). For the development of the theory, only their satisfying certain formal properties is essential. In fact, as it will appear clearly, only the properties as elements of a collection are significant, the individual properties are neglected.

Having established this peculiarity of the substitutions leads to more results on their product: $\theta^r\theta^s = \theta^s\theta^r = \theta^{r+s}$ and there exists a smaller integer n such that $\theta^{r+n} = \theta^n\theta^r = \theta^r \Rightarrow \theta^n = 1$, this n is called the order (*Ordnung*) of θ . From there, one can deduce that

³⁴ "Die nun folgenden Untersuchungen beruhen lediglich auf den beiden so eben bewiesenen Fundamentalsätzen und darauf, dass die Anzahl der Substitutionen eine endliche ist: Die Resultate derselben werden deshalb genau ebenso für ein Gebiet von einer endlichen Anzahl von Elementen, Dingen, Begriffen $\theta, \theta', \theta'', \dots$ gelten, die eine irgendwie definierte Composition $\theta\theta'$ aus θ, θ' zulassen, in der Weise, dass $\theta\theta'$ wieder ein Glied dieses Gebietes ist, und dass diese Art der Composition den Gesetzen gehorcht, welche in den beiden Fundamentalsätzen ausgesprochen sind. In vielen Theilen der Mathematik, namentlich aber in der Zahlentheorie und Algebra finden sich fortwährend Beispiele zu dieser Theorie; dieselben Methoden der Beweise gelten hier wie dort."

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The n substitutions $1, \theta, \theta^2, \dots, \theta^{n-1}$ are all different from one another. If $\theta^r = \theta^s$, then $r \equiv s \pmod{n}$, and reciprocally.³⁵ (*ibid.*, 63)

This can be shown to be true for negative powers as well, and one remarks that $\theta^{-1} = \theta^{n-1}$, the inverse (*reciproke*) substitution of θ .

Groups are, from Dedekind's viewpoint, the "most important concept of the theory of substitutions" (but not of Galois theory), and a group of substitutions is defined in the following way:

A complex³⁶ G of g different substitutions is called a *group* of degree g , when every arbitrary product of substitutions all contained in G , is again contained in G .³⁷ (*ibid.*, 64)

Dedekind's notion of group, like Cayley's, is a system (in this work, Dedekind only uses the word "*Complex*", and not "*System*", "*Inbegriff*" or "*Mannigfaltigkeit*") closed by a binary operation and which does not take into account the individual nature of its elements: the property of closure is the definitional property.

If one considers a group K of k distinct substitutions contained in a group G of g substitutions, then it is clear that g is divisible by k . Consequently

The group K shall be called a *divisor* of the group G .³⁸ (*ibid.*, 65)

Thus, to study relationships between groups, Dedekind introduces a notion of divisibility for groups, an operation between groups considered as objects of their own right rather than aggregates of things. This allows for an arithmetical reinterpretation of set-theoretic relationships.

Because of the analogy with number theory that he is building, Dedekind can employ methods of proof developed in number theory, such as the exhaustion method, which allows to show that if K is contained in G , then G can be decomposed:³⁹

$$G = K + K\theta_1 + K\theta_2 + \dots + K\theta_{h-1}$$

with $g = kh$. Dedekind can also carry on the analogy and give arithmetical results which are, in fact, describing inclusions between groups of substitutions:

³⁵ "Die n Substitutionen $1, \theta, \theta^2, \dots, \theta^{n-1}$ sind sammtlich von einander verschieden; ist $\theta^r = \theta^s$, so ist $r \equiv s \pmod{n}$, und umgekehrt."

³⁶ To say that A is a complex of substitutions $\theta, \theta', \theta'', \dots$ means that $A = \theta + \theta' + \theta'' + \dots$

³⁷ "Ein Complex G von g verschiedenen Substitutionen heisst eine Gruppe vom Grade g , wenn jedes beliebige Product aus Substitutionen, die alle in G enthalten sind, wieder in G enthalten ist."

³⁸ "Die Gruppe K soll ein Divisor der Gruppe G heissen."

³⁹ If A is a complex of substitutions $\theta, \theta', \theta'', \dots$, and φ another substitution, we have $A\varphi = \theta\varphi + \theta'\varphi + \theta''\varphi + \dots = (\theta + \theta' + \theta'' + \dots)\varphi$

The complex G of all the substitutions which are simultaneously contained in a sequence of groups G_1, G_2 , etc., forms a group; it must be called the *greatest common divisor* of these groups G_1, G_2 , etc.⁴⁰ (*ibid.*, 65)

With the notion of group as introduced by Dedekind, which seems to draw from both Cayley's and Galois's, it is possible to consider collections of "things" without taking into account, neither at the ground nor in the developments of the theory, the individual nature or properties of each object. Besides, the first elements of an arithmetical methodology introduced to consider certain properties of groups in analogy with number theory, seem to ease the investigation of group theory. With this approach, the collection of things is taken as a consistent whole, treated as a concrete object.

To work directly with infinite sets as if they were objects in themselves, and in particular numbers, is a very important and significant move in the history of mathematics, which has been underlined many times. Dugac, for example, wrote

A great step forward is made by this clear realization that, basically, one manipulates infinite sets as numbers.⁴¹ ([Dugac, 1976b], 23, my translation)

Dedekind's use of set-theoretic and infinitary methods has been widely discussed, and I will not develop this point.⁴² From Dedekind's viewpoint, forming collections of things sharing a common property (such as the closure by a binary operation) is a basic capacity of the human mind, as we will see later, and to consider the collection as a whole seems to be a natural consequence of the unity attributed to it by the definitional property.⁴³

The group, conceived as a united whole, appears to bear some similarities to numbers, since divisibility can be defined and methods from number theory can be imported, but which ones? Considering that only the inclusion of groups is linked to divisibility, it seems here that the possibility of ordering groups by inclusion, added to the peculiar divisibility relation between their cardinals, could be an essential factor for setting up the arithmetical terminology. In the following articles of §I, more properties of groups are studied and most of them make an essential use of the arithmetical methodology. Today's notion of normal subgroup is defined as "proper divisor"

⁴⁰ "Der Complex G sämtlicher Substitutionen, welche gleichzeitig in einer Reihe von Gruppen G_1, G_2 etc. enthalten sind, bildet eine Gruppe; diese soll der grösste gemeinschaftliche Divisor jener Gruppen G_1, G_2 etc. heissen."

⁴¹ "Un grand pas en avant est fait par cette nette prise de conscience que, au fond, on manipule les ensembles infinis comme des nombres."

⁴² See [Dugac, 1976b], [Tait, 1997], [Ferreirós, 2008], [Sinaceur and Dedekind, 2008], [Avigad, 2006], ...

⁴³ On Dedekind and set theory, see [Ferreirós, 2008].

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(*eigentlicher Divisor*): if G is a group and K a divisor of G such that

$$K = \theta_1^{-1}K\theta_1 = \theta_2^{-1}K\theta_2 = \dots = \theta_{h-1}^{-1}K\theta_{h-1}$$

then K is called a proper divisor of G (in modern terms, K is a normal subgroup of G). This section of the paper introduces all the properties of groups that will be necessary for Dedekind's remaining investigations. It will appear clearly that groups are essentially a tool, and a very efficient one, to study problems involving a polynomial equation.

§II is essentially concerned with substitutions of variables in rational functions, properties of symmetric functions, and theorems already proved, by Lagrange notably. The content of the paragraph does not differ from standard works on these matters, except for the last paragraph entitled by Scharlau "Program for a theory of fields of algebraic functions of one variable" (*Program für eine Theorie der algebraischen Funktionenkörper einer Veränderlichen*) and which hints at more general questions. Consider the general equation

$$x^m - \lambda_1 x^{m-1} + \lambda_2 x^{m-2} - \dots + (-1)^m \lambda_m = 0$$

where the $(\lambda_i)_i$ are "indeterminate magnitudes independent from each other." It is not necessary that the λ_i be numerical magnitudes, they could be rational functions, for example. If one wants to construct the "expression depending on" the $(\lambda_i)_i$, namely a linear combination of λ_i , which makes the polynomial on the left-hand side of the equality vanish, then the previous paragraph showed that finding this expression depends on the "decomposition of the entire group of all 1.2.3 . . . m substitutions" and that the solution of the equation is essentially linked to the group. If one wishes to consider such a general equation in which the λ_i are not fully independent from each other, for example if the λ_i are rational functions of a variable z , then the solution depends on a divisor of the said group (i.e., a subgroup). The roots of such an equation are themselves rational functions of z and

The way these functions change into each other, when the variable z continuously runs through a closed path of values, and whose position is given according to what was called ramification point by Riemann, will be completely determined by this group of substitutions.⁴⁴ ([Dedekind, 1856-1858], 79)

Dedekind does not justify or give any details about this, but seems to already have sufficiently general ideas about groups to consider that the methods

⁴⁴ "die Art, wie diese Functionen in einander übergehen, wenn die Variable z einen in sich zurücklaufenden Zug von Werthen stetig durchläuft, der seiner Lage gegen die von Riemann sogenannten Windungspuncte nach gegeben ist, wird durch diese Gruppe von Substitutionen vollständig bestimmt."

alluded to can be applied to *any* polynomial equation, including that whose coefficients are not numbers. In practice, an arithmetico-algebraic theory of algebraic functions is, as we saw, more complicated than what is proposed here. Nevertheless, it is striking that, in 1856-58, Dedekind is already considering this possibility. The main topic investigated in the rest of the paper is the case in which the coefficients λ_i in the general equation above are numbers with a definite value, and the general resolubility of polynomial equations is only considered in the short last paragraph.

The third and longest paragraph, is dedicated to “algebraic relationships between numbers”, that is, numbers related by equations. It is worth noticing that Dedekind immediately applies Galois theory to number theory. The algebraic resolution of equations does not seem to be of much interest for him, in and by itself, as a formal research. Equations as a tool to study relationships between numbers, on the other hand, are a key element of 19th century number theory, and consistently used in Dedekind's works. Equations express relations between numbers, be they coefficients or roots of the equations. This approach is a legacy of Gauss's *Disquisitiones Arithmeticae*. In this paragraph, Dedekind uses Galois's ideas and results in “higher algebra” to further apply them to number theory.

The starting point is to consider a polynomial equation $f(x)$ of degree m whose coefficients can be expressed as a rational combination of elements of a certain domain of (complex) numbers S (*gewisses Zahlengebiet S*). Dedekind studies the roots r_1, r_2, \dots, r_m of the equation f by means of their group of substitutions: if s is a rational function of the roots, and s' the “form into which s is changed by the substitution θ ”, then s' can be expressed as a rational function of the roots. Since it is completely determined by θ , it is denoted $s' = \theta(s)$. This process can be repeated $m!$ times, the number of elements of the group of substitutions, and

If s, s', s'' are any three roots of this equation, then one can set $s' = \theta(s)$, $s'' = \theta'(s)$, where the coefficients of the rational functions θ and θ' belong to the rational domain S .⁴⁵ (*ibid.*, 83)

A “rational domain” is a subdomain for the domain of complex numbers – it is the equivalent of the notion of a field of numbers, which Dedekind explicitly defines in 1871. The term “*rationale Gebiet*”, closer to the terminology later chosen by Kronecker, “domain of rationality” (*Rationalitäts-Bereich*), underlines the fact that the elements of the system are linked (or put together) by the basic operations of rational arithmetic, rather than insisting on the “unity” of the collection given by the closure by the four (arithmetical) operations, as intended with the choice of the word “*Körper*”. Unlike

⁴⁵ “Sind s, s', s'' irgend drei Wurzeln dieser Gleichung, so kann man $s' = \theta(s)$, $s'' = \theta'(s)$ setzen, worin die Coefficienten der rationalen Functionen θ und θ' in dem rationalen Gebiet S enthalten sind.”

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in his work in number theory, which I will consider in the next chapters, Dedekind does not insist on the “rational domain”. As Corry underlines

In later sections, he mentions this domain again several times, and indeed its actual role within the theory becomes increasingly clear. And yet, this, or any similar kind of domain, is never systematically treated in itself. Results concerning rational domains appear scattered, and they are proved whenever needed for a specific, immediate concern. ([Corry, 2004b], 80)

Group theoretical results are used to study properties of the equation and of its roots. For example, having shown that the substitutions of the roots of an equation $G(y)$ irreducible in the domain S form a group G , Dedekind shows that

Every rational function of the roots r , which for all substitutions of the group G just defined, keeps the same value, has a rational value. And reciprocally: every function of the roots r which has a rational value keeps the same value for all the substitutions in the group G .⁴⁶ (*op. cit.*, 84)

This allows to prove that for a certain equation f , there is only one group of substitutions, which can be called “the group of the equation $f(x)$ according to the domain S .”

Dedekind studies the relationships between G and a rational function of the roots and he shows that

The group of an irreducible equation whose roots are adequate⁴⁷ contains as much substitutions as the equation has roots. And reciprocally, if the group of an irreducible equation contains as much substitutions as the equation has roots, then the latter are all adequate.⁴⁸ (*ibid.*, 89)

The dependence of the relation of “adequacy” to the domain S is important, here, for it leads to the notion of adjunction. Quantities such as 2 and $\sqrt{8}$ cannot be expressed rationally as functions of each other in \mathbb{Q} , but they can be expressed rationally in $\mathbb{Q}(\sqrt{2})$ since $\sqrt{8} = 2\sqrt{2}$. Before introducing

⁴⁶ “Jede rationale Function der Wurzeln r , welche für alle Substitutionen der eben definirten Gruppe G denselben Werth behält, hat einen rationalen Werth. Und umgekehrt: jede rationale Function der Wurzeln r , welche einen rationalen Werth hat, behält für alle Substitutionen der Gruppe G denselben Werth.”

⁴⁷ Two magnitudes being adequate means that they can be rationally expressed in function of each other in the domain S .

⁴⁸ “Die Gruppe einer irreductibeln Gleichung, deren Wurzeln adäquat sind, enthält ebensoviele Substitutionen, als die Gleichung Wurzeln hat; und umgekehrt, hat die Gruppe einer irreductibeln Gleichung ebensoviele Substitutionen, wie die Gleichung Wurzeln, so sind diese letztere sammtlich adäquat.”

the notion of adjunction, Dedekind proves that the group of an equation is essentially linked to the domain S in which the equation is considered. Consider an equation $f(x) = 0$, and t a rational combination of its roots which belong to a certain group K included in G , the group of the equation. One can show that K is the group of the equation $f(x) = 0$ for a certain domain S' which contains all the quantities that can be expressed as rational combinations of t and elements of the initial domain S . Hence, S' is obtained by adjoining the quantity t to the initial rational domain of numbers S .

We can hence say that through the adjunction of the quantity t , that is by the extension of the domain S to all the quantities rationally representable by S and t , the earlier group G of the equation is reduced to the group K ; $s(?)$ satisfies a rational and irreducible equation of degree k related to the extended domain S' .⁴⁹ (*ibid.*, 91)

The difference of the terminology between Dedekind and Galois is significant in regard to Dedekind's comprehension and use of Galois theory. Galois's notion of adjunction consists in considering certain quantities as known and to adjoin them to the equation to be resolved. Dedekind's interpretation of the notion is in terms of extension of domains. It will appear, in the following pages of his paper, that Dedekind considers the notion of adjunction to be the most important one of the theory: the irreducibility of equations and the size of the group are explicitly reduced to that of (extensions of) domains of numbers. If Galois theory is to be studied as a method imported in number theory, the questions underlining the notion of adjunction (what is the nature of the adjunct quantity? how do we extend a domain of numbers?) become central.

Having defined the notion of adjunction, Dedekind can express and prove the theorem, essential in Galois's memoir, stating the reduction of the size of the group, when roots are adjoined to the rational domain:

If one adjoins all the h values of the function t , which belong to the divisor K of the group G , then the group of the equation is reduced to a proper divisor D of the group G , which is the greatest common divisor of the h conjugate groups

$$K, \theta_1^{-1}K\theta_1, \theta_2^{-1}K\theta_2, \dots, \theta_{h-1}^{-1}K\theta_{h-1}.$$
⁵⁰

⁴⁹ "Wir können daher sagen, dass durch Adjunction der Grösse t , d. h. durch Erweiterung des Gebiets S um alle durch S und t rational darstellbaren Grössen, die frühere Gruppe G der Gleichung sich auf die Gruppe K reducirt; s genügt einer in Bezug auf das erweiterte Gebiet S' rationalen und irreductibeln Gleichung vom Grade k ."

⁵⁰ "Adjungirt man sämmtliche h Werthe der Function t , welche zu dem Divisor K der Gruppe G gehört, so reducirt sich die Gruppe der Gleichung auf den eigentlichen Divisor D der Gruppe G , welche der grösste gemeinschaftliche Divisor der h conjugirten Gruppen $K, \theta_1^{-1}K\theta_1, \theta_2^{-1}K\theta_2, \dots, \theta_{h-1}^{-1}K\theta_{h-1}$ ist."

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(*ibid.*, 91)

Dedekind then gives a more general treatment of the previous considerations and explains the reduction of G into a divisor K by the adjunction of one root of the equation.⁵¹ These results are, for Dedekind, the core of Galois theory:

The preceding propositions give the essential aspect from which Galois theory is regarded.⁵² (*ibid.*, 95)

From Dedekind's viewpoint, the aim of Galois theory is to completely resolve an equation with coefficients in a "given rational domain S ", i.e., "for each of its roots, to find a precisely defined expression." But Dedekind explains that the goal is not reached when all roots can be expressed rationally as elements of S and a series of auxiliary quantities (*Hilfsgrößen*) themselves algebraically representable in S .⁵³ Rather, a domain is said to be given when it satisfies the following:

The system or domain S will be considered as given, when one can always decide whether an arbitrary rational equation with *proper* rational coefficients (see the part about the reciprocity between irreducible equations) between arbitrarily many of the quantities related to the domain S truly exists or not.⁵⁴ (*ibid.*, 96)

It is always possible to determine whether an equation is irreducible in a certain domain S , and the resolution of such equations lies in the process of adjunction of "auxiliary quantities", providing that it be done properly, that is,

for the same extended system S also to be considered as given, and then also that all the roots of the presented equation are actually contained in it.⁵⁵ (*ibid.*, 96)

The problem of the algebraic resolution of equations is reduced to that of the adjunction of an adequate number of auxiliary quantities. It can be

⁵¹However, the paragraph explaining the reduction of G into a proper divisor D by the adjunction of all roots of the equation is missing in the manuscript. ([Scharlau, 1981a], note 18, 92)

⁵²"Die vorhergehenden Sätze geben den wesentlichen Gesichtspunkt, von welchem aus die Galois'sche Theorie aufzufassen ist."

⁵³By algebraically representable in S , Dedekind means that the quantities are roots of an equation with coefficients in S .

⁵⁴"Das System oder Gebiet S wird als gegeben angesehen, wenn man stets entscheiden kann, ob eine beliebig rationale Gleichung mit eigentlich rationalen Coefficienten (siehe den Aufsatz über die Reciprocität zwischen irreductibeln Gleichungen) zwischen beliebig vielen dem Gebiet S angehörigen Grössen in der That besteht oder nicht."

⁵⁵"dass das um dieselben erweiterte System S auch als gegeben anzusehen ist, und dann auch wirklich alle Wurzeln der vorgelegten Gleichung in sich enthält."

shown to be equivalent to that of reducing the group of substitutions of the equation to the group composed of only the identity in the extended domain. Dedekind's explanations show that he considers the process of adjunction and the link between the equation and the domain in which it is studied to be central. These elements seem to constitute what Dedekind calls the "algebraic affinities between numbers", which will be of great importance in his later works. Adjunction, and its action on the domain of numbers considered, are a central part of the theory. Groups, on the other hand, do not hold the same status, they appear to be a mere tool, a part of what Dedekind calls later an auxiliary theory: an abstract (formal) theory interesting mostly (if not only) for its applications.

Dedekind's lecture had almost no immediate impact. However, the manuscript I just commented was read by Heinrich Weber whose work on Galois theory was, he claimed, deeply influenced by it. Dedekind studied thoroughly Galois's works and proposes, here, a more systematic and a broader treatment than any of the works done at the time. Dedekind's treatment builds on Galois's and Cayley's works and integrates them in his own epistemological framework, developing the theory with new (modified) definitions for the basic notions and with methods inspired by or analogous to elementary arithmetic.

4.3 An approach to higher congruences "rigorously tied to an analogy with elementary number theory"

Dedekind's attempts to develop an approach built on an analogy with elementary (rational) arithmetic are rooted in his very first works in algebra and number theory. In his work on Galois theory, Dedekind proposed for the first time to use arithmetical operations to treat set-theoretical relationships, inaugurating a method which will largely dominate his later works. The development of theories modeled on rational arithmetic is a very distinctive and important part of his algebraic works. The definition of new arithmetical relationships is such that they can be related to the old ones. Before fully developing this approach, Dedekind tried to develop an "analogy" between higher congruences and elementary numbers theory, in his first published paper in number theory, "*Abriß einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus*" (1857). This paper, in the continuity of Gauss's work on congruences, "later successfully revived by Galois, Serret, Schönemann"⁵⁶ explores the resolution of modular equa-

⁵⁶Dedekind is referring to [Galois, 1830], [Serret, 1854], [Schönemann, 1846a], [Schönemann, 1846b]. Note that Schönemann, a *Gymnasium* teacher and former student of Jacobi, studied Galois's works as well, but without making a very significant use them. After 1850, Schönemann was more interested in physics than in mathematics and did not publish any

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tions with a prime module p , with for example considerations on equations of the form

$$x^{p^v} - x \equiv 0 \pmod{p}$$

This paper is less interesting for its results than for the approach developed in it, which aims to provide

a simple and coherent presentation, which is rigorously tied to an analogy with elementary number theory. This analogy is, in fact, so thorough that with the exception of a few researches specific to our object, it only requires a change of words in the proofs from number theory.⁵⁷ ([Dedekind, 1857], 40, my translation)

Recall that Dedekind's first attempt to generalize Kummer's works on ideal numbers was by using higher congruences. While it is not what this 1857 paper is concerned with, it seems that the possibility to develop, here, an approach "*rigorously* linked" (my emphasis) to an analogy with rational arithmetic might have been an inspiration for Dedekind's later works.

4.3.1 Outlines of Dedekind's 1857 paper on higher congruences

Dedekind considers congruences between polynomials of one variable x with integral coefficients.⁵⁸ What is at the core of the investigation, here, are the "properties of such functions relating to a *modulus* which is a real *prime number* p ", that is a congruence relation defined in the following way: if A and B are two polynomials and p a prime number, $A \equiv B \pmod{p}$ means that the coefficients of $A - B$ are divisible by p , or again that the coefficients of a same power of x in each polynomial A and B are congruent (in the usual sense defined by Gauss) modulo p . This means, Dedekind writes, that this congruence is "just an expression for the *identity* $A = B + pC$ (*ibid.*, 41).

Congruences between polynomials are what is called "higher congruences". The usual rules of addition and multiplication of congruences hold, and it is also possible to define a notion of divisibility:

If A , B and C are three (polynomial) functions of x such that $A \equiv BC \pmod{p}$, then B , C (and all other functions congruent to them) are called *divisors* or *factors* of A (or any function congruent to A) according to the modulus p . The [following]

more papers in algebra.

⁵⁷"eine einfache zusammenhängende Darstellung (...), welche sich streng an die Analogie mit den Elementen der Zahlentheorie binden soll. Diese ist in der Tat so durchgreifend, daß es mit Ausnahme einiger unserem Gegenstand eigentümlicher Untersuchungen nur einer Wortänderung in den Beweisen der Zahlentheorie bedarf."

⁵⁸In this paragraph, "polynomial" always means polynomial of one real variable x with coefficients in \mathbb{Z} .

expressions are equivalent: A is a *multiple* of B, C ; or A is *divisible* by B, C .⁵⁹ (*ibid.*, 42)

Operations for classes of congruences are defined for a representative of the class, here A, B and C . Working with representatives of classes to avoid abstract considerations of the infinite classes was an approach initiated by Gauss, in his *Disquisitiones Arithmeticae*. Note that while Dedekind worked with groups as wholes, he is keeping, here, the Gaussian approach. Openly aiming for a development analogous to number theory, Dedekind transfers the operations and notions elaborated for numbers to the representatives of congruence classes of polynomials. The peculiar nature of the objects studied is only considered so as to assure that it is possible to define the new operations in a consistent way. Indeed, the possibility to define arithmetical operations is not taken for granted: before defining divisibility, Dedekind considers the “condition for the authorization (*Berechtigung*) for the division of a congruence by another one”, namely

If $AB \equiv 0 \pmod{p}$, then at least one of the two functions A, B is [such that] $A, B \equiv 0 \pmod{p}$; and moreover: If $AB \equiv A'B'$, and $A \equiv A'$ but not $\equiv 0 \pmod{p}$, then $B \equiv B' \pmod{p}$; so there is $AB \equiv AB'$ or $A(B - B') \equiv 0 \pmod{p}$.⁶⁰ (*ibid.*, 41-42)

It is clear, then, that only the consistent definition of operations is responsible for the (possibility of the) analogy. Dedekind explains that any result proved for the representatives hold for any function of the class, and hence for the whole class:

The preceding propositions correspond completely with that on the divisibility of number, insofar as whole system of infinitely many functions of one variable congruent modulo p behave like a single determinate number in number theory, as each individual function of one such system replaces completely any other arbitrary [function] of that same system in any relation; such a function if the representative of the class.⁶¹ (*ibid.*, 46-47)

⁵⁹ “Sind A, B, C drei solche Funktionen von x , daß $A \equiv BC \pmod{p}$, so heißen B, C (oder alle diesen kongruente Funktionen) Divisoren oder Faktoren von A (oder jeder mit A kongruente Funktion) in bezug auf den Modul p . Gleichbedeutend sind die Ausdrücke: A ist ein Multiplum von B, C ; oder: A ist teilbar durch B, C .”

⁶⁰ “Ist $AB \equiv 0 \pmod{p}$, so ist mindestens eine der beiden Funktionen $A, B \equiv 0 \pmod{p}$; und ferner: Ist $AB \equiv A'B'$, und $A \equiv A'$ nicht $\equiv 0 \pmod{p}$, so ist $B \equiv B' \pmod{p}$; denn es ist $AB \equiv AB'$, oder $A(B - B') \equiv 0 \pmod{p}$.”

⁶¹ “Die vorhergehenden Sätze entsprechen vollständig denen über die Teilbarkeit der Zahlen in der Weise, daß das ganze System der unendlich vielen einander nach dem Modulus p kongruente Funktionen einer Variablen sich hier verhält, wie eine einzige bestimmte Zahl in der Zahlentheorie, indem jede einzelne Funktion eines solchen Systems jede beliebige andere desselben Systems in jeder Beziehung vollständig ersetzt; eine solche Funktion ist der Repräsentant der Klasse... ”

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Pursuing the analogy with arithmetic, in the 1857 paper, Dedekind explains that it is possible to prove, for polynomials, theorems analogous to those in number theory.

From the definition of the multiple follows immediately the two following propositions: “If a function is the multiple of a second one, which is itself the multiple of a third one, which is itself the multiple of a fourth one, and so on, then the each former function in the sequence is a multiple of each further one. The sum and the difference of two multiples of one function is itself a multiple of this function.”⁶² (*op. cit.*, 43)

An algorithm analogous to Euclid’s algorithm to find the greatest common divisor of two numbers is given, from which the notion of greatest common divisor is derived. Dedekind considers two polynomials A and A' and whether they have a common divisor modulo p . It is possible to write

$$A \equiv A'Q + A'' \pmod{p}$$

with A'' and Q two new polynomials, both of degree lower than A and A' . If A'' is not $\equiv 0 \pmod{p}$, the operation can be repeated and the following system of congruences can be formed:

$$\left. \begin{array}{l} A \equiv QA' + A'' \\ A' \equiv QA'' + A''' \\ \dots \\ A^{(v-2)} \equiv Q^{(v-2)}A^{(v-1)} + A^{(v)} \\ A^{(v-1)} \equiv Q^{(v-1)}A^{(v)} \end{array} \right\} \pmod{p}$$

The degree of the polynomials is decreasing at each line, and “after a finite number of operations, it must happen that a function $A^{(v-1)}$ is divisible the following $A^{(v)}$ ” (*ibid.*, 45). This last polynomial $A^{(v)}$ is called the greatest common divisor of A and A' . Moreover, it is possible to determine two polynomials G and G' such that

$$GA + G'A' \equiv A^{(v)} \pmod{p}.$$

From there, a notion of relative primality between polynomials is also defined:

⁶² “Aus der Definition der Multipla ergeben sich unmittelbar die beiden folgenden Sätze: “Ist eine Funktion ein Multiplum von einer zweiten, diese ein Multiplum von einer dritten, diese von einer vierten usw., so ist jede frühere in der Reihe dieser Funktionen ein Multiplum von jeder späteren. - Die Summe und die Differenz zweier Multipla von einer Funktion sind selbst wieder Multipla derselben Funktion.”

If the greatest common divisor $A^{(v)}$ of the functions A and A' is of degree 0 (and $\equiv 1 \pmod{p}$ if it is primary⁶³), then A and A' are called relatively prime with each other.⁶⁴

Theorems analogous to number theory can be proved, for example:

If A, A' are two relatively prime functions and M is an arbitrary function, then each common divisor of the two functions AM, A' is also a common divisor of M, A' . (...) If M is also prime with A' , then the greatest common divisor of M and A' , and consequently also of that of AM and A' , is a function of degree 0, that is AM and A' are relatively prime to each other.⁶⁵ (*ibid.*, 44-45)

One can define the notion of a prime function modulo p :

A function whose only divisors modulo p are either itself or functions of degree 0 (i.e., units) or products congruent to either of them (then each function has all these divisors) is called (*irreducible* or) a *prime function* modulo p ; every other is called (*reducible* or) composed.⁶⁶ (*ibid.*, 45)

And familiar theorems can be re-stated, such as: if a product AB is divisible by a prime function P , then at least of one the factors A, B is divisible by P . Dedekind also gives a proposition similar to the decomposition into prime factors, whose truth he seems to consider evident:

It stands to reason that any arbitrary function M can be expressed as the product of powers of prime functions, which are incongruent to each other, and whose number is finite (if the degree of M is finite); and in fact, *essentially* only one of those representations is possible (...). One can suppose all the prime function primary, and then

$$M \equiv zA^a B^b C^c \dots \pmod{p}$$

⁶³A "primary function" is a function in which the coefficient of the term of highest degree x^n is $\equiv 1 \pmod{p}$.

⁶⁴"Ist der größte gemeinschaftliche Divisor $A^{(v)}$ der Funktionen A, A' vom Grade l Null (also $\equiv 1 \pmod{p}$, wenn er primär ist), so heißen A, A' relativ prim gegeneinander."

⁶⁵"Sind A, A' zwei relative Primfunktionen, und ist M eine beliebige Funktion, so ist jeder gemeinschaftliche Divisor der beiden Funktionen AM, A' zugleich gemeinschaftlicher Divisor von M, A' . (...) Ist auch M relativ prim gegen A' , so ist der größte gemeinschaftliche Divisor von M und A' , und folglich auch der von AM und A' eine Funktion vom Grade Null, d. h. AM und A' sind relativ prim gegeneinander."

⁶⁶"Eine Funktion, welche nach dem Modul p nur solche Divisoren hat, die entweder ihr selbst oder Funktionen vom Grade Null (d. h. Einheiten) oder Produkten aus beiden kongruent sind (denn jede Funktion hat alle diese Divisoren), heißt (irreduktibel oder) eine Primfunktion nach dem Modul p ; jede andere heißt (reduktibel oder) zusammengesetzt."

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where z is a unit, A, B, C etc. are incongruent primary prime functions, a, b, c are positive integral numbers.⁶⁷ (*ibid.*, 46)

Dedekind does not give proofs of this proposition, neither does he prove, for example, that any divisor of M can be expressed as a product of lower powers of the prime factors in the decomposition of M . To make the analogy with rational integers (more) operative, an additional notion of congruence is defined, a congruence with a double modulus: $A \equiv B \pmod{p, M}$, in which M is the representative of a class of functions, to designate the relation $A \equiv B - CM \pmod{p}$.

So, Dedekind here, defines a second level of congruences in order to be able to conduct more elaborated reasonings – yet still analogous to elementary number theory – in the higher level in which he already defined basic arithmetical properties. Here, the fact that he works directly on classes as if they were numbers is particularly significant. From there, Dedekind develops for classes of polynomials (but still working only with the representatives of classes) an approach analogous to Gauss's works in the *Disquisitiones Arithmeticae*. For example, he studies quadratic residues for the new notion of congruence: the congruence

$$y^2 \equiv A \pmod{P}.$$

where P is a prime function of degree π holds only when A is one of the roots of the congruence

$$y^{\frac{1}{2}(p^\pi-1)} \equiv 1 \pmod{P}$$

and tries to identify, given a certain function A , the prime functions P of which A is a quadratic residues and explains that

[this] question (...) which is important for the *theory of quadratic forms* (with functions of one variable x , will, in virtue of the preceding article, be reduced to the case in which A is a prime function R (of degree ρ). The analogous question in number theory, as is well-known, is answered with the so-called reciprocity law of Legendre (first proved by Gauss). This analogy, which up to now was preserved in all principles and proofs, does not leave

⁶⁷ "Es leuchtet ein, daß jede beliebige Funktion M sich darstellen läßt als Produkt aus Potenzen von Primfunktionen, welche untereinander inkongruent sind, und deren Anzahl eine endliche ist (wenn der Grad von M endlich ist); und zwar ist wesentlich nur eine einzige solche Darstellung möglich (...). Man kann die Primfunktionen sämtlich primär annehmen; ist dann

$$M \equiv zA^a B^b C^c \dots \pmod{p}$$

wo z eine Einheit, A, B, C etc. inkongruente primäre Primfunktionen, a, b, c etc. positive ganze Zahlen bedeuten."

doubt about the existence of a correspondant law in our theory. This theorem ends with the result

$$\left(\frac{P}{R}\right)\left(\frac{R}{P}\right) = \left(\frac{-1}{p}\right)^{\pi \cdot \rho}$$

where P, R denote two primary prime functions of degree π, ρ respectively, and $\left(\frac{-1}{p}\right) = (-1)^{\frac{1}{2}(p-1)}$ is the Legendre symbol.⁶⁸ (*ibid.*, 58-59)

Dedekind avowedly only “emphasized the key moments of the proofs” in this article, the details being considered “wearisome for the connoisseur of number theory”. This suggests that he considers the ‘good behavior’ of the analogy as a given fact. It seems that Dedekind let himself be guided by the analogy with rational number theory, which proved to be functioning so smoothly that the validity of the theorems in the new framework appeared as somewhat obvious. Of course, this only works inasmuch as the bases have been assured: it is possible to provide a proof that the analogy is well-founded (as it was the case in the notes about Galois theory). This example allows to see clearly how having defined operations enables to establish a (new) kind of arithmetic, developed following the model of elementary number theory. Arithmetic, here, is essentially attached to the *operations* and the possibility to establish theorems central to number theory, such as divisibility properties.

4.3.1.1 Conclusion

In the two works presented here, one can see the first steps of a certain arithmetical methodology, which Dedekind will adopt in algebraic number theory. As we saw in the first part of this dissertation, the arithmetical methodology developed will reveal to be rather fruitful (albeit not without limits). The “elementary number theory” taken as model, here, does not refer to the properties of the sequence of natural numbers, but to number theory as inaugurated by Gauss's *Disquisitiones Arithmeticae*, with the study of divisibility laws, congruences properties... These first examples suggest that arithmetic is intrinsically linked to the definition of the four rational operations and the possibility to prove a certain number of results, especially

⁶⁸ “[Diese] Frage (...) welche für die Theorie der quadratischen Formen (mit Funktionen einer Variablen x) von Wichtigkeit ist, wird vermöge des vorigen Artikels auf den Fall reduziert, in welchem A eine Primfunktion R (vom Grade ρ) ist. Die analoge Frage in der Zahlentheorie wird bekanntlich durch den (zuerst von Gauß bewiesenen) sogenannten Reziprozitäts-Satz von Legendre beantwortet. Diese Analogie, welche sich bisher in allen Prinzipien und Beweisen bewährt hat, läßt keinen Zweifel an der Existenz eines entsprechenden Satzes in unserer Theorie übrig. Dieses Theorem lautet in der Tat $\left(\frac{P}{R}\right)\left(\frac{R}{P}\right) = \left(\frac{-1}{p}\right)^{\pi \cdot \rho}$ worin P, R primäre Primfunktionen resp. von den Graden π, ρ , bedeuten, und $\left(\frac{-1}{p}\right) = (-1)^{\frac{1}{2}(p-1)}$ das Zeichen von Legendre ist.”

4.3 *Higher congruences rigorously analogous to elementary number theory*

pertaining to divisibility. The mathematical ground of the analogy is very clearly explained, however Dedekind does not give any epistemological explanation, any justification of his choice. In his subsequent works, many statements about epistemological values, such as rigor and uniformity, appear along the use of the arithmetical methodology.

In the two next chapters, I will investigate the increasing use of arithmetical methods and the possible benefits of using an arithmetical methodology, by analyzing the two first versions of algebraic number theory and the changes made from 1871 to 1877.

Part III

A strategical use of arithmetic?

Introduction to Part III

The theory of algebraic integers, first published by Dedekind in 1871 as part of a Supplement of Dirichlet's *Vorlesungen* contains the native soil of Dedekind's concepts of field, module and ideal. It is for the purpose of proving the validity for algebraic integers of the laws of divisibility known for rational integers, that Dedekind elaborates these concepts. Ideal are first introduced tools for the generalization of Kummer's works on ideal numbers. The arithmetical methodology outlined in the previous chapter is little by little taken advantage of in the successive rewritings of Dedekind's algebraic number theory. It appears that developed as an extension of Kummer's notion of ideal divisors, in 1871, Dedekind's ideals become, in 1877, a way of extending divisibility itself. To argue for this point, I will expose in detail both versions of the theory – albeit the first with slightly more details, since a contrast of the two versions by underlining the changes in the 1876-77 version is ultimately the goal of this Part.

Firstly, we will see how, in 1871, the set up of arithmetical operations for ideals allows to prove important theorems for algebraic numbers. In addition, I will insist on elements such as the aim of Dedekind's 1871 work, the role and status held by ideal and module theories, and the reception of his works, in order to put his work into its context. The many details given in the Chapter 5 will be particularly interesting when contrasted, in the next chapter, with the 1877 rewriting of ideal theory. In this new version, the change toward a theory of ideals using only the "simplest principles of arithmetic" is stronger. Indeed, in the second version of the theory (published two times, in French in 1876-77 in the *Bulletin des Sciences Astronomiques et Mathématiques*, and in German as a Supplement of Dirichlet's *Vorlesungen* in 1879), significant changes are made, and the theory appears to be more oriented towards a development following a strict analogy with rational number theory. Dedekind seems to take full advantage of the arithmetical relationships defined between ideals. Dedekind thus develops the theory with the same theorems and proofs, given at the level of ideals, as the well-known ones for rational numbers.

I will underline how the arithmetical components of his approach become more significant through the changes between the first and second version of the theory. This will lead me to consider questions about the definition of ideals and their arithmetical nature, so as to try to understand their pride of place in Dedekind's number theory. Finally, from the comparison of these two versions, and a link with *Algebraische Funktionen*, I will propose new elements to understand Dedekind's ideas about arithmetic. Indeed, it should be borne in mind that definition of divisibility between groups or ideals is not presented as a mere borrowing from arithmetic. In Dedekind's works, it is rather justified, or even proved, that it was a legitimate extension of the arithmetical (rational) notion of divisibility.

Chapter 5

Theory of algebraic integers in 1871

In 1836-37, Jacobi gave a course in number theory and introduced his lectures in the following way:

Number Theory in its present state consists of two big chapters, one of which may be called the theory of the solution of pure equations, the other the theory of quadratic forms.¹ ([Jacobi, 1836-1837], quoted and translated in [Goldstein and Schappacher, 2007a], 28)

As noted by Goldstein and Schappacher, Jacobi, who gave a proof of the bi-quadratic residue law using cyclotomy, seems to consider congruences as underlying both “chapters” of number theory. As we saw in Sec. 1.2.2, in 1854 these ideas were shared by Dirichlet, whose works were greatly responsible for making Gauss’s composition of forms more understandable. Dedekind’s first major contribution to number theory, an inquiry about binary quadratic forms, published as the Xth Supplement to Dirichlet’s lectures on number theory, is profoundly inscribed in this tradition.

The Xth Supplement, while often reduced – maybe because it was the only part published by Noether and Van der Waerden in Dedekind’s *Gesammelte Werke* – to the paragraphs concerned with fields and ideals of algebraic integer, was dedicated, as its name indicates, to binary quadratic forms. In this work, Dedekind introduces the notions of algebraic integer, field, module and ideal, so as to prove results about the arithmetical laws of algebraic numbers, and in particular to show that this extension of the concept of number preserved the laws of arithmetic known for rational integers.

Kummer, in his researches about cyclotomic numbers, began to investigate these questions and showed that the unique factorization in primes, the so-called fundamental theorem of arithmetic, failed for certain complex

¹ “Die Zahlentheorie auf ihrem jetzigen Standpunkte zerfällt in zwei große Kapitel, von denen das eine als die Theorie der Auflösung der reinen Gleichungen, das andere als die Theorie der quadratischen Formen bezeichnet werden kann.”

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numbers. In order to “save” the unique prime decomposition, he introduced the notion of “ideal numbers”. Ideal numbers were indeed able to restore this important property of numbers for the restricted domains investigated by Kummer. A general proof of the validity of the divisibility laws was still to be given – and this is the problem tackled by Dedekind in 1871, with the introduction of ideal theory as a way to generalize Kummer’s ideas.

The first version of the so-called ideal theory stays close to Kummer’s computational approach. It should be underlined that the paragraphs presenting the divisibility laws of algebraic integers (based on the notions of field, module, and ideal) are five paragraphs (§§159-163) in a work of twenty-five paragraphs in total. Moreover, ideals, modules and fields are only used starting in those five paragraphs, up to the end of the Supplement. From §145 to §158, Dedekind works on binary quadratic forms in customary Gaussian manner. In §§159-163, he presents the new concepts of field, module and ideal and gives a demonstration of the theorem of unique factorization for algebraic integers. From §164 to the end, Dedekind develops further considerations on ideal theory (composition and classes of ideals) and shows how it can be used in the study of composition of forms. Presented as embedded in the well established and widely studied theory of forms in the tradition of Gauss, Dirichlet, Lagrange and Kummer, ideal theory is essentially a tool, an element of an algebraic arsenal elaborated to provide new methods of proof, able to give new results as well as to enhance existing proofs.

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Before considering Dedekind’s theory of algebraic integers, let us consider the works on which it relies: Gauss’s “general arithmetic”, and Kummer’s ideal numbers. Dedekind was following these two authors and took part in a certain process of extension, of generalization of arithmetic. Here, “generalization” should be understood as an extension of the scope of arithmetic, of the objects of arithmetic. This extension goes with endeavors to extend essential properties of numbers as well, such as divisibility properties. With the extension of the notion of integer, from natural integers to Gaussian integers (to algebraic integers, in Dedekind’s and Kronecker’s works), one of the important questions to answer is that of the arithmetical laws verified by the enlarged concept of integer. Indeed, if one is to enlarge number theory (as understood by Jacobi and Dirichlet) by using a more general notion of integer, then it is necessary to understand their properties, so as to avoid, for example, to assume without ground that laws valid for rational integers still hold for complex integers.

5.1.1 Gauss's generalized arithmetic

Kurt Hensel, who was Kronecker's student, wrote in the introduction of Kronecker's lecture on number theory, the following account of Gauss's extension of the notion of integer:

In the introduction of his *Disquisitiones arithmeticae* Gauss fixes the domain of the natural integers as the field of arithmetic, but he himself was forced to extend this domain, as he added in the fifth section of this same work the realm of quadratic forms with two variables, in the seventh the functions of x which, once set equal to zero, produce the cyclotomic equation.² ([Kronecker, 1901], quoted and translated in [Goldstein and Schappacher, 2007b], 86)

Investigations about biquadratic residues led Gauss to consider complex numbers and to advocate their being considered as objects of arithmetic on their own right. As we saw in Sec. 3.2.2.2, Gauss considered and argued for complex integers as legitimate objects of number theory:

For the true grounding of the theory of biquadratic residues, one must extend the field of higher arithmetic, which has hitherto been confined to the real integers, into the imaginary integers, and must concede to the latter the same legitimacy as the former. As soon as one has seen this, that theory appears in an entirely new light, and its results acquire a startling simplicity.

But before the theory of biquadratic residues can be developed in this widened domain of numbers, the doctrines of higher arithmetic, which hitherto have been explored only for real numbers, must be extended as well. ([Gauss, 1831], 308-309)

Gauss's argument is, in fact, deeply rooted in the possibility to provide proofs of the reciprocity laws. Because complex integers allow to do so, they are to be taken not only as legitimate in higher arithmetic, but as allowing a "true foundation" of the theory of biquadratic residues:

As easily as all such special theorems are discovered by induction, so difficult it is to find a general law for these forms in the same way, even though several common features are obvious. And it is even more difficult to find the proofs of these theorems. (...)
One soon recognizes that totally new approaches are necessary

² "Gauss bestimmt in der Einleitung zu seinen 'Disquisitiones arithmeticae' das Gebiet der natürlichen ganzen Zahlen als das Feld der Arithmetik, aber er selbst war gezwungen, dieses Gebiet dadurch zu erweitern, daß er in der fünften Sektion desselben Werkes das Reich der quadratischen Formen von zwei Variablen, in der siebenten die Funktionen von x hinzunahm, welche gleich null gesetzt die Kreisteilungsgleichungen ergeben."

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to enter this rich domain of higher arithmetic, (...) that for the true foundation of the theory of biquadratic residues the field of higher arithmetic, which before had only extended to the real integers, has to be extended to also include the imaginary ones and that exactly the same right of citizenship has to be given to the latter as to the former. As soon as one has understood this, that theory appears in a totally new light, and its results acquire a most surprising simplicity.³ (*ibid.*, quoted and translated in [Goldstein and Schappacher, 2007a], 27)

Complex integers do not only provide means to develop proofs for biquadratic residues: one can also extend the essential notions of arithmetic to them, such as the arithmetic operations and the notions of primality, greatest common divisor, least common multiple, congruences, etc. One can prove as well that important theorems of rational arithmetic, such as Fermat's theorem or the quadratic reciprocity law are still valid for "the extended arithmetic."

5.1.2 Unique factorization in primes and ideal numbers

Kummer's invention of ideal numbers was made in the context of researches on reciprocity laws, but it was a (failed) attempt to solve Fermat's conjecture that brought it to light, in 1847. Fermat's famous conjecture was made in the margin of Diophantus's *Arithmetica* in 1637, and stated that

it is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as a sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain. (Quoted and translated in [Edwards, 2000], 2)

That is, for any integer n greater than 2, there doesn't exist any three positive integers x, y, z such that

$$x^n + y^n = z^n.$$

³ "So leicht sich aber alle der gleichen specielle Theoreme durch die Induction entdecken lassen, so schwer scheint es, auf diesem Wege ein allgemeines Gesetz für diese Formen aufzufinden, wenn auch manches Gemeinschaftliche bald in die Augen fällt, und noch viel schwerer ist es, für diese Lehrsätze die Beweise zu finden. ... Man erkennt demnach bald, dass man in dieses reiche Gebiet der höhern Arithmetik nur auf ganz neuen Wegen eindringen kann, ... dass für die wahre Begründung der Theorie der biquadratischen Reste das Feld der höhern Arithmetik, welches man sonst nur auf die reellen ganzen Zahlen ausdehnte, auch über die imaginären erstreckt werden, und diesen das völlig gleiche Bürgerrecht mit jenen eingeräumt werden muss. Sobald man diess einmal eingesehen hat, erscheint jene Theorie in einem ganz neuen Lichte, und ihre Resultate gewinnen eine höchst überraschende Einfachheit."

Since then, the problem held numerous mathematicians's attention. As is well-known, the attempts to prove Fermat's conjecture stimulated, or even impulsed, the development of large sections of number theory.

The part of the story which particularly interests us starts in 1847. More than 200 years after Fermat's conjecture, only a few special cases were proved (up to $n = 7$), using factorizations such as

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

On March 1st 1847, Gabriel Lamé presented a 'proof' of Fermat's conjecture to the Académie des Sciences, in Paris, in which he proposes to decompose " $x^n + y^n$ completely into linear factors" (*ibid.*, 76) so as to bypass the difficulty which he believed to result from the increasing degree of the factors in a factorization such as the one mentioned above. For this, he introduces roots of the unit, i.e., a complex number r such that $r^n = 1$ and he uses the algebraic identity

$$x^n + y^n = (x + y)(x + ry)(x + r^2y) \dots (x + r^{n-1}y)$$

with n odd. Edwards explains that

[Lamé] planned to show that if x and y are such that the factors $x + y, x + ry, \dots, x + r^{n-1}y, \dots$ are relatively prime, then $x^n + y^n = z^n$ implies that each of the factors $x + y, x + ry, \dots$ must itself be an n th power and to derive from this an impossible infinite descent. If $x + y, x + ry, \dots$ are *not* relatively prime, he planned to show that there is a factor m common to *all* of them so that $(x + y)/m, (x + ry)/m, \dots, (x + r^{n-1}y)/m, \dots$ are relatively prime and to apply a similar argument. (*ibid.*, 77)

However, as it was underlined immediately by Liouville, this attempt required the unique decomposition in prime factors to be valid for complex numbers and it was "by no means obvious that the needed techniques [could] be applied to the complex numbers that Lamé needed them for" (*ibid.*, 77). Lamé's attempt was then a failed one, and a few weeks later, Kummer wrote to Liouville to confirm that the unique factorization does not always hold for complex numbers:

As regards the elementary proposition, for these complex numbers, that a complex composed number can be decomposed into prime factors in but only one way, (...) I can assure you that it does not hold in general, as long as we deal with complex numbers that are in the form :

$$a_0 + a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1}$$

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However, it can be saved by introducing a new kind of complex numbers that I have called an ideal complex number.⁴ ([Kummer, 1847], 298)

Kummer included, in his letter, a copy of a memoir published in 1844 ([Kummer, 1844]), in which he proved the failure of the unique factorization precisely in cases which Lamé was considering. Liouville published Kummer's letter and a copy of his memoir, in his *Journal de Mathématiques Pures et Appliquées*, allowing a wider distribution of his results.

5.1.2.1 Kummer's ideal numbers

To preserve the unique factorization in primes, a fundamental property of numbers, Kummer introduces “ideal factors” which allow him to give a unique decomposition in primes for cyclotomic integers. Kummer's ideal factors appear as a generalization of the notion of divisor.⁵

It is important to underline that Kummer's work is underlaid by philosophical and mathematical conceptions very different from Gauss's (or Dedekind's). Indeed, Kummer, rather than presenting his approach as an *extension*, uses the metaphor of the tree:

The theory of complex numbers is fundamentally the same as the theory of these forms [i.e. of the homogeneous forms, which are decomposable in linear factors], and in this respect, it is part of one of the most beautiful branches of higher arithmetic.⁶ ([Kummer, 1851], 363, quoted and translated in [Boniface, Forthcoming])

The fact that complex numbers are not presented as an extension of rational integers implies their relative independence from them. Hence, Kummer's attempts to prove the validity of divisibility laws for complex integers is not sustained by a sort of principle of permanence, which demands that the laws from the restricted domain are still valid in the extended domain. Rather, it is linked to Kummer's deep convictions about the essential properties of

⁴ “Quant à la proposition élémentaire pour ces nombres complexes, qu'un nombre complexe composé ne peut être décomposé en facteurs premiers que d'une seule manière, [...] je puis vous assurer qu'elle n'a pas lieu généralement tant qu'il s'agit des nombres complexes de la forme

$$a_0 + a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1}$$

mais qu'on peut la sauver en introduisant un nouveau genre de nombres complexes que j'ai appelé nombre complexe idéal.” Kummer à M. Liouville.

⁵My presentation, here, will omit many details and subtleties of Kummer's theory. One can refer to [Edwards, 1980], [Edwards, 2000], [Boniface, 2004] and [Boniface, Forthcoming] for more details. I will rely on their works in this paragraph.

⁶ “La théorie des nombres complexes revient, au fond, à la théorie des formes, et, à cet égard, elle fait partie d'une des plus belles branches de l'Arithmétique supérieure”

numbers, and justified by the use of analogies.⁷ Kummer considered *arithmetic* to be restricted to rational integers, which he called “real integers”, and which were for him the only true, existing numbers. Investigations about complex integers were, as stated in the quote above, another branch of higher arithmetic.

For Kummer, a complex integer was “an integral function [i.e., a polynomial function] with integral coefficients of irrational roots of one or many algebraic equations, whose coefficients are also integral numbers” – those numbers were thus more general than Gauss’s. Kummer works essentially with roots of the unit, i.e., complex numbers α which satisfy the equation $\alpha^\lambda = 1$. Using the well-known results about the cyclotomic equations, one can see that a complex number $f(\alpha)$ can be written as

$$f(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{\lambda-2}\alpha^{\lambda-2}$$

where the a_i are integers. The complex numbers $f(\alpha), f(\alpha^2), \dots, f(\alpha^{\lambda-1})$, obtained by taking successively all the roots of the cyclotomic equations are called “conjugate complex numbers” (*nombres complexes conjugués*). The important remark, here, is that “the product of all complex numbers obtained by replacing the roots in them by conjugated roots (...) is a rational integer” ([Boniface, Forthcoming]). The idea is to use the norm of complex numbers, as proposed by Dirichlet: for a complex number denoted by $f(\alpha)$, the norm is

$$N(f(\alpha)) = f(\alpha)f(\alpha^2)f(\alpha^3)\dots f(\alpha^{\lambda-1}).$$

The interesting property of norms is that they are rational integers – or, as Kummer writes “delivered from all irrationality.” Consequently, Kummer considers complex numbers as factors of rational integers, making the decomposition into prime factors essential. Kummer then shows that for $\lambda = 5, 7, 11, 13, 17, 19$, any prime number $m\lambda + 1 < 1000$ is the norm of a complex number of the form $f(\alpha)$ and can thus be decomposed into $\lambda - 1$ factors. However, for $\lambda = 23$, this property fails for five of the thirteen prime numbers $m\lambda + 1 < 1000$. Using Gauss’s results on cyclotomic equations, Kummer shows that the norm of a root of the cyclotomic equation $\alpha^{23} = 1$ is of the form

$$\frac{a^2 + 23b^2}{4}$$

where a and b are natural integers. But, for example, there exist no such a and b for the prime number $47 = 2 \cdot 23 + 1$. From the impossibility to give

⁷Dedekind, while following closely Gauss’s ideas, does mention several times the “analogy” with rational number theory, displaying signs of the joined influences of Gauss and Kummer. However, his use of analogy appears to be different from Kummer’s, and closer to a kind of principle of permanence.

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a decomposition into $\lambda - 1$ linear complex factors for such prime numbers, Kummer deduces the failure of the unique decomposition in primes.⁸

Kummer's aim is then to develop the theory of complex numbers analogously to the theory of rational integers.⁹ The link between complex and rational numbers stated above, and the fact that only rational integers were considered as true numbers by Kummer, led him to consider that preserving the analogy was more important than the complex numbers themselves.

It is largely to be deplored that this property of real numbers [i.e., the rational integers] of being decomposable into prime factors, always in the same way for a given number, does not belong to complex numbers. If it were the case, all of the theory, which can only be elaborated with great difficulty, could easily be brought to its conclusion. For this reason, the complex numbers we consider seem imperfect, and this may generate doubt whether one should prefer to these numbers other complex numbers that could be found, and should not these other numbers be looked for, which would preserve the analogy with real integer numbers for this fundamental property. ([Kummer, 1847], 182, quoted and translated in [Boniface, Forthcoming])

This is related to the fact that, for Kummer, numbers as objects have permanent and accidental properties. Properties of the complex numbers which were different from those of rational integers were considered as “accidental”. These properties are seen as hiding the permanent properties and should be eliminated. Arithmetical properties are, on the other hand, permanent properties of numbers. On this point, again, Kummer's position is essentially different from Gauss's. Indeed, for Gauss, it was relationships between objects, rather than objects themselves, which were important.

In order to “save” the unique prime factorization for the prime numbers $m\lambda + 1$ into $\lambda - 1$ irreducible complex linear factors, Kummer proposes to feign the existence of such factors by introducing “ideal factors”

If p is a prime number of the form $m\lambda + 1$, then it can, in many cases, be represented as a product of $\lambda - 1$ complex factors $p = f(\alpha)f(\alpha^2)\dots f(\alpha^{\lambda-1})$. Where, however, a decomposition into existing complex factors is not possible, one has to introduce ideal prime factors in order to obtain it.¹⁰ (*ibid.*, 320, quoted and translated in [Boniface, Forthcoming])

⁸For more details, see [Boniface, Forthcoming] and [Edwards, 2000].

⁹Kummer drew analogies not only with arithmetic, but also with algebra, geometry and chemistry, as a way to justify the introduction of ideal factors. See [Hancock, 1928] and [Boniface, Forthcoming].

¹⁰“Ist p eine Primzahl von der Form $m\lambda + 1$, so lässt sie sich in vielen Fällen als Product von folgenden $\lambda - 1$ complexen Factoren darstellen: $p = f(\alpha)f(\alpha^2)\dots f(\alpha^{\lambda-1})$;

For example, in certain domains of complex numbers (e.g., of the form $x + i\sqrt{5}y$, with x and y real numbers and $i^2 = -1$), it can be relatively easily noticed that the unique factorization in primes fails. Kummer defines a number p as being prime if when p divides a product ab , then p divides a or p divides b . The equivalence between irreducibility (p can only be divided by 1 and itself) and primality ceases to be valid for cyclotomic integers, which is a very important component of the failure of unique factorization. To bypass this difficulty, Kummer introduces factors which are only defined by conditions of divisibility but do not actually *exist*. By doing so, Kummer introduces a more general notion of prime factor and proposes a solution to the failure of unique factorization.¹¹ However, Kummer's theory of ideal numbers is not *general* in the sense that it doesn't work for *any* domain of complex integers. In particular, his approach can not be generalized as such to all algebraic numbers.

The ideal factors are determined using congruences and divisibility tests. For Edwards, "Kummer described an ideal prime factor by defining precisely what it means to say that a number is divisible by it" ([Edwards, 1980], 325). For the example of 47, if p is the ideal factor of 47, and $\psi(\alpha)$ designates the product of all the conjugate complex linear factors which are *not* divisible by p , then $\psi(\alpha)$ is divisible by all the prime factors of 47 except p . Consequently, a product of $f(\alpha)\psi(\alpha)$ will be divisible by 47 if and only if p is contained in $f(\alpha)$ or, in another words:

A cyclotomic integer $f(\alpha)$ is said to be divisible by p if $f(\alpha)\psi(\alpha)$.
([Edwards, 1980], 325)

To test this condition of divisibility requires long complicated computations. It is important, here, to underline that Kummer does not define the ideal factors themselves, but only the divisibility of a cyclotomic integer by an ideal factor p . This definition by a condition of divisibility is one of the main points which Dedekind will try to avoid and to replace by another approach.

5.1.3 An example, explained by Dedekind.

In order to understand better what is at stake, I will give a little more details on a numerical example. This example will allow us to understand Kummer's approach, and Dedekind's reformulation of it. For this, I will use Dedekind's own explanation of an example, which he gave as an introduction to his ideal theory, in [Dedekind, 1876-1877]. Dedekind follows the method proposed by Kummer, since his aim is to illustrate the passage from ideal

wo aber eine Zerlegung in wirkliche complexe Primfactoren nicht möglich ist: dann sollen die idealen Primfactoren eintreten, um dieselbe zu leisten."

¹¹See the example I proposed p. 55 and the next paragraph.

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numbers to ideals ([Dedekind, 1876-1877], Chapter 2, §§7-12). I will follow Dedekind's text.

The example chosen by Dedekind is relatively simple. Let us consider the domain of numbers $\omega = x + y\theta$,¹² called \mathfrak{o} , where θ is a root of $\theta^2 + 5 = 0$ and x, y are rational integers. The four elementary operations of arithmetic can be defined as usual, in this domain. The norm of ω is, as defined by Dirichlet, the product of two conjugate numbers:

$$N(\omega) = (x + y\theta)(x - y\theta) = x^2 + 5y^2$$

Dedekind considers, here, the notion of decomposability, i.e., "a number (different from zero and ± 1) is called decomposable when it is the product of two factors, neither of which is a unit", and it is called "indecomposable" otherwise. Each decomposable number can be, then, expressed as the product of a finite number of indecomposable factors. Here, appears the property which one does not encounter in rational numbers theory, and which prompted the introduction of ideal factors by Kummer:

the same number is susceptible to several, essentially different, representations of this kind.¹³ ([Dedekind, 1876-1877], 87)

It is easy to see, Dedekind explains that each of the following numbers is indecomposable, by simple manipulations of their norm:

$$\begin{array}{cccc} a = 2 & b = 3 & c = 7 & \\ b_1 = -2 + \theta & b_2 = -2 - \theta & c_1 = 2 + 3\theta & c_2 = 2 - 3\theta \\ d_1 = 1 + \theta & d_2 = 1 - \theta & e_1 = 3 + \theta & e_2 = 3 - \theta \\ f_1 = -1 + 2\theta & f_2 = -1 - 2\theta & g_1 = 4 + \theta & g_2 = 4 - \theta \end{array}$$

However, when taking products of these numbers, one can see that the same number can be expressed in two or three different products of indecomposable numbers:

$$\begin{array}{l} a.b = d_1.d_2 ; b.b = b_1.b_2 ; ab_1 = d_1.d_1 ; a.c = e_1.e_2 \\ c.c = c_1.c_2 ; a.c_1 = e_1.e_1 ; b.c = f_1.f_2 = g_1.g_2 ; a.f_1 ; a.g_1 = d_1.e_2 \end{array}$$

Such an indecomposable number therefore does not possess the property which, in the theory of rational numbers, is characteristic of a prime number.¹⁴ (*ibid.*, 88)

¹²In this paragraph "number", unless stated otherwise, always designates a complex number.

¹³"un seul et même nombre est susceptible de plusieurs représentations de cette sorte, essentiellement différentes entre elles."

¹⁴"[U]n tel nombre indécomposable ne possède donc pas la propriété qui, dans la théorie des nombres rationnels, est tout à fait caractéristique pour un nombre premier."

Dedekind explains the idea of introducing ideal factors as “imagining” that the numbers a, b, c, a_1 , etc. are rational integers, and deducing a decomposition into primes from the relationships of their product. Feigning such a decomposition allows simple manipulations of the divisibility relationships and to express the numbers a, b, c, a_1 , etc. as products of five numbers $\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2$:

$$\begin{aligned} a &= \alpha^2 & b &= \beta_1\beta_2 & c &= \gamma_1\gamma_2 \\ b &= \beta_1^2 & b_2 &= \beta_2^2 & c_2 &= \gamma_1^2 & c_2 &= \gamma_2^2 \\ d_1 &= \alpha\beta_1 & d_2 &= \alpha\beta_2 & e_1 &= \alpha\gamma_1 & e_2 &= \alpha\gamma_2 \\ f_1 &= \beta_1\gamma_1 & f_2 &= \beta_2\gamma_2 & g_1 &= \beta_1\gamma_2 & g_2 &= \beta_2\gamma_1 \end{aligned}$$

We have, here, a decomposition into prime factors, in which the numbers “behave, in all questions of divisibility in the domain [considered], exactly as if they were composed, (...) of five different prime numbers.” The ideal factors, then, do have the character of prime numbers, that is, they divide a product only if they divide at least one of the factors.

How does one find these factors? For example, consider the following case: Two conjugate numbers are always congruent modulo 2, so one has

$$\omega^2 \equiv N(\omega) \pmod{2} \quad \text{and} \quad \omega^2\omega'^2 \equiv N(\omega)N(\omega') \pmod{2}$$

Hence, 2 divides the product $\omega^2\omega'^2$ and the product of two rational numbers $N(\omega)N(\omega')$, which implies that 2 divides at least one of the two norms, and consequently at least one of the two ω^2 and ω'^2 . In the mean time, if one chooses x and y to be odd numbers, then $\omega = x + \theta y$ is not divisible by 2, while its square is. Then, we “say that the number 2 behaves in our domain as though it were the square of a prime number α .” Although α does not exist, in the domain \mathfrak{o} , it can be introduced, as it was by Kummer, as an “ideal factor”. It is important, now, to be able to describe α by means of the existing numbers, in the domain, which is done by using congruences.

The same strategy can be applied to the other indecomposable numbers. After a close analysis of the domain \mathfrak{o} , Dedekind obtains a series of conclusions about the “divisibility laws” of the numbers:

- 1 Each positive rational prime number which is $\equiv 11, 13, 17, 19 \pmod{20}$ behaves like (actual) prime numbers.
- 2 The number θ whose square is -5 has the character of a prime number, and the number 2 behaves like the square of an ideal prime number α .
- 3 Each positive rational prime number which is $\equiv 1, 9 \pmod{20}$ can be decomposed into two different actually existing factors, which have the character of prime numbers.

5.1 Ideal numbers

- 4 Each positive rational prime number which is $\equiv 3, 7 \pmod{20}$ behaves like the product of two different ideal prime numbers.
- 5 Each actual number ω (different from zero and ± 1) either is one of the numbers mentioned above as having the character of a prime, or behaves as a unique product of actual or ideal prime factors for all questions of divisibility.

To obtain such results, it is necessary to develop a great amount of computations, and to use quadratic residues and binary quadratic forms. For Dedekind, those are unnecessary complications, which increase the risk of errors and hasty conclusions. Besides, although it is possible to “reach the proposed goal with all rigour,” Dedekind still advises for “the greatest circumspection”, since letting oneself being led by the analogy with rational number theory might lead to “premature conclusions”. Indeed, in following the analogy, one tends to trust that the same definitions hold and the results will be obtained, and to overlook certain definitions and proofs. Since the 1857 paper, then, Dedekind’s ideas have evolved from considering results arising from the analogy as evident, to the necessity of proving those results.

Here, the product of two arbitrary factors is a point of peculiar difficulty, and “cannot be exactly defined without going into minute detail.” Again, the risk of error is increased. Moreover, this approach demands to analyze independently each domain investigated, and in this domain to consider again multiple particular cases. Not only does it lack a certain uniformity in the methods of proof and definition, but there is also no guarantee that it can work in any domain, or in more general domains – and indeed, it does not.

Dedekind proposes to bypass the difficulties mentioned above by the following considerations. We saw that for a number $\omega = x + y\theta$ to be divisible by an ideal prime number α , it needs to satisfy the congruence $x \equiv y \pmod{2}$. This congruence gives us $x = y + 2z$ (with y and z rational integers), hence the numbers divisible by α are all of the form $2z + (1 + \theta)y$. Rather than considering the ideal prime number α , one can thus consider the system \mathfrak{a} of all the numbers of the form $2z + (1 + \theta)y$. These numbers form what Dedekind calls a “finite module” denoted by \mathfrak{m} , a system of (real or complex) numbers closed by addition and subtraction, and generated by 2 and $1 + \theta$. The same can be done for the other ideal prime numbers identified earlier.

Those systems enjoy an additional property: each product of a number in the system \mathfrak{m} by a number in the \mathfrak{o} studied is again a number in the system. Dedekind calls these systems “ideals”, and explains how they can be used to study the laws of divisibility of the domain \mathfrak{o} . In fact, as we will see, ideals can be used more generally to study the laws of divisibility of any domain of algebraic integers.

Let me, now that the main ideas of Dedekind's approach are clear, consider his 1871 theory of algebraic integers in more details. [Edwards, 1980] provides a thorough analysis of the texts of Kummer, Dedekind and Kronecker and the reformulation in modern mathematical terms which I will not give, in the following sections. My reading has been informed by Edwards's works, as well as by [Boniface, 2004] and the introduction of Avigad's translation of the Xth Supplement ([Avigad, 2004]). I will consider Dedekind's text in more details than these works, so as to highlight, in the next chapter, the changes to which the theory is subjected in 1876/77.

5.2 A new framework: Fields

Kummer noticed that ideal numbers are of great use for the study of forms, but he never published any work on the subject. The aim of Dedekind's Xth Supplement is to "introduce the reader to these new ideas." He explains in the preface to the second edition of the *Vorlesungen* that, in the Xth Supplement, he integrated the "general theory of ideals" to the theory of binary quadratic forms in order to "shed a new light on the subject matter of the book, from a higher viewpoint." Dedekind restricted himself to the basic principles but he expresses the hope that "the quest of characteristic basic concepts, which meets so great successes in other parts of mathematics, might not be completely unsuccessful with [him]"¹⁵ (Vorwort [Lejeune-Dirichlet, 1871], repr. in [Dedekind, 1932] III, 399-400, my translation). By the search of "characteristic basic concepts", Dedekind refers to the concepts of field and ideal. Dedekind considers these concepts, introduced in order to investigate the divisibility laws of algebraic integers, as founding the theory of algebraic numbers, and he proposes to use them in other areas of number theory, such as the theory of binary quadratic forms. For him, as he illustrates in each version of the theory, fields and ideals allow for a significant simplification of the theory and provide powerful methods of proofs.

5.2.1 An algebraic concept for arithmetical investigations

With the introduction of ideal theory, Dedekind succeeds, as he recalls in 1876-77, in "overcoming great difficulties" so as to establish a "rigorous and without exceptions" ideal theory ([Dedekind, 1876-1877], Introduction). Dedekind's work comes as a complement and a generalization of Kummer's

¹⁵ "Endlich habe ich in dieses Supplement eine allgemeine Theorie der Ideale aufgenommen, um auf den Hauptgegenstand des ganzen Buches von einem höheren Standpunkte aus ein neues Licht zu werfen; hierbei habe ich mich freilich auf die Darstellung der Grundlagen beschränken müssen, doch hoffe ich, daß das Streben nach charakteristischen Grundbegriffen, welches in anderen Teilen der Mathematik mit so schönen Erfolgen gekrönt ist, mir nicht ganz mißglückt sein möge."

5.2 A new framework: Fields

researches on ideal numbers exposed above. The approach he adopts in the Xth Supplement comes from the

conviction (...) that the study of the algebraic kinships between numbers is the most adequately (advantageously) grounded on a concept which is immediately tied in with the simplest principles of arithmetic.¹⁶ (Anzeige [Lejeune-Dirichlet, 1871], in [Dedekind, 1932] III, 399, my translation)

To do so, as I have mentioned earlier, he proposed to “introduce the reader to a higher domain in which algebra and number theory are joined with each other in the closest”¹⁷ (*ibid.*, 399) – a statement he makes again in the beginning of the paragraph on fields:

In our endeavors to introduce the reader to these new ideas, we are setting up for ourselves a somewhat higher standpoint and, from there, begin to introduce a new concept that seems to be well-suited to serve as a foundation for higher algebra and the related parts of number theory. ([Dedekind, 1871], transl. slightly altered, 12)

“Algebra” and “algebraic kinships” are, for Dedekind, related to equations. The algebraic relationships between numbers, studied already in the lectures on Galois theory, are the relationships between numbers established by polynomial equations. Here, Dedekind is working with “algebraic numbers”, that is, numbers which are defined as roots of a polynomial equation: a number θ is an algebraic number if it satisfies an equation

$$\theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n = 0 \quad (1)$$

of finite degree n and whose coefficients a_1, a_2, \dots, a_n are rational numbers. It will be called an *algebraic integer* if it satisfies such an equation (1), in which the coefficients a_1, a_2, \dots, a_n are integers.

For Dedekind, algebra is the theory of equations – a position which is not “modern” and which Dedekind still holds in 1894.¹⁸ In the notice for p. Bachmann’s *Die Lehre von der Kreisteilung und ihre Beziehungen zur Zahlentheorie*, ([Dedekind, 1873]), Dedekind explains the links between fields and equations, making their intimate relationship clear: a field can express

¹⁶“[D]rängte sich mir die Überzeugung auf, dass das Studium der algebraischen Verwandtschaft der Zahlen am zweckmässigsten auf einen Begriff gegründet wird, welcher unmittelbar an die einfachsten arithmetischen Prinzipien anknüpft.”

¹⁷“In den nun noch folgenden Paragraphen habe ich versucht, den Leser in ein höheres Gebiet einzuführen, in welchem Algebra und Zahlentheorie sich auf das Innigste miteinander verbinden.”

¹⁸Even though Dedekind’s theory of algebraic numbers becomes more “abstract” and “modern” through the rewritings, the ‘equation’ still holds a central and essential place in his approach.

the properties of an equation and relations between equations correspond to relations between fields.

As a matter of fact, from the seeds planted by Gauss, a science is born which one might designate as the science of the algebraic kinships of numbers or, if one wishes to use an expression I chose, as the science of the kinships between fields. Indeed, it is clear that the particular properties of an equation, the possibility that its solution arises from other equations, can first only be clearly known if one considers, beside its roots, also an infinity of other numbers, which are rationally derivable from one or several of them and whose collection (*Inbegriff*) forms precisely what I called a field, that is, a system of numbers which reproduce themselves by the four simplest rational operations of arithmetic. The properties of an equation will, by this conception, correspond to properties of the corresponding field. The relations between two equations are replaced by kinship between the fields.¹⁹ ([Dedekind, 1873], 409, my translation.)

The idea that algebraic relationships between numbers are best studied using fields, or on the basis of fields arose while he was working on Galois theory (as integrated in number theory), for his lectures in Göttingen. He explains that it was then that the “conviction that the algebraic kinships of numbers are the most appropriately grounded in a concept which is immediately tied with the simplest principles of arithmetic”, that is the concept of field²⁰ (Anzeige [Lejeune-Dirichlet, 1871], in [Dedekind, 1932] III, 400).

As I already alluded to, and as it will become clear, the field, a domain of numbers closed by the four operations of rational arithmetic, works as a

¹⁹ “*In der Tat ist aus dem von Gauss gelegten Keime eine Wissenschaft entstanden, welche man, (...) vielleicht als die Wissenschaft von der algebraischen Verwandtschaft der Zahlen oder, wenn man sich eines von mir gewählten Ausdruckes bedienen will, als die Wissenschaft von der Verwandtschaft der Körper bezeichnen könnte. Es zeigt sich nämlich, da die eigentümliche Beschaffenheit einer Gleichung, die Möglichkeit, ihre Auflösung auf die von anderen Gleichungen zurückzuführen, erst dann deutlich erkannt werden kann, wenn man außer ihren Wurzeln noch unendlich viele andere Zahlen betrachtet, welche aus einer oder mehreren von ihnen rational ableitbar sind und deren Inbegriff eben das bildet, was ich einen Körper nenne, nämlich ein System von Zahlen, die sich durch die vier einfachsten, rationalen arithmetischen Operationen immer wieder reproduzieren. Eigenschaften einer Gleichung werden bei dieser Auffassung zu Eigenschaften des entsprechenden Körpers, Beziehungen zwischen Gleichungen stellen sich dar als Verwandtschaft zwischen den Körpern.*”

²⁰ “*Im Laufe der Vorlesungen über Kreisteilung und höhere Algebra, welche ich zu Göttingen im Winter 1856-1857 vor den Herrn Sommer und Bachmann, im Winter 1857-1858 vor den Herrn Selling und Auwers gehalten habe, drängte sich mir die Überzeugung auf, daß das Studium der algebraischen Verwandtschaft der Zahlen am zweckmäßigsten auf einen Begriff gegründet wird, welcher unmittelbar an die einfachsten arithmetischen Prinzipien anknüpft. Den damals von mir benutzten Namen ‘rationales Gebiet’ habe ich später mit dem Worte ‘Körper’...*”

5.2 A new framework: Fields

framework for the study of number theory. Fields are essential insofar as they encompass the arithmetical relationships of numbers (by the definition of field itself), but also the algebraic relations of numbers. The “simplest principles of arithmetic”, then, especially if the fundamental concept is indeed that of field (which Dedekind repeats several times until 1894), seem to be the relationships established by the four operations of rational arithmetic, addition, multiplication, subtraction and division. A closer look at the theory of algebraic numbers will reveal that divisibility plays an important role.²¹

5.2.2 The concept of field in 1871

The “higher standpoint” which Dedekind proposes to adopt so as to develop Kummer’s ideas implies to introduce several new concepts, the first of which is the field (*Körper*):

By a field, we mean an infinite system of real or complex numbers, which is closed and complete in itself, so that the addition, subtraction, multiplication, and division of any two of these numbers always yields yet another number of the same system.²²
(*ibid.*, 12)

It was Dedekind’s intention, when choosing the term “*Körper*”, to describe the wholeness of a domain of numbers closed by the four rational operations, as he retrospectively explains in a footnote of the 1894 re-edition of Dirichlet’s *Vorlesungen*:

This name should, just as in the natural sciences, in geometry and in the life of human society, designate here too a system which possesses a certain completeness, perfection, closure, whereby it appears as an organic whole, als a natural unity.²³
([Dedekind, 1894a], in [Dedekind, 1932] III, 20)

²¹As it was explained in the *Habilitationsvortrag*, and will be mentioned again in later foundational works, the operations and their laws are, for Dedekind, guiding the development of arithmetic up to complex integers. Complex (and algebraic) integers have been defined in such a way that the laws established for addition, subtraction, and multiplication are (generally) valid. But it appears clearly with Kummer’s works that the divisibility laws don’t generalize as easily, which leads to an emphasis on divisibility.

²²“*Unter einem Körper wollen wir jedes System von unendlich vielen reellen oder komplexen Zahlen verstehen, welches in sich so abgeschlossen und vollständig ist, dass die Addition, Subtraction, Multiplication und Division von je zwei dieser Zahlen immer wieder eine Zahl desselben Systems hervorbringt.*”

²³“*Dieser Name soll, ähnlich wie in den Naturwissenschaften, in der Geometrie und im Leben der menschlichen Gesellschaft, auch hier ein System bezeichnen, das eine gewisse Vollständigkeit, Vollkommenheit, Abgeschlossenheit besitzt, wodurch es als ein organisches Ganzes, als eine natürliche Einheit erscheint.*”

Despite its non-mathematical aspect, this is an important part of Dedekind's concept of field and it seems to be underlying the difference of treatment between fields and ideals, modules or groups – the “conceptual difference” noticed by Corry. Note that the definition of a field explicitly mentions that the elements are numbers. While the definition as a domain closed by the operations can (and will) allow the indifference towards the elements, Dedekind doesn't seem interested in developing a theory of “abstract fields” – a point probably linked to the role of framework assumed by fields, also noticeable in the difference underlined between the “simplest field”, that of rational numbers, and the “largest field”, that of “all numbers.” ([Dedekind, 1871], 12)

One can define an inclusion relation between two or more fields, which is done by using the reformulation of inclusion in terms of divisibility already used for groups:

We call a field A a divisor of field M , and the latter a multiple of the former, when all the numbers in A are also found in M . One easily sees that the field of rational numbers is a divisor of every other field.²⁴ (*ibid.*, 12)

The “collection of numbers which are in two fields,” i.e., the intersection of two fields A and B is called their “*greatest* common divisor” D , since it satisfies the property that any divisor of A and B is necessarily a divisor of D (any field which is a subfield of A and a subfield of B is a subfield of D). In the same way, “there is always a field M which should be called the least common multiple of A and B , since it is a divisor of every other common multiple of both fields.”²⁵ These arithmetical properties appear as integral parts of the definition. They are neither proved, nor justified²⁶ – for example, the fact that the “greatest common divisor” is indeed a field isn't proved. Dedekind doesn't prove either the existence of the GCD or the LCM, and does not explain how to exhibit them. In addition, these arithmetical relationships will not be used in the theory, contrary to the ones which will be defined for modules or ideals. Divisibility properties for fields are immediately followed by the definition of a conjugate field:

If furthermore there corresponds to each number a in the field A a number $b = \varphi(a)$, in such a way that $\varphi(a + a') = \varphi(a) + \varphi(a')$

²⁴ “Wir nennen einen Körper A einen Divisor des Körpers M , diesen ein Multiplum von jenem, wenn alle in A enthaltenen Zahlen sich auch in M vorfinden; man findet leicht, dass der Körper der rationalen Zahlen ein Divisor von jedem andern Körper ist.”

²⁵ “Der Inbegriff aller Zahlen, welche gleichzeitig in zwei Körpern A, B enthalten sind, bildet wieder einen Körper D , welcher der größte gemeinschaftliche Divisor der beiden Körper A, B genannt werden kann (...); ebenso existiert immer ein Körper M , welcher das kleinste gemeinschaftliche Multiplum von A und B heißen soll.”

²⁶ Note, also, that the inclusion is inverted, here. While for groups, modules, and ideals, to divide is to contain, for fields, the divisor is contained is the multiple.

5.2 A new framework: Fields

and $\varphi(aa') = \varphi(a)\varphi(a')$, then the numbers b (assuming they do not all vanish) form a field B , which is conjugate to A , and arises from A from the *substitution* φ ; and conversely $B = \varphi(A)$ is *conjugate to* B .²⁷ (*ibid.*, 13)

The fields in which Dedekind will work are finitely generated fields, which he calls *finite fields*. Finite fields can be defined in different ways: as corresponding to an equation, as having a finite basis, as having a finite number of divisors. This last definition is the one that Dedekind considers as the “best” definition, i.e., the definition based on fundamental properties of the field concept. However, this definition is not the most effective one and appears to be difficult to use. To develop the theory in an efficient way – and notably by relying on already known things, Dedekind made a concession to his own principles, as he explains to Lipschitz, when discussing the publication of the second version of the theory:

The simplest theories are always rapidly disfigured by the unnecessary immixtion of forms of representation, which should actually be the result, nor the instrument (means) of the theory. In the same way, I disfigure, in the introduction, the concept of a finite field Ω , by giving it a presentation (*Darstellungform*), in which all the numbers of the field are included, and which could as well be replaced by an infinite number of other [equivalent] presentations²⁸ (...) In principle, the definition given in the §159 of the theory of numbers is by far preferable: “a finite field is such that it only has a finite number of divisors”, or again the following equivalent definition: “a finite field is a field such that it only contains a finite numbers of mutually independent numbers.” But I have mad this concession, in order to borrow as little as possible from the general theory of fields, and in order to make the link with generally known things.²⁹ ([Lipschitz and

²⁷ “Entspricht ferner einer jeden Zahl a des Körpers A eine Zahl $b = \varphi(a)$ in der Weise, dass $\varphi(a + a') = \varphi(a) + \varphi(a')$, und $\varphi(aa') = \varphi(a)\varphi(a')$ ist, so bilden die Zahlen b (falls sie nicht sämmtlich verschwinden) ebenfalls einen Körper $B = \varphi(A)$, welcher mit A conjugirt ist und durch die Substitution φ aus A hervorgeht; dann ist rückwärts auch $A = \psi(B)$ mit B conjugirt.”

²⁸This is a reference to the choice of a variable to describe the elements of the finite field.

²⁹ “[F]ast immer wird die einfache Theorie verunziert durch unnöthige Einnischung der Darstellungsformen, welche doch eigentlich nur Resultat, nicht Hilfsmittel der Theorie sein sollten. In ähnlicher Weise verunziere ich in der Einleitung den Begriff eines endlichen Körpers Ω dadurch, dass ich eine Darstellungsform angebe, in welcher alle Zahlen des Körpers enthalten sind und welche ebenso gut durch unendlich viele Darstellungsformen ersetzt werden könnte (...). Principiell ist daher die in der Zahlentheorie §159 (1871) gegebene Definition bei Weitem vorzuziehen ‘ein endlicher Körper ist ein solcher, der nur eine endliche Anzahl von Divisoren besitzt’ oder auch die hiermit abermals äquivalente Definition: ‘ein endlicher Körper ist ein solcher Körper, welcher nur eine

al. 1986], 60, my translation)

Finite fields are, in other words, fields with a finite basis³⁰:

[I]n a field Ω of the indicated type [i.e., finite], it is possible to choose a finite number (*Anzahl*) n of independent numbers $\omega_1, \omega_2, \dots, \omega_n$ so that every number ω of the field can be represented in the form

$$\omega = h_1\omega_1 + h_2\omega_2 + \dots + h_n\omega_n = \sum h_i\omega_i$$

in a unique way, where h_1, h_2, \dots, h_n denote *rational numbers*. We call the number n the *degree*, the system of n independent numbers ω_i a *basis of the field* Ω , and the n numbers h_i the *coordinates* of the number ω corresponding to this basis.³¹ ([Dedekind, 1871], 13)

Dedekind proves that the numbers ω defined by their coordinates do satisfy the properties of closure by the four operations: the addition and subtraction are obvious manipulations of sums. For multiplication and division, Dedekind uses an analytical treatment of forms. In modern terms, the field Ω is considered as a vector space over \mathbb{Q} and Dedekind studies its multiplicative ‘structure’ by looking at the product $\omega_k\omega_j$ of two elements of the basis expressed in the basis ω_i ([Avigad, 2004]). This is done in the framework of quadratic forms, with crossed uses of methods from analysis and determinants. The rest of the paragraph continues the analytic considerations on forms, with an approach close to that of Jacobi’s and Dirichlet’s and traditional for a work on quadratic forms in 1871.³²

In the course of these considerations, Dedekind is able to show that any number of the finite field Ω considered “is the root of an equation of degree n with rational coefficients” which does not depend on the basis chosen.³³ He has thus made explicit that the numbers contained in Ω are algebraic numbers. The notion of algebraicity as being the root of a polynomial equation was sufficiently well-known for Dedekind to not feel the need to give

endliche Anzahl von einander unabhängiger Zahlen enthält’. Aber ich habe diese Concession gemacht, um aus der allgemeinen Theorie der Körper möglichst wenig zu entlehnen und um an allgemein bekannte Dinge anzuknüpfen.”

³⁰In other words, again, the degree of the equation which generates the field is finite.

³¹“aus einem Körper Ω von der angegebenen Art nur eine endliche Anzahl n von unabhängigen Zahlen ω_i sich auswählen lässt, dass also jede Zahl ω des Körpers stets und nur auf eine einzige Art durch die Form $\omega = h_1\omega_1 + h_2\omega_2 + \dots + h_n\omega_n = \sum h_i\omega_i$ darstellbar ist, wo h_1, h_2, \dots, h_n rationale Zahlen bedeuten. Wir wollen die Zahl n den Grad, ferner den Complex der n unabhängigen Zahlen ω_i eine Basis des Körpers Ω , und die n Zahlen h_i die dieser Basis entsprechenden Coordinaten der Zahl ω nennen.”

³²Jeremy Avigad’s introduction to his translation of §§159-163, [Avigad, 2004] provides some really clear explanations of this passage.

³³This allows for the definition to keep its desired generality, even though it is based on the choice of a basis.

5.3 Algebraic integers, a new and more general notion of integer

explanations when he talked about “algebraic numbers”.³⁴ The essential notion, for the theory, is that of *integer* defined in the next paragraph.

5.3 Algebraic integers, a new and more general notion of integer

5.3.1 Algebraic integers in 1871

To be able to find the right generalization of the notion of integer, i.e., the notion of integer for algebraic numbers, is a crucial step: the generalization of Kummer’s theory of ideal numbers has, at its core, the study of divisibility properties and requires thus to identify the appropriate notion of integer, but also of divisibility and of primality. Dedekind and Kronecker solved the question of integers in the same way, but, as underlined by Edwards ([Edwards, 1980], 332), there is no trace of their respective elaboration of the idea. Neither Dedekind nor Kronecker explained how they were able to identify this particular definition as the right one.³⁵

Algebraic integers are defined in the §160 (*Ganze algebraische Zahlen*) as elements of a field Ω of algebraic numbers.³⁶

We now want, to approach closer our actual subject, to call a number α an *algebraic integer* when it is the root of an equation whose coefficients are rational integers.³⁷ ([Dedekind, 1871], transl. modified, 25)

Algebraic integers are a more general notion of integral numbers, since any rational integer is also an algebraic integer.³⁸ In the 1877 version, Dedekind explicitly states that they are the “most general concept of integer”. Since the notion, here, is an extended concept of number, the study of its properties must involve redefinition of arithmetical operations and proofs of the validity of their laws. Addition, subtraction and multiplication involve

³⁴Dedekind makes a direct reference to Liouville’s works on transcendental numbers, in the next paragraph. Algebraic numbers as numbers solution of a polynomial equation are already a familiar notion.

³⁵Dedekind mentioned, in the 1894 version of his theory, that he doesn’t “know whether this appellation was already used before [his 1871 Xth Supplement] in this sense.” And in fact, while writers such as Dirichlet, Eisenstein or Hermite do work with monic polynomials with integer coefficients and take advantage of the closure by multiplication, they do not consider the question of identifying the criterion for defining the appropriate concept of integer for algebraic numbers (see [Edwards, 1980]).

³⁶Algebraic numbers, writes Dedekind “also clearly constitute a field.” The proof is left to the reader.

³⁷“Wir wollen nun, indem wir unserem eigentlichen Gegenstande näher treten, eine Zahl a eine ganze algebraische Zahl nennen, wenn sie die Wurzel einer Gleichung ist, deren Coefficienten rationale ganze Zahlen sind.”

³⁸Dedekind chooses to call algebraic integers only “integers”. I will let this in quotations, but won’t follow him in the commentary, for clarity reasons.

simple manipulations of polynomials, and the kind of computations developed in the preceding paragraph. For divisibility, the definition is relatively orthodox, but it will be necessary to do more work on certain associated notions such as primality.

The integer α is said to be divisible by the integer β , or a multiple of β , if the quotient $\frac{\alpha}{\beta}$ is also an integer.³⁹ (*ibid.*, 26)

There is a natural notion of congruence for algebraic integers: " $\alpha \equiv \beta \pmod{\gamma}$ when $\alpha - \beta$ is divisible by γ "⁴⁰ (*ibid.*, 26) for which all theorems proven in rational integers "remain valid."

Since it is "well-known that the concepts of divisibility and multiplicity of the rational integers transfer directly to" polynomials, it is possible to give an algorithm perfectly analogous to Euclid theorem for finding the GCD of two algebraic integers, and establish the following result: *α root of the irreducible equation $\varphi(\alpha) = 0$ in a field K if and only if all the coefficients of φ are algebraic integers.* This result allows to consider polynomials with algebraic integers as coefficients. The well known rules of manipulations of polynomial equations also allow to prove that any algebraic number can be transformed into an algebraic integer by multiplication by a rational integer h and its multiples.

5.3.2 The redefinition of primality

In rational number theory, a prime number p is defined as being a number, different from 1, which is only divisible by 1 and by itself. Such a number is said to be irreducible or indecomposable. However, in domains such as those of cyclotomic integers or finitely generated fields of algebraic numbers, prime numbers cannot be defined in this way, for

one realizes at once that no such number exists. For, if α is an integer but not a unit, then α has infinitely many essentially different divisors, e.g., the numbers $\sqrt{\alpha}$; $\sqrt[3]{\alpha}$, $\sqrt[4]{\alpha}$ and so forth, which (...) are integers.⁴¹ (*ibid.*, 28)

The first step towards the redefinition of primality, proposed by Dedekind, is the redefinition of relative primality. It is not possible "*for the moment*" to define the relative primality of two numbers α and β by means of greatest common divisor. However, our experience in rational number theory teaches us that

³⁹ "Die ganze Zahl α heisst theilbar durch die ganze Zahl β , oder ein Multiplum von β , wenn der Quotient $\frac{\alpha}{\beta}$ ebenfalls eine ganze Zahl ist."

⁴⁰ "Ebenso setzen wir $\alpha \equiv \beta \pmod{\gamma}$, wenn $\alpha - \beta$ durch γ theilbar ist."

⁴¹ "erkennt man sofort, dass gar keine solche Zahl existirt; ist nämlich α eine ganze Zahl, aber keine Einheit, so besitzt sie immer unendlich viele wesentlich verschiedene Divisoren, z. B. die Zahlen $\sqrt{\alpha}$; $\sqrt[3]{\alpha}$, $\sqrt[4]{\alpha}$, welche (...) ganze Zahlen sind."

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several theorems followed from this definition, each of which, conversely, completely characterized the behavior of relatively prime numbers, without assuming knowledge of their divisors.⁴² (*ibid.*, 28)

The theorem chosen, which is at the basis of Kummer's characterization of primality as well, is the following: *if any number divisible by both a and b is divisible by ab , then a and b are relatively prime.*

These considerations lead us to put forward the following more general definition for the domain of algebraic integers:

*Two nonzero integers α , β are said to be relatively prime if every number that is divisible by α and β is also divisible by $\alpha\beta$.*⁴³ (*ibid.*, 28, original emphasis)

To use this property as definition allows for a more general characterization, because it is valid for *algebraic integers* as well as for rational integers, that is, two rational integers “relatively prime (...) in the old sense of the phrase (...) remain relatively prime in the new sense” (*ibid.*). To establish the new definition of relative primality, one has identified where the old definition failed – the assumption of the knowledge of the divisors – and identified an equivalent (in the formal sense) property whose validity remains in the extended domain of numbers. Since this property “characterizes completely the behavior” of two relatively prime numbers, it can be considered as a definitional property. What should decide of its appropriateness as a definition is its fruitfulness and the scope of its validity. It is easy to see that while the two properties were equivalent in the restricted domain of rational integers, the property giving the old definition is a consequence of the new one, for algebraic integers. It highlights the fact that the new definition is a necessary condition for relative primality. As Tappenden underlined, with this new definition, “[t]he most significant facts about prime numbers turn out to depend on [the property taken as the new definition]” ([Tappenden, 2008a], 268). Not only is this property more generally valid, which implies that all the theorems deducible from the old definition are also provable from the new definition, but a considerable amount of additional (more general) theorems follow from the new definition as well. For these reasons, the new definition appears to be the ‘right’ definition.

⁴²“*aber aus dieser Definition ergaben sich mehrere Sätze, deren jeder umgekehrt das Verhalten zweier relativen Primzahlen vollständig charakterisirt, ohne die Kenntniss ihrer sämtlichen Divisoren vorauszusetzen.*”

⁴³“*Diese Betrachtung veranlasst uns, folgende für das Gebiet aller ganzen algebraischen Zahlen gültige Erklärung aufzustellen.*

Zwei von Null verschiedene ganze Zahlen α , β heißen relative Primzahlen wenn jede durch α und β theilbare Zahl auch durch $\alpha\beta$ theilbar ist.”

With this redefinition of relative primality, the usual criteria for determining whether two integers are relatively prime can not be applied anymore, so Dedekind proposes a “test for relative primality” in a certain field K . Primality tests hold an important role in the rest of Dedekind’s theory. This shows that in 1871 Dedekind’s approach was still relatively close to Kummer’s. To test whether two algebraic integers are relatively primes, Dedekind proposes the following. In a field K , consider two algebraic numbers α^{-1} and β^{-1} .⁴⁴ It was proven earlier that it is always possible to change any algebraic number into an algebraic integer by multiplication by a rational number. Let a and b be the two smallest rational numbers such that $a\alpha^{-1}$ and $b\beta^{-1}$ are algebraic integers.

If it turns out that a and b are relatively prime, then we know that α and β are relatively prime numbers.⁴⁵ ([Dedekind, 1871], 29)

However, the criterion is not fully satisfactory, for it is not a necessary and sufficient condition: it is possible for α and β to be relatively prime algebraic numbers, without “the smallest rational multiples a, b ” to be relatively primes themselves. For example, $\alpha = 2 + i$ and $\beta = 2 - i$ are relatively primes, but the rational multiples⁴⁶ are $a = b = 5$, and hence are not relatively primes. To correct the insufficiency of this test, one has to consider the following theorem:

If two integers α, β satisfy the test for relative primality with respect to a field K to which they themselves belong, i.e., if each number in K that is divisible by both α and β is also divisible by $\alpha\beta$, then α, β are in fact relatively prime.⁴⁷ (*ibid.*, transl. slightly altered, 29)

Dedekind’s proof of this theorem involves the irreducible equation of a number ω divisible by α and β in a field K , with manipulation of the coefficients, in order to get to the conclusion that ω' such that $\omega = \alpha\beta\omega'$ can also be expressed as the root of an equation whose coefficients are integers of K . Consequently,

in order to understand the behavior of two integers α, β with respect to each other, it suffices to consider the smallest field K to which they both belong. And it is easy to see that these fields

⁴⁴i.e., such that $\alpha\alpha^{-1} = \epsilon, \beta\beta^{-1} = \epsilon$, where ϵ is a unity of the field K .

⁴⁵“zeigt sich nun, daß a und b relative Primzahlen sind, so sind auch α und β gewiss relative Primzahlen.”

⁴⁶ $\alpha^{-1} = \frac{1}{5}(2 - i)$, hence the smallest rational number a such that $a\alpha^{-1}$ is an integer, is 5.

⁴⁷“Wenn zwei ganze Zahlen α, β sich in einem Körper K , dem sie selbst angehören, als relative Primzahlen bewähren, d. h. wenn jede durch α und β theilbare Zahl des Körpers K auch durch $\alpha\beta$ theilbar ist; so sind α, β wirklich relative Primzahlen.”

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are always of the kind we considered in the preceding section [i.e., finite fields].⁴⁸ (*ibid.*, 29)

In the following, one will restrict the considerations to the algebraic numbers in a finite field Ω . An algebraic integer should be understood as an integer of the field Ω . The system of all integers of Ω is denoted by \mathfrak{o} and it is easy to show that it is closed by addition and subtraction.

A more effective criterion to test the relative primality of two algebraic integers is given a little further in the Supplement. For this, the notion of complete system of residues modulo a certain algebraic integer μ is necessary. The notion of complete system of residue modulo a rational integer was given in the §18 of the *Vorlesungen*: a complete system of residues modulo k is defined by picking out k numbers a_0, a_1, \dots, a_{k-1} such that for each i , $a_i \equiv i \pmod{k}$. For algebraic integers, ω and ω' in algebraic integers in the finite field K are called congruent with respect to a third algebraic integer μ if and only if the quotient $\frac{\omega-\omega'}{\mu}$ is again in \mathfrak{o} .⁴⁹ To determine whether two algebraic integers θ and μ are relatively prime,

one evidently only has to run ω through a complete system of residues $(\text{mod } \mu)$, and determine how often $\theta\omega \equiv 0(\text{mod } \mu)$. If it turns out that this happens only when $\omega \equiv 0(\text{mod } \mu)$, then each integer $\theta\omega$ that is divisible by θ and μ is also divisible by $\theta\mu$, whereby θ and μ are relatively prime. (...) If θ is relatively prime to μ (e.g., $\theta = 1$), then $\theta\omega$ runs through a complete residue system $(\text{mod } \mu)$ simultaneously with ω .⁵⁰ (*ibid.*, transl. slightly altered, 36)

Dedekind's method, here, is very close to the one used in Kummer's divisibility tests.⁵¹

⁴⁸ "Hieraus geht hervor, dass man, um das gegenseitige Verhalten zweier ganzen Zahlen α, β zu untersuchen, nur den kleinsten Körper K zu bilden braucht, welchem sie beide angehören; und dieser Körper ist, wie man leicht erkennt, immer von der im vorigen Paragraphen betrachteten Beschaffenheit."

⁴⁹The notion of congruence is first defined by means of *modules*. A module is "a system \mathfrak{a} of real or complex numbers α , whose sums and differences themselves belong to \mathfrak{a} " ([Dedekind, 1871], 30). I will say more about modules in next paragraph. Two algebraic numbers ω and ω' are called "congruent with respect with respect to \mathfrak{a} " (*congruent nach* \mathfrak{a}) when their difference is contained in \mathfrak{a} . In particular, the system \mathfrak{o} of all integers of a finite field Ω is a module, which gives the definition of congruence of two integers with respect to a third integer.

⁵⁰"braucht man offenbar ω nur ein vollständiges Restsystem $(\text{mod. } \mu)$ durchlaufen zu lassen und nachzusehen, wie oft $\theta\omega \equiv 0(\text{mod } \mu)$ wird; zeigt sich, dass dies nur dann eintritt, wenn $\omega \equiv 0(\text{mod } \mu)$ ist, so ist also jede durch θ und μ theilbare ganze Zahl $\theta\omega$ auch theilbar durch $\theta\mu$, mithin sind θ, μ relative Primzahlen (...). Ist θ relative Primzahl zu μ (z. B. $\theta = 1$), so durchläuft $\theta\omega$ gleichzeitig mit ω ein vollständiges Restsystem $(\text{mod. } \mu)$."

⁵¹See also [Edwards, 1980].

The analogy with rational numbers guides the questions and problems considered by Dedekind.⁵² The natural question that appears “[i]f one pursues the analogy with the theory of rational numbers further” is that of the decomposition of numbers of \mathfrak{o} into factors themselves belonging to \mathfrak{o} . While the possibility to decompose a certain number in infinitely many different ways highlighted “in the infinite field of *all* algebraic numbers” disappears, there is still another problem left. Indeed, it is not guaranteed that the representation of a decomposable number (i.e., a number that can be written as the product of at least two numbers in \mathfrak{o}) as a product of indecomposable numbers⁵³ is unique. This, according to Dedekind, “contradicts so many notions governing the character of prime numbers that we therefore can no longer recognize indecomposable numbers as prime” (*ibid.*, 37). Indecomposability cannot be considered as an essential, definitional property of prime numbers, and the situation cannot be handled as the question of relative primality was. The criterion for primality which “preserve[s] the character of primes” needs to be “more robust” (*kräftiger*) and the “behavior as modulus” of the numbers provides it:

An integer μ , which is not a unit, will be called a prime number if every product $\eta\rho$ that is divisible by μ , has at least one factor η or ρ that is divisible by μ .⁵⁴ (*ibid.*, 37)

How does one find a decomposition in primes for algebraic integers? Following Kummer’s approach, Dedekind explains how to identify “actual or *ideal* i.e., fictional factors” for a certain algebraic number μ . For any non prime algebraic number (which is not a unit), it is possible to find two algebraic numbers η and ρ themselves not divisible by μ , but whose product is divisible by μ . Consider ν the greatest common divisor in \mathfrak{o} of η and $\mu = \nu\mu'$ such that $\frac{\eta}{\nu}$ and $\frac{\mu}{\nu}$ are relatively prime. Then, $\nu\mu'$ is the decomposition of μ . The key is then to determine μ' , which is done with a congruence condition: μ' is such that all roots α' of $\eta\alpha' \equiv 0 \pmod{\mu}$ are divisible by μ' , and any α' divisible by μ' satisfies the congruence. Conversely, if there is in \mathfrak{o} a certain number μ' which divides all and only the roots of $\eta\alpha' \equiv 0 \pmod{\mu}$, then μ' also divides μ and $\nu = \frac{\mu}{\mu'} = \text{gcd}(\eta, \mu)$. Those considerations, still very faithful to the spirit of Kummer’s ideal numbers will disappear in the later versions of Dedekind’s theory.

It is not always possible to find such a number μ' in \mathfrak{o} , which is why Kummer introduced the idea of ideal factors, explains Dedekind:

[Kummer] came upon the fortunate idea of nonetheless feigning (*fingieren*) such numbers μ' and introducing them as *ideal* num-

⁵²See also [Schlimm, 2008] on analogies.

⁵³Such a decomposition is always possible.

⁵⁴“Eine ganze Zahl μ welche keine Einheit ist, soll eine Primzahl heißen, wenn jedes durch μ theilbare Product $\eta\rho$ wenigstens einen durch μ theilbaren Factor η oder ρ besitzt.”

5.4 The ideal numbers's new clothes

bers. The *divisibility* of a number α' by these ideal numbers μ' depends entirely on whether α' is a root of the congruence $\eta\alpha' \equiv 0 \pmod{\mu}$, and consequently these ideal numbers are only treated as divisors or as moduli.⁵⁵ (*ibid.*, transl. slightly altered, 38)

Although there is, in principle, “absolutely no problems with this manner of introducing” ideal factors, Dedekind considers that transferring directly “usual concepts of the *actual* numbers [to the fictional ones] can, initially, easily evoke mistrust of the certainty of proofs”⁵⁶ (*ibid.*, 38). To avoid this, Dedekind proposes a new formulation for Kummer’s idea: instead of considering fictional numbers, he proposes to “always consider *systems* of actual numbers.” Thus, rather than considering a (actual or ideal) factor μ' , Dedekind will consider the system of all numbers (in \mathfrak{o}) which are divisible by μ' , which he calls an *ideal*, in reference to Kummer’s ideal numbers. Importantly, to each ideal corresponds an ideal factor and conversely.

5.4 The ideal numbers's new clothes

The invention of ideals is, for Dedekind, a “means of clothing the theory in a different grab.” In 1871, Dedekind’s theory of algebraic numbers is developed following lines very close to Kummer’s theory, in particular with an intensive use of effective divisibility tests, which we encountered in the paragraph above and which hold an important place in the development of ideal theory as well. Although Dedekind did mention, in the preface of the *Vorlesungen*, that his work had been essentially guided by the research of new grounding concepts, he does not present his approach as departing essentially from Kummer’s.

Ideals will appear as being the central notion (and essential tool) of the theory. They are themselves intimately linked to the (study of) algebraic numbers, since they uncover their divisibility laws:

Kummer, through the creation (*Schöpfung*) of ideal numbers, opened a new path which not only allows for a more convenient manner of expression, but also leads to a deeper insight into the true nature of algebraic numbers.⁵⁷ (*ibid.*, transl. slightly

⁵⁵ “so kam [Kummer] auf den glücklichen Gedanken, trotzdem eine solche Zahl μ zu fingiren und dieselbe als ideale Zahl einzuführen; die Theilbarkeit einer Zahl α' durch diese ideale Zahl μ' besteht lediglich darin, dass α' eine Wurzel der Congruenz $\eta\alpha' \equiv 0 \pmod{\mu}$ ist, und da diese idealen Zahlen in der Folge immer nur als Theiler oder Moduln auftreten.”

⁵⁶In the subsequent version of ideal theory, Dedekind will be more critical towards Kummer’s approach, mentioning the difficulties encountered when trying to generalize it and the possibility of errors in the proofs.

⁵⁷ “Kummer, durch die Schöpfung der idealen Zahlen, einen neuen Weg betreten [hat], welcher nicht nur zu einer sehr bequemen Ausdrucksweise, sondern auch zu einer tieferen Einsicht in die wahre Natur der algebraischen Zahlen führt.”

altered, 12)

So as to develop the theory as generally and rigorously as wanted, Dedekind places himself in the “higher level” of ideals.

5.4.1 Module theory in 1871

In order to ease the development of the study of algebraic integers and the divisibility of ideals, Dedekind introduces “some very general observations” (*sehr allgemeine Betrachtung*) about modules, a “separate inquiry [which] will be of great use to us in [his] subsequent topic of study, as well as for many others”.⁵⁸

Module theory is explicitly presented as an independent and general theory (it is valid for any module, and it does not depend on the nature of the elements or on a particular representation of either numbers or modules), which is going to be *used* for the benefits of number theory. It will quickly appear to be at the core of the algebraic arsenal that Dedekind is developing. In 1871, the paragraph dedicated to modules is short (less than 3 pages), and the theory is not clearly motivated. It seems to mostly shorten certain proofs, but not to the point that it should justify a new concept, let alone an auxiliary theory. The usefulness of module theory, its scope and its relation to ideal theory are not very clear in 1871. Module theory will, in Dedekind’s subsequent works gains in detail and clarity. While keeping its status of an auxiliary theory, it will appear to be better integrated in the whole work, notably because of the emphasis put on the (structural) links between ideals and modules, which is absent here.

A module is a “system \mathfrak{a} of real or complex numbers α whose sums and differences themselves belong to \mathfrak{a} ”⁵⁹ (*ibid.*, 30). Modules are, as their name indicates, intimately related to congruences.⁶⁰ Previous works on higher congruences have paved the way for considerations about congruences of algebraic numbers.⁶¹ Congruences are central in the 1871 theory, and systematically put forward, suggesting a strong link between Dedekind’s works Kummer’s (and Gauss’s).

A notion of congruence with respect to a module is introduced as an integral part of the definition, so that modules appear as extending the idea of modulus:

⁵⁸ “welche für die nachfolgenden, sowie für viele andere, unserem Gegenstande fremde Untersuchungen von grossem Nutzen ist.”

⁵⁹ “Ein System \mathfrak{a} von reellen oder complexen Zahlen α , deren Summen und Differenzen demselben System \mathfrak{a} angehören, soll ein Modul heissen. . . ”

⁶⁰The word “Modul” is used (along with “Modulus”) in German, since Gauss, to designate the modulus of the congruence.

⁶¹Note that there is a stronger relationship between ideal theory and higher congruence theory, which is explained in detail in “Über den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen” ([Dedekind, 1878]).

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[W]hen the difference of two numbers ω, ω' is contained in \mathfrak{a} , we will call them congruent with respect to \mathfrak{a} , and denote this with the congruence

$$\omega \equiv \omega' \pmod{\mathfrak{a}}$$

Such congruences can be added and subtracted, and therefore also multiplied by an arbitrary rational integer, like equations.⁶² (*ibid.*, transl. slightly altered, 30)

At least, it should be clear that modules are extending the notion of modulus once divisibility for modules is defined. In subsequent versions of the theory, the order of presentation is reversed, and divisibility is introduced before congruences. With such a reformulation, the theory might move away from its Kummerian inspiration, but the transposition of arithmetic operations to modules (and ideals) seems easier to understand.

The definition of a notion of divisibility for modules is similar to that defined for groups in the lectures on Galois theory:

If all the numbers of a module \mathfrak{a} are also numbers of a module \mathfrak{d} , then \mathfrak{a} is called a multiple of \mathfrak{d} , and \mathfrak{d} a divisor of \mathfrak{a} . Alternatively, we say \mathfrak{d} divides \mathfrak{a} (*gehe in \mathfrak{a} auf*), or \mathfrak{a} is divisible by \mathfrak{d} . From any congruence $\omega \equiv \omega' \pmod{\mathfrak{a}}$, it follows that also $\omega \equiv \omega' \pmod{\mathfrak{d}}$.⁶³ (*ibid.*, 30)

The divisibility for modules is, then, an inclusion relation: \mathfrak{d} divides \mathfrak{a} means that \mathfrak{a} is included in \mathfrak{d} . Dedekind will say, in later version of the theory, that a module \mathfrak{a} divides a number α when $\alpha \in \mathfrak{a}$, since it is equivalent to say that the module $[\alpha]$ generated α is included in \mathfrak{a} . Then, one understands why $\omega - \omega'$ contained in \mathfrak{a} is expressed as a congruence according to \mathfrak{a} :⁶⁴

$$\omega \equiv \omega' \pmod{\mathfrak{a}} \iff \omega - \omega' \in \mathfrak{a} \iff \mathfrak{a} \text{ divides } \omega - \omega'.$$

Divisibility considerations for modules are pursued a little further with the notions of GCD and LCM. For \mathfrak{a} and \mathfrak{b} two modules, Dedekind defines the LCM of \mathfrak{a} and \mathfrak{b} as the collection of all the numbers contained both in \mathfrak{a} and \mathfrak{b} (i.e., $\text{LCM}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} \cap \mathfrak{b}$). The GCD of \mathfrak{a} and \mathfrak{b} is the module composed by

⁶² “[W]enn die Differenz zweier Zahlen ω, ω' in \mathfrak{a} enthalten ist, so wollen wir sie congruent nach \mathfrak{a} nennen und dies durch die Congruenz $\omega \equiv \omega' \pmod{\mathfrak{a}}$ andeuten. Solche Congruenzen können addirt, subtrahirt und folglich auch mit beliebigen ganzen rationalen Zahlen multiplicirt werden, wie Gleichungen.

⁶³ “Wenn alle Zahlen eines Moduls \mathfrak{a} auch Zahlen eines Moduls \mathfrak{d} sind, so heisse \mathfrak{a} ein Vielfaches von \mathfrak{d} , und \mathfrak{d} ein Theiler von \mathfrak{a} ; oder wir sagen auch, \mathfrak{d} gehe in \mathfrak{a} auf, \mathfrak{a} sei theilbar durch \mathfrak{d} . Aus jeder Congruenz $\omega \equiv \omega' \pmod{\mathfrak{a}}$ folgt auch $\omega \equiv \omega' \pmod{\mathfrak{d}}$.”

⁶⁴ Dedekind draws this from considerations on bases of modules, however he does not make the link explicit himself. As mentioned earlier, the exposition of module theory is rather short and motivated essentially by the possibility to shorten proofs and to hide away certain computations.

all the numbers of the form $\alpha + \beta$ where α and β respectively run through all the numbers of \mathfrak{a} and \mathfrak{b} .

Dedekind defines a finite module, which is a module generated by n numbers $\omega_1, \dots, \omega_n$, i.e., the numbers of the form

$$\omega = h_1\omega_1 + \dots + h_n\omega_n$$

in which h_1, \dots, h_n “run through the rational integers”. The finite system of numbers $\omega_1, \dots, \omega_n$ is called a basis of the module “whether they are dependent or independent”. Finite modules hold the key to the central results of this paragraph. Indeed, if a certain finite module \mathfrak{o} can be transformed in another finite module \mathfrak{m} by multiplication by nonzero rational numbers, “then \mathfrak{o} contains only a finite number of incongruent numbers (mod \mathfrak{m}).” It is important to be able to determine the exact number of incongruent numbers of \mathfrak{o} (mod \mathfrak{m}). Indeed, consider \mathfrak{a} the least common multiple of \mathfrak{o} and \mathfrak{m} . For ω, ω' in \mathfrak{o} , we have $\omega \equiv \omega' \pmod{\mathfrak{m}} \iff \omega \equiv \omega' \pmod{\mathfrak{a}}$. The task is, then, to determine of how many classes modulo \mathfrak{a} consists \mathfrak{o} and this number is “equal to the determinant of the system of coefficients of the n particular numbers (...) which form the basis of” the LCM \mathfrak{a} . Determinants are important in a number of proofs, but more significantly Dedekind is able to prove that

[i]f μ is a nonzero integer of the field Ω , then the number of integers of the field that are incongruent with respect to μ is equal to the absolute value of the norm of the modulus μ .⁶⁵
(*ibid.*, 35)

This result is used to define the norm of modules: the number of incongruent numbers modulo \mathfrak{a} “is called the norm of \mathfrak{a} .”

5.4.2 Ideals in 1871

In order to properly generalize Kummer’s results and prove the general validity of the fundamental theorem of arithmetic, Dedekind introduces the notion of ideal, which remodel and generalize Kummer’s ideal factors. In order to demonstrate the validity of the (old) divisibility laws in the extended domains of (algebraic) numbers, Dedekind builds upon Kummer’s works on ideal factors. But to be able to provide a suitable generalization of Kummer’s works demands to make some changes. Besides, from Dedekind’s viewpoint, in Kummer’s approach, insofar as it “feigns” the existence of divisors, “the immediate transfer of the usual concepts of the actual numbers can, initially, easily evoke mistrust of the certainty of the proof” (*ibid.*, 38).

⁶⁵ “Ist μ eine von Null verschiedene ganze Zahl des Körpers Ω , so ist die Anzahl der nach dem Modul μ incongruenten ganzen Zahlen des Körpers gleich dem absoluten Werth der Norm des Moduls μ .”

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Dedekind replaces Kummer's ideal numbers by the notion of ideal, to bypass the difficulties and establish a surer basis for the theory. Ideals are systems of (infinitely many) actually existing numbers, and do not require the creation of new entities, such as Kummer's ideal factors. Rather than an ideal factor p , Dedekind considers the system of all existing numbers in the domain of integers that are divisible by p and calls it an ideal (even though it is a system of numbers which actually exist). The choice of terminology is meant to display that Dedekind is following Kummer's works, but as underlined by Klein, in [Klein, 1979], who "always found Dedekind's terminology unpleasant":

it is altogether unintuitive. He calls these sets *ideals*, and, when an "actual" common factor exists, *principal ideals*! (For example, $2\mu + 2\nu\sqrt{-5}$ is a principal ideal, since 2 is an actual factor.) He should have spoken of "reals". For it is a question of an aggregate of numbers that are actually present in the given integral domain. ([Klein, 1979], 304)

The aim of Dedekind's ideal theory, more precisely, is to prove the validity of the known theorems and laws of arithmetic for algebraic integers, by transferring the usual concepts of rational arithmetic to ideals. Thus, he introduces notions such as divisibility of ideals, prime ideal, greatest common divisor and least common multiple of ideals. Dedekind's ideals can be regarded as extending or generalizing the notion of divisibility. The generalization (which implies to give up previous definitions) is carried so as to assure the unconditional validity of fundamental laws of arithmetic. That particular point is made particularly clear in Dedekind's long and explicit introduction to the second version of the theory:

(...) these ideal numbers are introduced with no other goal than restoring the laws of divisibility in the numerical domain \mathfrak{o} [of algebraic integers] to complete conformity with the theory of rational numbers, (...) as we pursue the goal of restoring the laws of divisibility in the domain \mathfrak{o} to complete conformity with those ruling the domain of rational integers, by introducing ideal numbers and a corresponding language...⁶⁶ ([Dedekind, 1876-1877], 59-60)

In 1871, the study of algebraic integers is mostly embedded in the larger theory of binary quadratic forms. The question of the laws of divisibility,

⁶⁶ "(...) l'introduction des nombres idéaux n'a pas d'autre but que de ramener les lois de la divisibilité dans le domaine numérique \mathfrak{o} [des entiers algébriques] à une complète conformité avec la théorie des nombres rationnels (...) nous poursuivons le but de ramener généralement, par l'introduction des nombres idéaux et d'un mode de langage correspondant, les lois de la divisibilité dans le domaine numérique \mathfrak{o} à une complète conformité avec celles qui règnent dans le domaine des nombres entiers rationnels (...)"

then, is only presented as the desire to have, for algebraic integers, the conservation of certain properties of the divisibility of rational integers, in order to ease further developments of binary quadratic form theory.

The subject of Dedekind's inquiry is thus "the theory of the numbers of \mathfrak{o} ", the set of all algebraic integers of a finite field Ω . The core novelty of his approach is that he *grounds* (*gründen*) it on the new concept of ideal. This means that the main results of the theory will be deduced from the concept of ideal. Ideal theory assures that the theory of the numbers of \mathfrak{o} be developed with the sought degree of rigor and generality. Ideal are defined by two necessary and sufficient conditions:

A system \mathfrak{a} of infinitely many numbers contained in \mathfrak{o} will be called an ideal if it satisfies the following pair of conditions:

- I. The sum and difference of any two numbers in \mathfrak{a} is again a number in \mathfrak{a} .
- II. Each product of a number in \mathfrak{a} and a number in \mathfrak{o} is again a number in \mathfrak{a} .⁶⁷ (*ibid.*, 38)

The formal definition is immediately followed by that of the divisibility relation between a number and an ideal. Arithmetic properties are presented, as part of the definition, as immediately relevant and apparently essential properties of ideals:

If α is contained in [the ideal] \mathfrak{a} , we will say that α is *divisible by* \mathfrak{a} , and that \mathfrak{a} *divides* α , since by this manner of expression we gain in facility.⁶⁸ (*ibid.*, 452, transl. slightly altered, 38)

Note that Dedekind explicitly introduces this arithmetical terminology as a characterization of set-theoretic relationships because it makes things *easier*. The movement of re-definition of arithmetical relationships is a two-fold move: first, the divisibility of algebraic numbers is defined in terms of set-theoretic relations between ideals and these relations between ideals are then expressed in arithmetical terms. Dedekind's 'algebraic theory of numbers' develops grounded on an arithmetic 'of a higher level', the arithmetic between algebraic concepts.

Dedekind can now give a definition of the notion of congruence with respect to an ideal \mathfrak{a} , similar to that of modules. For two numbers ω, ω' in \mathfrak{o} ,

⁶⁷ "Ein System \mathfrak{a} von unendlich vielen in \mathfrak{o} enthaltenen Zahlen soll ein Ideal heißen, wenn es den beiden Bedingungen genügt: I. Die Summe und die Differenz je zweier Zahlen in \mathfrak{a} sind wieder Zahlen in \mathfrak{a} . II. Jedes Product aus einer Zahl in \mathfrak{a} und einer Zahl in \mathfrak{o} ist wieder eine Zahl in \mathfrak{a} ."

⁶⁸ "Ist α in \mathfrak{a} enthalten, so sagen wir, α sei theilbar durch \mathfrak{a} , \mathfrak{a} gehe in α auf, weil die Ausdrucksweise hierdurch an Leichtigkeit gewinnt."

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$\omega \equiv \omega' \pmod{\mathfrak{a}}$ means that $\omega - \omega'$ is contained in \mathfrak{a} . Since the properties for this new notion of congruence are the same as for the one used in rational arithmetic, one can define the notion of *class* “in such a way that two congruent numbers are put in the same class and any incongruent numbers are put in different classes” (*ibid.*, 38-39). An ideal can then always be divided into classes (mod \mathfrak{a}) of incongruent numbers: if μ is a nonzero number in \mathfrak{a} , any two numbers congruent (resp. incongruent) modulo μ are by definition also congruent (resp. incongruent) modulo \mathfrak{a} .

[i]t thereby follows immediately that \mathfrak{a} consists of one or more classes (mod μ).⁶⁹ (*ibid.*, 39)

The number of incongruent numbers modulo \mathfrak{a} is said to be the norm of an ideal \mathfrak{a} , $N(\mathfrak{a})$.⁷⁰

The system of all the numbers in \mathfrak{o} divisible by a certain number η in \mathfrak{o} forms an ideal called principal ideal and denoted $\mathfrak{i}(\eta)$. This special instance of the concept of ideal will prove essential in the rest of the theory

Dedekind develops what appears to be the first elements of an arithmetic of ideals, with a notion of divisibility between ideals:

If every number of an ideal \mathfrak{a} is also in an ideal \mathfrak{d} , then clearly \mathfrak{d} consists of one or more classes \mathfrak{a} , and we will say that \mathfrak{a} is a multiple of \mathfrak{d} or is divisible by \mathfrak{d} ; and that \mathfrak{d} is a divisor of \mathfrak{a} , or \mathfrak{d} divides \mathfrak{a} .⁷¹ (*ibid.*, 39)

Like for groups or higher congruences, the use of the arithmetical terminology is justified by certain existing relations reinterpreted in arithmetical terms. Here, \mathfrak{d} is composed of a number of (congruence) which is a multiple of the number of classes of \mathfrak{a} , which justifies the reinterpretation of inclusion as an arithmetical relation.

Each ideal has only a finite number of divisors, and the relation of divisibility is transitive. One can easily observe that \mathfrak{o} divides any ideal and is, in fact, the unit for divisibility. Since, in this version of ideal theory, Dedekind does not mention the fact that every ideal is a module, he needs to unfold the basic arithmetical notions once again (this time, with a justification):

The system of those numbers that are contained in each of two ideals \mathfrak{a} , \mathfrak{b} is the least common multiple \mathfrak{m} of \mathfrak{a} , \mathfrak{b} , insofar as every common multiple of \mathfrak{a} , \mathfrak{b} is divisible by the ideal \mathfrak{m} . If α

⁶⁹ “woraus zugleich folgt, dass \mathfrak{a} aus einer oder mehreren Classen (mod. μ) besteht.”

⁷⁰ Notably, if one considers a module (resp. ideal) generated by a number μ , then its norm is $\pm N(\mu)$.

⁷¹ “Wenn alle Zahlen eines Ideals \mathfrak{a} auch in einem Ideal \mathfrak{d} enthalten sind, so besteht offenbar \mathfrak{d} aus einer oder mehreren Classen (mod. \mathfrak{a}), und wir wollen sagen, \mathfrak{a} sei ein Multiplum von \mathfrak{d} oder theilbar durch \mathfrak{d} , \mathfrak{d} sei ein Theiler von \mathfrak{a} oder gehe in \mathfrak{a} auf.”

runs through all the numbers of \mathfrak{a} , and β runs through all the numbers of \mathfrak{b} , then the system of all numbers $\alpha + \beta$ is the greatest common divisor \mathfrak{d} of the ideals \mathfrak{a} , \mathfrak{b} , since each common divisor of \mathfrak{a} , \mathfrak{b} divides the ideal \mathfrak{d} .⁷² (*ibid.*, 40)

Dedekind pursues the development of arithmetical notions for ideals and introduces the notion of primality for ideals, which is (contrary to that for algebraic integers) similar to primality for rational integers: a prime ideal \mathfrak{p} different from \mathfrak{o} is an ideal with no other divisors than \mathfrak{o} and \mathfrak{p} itself. It seems that the similarity of the definition of prime ideal with that of prime number in rational number theory is the first sign that studying arithmetical laws for ideals should be “easier” than for algebraic numbers. Prime ideals are fully defined with the following theorems:

If $\eta\rho \equiv 0 \pmod{\mathfrak{p}}$, then at least one of the two numbers η , ρ , is divisible by \mathfrak{p} . (...) If every product which is divisible by an ideal \mathfrak{p} (which is distinct from \mathfrak{o}) has at least one factor that is divisible by \mathfrak{p} , then \mathfrak{p} is a prime ideal.⁷³ (*ibid.*, 40)

The proofs combine divisibility between a number and an ideal, divisibility between ideals, and congruences. An ideal \mathfrak{a} which is not prime is called a composite ideal, and one can always find two numbers η and ρ not divisible by \mathfrak{a} but whose product $\eta\rho$ is divisible by \mathfrak{a} . Overall, writes Dedekind, “the general theorems [for rational number theory] on congruences with respect to the modulus \mathfrak{p} carry over without difficulties” (*ibid.*, 41).

5.4.3 The proofs of the arithmetical propositions

The core idea of Dedekind’s approach is to transfer the study of divisibility laws of algebraic integers to that of ideals, thus bypassing the difficulty which led to the use of ideal numbers by Kummer. For this, Dedekind introduces a *temporary* notion, simple ideals (which will disappear in subsequent versions of the theory) which are formed by certain roots of congruences. Let η be a number *not* divisible by an ideal \mathfrak{a} , then there always exists a number ν such that all the solutions π of the equation $\nu\pi \equiv 0 \pmod{\mathfrak{a}}$ form a prime ideal. For μ in \mathfrak{o} , one can consider the roots π of the congruence $\nu\pi \equiv 0 \pmod{\mu}$, which also form a prime ideal. Prime ideals which “consist in” such system

⁷² “Das System aller derjenigen Zahlen, welche gleichzeitig in zwei Idealen \mathfrak{a} , \mathfrak{b} enthalten sind, ist das kleinste gemeinschaftliche Multiplum \mathfrak{m} von \mathfrak{a} , \mathfrak{b} , insofern jedes gemeinschaftliche Multiplum von \mathfrak{a} , \mathfrak{b} durch das Ideal \mathfrak{m} theilbar ist. Durchläuft α alle Zahlen in \mathfrak{a} , β alle Zahlen in \mathfrak{b} , so ist das System aller Zahlen $\alpha + \beta$ der größte gemeinschaftliche Theiler \mathfrak{d} der Ideale \mathfrak{a} , \mathfrak{b} , weil jeder gemeinschaftliche Theiler von \mathfrak{a} , \mathfrak{b} in dem Ideale \mathfrak{d} aufgeht.”

⁷³ “Ist $\eta\rho \equiv 0 \pmod{\mathfrak{p}}$, so ist wenigstens eine der beiden Zahlen η , ρ durch \mathfrak{p} theilbar (...). Enthält jedes durch ein (von \mathfrak{o} verschiedenes) Ideal \mathfrak{p} theilbare Product mindestens einen durch \mathfrak{p} theilbaren Factor, so ist \mathfrak{p} ein Primideal.”

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of roots are “temporarily” (*vorläufig*) called “*simple ideals*”. As underlined by Edwards and Avigad, this provides an effective divisibility test similar to what can be found in Kummer’s theory: π is divisible by the (ideal) factor corresponding to μ and ν if and only if it satisfies the associated congruence. This divisibility test can be extended to divisibility by powers of μ and ν : for a positive rational integer r , the solutions ρ of the equation

$$\rho\nu^r \equiv 0 \pmod{\mu^r}$$

form an ideal “called the r -th power of \mathfrak{p} ” and denoted \mathfrak{p}^r , thus defining powers of ideals.

One can see, especially with “simple ideals”, that Dedekind is transferring Kummer’s ideas to the “higher level” of his notion of ideal. From there, it is possible to give a series of theorems, analogous to some theorems for prime numbers in rational number theory, and which will lead to a test of divisibility by powers of such simple ideals:

If $s \geq r$, then \mathfrak{p}^s is divisible by \mathfrak{p}^r . (...)

If ρ is a nonzero number in \mathfrak{o} , then there is always a highest power of \mathfrak{p} that divides ρ . (...)

If $\mathfrak{p}^r, \mathfrak{p}^s$ are the highest powers [of \mathfrak{p}] dividing ρ, σ respectively, then \mathfrak{p}^{r+s} is the highest power of \mathfrak{p} dividing $\rho\sigma$. (...)

Each power \mathfrak{p}^r of a simple ideal \mathfrak{p} is not divisible by any prime ideal other than \mathfrak{p} .⁷⁴ (*ibid.*, 42-43)

The proofs are very short and only use divisibility and congruence properties. Dedekind explicitly underlines the analogy with prime rational numbers, before introducing a new and central theorem:

The importance of the simple ideals and their analogy with the rational primes comes immediately to the fore in the following main theorem: *If all the powers of simple ideals that divide a nonzero number μ also divide a number ν , then ν is divisible by μ .*⁷⁵ (*ibid.*, 43)

This theorem tells us that “every principal ideal $i(\mu)$ is the least common multiple of all the powers of simple ideals that divide μ ” and an important corollary of this result is that

⁷⁴ “Ist $s \geq r$, so ist \mathfrak{p}^s theilbar durch \mathfrak{p}^r . (...) Ist ρ eine von Null verschiedene Zahl in \mathfrak{o} , so giebt es immer eine höchste in ρ aufgehende Potenz von \mathfrak{p} . (...) Sind $\mathfrak{p}^r, \mathfrak{p}^s$ resp. die höchsten in ρ, σ aufgehenden Potenzen, so ist \mathfrak{p}^{r+s} die höchste in $\rho\sigma$ aufgehende Potenz von \mathfrak{p} . (...) Jede Potenz \mathfrak{p}^r eines einfachen Ideals \mathfrak{p} ist durch kein von \mathfrak{p} verschiedenes Primideal theilbar.”

⁷⁵ “Die Wichtigkeit der einfachen Ideale und ihre Analogie mit den rationalen Primzahlen tritt unmittelbar hervor in dem folgenden Hauptsatz: Wenn alle in einer von Null verschiedenen Zahl μ aufgehenden Potenzen einfacher Ideale auch in einer Zahl ν aufgehen, so ist ν durch μ theilbar.”

Every prime ideal \mathfrak{p} is a simple ideal.⁷⁶ (*ibid.*, 44)

Once this corollary is proved, the name “simple ideals” is completely abandoned, and Dedekind “speak[s] only of prime ideals” (*ibid.*).

Basically, simple ideals are only prime ideals represented in a certain way.⁷⁷ The important role they play, in 1871, contradicts the methodological principles often stated by Dedekind: to avoid proofs based on a definite representation, and to rather use their “fundamental characteristics.” Indeed, while the name “simple ideal” is given up, the proofs given for the next theorems – and notably the proof of the unique factorization theorem – are essentially based on the congruence associated with the definition of simple ideals.

The link then tied between prime ideals and the ideal factors appearing in the definition of simple ideals implies that any non-prime ideal can be represented (through prime ideals) with the appropriate μ and ν . Thus, the link between ideals and Kummer’s ideal numbers is proved: an ideal \mathfrak{m} is the system of numbers divisible by a certain (ideal or actual) number μ , i.e., “each system enjoying [the new definition of an ideal] is also an ideal [number]” in Kummer’s sense, which was, for Dedekind “a fact of highest importance” ([Dedekind, 1876-1877], 59-60).

The core result, here, is:

If all the powers of prime ideals that divide an ideal \mathfrak{m} also divide a number η , then η is divisible by \mathfrak{m} .⁷⁸ (*ibid.*, 44)

From there, the similarity of the laws of divisibility between ideals and rational numbers becomes clear, since corollaries of this theorem are theorems analogous to those found in rational number theory with rational numbers:

Every ideal is the least common multiple of all the powers of prime ideals that divide it. This corresponds to the fundamental theorem of rational number theory on the composition of numbers from primes (§8). It follows that every ideal \mathfrak{m} is *completely determined* once the highest powers \mathfrak{p}^e , $\mathfrak{p}'^{e'}$, $\mathfrak{p}''^{e''}$, ... of prime ideals that divide it are given.⁷⁹ (*ibid.*, 44)

With the approach adopted by Dedekind, here, the factorization theorem is merely a corollary of a theorem regarding divisibility of numbers by ideals,

⁷⁶ “Jedes Primideal \mathfrak{p} ist ein einfaches Ideal.”

⁷⁷ [Edwards, 1980]

⁷⁸ “Wenn alle in einem Ideal \mathfrak{m} aufgehenden Potenzen von Primidealen auch in einer Zahl η aufgehen so ist η theilbar durch \mathfrak{m} .”

⁷⁹ “Jedes Ideal ist das kleinste gemeinschaftliche Multiplum aller in ihm aufgehenden Potenzen von Primidealen. Er entspricht durchaus dem Fundamentalsatze der rationalen Zahlentheorie über die Zusammensetzung der Zahlen aus Primzahlen (§8); denn ihm zufolge ist jedes Ideal \mathfrak{m} vollständig bestimmt sobald die höchsten in \mathfrak{m} aufgehenden Potenzen \mathfrak{p}^e , $\mathfrak{p}'^{e'}$, $\mathfrak{p}''^{e''}$, ... von Primidealen gegeben sind.”

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and whose proof is made using the congruence relations for simple ideals. Dedekind still works in a framework close to Kummer's, and in fact, as underlined by Edwards:

[Dedekind's] method of proof of the fundamental theorem was to introduce temporarily simple ideals that are essentially the same as Kummer's "ideal prime factors" and are represented in a specific way, to prove the theorem, and only then to show that every prime ideal is simple. ([Edwards, 1980], 346-347)

In the subsequent versions of the theory, simple ideals are not introduced. The congruence defined by ν and π , while they still play a role in the theory, are only presented as auxiliary propositions – and certainly don't enjoy the "privilege" of being given a definition, as Avigad remarks ([Avigad, 2006], 22). Finally, note that in this presentation of the theory, one can even question whether the factorization theorem is still a fundamental theorem of arithmetic.

It should be underlined that Dedekind only defined multiplication of ideals in the very end of §163. In particular, the proof of the theorem of decomposition of ideals in primes does not require the (definition of the) product of ideals. Indeed, the proof is based only on powers of prime (simple) ideals defined with congruences. The definition of the multiplication of ideals is given "almost as an afterthought" as Edwards writes, and plays little, if any, role in the theory.

If one multiplies all the numbers of an ideal \mathfrak{a} with all the numbers of an ideal \mathfrak{b} , these products and their sums form an ideal that is divisible by \mathfrak{a} and \mathfrak{b} , which will be called the product of \mathfrak{a} and \mathfrak{b} and denoted \mathfrak{ab} .⁸⁰ ([Dedekind, 1871], 45)

The primitive notions, in 1871, are congruences and primality ("prime powers"). Arithmetical notions, such as the product or the relative primality of ideals, are only stated at the end of the theory, and seem to be merely side-effects of the preceding developments. In fact, an approach *à la* Kummer as adopted here does not require the multiplication of ideals (or ideal numbers), and the multiplication and arithmetical notions linked to it, such as relative primality, appear as the aftermath of the prime factorization theorem. Once the relative primality of ideals is defined,⁸¹ one can transfer, or translate in the theory of ideals, all the "theorems of rational number theory on relative primality" (*ibid.*). Despite stating that theorems can be transferred without

⁸⁰ "Multipliziert man alle Zahlen eines Ideals \mathfrak{a} mit allen Zahlen eines Ideals \mathfrak{b} , so bilden diese Produkte und deren Summen ein durch \mathfrak{a} und \mathfrak{b} teilbares Ideal, welches das Produkt aus den Factoren \mathfrak{a} und \mathfrak{b} heißen und mit \mathfrak{ab} bezeichnet werden soll."

⁸¹ Two ideals \mathfrak{a} and \mathfrak{b} are called relatively prime if their greatest common divisor is $= \mathfrak{o}$.

difficulties, Dedekind does not state any of the theorems, except for this one, which gives another congruence property:

If \mathfrak{a} and \mathfrak{b} are relatively prime ideals, and μ, ν are two given numbers, then there is always exactly one class of numbers $\eta \pmod{\mathfrak{ab}}$ that satisfy the conditions $\eta \equiv \mu \pmod{\mathfrak{a}}$ and $\eta \equiv \nu \pmod{\mathfrak{b}}$.⁸² (*ibid.*, 46)

The ‘arithmetic of ideals’, here, is neither fully exploited nor fully developed, but the essential idea (namely the divisibility of ideals as the key to developing the theory) is present. In the following paragraphs of the Xth Supplement, Dedekind does use the definition of the operations between ideals to define notions such as classes of ideals, analogous to that of classes of number for congruences. On the basis of this ‘arithmetic of ideals’, Dedekind elaborates tools, in analogy to number theory, to study, properties of ideals as objects in their own right.

5.5 A short comparison with Kronecker’s approach

I will propose to compare briefly Dedekind’s approach to Kronecker’s take on the similar problem, the generalization of Kummer’s results. Kronecker’s theory was announced as a forthcoming work by Kummer in 1857 but was only published in 1881 and 1882. My presentation of Kronecker’s ideas will be very fragmentary. I will rely on [Boniface, 2004] in great part, as well as on [Edwards, 1980] and [Smadja, 2002], which all propose more complete commentaries.⁸³ I will also use Dedekind’s “*Bunte Bemerkungen*” to stress the differences in their approach, from Dedekind’s own view.

I propose to consider Kronecker’s work on algebraic magnitudes in this chapter, despite the twenty years between Dedekind’s and Kronecker’s publications because it seems more consistent to compare Kronecker’s approach, with Dedekind’s first handling of the problem, which stays relatively close to Kummer’s approach. In any case, divergences between Kronecker’s and Dedekind’s approaches will only increase with each rewriting of Dedekind’s theory of algebraic integers, since Dedekind will move away from the more computational aspects of Kummer’s ideal numbers. This presentation will allow me to highlight some specific features of Dedekind’s approach. In presenting the next versions of Dedekind’s theory, I would like to concentrate on the changes Dedekind inflected on his own works.

⁸²“Sind \mathfrak{a} , \mathfrak{b} relative Primideale, und μ, ν zwei gegebene Zahlen, so giebt es immer eine und nur eine Classe von Zahlen $\eta \pmod{\mathfrak{ab}}$, welche den Bedingungen $\eta \equiv \mu \pmod{\mathfrak{a}}$, $\eta \equiv \nu \pmod{\mathfrak{b}}$ genügen.”

⁸³For more on Kronecker vs. Dedekind, one can refer to these works and to [Edwards, 1992], [Edwards, 1983] and [Edwards, Neumann, and Purkert, 1982]. A thorough comparison of Dedekind’s (and Weber’s) and Kronecker’s approaches, whose groundwork has been done by Edwards, would require a level of details about Kronecker’s works which falls out of the scope of my inquiry.

5.5 A short comparison with Kronecker's approach

5.5.1 Kronecker's algebraic magnitude and the *Rationalitäts-bereich*

For Kronecker, magnitudes (*Größe*)⁸⁴ have different genres (*Gattung*)⁸⁵, depending on the way in which they are generated. For example, the irrational $\sqrt{2}$ belongs to a particular genus and is “‘conceptually’ very far from any rational number as close as possible of the square root of two”⁸⁶ ([Kronecker, 1881], 250). This clear-cut differentiation between conceptual genres of magnitudes comes from a desire to build up a classification of magnitudes independent of their measure (size). Thus, Kronecker criticizes the tendency to adopt an idea of neighborhood for magnitudes whose measures are close – such as $\sqrt{2}$ and a rational number q as close as possible of $\sqrt{2}$ – but whose algebraic properties are fundamentally different. The concept of *Körper*, which holds strong spatial connotations is explicitly rejected by Kronecker:

I consider appropriate to avoid, in the terminology, the expressions with a decidedly spatial connotation, and to only use the hardly avoidable general expressions, such as this work “domain”, or another general idea, which has already nearly lost its original spatial meaning by its many applications in common language uses.⁸⁷ (*ibid.*, 250, my translation)

Kronecker, instead, uses the notion of domains of rationality:

The “domain or rationality ($\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$)” contains, as one can already see in the denotation, all these magnitudes which are rational functions of the magnitudes $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ with integral numbers as coefficients. This stipulation that the coefficients be integral numbers, is only added at for the beginning, in order to eliminate any misunderstanding. (...) By “domain or rationality ($\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$)”, all the rational functions of the element \mathfrak{R} must be conceptually united – but only for easing the mode of expression in the presentation of the theory – ad in the same way, one can furthermore also carry out the classification of magnitudes in closed circles or categories, according to

⁸⁴In [Edwards, Neumann, and Purkert, 1982], Dedekind puts “(?)” after uses of the word “*Größe*”.

⁸⁵Kronecker indicated *genus* as the Latin equivalent of “*Gattung*”. See [Goldstein and Schappacher, 2007b], Note 68, p. 83.

⁸⁶“die zu einer besonderen *Gattung* gehörige Grösse $\sqrt{2}$ ‘begrifflich’ weit ab von irgend einer der Quadratwurzel aus zwei noch so nahe liegenden rationalen Zahl”

⁸⁷“halte ich es für angemessen, in der Terminologie die Ausdrücke mit entschieden räumlichem Gepräge zu vermeiden und nur solche, kaum zu umgehende allgemeine Ausdrücke — wie eben jenes Wort ‘Bereich’ — oder allgemeine Bilder zu gebrauchen, welche die ursprünglich räumliche Bedeutung bei ihrer vielfachen Verwendung im gewöhnlichen Sprachgebrauche schon fast verloren haben.”

determinate common properties to be exposed.⁸⁸ (*ibid.*, 250)

It is important to underline that “magnitudes” for Kronecker have only an arithmetico-algebraic meaning (in particular, they do not refer to any consideration of size or measure):

The expression “magnitude” is, here, to be taken in the broadest arithmetico-algebraic sense, and one must, generally, also understand “rational functions of indeterminate magnitudes”, the so-called “forms of arbitrarily many variables”, and so forth, as being magnitudes, to which the concept of measure, to be “greater or smaller”, are completely foreign.⁸⁹ (*ibid.*, 249-250)

This implies an enlargement of the notion of magnitude to *any* algebraic expression, i.e., rational function with integral coefficients. The standpoint taken by Kronecker in [Kronecker, 1881] is considerably wider than Dedekind’s. It is more general in the sense that it considers a larger scope of objects. In fact, Kronecker’s researches are valid for algebraic functions as well, and many of the remarks here can be applied to Dedekind and Weber’s paper on algebraic functions.

Kronecker’s ideas about magnitudes are directly linked to the importance of computations in Kronecker’s mathematics, often emphasized by commentators. Kronecker, who walks in Kummer’s footsteps, has a position diametrically opposed to Dedekind’s, and considers that concepts are (must be) the result of computations. David Reed underlines it clearly:

Since [Kronecker] is of the view that numbers are merely systems of representations with which calculations are done, the essence of number theory is calculation. The means by which such calculation is performed may be (in modern terminology) strictly algebraic or may involve analytic tools from calculus or function theory. ([Reed, 2003], 98)

⁸⁸ “Der ‘Rationalitäts-Bereich ($\mathfrak{R}, \mathfrak{R}', \mathfrak{R}''$, ...)’ enthält, wie schon die Bezeichnung deutlich erkennen lässt, alle diejenigen Grössen, welche rationale Functionen der Grössen $\mathfrak{R}, \mathfrak{R}', \mathfrak{R}''$, ... mit ganzzahligen Coefficienten sind. Diese Bestimmung, dass die Coefficienten ganzzahlig sein sollen, ist nur hier im Anfange, um jedes Missverständniss auszuschliessen, hinzugefügt. (...) Durch den ‘Rationalitäts- Bereich ($\mathfrak{R}, \mathfrak{R}', \mathfrak{R}''$, ...)’ sollen die sämtlichen rationalen Functionen der Elemente \mathfrak{R} – nur zur Erleichterung der Ausdrucksweise bei der Darstellung der Theorie — begrifflich zusammengefasst werden, und in derselben Weise soll auch noch weiterhin die Einordnung von ‘Grössen’ nach bestimmten, besonders darzulegenden, gemeinsamen Eigenschaften in geschlossene Kreise oder Kategorien erfolgen.”

⁸⁹ “Der Ausdruck ‘Grösse’ ist hierbei in der weitesten arithmetisch-algebraischen Bedeutung zu nehmen, und es sind im Allgemeinen auch Grössengebilde wie ‘rationale Functionen unbestimmter Grössen’, sogenannte ‘Formen beliebig vieler Veränderlicher’ u. s. w. mit darunter zu verstehen, denen der Begriff der Maassgrösse, der des ‘grösser oder kleiner Seins’ gänzlich fremd ist.”

5.5 A short comparison with Kronecker's approach

Boniface corrects slightly Reed's statement about Kronecker's view of numbers. She explains that for Kronecker, the only numbers are the natural integers and if the entities formed by the algebraic or analytic expressions are arithmetical, it does not mean that they are numbers. Kronecker's strong ontological commitment towards natural numbers should be understood in relation to his idea that mathematics must be treated as a natural science. The objects of mathematics should thus be objects consistent with experience. (see Sec. 7.1.3.) This is the essential reason behind the primacy of natural integers: they are, for Kronecker, the only objects consistent with mathematical experience. Kronecker's philosophical position cannot be detached from the mathematical methods that he develops to support it, as Vládut explains:

Kronecker's views are quintessentially expressed in two ideas. The first is the rejection of actual infinity as mathematical reality. The second is also prohibitive in nature and can best be expressed by his well-known dictum "God Himself made the integers – everything else is the work of men". This leads to the prohibition against using mathematical notions and methods which cannot be reduced to the arithmetic of integers. Despite their prohibitive form, these statements carry a powerful positive charge. (...) The desire to argue "arithmetically" influenced Kronecker's works, many of which were based on the method of computation of some mathematical objects. ([Vládut, 1991], 11-12)

The point made by Vládut, here, is that methods express and develop the philosophical position, and that one cannot be understood one without the other – a point which is certainly valid for Dedekind, and probably many other mathematicians as well. For Kronecker, to effectively put into action his philosophy means to elaborate methods, algorithms, forms of presentations of the mathematical content which yield the arithmetical meaning.⁹⁰

In order to obtain actually fruitful mathematics, "computations" here does not refer to mere numerical computations, but to a larger understanding including computations on and with algebraic expressions. While algebraic expressions are not, properly speaking, numbers, they are integrated into what Kronecker calls "general arithmetic" which contains algebra and analysis. This marks Kronecker's peculiar approach of the "arithmetization of mathematics" which, for him, consists in bringing all mathematical dis-

⁹⁰An important example of this is Kronecker's rejection of the irrationals. For him, irrational numbers are just a step in the computation, and rather than numbers, they are symbols to which no operational meaning can be attached. They cannot, thus, be considered as numbers. Algebraic irrationals are attached to polynomials. Transcendental numbers are not considered as a part of arithmetic, for they are generated by means and processes belonging to geometry.

ciplines (algebra, analysis) under the dominance of arithmetic.⁹¹ Kronecker proposes to extend arithmetic to “integral algebraic functions of one indeterminate variable with integral coefficients”. These integral functions are the integral magnitudes of rational domains.

Kronecker’s rationality domain is a concept of larger extent than Dedekind’s concept of field, since there are no restrictions on the choice of rational magnitudes. Moreover, it leads immediately to the idea of adjunction of magnitudes, a point less easy to see in Dedekind’s notion of field, especially in the early exposition.

A rationality domain is, in general, a domain of magnitudes *arbitrarily* delimited but only as wide as the concept allows. Since a rationality domain is only enlarged by the addition of arbitrarily chosen elements \mathfrak{R} , any arbitrary expansion of its limits requires in the same time, that all the magnitudes which are which are rational expressions of the new element be captured. But there is also naturally limited domains of rationality, as the domain of the usual rational numbers, which is contained, as an absolute domain, in all the rationality domains, and as we indicated with the denotation $\mathfrak{R} = 1$, which represents to a certain extent the absolute unite of the concept of rationality domain. The domain of the rational functions of $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ is also naturally limited, if the $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ represent all independent variables; the domain of rational numbers is contained in it as is, in general, that of the rational functions of a part of $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ ⁹² ([Kronecker, 1881], 255)

A domain of rationality is thus a domain formed by all rational functions of any magnitude (with integral coefficients). The “magnitudes” $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$

⁹¹A more subtle explanation of Kronecker’s conception of arithmetization can be found in [Smadja, 2010], in which it is explained that “Kronecker’s conception of arithmetization as a process aiming at disclosing the ‘arithmetical essence’ which is enshrined in analytic formulas” ([Smadja, 2010], 62). Kronecker’s arithmetization presents itself, according to I. Smadja, “as an endeavour to grasp mathematical concepts in terms of invariants thus reflecting non arithmetical properties in arithmetical ones” (*ibid.*, 67).

⁹²“*Ein Rationalitäts-Bereich ist im Allgemeinen ein willkürlich abgegrenzter Grössenbereich, doch nur, so weit es der Begriff gestattet. Da nämlich ein Rationalitäts-Bereich nur durch Hinzufügung beliebig gewählter Elemente \mathfrak{R} vergrößert werden kann, so erfordert jede willkürliche Ausdehnung seiner Begrenzung zugleich die Umschliessung aller durch das neue Element rational ausdrückbaren Grössen. Es giebt aber auch natürlich abgegrenzte Rationalitäts-Bereiche, so das Reich der gewöhnlichen rationalen Zahlen, welches als das absolute in allen Rationalitäts-Bereichen enthalten ist und, wie es durch $\mathfrak{R} = 1$ bezeichnet worden, auch gewissermassen die absolute Einheit des Rationalitäts-Begriffs repräsentiert. Auch das Reich der rationalen Functionen von $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ wenn diese sämtlich unabhängige Variable bedeuten, ist ein ‘natürlich’- abgegrenztes; es ist darin das Reich der rationalen Zahlen sowie überhaupt das der rationalen Functionen von einem Theile der Variablen $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$ mit enthalten.*”

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can be considered as “variables” as well as “magnitudes” properly speaking. In the first case, they are considered as mere symbols with no meaning attached, and in the second case, they have the algebraic meaning given to algebraic quantities. The domain of rationality is constituted by the successive adjunction of these irrational (algebraic) magnitudes, which should thus be considered in relation with an auxiliary equation of an indeterminate variable y , a resolvent, which gives them their algebraic meaning.⁹³ Each root of an algebraic equation with coefficients in the domain of rationality ($\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$) is considered as an algebraic function of the magnitudes $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \dots$. The decomposition and the irreducibility of algebraic functions are then dependent of the rationality domain, and each time one adjoins a magnitude \mathfrak{R} to the domain, a new rationality domain is obtained.

As underlined by Reed, Kronecker does not look, in his rationality domains, for the “completeness”, the property of being made a “whole” by the closure by the four arithmetic operations, that Dedekind valued in the concept of field.

Kronecker's ‘domains’ are simply realms in which calculations of certain types are possible. At no time does Kronecker try to view his domains as ‘completed wholes’; his interest in introducing them is not to provide new types of numbers but rather to permit complex types of calculation to be undertaken in a systematic manner. ([Reed, 2003], 99)

Kronecker's rationality domain is not conceived as a framework, or an algebraic structure, but as an extension of the initial domain.

Finally, it should be clear, then, as Edwards likes to remind us, that Kronecker's *Rationalitäts-bereich* is a more general notion than Dedekind's field. This emphasizes the idea that the so-called “conceptual approach” is not the only way for a significant gain of generality. Kronecker's domains of rationality are more general simply because of the generality of Kronecker's notion of magnitude. Dedekind made the following remark in relation to the generality of Kronecker's theory:

The common treatment of the theory of algebraic numbers with the theory of algebraic functions of *variables* is somewhat attractive, still it seems to be more advantageous, in the theory of the latter, to give up completely the restriction of constant coefficients to rational or algebraic numbers, because they are of completely secondary importance for the actual principal point of the theory.⁹⁴ ([Edwards, Neumann, and Purkert, 1982], 55)

⁹³For example, for the equation $x^2 - 2 = 0$, whose roots are $x_{1,2} = \pm\sqrt{2}$, the domain will be extended from R to $R(\sqrt{2})$.

⁹⁴“Die gemeinschaftliche Behandlung der Theorie der algebraischen Zahlen und der algebraischen Functionen von Variablen hat zwar etwas Anziehendes, doch scheint es mir

5.5.2 An overview of Kronecker's integral algebraic magnitudes and divisors

Kronecker and Dedekind, despite the many mathematical and methodological differences between their works, gave the same definition of algebraic integers. Kronecker, however, approaches the question from a very different angle. Following his considerations on algebraic magnitudes, he starts from a wider viewpoint, and proceeds in two steps. First, he introduces the notion of “integrity domain”: the domain $[\mathfrak{R}', \mathfrak{R}'', \dots]$ composed of the integral functions with integral coefficients in the magnitudes $\mathfrak{R}', \mathfrak{R}'', \dots$.⁹⁵ Then, he gives a simple definition of integral algebraic magnitudes:

A magnitude x must be called an “integral algebraic function of the variable \mathfrak{R} ” or a “integral algebraic magnitude”, if it satisfies an equation in which the coefficient of the higher power of x is 1, and the remaining coefficients are all integral functions with integral numbers of the variable \mathfrak{R} as coefficients, and so are magnitudes of the domain $[\mathfrak{R}', \mathfrak{R}'', \dots]$.⁹⁶ ([Kronecker, 1899] II, 260, my translation)

The addition and multiplication of any two integral algebraic magnitudes are again algebraic magnitudes.⁹⁷

In the case $\mathfrak{R} = 1$, we have algebraic integers (*ganze algebraische Zahlen*), which are thus a special case of the magnitudes considered by Kronecker. There is, here again, a notable difference with Dedekind in the generality of the concepts adopted as basic ones: while Dedekind focuses on algebraic numbers alone, Kronecker works with a general concept of algebraic magnitudes. This seems to be a side-effect of the profound differences between Kronecker's and Dedekind's conceptions of arithmetic. The “general Arithmetic” which Kronecker wants to develop is based on an “enlargement” of Gauss's notion of congruence and includes formal algebraic manipulations and formulae for the study of polynomials and algebraic functions.

Kronecker considers the divisibility of algebraic integral magnitudes. For this, he introduces the greatest common divisor of algebraic magnitudes which, he states, “transfers this notion from the sphere of mere abstraction to

vortheilhafter, bei der Theorie der letzteren die Beschränkung der constanten Coefficienten auf rationale oder algebraische Zahlen ganz aufzugeben, weil sie von ganz untergeordneter Bedeutung für die eigentlichen Hauptpunkte der Theorie ist.”

⁹⁵This domain is a ring, a property noticed by Kronecker, although without using the term “ring”, of course.

⁹⁶“Eine Grösse x soll eine ‘ganze algebraische Function der Variabeln \mathfrak{R} ’ oder eine ‘ganze algebraische Grösse’ genannt werden, wenn sie einer Gleichung genügt, in welcher der Coefficient der höchsten Potenz von x gleich Eins ist, und die übrigen Coefficienten ganze ganzzahlige Functionen der Variabeln \mathfrak{R} , also Grössen des Bereichs $[\mathfrak{R}', \mathfrak{R}'', \dots]$.”

⁹⁷In modern terms, Kronecker's integral algebraic magnitudes are integers in a ring of polynomials $\mathbb{Z}[X]$ with integral coefficients, where $X = \{X_i\}$ rational functions with coefficients in \mathbb{Z} .

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the reality of algebraic formations (*Gebilde*).” The idea is the following: if x and x' are two algebraic magnitudes, then x is said to divide x' if the quotient $\frac{x'}{x}$ is an integral algebraic magnitude. Kronecker proposes to determine the greatest common divisor without symbolism or abstraction, but by providing effective computations to exhibit the corresponding algebraic form. To do so, let x, x', x'', \dots be integral algebraic magnitudes (i.e., integers in a ring $\mathbb{Z}[X]$). Kronecker considers a linear function of these magnitudes

$$x + ux' + u''x'' + \dots$$

with u', u'', \dots magnitudes. The norm of this linear function is a polynomial N with coefficients in the base domain (i.e., $\mathbb{Z}[X]$), in which we know how to find the GCD of two or more elements. In particular, one can give the GCD of all the coefficients of N . The division of N by this GCD yields a polynomial in u', u'', \dots whose coefficients are all relatively prime. Kronecker designates it as $Fm(x + u'x' + u''x'' + \dots)$ and calls primitive form. Kronecker defines the GCD of any integral algebraic magnitudes as

$$\frac{x + u'x' + u''x'' + \dots}{Fm(x + u'x' + u''x'' + \dots)}$$

Kronecker calls this the “divisor of $[x + u'x' + u''x'' + \dots]$ ” or “module $[x + u'x' + u''x'' + \dots]$ ”.

To be able to provide this explicit representation is essential, from Kronecker's viewpoint. Moreover, this form will allow, when the divisor is an ideal number, to avoid the “ideality” of those factors by providing a concrete algebraic form to the ideal numbers (see [Boniface, 2004]).

Following the considerations about the divisibility of algebraic magnitudes, Kronecker generalizes the notion of divisor (*Divisor*) to any algebraic form in the following way:

If an integral algebraic form is divided by the primitive form from which its norm is derived, then the quotient represents a general “algebraic module or divisor” whose “elements” are formed by the coefficients of the algebraic form.⁹⁸ ([Kronecker, 1881], 303)

The extension of the concept of *Divisor* goes with an extension of *divisibility*, which allows for the concept of divisor to “take all its meaning (that of *Theiler*)” according to Boniface. Kronecker gives the following definitions for divisibility:

⁹⁸ “Wird eine ganze algebraische Form durch diejenige primitive Form dividirt, deren abgeleitete die Norm der algebraischen Form ist, so repräsentirt der Quotient einen allgemeinen ‘algebraischen Modul oder Divisor’, dessen ‘Elemente’ durch die Coefficienten der algebraischen Form gebildet werden.”

An integral algebraic form is divisible by an algebraic divisor when the quotient is itself an integral algebraic form as well.⁹⁹ (*ibid.*, 304)

Kronecker says that an algebraic divisor is “divisible” by another algebraic divisor when “the form which constitutes the numerator of the first is divisible by the latter”¹⁰⁰ (*ibid.*, 304).

Kronecker defines a notion of equivalence between divisors, which allows to show that any divisor is equivalent to a divisor constituted from a linear form. Thus it is possible to only use those simpler kind of divisors.

Two algebraic divisors are “absolutely equivalent” when each of the two is divisible by the other. An algebraic divisor is equivalent to one, and thus not a divisor in the proper meaning of the word, if and only if the integral algebraic form from which it is formed is primitive.¹⁰¹ (*ibid.*, 304)

Dedekind remarks, in his *Bunte Bemerkungen* that when Kronecker “must talk about” two absolutely equivalent algebraic divisors, his theory in fact only needs to refer to “ideal”, “whose invariant nature lies immediately in the definition”.¹⁰²

To obtain the proof of unique factorization, Kronecker introduces a notion of irreducibility of divisors, which for him is the same as primality, relatively to the domain to which this divisor belongs and can find again the equivalence between irreducibility and primality. The unique factorization in prime elements can only be proved, for Kronecker, by exhibiting explicitly all the prime factors of a certain algebraic magnitude. For this, it is sufficient to exhibit the prime factors of its norm, which implies to try to find the irreducible divisors of an integral and irreducible rational magnitude of the initial domain $(\mathfrak{R}', \mathfrak{R}'', \dots)$. For this, one has to effectively perform the division, until the complete decomposition of an algebraic divisor of a certain genus is obtained.

⁹⁹ “Eine ganze algebraische Form ist durch einen algebraischen Divisor theilbar, wenn der Quotient ebenfalls eine ganze algebraische Form ist.”

¹⁰⁰ “Ein algebraischer Divisor soll als theilbar durch einen anderen Divisor bezeichnet werden, wenn die Form, welche den Zähler des ersteren bildet, durch den letzteren theilbar ist.”

¹⁰¹ “Zwei algebraische Divisoren sind ‘absolut äquivalent’, wenn jeder von beiden durch den anderen theilbar ist. Ein algebraischer Divisor ist dann und nur dann äquivalent Eins, also überhaupt kein Divisor in der eigentlichen Bedeutung des Wortes, wenn die ganze algebraische Form, aus welcher derselbe gebildet worden, primitiv ist.”

¹⁰² “Wo Kronecker von zwei ‘absolut äquivalenten’ algebraischen Divisoren sprechen muß, ist bei mir immer nur von dem einzigen ‘Ideal’ die Rede, dessen invariante Natur unmittelbar in der Definition liegt.”

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For Dedekind the effective construction of prime factorization is not an issue. Rather, he wishes to provide an independent proof, based on “invariant definitions,” to use Dedekind's own words in the *Bunte Bemerkungen*.

5.5.3 Dedekind on Kronecker's reading of ideal theory

Kronecker does comment on the differences between his approach and Dedekind's. He explains that while he started from “the system of elements themselves as divisor system” (i.e., modular system), Dedekind only considers the case $R = 1$, and that the linear forms are the ground of Dedekind's “conception”. This last remark did not please Dedekind, who irritatedly remarked that this would mislead the readers and wrote:

Where does this come from? The origin of my theory is rather formed solely by such concepts (field, module, ideal) which from the first are defined in an invariant way, that is, independently of any contingent form of representation.¹⁰³ ([Edwards, Neumann, and Purkert, 1982], 63)

Beyond Dedekind's recurring insistence about his approach being independent of any form of explicit representation (a characteristic that an approach based on “linear forms” would lose), it seems that Dedekind and Kronecker simply don't agree on what forms the “origin” of a theory. Kronecker sees in the “origin” of the theory the fact that Dedekind investigates a certain restricted kind of “algebraic form”. Dedekind, however, sees as the “origin” of his theory the fundamental concepts from which the results will be deduced – and which have been elaborated precisely so that the consideration of “linear forms” does not enter into account.¹⁰⁴

Kronecker continues his short commentary of Dedekind's approach and states that Dedekind considers “the collections of numbers actually divisible by an ideal divisor” as the center of his researches.¹⁰⁵ While it was the initial idea for the introduction of ideals, it is not, Dedekind insists, the fundamental characteristic of ideals. Kronecker's understanding of Dedekind's ideals¹⁰⁶ implies that Dedekind defined ideals in relation to Kummer's ideal

¹⁰³ “*Wo geschieht denn das ? Den Ausgangspunkt meiner Theorie bilden vielmehr lediglich solche Begriffe (Körper, Modul, Ideal), welche von vornherein auf invariante Weise definiert werden, d.h. unabhängig von jeder zufälligen Darstellungsform.*”

¹⁰⁴This is the same kind of misunderstanding as regarding the “abstraction” of their respective methods. Kronecker considers his approach as more concrete and Dedekind's way too abstract because of the use of actual infinities and the non-constructive methods. Dedekind, on the other hand, considers his approach more concrete because ideals are constituted of actually existing numbers, while Kronecker develops purely formal (and thus abstract) methods. Both are defending their approach as being more “concrete”.

¹⁰⁵ “*den Inbegriff der durch einen idealen Divisor theilbaren wirklichen Zahlen an die Spitze der Entwicklung*”

¹⁰⁶More precisely, it is Dedekind's reading of Kronecker's understanding.

factors, that each ideal is defined *via* the corresponding ideal factor. For Dedekind, however, ideals are to be considered as having an “*independent, invariant definition*” given by the properties of closure by arithmetical operations. The actual definition of the concept ought to be clearly discerned from the “historical presentation of the chain of thought”, as told in the introduction of the 1877 version of ideal theory, in which Dedekind explains how he arrived to the elaboration of the concept of ideal.¹⁰⁷

5.6 Role and status of ideals

The elements of Dedekind’s Xth Supplement exposed above constitute the very basis of ideal theory, but are a small part of the Supplement. In order to adjust the focus and clarify the place and role of ideal theory in Dedekind’s works, I will propose a quick overview of the following paragraphs.¹⁰⁸ This will allow to emphasize the fact that ideal theory is, for Dedekind, a new way to lay the ground for number theory and, notably, the study of binary quadratic forms. Ideal theory is not the object of study itself, and the investigations about algebraic integers are developed essentially so that the laws of divisibility of algebraic numbers are proved and well-known, so that the further investigations are eased – a point stated very explicitly in the second version of the theory. If Dedekind later restricted the focus and published his algebraic integer theory as an independent paper, dealt with separately from binary quadratic forms, he nevertheless always took the time to highlight the possible applications. Thus, ideal and module theories appear to be important because of the large potential of applications and of new results provided.

5.6.1 The last paragraphs of Dedekind’s Xth Supplement

In Dedekind’s *Gesammelte Werke* are reproduced only the §159 to §163 of the Xth Supplement (as well as in van der Waerden publication’s of the Xth Supplement) which, in Dirichlet’s *Vorlesungen* consists in the §§145-170. The paragraphs §§159-163 contain the essentials of the theory of algebraic integers, and the basic elements of ideal theory. Maybe for this reason, commentators have omitted to consider the content of the previous and following paragraphs. By doing so, they appear to overlook the fact that Dedekind’s ideal theory is, at first, only a small part of a very long supplement dedicated to binary quadratic forms — a standard subject for number theory, in 1870. Even if he adopts a singular position on actual infinities, Dedekind

¹⁰⁷ “*Vielleicht ist diese Bemerkung Kroneckers hervorgerufen durch die historische Darstellung des Gedankengangs, der mich auf meinen Idealbegriff geführt hat (Introduction der franz. Schrift, S. 8-10)!*”

¹⁰⁸The references will be to the 1871 edition of Dirichlet’s *Vorlesungen*.

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does investigate orthodox number theoretical questions. Thus, the status of ideal theory as a new grounding concept is clearly stated. But it also shows that ideal theory has the important advantage to be a powerful tool for further developments of number theory.

After the §163, Dedekind develops, for seven paragraphs (almost forty pages), more considerations about ideal theory in relation with binary quadratic forms. His theory does not stop, abruptly, after a few propositions stating the similarity between arithmetic properties of ideals and arithmetic properties of rational integers, as the version reproduced by Noether in Dedekind's *Gesammelte Werke* suggests. Dedekind also shows how ideal theory is useful and efficient for the study of decomposable forms, the theory of units, the quadratic fields.

A notable example of what could be missed by overlooking the subsequent paragraphs in the Supplement X, is the definition of the concept of *Ordnung*, which, unlike what has been said, is already given in that version.¹⁰⁹ In order to be more thorough, although without claims of exhaustiveness, I will pursue the reading of the Xth Supplement a little further than the §163. Some of the elements that could seem to be additions to the second version are already present in the 1871 version. Their appearing in the paragraphs on algebraic integers is a result of the reorganization of the theory in 1876.

5.6.1.1 Equivalence of ideals

The §164 defines an equivalence relation between ideals. More precisely, Dedekind gives rules to divide the collection of all the ideals of a field Ω into classes: let E be the system of all principal ideals, which satisfies three properties:

- (i) Closure by multiplication.
- (ii) If \mathfrak{e} and $\mathfrak{e}\mathfrak{e}'$ are in E , then \mathfrak{e}' is in E .
- (iii) For any ideal \mathfrak{a} , there exists an ideal \mathfrak{m} such that $\mathfrak{a}\mathfrak{m}$ is in E .

Two ideals \mathfrak{a} and \mathfrak{a}' are called equivalent (*äquivalent*) if there exists an ideal \mathfrak{m} for which $\mathfrak{a}\mathfrak{m}$ and $\mathfrak{a}'\mathfrak{m}$ are in E . A class is the collection (*Inbegriff*) of every ideals equivalent to a determinate ideal. One can define the class of a product of ideals as the product of the classes of each ideal and study some properties of the product of classes (E is the unit for this product). Two important results are that the number of classes of ideals – i.e., the number of non-equivalent ideals – is finite, and the fact that every ideal \mathfrak{a} can be transformed into a principal ideal by exponentiation.¹¹⁰ Dedekind is thus

¹⁰⁹An *Ordnung*, defined below, is formally similar to a ring. It is not, however, used by Dedekind as modern algebra uses rings. The role and status of *Ordnung* seem to change from one version to another.

¹¹⁰At the end of §163, Dedekind also proved that an ideal can be transformed into a

defining, for ideals, relations comparable to congruences, in order to obtain a notion of class for ideals comparable to that of class for numbers. He then (once again) transfers, or translates, familiar ideas of number theory to the level of ideals, with an approach reminiscent of the one used in his 1857 paper, in which he defined a new level of congruences with a double modulus.

Dedekind applies methods similar to those he used for numbers to systems of numbers conceived as objects in themselves. Apparently without restrictions, he manipulates infinite systems of systems – in fact infinite systems of infinite systems – in the same way that he manipulated systems of numbers. It seems that, for Dedekind, having well-defined notions of division and multiplication for ideals allows to extend the idea that guided the definition of these systems, i.e., to collect elements satisfying a condition of closure by binary operations. Hence, providing that one (doesn't have any problem with completed infinity and) agrees to see ideals as an object *per se* rather than an aggregate of things (which is Dedekind's case), it becomes possible to consider a collection of ideals, defined by a condition of closure by multiplication and division.

The subsequent paragraphs illustrate the practical aspect of Dedekind's ideal theory. Ideals are not valuable only as a rigorous and general means to prove the unique factorization theorem for algebraic integers, but also for their fruitfulness and the fact that they can provide easier, more efficient methods for higher number theory (e.g., for the decomposition of forms). This was a point that Kummer had noticed himself, when he first introduced ideal factors, but he never pursued this path. Edwards explains that

[a]lready in his first publication ([Kummer, 1846]) on the theory of “ideal complex numbers”, Kummer spoke of generalizing the theory to “complex numbers of the form $x + y\sqrt{D}$ and of the relation of this theory to Gauss's very intriguing and important theory of composition of binary quadratic forms. In his later work he never returned to this subject, but he did generalize the theory in other ways; he gave a complete theory of ideal complex numbers for cyclotomic integers in the case where 2 is not a prime ([Kummer, 1856]), and in his *chef d'oeuvre*, the proof of the higher reciprocity law for regular prime exponents [Kummer, 1859], he developed the theory for extensions of the cyclotomic integers obtained by the adjunction of a λ th root of a cyclotomic integer. ([Edwards, 1980], 328)

Dedekind explains in §165, that the theory of ideals of a field Ω is immediately related to the theory of decomposable forms corresponding to Ω .

principal ideal through the multiplication by the appropriate ideal.

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The theory of the ideals of a field Ω is immediately connected together with the theory of the decomposable forms which correspond to the same field. We restrict ourselves, thereupon, to indicate the main features of this connection.¹¹¹ ([Dedekind, 1871], 465, my translation)

Dedekind, for lack of space, only sketches the outlines of this relation, but still gives a first grasp of how the study of forms can be eased by the use of ideal theory.

For the general case of quadratic forms, Dedekind explains that it is “convenient” (*zweckmässig*) to extend, to broaden (*erweitern*) the concept of ideal. The “extension” of the concept of ideal consists in the following. Let \mathfrak{a} be a system of integers α of Ω such that

- (i) \mathfrak{a} is closed by addition, subtraction and multiplication;
- (ii) there exists n (linearly) independent numbers in \mathfrak{a} , which is the equivalent to: every number in Ω can be transformed in a number of \mathfrak{a} through multiplication by a non-zero rational number.

Among the systems satisfying these conditions, the most important one is the system of all the numbers which “through multiplication by all numbers of the ideal \mathfrak{a} are transformed in numbers of the same ideal \mathfrak{a} .” This system is an ideal which contains the number 1 and is called “order of \mathfrak{a} ” (*Ordnung von \mathfrak{a}*) or “unit ideal” (*Einheitideal*) for the numbers it contains satisfy Dirichlet’s theory of units. Although formally equivalent to our modern concept of ring, the notion of *Ordnung* has little in common with it: neither its genesis, nor its use or its role or even its relation with ideals or fields are the same. The *Ordnung*, in 1871, is essentially a refocus of the domain in which Dedekind works for the study of binary quadratic forms. What is fundamental in Dedekind’s reflection is the distinction of several different *Ordnungen* and the immediate understanding that the one that matters is the *Ordnung* of all the integers of the field. But Dedekind is very clear, here, that what is interesting is not the “general theory of ideals” but the composition of forms, in the lines of Dirichlet’s works.

My point, here, is not to enter the subtleties of the “equivalence” between forms and ideals of Ω , but to underline the use of ideal theory to develop the study of composition forms. In fact, ideal theory is clearly embodied in the arithmetical investigations that presided at its birth. It is also used, notably, to study Dirichlet’s theory of units. Recall that Dedekind did claim, in the preface of the book, that his new theory was here to help shed light “from a higher standpoint” on the subject matter of the *Vorlesungen*. Considering how rich Dirichlet’s *Vorlesungen* are, it seems that having proved that the

¹¹¹ “Die Theorie der Ideale eines Körpers Ω hängt unmittelbar zusammen mit der Theorie der zerlegbaren Formen welche demselben Körper entsprechen; wir beschränken uns hier darauf, diesen Zusammenhang in seinen Grundzügen anzudeuten.”

laws of divisibility of algebraic numbers would not suffice to provide, by itself, the said light from a higher standpoint. In fact, Dedekind is putting a lot of effort into showing the fruitfulness (or potential fruitfulness) of ideal theory for number theoretical researches.

5.6.2 Auxiliary theories

Ideal theory, presented as a new way to ground the theory of algebraic integers, is an independent, “invariant” theory (the individual nature of the elements do not enter into account) and, as module theory, can be *applied* in different areas of mathematics. In algebraic number theory, as in algebraic function theory, ideals are not the subject of study for Dedekind, but a means to study the arithmetic of algebraic integers or functions. Ideals should allow to develop number theory or a sound basis for the Riemann surface. It is an auxiliary theory which *founds* the theory of algebraic numbers.

As Corry brings out, it seems that Dedekind makes a sharp distinction between groups (or later modules, ideals) and fields of numbers: groups are part of a more general (formal) theory and susceptible to be applied in diverse contexts. For Corry, this denotes “different conceptual levels of reference” and an important conceptual difference between the concepts studied (e.g., domains of numbers) and the tools used for it (e.g., groups). Indeed, on one hand, as we saw in the part on *Algebraische Funktionen* and as will be clear in the paragraphs below, there is an important contrast between domains (fields) of numbers and groups (or modules, ideals). Domains of numbers or functions are a framework for the study of their elements, while groups, modules or ideals are part of auxiliary theories, which can be, under certain conditions, applied in a certain number of different mathematical areas. On the other hand, as underlined by Corry,

the questions raised by Dedekind about the [domains of numbers] concern properties of their elements, his questions about groups concern the groups themselves, seen as autonomous mathematical entities, rather than their elements. ([Corry, 2004b], 81)

This distinction is a very consistent part of Dedekind’s writings.

Group theory and module theory are two additional examples of “auxiliary theories”, and are explicitly (and even more so in the following versions of algebraic number theory) presented as independent theories susceptible to be applied in many different areas of mathematics and which are, in fact, essentially interesting for the new possibilities of proofs and of further developments provided. Remember that at the end of the paragraph about groups, in the 1856-58 lectures on higher algebra, Dedekind concludes by explaining how the fact that the researches don’t take the individual nature of the elements allows for general results, which are embedded in a theory

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of larger scope (see the quote p. 250). Modules, in 1871, are presented in the same way:

we now interpose some very general observations. This separate inquiry will be of great use to us in our subsequent topic of study, as well as for many others.¹¹² ([Dedekind, 1871], 30)

In 1877, Dedekind explicitly states that module theory has been adapted to the specific number theoretical research in which it is used:

The researches in this first chapter have been expounded in a special form suited to our goal, but it is clear that they do not cease to be true when the Greek letters denote not only numbers, but any objects of study, any two of which α, β produce a determinate third element $\gamma = \alpha + \beta$ of the same type, under a commutative and uniformly invertible operation (composition), taking the place of addition.¹¹³ ([Dedekind, 1876-1877], 82)

As we saw in the first part of this dissertation, module and ideal theories are still playing this role in 1882. More importantly, these auxiliary theories are never developed for their own sake.

We also saw that ideals and modules can be transferred to algebraic function theory and used in a similar way, but that this possibility is not taken as self-evident. All the results are proved again in the new framework, highlighting how these auxiliary theories are used embedded in a certain mathematical context. This stresses the *localized* aspect of Dedekind's approach, that is, the development of conceptual arsenals in determinate frameworks, and whose result is an increase of the uniformity of treatment in the theory that constitutes the said framework. In order to obtain a theory valid for *any* relevant element, for example *any* algebraic number in a finite field Ω , Dedekind develops the theory on the basis of tools defined for uninterpreted elements¹¹⁴ and which are developed as injected in the theory, so as to serve its development. The core idea, then, is the importance to identify the appropriate basic tools and concepts to develop a theory.

¹¹² "schalten wir hier eine sehr allgemeine Betrachtung ein, welche für die nachfolgenden, sowie für viele andere, unserem Gegenstande fremde Untersuchungen von grossem Nutzen ist."

¹¹³ "Les recherches dans cette première Section ont été exposées sous la forme spéciale qui répond à notre but, mais il est clair qu'elles ne cessent en rien d'être vraies, quand les lettres grecques désignent, non plus des nombres, mais des éléments quelconques, objets de l'étude que l'on poursuit, dont deux quelconques α, β par une opération commutative et uniformément inversible (composition), tenant la place de l'addition, produiront un élément déterminé $\gamma = \alpha + \beta$ de la même espèce."

¹¹⁴ Note that it is not the case for fields, which are collections of numbers which are "precisely (...) a system of numbers which reproduce themselves by the four simplest rational operations of arithmetic."

Ideals are grounding the theory of algebraic integers in the sense that everything can be *deduced from* ideals. The idea that a (more) general (auxiliary) theory can be used to provide a basis and tools for the development of a specific theory is widely used in Dedekind's works. This is particularly striking with the use of ideals to develop the theories of algebraic numbers and algebraic functions, and can be seen in the use of sets and mappings for the definition of natural numbers.

Interestingly, this approach appears, for Dedekind, to assure that the investigated theory be developed following strict epistemological demands: uniformity, rigor, no reference to *Darstellungsformen*, but also no introduction of external or "foreign" elements in the theory. A typical example of "foreign elements" are geometrical considerations in non-geometrical contexts, such as function theory or the theory of real numbers. In particular, the theory of algebraic functions is considered as developed "in its own sphere" with the algebraic treatment based on ideal theory given.

To develop the investigated theory with such auxiliary theories should, according to Dedekind, enable to elaborate the basic definitions as well as the proofs of the theory with a greater rigor and a more uniform approach – a point which we will consider in more details later. That particular aspect becomes increasingly important in further works of Dedekind: subsequent versions of algebraic integer theory are developed to follow the epistemological values more closely, and the 1882 paper with Weber is entirely designed to provide a satisfactory definition of the point of a Riemann surface and the related basic notions.

5.7 Remarks on the reception and criticisms

If Dedekind's theory of ideals has now been more than widely accepted, the reception of his works was rather fresh, when there was any reception at all.¹¹⁵ Dedekind did not publish any important work before 1870,¹¹⁶ and an actual, more widely spread, reception of Dedekind's work only starts around 1880. Most of the criticisms, then and now, are focused on the set-theoretical, infinitist (and non-constructivist) aspects of Dedekind's approach.¹¹⁷

Edwards explains that

[t]he familiarity of the set-theoretic approach today should not obscure the fact that Dedekind's approach was a major departure

¹¹⁵Even Frobenius, with whom Dedekind had a long and very fruitful correspondence, said, in 1893, that Dedekind was unnecessarily visiting "too abstract corners."

¹¹⁶Recall what Clebsch said about Dedekind, in 1868: "[I]t will not be easy to find somebody suitable who has, in the same time, accomplished something from a scientific point of view (among which I can, unfortunately, not count Dedekind)."

¹¹⁷I will say more about Dedekind's 'methodological' choices in the next chapter.

5.7 Remarks on the reception and criticisms

from established practice at the time. At a time when the notion of a completed infinity was far from uncontroversial, it must have been jarring to many of his readers to have Dedekind define his basic concept, his *Grundbegriff*, the ideal, to be an infinite set (or system) without any indication at all as to how that set may be generated or described. ([Edwards, 1980], 348)

Dedekind's decision to publish his theory as a supplement of Dirichlet's *Vorlesungen* was motivated by the desire to give his work a wider audience. However, as was suggested by Edwards, this might have been a strategic mistake from Dedekind:

The location of this sophisticated, demanding, and highly original material as the very last item in a book which is for the most part an expository and rather elementary account of classical number theory might be expected to have a discouraging effect. Inexpert readers would probably not get to the end of the book, and experts would probably not expect to find important new material in such a place and therefore would not look. ([Edwards, 1980], 349)

Dedekind's Xth Supplement, and its subsequent versions, are indeed more research works than "lessons". Another point that Dedekind seems to have overlooked are the difficulties inherent to the set-theoretical approach adopted. The contrast with the usual mathematical practices in the 19th century, especially the use of actual infinities, renders his work all the more difficult for his potential readers.

For the lack of immediate reception of his algebraic number theory, Dedekind blames the exposition of the theory. In 1876, in a letter to Rudolf Lipschitz, who offered to publish his theory of algebraic numbers in the *Bulletin des Sciences Mathématiques et Astronomiques*, he wrote that he suspected the readers were "deterred" by the "excessive brevity and terseness" of it and, before Lipschitz's proposal, started to work on a completely reshaped version. Dedekind considers the new version to be essentially a change of presentation, offering more details and "a somewhat improved form of the essential foundation" (i.e., the theory of ideals in the field Ω). As we will see later, the second version of his work is focused on divisibility laws of algebraic integers and differs in fact rather significantly from the 1871 one.

Kummer rejected completely Dedekind's works, as did Kronecker, creating a difficult context for the reception of Dedekind's works. To Lipschitz, Dedekind expressed his happiness to be read:

Your letter brought me great and unexpected joy, since for years I had more or less given up hope of interesting anybody in my

general theory of ideals. With the exception of Professor H. Weber in Königsberg, with whom I have worked closely as editor of the forthcoming collected works of Riemann, and who has expressed his intention to acquaint himself with this theory, you are the first, not merely to show interest in the subject, but also in such a practical way, that it revives hope that my work may not have been in vain. I thought that the inclusion of this investigation in Dirichlet's *Zahlentheorie* would be the best way to attract a wider circle of mathematicians to the field, but little by little I have become convinced that the presentation itself is to blame for the failure of this plan. (Translated in [Stillwell, 1996], 48-49.)

Dedekind related his adoption of infinite sets conceived as “wholes” to Gauss's notion of congruence class. However, unlike Gauss (and except for [Dedekind, 1857]), he did not work with a representative of the class that would avoid the assumption of complete infinities. Rather, Dedekind works directly with the systems themselves. Dedekind's infinitary set-theoretical methods were considered too abstract if not altogether unacceptable by many of his contemporaries. This is, of course, notably Kronecker's position, who considers Dedekind's works too abstract and his terminology too difficult to understand. In 1880, Kronecker wrote the report for Dedekind's nomination at the Berlin Akademie der Wissenschaften, underlining the importance of Dedekind's edition of Gauss's and Riemann's works and of Dirichlet's lectures on number theory. He also considered that Dedekind's works in algebraic number theory is what justified his entry at the Akademie, despite the fact that the methods chosen made the work difficult to judge.¹¹⁸ As underlined by Dugac, Dedekind had been active for almost thirty years and while he entered Göttingen Akademie in 1862, his nomination at the Berlin Akademie der Wissenschaften came rather late in his career. It is especially late compared to mathematicians of the same generation, who entered the Akademie ten to twenty years before he did: Riemann in 1859, Kronecker in 1861, Heine in 1863, Clebsch in 1868, Lipschitz in 1872.

5.7.1 Later criticisms

Some later mathematicians strongly criticized Dedekind's approach as well. André Weil was notably a strong advocate of Kronecker's approach. Hermann Weyl, in his *Algebraic theory of numbers*, explicitly states his pref-

¹¹⁸Kronecker also claims that he obtained all these results long before Dedekind. For more on the dispute of priority between Kronecker and Dedekind, see [Dugac, 1976b], 73-74 and [Hawkins, 2008]. Note that Kronecker did not mention *Stetigkeit* in the report, which shouldn't come as a surprise, since he rejected irrational numbers as well as infinitary methods.

5.7 Remarks on the reception and criticisms

erence for Kronecker’s approach. In a paragraph entitled “Our disbelief in ideals”, he explains that

The notion [of ideal] was first introduced by Dedekind for the arithmetics of algebraic number fields. Dedekind substituted for a divisor \mathcal{A} the ideal of the integers divisible by \mathcal{A} e.g, in the ring of ordinary integers, [he replaces] the divisor 3 by the ideal of numbers

$$\dots, -9, -6, -3, 0, 3, 6, 9, \dots$$

Instead of saying that α is divisible by the divisor \mathcal{A} , he says that α is an element of the set or ideal \mathcal{A} . I prefer to stick to the more suggestive divisor terminology. ([Weyl, 1940], 36)

He goes on, a couple of pages later, arguing that for his aim, namely “to secure the law of unique decomposition” in primes for algebraic numbers, he must “reject Dedekind’s notion of ideal as a universal solution” and adopt “an axiomatic approach”:¹¹⁹

after setting down in the next section our axioms including the law of unique decomposition, we shall endeavor to show that once these axioms are granted in the ground field K , one can extend the basic concepts of integers and divisors to any finite field over K without invalidating the axioms. This is accomplished by following Kroneker’s idea of adjoining indeterminates rather than by Dedekind’s procedure. (*ibid.*, 38)

For the sole purpose of studying divisibility of algebraic integers, Kronecker’s and Dedekind’s theories are equivalent, and Weyl explains that bridges exist between them. It is only a question of preference, then:

As both theories are actually equivalent one can dissent about questions of convenience only. To my judgment the odds are here definitely against Dedekind. (*ibid.*, 75)

Two notable characteristics give the advantage to Kronecker’s approach. The first, and most closely related to Dedekind’s infinitary set-theoretic approach, is the finiteness of the methods:

Kronecker’s criterion of divisibility is one decidable by finite means, while Dedekind’s criterion refers to the infinite set of all possible integers. This has further awkward consequences. . . (*ibid.*, 67)

¹¹⁹Note that the “axiomatic method” is incidentally pointed out by Avigad as an essential component of Dedekind’s methods. It is even pointed as one of the elements for the efficiency and fruitfulness of Dedekind’s approach. Beyond a possible disagreement between Weyl and Avigad about what the “axiomatic approach” is supposed to be, this reveals a certain vagueness in the criterion itself. I will come back to Avigad’s analysis of Dedekind’s “methodology” in the next chapter.

And the second one is the greater generality of Kronecker's divisor theory, since for the criterion of divisibility "in Kronecker's theory, the embedding field (...) is irrelevant for the definition."

Outside of the scope of what can properly speaking said to be a "reception" of Dedekind's works, one of the harshest criticisms of Dedekind's approach comes from Harold Edwards, who claims, not far from Weyl's statements, that everything in Dedekind's theory can be translated into Kronecker's divisor theory.¹²⁰ And since everything done in Dedekind's theory of algebraic numbers can be reformulated in finitistic ways, there is no necessity for Dedekind's methods. For Edwards, Dedekind's choice to use infinite sets is likely to be motivated by "something from analysis" ([Edwards, 1983], 12), that is, by Dedekind's definition of real numbers by means of cuts. Edwards argues that because Dedekind was aware that his definition of irrational number could not be developed by finitistic, constructivist methods, he probably chose to "fly in the face of the doctrine against completed infinities" rather than trying to develop a constructive approach for his algebraic number theory.¹²¹

Edwards explains that replacing Kummer's ideal factor p by the set of all integers divisible by p amounts to replace "a very explicitly defined object (in Dedekind's view, too explicitly defined, because the definition involves a particular representation) with a very vague one" ([Edwards, 1980], 348).

What kinds of "systems" of integers are to be allowed as ideals? How are they to be described? Has one defined an ideal if the determination of whether a particular integer belongs to it depends on solving, say, the Goldbach conjecture? Dedekind does not address these questions at all. (*ibid.*, 348)

For Edwards, Dedekind's methodological demands are not a sufficient ground for the "revolutionary change" that ideals are, especially insofar as

[It] seems (...) that Dedekind's objection to ideal numbers was, even at the outset, philosophical and aesthetic and did not stem from any logical or practical difficulty with ideal complex numbers. (*ibid.*, 348)

¹²⁰As underlined by Avigad, in [Avigad, 2006], while this is true today, one can doubt that Dedekind and Kronecker would indeed have had the tools to do so.

¹²¹Edwards's rather uncharitable hypothesis seems, however, doubtful. Indeed, Dedekind already used infinite sets in his lectures on Galois theory, in 1856-58. Moreover, Dedekind did not take particular pride in his theory of irrational numbers. While he did consider it as necessary to fill the lack of a rigorous definition for the linear continuum, he thought of it as a simple, uncomplicated work, which many other mathematicians could have written – and in fact, Cantor's definition published the same year was, for him, as acceptable as his own. See also [Ferreirós, 2008].

5.8 Conclusion. From ideal numbers to ideals to arithmetic of ideals

For example, Edwards wonders¹²² why Dedekind prefers to use ideals, if the fact that each ideal corresponds to an ideal factor is such a crucial point. In fact, Dedekind's concern about the correspondence between ideal numbers and ideals shows that he is engaged in a generalization of Kummer's approach. For Dedekind, for the generalization to be acceptable, one has to make sure that the restricted theory stays valid in the more general theory.

I will consider the reasons of Dedekind's preference for ideals over an approach closer to Kummer's ideal factors in the next chapter. Avigad ([Avigad, 2006]) provided some hints on these matters, and I will propose to deepen his reflection in the next chapter.

5.8 Conclusion. From ideal numbers to ideals to arithmetic of ideals

I tried to show, in this chapter, that the idea on which ideal theory relies is the transfer of ideal numbers's properties to the "higher level" of ideals, setting up the first elements of an arithmetic of ideals. But in 1871, unlike in the 1877 and 1879 versions, Dedekind does not fully develop ideal theory as a new level of arithmetic similar or analogous to rational arithmetic, he rather transfers the properties characterizing ideal numbers to the level of ideals. Dedekind, here, appears to be engaged in a generalization of Kummer's works, in which the core move is the objectification of ideals, in order to work in the "higher level" of collections of numbers. The "analogy" with rational number theory is however not clear.

According to Ferreirós, it was because of editorial imperatives, that Dedekind's theory was published under this form:

Due to the fact that the second edition of Dirichlet's *Vorlesungen* was scheduled for late 1870, and the notion of ideal was first formulated in August of that year, he had to work under great time pressure in developing his new ideas. It was impossible to adapt the development of the theory to the set-theoretical characteristics of its central notion, as his methodological preference for a "pure" development of the theory demanded. ([Ferreirós, 2008], 111)

It is true that Dedekind complained a lot about editorial issues and lack of time. However, we will see that his rewriting of his own theory implies not a more set-theoretical approach but a more arithmetical treatment. In addition, the justification for the new version of the theory is not about

¹²²Edwards dissatisfaction with Dedekind's theory increases with each subsequent version, since the abstract set-theoretic methods only play a greater role in them. I will not address the question of which version of the theory is "better".

“purity” but rather related to the desire to simplify it, notably by giving it a more satisfying foundation.¹²³

The epistemological requisites often expressed by Dedekind are, as it should be clear, guiding the “*Gedankengang*” and the choices made for the definition and treatment of ideals. A more accurate understanding of (what Dedekind saw as) the epistemological advantages of ideal theory should be gained by considering the uses of the theory, as well as the changes inflected to it by Dedekind – especially insofar as they are considered as improvements) I will suggest that this aspect of ideal theory, from 1871 to (1882 and) 1879 is linked to its relation with arithmetic.

¹²³Note that “set theory” properly speaking did not exist by the 1870s. The closest to it was Cantor’s works, which would certainly not have been of any use, here.

Chapter 6

Towards a more arithmetical theory of algebraic numbers?

We saw in the previous chapter how Dedekind introduced the notion of ideal as a new ground for the theory of algebraic integers, and developed ideal theory on the basis of the definition of notion of divisibility for ideals. Dedekind's 1871 algebraic integer theory proposes an extension of Kummer's ideal numbers in order to prove the general validity of divisibility laws for algebraic integers, as part of investigations on the decomposition of binary quadratic forms. In 1876-77 and 1879, he published a revised version of his theory, in which while the essential ideas are similar, their articulation and the treatment of certain parts of the theory change significantly. Dedekind's rewriting of his theory aimed at providing a clearer theory, in which the basis would be laid more carefully, so that the reader could tame the new theory with less difficulties. The most perspicuous change from the first version of the theory to the second one is a profound restructuration. This restructuration stresses the importance of ideals as objects of arithmetic. From, Dedekind's viewpoint, the subsequent versions of his ideal theory are a reformulation: nothing fundamental as been changed in the theory itself, it is a more detailed, more carefully unfolded version of the theory. Incidentally, as we will see, it brings out, in 1877, the more elementary arithmetical nature of the research.

In this chapter, I will proceed in three times. In the first section, I will propose to investigate in more detail the reasons that, for Dedekind, justified the pride of place of ideal theory against Kummer's ideal numbers and argue that the possibility to define them as *arithmetical* elements for which rational operations can be defined is a decisive property. In the second section, I will give an analysis in more details of the 1877 version of the theory of algebraic integers, in particular as regards the arithmetical relationships defined between ideals and the endeavors to develop a theory of ideals following the same lines as elementary rational number theory. For this, the

study of the 1877 and 1879 version of the theory will reveal to be more than useful, insofar as it relies more deeply on elementary arithmetic and exhibits a clear desire to develop ideal theory analogously or similarly to elementary rational number theory. This will be the focus of my last section.

6.1 Ideals and arithmetic of ideals

So as to understand the ideals' pride of place for Dedekind, I will start from Avigad's reflections in [Avigad, 2006] about the definition of ideals, and try to clarify some specific points. In particular, I will consider the question of the "objectification" of ideals, that is, the consideration of ideals as objects in themselves, as wholes rather than as aggregates. Where Avigad bases his analysis of introduction of ideals on the idea of "axiomatic and set-theoretic methods", my goal will be to understand the gains of such an objectification without resorting to categories external to Dedekind's writings. This will lead me to consider how ideal theory is developed and how ideals are *used* in algebraic number theory.

The definition using necessary and sufficient conditions is, as stated by Dedekind, a central feature of ideals insofar as it allows to avoid considering anything else than the system of elements themselves. In particular, this method of definition avoids relating ideals to ideal factors or to arbitrary *Darstellungsformen*. This point is essential for the possibility to consider ideals "as wholes" and for the set up of arithmetical relationships between ideals. Indeed, the operations should, just as the definition itself, allow to develop the theory as an "independent" theory and in an "invariant" way. Considering what can be done with ideals-as-objects, I will propose that the definition of ideals as arithmetical objects and the possibility to define arithmetical operations for ideals play an essential role. A complete understanding of the concept of ideal can only be achieved by analyzing the ideals's role in the theory of algebraic integers and not by relying solely on their definition.

6.1.1 A note about Avigad's article

In his 2006 paper, "Methodology and metaphysics in the development of Dedekind's theory of ideals", Avigad uses Dedekind as a historical case study to investigate questions about mathematical understanding: How does one increase our understanding of a theorem or a proof? How can one identify the "right" definition? What is the informative value of a certain method of definition or proof, and what makes it better than another one? Dedekind's works are of particular interest for Avigad, because Dedekind's approach appears intricately tied with "general philosophical views and methodological concerns" ([Avigad, 2006], 160) and has had a noticeable impact on

6.1 Ideals and arithmetic of ideals

mathematics and mathematical practice:

His work has certainly had a tangible effect on mathematics, inaugurating practices that were to proliferate in the decades that followed. These include the use of infinitary, set-theoretic language; the use of non-constructive reasoning; axiomatic and algebraic characterization of structures, and a focus on properties that can be expressed in terms of mappings between them; the use of particular algebraic structures, like modules, fields, ideals, and lattices; and uses of algebraic mainstays like equivalence relations and quotient structures. (*ibid.*, 160)

Avigad is interested in Dedekind's "methodology", that is in "more distinctly mathematical concerns (...) and the mathematical ramifications of the methods" versus more "metaphysical" preoccupations regarding the mathematical ontology.

Avigad's analysis of the development of ideal theory is focused on the "set-theoretical" and "axiomatic" aspects of Dedekind's approach. The idea advanced by Avigad is that the "axiomatic" and "set-theoretic" approach developed by Dedekind, which Avigad matches broadly with the "conceptual approach", holds the key to the resolution of the difficulties or defects of Kummer's ideal numbers, in particular because it allows to consider ideals as objects in themselves, to objectify ideals. Avigad lists four salient characteristics of set-theoretic mathematics: (1) predicates and properties can be treated as objects; (2) uniform definitions of mathematical domains can be given with reference to "arbitrary" elements and mappings; (3) the results obtained "hold more generally", are independent of "manner of representing elements" and can be applied to particular cases; (4) they suppress "tedious calculational information" (*ibid.*, 185-186). Avigad furthermore identifies the benefits of an axiomatic analysis as being: (1') results stated with a greater generality; (2') suggestions of "appropriate generalizations" and "appropriate definitions"; (3') possibility to transfer prior results and/or adapt prior proofs in a new setting; (4') the presentation is simplified "by removing irrelevant distractions".

A point that Avigad does not seem to notice is that if these elements appear to fit Dedekind's approach, it is because he proposed to set up what he calls a "higher level" *in order* to provide general, uniform definitions answering simplicity and rigor requirements and following his desire to get rid of long tedious computations.

It seems that Avigad's statements should be moderated. First, these "new practices" were emerging before Dedekind's works: non-constructive reasonings can notably be found in Dirichlet's and Riemann's works, and infinitary set-theoretic language is also used by Riemann. Secondly, the "tangible effect" was far from an immediate one – in both meanings of the term: it took a long time, and it took the mediation of Hilbert, Noether

and others early 20th century algebraists. Thirdly, the ideas alluded to by Avigad do not all appear as clearly and explicitly as he is suggesting, and in particular not with the level of generality suggested – especially as far as “algebraic structures” are concerned.

For Avigad, the after-effects of Dedekind’s innovations in algebraic number theory are so important, “so far-reaching”, that it is difficult to explain them in any other way than coming from “a fundamentally different conception of what it means to do mathematics.” But Avigad doesn’t seem to acknowledge that contemporary mathematics does not use Dedekind’s concepts as he introduced them. On one hand, Dedekind never introduced abstract ideals, fields, modules, etc., but always ideals (or fields or modules) *of numbers*. He never developed field theory or ideal theory for their own sake, they were tools for algebraic number or algebraic function theories. On the other hand, the peculiar arithmetical development proposed by Dedekind shows, as I hope to make clear, an additional difference with abstract algebra.

In fact, Avigad’s analysis seems to be investigating the consequences of Dedekind’s works in Dedekind’s works themselves: trying to explain the efficiency of Dedekind’s “methodology”, he puts the consequences in place of the causes.¹

The reference to an “axiomatic method”, in Avigad’s paper, designates essentially the analysis of a mathematical problem or notion so as to identify necessary and sufficient conditions to solve the problem or define the notion. This approach is considered, following Hilbert, as axiomatic. It is clear that Dedekind does put a lot of effort in identifying and isolating the “essential characteristics” of concepts, properties or laws which are mutually independent and form the minimal necessary conceptual basis for a concept. This approach will be typical of the (Hilbertian) “axiomatic method” that Avigad is considering² and which is guided by the following principle:

One should take great care to identify the axiomatic features of the domain in question that are in play at each stage of the development of a theory. ([Avigad, 2006], p. 181)

¹To be clear, Avigad’s analysis is certainly of great philosophical significance. Dedekind’s works, at the turning-point of fundamental changes in mathematics, give the possibility to see, and to a certain extent to understand, the profits and losses induced by the changes in the practices and the conception in mathematics, as well as the reasons for these changes. But if the focus is to be on understanding the introduction, by Dedekind, of what Avigad considers to be axiomatic and set-theoretic methods in algebraic number theory, then looking in the direction of future consequences presents the risk of retrospectively projecting on Dedekind, conceptions which might not have been his own.

²And in fact, Sieg has argued in several places that Dedekind was to some extent a precursor of Hilbert on these matters – but it is not the point, here.

6.1 Ideals and arithmetic of ideals

Dedekind almost never uses the word “axiom”. He does, however, make a point in identifying the “characteristic properties” of concepts which he explicitly states as a definition by necessary and sufficient conditions. The emergence of the “axiomatic approach” is sensible in the ‘modernity’ of Dedekind’s approach here. This being said, the analysis of a mathematical situation is not inherent to the (modern) “conceptual approach”, and can be spotted in Kronecker’s writings as well:

The extremely simple principles on which Gauss’s method [of classification of binary quadratic forms] rests can not only be applied in the place mentioned above, but also in many others and, in particular, already in the most elementary parts of number theory. This fact suggests, and it is easy to convince oneself of it, that these principles belong to a more general and more abstract realm of ideas. It seems therefore to be appropriate to free the further development of the latter from all inessential restrictions, so that one is then spared from having to repeat the same argument in the different cases of application. The advantage comes to the fore already in the development itself, and the presentation (if it is given in the most general way possible) thereby gains in simplicity and clarity, since it clearly exhibits what alone is essential. (in [Kronecker, 1899] I, 274-275, transl. in [Schlimm, 2005] and quoted in [Avigad, 2006], 168)

It is prompted by a desire of clarity and exhaustiveness that Dedekind develops his approach, which will later be extended in two directions: Hilbertian axiomatics and modern algebra. The underlying ‘modernity’ of Dedekind’s approach, palpable in his influence on Hilbert’s and Noether’s works, is not to be discussed (or underestimated) here. Rather, I would like to underline that the said modernity should not be justified or explained by using properties of methods and conceptions which constitute the core of the reappropriation of Dedekind’s ideas by later authors.

Avigad’s explanation is made in his own categories, as a case study for philosophy of mathematics, while what I hope to achieve is an explanation strictly relevant to the framework of Dedekind’s works. Thus, on the one hand, I do not have the ambition of providing (more) general philosophical considerations which might apply outside of a Dedekindian framework; and on the other hand, it will be important for my purpose to resist the temptation to see Dedekind’s “modernity” as meaning that he is one of our own, thus using today epistemological categories to understand the articulations and motivations of his mathematical and methodological choices.

Before considering the objectification of ideals, I would like to clarify some terminology matters. Avigad makes a distinction between generality and uniformity:

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Generality: the theory should apply to rings of integers beyond the ordinary integers, Gaussian integers, cyclotomic integers, and rings of quadratic integers that were useful to Euler.

Uniformity: one theory should cover all these cases, and, indeed, one definition of the ideal divisors should account for all the ideal divisors in a given ring of integers. Furthermore, as much as possible, proofs should cover all situations uniformly, without case distinctions. (*ibid.*, 171)

I will rather use a distinction between generalization and uniformity (as a certain form of generality). To differentiate in this way allows to highlight that aiming for a general theory (in Avigad's first sense) is part of a process, an important point to understand more precisely Dedekind's works. It shows how Dedekind seems to place his own contribution to the developments of mathematics. Generalization does not require uniformity: the approach using higher congruences, which Dedekind gave up, was an attempt to the same generalization as ideal theory – that of Kummer's results. Uniformity is a different preoccupation, concerned with the choices made to elaborate definitions and methods of proof, and is linked to epistemological values guiding his conception of what (good) mathematics should be. The requirement for uniformity runs through all of Dedekind's works: uniformity is a requirement for the definition of natural numbers as well, for example.

6.1.2 The definition of ideals and ideals as objects

The definition of ideals plays a crucial role in the set up of the theory, since they are presented as grounding concepts of the theory of algebraic numbers, that is, all results are derived from the notion of ideals. As was mentioned before, Dedekind was keenly aware of the role played by definitions in shaping a theory, and carefully chose which basic concept(s) to use, and how to present their definition(s). I will suggest that arithmetic being *arithmetical* elements was a crucial point for Dedekind.

Dedekind's endeavours to satisfy requisites of rigor, uniformity, or even simplicity, are often expressed out loud by Dedekind, especially as regards his attempts to generalize Kummer's theory. As I have mentioned, Dedekind explains in the introduction of the second version of the algebraic number theory, his first attempts using higher congruences were unsuccessful (see the quotation on p. 56). The invention of the concept of ideal, Dedekind explains, is the key element of his ability to propose a "general exceptionless theory" of algebraic integers. Indeed, for him, the greatest advantage of the concept of ideal is "its extreme simplicity": it does not require any creation of new entities (such as ideal factors), but only the consideration of "a system of actual numbers".

6.1 Ideals and arithmetic of ideals

According to Edwards, Dedekind is mistaken in pointing out the introduction of ideals as the essential unlocking moment for the developments of the theory. For him, the key moment is, in fact, the identification of algebraic integers as the new concept of integer, and all difficulties faced by Dedekind were coming from this point. It is not clear whether Dedekind had the concept of algebraic integer, when he tried to generalize Kummer's approach using higher congruences, but Dedekind never alluded to any difficulty coming from the definition of algebraic integers. Note that it also seems unlikely that Dedekind, who attached such great importance to identifying the ground concepts of a theory and to proceeding step by step, would have tried to generalize Kummer's approach without a good idea of the general concept of integer needed for it. He does however mention, at several points, the importance of the concept of ideal for his approach, as well as that of identifying the right domain of integers in which the laws of divisibility should be proven (i.e., the ring of integers, which Dedekind calls *Ordnung*). The quotation of Dedekind's letter to Lipschitz (given on p. 341) which Edwards uses to support his point seems to be, in fact, a reference to the notion of *Ordnung*, which, as we will see, plays an important role in the reformulation of the theory in 1877.

An essential part of Dedekind's definition of ideals and development of ideal theory is the consideration of infinite systems as united wholes. The use of complete infinities was a contentious point for Dedekind's contemporaries, and often taken as a ground for a set-theoretic reading of his works. While it is one of the most striking aspects of Dedekind's approach, the use of infinite sets is not something Dedekind comments upon very much. In fact, from his standpoint, the use of infinite sets appears, in a way, like a rather natural thing to do. We will see that one can understand, when reading *Zahlen* and looking back at his previous algebraic works, that the consideration of infinite sets is probably less a blind spot for Dedekind than it is related to the operations of thought, as Ferreirós notices:

Dedekind's acceptance of the infinite does not have the appearance of a more or less *ad hoc* position (...). It rather looks like a deep-rooted conviction: infinite sets seemed to him perfectly acceptable objects of thought, that involve no contradiction, and that play a crucial role in mathematics. ([Ferreirós, 2008], 110-111)

In the letter to Lipschitz quoted on p.44, Dedekind explains his use of infinite sets in ideal theory. He explains the relation between his conception of ideals and Gauss's notion of class in the *Disquisitiones Arithmeticae*. As Ferreirós underlines, Gauss did not use actual infinities, he even carefully avoided them (by choosing a representative for classes)³ but that point does

³In a letter to Schumacher from 1831, referring to limits Gauss wrote that he "object[ed]"

not seem to bother Dedekind. In fact, the core point of Dedekind’s parallel with Gauss’s method is not the actual infinity of collections, it is the act of putting together all functions or numbers satisfying a property into a collection that one subsequently considers “as one whole”, i.e., as a class or a group or an ideal. The transfer to the “higher level” and the consideration of systems as united wholes avoid the choice of one particular representative – even if the choice is itself indifferent. As I have noted in the first chapter of this dissertation, this remark about ideals is following Dedekind’s mention of his own works as adopting an approach inspired by Riemann. We have seen the influence of Riemann’s works on Dedekind’s ideas (see also [Ferreirós, 2008]). Using infinite systems seems to be an instance of Dedekind’s peculiar way of joining Gauss’s and Riemann’s influences. Indeed, unlike Gauss, Riemann accepted actual infinities – and in fact, the “class of algebraic functions” is an infinite system.

In [Dedekind, 1894b], Dedekind explains that there are “impartial” reasons to prefer his ideal theory, and mentions Gauss again by quoting Gauss’s famous remark about Waring and Wilson’s difficulties to prove a theorem about divisibility of numbers (see p. 1.1.1). While the authors, Gauss wrote, thought that it was because “no notation can be devised to express a prime number”, Gauss states that the difficulties were caused by their focusing on the wrong problem. For him such “truths” must be “be drawn out of notions [*notionibus*] not out of notations.” Dedekind considered this statement to be “a great scientific thought”:

the decision for the internal in contrast to the external. This contrast also recurs in mathematics in almost all areas; it suffices to think of function theory, and Riemann’s definition of functions through internal characteristic qualities, from which the external forms of representation flow with necessity. ([Dedekind, 1894b], 54–55, partially translated in [Tappenden, 2005a], 8-9)

Dedekind saw, in his approach, the best way to focus on “the internal in contrast to the external”, because it did not involve indeterminate variables nor any external use or explicit representations: it only used systems of numbers.

We saw that in order to generalize Kummer’s ideal factors, Dedekind chooses to consider the system of all integers divisible by these factors. Avigad explains this point as being founded on the following remark:

[i]n a sense, replacing the predicates P_α [“is divisible by p ”] by the systems S_α of integers that satisfy them is mathematically

above all the use of an infinite magnitude as if it were complete, which is never permitted in mathematics. The infinite is only a *façon de parler*, when we are properly speaking about limits that certain relations approach as much as one wishes, while others are allowed to increase without limit.” ([Gauss, 1863-1874, 1900-1917], VIII, 216)

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inert. All the effects that mathematical objects can have on mathematical discourse are mediated by the roles they play in assertions; so if all references to ideal divisors are expressed in terms of the property of dividing an element of the ring, it makes little difference as to whether one takes ‘ x has property P_α ’ or ‘ x is an element of the set S_α ’ to stand duty for ‘ α divides x .’ ([Avigad, 2006], 172-173)

While Dedekind did not express the “mathematical inertia” in these modern set-theoretic terms, he did take the time to explain, in the introduction of the 1877 version of ideal theory, why it was mathematically equivalent to consider ideals rather than ideal numbers. The desire to provide a definition “exact and covering all the ideal numbers that must be introduced in a particular numerical domain \mathfrak{o} ” ([Dedekind, 1876-1877], transl. slightly altered, 57) is explicit and leads to the re-modeling of Kummer’s initial idea.

However, recall that Dedekind insisted, in his *Bunte Bemerkungen*, that ideals as “the system of numbers divisible by a factor p ” does not constitute the essential, definitional property of ideals. Rather, it is the step of the “*Gedankengang*” towards ideals defined as sets closed by certain operations. To *define* ideals as the system of numbers divisible by a factor p , even if it is done for any such p , would attach the definition to a particular number. Dedekind was able to prove that the properties of closure (see p. 300) are necessary and sufficient conditions for a system to be an ideal. This definition is much preferable because it does not require “the admission of ideal numbers.”

We know that in 1871, and in later versions of the theory of algebraic integers, Dedekind wants a theory of ideals that can yield an independent, more general (covering all cases) and more rigorous study of divisibility for algebraic integers. For Dedekind, in order to assure a uniform and rigorous approach, one has to focus on the “characteristic properties” of the notions investigated. This is a rather vague claim, which Avigad identifies as “Dedekind’s emphasis on conceptual reasoning”. For Avigad, it constitutes the core of what he calls the “methodology” developed by Dedekind: choices in the elaboration of definitions of concepts and methods of proofs which are guided by epistemological requisites. What exactly do these vague methodological claims mean?

In [Dedekind, 1877] and [Dedekind, 1878], Dedekind gives details, in a retrospective explanation and justification of his methods, about how and why he came to introduce ideals. One can find the usual statements about the importance to try, as much as possible, to avoid taking any explicit representations (notations) or computations – especially insofar as they should require to chose a certain variable – as the basis of the definitions. Neither notations nor algorithms are, for Dedekind, giving any insight on the

essential, characteristic properties of the notions. They are rather hiding them. This is an important reason for not pursuing the approach using higher congruences, which Dedekind tried first, as I have mentioned earlier. This approach was explicitly based on a definite equation, and thus hiding the possibility to define independent concepts. As Dedekind showed, ideals and their divisibility can be defined in a way that is uniformly valid for any field of algebraic numbers: to transfer the divisibility properties to ideals, for example to define primality as a property of ideals, allows to study divisibility laws without reference to a specific number or equation (see p. 263), and in fact without reference to the domain of numbers itself. This allows to satisfy the other important requisite, from Dedekind's viewpoint: by not taking into consideration the particular properties of domains of numbers, one can have a definition which encompasses all the cases possible, and in particular which avoids the distinctions of several cases (a drawback that the approach with higher congruences did not avoid). Such a definition allows to see, or to better understand, what is, indeed, essential to the concept defined, rather than some kind of second effect of a particular representation or algorithm.⁴

In his reflection about the appropriate general definition of ideal numbers, Dedekind makes it clear that the possibility to define their multiplication is a decisive point. Dedekind observes that the definition of ideal numbers is assured (i.e., it is necessary and sufficient to do so) by establishing "once and for all the common characteristic of the properties A, B, C, \dots that always, and only them, serve to introduce determinate ideal numbers."⁵ For the multiplication of two ideal numbers, it is necessary and sufficient

to indicate generally how one can deduce from two of these properties A, B to which correspond two particular ideal numbers, the property C corresponding to their product.⁶ (*ibid.*, transl. modified, 57)

For Dedekind, this "leads naturally" to consider the system of all numbers divisible by a certain ideal factor:

Since such a characteristic property A serves to define, not an ideal number itself, but only the divisibility of the numbers in \mathfrak{o}

⁴Another good example of this point is the definition of irrational numbers by means of cuts: the cuts allow to characterize real numbers as elements of the linear continuum, and because it is valid for *all* real numbers, it is, for Dedekind, a more essential characteristic than their being, for example, logarithms or roots of equations.

⁵"il sera nécessaire et suffisant d'établir une fois pour toutes le caractère commun de toutes les propriétés A, B, C, \dots , qui toujours, et elles seules, servent à l'introduction de nombres idéaux déterminés"

⁶"d'indiquer généralement comment de deux de ces propriétés A, B auxquelles correspondent deux nombres idéaux déterminés, on pourra déduire la propriété C qui doit correspondre au produit de ces deux nombres idéaux."

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by the ideal number, one is naturally led to consider the system \mathfrak{a} of all these numbers α of the domain \mathfrak{o} which are divisible by a particular ideal number.⁷ (*ibid.*, transl. slightly altered, 58)

For “each particular ideal number there corresponds a particular ideal \mathfrak{a} ” (*ibid.*). Since the ideal contains *all* the numbers divisible by the corresponding ideal factor, we have that “conversely the property A , i.e., the divisibility of a number α by an ideal number, consists uniquely in that α belongs to the corresponding ideal \mathfrak{a} ” (*ibid.*, 58, transl. modified). This equivalence allows to consider, then, the ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ in place of the properties A, B, C . Avoiding any essential reference to a definite equation, the definition of ideals and their multiplication given here is a uniform definition in the framework of the theory investigated and meets Dedekind’s requirements. Because of the standpoint adopted by Dedekind, the proof of the validity of the divisibility laws given with ideal theory is valid for the ordinary integers, as well as for Gaussian integers, cyclotomic integers, etc. – in fact, for *any* algebraic integer.

Ideals allow, according to Dedekind, to eliminate the problems inherent to the absence of a general definition for Kummer’s ideal numbers and their multiplication. The fact that ideals are considered as wholes, rather than aggregates of things, that is, as objects in their own right gives Dedekind the possibility to define operations for these new objects, which is the move that allows to solve, or at least bypass, the difficulties in Kummer’s approach. The set-theoretic definition of ideals does not solve the problems by itself. Rather, the set up of an arithmetical framework allowing to develop arithmetical methods of proofs do.

How do ideals solve the problems of Kummer’s approach? After an analysis of the problem investigated and the properties of numbers considered and used, the elaboration of the new definition is made in several steps. Firstly, Dedekind identifies the characteristic property common to all objects considered (here, the divisibility by a certain factor p). Secondly, on this basis, he elaborates a concept, the concept of ideal, that allows to express or state these properties in a general, rigorous and contextually relevant way. That is, the new concept should be a legitimate object of the theory investigated (a number-theoretical object, in the case of algebraic number theory) based upon non-external notions. This implies that it is preferably defined on the basis of “earlier concepts” (i.e., logically and conceptually anterior notions, sets, and arithmetical operations, in the case of ideals) intern to the theory of numbers. And finally, Dedekind provides a definition for this

⁷ “Comme une telle propriété caractéristique A sert à définir, non un nombre idéal lui-même, mais seulement la divisibilité des nombres contenus dans \mathfrak{o} par un nombre idéal, on est conduit naturellement à considérer l’ensemble \mathfrak{a} de tous ces nombres α du domaine \mathfrak{o} qui sont divisibles par un nombre idéal déterminé.

new concept that allows to develop consistent and fruitful methods of proof, which reduces the length of computations. For ideals, the last requirement is met with the definition of operations between ideals which allow to transfer the computations from the level of numbers to the higher level of ideals. This allows to hide away computations by reducing them to simpler ones: computations between ideals which are similar to elementary number theory (as I will develop in the next section). Let me now consider this aspect more precisely.

6.1.3 Arithmetic of ideals

A property of ideals that has not been considered in the previous paragraph is the definition of operations and the desire to define *arithmetical* elements. The use of infinite collections allows to avoid introducing foreign elements. Note that Dedekind does not use this strategy only to replace ideal numbers: it is also the method adopted in *Stetigkeit*, in 1872, to define real numbers and in manuscripts to define other extensions of the natural numbers (see [Sieg and Schlimm, 2005] and [Schlimm, 2000]).

Dedekind is explicitly drawing links between the definition of ideal numbers in algebraic number theory and that of irrationals, in a lengthy footnote to the paragraph of [Dedekind, 1876-1877] commented just above. In this footnote, he comments on his chosen method by explaining that the demands for a general and exact definition of all ideal numbers in the domain of integers and their multiplication are “legitimate or even necessary” when one “introduces or creates new arithmetic elements.” The introduction of ideal numbers should be done following the same guidelines as that given for real numbers which are:

As first demand, I reckon that arithmetic must be kept free from intermixture with foreign elements, and for this reason I reject the definition of real number as the ratio of two quantities of the same kind. On the contrary, the definition or creation of irrational number ought to be based on phenomena one can already observe clearly in the domain R of rational numbers. Secondly, one should demand that all real irrational numbers be engendered simultaneously by a common definition, and not successively as roots of equations, as logarithms, etc. Thirdly, the definition should be of a kind which also permits a perfectly clear definition of the calculations (addition, etc.) one needs to make on the new numbers. (*ibid.*, 57-58, footnote, transl. slightly altered)

The parallel to real numbers suggests that the core demand of the approach adopted to define ideals, which are to replace ideal numbers, is related to

6.1 Ideals and arithmetic of ideals

epistemological requirements regarding the definition of *arithmetical* elements. Dedekind's requirements involve requisites of uniformity of the definition, a certain kind of purity, and the demand that the operations of arithmetic be considered as an indispensable part of the definition of these "arithmetical elements".

Once the definition for the (new) fundamental concept, here ideals, is identified, the appropriate methods of proofs should be developed. The possibility to develop such methods and to give proofs which are, like the new definition, clear, uniform, and rigorous, is a good way to test whether the concept chosen is, indeed, the right one. In the case of ideals, the generalization of Kummer's result became doable with the new concept, but developing rigorous, acceptable proofs was not without considerable difficulties, as Dedekind explains in the letter to Lipschitz from 29 April 1876:

only after inexpressible efforts that I succeeded in moving forward step by step and finally filling all the gaps; I continually had the feeling that I was hanging from a ladder, that I might not succeed in reaching the next rung, and if I did not have before me now in published or written form my presentation of the theory from that time it would give me great difficulty all over again to put every little step of the proof together in the right order so that the goal would truly be reached.⁸ ([Lipschitz and al. 1986], 466, translated in [Edwards, 1980] 347-348)

When considering ideals as "wholes", one can define operations between them, in particular, a divisibility relation which will be essential for the proofs. As we saw in 1871 (and in 1882), Dedekind sets up arithmetic operations for ideals, shifting the study to the "higher level" of ideals. This allows him to transfer all considerations about divisibility of algebraic integers, to the study of divisibility between ideals. Ideals, thus, provide in the same time the possibility to define arithmetical objects with rigorously defined operations (a necessity for any introduction of new arithmetical elements), and to work with, so to speak, actual objects, rather than *Darstellungsformen*, a point underlined by Avigad. This is what Dedekind, in the *Bunte Bemerkungen*, refers to as his approach being more "concrete" than Kronecker's formal algebraic methods.

⁸ "Obgleich damals das zu erreichende Ziel stets klar vor mir lag, so ist es mir doch erst nach wirklich unsäglichen Anstrengungen gelungen, Schritt für Schritt vorwärts zu kommen und endlich jede Lücke auszufüllen; ich hatte fortwährend das Gefühl, an einer Leiter zu hängen mit der Furcht, dass es mir nicht mehr gelingen würde, die folgende Sprosse zu erreichen, und wenn ich meine damalige Darstellung dieser Beweise nicht gedruckt oder geschrieben vor mir hätte, so würde es mir jetzt abermals eine grosse Mühe machen, alle Beweismittelchen, jedes am rechten Orte wieder so zusammenzufügen, dass das Ziel wirklich erreicht würde."

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Hence, the development of ideal theory is made by defining, so to speak, a new layer of arithmetic in the “higher level” – the arithmetic of ideals, sets of numbers. At this higher level, in the new layer of arithmetic, Dedekind can prove the validity of the laws of arithmetic for ideals by using solely the notion of divisibility analogously to what one is used to do in rational numbers theory. This allows him to prove the general validity of the laws of arithmetic at the level of algebraic numbers, insofar as there is a one-to-one correspondence between numbers and ideals. The divisibility between ideals is thus said to be equivalent to or to “coincide with” the divisibility between algebraic integers. It suffices, then, to prove the validity of laws of divisibility for ideals (in the “higher layer”), so as to have proven them for algebraic integers. Thus, ideals hide away long arduous computations by replacing them with simple(r) ones, insofar as the arithmetic of ideals is analogous to that of rational integers. But the analogy is not presupposed, as it was with Kummer’s ideal numbers, it is *proved* as an integral part of the theory. Rather than developing laborious proofs based on complicated computations, Dedekind can unfold a very complex and rich theory by using, essentially, the “simplest principles of arithmetic”. The possibility to treat ideals as arithmetical objects, that is, to define operations and study divisibility properties of ideals, appears to be an essential point for their pride of place.

This possibility is, however, not fully exploited in the 1871 version of the theory. One can distinguish a subtle change in Dedekind’s idea of what ideals could and should do, how ideals can and should be used in the second version. Indeed, we saw in the previous chapter that ideals are introduced, in 1871, as an extension of Kummer’s ideal numbers. They are, thus, studied following the spirit of Kummer’s theory with divisibility conditions and effective tests of divisibility – in particular, simple ideals, which play a crucial role in the 1871 version, are ideals formed by certain roots of congruences which provide an effective divisibility test similar to Kummer’s.

Dedekind took the opportunity of Lipschitz’s invitation, in 1876, to publish his theory of algebraic integer in the *Bulletin des Sciences Mathématiques et Astronomiques* to propose a new version of his theory which, he hoped, would resolve the lack of readership encountered by his 1871 theory. Dedekind then proposed a completely reshaped version, in which a considerable amount of developments are added: the links with Kummer’s ideal numbers and the ideas preceding the introduction of the concept of ideal are explained in detail and a numerical example is given; the theory of modules and the essential ideas about fields of algebraic integers are developed and detailed, making it possible for the reader to see more easily the inner articulation of the theory. More importantly, Dedekind proposes “a somewhat improved form of the essential foundation”, that is, of the theory of ideals (in finitely generated fields of algebraic numbers). The improvements in

6.2 In 1877, a “more arithmetical” version of algebraic number theory?

question appear to reside in the restructuration which yields a more arithmetical form of his argument: ideal theory is developed following the lines of rational number theory.

The 1877 paper consists essentially in a rewriting of the §§159-163 in the first version, dedicated to the divisibility properties of algebraic integers. Recall that Dedekind considered the proof that the laws of divisibility of algebraic integers are the same as that of rational integers to be, for the beginner, “the biggest difficulty” to enter binary quadratic form theory. For him, the “certainty that [the] general laws [of arithmetic] actually exist” is essential to ease the readers’s way into the theory of binary quadratic forms. In order to make this essential part of his work more approachable for the readers, Dedekind takes many precautions so as to follow a path rigorously analogous to elementary number theory.

When writing the second version of his theory, Dedekind is already well-aware of the similarity between the divisibility laws of algebraic and rational integers. The possibility to develop a completely analogous approach seems, thus, justified by the knowledge – rigorously proved knowledge – that the laws are, indeed, similar. The idea of a “rigorous analogy” with rational number theory is not foreign to Dedekind. Unlike in 1857, though, the said analogy is not considered as being obvious at all: the analogy is mathematically proved and the methods of proofs themselves are developed in a way that makes the analogy all the more clear and convincing.

The treatment of ideal theory as an extension of rational number divisibility *per se* is proposed only in the second version. For example, in 1871, Dedekind characterized prime ideals on the basis of simple ideals, and with theorems such as:

If $\eta\rho \equiv 0 \pmod{\mathfrak{p}}$, then at least one of the two numbers η, ρ , is divisible by \mathfrak{p} . ([Dedekind, 1871], 40)

While, as we will see, in 1877, the definition of prime ideals relies solely on the idea that a prime ideal \mathfrak{p} is an ideal divisible only by the unit and by itself, and on properties such as Euclid’s lemma. It seems, then, that the process of rewriting from 1871 to 1877 is a result of a re-reading of his own theory with the assurance that the divisibility laws are indeed similar between ideals (and therefore algebraic integers) and rational integers – then, shouldn’t it be possible to develop the theory itself following the same lines?

6.2 In 1877, a “more arithmetical” version of algebraic number theory?

The second version of Dedekind’s algebraic integer theory was published in five parts between 1876 and 1877, in the *Bulletin des Sciences Mathématiques et Astronomiques*, which was edited by Darboux, Houël and Tannery.

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Following the invitation sent by the editors via Lipschitz in the letters I mentioned earlier, Dedekind proposes this rewritten version, which Houël translated into French.

The version published in French is reproduced “mainly word for word” ([Lejeune-Dirichlet, 1879], viii) in the third edition of Dirichlet’s *Vorlesungen*. The differences are essentially due to the different contexts of publication. I will mostly use the 1876-77 version and consider the 1879 version at the end of the section.

6.2.1 Properties of algebraic numbers

As it should be clear, some parts of Dedekind’s theory are not subjected to important changes. In particular, the sections concerned with properties of algebraic numbers and fields are not significantly modified. They are, however, expanded and I will expose quickly, here, the developments given.

The concept of field, a “new fundamental concept” for algebra, was given in the first paragraph of the 1871 theory of algebraic numbers, as stating a general framework for the theory, and the algebraic numbers were, later on, proven to form a field. In 1877, Dedekind takes up the question differently. Rather than to begin with the general definition of a field and then consider the algebraic numbers, he starts with the algebraic numbers (and the wide domain of all algebraic numbers), and subsequently narrows the focus by defining the notion of finite field (Ω). Of course, the definitions themselves do not change.

Dedekind presents, in the introduction of the 1877 article, the algebraic integers as “the most general notion of integer”:

The broadest generalisation of the notion of integer is the following. A number θ is called an algebraic number if it satisfies an equation

$$\theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + \dots + a_{n-1}\theta + a_n$$

with finite degree n and rational coefficients $a_1, a_2, \dots, a_{n-1}, a_n$. It is called an *algebraic integer*, or simply an integer, when it satisfies an equation of the form above in which all the coefficients $a_1, a_2, \dots, a_{n-1}, a_n$ are rational integers.⁹ ([Dedekind, 1876-1877], 53-54)

⁹ “La plus haute généralisation de la notion du nombre entier consiste dans ce qui suit. Un nombre θ est dit un nombre algébrique, lorsqu’il satisfait à une équation $\theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + \dots + a_{n-1}\theta + a_n$ de degré fini n et à coefficients rationnels $a_1, a_2, \dots, a_{n-1}, a_n$, il est dit un nombre entier algébrique, ou plus brièvement un nombre entier, lorsqu’il satisfait à une équation de la forme ci-dessus, dans laquelle les coefficients $a_1, a_2, \dots, a_{n-1}, a_n$ sont tous des nombres entiers rationnels.”

6.2 In 1877, a “more arithmetical” version of algebraic number theory?

In order to introduce the concept of field as a concept allowing to frame the theory in an appropriate way, Dedekind starts by considering algebraic integers in general (i.e., all of them). They reproduce by addition, multiplication and subtraction; a number satisfying an equation with algebraic integers as coefficients is itself an algebraic integer; and one can define a notion of divisibility for integers:

We say that an integer α is divisible by an integer β when we have $\alpha = \beta\gamma$ with γ being itself an integer.¹⁰ (*ibid.*, transl. slightly altered, 105)

The definition is very standard and Dedekind gives a series of usual properties, such as the transitivity of divisibility, the existence of units, the existence of a GCD and of a Bezout identity. Finally, in the domain of all algebraic integers, every algebraic integer can be decomposed in infinitely many different ways, which implies that no algebraic integer has “the character of prime numbers.” Dedekind showed, in the Section 2, in which he exposed Kummer’s theory and how to replace it by ideals (see Sec. 5.1.3), that for algebraic integers certain “indecomposable number(s) (...) [do] not possess the property which, in the theory of rational numbers, is characteristic of a prime number(s)¹¹” (*ibid.*, 88) and explained that prime numbers can still be characterized “by their behavior as divisors”. In particular, questions about the relative primality of algebraic integers are not studied in depth, here, and the divisibility tests disappear completely. It suffices to know that the property of being indecomposable is not an essential property of prime numbers. Everything else will be left to ideal theory.

The existence of infinitely many essentially different decompositions for any algebraic integer, however, can be fixed by considering only the algebraic integers contained in a finite field.¹² Finite fields, whose definition is attached to a certain equation, are introduced as intimately linked to the nature of the investigation, here. They allow to reframe the domain of numbers in a way that eliminates undesirable properties.

A field is defined as a system of numbers closed by the four rational operations. A finite field Ω of degree n is a field obtained by an irreducible polynomial equation $f(\theta) = 0$ of degree n . The numbers ω of Ω are of the form $\omega = \psi(\theta)$, where

$$\psi(t) = b_0 + b_1t + \dots + b_{n-1}t^{n-1}$$

¹⁰*Nous dirons qu’un nombre entier α est divisible par un nombre entier β , lorsqu’on aura $\alpha = \beta\gamma$, γ étant également un nombre entier.*

¹¹That is, “a prime cannot divide a product of two or more factors without dividing at least one of the factors” (*ibid.*, 56)

¹²Stillwell translates “corps fini” (“*endliche Körper*”) as “field of finite degree”, to avoid confusion with our modern notion finite field. I will however keep Dedekind’s terminology. I will not have to consider (our modern notion of) finite fields, and I trust that the reader will not be confused.

is a polynomial of degree $< n$ with rational coefficients. The system of numbers $1, \theta, \theta^2, \dots, \theta^{n-1}$ is an irreducible system (i.e., its elements are linearly independent) and form a basis of Ω . Dedekind, then, studies the “the extent and nature” of the concept of field by first investigating the internal structure of his system Ω by algebraic considerations. The (general) concept of field is thus studied through investigations of the properties of the numbers (and the equations which define them). The fact that the field Ω is attached to a certain equation is a “concession” made to develop field theory in relation to “generally known things” which is authorized by the possibility to prove that “the content in numbers of the field stays absolutely unchanged” if one were to consider another “means of expression” (*Ausdrucksmittel*) of the numbers (in letter to Lipschitz, [Lipschitz and al. 1986], 60).

Dedekind continues the presentation of the concept of field with more general considerations, that is, working in any (not necessarily finite) field. The important notion, here, is that of conjugate fields. While in 1871, the notion of conjugate fields was mostly mentioned in passing, a substantial paragraph is devoted to it here. Dedekind considers the well-known notion of substitution to be “an act by which objects or elements being studied are replaced by corresponding objects or elements,”¹³ and proposes to call a “permutation”¹⁴ a substitution between the numbers of Ω and another set of numbers Ω' such that:

By a *permutation* of Ω , we mean a substitution for which a determined number of Ω

$$\alpha, \beta, \alpha + \beta, \alpha - \beta, \alpha\beta, \frac{\alpha}{\beta}$$

is changed into a corresponding number

$$\alpha', \beta', (\alpha + \beta)', (\alpha - \beta)', (\alpha\beta)', \left(\frac{\alpha}{\beta}\right)'$$

in such a way that the conditions

- (1) $(\alpha + \beta)' = \alpha' + \beta'$
- (2) $(\alpha\beta)' = \alpha'\beta'$

¹³Dedekind, then, calls substitution a notion extremely close to what he will later call a “mapping” (*Abbildung*). Note that he considers it to be an *action* made by the mathematician.

¹⁴Stillwell translates this as “isomorphism”, here again I will keep Dedekind’s own terminology. It is also important insofar as it suggests that Dedekind was generalizing well-known concepts of algebra and drawing from his experience in Galois theory.

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are satisfied and the substitute numbers α', β', \dots are not all zero.¹⁵ (*ibid.*, 108-109, transl. modified)

While the permutation is defined only for numbers, the nature of the elements do not enter into account, and the generality of the definition is noteworthy. The first important result is that the numbers of the domain Ω' also form a field. The second is the bijectivity of the permutation, which allows to show that the permutation can be “reversed univocally” into a permutation from Ω' into Ω . Such permutation are called inverse of one another. Ω and Ω' are called conjugate fields and two corresponding numbers are called conjugate numbers. Only to deal with the composition of permutation and the inversion of the composition, does Dedekind introduce a notation for the permutations: if P is a permutation of Ω into Ω' and P' a permutation of Ω' into a third field Ω'' , then PP' is a permutation of Ω into Ω'' .

6.2.2 The ideals’s makeover

So as to be able to provide a theory that would be simpler, easier to read and understand, and following more closely his own methodological requirements, Dedekind proposes a new version of the theory of ideals. In this new version, he unfolds the laws of divisibility of ideals by highlighting the “analogy” with rational number theory in what he considered, in 1876, as the clearest, most general and rigorous way possible. Here, “analogy” should not be taken as a heuristic metaphor, but as a way to underline that the laws are *similar*: provided that one defines arithmetic operations for ideals, then the laws are *analogous* for ideals and for rational integers. Dedekind proves each and every result, and it is assured that he does not solely rely on the analogy without ground to it. There is thus not doubt raised on neither the validity nor the rigor of his results. Dedekind states very explicitly, here, that the goal of the theory of algebraic integers is to exhibit the similarity of the laws of divisibility. In fact, it seems that he carefully exposed his theory so that the said similarity does not appear as coincidental. Rather, the long preliminaries and the efforts deployed to set up the arithmetic of ideals allows for the arithmetical laws to follow seemingly with little difficulties.

Dedekind affirms that in substance, the 1877 theory is the same as in 1871, and that the changes are made only to the “external form”. Indeed, it is clear that the definition of the basic concepts did not change and the *results* are essentially similar. However, some significant differences appear.

¹⁵ [N]ous entendons par une permutation de Ω une substitution par laquelle un nombre déterminé contenu dans Ω , α , β , $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, $\frac{\alpha}{\beta}$ se change en un nombre déterminé correspondant α' , β' , $(\alpha + \beta)'$, $(\alpha - \beta)'$, $(\alpha\beta)'$, $(\frac{\alpha}{\beta})'$. Et cela de telle manière que les deux conditions (1) $(\alpha + \beta)' = \alpha' + \beta'$ (2) $(\alpha\beta)' = \alpha'\beta'$ soient remplies, et que les nombres substitués α' , β' , \dots ne s'annulent pas tous.”

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The emphasis put on divisibility relations between ideals, as a generalization of rational arithmetic divisibility comes with a significant shift in what constitute the center of the theory and what the results are identified as essential. Most notably, Dedekind states that the “main difficulty to overcome” is to show the mutual dependency of divisibility and multiplication of ideals, that is, to prove that the divisibility as inclusion corresponds to the “arithmetical” notion of divisibility – a point to which little attention was paid in 1871. Dedekind explores and unfolds the elementary arithmetical laws of ideals, in a way that allows him to demonstrate that ideals and thus algebraic numbers satisfy the divisibility laws valid in rational number theory by means of analogous notions and methods of proofs.

There are no change in the definition of ideals. Insofar as ideals are conceived as grounding concepts for number theory, it seems that developing ideal theory on the grounds of divisibility and multiplication presents the advantage of a certain conceptual elementarity (similar to that of division and multiplication of numbers in rational number theory). From this viewpoint, then, the changes can be considered as improvements of the initial theory. The question of what is considered properly “arithmetical”, here, would be a point of dissension between Dedekind and many other writers, among which Kronecker. Nevertheless, we have identified, earlier, that arithmetic appeared to be intimately linked to the four rational operations, in particular, the study of divisibility relationships. We also saw that the theory should be set up in such a way that all “formal” considerations (as one can find in Kronecker’s works) be avoided. Dedekind’s strategy, then, consists in establishing a generalized notion of divisibility through the building of a “higher level” in which collections of actually existing numbers are considered as “wholes”, as objects in themselves, subjected to the new arithmetic operations.

In addition, one should not overlook the fact that the changes embedded in the modifications seem, for Dedekind, to allow for a deeper, more solid grasp of the theory: the recourse to arithmetic of ideals seems to have effectively strengthened the understanding of both ideals and algebraic integers. Of course, the use of arithmetical operations for objects that are not numbers is not new for Dedekind. The development of an arithmetic of ideals, here, appears as a consolidation and considerable development of an already established way of working, which Dedekind adopted as early as 1856 during his lectures on Galois theory.

6.2.2.1 Modules

The theory of modules is given considerably more developments, in 1877, than it was in 1871: the 3 pages long §161 of the Xth Supplement X becomes

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a preamble of 25 pages.¹⁶

Dedekind carefully underlines the independence of the theory. The theory of “systems of numbers reproducing themselves by addition and subtraction” is, as we saw in the previous chapter, an auxiliary theory, a general and wide theory likely to be used in several research fields, and mostly interesting for its *applicability*. In 1877, Dedekind relies more on module theory, and notably, he explicitly states that since “all ideals are modules”, everything proven for modules is also valid for ideals.

Dedekind explicitly develops the bases of the theory, giving a “small number of very simple theorems” – nothing unneeded for his purpose. The section opens with a paragraph exposing the “modules and their divisibility”:

A system \mathfrak{a} of real or complex numbers will be called a module when all the sums and differences of these numbers also belong to \mathfrak{a} . (...) A module \mathfrak{a} will be called divisible by the module \mathfrak{b} or a multiple of \mathfrak{b} , and \mathfrak{b} a divisor of \mathfrak{a} , when all the numbers in the module \mathfrak{a} are also in the module \mathfrak{b} .¹⁷ ([Dedekind, 1876-1877], 61-62)

The LCM and GCD of modules are defined in the same way than in 1871 – albeit with more details, following Dedekind’s worry to provide the clearest possible exposition of his theory. Dedekind also introduces the notation $[\alpha]$ for the module formed by all the numbers of the form $x\alpha$, with x a rational integer and α an algebraic number, that is the module generated by α .¹⁸

Note that congruences according to a module¹⁹ are not a part of the definition of modules anymore. A full section is dedicated to them. Dedekind gives a series of simple – and usual – properties on congruences and classes of numbers, and considers in more details the “complete residue system according to a module \mathfrak{a} ” used to define the norm of modules (and ideals).

After the congruences, Dedekind gives his research a more linear orientation with the study of “finite modules”, i.e., modules with a finite basis.

Let $\beta_1, \beta_2, \dots, \beta_n$ be particular numbers. All the numbers

$$\beta = \gamma_1\beta_1 + \gamma_2\beta_2 + \dots + \gamma_n\beta_n$$

¹⁶This tendency will strengthen in the 1894 version of the theory, in which preliminary considerations in module theory (§§168-175) take 47 pages.

¹⁷“Un système \mathfrak{a} de nombres réels ou complexes sera dit un module quand toutes les sommes et les différences de ces nombres appartiendront à ce même système \mathfrak{a} . (...) Un module \mathfrak{a} sera dit divisible par le module \mathfrak{b} ou un multiple de \mathfrak{b} et \mathfrak{b} un diviseur de \mathfrak{a} , quand tous les nombres du module \mathfrak{a} seront contenus aussi dans le module \mathfrak{b} .”

¹⁸A few lines are dedicated to the consideration of this particular type of modules at the end of each proposition. For example, if α is a number of the module \mathfrak{a} , then $[\alpha]$ is divisible by \mathfrak{a} . To this special case of module will correspond the notion of principal ideal.

¹⁹ ω and ω' are said to be congruent according to a module \mathfrak{a} if their difference $\omega - \omega'$ is in \mathfrak{a} .

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where $\gamma_1, \gamma_2, \dots, \gamma_n$ are arbitrary rational integers, evidently form a module, which we call a finite module²⁰ $[\beta_1, \beta_2, \dots, \beta_n]$. The complex of constants $\beta_1, \beta_2, \dots, \beta_n$ will be called the *basis* of the module.²¹ (*ibid.*, transl. slightly altered, 67)

This notation is new in Dedekind's exposition of the theory and abundantly used in the subsequent section of the paper in which the "germs of ideal theory" are exposed (which I used in Sec. 5.1.3).

6.2.2.2 First introduction of ideals

In the said section, Dedekind's considers ideals as finite modules with a supplementary property, namely that each product of a number in the module \mathfrak{a} by a number in the system of integers \mathfrak{o} is again a number in the module \mathfrak{a} – that is, this module is an ideal. Dedekind works, for this short paragraph, with explicit expressions of each module studied. Indeed, recall that the idea is to replace an ideal factor α , defined by conditions of divisibility, by an ideal defined as the system \mathfrak{a} of all numbers divisible by α . For the example of the number 2, the system \mathfrak{a} will be the system of numbers of the form $(1 + \theta)y + 2z$, in which one recognizes a "finite module with basis consisting of the two independent numbers 2 and $1 + \theta$," hence $\mathfrak{a} = [2, 1 + \theta]$.

To explain how "the theory of the numbers of the domain \mathfrak{o} can be founded on the notion of ideal", Dedekind proposes a notion of divisibility for ideals:

Just as in the theory of modules (§1, 2), we say that an ideal \mathfrak{m}'' is divisible by an ideal \mathfrak{m} when all numbers in \mathfrak{m}'' also belong to \mathfrak{m} . It follows that a principal ideal $\mathfrak{o}\mu''$ is divisible by a principal ideal $\mathfrak{o}\mu$ if and only if the number μ'' is divisible by the number μ . Thus the theory of divisibility of numbers is contained in the theory of divisibility of ideals. One sees immediately that the necessary and sufficient conditions for the ideal $\mathfrak{m}'' = [m''a'', m''(b'' + \theta)]$ to be divisible by the ideal $\mathfrak{m} = [ma, m(b + \theta)]$

$$m''a \equiv m''a \equiv m''(b'' - b) \equiv 0 \pmod{ma}.^{22}$$

²⁰I will not use Stillwell's "finitely generated module", here, for the same reasons than for finite fields.

²¹"Soient $\beta_1, \beta_2, \dots, \beta_n$ des nombres déterminés; tous les nombres de la forme $\beta = \gamma_1\beta_1 + \gamma_2\beta_2 + \dots + \gamma_n\beta_n$, $\gamma_1, \gamma_2, \dots, \gamma_n$ désignant des nombres rationnels entiers arbitraires, constituent évidemment un module, que nous appelleront module fini et que nous désignerons par $[\beta_1, \beta_2, \dots, \beta_n]$; le complexe de constantes $\beta_1, \beta_2, \dots, \beta_n$ sera dit la base du module."

²²"Nous dirons, absolument comme dans la théorie des modules (§ 1, 2), qu'un idéal \mathfrak{m}'' est divisible parmi idéal \mathfrak{m} , quand tous les nombres du premier seront contenus aussi dans

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(*ibid.*, 98)

If ideals are introduced in this way, then the multiplication of ideals is defined by:

if μ runs through the numbers in the ideal \mathfrak{m} , and μ' through the numbers in the ideal \mathfrak{m}' , then all the products $\mu\mu'$ and their sums form an ideal \mathfrak{m}'' called the product of the factors $\mathfrak{m}, \mathfrak{m}'$ and denoted by $\mathfrak{m}\mathfrak{m}'$.²³ (*ibid.*, 99)

However, in this context, the multiplication relies on the basis of the finite modules, and will require many computations and considerations about the determinant of systems of numbers.

Dedekind quickly sums up the principal results of the theory:

Combining the theorem just proved with the preceding theorems on the connection between divisibility and multiplication of ideals, and bearing in mind that \mathfrak{o} is the only ideal with norm 1, we arrive, by exactly the same reasoning as in the theory of rational numbers, at the following theorem: “Each ideal different from \mathfrak{o} is either a prime ideal or else uniquely expressible as a product of finitely many prime ideals.”²⁴ (*ibid.*, 102)

Dedekind took the opportunity of exposing the “germs of ideal theory” to explain his own invention through a presentation that appears less ‘abstract’ and directly linked with congruences. It may, then, seem more familiar to the readers and it allows Dedekind to clarify the concept of ideal, before exposing his theory. In fact, Dedekind’s careful and extremely clear presentation does not only demonstrate the teaching skills he was famous for, it is also carried to the point where he illustrates himself what *should not* be done from a methodological viewpoint. By doing so, he enlightens how he construed ideals on the basis of deep methodological and epistemological

le second. D’après cela, un idéal principal $\mathfrak{o}\mu''$ sera toujours divisible par un idéal principal $\mathfrak{o}\mu$ dans le cas, et seulement dans ce cas, où le nombre μ'' sera divisible par le nombre μ , de là résulte que la théorie de la divisibilité des nombres est contenue dans celle des idéaux. Les conditions nécessaires et suffisantes pour que l’idéal $\mathfrak{m}'' = [\mathfrak{m}''a'', \mathfrak{m}''(b'' + \theta)]$ soit divisible par l’idéal $\mathfrak{m} = [\mathfrak{m}a, \mathfrak{m}(b + \theta)]$ consiste, comme on l’aperçoit immédiatement dans les trois congruences $\mathfrak{m}''a \equiv \mathfrak{m}''a'' \equiv \mathfrak{m}''(b'' - b) \equiv 0 \pmod{\mathfrak{m}a}$.”

²³ “Si μ parcourt tous les nombres de l’idéal \mathfrak{m} , et de même μ' tous les nombres de l’idéal \mathfrak{m}' , tous les produits $\mu\mu'$ et leurs sommes formeront un idéal \mathfrak{m}'' , qui sera dit le produit des facteurs $\mathfrak{m}, \mathfrak{m}'$, et que l’on désignera par $\mathfrak{m}\mathfrak{m}'$.”

²⁴ “En combinant le théorème que nous venons de démontrer avec les théorèmes précédents relatifs à la dépendance entre les notions de divisibilité et de multiplication des idéaux, et ayant égard à ce que, en dehors de \mathfrak{o} , il n’existe aucun autre idéal dont la norme soit = 1, on arrive, par les mêmes raisonnements que dans la théorie des nombres rationnels, au théorème suivant : ‘Tout idéal différent de \mathfrak{o} ou est un idéal premier, ou peut se mettre, et cela d’une seule manière, sous la forme d’un produit d’un nombre fini d’idéaux premiers.’”

considerations. Indeed, Dedekind closes the section by remarking that a lot of the demonstrations given rely too much on the square-bracket notation and the “effective realization of multiplication.” If the computations related to the multiplication of ideals appear, here, in relatively moderate proportions, it is mostly linked to the fact that the example used by Dedekind is rather simple. However, Dedekind warns us, their length and difficulty will increase considerably, insofar as the multiplication of ideals, here, “coincides with the composition of binary quadratic forms as taught by Gauss,” and if one were to try to extend it to a genuinely general treatment of all the fields, it would lead to immense – even “insurmountable” – difficulties.

If we wanted to treat fields Ω of arbitrary degree in the same way, then we would run into great difficulties, perhaps insurmountable ones. Even if there were such a theory, based on calculation, it still would not be of the highest degree of perfection, in my opinion. It is preferable, as in the modern theory of functions, to seek to infer the proofs immediately from the fundamental characteristics of concepts, rather than from calculation, and indeed to construct the theory in such a way that it is able to predict the results of calculation (for example the composition of decomposable forms of all degrees).²⁵ (*ibid.*, transl. slightly altered, 102)

We find, here, the usual “red-letter clichés” describing the so-called “conceptual approach”, based on “characteristic fundamental concepts” rather than computations. I will present, in the next section, how the “conceptual” theory is developed based on arithmetical relationships.

6.2.2.3 Ideals and their divisibility

Ideal theory opens with the definition of ideals and their divisibility and starts by following lines similar to the paragraph on modules: the formal definition of ideals, the principal ideal $\mathfrak{o}\mu$, the definition of divisibility between ideals, divisibility between numbers and ideals, the notions of GCD and LCM of ideals, congruences and prime ideals.

Dedekind notices that whenever, in a “finite field” Ω , every ideal is a principal ideal, then every *indecomposable* number is also a *prime* number,

²⁵ “Si l’on voulait traiter de la même manière tous les corps Ω de degré quelconque, on se heurterait à de grandes difficultés, peut-être insurmontables. Mais, lors même qu’il n’en serait pas ainsi, une telle théorie, fondée sur le calcul, n’offrirait pas encore, ce me semble, le plus haut degré de perfection ; il est préférable, comme dans la théorie moderne des fonctions, de chercher à tirer les démonstrations, non plus du calcul, mais immédiatement des concepts fondamentaux caractéristiques, et d’édifier la théorie de manière qu’elle soit, au contraire, en état de prédire les résultats du calcul (par exemple, la composition des formes décomposables de tous les degrés).”

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that is, the same “laws of divisibility of numbers” as in rational number theory are valid. Coming to the point of divisibility, Dedekind explains that since every ideal is a module, one can transfer everything that has been defined and proved in the case of modules. This explains the inverted weight of divisibility in module and ideal theories in 1877 and 1871. Hence, all propositions about divisibility for ideals are equivalent to those given in module theory and are not stated again.

The equivalence between the divisibility of numbers and the divisibility of ideals, an important point for the development of the theory, is given very explicitly by Dedekind. On one hand, we have that a number α is divisible by an ideal \mathfrak{a} when $\alpha \in \mathfrak{a}$, or again the principal ideal $\mathfrak{o}\alpha$ is divisible by \mathfrak{a} . On the other hand, an ideal \mathfrak{a} is said to be divisible by a number η when \mathfrak{a} is divisible by the principal ideal $\mathfrak{o}\eta$. It is easy to see that the divisibility between $\mathfrak{o}\mu$ and $\mathfrak{o}\eta$ is “completely identical” to that of the numbers μ and η . Consequently, the laws of the divisibility of the numbers of \mathfrak{o} are completely *contained in* the laws of divisibility of ideals. Therefore, every proposition of arithmetic for algebraic integers can be expressed in terms of ideals. The focus of the theory is thus completely shifted to the arithmetic of ideals, and divisibility conditions for numbers are reduced to that of ideals.

We saw that one problem for prime algebraic integers was the loss of the equivalence between indecomposability and primality, leading to adopt the so-called Euclid’s lemma²⁶ as the definitional property of primes. For prime ideals, the problem disappears and the definition of prime elements can be completely analogous to that in rational number theory:

An ideal \mathfrak{p} is called prime when it is different from \mathfrak{o} and divisible by no ideals except \mathfrak{o} and \mathfrak{p} .²⁷ (*ibid.*, 123)

Note that prime ideals are defined immediately, and the “simple ideals”, which played an important role in the 1871, do not appear in the theory.

There is a first analog to Euclid’s lemma with the divisibility of numbers by a prime ideal²⁸:

If neither of the two numbers η, ρ is divisible by the prime ideal \mathfrak{p} , then their product $\eta\rho$ will not be divisible by \mathfrak{p} .²⁹ (*ibid.*, 123)

Dedekind also gives several propositions stating simple arithmetical properties of numbers and prime ideals which highlight the similarity with rational number theory, such as:

²⁶If a prime number p divides the product ab , then it must divide at least a or b .

²⁷“Un idéal \mathfrak{p} est dit un idéal premier quand il est différent de \mathfrak{o} et qu’il n’admet comme diviseur aucun autre idéal que \mathfrak{o} et \mathfrak{p} .”

²⁸Euclid’s lemma for ideals is given a few pages later, once the multiplication of ideals is defined.

²⁹“Si aucun des deux nombres η, ρ n’est divisible par l’idéal premier \mathfrak{p} , leur produit $\eta\rho$ ne sera pas non plus divisible par \mathfrak{p} .”

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Each ideal \mathfrak{a} different from \mathfrak{o} is divisible by a prime ideal.³⁰ (*ibid.*)

The norm of ideals is defined using “the number of different classes, modulo the ideal \mathfrak{a} ”, which is equivalent to the number of incongruent numbers with respect to \mathfrak{a} . The complete system of residues was defined for modules. The number of incongruent numbers of \mathfrak{b} according to (modulo) \mathfrak{a} is denoted $(\mathfrak{b}, \mathfrak{a})$. The norm of the ideal \mathfrak{a} is the number of incongruent numbers in \mathfrak{o} with respect to \mathfrak{a} . It is designated by $(\mathfrak{o}, \mathfrak{a}) = N(\mathfrak{a})$.

6.2.2.4 Multiplication of ideals and “the difficulty of the theory”

The most significant modification appears with the multiplication of ideals. We saw that in 1871 the definition of the multiplication is given after the demonstration of the existence of unique factorization. It did not play any significant role. In the 1877 version, multiplication of ideals plays an important part in developing the theory. Following the desire to develop a theory analogous to elementary rational number theory based on what appears as a generalization of elementary arithmetic, the multiplication of ideals is placed at the core of the research. The distance with Kummer’s approach is explicitly taken, here. The significant change in the structure of the theory, bringing to the fore the change of focus of the theory from divisibility conditions for (ideal) numbers to the laws of divisibility of ideals.

For Dedekind, by moving the definition of multiplication of ideals at the heart of the theory, a certain gain of clarity in the theory is enabled, and it seems related to the fact that the construction of the theory of ideals appears to be truly based on the simplest arithmetical principles. The gain of clarity partly comes from the impression of familiarity offered by the analogical development – and, again, the systematic proof of every proposition allows to avoid doubts about the transfer of arithmetic properties from rational numbers to ideals. But more importantly, the gain of clarity seems to be linked to the simplicity gained by using generalizations of (elementary) arithmetical notions rooted in rational arithmetic.

Recall that the definition of the multiplication of ideals (which does not change) is the following:

If α runs through all the numbers in an ideal \mathfrak{a} , and β runs through those of an ideal \mathfrak{b} , then all the products of the form $\alpha\beta$, together with their sums, form an ideal \mathfrak{c} .³¹ (*ibid.*, 125)

³⁰ “Tout idéal \mathfrak{a} différent de \mathfrak{o} est divisible au moins par un idéal premier.”

³¹ “Si α parcourt tous les nombres d’un idéal \mathfrak{a} , et de même β tous les nombres d’un idéal \mathfrak{b} , tous les produits de la forme $\alpha\beta$ et toutes les sommes de ces produits formeront un idéal \mathfrak{c} .”

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The multiplication of ideals satisfies the usual properties of multiplication: existence of a unit (\mathfrak{o}), commutativity, associativity, possibility to extend the definition to as many factors as wanted. . . And it is also possible to define powers of ideals in terms of multiplication, without the recourse to simple ideals and/or divisibility tests:

If all the m factors = \mathfrak{a} , their product will be called the m th power of \mathfrak{a} , and we denote it by \mathfrak{a}^m . By also putting $\mathfrak{a}^0 = \mathfrak{o}$ and $\mathfrak{a}^1 = \mathfrak{a}$, we thus have in general $\mathfrak{a}^r \mathfrak{a}^s = \mathfrak{a}^{r+s}$.³² (*ibid.*, 125)

A series of (simple) propositions about divisibility properties follows, which exhibit the “resemblance” with rational number theory even more strikingly:

- \mathfrak{a} divides $\mathfrak{a}\mathfrak{b}$ and \mathfrak{b} divides $\mathfrak{a}\mathfrak{b}$;
- if \mathfrak{a} divides \mathfrak{a}' and \mathfrak{b} divides \mathfrak{a}' , then $\mathfrak{a}\mathfrak{b}$ divides $\mathfrak{a}'\mathfrak{b}'$;
- Euclid’s lemmas for ideals: if neither \mathfrak{a} nor \mathfrak{b} are divisible by a prime ideal \mathfrak{p} , then $\mathfrak{a}\mathfrak{b}$ is not divisible by \mathfrak{p} .

The analogy with the divisibility laws of rational numbers is still not “complete”, here, since there seems to be no connection between multiplication and division of ideals. This point, crucial for Dedekind’s purpose, is delicate to prove, and Dedekind considered it to be “the main difficulty of the theory [of ideals]”. For the similarity of the arithmetical laws to be complete, one ought to show that division and multiplication are inverse operations of each other. But the divisibility relation defined between ideals is a relatively weak property (the inclusion of an ideal in another one). Hence, Dedekind writes, it is “not in the least easy” to prove that if an ideal \mathfrak{a} divides an ideal \mathfrak{b} , then there exists an ideal \mathfrak{c} such that $\mathfrak{a}\mathfrak{c} = \mathfrak{b}$.

This difficulty, which is the greatest and really the only one presented by the theory, cannot be surmounted by the methods we have employed thus far, and it is necessary to examine more closely the reason for this phenomenon, because it is connected with a very important generalization of the theory.³³ (*ibid.*, 126)

In order to bypass the difficulty, Dedekind proposes to act directly on the domain \mathfrak{o} itself. He notices that “every definition keeps a determinate meaning” and “all demonstrations of theorems keep their full strength” if one restricts the hypotheses on \mathfrak{o} in the following way:

³² “Si tous les m facteurs sont = \mathfrak{a} , leur produit sera dit la m -ième puissance de \mathfrak{a} , et on le représentera par \mathfrak{a}^m ; en posant de plus $\mathfrak{a}^0 = \mathfrak{o}$ et $\mathfrak{a}^1 = \mathfrak{a}$, on aura en général $\mathfrak{a}^r \mathfrak{a}^s = \mathfrak{a}^{r+s}$.”

³³ “Cette difficulté, la plus grande et, à proprement parler, la seule que présente la théorie, ne peut en aucune manière être surmontée à l’aide des seuls moyens de démonstration que nous avons employés jusqu’ici, et il faut que nous examinions ici d’un peu plus près la raison de ce phénomène, parce que celui-ci se rattache à une généralisation très-importante de la théorie.”

[O]ne no longer supposes that the domain \mathfrak{o} consists of *all* integers in the field Ω . The only properties of \mathfrak{o} really needed are the following:

- (a) The system \mathfrak{o} is a finite module $[\omega_1, \dots, \omega_n]$ whose basis is also a basis for the field Ω .
- (b) The number 1 is in \mathfrak{o} , hence so are all the rational integers,
- (c) Each product of two numbers in \mathfrak{o} is also in \mathfrak{o} .³⁴ (*ibid.*, transl. slightly altered, 127)

This is the definition of an order (*Ordnung*). In this version of the theory, the concept of *Ordnung* still does not hold the same place in Dedekind's theory than in the modern conception of a ring. The definition is basically the same than in 1871, but the context in which the *Ordnung* is introduced changes significantly, as well as the role attributed to it. In 1871, it was an "extension of the concept of ideal", which allows to apply ideal theory to the study of decomposable forms. In 1877, the *Ordnung* is a restriction of the hypotheses on the domain of integers of the field, enabling to prove the reciprocity of the operations of divisibility and multiplication of ideals. An order \mathfrak{o} only contains integers of Ω but it does not *necessarily* contains *all* the integers of the field.

Having changed the domain in which he works, Dedekind consequently redefines divisibility: a number α (in \mathfrak{o}) is divisible by a number μ (in \mathfrak{o}) *only* when there exists a number ω such as $\alpha = \mu\omega$ with ω in \mathfrak{o} . The congruence is slightly modified in the same way. It is "easy to recognize that" everything that has been defined and proved still holds when considering the notion of "number" to designate numbers of \mathfrak{o} . Consequently, all the definitions and proofs given in the preceding sections, in particular the theory of ideals, are still valid.

When working in an *Ordnung*, the theory of ideals developed is a *particular* theory of ideals, but up to the point of development reached in the previous paragraphs, it is the same for every *Ordnung*. At that step, however, one should chose between two paths: either one can develop the theory of the ideals of the *Ordnung* containing all the integers of Ω , or one can develop a more general theory of ideals in *any* *Ordnung*. It is the first possibility which, here, interests Dedekind, for it "leads to general laws, which don't suffer exception and coincide completely with the laws of divisibility

³⁴ "[O]n ne suppose plus que le domaine désigné par \mathfrak{o} comprenne tous les nombres entiers du corps Ω . Les propriétés du système \mathfrak{o} sur lesquelles on s'est appuyé se réduisent en réalité aux suivantes : (a) Le système \mathfrak{o} est un module fini $[\omega_1, \dots, \omega_n]$ dont la base forme en même temps une base du corps Ω . (b) Le nombre 1, et par suite aussi tous les nombres rationnels entiers sont contenus dans \mathfrak{o} . (c) Tout produit de deux nombres du système \mathfrak{o} appartient au même système \mathfrak{o} ."

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for the rational numbers.” The second possibility, the theory of ideals of any *Ordnung*, demands a “restriction of the notion of ideal” and is not the one Dedekind is interested in, although it is “necessary for the development of number theory.”³⁵

Through the introduction of the notion of *Ordnung*, Dedekind modifies the framework of the theory, which leads him to reshape slightly the notion of number so as to fit in the new framework and to maintain the validity of propositions and theorems. This suggests that arithmetical properties are intrinsically related to the domain in which they are studied. Dedekind’s explanation of which theory of *Ordnungen* he chooses to develop shows that his focus, in this case, is on the divisibility laws of (the) algebraic integers in a certain field. This supports the idea that Dedekind is looking for a uniform (valid for all the cases concerned) treatment of a certain problem – here, the validity of the divisibility laws for algebraic integers, the most general concept of integer.

From there, the proof of the “great difficulty” will be greatly eased. In fact, with all the preliminary developments, the proofs of the divisibility laws are following rather easily (the proof of the “great difficulty” is no longer than four lines) and the arithmetical setting of ideal theory allows to give proofs strikingly similar to that of rational number theory.

6.2.2.5 Propositions of arithmetic

In the following paragraphs, Dedekind proposes proofs of several propositions and theorems stating divisibility properties of ideals. As I mentioned, simple ideals are not introduced, here. They were undeniably unsatisfying with respect to some of Dedekind’s methodological requisites: a substantial part of the theory was based on a particular representation of ideals, and the notion was simply given up in the course of the theory. Nevertheless, the divisibility properties between algebraic numbers, that served to defined simple ideals are still needed, such as :

Let μ, ν be two nonzero numbers in \mathfrak{o} , with ν not divisible by μ .

Then there are two nonzero numbers κ, λ in \mathfrak{o} such that

$$\frac{\kappa}{\lambda} = \frac{\nu}{\mu}$$

and κ^2 is not divisible by λ .³⁶ ([Dedekind, 1876-1877], 129)

³⁵For example, when Ω is a quadratic field, the theory is the “same as the theory of orders of binary quadratic forms” as seen in 1871 and in “*Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers*” ([Dedekind, 1877]).

³⁶“Soient μ, ν deux nombres de \mathfrak{o} , différents de zéro, ν n’étant pas divisible par μ , il existe toujours dans \mathfrak{o} deux nombres κ, λ différents de zéro, et tels que l’on ait $\frac{\kappa}{\lambda} = \frac{\nu}{\mu}$ et que κ^2 ne soit pas divisible par λ .”

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Dedekind condenses them in “auxiliary propositions” to be used in the proofs.

The propositions and theorems stating divisibility properties of ideals, in §25, are presented as the “complement” needed for the theory of ideals in \mathfrak{o} to be completely analogous to rational number theory. The theorems given are relatively simple ones, since the aim is to prove that the arithmetical behavior of the ideals (and therefore the numbers) of \mathfrak{o} is similar to that of rational numbers.

In 1871, the unique factorization theorem was a corollary of the main results of ideal theory. Here, it appears, as one of many arithmetical properties and does not seem more important than the other propositions. The initial focus of Kummer’s theory was changed by Dedekind, who is not pre-occupied by properties of (ideal or existing) divisors, but has thoroughly adopted the viewpoint of the study of (elementary) arithmetic at the level of ideals. According to Edwards, the unique factorization theorem, in this version, seems *ad hoc* and unmotivated. He considers that Dedekind’s theory of algebraic numbers has become too abstract, too long, too technical and has moved too far away from its initial purpose.

As Dedekind underlined in the introduction of the *Vorlesungen* that the purpose of ideal theory is to shed light “from a higher viewpoint” to all number theory. Dedekind’s theory aims at establishing the validity of fixed general laws for the “phenomenon of divisibility” of numbers in *every* domain \mathfrak{o} composed by integers of *any* finite number field Ω of degree n . Since there exists an infinite variety of such fields Ω with all their “own special number theory,” the laws (and definitions) need to be perfectly general if they are to be valid for each and every one of these theories. This is the case of Dedekind’s theory of algebraic numbers, as he underlines himself (at the beginning of §27, for example).

For Dedekind, such a “certainty that these general laws actually exist” allows and eases more and more further developments. Moreover, if the certainty that the unique factorization in primes and other good properties of rational numbers are still valid for algebraic integers is an essential point for the development of number theory, Dedekind also seems to believe that he found, with ideals, a tool enabling considerable simplifications and new developments. Without trying to judge whether the distance from Kummer’s works is or is not unnecessary, it should be underlined that the distance taken from Kummer’s approach is not the mere consequence of Dedekind’s philosophical dicta, but the result of a modulation in the purpose of the theory.

In §25, Dedekind gives nine “laws” to describe the divisibility in \mathfrak{o} . With respect to the notion of divisibility defined, the laws come across as familiar ones, for they offer a strong parallelism with rational number theory – as

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wanted by Dedekind.

Two important properties, for the proof of the unique factorization theorem, are the following:

Each prime ideal \mathfrak{p} can, through multiplication by an ideal \mathfrak{b} , be transformed into a principal ideal. (...)

If the ideal \mathfrak{a} is divisible by the prime ideal \mathfrak{p} then there is one and only one ideal \mathfrak{a}' such that $\mathfrak{p}\mathfrak{a}' = \mathfrak{a}$, and at the same time we will have $N(\mathfrak{a}') < N(\mathfrak{a})$.³⁷ (*ibid.*, transl. slightly altered, 129-130)

Dedekind’s efforts to transfer the study of arithmetic to the level of ideals has further consequences than the possibility to highlight the similarity between divisibility properties of ideals (and consequently, algebraic integers) and rational numbers: it enables to provide proofs that are analogous to proofs in rational number theory. That point is especially noticeable in the proof of the unique factorization theorem.

In 1871, the proof was based on the notion of simple ideal and on tests of divisibility by powers of ideals. It was a consequence of the theorem stating that every ideal is the LCM of all the powers of prime ideals that divide it. If the analogy with rational number theory was mentioned, it was barely followed in the definitions, and certainly not followed in the proofs.

In 1877, the theorem of unique factorization itself is expressed in a way that parallels perfectly rational number theory:

Each ideal \mathfrak{a} different from \mathfrak{o} is either a prime ideal or else expressible as a product of prime ideals, and in only one way.³⁸ (*ibid.*, 130)

The demonstration itself follows the analogy with rational number theory. The core idea and the main principles of the proofs are the same as the usual proof given in the case of the rational integers:

Let \mathfrak{a} be an ideal different from \mathfrak{o} . There exists a prime ideal \mathfrak{p}_1 which divides \mathfrak{a} , and we have $\mathfrak{a} = \mathfrak{p}_1\mathfrak{a}_1$ with $N(\mathfrak{a}_1) < N(\mathfrak{a})$.

If $N(\mathfrak{a}_1) = 1$, then $\mathfrak{a}_1 = \mathfrak{o}$, which means that $\mathfrak{a} = \mathfrak{p}_1$ and \mathfrak{a} is a prime ideal. If not, then one can put $\mathfrak{a}_1 = \mathfrak{p}_2\mathfrak{a}_2$ with \mathfrak{p}_2 prime and $N(\mathfrak{a}_2) < N(\mathfrak{a}_1)$.

If $N(\mathfrak{a}_2) > 1$, one can go on in the same way until among the ideals \mathfrak{a}_i (with decreasing norms), there is one such as $\mathfrak{a}_m = \mathfrak{o}$. Therefore,

$$\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_m$$

is a product of m prime ideals.

³⁷ “Tout idéal premier \mathfrak{p} peut, au moyen de la multiplication par un idéal \mathfrak{b} , être changé en un idéal principal. (...) Si l'idéal \mathfrak{a} est divisible par l'idéal premier \mathfrak{p} , il existera un idéal \mathfrak{a}' , et un seul, tel que l'on aura $\mathfrak{p}\mathfrak{a}' = \mathfrak{a}$, et en même temps on aura $N(\mathfrak{a}') < N(\mathfrak{a})$.”

³⁸ “Tout idéal \mathfrak{a} différent de \mathfrak{o} est lui-même un idéal premier, ou bien il peut se mettre sous la forme d'idéaux tous premiers, et cela d'une seule manière.”

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If, in the same time, one has $\mathfrak{a} = \mathfrak{q}_1 \mathfrak{q}_2 \dots \mathfrak{q}_r$, with \mathfrak{q}_j prime ideals, then \mathfrak{q}_1 divides $\mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_m$. Consequently (§22, 3.), one of the factors, say \mathfrak{p}_1 , must be divisible by \mathfrak{q}_1 . Since \mathfrak{p}_1 is prime and \mathfrak{q}_1 is different from \mathfrak{o} , $\mathfrak{p}_1 = \mathfrak{q}_1$.

One has then $\mathfrak{p}_1(\mathfrak{p}_2 \dots \mathfrak{p}_m) = \mathfrak{p}_1(\mathfrak{q}_2 \dots \mathfrak{q}_m)$, i.e., (§23, 3.) $\mathfrak{p}_2 \dots \mathfrak{p}_m = \mathfrak{q}_2 \dots \mathfrak{q}_m$, and the procedure can be repeated until the unicity is proven, “absolutely like in rational number theory.”

The theorem of unique factorization can now be used in proofs. For example, it allows to prove, with the first proposition given above, that any ideal \mathfrak{a} can be transformed into a principal ideal through multiplication by a certain (determined) ideal \mathfrak{m} . This property plays an essential role in the proof of the “main difficulty” of the theory: *if an ideal \mathfrak{c} is divisible by an ideal \mathfrak{b} , then there exists one and only one ideal \mathfrak{b} satisfying the condition that $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$.*³⁹ The proof is extremely simple. One chooses \mathfrak{m} such that $\mathfrak{a}\mathfrak{m}$ is a principal ideal $\mathfrak{o}\mu$. If \mathfrak{a} divides \mathfrak{c} , then $\mathfrak{a}\mathfrak{m}$ divides $\mathfrak{c}\mathfrak{m}$. So, there is a certain ideal \mathfrak{b} such that $\mathfrak{c}\mathfrak{m} = \mu\mathfrak{b}$. From the multiplication by \mathfrak{a} , it follows $\mu\mathfrak{c} = \mu\mathfrak{a}\mathfrak{b}$ and therefore $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$.

The reciprocity between multiplication and division was the crucial point left is to demonstrate. It shows that the divisibility defined for ideals (as an inclusion relation) corresponds to the arithmetical (number theoretical) notion of divisibility, in which division and multiplication are inverse operations of each other. Hence, the multiplication and division of ideals correspond to the multiplication and division of rational numbers – one can, thus, actually do arithmetic with ideals

In 1871, this property was given just after the definition of the multiplication of ideals, at the very end of the study of divisibility properties. The link between multiplication and division did not appear as an important matter, then, and its proof was not (presented as) a difficult conquest.⁴⁰

As it should be clear, a crucial difference between the 1871 and the 1877 versions, is the nature of the proofs given. As we saw, the proofs, in 1871, were tightly linked to tests of divisibility by powers of prime ideals, which contravened the demand that the theory not be based on computations, and also relied heavily on a particular representation of ideals (the simple ideals).

We know that Dedekind considers that a theory should be built on “characteristic properties”, so that the computations follow from the definitions – or better, the definitions can allow to predict results of computations.

³⁹This theorem is given together with the following proposition: If the product $\mathfrak{a}\mathfrak{b}$ is divisible by the product $\mathfrak{a}\mathfrak{b}'$, then \mathfrak{b} will be divisible by \mathfrak{b}' and from $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{b}'$, it follows that $\mathfrak{b} = \mathfrak{b}'$.

⁴⁰In the same time, one should not exclude the fact that the multiplication of ideals appeared at the very end of the study of arithmetical properties of ideals could be related to the “difficulty” of providing the property looked for – although Dedekind did not allude to any difficulty.

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In 1871, computations appeared to carry the theory to the definition of multiplication of ideals. In 1877, on the other hand, after the restructuring of the theory, in which the multiplication of ideals is given not only a more important role but also a role closer to that of multiplication in rational number theory, the theory is not encumbered by long computations. When the multiplication of ideals gains its place as an essential property, it offers Dedekind the possibility to give several definitions and properties in a considerably simpler way, such as power of ideals or relative primality (which doesn't need its own subparagraph anymore). Moreover, Dedekind can reduce the uses of congruences in definitions and proofs.

The changed structure of the theory provides a theory based on arithmetical notions which are familiar and whose rigor is assured. The arithmetic of ideals, introduced as a way to make things easier, is a way to bypass the difficulties such as the long tedious computations that Dedekind dislikes so much. It allows to simplify the inferences by reducing the complicated computations to very simple ones: manipulations of arithmetical operations similar to elementary number theory. It allows to reduce the chains of inferences and the length of computations by transposing the study of arithmetic to systems of elements, that is, to the “higher level”.

6.2.3 Outlines of possible applications

In the 1871 version, we saw that congruences played an essential part in the theory. Not only were they an integral part of the definition of ideals, they also had a preponderant role to play in most of the theorems and proofs, through simple ideals and the divisibility tests associated to them. In this, Dedekind was following Kummer's lead. In the 1877 version, congruences are mostly used for the definition of the norm of ideals, and are not given the kind of fundamental role they had in 1871. Once the divisibility laws for ideals are proven, however, Dedekind develops considerations further on congruences for ideal theory.

The study of congruences properties is shifted to the end of the presentation. In 1877, congruences are further developments of ideal theory – Dedekind explains that congruences are “important for ideal theory” – rather than its ground. They are also a good illustration of the efficiency of the new concept. Indeed, Dedekind gives new versions of already well-known theorems, such as the Chinese remainder theorem or Fermat's theorem, whose proofs, using ideal theory, are more general since they are valid for *any* algebraic integer.

Congruences for ideals are a powerful tool for more developed parts of number theory. Dedekind thus transfers the typically Gaussian notion of congruence to the level of ideals. Congruences in ideal theory allow to reformulate results about binary forms and some of Dirichlet's theorems. This

suggests that the scope of their potential applications is quite wide. For example, one can define classes of ideals, and determinate the number of classes of any field Ω . The equivalence relation between ideals is the same as the one defined in 1871 (see p. 317). Dedekind explained that the theory of quadratic fields “completely coincides” with the theory of binary quadratic forms, and that the problem of the number of classes of ideals is equivalent to Dirichlet’s results in *Recherches de diverses applications de l’analyse infinitésimale à la théorie des nombres* ([Lejeune-Dirichlet, 1839-1840]). If “everything [is] expressed in the terminology of ideals”, the solution lies in studying the function

$$\sum \frac{1}{N(\mathfrak{a})^s} = \prod \frac{1}{1 - \frac{1}{N(\mathfrak{p})^2}}$$

where $s-1$ is positive and infinitely small (one can recognize, here, Dedekind’s ζ -function). The sum applies to every ideal \mathfrak{a} and the product to every prime ideal \mathfrak{p} in a (quadratic) field Ω . The equality is an “immediate consequence of the divisibility laws” proved in §25. Dedekind does not give more details, but mentions that ideal theory can “simplify considerably” the works of Eisenstein and Kummer as well as Dirichlet’s “theorem on primes in arithmetic progressions” whose proof takes pages of computations and analytic considerations.⁴¹ Dedekind states his conviction that his researches are able to bring “important progress”. In fact, in the §167 of the 1871 edition of Dirichlet’s *Vorlesungen*, Dedekind did succeed in accomplishing “in a general manner, a piece of this study for any field Ω ” already in 1871 (§167), but a complete solution is still unreachable and “one has, for now, to limit oneself to study new particular cases.”

6.2.4 ... and in 1879

In 1879 appeared a third edition of Dirichlet’s *Vorlesungen über Zahlentheorie*, in which Dedekind reproduces the new version of the theory of algebraic integers. Between 1876, when algebraic integer theory was rewritten, and 1879, Dedekind and Weber wrote *Algebraische Funktionen*, using Dedekind’s ideal theory to give a new definition of the Riemann surface. The imminent publication of this paper was, for Dedekind, a core motivation for updating the supplement of Dirichlet’s *Vorlesungen* and publish the new version of ideal theory in German. It is published as the XIth Supplement entitled “Über die Theorie der ganzen algebraischen Zahlen” (and “Allgemeine Zahlentheorie” in the headers).

⁴¹Dirichlet’s theorem on primes in arithmetic progressions states that “any unbounded (*unbegrenzte* arithmetic progression whose first member and difference are without common factor [i.e., $a + nd$ such that a and d are relatively prime] contains infinitely many prime numbers.” It is proven by introducing Euler’s analytical researches, notably Euler product, into Gaussian number theory.

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In particular, one has to mention the enlarged presentation of the theory of ideals contained in the last supplement, which I first published in the second edition, but in such a compact form that the wish for a more detailed treatment was expressed to me from several parts. I have met this demands with even more pleasure, for a research accomplished together with my friend H. Weber in Königsberg, which will soon be published, yields that these principles can successfully be transferred to the theory of algebraic functions.⁴² ([Lejeune-Dirichlet, 1879], Vorwort, vii-viii, my translation)

Hence, the 1879 version of the theory is not another rewriting of Dedekind’s theory of ideals. Rather it is Dedekind’s renewed attempt at diffusing his approach, which the publication of *Algebraische Funktionen* made all the more important. The theory published in the Supplement XI is for the most part similar to that published in 1876/77:

The new presentation, which is mainly borrowed word for word from a paper published three years ago *Sur la théorie des nombres entiers algébriques* demands admittedly a much bigger space. But this will certainly be excused, when one is assured, as I hope, that the homogeneous character of the whole work has in no way be damaged.⁴³ (*ibid.*, viii, my translation)

The small changes made to the theory are essentially due to the context of publication. In particular, the introductory section and the section about the “seeds of ideal theory” are replaced by a detailed account of the theory of Gaussian integers and of the decomposition of numbers into indecomposable factors (and its failure).

The most noticeable addition is, in fact, the first appearance in Dedekind’s number theoretical works of the notion of *Abbildung*. Introduced for the study of conjugate fields, it is already very close to the notion defined in

⁴² *Besonders zu erwähnen ist die in dem letzten Supplemente enthaltene breitere Darstellung derselben Idealtheorie, welche ich zuerst in der zweiten Auflage, aber in so gedrängter Form veröffentlicht habe, dass der Wunsch nach einer ausführlicheren Behandlung von mehreren Seiten gegen mich ausgesprochen ist. Ich bin dieser Aufforderung um so lieber nachgekommen, als eine von meinem Freunde H. Weber in Königsberg in Gemeinschaft mit mir ausgeführte Untersuchung, welche demnächst erscheinen wird, das Resultat ergeben hat, dass dieselben Principien sich mit Erfolg auf die Theorie der algebraischen Funktionen übertragen lassen.*”

⁴³ *“Die neue Darstellung, in welcher Manches aus der vor drei Jahren erschienenen Schrift Sur la théorie des nombres entiers algébriques wörtlich entlehnt ist, hat nun freilich einen viel grösseren Raum erfordert; doch wird dies wohl Entschuldigung finden, wenn man, wie ich hoffe, sich davon überzeugt, dass der einheitliche Charakter des ganzen Werkes keineswegs Schaden gelitten hat.”*

Zahlen in 1888:⁴⁴

It happens very frequently, in mathematics and other sciences, that when we find a system Ω of things or elements ω , each definite element ω is replaced by a definite element ω' which is made to correspond to it according to a certain law; we use to call such an act a substitution, and we say that by means of this substitution the element ω is transformed into the element ω' , and also the system Ω is transformed into the system Ω' of the elements ω' . Terminology becomes somewhat more convenient if, as we shall do, one conceives of that substitution as a mapping of the system Ω , and accordingly one calls ω' the image of ω , and also Ω' the image of Ω .⁴⁵ (*ibid.*, 469-470, transl. in [Ferreirós, 2008], 89)

Although, in this context, *Abbildungen* will only be applied to the study of conjugate fields of numbers, the generality of the notion can be detected in this presentation, which introduces the terminology used in *Zahlen* to define the general notions of sets and mappings (see Chapter 8). Moreover, the notion presented is for Dedekind, a frequent notion “in mathematics and other sciences”, which is related to the idea expressed in a footnote that *Abbildungen* are the expression of a “mental faculty of comparing a thing ω with a thing ω' , or relating ω with ω' , or making ω' correspond to ω , without which no thinking at all is possible”⁴⁶ (*ibid.*, transl. slightly altered).⁴⁷ This echoes Dedekind’s idea of science as an activity of the human understanding and built on “mental faculties” or actions of thought. The capacity to map systems onto each others is the foundation upon which “rests also the entire science of numbers,” continues Dedekind in the footnote, announcing a publication explaining this point – which, as we know, only happened in 1888.

For the definition of conjugate fields, Dedekind gives the conditions which define a morphism between fields and which correspond to that defining the

⁴⁴The first draft of *Zahlen*, which Dedekind dates from 1872-1878, contains a notion of *Abbildung* close to that one.

⁴⁵“*Es geschieht in der Mathematik und in anderen Wissenschaften sehr häufig, dass, wenn ein System Ω von Dingen oder Elementen ω vorliegt, jedes bestimmte Element ω nach einem gewissen Gesetze durch ein bestimmtes, ihm entsprechendes Element ω' ersetzt wird; einen solchen Act pflegt man eine Substitution zu nennen, und man sagt, dass durch diese Substitution das Element ω in das Element ω' , und ebenso das System Ω in das System Ω' der Elemente ω' übergeht. Die Ausdrucksweise gestaltet sich noch etwas bequemer, wenn man, was wir thun wollen, diese Substitution wie eine Abbildung des Systems Ω auffasst und demgemäss ω' das Bild von ω , ebenso Ω' das Bild von Ω nennt.*”

⁴⁶“*Fähigkeit des Geistes, ein Ding ω mit einem Ding ω' zu vergleichen, oder ω auf ω' zu beziehen, oder dem ω ein ω' entsprechen zu lassen, ohne welche ein Denken überhaupt nicht möglich ist*”

⁴⁷For Ferreirós, this shows that “Dedekind is already here suggesting his logicistic standpoint”, in which mappings form the foundation of all mathematics.

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notion of point of a Riemann surface in 1882. These conditions are presented as answering the question

whether it is possible to represent the numbers ω of the field Ω by the numbers ω' , in such a way that all the rational relations existing between the numbers ω are completely transferred to the image ω' .⁴⁸ (*ibid.*, 470, my translation)

In other words, one wants to preserve the relations established by the “rational operations”, a necessary and sufficient condition expressed in the list of equalities given on p. 346. It is explicit that permutations are a certain kind of *Abbildungen*. This suggests that mappings were probably conceived as an extrapolation of the concept of permutation.

Recall that the notion of point, in algebraic function theory (written in 1878-79) used this kind of morphism between a field of functions and a field of “constants” (the Riemann sphere). It is possible that this use of mappings played an important role in the development of the idea of *Abbildung*. From the mapping of one field of number onto another, in the previous version of algebraic number theory, one goes to, in 1882, mappings (satisfying the conditions for the definition of conjugate fields) between functions and numbers, and one-to-one correspondences between ideals and points.

The notion of *Abbildung* also appears in Riemann’s works (as well as in Clebsch’s, later), and plays a crucial role in Riemann’s 1851 dissertation on the theory of functions:

One can picture the dependence of the magnitude w to z as a mapping (*Abbildung*) of the plane A onto the plane B .⁴⁹ ([Riemann, 1851], 5)

Ferreirós, when studying the notion of mapping in Dedekind’s works, underlines that it “emerged in his work of the 1850s” (i.e., in the lectures on Galois theory) but that it “is obscured by the fact that Dedekind used a rather strange name for maps, namely ‘substitutions,’ but there is substantial evidence that avails a set-theoretical interpretation of this notion” ([Ferreirós, 2008], 88). For Ferreirós, the “substantial evidence” consists essentially in the fact that Dedekind defines the substitutions in his works on groups independently from the individual nature of the elements. However, it should be neither “strange” nor “obscure” that Dedekind uses the words “substitution” and “permutations” in Galois theory and in number theory in general – even in the 1894 version of algebraic number theory, in

⁴⁸ “ob es möglich ist, die Zahlen ω des Körpers Ω in der Weise durch Zahlen ω' abzubilden, dass alle zwischen den Zahlen ω bestehenden rationalen Beziehungen sich vollständig auf die Bilder ω' übertragen”

⁴⁹ “Man wird sich also diese Abhängigkeit der Grösse w von z vorstellen können als eine Abbildung der Ebene A auf der Ebene B ”

which Dedekind explicitly prefers to use “permutation” – for this was well-established terminology in number theory. The general notion of *Abbildung* is not a pre-existing idea underlining all his works and that he tried to make clear battling with “obscure” terminology. Rather, it is a notion *to which* Dedekind arrived through his works in algebraic number theory, and even in the use of correspondences in the definition of a point of a Riemann surface.⁵⁰

Mappings are eminently important in Dedekind’s mathematics because they play an essential role in the definition of natural numbers, which were an ongoing interest and subject of research for Dedekind in the 1870s. This suggests an indirect influence of the work on algebraic number and function theories on the definition of numbers.⁵¹

6.3 Arithmetical strategies?

In the following section, I will emphasize certain characteristics of Dedekind’s approach, in particular in ideal theory. As a first step towards proposing an interpretation of Dedekind’s resort to arithmetic, I would like to offer a recapitulation of the preeminence of arithmetic in Dedekind’s practice between 1871 and 1882. I will propose, in this section, to assemble the most prominent examples of Dedekind’s practice, to support my claim, and to highlight the steps towards an “arithmetical” treatment of algebraic function theory.

Before underlining the textual elements in Dedekind’s works, I want to underline that this arithmetical approach was very peculiar to Dedekind. Indeed, even Weber, who was the mathematician working the closest to Dedekind, adopts a different approach in his *Lehrbuch der Algebra*.

Weber used the notion of “*Functionale*” in the *Lehrbuch*, rather than working directly with ideals.⁵² He does so when exposing the theory of algebraic functions, but also when exposing the theory of algebraic integers. For algebraic integers, the introduction of Dedekind’s concept of ideal is followed immediately by the following properties:

One can refer functionals and ideals to each other in such a way that the following rules prevail:

1. Any integral function corresponds to a determinate ideal, and associate functionals correspond to the same ideal.
2. Any ideal corresponds to infinitely many but only to associate integral functionals.
3. The product of two or more integral functionals corresponds

⁵⁰See also [Sieg and Schlimm, 2014].

⁵¹I will consider this point when presenting the definition of the natural numbers, in the last chapter.

⁵²A functional is an algebraic function that can be written as the quotient of two algebraic integral functions.

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to the product of the corresponding ideals.

4. An integral number corresponds to a principal ideal.

5. The unit corresponds to the ideal \mathfrak{o} .⁵³ ([Weber, 1895-96] II, 548)

In particular, the prime ideals correspond to prime functionals, and “the decomposition of ideals into prime factors, and actually the laws of divisibility of ideals, ensue in complete agreement from the corresponding theorems of the theory of functionals”⁵⁴ (*ibid.*, 550).

All considerations of divisibility are thus completely taken care of with the notion of *Functionale* and Dedekind’s arithmetic of ideals is not developed.

6.3.1 From 1871 to 1877

In 1871, in place of the ideal factors defined only by conditions of divisibility, Dedekind introduced ideals, first conceived as the collection of actually existing numbers that can be divided by an existing or ideal factor p . By this move, not only does Dedekind avoid the unnecessary introduction of new entities, but he creates a new level in which the arithmetical investigations can be conducted using only the “simplest principles of arithmetic”. Ideals, defined by conditions of closure by arithmetic operations and thus independent of any particular factor (contrary to the way in which Dedekind first came to the idea), are considered as “wholes”, as objects in themselves, which enables the definition of arithmetic operations *between ideals*. This definition is made in a way that is consistent with the divisibility of numbers themselves. Indeed, recall that one has:

- An ideal \mathfrak{a} is divisible by an ideal \mathfrak{b} when $\mathfrak{a} \subset \mathfrak{b}$;
- A number α is divisible by an ideal \mathfrak{a} when $\alpha \in \mathfrak{a} \Leftrightarrow \mathfrak{o}\alpha \in \mathfrak{a}$;
- An ideal \mathfrak{a} is divisible by a number α when \mathfrak{a} is divisible by $\mathfrak{o}\alpha$ i.e., $\mathfrak{a} \in \mathfrak{o}\alpha$;
- A number α is divisible by a number β when $\mathfrak{o}\alpha$ is divisible by $\mathfrak{o}\beta$ i.e., $\mathfrak{o}\alpha \subset \mathfrak{o}\beta$; in particular \mathfrak{a} is divisible by \mathfrak{b} , then for any $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$, β divides α .

⁵³ “Man kann nun die Ideale und Functionale in der Weise auf einander beziehen, dass dabei folgende Gesetze obwalten:

1. Jedem ganzen Functional entspricht ein bestimmtes Ideal, und associirten Functional entspricht dasselbe Ideal.
2. Jedem Ideal entsprechen unendlich viele, aber nur associirte ganze Functionale.
3. Dem Producte zweier oder mehrerer ganzer Functionale entsprechen die Producte der den Factoren entsprechenden Ideale.
4. Einer ganzen Zahl entspricht ein Hauptideal. 5. Den Einheiten entspricht das Ideal \mathfrak{o} .”

⁵⁴ “die Zerlegung der Ideale in Primfactoren und überhaupt die Gesetze der Theilbarkeit der Ideale ergeben sich in völliger Uebereinstimmung mit den entsprechenden Sätzen aus der Theorie der Functionale.”

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The set up of a new level of divisibility for ideals allows to prove that theorems and propositions about the divisibility of rational integers are still valid for algebraic integers, by transferring the study of their divisibility to that of ideals. As it will be stated in the introduction of *Algebraische Funktionen*, it allows to “calculate” with ideals as with numbers. For this reason, ideals form a new “ground” for the theory of algebraic numbers, insofar as all questions about properties of algebraic numbers can be transferred, or reduced to questions about ideals. With the development of appropriate methods of proof, which is allowed by the new arithmetic operations defined for ideals, all results about algebraic numbers are thus deduced from the sole concept of ideal.

In 1871, this is done by the consideration of a certain type of ideal, simple ideals which are defined by a divisibility test⁵⁵ and are used for most of the proofs. Simple ideals allow to transfer to ideals the divisibility tests used by Kummer for the definition of ideal factors. This way, Dedekind can generalize Kummer’s ideas to the level of ideals and thus provide a proof of the general validity of the laws of divisibility for ideals, and consequently for algebraic integers in a finitely generated field. With this move, even though congruences and divisibility tests still play an important role in the theory, the length of computations is considerably reduced. More importantly, it is not necessary to rely on a definite equation for the investigation of the divisibility properties, which would restrain the research to definite representations of numbers. This frees the investigation of divisibility properties from the consideration of special cases. The “analogy” with rational numbers is proved with the help of the invariant, independent theory that is ideal theory. For this reason, the theory, valid for *any* algebraic integer, is more uniform, more general.

Thus, in 1871, Dedekind works with relations between numbers and ideals using congruences conditions, and proves results about the divisibility of numbers by ideals – the core result having, as a corollary, the fact that “every ideal is completely determined once the highest powers (...) of prime ideals that divide it are given”, that is, an ideal has a unique prime factorization. The 1871 Xth Supplement proved that as far as divisibility is concerned, ideals have a behavior similar to that of rational numbers, satisfying properties such as the equivalence between being indecomposable (divisible by the unit and itself) and being prime (satisfying Euclid’s lemma).

With this certainty at hands, in the second version of the theory, which was written in 1876 to provide a more readable, simpler theory and to improve the development of ideal theory, Dedekind completely erases the role played by numbers and works only with divisibility relations between ideals

⁵⁵“For μ in \mathfrak{o} , the roots π of the congruence $\nu\pi \equiv 0 \pmod{\mu}$ form a prime ideal, this system of roots is called a “simple ideal”.

6.3 *Arithmetical strategies?*

themselves. He proposes a development which completely gets rid of divisibility tests and fully takes advantage of the now rigorously proved similarity of the divisibility properties between ideals and rational numbers. This allows to bring to the fore the inherently arithmetical nature of ideals, by developing their theory with the same ideas, analogous methods of proofs and similar results, as the ones which guided the development of elementary arithmetic. From an extension of Kummer's approach, Dedekind passes to an extension of divisibility itself.

If the theory is supposed to be easier to approach for the readers, it is firstly because the technical preliminaries are carefully developed and explained. The "improvement" of ideal theory, on the other hand, appears to be essentially related to the new form given to ideal theory. In 1877, the results as well as their proofs are given a form embracing completely the "simplest principles of arithmetic", which were affirmed to be at the basis of Dedekind's approach. The results, while being far from obvious ones, have a form so closely related to elementary arithmetic that they seem, themselves, easy to prove and understand: the ideals have a unique decomposition in primes, just like rational numbers do, and this can be proven with methods of proof developing a new arithmetic perfectly analogous to elementary rational number theory. If the theory is improved, it seems to be because the change towards a more arithmetical theory allows to highlight the arithmetical nature of ideals by developing the theory on the basis of notions and methods which are not only familiar, but whose rigor is assured.

At the expense of long, technical preliminaries, the transfer of arithmetic investigations to the less complicated and more general arithmetic of ideals allows to develop number theory by purely arithmetical means, to avoid lengthy computations and replace them by simpler ones, and thus to keep the treatment of divisibility relatively simple by its similarity with elementary arithmetic. Dedekind can thus establish, at the level of ideals, results such as:

Rational numbers	Ideals
A prime number p is an integer different from 1 and divisible only by 1 and p .	A prime ideal \mathfrak{p} is an ideal different from \mathfrak{o} and divisible only by \mathfrak{o} and \mathfrak{p} .
Each rational integer greater than 1 is divisible by a prime number.	Each ideal \mathfrak{a} different from \mathfrak{o} is divisible by a prime ideal.
If a prime number p divides the product ab , p divides a or b .	If a prime ideal \mathfrak{p} divides the product $\mathfrak{a}\mathfrak{b}$, then \mathfrak{p} divides either \mathfrak{a} or \mathfrak{b} .

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Every integer greater than 1 is either a prime number or can be written as the product of prime numbers, and in only one way.	Each ideal \mathfrak{a} different from \mathfrak{o} is either a prime ideal or else expressible as a product of prime ideals, and in only one way.
An integer a is divisible by an integer b if there is a unique integer c such that $a = bc$	If an ideal \mathfrak{a} is divisible by an ideal \mathfrak{b} , then there exists one and only one ideal \mathfrak{c} satisfying the condition that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$.
For a and b two relatively prime integers, then for any integers c_1, c_2 the system of congruences $x \equiv c_1 \pmod{a} \quad x \equiv c_2 \pmod{b}$ has a solution and all solutions of this system are such that $x \equiv c_1 \pmod{a} \iff x \equiv c \pmod{ab}$ $x \equiv c_2 \pmod{b} \iff x \equiv c \pmod{ab}$	If \mathfrak{a} and \mathfrak{b} are two relatively primes ideals, then for any ϱ, σ in \mathfrak{o} , the system of congruences $\omega \equiv \varrho \pmod{\mathfrak{a}} \quad \omega \equiv \sigma \pmod{\mathfrak{b}}$ “always has roots ω , and all these roots come under the form $\omega \equiv \tau \pmod{\mathfrak{a}\mathfrak{b}}$ where τ represents a class of numbers modulo $\mathfrak{a}\mathfrak{b}$ which is determined by ϱ and σ , or by their corresponding classes modulo \mathfrak{a} and \mathfrak{b} respectively.” ([Dedekind, 1876-1877], 134)

Note that Dedekind also gives theorems valid simultaneously for numbers and for ideals:

An ideal \mathfrak{a} (or a number α) is divisible by an ideal \mathfrak{d} (or a number δ) if and only if each power of a prime ideal which divides \mathfrak{d} (or δ) also divides \mathfrak{a} (or α).⁵⁶ ([Dedekind, 1876-1877], 133)

Dedekind fully develops in 1877 an approach that had been underlying his works since 1856, in which the operations of rational arithmetic are extended so that properties pertaining to “algebraic kinship of numbers” can be reformulated into extended versions of the operations of rational arithmetic.

The definition of a divisibility relation between groups was justified, in 1856, by the divisibility between their numbers of elements (see p. 251). For ideals, the “historical” narration of Dedekind’s “*Gedankengang*” to the development of the concept of ideal allows to understand how ideal theory came to be developed, as well as why ideals are considered as truly

⁵⁶ “Un idéal \mathfrak{a} (ou un nombre α) est toujours, et seulement alors, divisible par un idéal \mathfrak{d} (ou un nombre δ), quand toutes les puissances d'idéaux premiers qui divisent \mathfrak{d} (ou δ) divisent aussi \mathfrak{a} (ou α).”

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arithmetical objects. It thus justifies the attempts to develop ideal theory arithmetically.

In the development of algebraic number theory, however, it is for *modules* that the arithmetical operations and the arithmetical methodology are first introduced. As a part of algebraic number theory, they allow for an extension of the notion of modulus, and thus an extension of the notion of congruence, however they are clearly presented as constituting an independent auxiliary theory that can be applied for numbers or other elements. The definition of arithmetic operations for modules is not given with the same kind of justification as groups or ideals. It is however possible that the ‘structural’ similarities between ideals and modules suggested the development of the same approach for the more general theory of modules. Thus, module theory, while it does not present the same properties as number theory, can be developed by arithmetical means. This points to the possibility to use arithmetical operations not only for objects that are not numbers, but also in non-arithmetical settings.

6.3.2 1877 and 1882

In the higher level of arithmetic defined by ideals, the theorems about divisibility are proved independently of the nature of the elements forming the ideals. If this point was seen as one of the essential benefits for the initial introduction of ideals, it also presents the great advantage to enable the transfer to algebraic function theory. The transfer of methods from fields of algebraic numbers to fields of algebraic functions is, as we saw in Chapter 2, based on the following correspondence:

rational integers	→	polynomials
rational numbers	→	rational functions
algebraic numbers	→	algebraic functions
algebraic integers	→	integral algebraic functions

On this basis, the methods used by Dedekind for the theory of algebraic numbers, in 1877, can be completely translated into methods for the study of fields of algebraic functions, enabling Dedekind and Weber to study algebraic functions using, again, only the “simplest principles of arithmetic”. The transfer of methods itself sets a process in several steps:

1. The ground of ideal theory is developed in function fields;
2. Function fields are studied further with the introduction, notably, of ramification ideals;
3. Points are defined in such a way that there exists a one-to-one correspondence between prime ideals and points, and between composite ideals and complexes of points, allowing the development of arithmetic notions for points and polygons;

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4. An equivalence relation is defined for polygons, leading to the definition of classes of polygons which can be gathered into families, for which arithmetical notions are introduced as well.

The last point is repeated several times, for the study of differentials, for example.

This leads to a presentation of the theory of functions which, in a first time, is identical almost word for word, with algebraic number theory (and, transitively, with rational number theory):

Algebraic numbers	Algebraic functions
<p>A real or complex number θ is called an algebraic number if it satisfies an equation</p> $(1) \theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n = 0$ <p>of finite degree n and with rational coefficients a_1, \dots, a_n.</p>	<p>A variable θ is called an algebraic function of an independent variable z if it satisfies an irreducible equation</p> $(1') a_0\theta^n + a_1\theta^{n-1} + \dots + a_{n-1}\theta + a_n = 0$ <p>where the coefficients a_0, a_1, \dots, a_n are polynomials in z without common divisors.</p>
<p>If in the equation (1), the coefficients are rational integers, θ is called a rational integer.</p>	<p>If in the equation (1') (of lowest degree possible), the coefficients are polynomials in z, θ is called an integral algebraic function.</p>
<p>A field A is an infinite system of real or complex numbers, which is closed and complete in itself, so that the addition, subtraction, multiplication, and division of any two of these numbers always yields yet another number of the same system A.</p>	<p>A field of algebraic functions is a system of such functions with the property that application of the four fundamental operations of arithmetic (<i>Spezies</i>) to the functions of this system always leads to functions of the same system.</p>
<p>A system \mathfrak{a} of infinitely many numbers in \mathfrak{o} will be called an ideal if it satisfies:</p> <p>I. The sum and difference of any two numbers in \mathfrak{a} is again a number in \mathfrak{a}.</p> <p>II. Each product of a number in \mathfrak{a} and a number in \mathfrak{o} is again a number in \mathfrak{a}.</p>	<p>A system \mathfrak{a} of functions of z in \mathfrak{o} is called an ideal if it satisfies:</p> <p>I. The sum and difference of any two functions of \mathfrak{a} are again functions of \mathfrak{a}.</p> <p>II. The product of any function in \mathfrak{a} by any function in \mathfrak{o} is again a function of \mathfrak{a}.</p>

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If μ is a non-zero number in \mathfrak{o} , the system $\mathfrak{o}\mu$ of all integral numbers divisible by μ is an ideal, called a principal ideal.	If μ is a non-zero function in \mathfrak{o} , the system $\mathfrak{o}\mu$ of all the integral functions divisible by μ is an ideal, called a principal ideal.
An ideal \mathfrak{m} is said to be divisible by an ideal \mathfrak{a} when all the numbers contained in \mathfrak{m} are also contained in \mathfrak{a} .	An ideal \mathfrak{a} is said to be divisible by an ideal \mathfrak{b} if every function of \mathfrak{a} is also contained in \mathfrak{b} .
A principal ideal $\mathfrak{o}\mu''$ is divisible by a principal ideal $\mathfrak{o}\mu$ if and only if the number μ'' is divisible by the number μ .	A principal ideal $\mathfrak{o}\mu$ is divisible by a principal ideal $\mathfrak{o}\nu$ if and only if the integral function μ is divisible by the integral function ν .
A number α in \mathfrak{o} is divisible by an ideal \mathfrak{a} when $\alpha \in \mathfrak{a}$, or again the principal ideal $\mathfrak{o}\alpha$ is divisible by \mathfrak{a} .	A function α in \mathfrak{o} is divisible by the ideal \mathfrak{a} if the principal ideal $\mathfrak{o}\alpha$ is divisible by \mathfrak{a} or if α is a function of \mathfrak{a} .

Dedekind and Weber do not have the conceptual apparatus that constitutes modern algebra. For Dieudonné, this implies that they do not truly understand why their approach is efficient. If the “independence” and “invariance” of ideal (and module) theory is unambiguous, for Dedekind, the methodic resort to arithmetic is particularly interesting insofar as the authors are developing ideal theory as embedded in peculiar frameworks, such as algebraic function theory, in the flesh of mathematics so to speak. The (almost literal) transfer of methods modeled on the well-known methods of rational number theory from algebraic number theory to algebraic functions and the development of the theory of ideals of functions modeled on the 1877 theory entail that algebraic function theory is developed arithmetically, following the lines of rational number theory. Algebraic function theory can be developed with an arithmetical approach *independently* of the non-numerical context.

Then, the results of ideal theory analogous to rational number theory, stated in the first table, can be found for algebraic functions as well (see section 2.3.2), which comes from the fact that ideals are used *as wholes*: their content is indifferent for the statement of the properties as well as for their proofs.

Not only can statements such as the unique factorisation theorem be proven for ideals of algebraic functions, but less elementary results can be stated for algebraic functions as well, such as the Chinese remainder theorem, which was already stated for ideals of numbers: for \mathfrak{a} , \mathfrak{b} , \mathfrak{c} , ... two by two relatively prime ideals, it is always possible to find ω in \mathfrak{o} such that

$$\omega \equiv \lambda \pmod{\mathfrak{a}}, \quad \omega \equiv \mu \pmod{\mathfrak{b}}, \quad \omega \equiv \nu \pmod{\mathfrak{c}} \quad \dots$$

with λ , μ , ν , ... in \mathfrak{o} . This result, as we saw, is an essential step on the road to the definition of the ramification ideal. The way to the definition of a concept which appears, at first, to be deeply related to the Riemann

surface, is deeply arithmetical.

Dedekind transfers the concept of ramification ideal back to number theory, under the name “fundamental ideal” (*Grundideal*), in a paper published in 1882, “*Über die Diskriminanten endlicher Körper*” ([Dedekind, 1882]). The “fundamental ideal” corresponds to what we now call the “relative different”⁵⁷ and is defined as the ideal whose norm is the discriminant of the algebraic number field Ω investigated. For Dedekind, this property gives this peculiar ideal its name, since the discriminant is called the “fundamental number” (*Grundzahl*). By this sole detail, one can recognize that for Dedekind, to transfer back into number theory certain results found while working on algebraic function theory, does not amount to merely applying them. Rather, he injects them and adapts them to the context of the investigation. Dedekind, in fact, doesn’t seem to be transferring the (at that point, Riemannian) idea of ramification to fields of algebraic numbers. The introduction of the fundamental ideal is motivated by the possibility offered to study more finely the composition of the discriminant of a finitely generated field and the prime ideals.

The property allowing to correlate the fundamental ideal to the ramification ideal is the following: Let \mathfrak{d} denote the fundamental ideal, then

If \mathfrak{p} is any prime ideal, p a rational prime number divisible by \mathfrak{p} and p^e the highest power of \mathfrak{p} which divides p , then the fundamental ideal \mathfrak{d} is always divisible by \mathfrak{p}^{e-1} .⁵⁸ ([Dedekind, 1882], 393)

Indeed, recall that the ramification ideal \mathfrak{z} is defined as being the product

$$\mathfrak{z} = \prod \mathfrak{p}^{e-1}$$

of all the prime ideals \mathfrak{p} for which a certain power e divides $N(\mathfrak{p})$. In *Algebraische Funktionen*, the essential difference between ideal theory in algebraic function fields and in algebraic number fields was the norm of a prime ideal: for algebraic functions, the norm of a prime ideal is always of first degree (i.e., a linear factor), while in number theory, the norm of a prime ideal \mathfrak{p} is a rational integer divisible by \mathfrak{p} and of the form p^f with $f \geq 1$. Hence, the characterization of the ramification / fundamental ideal appears in the property stated above: \mathfrak{d} is a product $\prod \mathfrak{p}^{e-1}$ of prime ideals \mathfrak{p} such that e is the highest power dividing p with $N(\mathfrak{p}) = p^f$.

The investigations, here, show that the “analogy” between numbers and functions are not merely inspiring the development of algebraic function

⁵⁷The name “different” was introduced by Hilbert in the *Zahlbericht*.

⁵⁸“Ist \mathfrak{p} ein beliebiges Primideal, p die durch \mathfrak{p} teilbare rationale Primzahl, und p^e die höchste in p aufgehende Potenz von \mathfrak{p} , so ist das Grundideal \mathfrak{d} allemal teilbar durch \mathfrak{p}^{e-1} (...).”

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theory, but the researches in one theory sustain researches in the other one. The concept of ideal, albeit a grounding concept, is still an adaptable concept developed in a context-dependent way. This is made possible by what Dedekind calls the “independence” and the “invariance” of the concept, allowed by its definition and by its development as an arithmetic of a “higher level”.

It seems, thus, that the (extended) rational operations provide a highly efficient tool. I will argue, in the next chapters, that this is so in relation to their rigor and potential generality. Once given the first definition of arithmetic operations for ideals, the fact that Dedekind thinks of sets as “closed bags” entails that there is no reason not to extend this strategy to other contexts, as long as one can prove the consistency of an ‘arithmetical’ approach. And not only is this strategy transferable to ideals of algebraic functions, it will also be transferable to points and polygons on the Riemann surface.

Indeed, as we saw, an arithmetic analysis of fields of functions is not the ultimate aim of Dedekind and Weber’s approach. This stresses the importance of choosing to found the theory on an arithmetical ground. But additionally, it highlights the use of arithmetical notions for the development of the theory outside of the scope of the mere “analogy” between fields of numbers and fields of functions.

Recall that once the concept of point of a Riemann surface has been defined with the proof that each point corresponds biunivocally to a prime ideal, Dedekind and Weber explain that one can define the equivalent of a prime decomposition for polygons, in which points play the role of prime factors:

$$\mathfrak{A} = \mathfrak{P}^r \mathfrak{P}_1^{r_1} \mathfrak{P}_2^{r_2} \dots$$

The one-to-one correspondence between points and prime ideals entails a one-to-one correspondence between polygons and composite ideals and, consequently, the validity of the laws of divisibility of ideals for polygons. From there, Dedekind and Weber define the GCD and LCM of two polygons \mathfrak{A} and \mathfrak{B} in the following way: for \mathfrak{A} , \mathfrak{B} polygons such that

$$\mathfrak{A} = \prod \mathfrak{P}_i^{r_i} \quad ; \quad \mathfrak{B} = \prod \mathfrak{P}_j^{s_j}$$

their GCD and LCM are given by

$$GCD(\mathfrak{A}, \mathfrak{B}) = \prod \mathfrak{P}_k^{\min(r_k, s_k)} \quad ; \quad LCM(\mathfrak{A}, \mathfrak{B}) = \prod \mathfrak{P}_k^{\max(r_k, s_k)}$$

This corresponds to the definition of the GCD and the LCM of two numbers by powers of prime numbers in their prime decompositions: for a , b natural numbers, such that

$$a = \prod p_i^{r_i} \quad b = \prod p_j^{s_j}$$

their GCD and LCM are given by

$$GCD(a, b) = \prod p_k^{\min(r_k, s_k)} \quad ; \quad LCM(a, b) = \prod p_k^{\max(r_k, s_k)}$$

This characterization was given for GCD and LCM of ideals as well:

If \mathfrak{a} , \mathfrak{b} are two ideals

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \dots \mathfrak{p}_r^{e_r} \quad ; \quad \mathfrak{b} = \mathfrak{p}_1^{f_1} \mathfrak{p}_2^{f_2} \dots \mathfrak{p}_r^{f_r}$$

(where some of the e, f exponents can be zero), then one obtains the greatest common divisor and the least common multiple of \mathfrak{a} and \mathfrak{b} in the form

$$\mathfrak{p}_1^{g_1} \mathfrak{p}_2^{g_2} \dots \mathfrak{p}_r^{g_r}$$

by taking g_1, g_2, \dots, g_r to be the least members of the numbers $e_1, f_1; e_2, f_2; \dots; e_r, f_r$ in the former case, and the greatest in latter case.⁵⁹ ([Dedekind and Weber, 1882], transl. slightly altered, 74)

Dedekind's approach, then, is characterized by the transfer of arithmetical notions and tools as a way to express relations between the mathematical objects studied. This strategy comes with the set up of a "higher level", such as the "level" of ideals, and allows to translate in the "higher level" of collections of numbers (or functions) certain relations of the numbers. Dedekind can "calculate" with objects of the higher level just like in rational arithmetic. In the cases where correspondences can be shown to exist between two kinds of objects (algebraic numbers and functions and ideals, ideals and polygons, ...), the validity of the laws in one domain entails the validity of the same laws in the other domain.

The elaboration of "higher levels" can be reiterated if needed. Indeed, Dedekind defines equivalence relations for ideals, or for polygons, for which the same kind of arithmetical approach is developed. Two polygons (with the same number of points) are said to be equivalent when their quotient is a function in the field Ω . According to this equivalence relation, polygons can be organized into classes:

All the polygons $\mathfrak{A}', \mathfrak{A}'', \dots$ equivalent to a given polygon \mathfrak{A} constitute a *polygon class* A . (...) Each polygon belongs to one and only one class. (*ibid.*, 105)

⁵⁹ "Sind \mathfrak{a} , \mathfrak{b} zwei Ideale $\mathfrak{a} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \dots \mathfrak{p}_r^{e_r}$; $\mathfrak{b} = \mathfrak{p}_1^{f_1} \mathfrak{p}_2^{f_2} \dots \mathfrak{p}_r^{f_r}$ (worin die Exponenten e, f auch zum Teil Null sein können), so erhält man den größten gemeinschaftlichen Teiler und das kleinste gemeinschaftliche Vielfache von \mathfrak{a} und \mathfrak{b} in der Form $\mathfrak{p}_1^{g_1} \mathfrak{p}_2^{g_2} \dots \mathfrak{p}_r^{g_r}$ wenn man für g_1, g_2, \dots, g_r für ersteren die kleinsten, für letzteres die größten unter den Zahlen $e_1, f_1; e_2, f_2; \dots; e_r, f_r$ nimmt."

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For polygons classes, Dedekind and Weber introduce the notions of product and divisor, in accordance with products and divisibility of polygons. These notions can be extended to families of polygons and of classes (the divisor of the family is the GCD of the elements in the family), and be used, for example, to reduce the dimension of a family. These ideas and the relationships established by the arithmetical treatment of polygons, classes and families are used for the study of differentials of the field without any considerations about limits and continuity.

It is not the first time that Dedekind defines equivalence relation for elements of the “higher level”. Indeed, in 1871, he proposed to consider an equivalence relation between ideals: two ideals \mathfrak{a} and \mathfrak{a}' are called equivalent if there exists an ideal \mathfrak{m} for which $\mathfrak{a}\mathfrak{m}$ and $\mathfrak{a}'\mathfrak{m}$ are in a system of ideals E satisfying (i) closure by multiplication, (ii) if \mathfrak{c} and \mathfrak{c}' are in E , then \mathfrak{c}' is in E and (iii) for any ideal \mathfrak{a} , there exists an ideal \mathfrak{m} such that $\mathfrak{a}\mathfrak{m}$ is in E . This led to the definition of classes of ideals (the collection of all ideals equivalent to a certain ideal) and of the product of classes.

Classes of ideals are also present in 1877, in a slightly different form. Dedekind proved earlier in the paper that each ideal in \mathfrak{o} can be transformed into a principal ideal through the multiplication by a certain ideal. Then, two ideals \mathfrak{b} and \mathfrak{b}' can be said to be equivalent if they are changed into principal ideals via the multiplication by a same ideal \mathfrak{m} , i.e., $\mathfrak{b}\mathfrak{m} = \mathfrak{o}\mu$ and $\mathfrak{b}'\mathfrak{m} = \mathfrak{o}\mu'$. And of course, this leads to the definition of classes of ideals, and their product.

For further developments of the theory investigated, Dedekind seems to reproduce (*mutatis mutandis*) Gauss’s introduction of the notion of congruence by shifting the investigations to classes of ideals (or polygons). He thus sets up another higher level, in which he extends the previous considerations, or familiar notions such as that of divisor.⁶⁰

The set up of an arithmetical framework for the study of polygons, which are systems of the constitutive elements of the concept of Riemann surface is (mathematically) justified by the biunivoque correspondence that has been proved to exist between ideals and complexes of points. Does the existence of a one-to-one correspondence implies that similar methods *must* be developed or does it *authorize* to do so, thereby offering the possibility to develop an arithmetical approach for Riemann surfaces? But then again, what would be the benefits of developing an arithmetical approach for a theory that seems to be so remote from arithmetic? And to what extent can such an approach be developed without it being a reduction to natural numbers? Such a reduction is a move which Dedekind does not encourage, as is well known:

⁶⁰Recall that already in 1857, Dedekind proposed to define a new notion of congruence, which had two moduli, one of which was a class of polynomials. See p. 263.

[I]t appears as something self-evident and not new at all that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers – a declaration I have also heard repeatedly from the lips of Dirichlet. But I see nothing meritorious – and this was just as far from Dirichlet’s thought – in actually performing this wearisome circumlocution and insisting on the use and recognition of no other than natural numbers.⁶¹ ([Dedekind, 1888], transl. slightly altered, 792)

It seems that to demand that similar methods be developed if one can exhibit a one-to-one correspondence between domains would proceed from a reductionist approach, which is not Dedekind’s aim. In fact, the set up of arithmetical methods and the translation of relations between objects in arithmetical terms appears as a clear instance of giving an arithmetical form to arguments without a reductionist perspective.

6.3.3 Conclusion: What role and status for arithmetic?

Presented as a change of presentation whose aim was to make the theory more readable, the second version of Dedekind’s theory of algebraic integers is given a development following very closely that of rational number theory. The desire to obtain a theory perfectly analogous to rational arithmetic is explicitly stated by Dedekind, and he puts a lot of effort into developing methods of proof allowing him to expose his theory in a way that reproduces rational number theory. Rational arithmetic, then, appears to take a more important role in the methods developed by Dedekind: the goal is not solely to prove the similarity of certain theorems about divisibility anymore, but to define a higher level of arithmetic, the arithmetic of ideals, which can be identical to rational arithmetic. Dedekind seems, then, to propose an extension of the operations of rational arithmetic and, by the importance given to the possibility to provide a perfectly analogous development, points out the primacy of arithmetic for his approach.

I have mentioned, on p. 186, Avigad’s remark about Dedekind’s “insistence that definitions and methods of proof used in an extended domain should parallel” the ones given in restricted domains, and how this does not seem to completely apply to Dedekind’s approach and will not repeat them here. Rather than, as Avigad suggests, a mere “enjoyment” of finding and

⁶¹ “[E]rscheint es als etwas Selbstverständliches und durchaus nicht Neues, daß jeder auch noch so fern liegende Satz der Algebra und höheren Analysis sich als ein Satz über die natürlichen Zahlen aussprechen läßt, eine Behauptung, die ich auch wiederholt aus dem Munde von Dirichlet gehört habe. Aber ich erblicke keineswegs etwas Verdienstliches darin – und das lag auch Dirichlet gänzlich fern –, diese mühselige Umschreibung wirklich vornehmen und keine anderen als die natürlichen Zahlen benutzen und anerkennen zu wollen.”

6.3 *Arithmetical strategies?*

pointing out “parallel developments” between elementary number theory and higher congruences or ideal theory, it seems that Dedekind is putting a lot of effort in having ideal theory following similar developments as elementary number theory. For Avigad, the endeavors to develop arithmetical theories are only based on the fact that “[t]he methodological benefits are clear, since it is often easy and efficient to reuse, adapt, and extend familiar modes of reasoning” ([Avigad, 2006], 171). Yet, Avigad’s explanation only seems to suggest that one uses arithmetic because it is “easy and efficient” but does not address the epistemic benefits of using arithmetic: why is an arithmetical approach considered as the best solution for providing more rigorous and/or uniform definitions and treatments of theories?

Besides, what are the properties of arithmetic making it Dedekind’s weapon of choice for developing new concepts and/or rewriting theories? Is the possibility to define new arithmetic operations at “higher levels” or in different settings (such as polygons in algebraic function theory) an inherent property of the operations of the “science of numbers”, or is it a specificity acquired by experience, through its use in algebraic number theory?

Arithmetic, in what has been described up to now, seems to play both the role of a foundation, e.g., with its role in the development of ideal theory as a new grounding concept, and of a tool, as a means to develop the researches on Riemann surfaces, for example. This underlines that arithmetic, from Dedekind’s viewpoint is susceptible to be operative in many areas of mathematics. Besides, insofar as algebraic function theory is not considered as being itself (revealed as) an arithmetical theory, it also suggests that the possibility to resort to arithmetical notions to fulfill the desire to provide a “new grounding” and a more rigorous, more uniform theory is not related to a desire to reduce mathematics to arithmetic.

Rather, operations of arithmetic appear to be part of a conceptual arsenal that Dedekind considers to be usable in different areas in mathematics in such a way that it is presented as allowing a greater rigor, and to some extent a greater simplicity or clarity in the concepts. In this conceptual arsenal, one can find arithmetic operations, but also systems and mappings, which are for Dedekind the most general and most fundamental notions in mathematics – and in human thinking in general. Even if I have not underlined their uses as much as I have underlined the use of arithmetic in Dedekind’s works, the preeminence of systems in Dedekind’s mathematics is relatively obvious (and mappings can be detected, notably in the definition of conjugated fields). I will, however, come back to this point later on, when their importance and their status in mathematics will have been made clearer. Indeed, systems and mappings, in addition to being used throughout mathematics by Dedekind, serve as a foundation for the definition of natural numbers, to which I will dedicate Chapter 8.

Chapter 6 - Towards a more arithmetical theory of algebraic numbers?

The crucial and central role played by elementary arithmetic operations in Dedekind's most elaborated mathematical works suggests that arithmetical operations are given a very particular and active role in the development of mathematics. I will suggest, in the next Part of this dissertation that arithmetic, in Dedekind's mathematics, is involved in the development of mathematical knowledge. *Algebraische Funktionen* presents itself as a strong illustration of how arithmetic acts at several levels, how concepts such as ideals are devised on arithmetical grounds, and how the resort to arithmetic allows to develop a certain theory on more rigorous grounds, leading for example to the first actual complete proof of the Riemann-Roch theorem.

I would like to defend the view that arithmetic operations appear to be used as an epistemic tool by Dedekind. That is, the four elementary operations of arithmetic are involved in the process of knowledge production, they are a medium, a tool at the mathematician's disposal for the development of mathematical knowledge: they are used to design and/or study new concepts that will be involved in the development of the theory studied. This epistemic tool allows Dedekind to develop complex and elaborated theories based on elementary notions of arithmetic (in particular divisibility).

But why would Dedekind take such winding roads to be able to develop a theory using operations of arithmetic? If it is an issue of rigor, then why should arithmetic be taken as a so much more appropriate means to reach the goal, that it would require detours such as the ones taken in 1882? If the desire to provide an arithmetical theory can be easily understood in 1871 and 1877, it is less obvious in 1882, once the transfer of ideal theory is achieved. Moreover, note that Dedekind always justifies his use of arithmetical operations and carefully provides proofs that an arithmetical approach can be seen as consistent. We are, thus, not in presence of an act of faith about the supremacy of arithmetic, but a deeply reflected upon strategical use of arithmetical notions.

Part IV

Numbers, arithmetic and
mathematical practice

Introduction to Part IV

Having brought to the fore the preeminence of arithmetic and arithmetical methods in Dedekind's works, I would like to propose an account of his approach which could take into account two sides of the story: the uses of arithmetic to develop new concepts and the foundations of arithmetic.

First, it will be useful to place Dedekind into a part of the epistemological and mathematical contexts in which he wrote and which I have not considered yet, namely the so-called "arithmetization of mathematics". The arithmetization movement illustrates clearly how the desire to identify the appropriate methods in mathematics became an important issue in the 1870s. It is one of Dedekind's central arguments for his own works against rival approaches – the most notable opposition being obviously that with Kronecker, which we have looked at before and will keep on considering as the need arises. Arithmetization will provide the basis for an epistemological account of Dedekind's uses of arithmetic, which I will develop in the last section of this chapter. Indeed, with all the evidences at hand, I will highlight how Dedekind designed a strategy answering a certain desire for rigor, which shall allow to make sense of Dedekind's recurrent use of arithmetic. The idea of "arithmetization", here, should not be understood as a mere rigorization of the foundation of analysis, which it has become synonymous with, but rather as argued by Otte and Jahnke, as a desire to "reshape and reformulate" mathematical theories:

An attempt for better comprehending the change in mathematics' understanding of its object field, which began to take place towards the turn to the 19th century, first of all requires tackling the problem and the context of the "arithmetization of mathematics". It must be said at once that this arithmetization was not only, as is often supposed, a matter of founding infinitesimal calculus anew, but rather of reshaping and reformulating mathematics as a whole. ([Jahnke and Otte, 1981], 28)

As an "arithmetization", the approach adopted by Dedekind – the questions investigated and the desire to rewrite definitions within a different conceptual framework, in particular – is far from isolated. The conception of arithmetic appears as a crucial part of what distinguishes mathematicians working in the same field and providing answer with such strikingly different forms and methods as, for example, Dedekind and Kronecker.

If one is to shed light on Dedekind's arithmetization strategy, a clear understanding of what arithmetic is and can do is indispensable. As I have mentioned several times, arithmetic is presented by Dedekind as the "science of numbers", whose first objects are the natural numbers and is in the mean time said to present with an immense wealth of laws and limitless possibilities of application. Hence, following the investigations on arithmetization, I

will inquire into how these two parts relate. What relations, if any, are there between the uses of arithmetic in Dedekind's mathematical works and his definition of natural numbers (and with them, the first definition of arithmetic)?

To make sense of such questions, Dedekind's concept of (natural) number must be considered closely, for arithmetization as "putting into arithmetical form" is intrinsically related to a certain conception of arithmetic and number. Jahnke and Otte related the development of arithmetization to the rise of a purely symbolical concept of number. Gray in [Gray, 2008b] talks about "the erosion of the concept of number." But while true for the 'bigger picture', these statements do not apply to Dedekind, whose insistence on the creation of numbers suggests that his concept is neither formal nor eroded.¹ This point, essential to understand his book, was made very clear by Dedekind and was, in fact, a widespread conception about numbers and arithmetic.

The foundation of arithmetic was a long lasting reflection for Dedekind. Questions about the principles lying at the basis of natural numbers were "not settled" during the 1870s for Dedekind, and according to Sieg and Schlimm, "on the contrary, Dedekind struggled with it intermittently over the next six years. The intense work is reflected in the manuscript from 1872/1878 (in [Dugac, 1976b], 293-309) which was the very first draft for the 1888 essay on the nature and meaning of numbers" ([Sieg and Schlimm, 2005], 129). Chronologically speaking, the "struggles" through which Dedekind went correspond broadly to the writings of the 1876-77 version of algebraic number theory and the correspondence with Weber for the writing of *Algebraische Funktionen*. In the next and last chapter of this dissertation, I will consider the possibility that what can be achieved with arithmetical methods, using arithmetical operations, might have had an effect on Dedekind's reflection on the number concept.

One of the motivations of the last chapter will be to replace Dedekind's definition of natural numbers into the Dedekindian mathematical corpus, rather than to consider it as an isolated philosophical work. On one hand, because *Zahlen* has only often been taken as a program comparable to that of Frege or Peano, it seems important to underline the specificity of its aim, of its focus. On the other hand, to consider the definition of natural numbers in perspective of Dedekind's other works will imply to consider not only how *Zahlen* can shed light on Dedekind's other works, but also how Dedekind's previous mathematical researches can shed light on Dedekind's

¹In fact, his insistence on creating new domains numbers at each definition, rather than considering the cuts or the simply infinite systems as the numbers, is an argument against an actually structuralist or logicist interpretation of Dedekind's concept of number. Numbers are, indeed, defined by logical means and as elements of a structure, but they are, for Dedekind, mathematical entities created by the human mind.

6.3 Introduction to Part IV

definition of natural numbers. Not only is *Zahlen* a very singular book in the mathematical landscape of 1880, but it also took Dedekind years of reflection. The strong inner unity of Dedekind's works, which he underlined himself, and the dominance of arithmetical methods suggest that Dedekind's ideas grew nurtured by the mathematical practice.

Dedekind's definition of natural numbers is grounded on the concepts of systems and mappings, and only makes use of these concepts. However, a completely set-theoretical reading of Dedekind's conception of arithmetic and numbers would project on Dedekind, ideas that were developed decades after his own works. I will propose that the 'set-theoretical' definition of natural numbers is related to Dedekind's conception of rigor and to the idea of numbers as creation of the mind, inasmuch as the concept of system and mappings are mathematical representations of the fundamental operations of thought: to collect things with a common property and to relate things to things. On the other hand, I would like to suggest that the grounding of arithmetic, that is, the definition of natural numbers and the first introduction of the operations for the natural numbers, plays a significant role in accounting for Dedekind's conception and use of arithmetic in other works. Recall Dedekind's statements about what properties an "arithmetical" theory should possess, for example in his *Bunte Bemerkungen* to Kronecker's *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*:

Neither from this introduction nor from the essay itself is it clear why exactly this theory should be called an *arithmetical* one. Under this name, one should suspect that the consideration of the realm of numbers (the absolute constants) would form the main foundation, it is by no means the case. I would very much like to rather call this theory *formal*, because it predominantly is based on the "auxiliary methods of indefinite coefficients" (pp. 47, 48, 69) and the "association of forms" (formed by these coefficients or auxiliary variables u, u', u'') (§15 and §22, pp. 93-96). ([Edwards, Neumann, and Purkert, 1982], 54)

Chapter 7

Arithmetic, arithmetization and extension of the number concept

The arithmetization of analysis and Dedekind's *Stetigkeit und irrationale Zahlen* have been studied in detail by many authors. I do not pretend that I will bring new lights on Dedekind's definition of irrational numbers. Rather, I want to propose to consider this work as a particularly clear and eloquent example of Dedekind's strategy to obtain a certain rigor in mathematics. *Stetigkeit* aims very explicitly at providing an arithmetical and rigorous definition of a certain mathematical notion (continuity) that will provide a better understanding of the concept – even, to a certain extent, according to Dedekind, a clearer view of its nature. Moreover, it is an important work, in Dedekind's corpus, and as such deserves attention for a comprehensive understanding of Dedekind's ideas.¹

Arithmetization was described by Klein as a process of rigorization. I will propose, in the following, that the rigor, which Klein associated with a certain logical rigidity, was in Dedekind's mathematics a structuring element for his practice of mathematics. Moreover, rigor for Dedekind appears to be an active part of the creative activity of mathematicians. Arithmetization in Dedekind's works is a strategy that allows him to set up a mathematical arsenal built upon well-defined concepts or even unprovable premisses, tailored to reach more rigorous and uniform bases for certain theories, and

¹Note that I will not consider questions such as the axiomatic character of Dedekind's definition. On these matters, one can refer to [Sieg and Schlimm, 2005]. Sieg and Schlimm's rich paper studies several of Dedekind's published and unpublished foundational writings and provides a fine analysis of Dedekind's growingly axiomatic conception. It should be underlined, however, that the fact that the definition answers criteria of being axiomatic does not, however, imply that Dedekind is engaged in an axiomatization of the real numbers. Rather, he is providing a definition that creates a new concept. The way in which he chooses to formulate this definition does exhibit signs a certain axiomatic thinking. Sieg and Schlimm point out this particularity of Dedekind's approach as the co-existence of the axiomatic and 'genetic' approaches. One can also refer to [Sinaceur and Dedekind, 2008], in particular pp. 107-110.

7.1 Arithmetization of Analysis in the 19th century

which can be sufficiently powerful and general that they don't only offer a good foundation but also the possibility to stimulate and ease the development of entire parts of mathematics. This last point can be seen, for example, with ideal theory, rather than higher congruences, as a grounding concept for the theory of algebraic numbers. Dedekind's rigor, rather than a purely logical ideal of, embraces the demands of mathematical practice.

In this, arithmetic plays a crucial role, for it is taken as a means to reach the rigorous theory aimed for in many of Dedekind's most important works.² To characterize Dedekind's approach as an "arithmetization" offers the possibility of a global understanding of his approach of which it can also be made sense on a more local scale, with the possibility to understand recurrent elements of practice and particular approaches adopted by Dedekind.

In this chapter, I will propose to consider the question of "arithmetization" in its historical context, from the reading made by Klein to the initial sense of "arithmetize" in Kronecker's works via other instances of the so-called "arithmetization of analysis". This shall allow me to show how Dedekind's approach is both singular and embedded in a widespread line of ideas. Then, I will propose to take seriously the idea of "arithmetization" as "putting into an arithmetical form" and try to show how this can provide a more precise and more thorough account of Dedekind's approach than the mere adjective of "conceptual". The idea of a strategy of putting definitions or proofs into an arithmetical form, in Dedekind's works, should not be taken as a reductive argument. As underlined by Jahnke and Otte, to consider arithmetic "as a foundational science of mathematics does not mean that arithmetic constitutes the actual subject matter of mathematics." In an "arithmetization", arithmetic is taken as a tool, even to some extent, following Jahnke and Otte, as a "means of objectifying mathematical thought" ([Jahnke and Otte, 1981], 29).³

7.1 Arithmetization of Analysis in the 19th century

The notions of number and of magnitude, during the 18th and 19th centuries, underwent many modifications of scope and meaning, for example with Gauss's generalized arithmetic and Kummer's ideal factors, with Dirichlet's introduction of analytical methods in number theory, with Riemann's

²Of course, it is not the case for the definition of natural numbers, since at that point of conceptual development, arithmetic does not exist. I will not be considering the foundation of natural numbers, in this chapter.

³My argument (and more broadly, in fact, my goal) will differ on several points from the analysis given by Jahnke and Otte, in particular because one of their core arguments concerns the idea that "numbers are no longer interpreted as objects, but as pure symbols, as 'marks'," a point that is not valid in Dedekind's case. In particular, rather than talking about the status of numbers, here, I will talk about *arithmetic*.

generalized concept of magnitude in 1854, or again, outside of Germany, with the development of a purely symbolical algebra in Cambridge, where the symbols manipulated in algebraic formulae became completely devoid of any (need for an) interpretation in terms of numbers or magnitudes. All these points were not susceptible to be seized by the classical notions of number and magnitude.

In addition, the foundations of Analysis were themselves submitted to many doubts, from Lagrange's desire to get rid of infinitesimals to Cauchy's restructuration of Analysis in pursuit of a certain Euclidean ideal of rigor. In particular, continuity was at the basis of analytical investigations and still lacked a rigorous definition. To be able to provide a solid foundation to continuity was in particular important, for an appropriate characterization of the notion of limit, and led to the demand that continuous magnitudes be rigorously and consistently defined. Many mathematicians during the second half of the 19th century took interest in this problem. According to their (sometimes very) different interests and viewpoints in mathematics, the problem had different forms. For example, Weierstrass was mainly interested in the long lasting project of rigorizing analysis, whereas Dedekind was motivated by the desire to provide a purely arithmetical foundation to (linear) continuity in order to be able to define analytical notions like limits.

7.1.1 Klein's idea of arithmetization

In 1887, Leopold Kronecker, published his famous essay on the concept of number, *Zum Zahlbegriff* ([Kronecker, 1887]), in which he introduces the term "to arithmetize":

And I also believe, that we will, one day, succeed in "arithmetizing" the entire content of these mathematical disciplines [i.e. Algebra and Analysis], that is to found them exclusively and purely on the concept of number in its narrowest sense, and thus, to take off the modifications and extensions of this concept which are in most cases caused by the applications to geometry and mechanics.⁴ ([Kronecker, 1887], 338-339, my translation)

By this, he expresses his wish to found mathematics on the sole concept of natural number. His aim, then, is a *reduction* of analysis, algebra and arithmetic to the concept of number "in its narrowest sense." In Kronecker's

⁴"Und ich glaube auch, dass es dereinst gelingen wird, den gesamten Inhalt aller dieser mathematischen Disciplinen zu 'arithmetisieren' d. h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Erweiterungen dieses Begriffs wieder abzustreifen welche zumeist durch die Anwendungen auf die Geometrie und Mechanik veranlasst worden sind."

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works, the commitment to (admitting only) the natural numbers is significant and accompanied by strict constructivist demands.⁵

In 1895, Klein used the expression “the arithmetizing of mathematics” (*Arithmetisierung der Mathematik*) in his famous 1895 address to the Kgl. Gesellschaft der Wissenschaften in Göttingen ([Klein, 1895]). The phrase borrowed from Kronecker gives up the initial idea to give an arithmetical form to all pure mathematics. Klein’s standpoint on the movement of arithmetization is that it is a rigorization process arising in analysis to put things in order after the discoveries made during the 18th century. Klein explicitly dismisses the “mere putting of the argument into arithmetical form” as unimportant, and considers that only the rigid logic behind the argument is significant:

I consider that the essential point of the argument is not to be found in the arithmetical form of this development of thought, but in the more rigid logic obtained by means of this form.⁶
(transl. slightly altered, in [Ewald, 2005], 967)

For Klein, the movement of “arithmetization of analysis” is a reaction against the “century of discoveries” that preceded. In his view, it is facing the many evolutions mathematics was subjected to after the 18th century, that a “more critical spirit asserted itself” and demanded that the innovations of the previous century be given a “logical justification”. An increased sense of rigor made its way into mathematics with the works of Gauss, Cauchy and Dirichlet, but “this was not the end of the matter”, Klein explains. One can still find, in Gauss’s works, the acceptance of space as given by intuition, a point against which most of the “arithmetizers” will protest, especially insofar as

space intuition had led to the too hasty assumption of the generality of certain theorems which are by no means general. Hence arose the demand for *exclusively arithmetical methods of proof*; nothing shall be accepted as a part of the science unless its rigorous truth can be clearly demonstrated by the ordinary operations of analysis.⁷ (*ibid.*, 966, original emphasis)

⁵As I have mentioned earlier, Kronecker’s arithmetization is linked to many deep philosophical ideas, and I will come back to some points of his approach later.

⁶“*Indem ich als das Wesen der Sache nicht die arithmetische Form der Gedankenentwicklung ansehe, sondern die durch diese Form erreichte logische Verschärfung, ergibt sich die Forderung.*”

⁷“*die Raumschauung dazu geführt hatte, in übereilter Weise Sätze als allgemeingültig anzusehen, die es nicht sind. Daher die Forderung ausschließlich arithmetischer Beweisführung. Als Besitzstand der Wissenschaft soll nur angesehen werden, was durch Anwendung der gewöhnlichen Rechnungsoperationen als identisch richtig klar erwiesen werden kann.*”

Klein's examples are Weierstrass and the famous Weierstrassian rigor (“*Weierstrass'sche Strenge*”), Kronecker and Peano.⁸ Klein also mentions Cantor, later on. Cantor wrote to Klein about arithmetization, stressing his general agreement with Klein's analysis, but disagreed with “the preponderance given to Weierstrass in the arithmetization of mathematics”, for one should “separate what Weierstrass actually did from the myth in which his students enwrapped him, so to speak a thick fog for the stabilization and elevation of their own reputations” (in [Dugac, 1976b], 165).

Klein does not consider that to completely reject intuition is a fruitful approach, but it does not imply that he considers the “arithmetization of mathematics” as unimportant or unnecessary. On the contrary, he clearly states that the rigorization of the foundation of analysis has important benefits: on the basis of a solid “logical” foundation given to analysis, it becomes possible to “subject the remaining divisions of mathematics to a fresh investigation.” This must however be nuanced, since, according to Klein, “it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, intuition has its special province.”

Insofar as Klein sees arithmetization as only a question of *logic*, the underlying logic of arithmetic that must always – according to (Klein's vision of) the arithmetizers – be used in mathematics, it doesn't matter for him that these mathematicians hold very different positions on mathematics.⁹ To disregard the *arithmetical* part of *arithmetizing* allows Klein to give a rather uniform presentation (*the* arithmetization) of a movement embedding several different practices (of Weierstrass, Peano, Kronecker, Cantor, or Dedekind). By doing so, Klein reduces an approach heavy with epistemological requisites to a rather narrow demand of rigid logic to secure the argument. But by doing so, he also seems to enlarge the arithmetization approach to any mathematician who demands a more rigorous definition of the fundamental notions of analysis – as is clear from the list of people mentioned. With the very wide view adopted by Klein, Cantor recognized himself as a member of a movement whose first representative would be... Kronecker!

It is clear that if one takes a closer look at the justifications given by the actors for their “arithmetization”, there is a myriad of reasons, from the mere reject of intuition to strong ontological commitments. If the actors all preach rigor, it is not obvious that they all have the same ideas about rigor and the ways to get to it.

⁸Peano, who gave the first example of a space-filling curve, was strongly involved in rigorizing analysis. He also proposed an axiomatization of arithmetic and the foundation of analysis. see Sec. 8.3.

⁹In fact, the said differences are probably of no significance for Klein.

7.1 *Arithmetization of Analysis in the 19th century*

For Petri and Schappacher, Klein’s speech “also marked the beginning of the nostrification of arithmetization by the newly emerging mathematical centre at Göttingen” ([Schappacher and Petri, 2007]). They explain that

[h]ardly used today, the term “arithmetization” (...) was in use around 1900 as a generic description of various programmes which provided non-geometrical foundations of analysis, or other mathematical disciplines. ([Schappacher and Petri, 2007], 343)

Historians (and philosophers) of mathematics also followed Klein’s lead. While it is acknowledged that arithmetization involves many different practices, it is essentially seen and studied as a large rigorization process in which any non-geometrical characterizations of the foundations of analysis can be included. Arithmetization has been largely commented by historians of mathematics.¹⁰ These studies are usually a succession of presentations of the works of several “arithmetizers”: Weierstrass, Cantor, Kronecker, Dedekind, sometimes Méray, Peano, sometimes also Frege. But the phrase “arithmetization” seems to have lost the ‘putting into arithmetic form’ part of the approach, which did constitute the core of the idea when it was first introduced by Kronecker in 1887. By sticking to a “rigorization” idea of arithmetization and omitting to consider that part of arithmetization in which an actually arithmetical form is given to the argument, some fairly important components of “arithmetization” might have been overlooked. In particular, one may miss a finely tuned understanding of the conception of arithmetic sustaining arithmetization and its relations with the rigor pursued by arithmetizers, which holds a key to the differences behind their approaches.

7.1.2 Examples of “Arithmetization of Analysis”

The successive extensions of the domains of numbers were already recognized by Gauss as a driving force in the development of mathematics, as Boniface explains:

Indeed, for Gauss, the successive enlargements of the concept of number to negative numbers first, then to fractional numbers and finally to irrational numbers and to complex numbers are driving the development of pure mathematics. One can thus

¹⁰[Dugac, 1970], [Dugac, 1973], [Dugac, 1976a], [Jahnke and Otte, 1981], [Epple, 2003], [Boniface, 2002], and [Schappacher and Petri, 2007], in particular. I will not comment further these studies, for my aim is not to oppose any of them, but to consider the idea of “arithmetization” under a slightly different light. What I will say doesn’t invalidate anything in these works. Rather, I propose to first narrow the understanding of “arithmetization” from “rigorization” to “putting into an arithmetical form”, which will in return allow to see how this strategy is applied in other areas than the sole “Science of quantity”.

find, in Gauss[’s thinking] both the conception of arithmetic limited to the sole positive integers, and that of a number concept enlarged to rational, irrational and complex numbers, and which is concerned not only with arithmetic but also with algebra and analysis.¹¹ ([Boniface, 2010])

The desire to identify the most appropriate methods in mathematics which developed in the 1870s affected the definition of the domains of numbers as well as many other parts of mathematics – some of which we saw in earlier chapters of this dissertation.

In the commentaries mentioned above, as well as in [Ferreirós, 2008] or again [Belna, 1996], one can find a great amount of details and comparisons between the approaches. Before presenting Dedekind’s definition of the real numbers, I will propose a detour by an alternative definition of the irrational magnitudes: the definition as limits of Cauchy sequences Cantor.¹²

It will not be my aim, here, to systematically compare the different “arithmetizations”. I do, however, wish to underline the fact that the desire of rigor, which prevails the definitions, entails answers which are epistemologically, methodologically and mathematically very different. This shall allow me to highlight certain central features of Dedekind’s approach in *Stetigkeit* and more broadly. On this ground, I will propose a characterization of Dedekind’s methods and ideas that would be coherent as regards the unity of Dedekind’s corpus, able to consistently describe and explain his works both globally and locally – and related to the historical context.

The “arithmetization of Analysis” is usually closely related to the name of Weierstrass, who introduced rigorous definitions and methods of proof in Analysis and cleared “many ambiguities in real and complex function theory by counterexamples and rigorous exposition” ([Schappacher and Petri, 2007], 351). In the case of Weierstrass’s association to the arithmetization of Analysis, the term “arithmetization” is quasi synonymous with “rigorization”, as in the Kleinian understanding of “arithmetization”. Weierstrass, during his lectures in Berlin, introduced the rigorous definitions of basic

¹¹ “*Pour Gauss, en effet, les élargissements successifs du concept de nombre, aux nombres entiers négatifs d’abord, puis aux nombres fractionnaires et enfin aux nombres irrationnels et aux nombres complexes, sont le moteur du développement des mathématiques pures. On trouve ainsi chez Gauss à la fois la conception d’une arithmétique limitée aux seuls entiers positifs et celle d’un concept de nombre élargi aux nombres rationnels, irrationnels et complexes, et qui concerne donc non seulement l’arithmétique, mais aussi l’algèbre et l’analyse.*”

¹²Charles Méray (1835-1911) independently gave a similar definition in 1869. Méray wanted to give an appropriate characterization of sequences of numbers whose limit was not a rational number. For works on Méray see [Dugac, 1970] and [Boniface, 2002]. Méray’s was the first mathematician to publish a consistent theory of irrational magnitudes in 1869. His definition is broadly similar to the one independently given by Cantor in 1872. Yet, Méray’s model for rigor is Algebra, and not Arithmetic (as it was for Lagrange).

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analytical notions with δ and ε , and gave one of the first rigorous definition of irrational numbers. He was also the first to exhibit a pathological (everywhere continuous and nowhere differentiable) function.

For the definition of irrationals, Weierstrass considers “aggregates” of numbers, in particular of positive fractions $\frac{1}{n}$. Positive rational numbers are finite aggregates and positive irrational numbers are arbitrary “numerical magnitudes” thought of as infinite aggregates of fractions $\frac{1}{n}$. For both finite and infinite aggregates, an order relation and an equality relation (as equivalence classes for infinite aggregates) can be defined.

Having a rigorous foundation for irrational numbers allowed Weierstrass to develop analytic functions on a rigorous basis since, with a rigorous definition of irrational numbers, he was able to provide appropriate proofs of propositions about limits of number and function sequences.

The approach adopted by Weierstrass is said to be “purely arithmetical” because it starts from the rational numbers themselves, rather than being grounded in the “science of quantity” or in geometry. Weierstrass’s rigorization and arithmetization of the foundation of Analysis had a great influence on many of his students, among which Cantor.

7.1.2.1 Definition of irrationals as limits by Cantor

Cantor, in 1872, defined irrational numbers as limits of what we call Cauchy sequences. This definition of irrationals was known by Dedekind through its first use by Heine in 1872, who was openly inspired by Cantor. Cantor’s definition of the irrationals was published a few months later, and Dedekind only read it when he was writing the introduction for *Stetigkeit und irrationale Zahlen*, as he explains himself.

Cantor’s theory of irrational magnitudes is first published in his researches on trigonometric series in 1872, “*Über die Ausdehnung eines Satzes aus der trigonometrischen Reihen*” ([Cantor, 1872]).¹³

The 1872 paper presents the definition of irrational magnitudes as essentially a way to highlight the researches on trigonometric series in the rest of the paper.¹⁴ Cantor’s definition, in 1872, is necessary for the result he aims to prove. Like Dedekind, Cantor underlines that his definition could be a good basis for the development of infinitesimal analysis.

¹³A more complete version with a comparison with Dedekind’s and Weierstrass’s theories is published in 1883, in *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, which I will not consider in detail here. Expositions of Cantor’s approach in 1872 and 1883 can be found most of the studies cited at the beginning of the section. [Epple, 2003], [Jahnke and Otte, 1981] and [Boniface, 2002] have particularly informed my reading of Cantor.

¹⁴In 1883, the definition holds a fundamental place in the theory. It is presented as aiming at a generalization of the concept of integral number, so that it can pave the way to a generalization to transfinite numbers.

When one defines the irrational magnitudes as limits of a number sequence, there is always a risk, of which Cantor is perfectly aware, that one would run into a vicious circle and presuppose the existence of the said irrationals whose existence one wishes to prove. Cantor uses a formal approach and designate the limit as a symbol, a certain “determinate limit b ” which will be defined subsequently.

Cantor considers the rational numbers as given and wants to develop a “larger concept of numerical magnitude”. These numerical magnitudes are defined by considering a sequence of rational numbers a_n such that for any natural number m and for any positive (rational) number ε , when n tends to infinity:

$$|a_{n+m} - a_n| < \varepsilon$$

(i.e. a_n is a Cauchy sequence). Cantor followed Weierstrass’s lectures in Berlin, in which the latter gave a sound basis to the notion of limit, with the definition of infinitely small variations of a variable by means of δ and ε .

I express this property of the sequence $[(a_n)]$ by the words: “The sequence $[(a_n)]$ has a determinate limit b .”

These words thus do not have, at first, any other sense than to be expression of this property of the sequence, and from the fact that we relate the sequence $[(a_n)]$ to a *particular sign* b , it follows that one can form, for distinct sequences of this kind, distinct symbols (signs?) b, b', b'', \dots ¹⁵ ([Cantor, 1872], 93)

It is possible to compare sequences, for example for two sequences (a_n) and (a'_n) with limits b and b' , there are three mutually exclusive possibilities:

- (1) $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a'_n| < \varepsilon$ [i.e. the sequences are equivalent]
- (2) $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a'_n| > \varepsilon$ [i.e. $(a_n) > (a'_n)$]
- (3) $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a'_n| < -\varepsilon$ [i.e. $(a_n) < (a'_n)$]

These three possibilities thus allow to define an order for the limits themselves:

- (1) means that $b = b'$
- (2) means that $b > b'$
- (3) means that $b < b'$

Hence, the order between the magnitudes b is defined by the order between the general terms of the sequences.

¹⁵ “Diese Beschaffenheit der Reihe $[(a_n)]$ drücke ich in den Worten aus: ‘Die Reihe $[(a_n)]$ hat eine bestimmte Grenze b .’

Es haben also diese Worte zunächst keinen anderen Sinn als den eines Ausdruckes für jene Beschaffenheit der Reihe, und aus dem Umstande, daß wir mit der Reihe $[(a_n)]$ ein besonderes Zeichen b verbinden, folgt, daß bei verschiedenen derartigen Reihen auch verschiedene Zeichen b, b', b'', \dots zu bilden sind.”

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The same can be done for numerical magnitudes b and rational numbers a by defining a relation between the general term of the sequence and the rational number a .¹⁶ This allows to make clear that any rational number can also be defined by a Cauchy sequence. This is an important point, because it legitimates the use of the term “limit” for the magnitudes b : from (1') follows that if b is the limit of a_n , then $b - a_n$ “becomes infinitely small when n increases.”

In addition, it allows to extend the elementary arithmetical operations of rational numbers to the “numerical magnitudes b ”, which plays an essential role in giving them the said legitimacy. Everything is given using the general terms of the sequences of which the magnitudes b are limits:

(i) $b \pm b' = b''$ means that $\lim(a_n \pm a'_n - a''_n) = 0$

(ii) $bb' = b''$ means that $\lim(a_n a'_n - a''_n) = 0$

(iii) $\frac{b}{b'} = b''$ means that $\lim(\frac{a_n}{a'_n} - a''_n) = 0$

One can define the operations for a “numerical magnitude” and a rational number using these relations and the relation (1') giving the equality between a rational numbers and a magnitude b .

Finally, it is possible to generalize these results to any finite number of elementary operations, that is, to define a rational function of “numerical magnitudes”, which is thus a relation between the sequences of which the b are limits.

The collection of “numerical magnitudes” b forms a set B that can be adjoined to the set A of rational numbers and which has been “generated by A ”. It is possible to reiterate the operation by considering the set constituted of A and B and the sequences “of numerical magnitudes of the domains A and B ” which are not all in A and verify the Cauchy criteria. These sequences have a limit c , and form a set C . In C , one can give the magnitudes c the same treatment as the magnitudes b , and (re)define the equality, the order and the elementary operations. The process can be repeated as much as wanted and obtain sets of different “species” according to whether they have been generated by a number sequence or by a sequence composed of numbers and limits of sequences (which are considered as sequences of order 2 for C , and so on). The number in each set are all numerical magnitudes of the same nature but are thought of by Cantor as conceptually different.¹⁷

Cantor explains that when going from the set A to the set B , one obtains a larger set (A is included in B), but when going from the set B to the set

¹⁶That is, $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a| < \varepsilon$ means that $b = a$

$\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a| > \varepsilon$ means that $b > a$.

And $\forall \varepsilon > 0, \forall n \in \mathbb{N}, |a_n - a| < -\varepsilon$ which means that $b < a$.

¹⁷For Cantor, this conceptual difference is essential, because it allows to “acquire an extraordinarily flexible and at the same time intelligible idiom for describing in the simplest and most significant way the richness of the protean and often complicated webs of analysis. One also gains clarity and lucidity, which, in my opinion, is not to be undervalued.” ([Cantor, 1883], 901) It is, for him, the main superiority of his theory over Dedekind’s.

C , one obtains the same set. In a sense, Cantor writes, “the domains $[B$ and $C]$ coincide”, whereas B contains real numbers that are not in A . In short, then, and even if the notion is not very clear or explicit in Cantor’s text, B is complete, contrary to A .

Cantor, like Dedekind, underlines the fruitfulness of his approach (with an allusion to transfinite numbers):

All the results of Analysis (with the exception a few known cases) can be reduced to the form of such equalities, even if (which can here only be alluded to with regards to these exceptions) the concept of number, as far as it is developed here, carries the germs of a extension in itself necessary and absolutely infinite.¹⁸
(*ibid.*, 95)

For example, if one considers the equality $\sin(\frac{\pi}{2}) = 1$, it is possible to show that $\sin(\frac{\pi}{2})$ is determined by the fundamental sequence of second order (since $\frac{\pi}{2}$ is itself written as a convergent series and so is \sin) and the equality $\sin(\frac{\pi}{2}) = 1$ is about the relation between a rational number and a number defined as a sequence of second order.

Cantor briefly considers an analogy with the line, which for him allows to give a certain “concreteness” (*Gegenständlichkeit*) to the “numerical magnitudes” which only have been defined formally. The idea is the following: one establishes a correspondence between the points of the line and the numerical magnitudes with a notion of distance, so that to any point corresponds a numerical magnitude. The converse correspondence (i.e. to any numerical magnitude corresponds a point) is taken as an axiom. Thus, one gains “a certain concreteness of the numerical magnitudes, from which they are nevertheless completely independent.”

Note that Cantor does not define continuity itself in 1872. He does define it in 1883. Cantor explains that he wants to develop continuity “in the briefest way possible” and only using set theory. Starting from the notion of a dense set, Cantor proposes the notion of “connected” (*zusammenhängenden*) set, which amounts to a condition of (what we nowadays call) connectedness and, thus is a metrical characterization of continuity.

For Dedekind, the characterization of continuity comes with the definition of irrational numbers and ought to be done by purely arithmetical ways. In particular, one should avoid the use of notions foreign to arithmetic, i.e. external to the systems of numbers and the elementary operations, especially insofar as they are, like limits or metrics, related to continuity itself

¹⁸ “Auf die Form solcher Gleichsetzungen lassen sich die Resultate der Analysis (abgesehen von wenigen bekannten Fällen) zurückführen, obgleich (was hier nur mit Rücksicht auf jene Ausnahmen berührt sein mag) der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt.”

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and will benefit from a rigorous, independent foundation as the one provided by Dedekind.

7.1.3 Kronecker's "arithmetization"

Kronecker's conception of number (published in 1887) is notoriously very different from Dedekind's, even though they both consider arithmetic as being at the basis of mathematics, displaying two very different sorts of what could be called "arithmetism".¹⁹ By exposing Kronecker's ideas, I hope that the contrast can highlight the peculiar role of natural numbers and arithmetic in Dedekind's mathematics. As the one who coined the term "to arithmetize" and a typical representative of arithmetization construed in a reductionist way, it will be useful to consider Kronecker's approach so as to contrast it with Dedekind.

Kronecker's idea of "arithmetization", introduced in 1887 in *Über den Zahlbegriff* is rather different from the use made by later Klein since for Kronecker, the arithmetical form of the argument is its very core. In the 1887 paper and in his 1891 lecture on the same subject, in Berlin, Kronecker develops his ideas about the concept of number and its place and role in mathematics. The important dissimilarities with Dedekind's approach should allow to understand better, by contrast, Dedekind's approach. Dedekind's and Kronecker's works were rooted in "Arithmetic Algebraic Analysis", following the works of the same mathematicians. This draws them to investigate similar questions, but the strong (and strongly opposed) epistemological demands which shape their respective methods lead them to very different solutions despite their common inspirations.

For Kronecker, mathematics is a natural science, or an "experimental science", in the following sense:

Mathematics is to be handled as a natural science, because its objects (*Gegenstände*) are as real as those of its sister sciences.²⁰
([Kronecker, 1891], 232)

His position is inspired by Kirchhoff, as he explains:

It has often been said, that mathematics must begin with definitions, and the propositions mathematics must be deduced from

¹⁹[Boniface, 2005], [Boniface, 2004] are the main sources for Kronecker's concept of number. The introductions of [Boniface, 1999] and [Kronecker, 1891] also provide useful details. One can also refer to [Vlăduț, 1991] and [Smadja, 2002].

²⁰"*Die Mathematik ist wie eine Naturwissenschaft zu behandeln, denn ihre Gegenstände sind ebenso wirklich wie diejenigen ihrer Schwesterwissenschaft.*" The references for Kronecker 1891 lecture are to Boniface and Schappacher's edition of the manuscript, in [Kronecker, 1891].

those definitions together with the postulated fundamental principles. But the definitions are already, in themselves, an impossibility, for, as Kirchhoff used to say, every definition needs its own concepts, which needs again to be defined, and so forth.²¹ (*ibid.*, 225)

Therefore, any attempt to give a logical or philosophical foundation (doomed to be imprecise) to mathematics is, for Kronecker, a mistake. Kronecker considers that *phenomena* should be the basis of mathematics, rather than definitions. In the statements alluded to by Kronecker, Kirchhoff was considering mechanics and other natural sciences, which were supposed to describe “simply and completely” the phenomena observed. In the case of mathematics, Kronecker explains that mathematicians should “describe the phenomena simply and completely” in the same way. Hence, phenomena yield the basic concepts, and the principles are given by experience (and can be modified during the course of the scientific developments). The famous statement that “God created the integers, all the rest is the work of man” should, thus, be understood in this light. Indeed, the natural numbers are, for Kronecker, the only mathematical objects consistent with the mathematical experience, they are therefore the basis of mathematics and do not *need* to be defined: natural numbers are given, and are the only given objects of mathematics, everything else has to be defined on this basis.

The essential difference between mathematics and other natural sciences is the nature of the reality of their respective objects. Indeed, to state that mathematics should be treated as a natural science does not amount to the adoption of an empirical viewpoint, for Kronecker, and it certainly does not amount to consider mathematics as anything else than a pure science.

In fact, arithmetic is in the same correlation with the two other mathematical disciplines, geometry and mechanics, as the entirety of the mathematical sciences is to astronomy and the other natural sciences; arithmetic also renders diverse services to geometry and mechanics, and receives in return from its sister disciplines an abundance of stimulations.²² (*ibid.*, 252-253)

On this matter, Kronecker quotes a famous statement by Gauss:

²¹ “Man hat häufig gesagt, die Mathematik müßte mit Definitionen beginnen, und aus ihnen zusammen mit den postulierten Grundsätzen seien die mathematischen Sätze abzuleiten. Nun sind aber Definitionen an sich schon eine Unmöglichkeit, wie Kirchhoff zu sagen pflegte, denn jede Definition braucht ihre Begriffe, welche wieder zu definieren sind u.s.f.”

²² “In der That steht die Arithmetik in ähnlicher Beziehung zu den anderen beiden mathematischen Disciplinen, der Geometrie und Mechanik, wie die gesammte Mathematik zur Astronomie und den anderen Naturwissenschaften; auch die Arithmetik erweist der Geometrie und Mechanik mannigfache Dienste und empfängt dagegen von ihren Schwester-Disziplinen eine Fülle von Anregungen.”

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Mathematics is the queen of the sciences and arithmetic is the queen of mathematics. From time to time it condescends to pay service to astronomy and other natural sciences, but it ranks first in all circumstances.²³ (*ibid.*, 252)

However, Kronecker does not adopt Gauss's position: he does not consider mathematical objects as abstract, but as resulting from mathematical experience. The objects of mathematics are "mathematical phenomena" (e.g. computations, concrete algebraic expressions), which are thus neither in nature nor in the human mind.

This position on mathematics implies a very strong realism, for the objects of mathematics are considered as real as the objects of any natural science and, for Kronecker:

That [the objects of mathematics are real] is true can be felt by any person who talks about mathematical 'discoveries', and not about mathematical 'inventions' (*Erfindungen*), for only what already really exists can be discovered, and only what is produced by the human mind is called 'invention'. Therefore, mathematicians 'discover' results via the methods they 'invent' for these purposes.²⁴ (*ibid.*, 232-233)

If the mathematician is free to invent, then, it is only to invent new methods of discovery – certainly not new concepts, as advocated for by Dedekind (or Cantor). Kronecker's realism is, as is well known, extremely restrictive and only accepts positive integers in arithmetic. Because of this restriction, at the core of his project of a General Arithmetic, Kronecker is led to develop an arsenal of methods allowing him to only use natural numbers. The objects of mathematics, then, as "mathematical phenomena", take the form of concrete algebraic expressions, which exist neither in nature nor in the human mind. In particular, they are not abstract objects, unlike what Gauss or Dedekind would state.

For all these reasons, Kronecker is not interested in finding basic principles, fundamental concepts, ultimate premisses, as did Dedekind, Frege, or Hilbert. A foundation of mathematics can only be built, for Kronecker, in the mathematics themselves. A logical or philosophical foundation cannot be accepted. This idea of a purely mathematical foundation is linked

²³ "Die Mathematik ist die Königin der Wissenschaften und die Arithmetik die Königin der Mathematik. Diese lasse sich dann öfter herab, der Astronomie und anderen Naturwissenschaften einen Dienst zu erweisen, doch gebühre ihr unter allen Verhältnissen der erste Rang."

²⁴ "Daß dem so ist, fühlt ein jeder, der von mathematischen 'Entdeckungen', nicht aber von mathematischen 'Erfindungen' spricht. Denn entdeckt kann doch nur dasjenige werden, was bereits wirklich existiert; was aber der menschliche Geist aus sich hervorbringt, das heißt 'Erfindung'. Daher 'entdeckt' der Mathematiker die Resultate durch Methoden, welche er zu diesem Behufe 'erfunden' hat."

to a demand of mutual airtightness of the disciplines.²⁵ One important consequence of this requirement is the status of irrational magnitudes for Kronecker. Indeed, from his viewpoint, irrationals are part of geometry – especially transcendental irrationals – and have no right to be considered as part of arithmetic.

In [Kronecker, 1887], Kronecker proposes a project of arithmetization which, unlike the works which were presented earlier, is a project which aims ultimately at a reduction of all (pure) mathematics to arithmetic. Here, by “arithmetic”, one should understand the “general arithmetic” mentioned above:

The word “arithmetic” must not be understood in its conventional limited sense, though, rather one must understand all the mathematical disciplines, with the exception of geometry and mechanics, hence specifically algebra and analysis.²⁶ (*ibid.*, 253)

Kronecker’s general arithmetic encompasses the whole of pure mathematics, and his program of arithmetization relies on the desire to give back its true meaning to the concept of number, to retrieve its true nature: the natural integer. Pure mathematics should, then, be developed on this sole basis:

And I also believe, that one day we will be able to “arithmetize” the entire content of the mathematical disciplines, that is, to ground them on the concept of number taken in its narrowest sense, and hence to peel off all the modifications and extensions of this concept, which were most often prompted by applications to geometry and mechanics.²⁷ (*ibid.*, 253)

Kronecker insists on the importance to avoid the gratuitous introduction of new concepts, and to respect a principle of ontological economy: only the positive integers are acceptable. However, negative integers and rational numbers can be useful in mathematics – as long as they are defined on

²⁵The demand that mathematics be founded in itself is linked to the idea that scientific disciplines ought to be clearly separated. In particular, one should not try to find a foundation for a discipline in another discipline. This does not mean that all applications are bad, but that to apply, for example, mathematics to astronomy, one should adapt the concepts considered. In particular, non-algebraic irrational numbers are, for Kronecker, geometrical concepts and therefore do not have a place in arithmetic.

²⁶“Dabei ist aber das Wort ‘Arithmetik’ nicht in dem üblichen beschränkten Sinne zu verstehen, sondern es sind alle mathematischen Disciplinen mit Ausnahme der Geometrie und Mechanik, also namentlich die Algebra und Analysis, mit darunter zu begreifen.”

²⁷“Und ich glaube auch, dass es dereinst gelingen wird, den gesamten Inhalt aller dieser mathematischen Disciplinen zu ‘arithmetisieren’ d. h. einzig und allein auf den im engsten Sinne genommenen Zahlbegriff zu gründen, also die Modificationen und Erweiterungen dieses Begriffs wieder abzustreifen, welche zumeist durch die Anwendungen auf die Geometrie und Mechanik veranlasst worden sind.”

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the basis of natural numbers only and in a purely arithmetical way. Hence, rather than introducing new concepts, one replaces negative and fractional numbers by congruences. The fact that it is possible to replace negative and fractional numbers by congruences is, for Kronecker, evidence that it is unnecessary to introduce new concepts. The negative and fractional quantities will be called magnitudes rather than numbers. Kronecker, in 1887, does consider algebraic irrational magnitudes, transcendentals are however clearly excluded from arithmetic.

The introduction of new tools, new methods, providing the “mathematical phenomena” to investigate should respect Kronecker’s requirements of constructivity and finitism:

The point of view on which I disagree with most mathematicians resides in the basic assertion that mathematics and the natural sciences — which have recently been separated by this name from the remaining sciences, the so-called sciences of the mind (*Geisteswissenschaften*) — must not only be free of contradiction, but must also result from experience and, what is even more essential, must dispose of a criterium by which one can decide, for each particular case, whether the presented concept is to subsume, or not, under the definition. A definition which does not achieve this, can be advocated by philosophers or logicians, but for us mathematicians, it is a bad nominal definition. It is worthless.²⁸ (*ibid.*, 240, transl. in [Boniface, 2005], 149)

In particular, a proof of existence is only rigorous and valid if it allows to explicitly exhibit, to construct the object whose existence is proved. The demands of constructivity and finitism are guiding the elaboration of methods and algorithms for Kronecker’s mathematics. Algebraic expressions have, for Kronecker, an arithmetical existence, although they are not numbers, and are integrated in what he calls “General(ized) arithmetic”. For Kronecker’s mathematics, the concrete algebraic expressions take a greater importance than the result obtained.

²⁸ “*Der Standpunkt, welcher mich von vielen andern Mathematikern trennt, gipfelt in dem Grundsatz, dass die Definitionen der Erfahrungswissenschaften, — d.h. der Mathematik und der Naturwissenschaften, welche man neuerdings unter jenem Namen von den übrigen Wissenschaften, den sondern Geisteswissenschaften trennt, — nicht bloss in sich widerspruchsfrei sein müssen, sondern auch der Erfahrung entnommen sein müssen, und was noch wesentlicher ist, das Kriterium mit sich führen müssen, durch welches man für jeden speziellen Fall entscheiden kann, ob der vorliegende Begriff unter die Definition zu subsumieren ist, oder nicht. Eine Definition, welche dies nicht leistet, mag von Philosophen oder Logikern gepriesen werden, für uns Mathematiker ist sie eine bloße Wortdefinition und ohne jeden Wert.*”

7.1.3.1 The number concept

According to Kronecker, the number concept is linked to counting (*zählen*) and must stay as such. The starting point for a characterization of the concept of number are the ordinal numbers:

I find the natural starting-point for the development of the number concept in the *ordinal numbers*. With them, we possess certainly a supply of denotations ordered according to a fixed succession, which we can attach to a collection (*Schaar*) of different objects and for us distinguishable at the same time.²⁹ (*ibid.*, 253-254)

The “supply of denotations”, Kronecker explains, is “always sufficient” because it is an “ideal” (*ideelle*) supply, and the laws of formations of numbers allow us to have the “capacity” to answer the need of an inexhaustible supply. These “*Bezeichnungen*” are called “quantities of objects”.

The “totality of the denotations used” (*Gesammtheit der verwendeten Bezeichnungen*) is assembled into the concept of “quantity of objects” which constitute the collection. The “expression of this concept” is linked to the “denotation” of the last element – which is always possible since the elements in the collection have been ordered.

Hence, for example, in the collection of letters (a, b, c, d, e), one can attribute the denotation “first” to the letter a , the denotation “second” to the letter b , etc., and finally the designation “fifth” to the letter e . The totality of the ordinal numbers used here, or the “number” of letters a, b, c, d, e be denoted accordingly in relation to the last ordinal number used, by the number “five”.³⁰ (*ibid.*, 254)

Cardinal numbers are, for Kronecker, numbers properly speaking and are defined by considering the “collection” (*Schaar*) formed by “a given ordinal (the n th) and all the numbers preceding it”. The “quantity” of these numbers will be expressed by the “cardinal number n ” which “corresponds to the n th ordinal number”. And these cardinal numbers are the ones which are simply called “numbers”.

²⁹ “Die naturgemässen Ausgangspunkt für die Entwicklung des Zahlenbegriffs finde ich in den Ordnungszahlen. In diesen besitzen wir einen Vorrath gewisser, nach einer festen Reihenfolge geordneter Bezeichnungen, welche wir einer Schaar verschiedener und zugleich für uns unterscheidbarer Objecte beilegen können.”

³⁰ “So kann z. B. in der Schaar der Buchstaben (a, b, c, d, e) dem Buchstaben a die Bezeichnung als ‘erster’, dem Buchstaben b die Bezeichnung als ‘zweiter’ u. s. f. und endlich dem Buchstaben e die Bezeichnung als ‘fünfter’ beigelegt werden. Die Gesammtheit der dabei verwendeten Ordnungszahlen oder die ‘Anzahl’ der Buchstaben a, b, c, d, e kann demgemäss in Anknüpfung an die letzte der verwendeten Ordnungszahlen durch die Zahl ‘Fünf’ bezeichnet werden.”

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One can define an order relation for the numbers: a number m is “less” than a number n if the “ordinal number belonging to m precedes the ordinal number belonging to n ”. Then,

The so-called sequence of natural numbers is nothing else than the sequence of the correspondent ordinal numbers.³¹ (*ibid.*, 255)

The counting of objects is tantamount to giving them a certain order. And, of course, the quantity of numbers is independent of the order in which the objects are counted: if the elements are permuted and re-ordered differently, the “quantity” of objects is unchanged – “the result of counting is independent of the order followed in the counting.”

The “number” of objects of a collection is thus a property of the collection as such, that is, of the totality of objects thought independently from any determined order.³² (*ibid.*, 256)

So as to characterize more precisely the notion of cardinal number, Kronecker uses the notion of equivalence (*Aequivalenz*) and invariant (*Invariante*) introduced by Gauss (since arithmetic can only be founded on itself). Kronecker calls two systems “equivalent” if (in modern terms) there is a one-to-one correspondence between them:

But if now any two systems (a, b, c, d, \dots) , (a', b', c', d', \dots) are said to be “equivalent” as soon as it is possible to transform one into the other by successively replacing each element of the first system by an element of the second system, then the necessary and sufficient condition of the equivalence of these two systems is the equality of the number of their elements, and the number of elements of a system (a, b, c, d, \dots) is characterized hereafter as the only “invariant” of all these mutually equivalent systems.³³ (*ibid.*, 256)

The notion of “invariant”, which comes from the theory of algebraic forms, allows to avoid the use of the notion of equivalence classes. The cardinal number, i.e. the quantity of elements, as an invariant is a representative of

³¹ “Die sogenannte natürliche Reihenfolge der Zahlen ist nichts Anderes als die Reihenfolge der entsprechenden Ordnungszahlen.”

³² “Die ‘Anzahl’ der Objecte einer Schaar ist also eine Eigenschaft der Schaar als solcher, d. h. der unabhängig von irgend einer bestimmten Anordnung gedachten Gesammtheit der Objecte.”

³³ “Wenn nun aber irgend zwei Systeme (a, b, c, d, \dots) , (a', b', c', d', \dots) ‘äquivalent’ genannt werden, sobald es möglich ist, das eine in das andere dadurch zu transformiren, daß man der Reihe nach jedes Element des ersten Systems durch je eines des zweiten Systems ersetzt, so besteht die nothwendige und hinreichende Bedingung für die Aequivalenz zweier Systeme in der Gleichheit der Anzahl ihrer Elemente, und die Anzahl der Elemente eines Systems (a, b, c, d, \dots) charakterisirt sich hiernach als die einzige ‘Invariante’ aller untereinander äquivalenten Systeme.”

the equivalence class, which provides a more concrete basis to the number concept – and in particular does not require the consideration of infinite collections, which Kronecker considers to be too abstract.

Operations between numbers are defined in a rather traditional way, and show how Kronecker takes the mathematical experience to be at the ground of arithmetic. To define addition, one “takes numbers themselves as the object of counting”: one can “keep counting n_2 from $n_1 + 1$ ”:

We can, then, for example, keep counting n_2 further from the number $n_1 + 1$, that is, to collect exactly as much numbers following immediately n_1 that are needed to form a collection whose number is n_2 . This “keep counting” is called “to add the numbers n_2 to the numbers n_1 ” and these numbers s to which one arrives by this “keep counting” is called “the result of the addition” or “the sum of n_1 and n_2 and is represented by $n_1 + n_2$.³⁴ (*ibid.*, 257)

The commutativity of addition is immediate, with this definition. Kronecker proves the commutativity of addition for any number of numbers, by considering the equality $n_1 + n_2 + \dots + n_r = n_\alpha + n_\beta + \dots + n_\rho$ and using permutations of indexes.

The multiplication consists simply in considering “all the terms $n_1 + n_2 + \dots + n_r$ to be equal to a same number n ”, and this peculiar addition will be called “the multiplication of the numbers n by the number r ” and denoted by $n_1 + n_2 + \dots + n_r = rn$. For the commutativity of multiplication, Kronecker also resorts to permutations.

Kronecker then immediately goes on to introduce the “literal computations” (*Buchstabenrechnung*) which he proposes to use in place of negative, rational and algebraic numbers.

7.1.3.2 Negative and rational numbers

For Kronecker, if one were to enlarge the concept of number to negative and rational numbers, relatively to subtraction and division, then the concept would be devaluated, denatured. To assure the possibility of applying operations without restriction, Kronecker turns towards literal calculus. With the introduction of an indeterminate variable, one can consider the “general arithmetic of indeterminate functions with integral coefficients” and through

³⁴ “Man kann also z. B. von der Zahl $n_1 + 1$ an um n_2 weiter zählen, d. h. genau so viele von den auf die Zahl n_1 zunächst folgenden Zahlen zu einer Schaar zusammenfassen, dass deren Anzahl n_2 , beträgt. Dieses ‘weiter Zählen’ heisst: ‘zur Zahl n_1 die Zahl n_2 addiren’, und diejenige Zahl s , zu welcher man bei jenem weiter Zählen gelangt, heisst das ‘Resultat der Addition’ oder die ‘Summe von n_1 und n_2 und wird durch $n_1 + n_2$, dargestellt.”

7.1 Arithmetization of Analysis in the 19th century

this, “avoid all concepts foreign to the pure arithmetic: negative, fractional, real numbers, and imaginary algebraic numbers”.³⁵ As Boniface explains, the use of indeterminate variables is, for Kronecker, “a very fruitful purely arithmetical method, which exempts from resorting to the concepts” of negative and rational numbers ([Boniface, 2005], 147). Algebraic expressions, though not numbers properly speaking, are integrated into what Kronecker calls “general arithmetic”.

To understand the link between this and the previous statements about Kronecker’s strong ontological position, it should be underlined that the philosophical position, here, cannot be detached from the mathematical methods that he develops to support it. As was explained earlier,³⁶ Kronecker’s philosophy of mathematics is effectively developed by the elaboration of methods, algorithms, concrete algebraic expressions, and explicit modes of representation of the mathematical content, which yield the arithmetical meaning. Kronecker’s rejection of the irrational numbers is related to these ideas: because the computations play such a fundamental role, irrational numbers are just a step in the computation, a symbol to which no operational meaning can be attached, which forbids to consider them as numbers.

This must also be linked to Kronecker’s demands of constructivity: a proof of existence is only valid if it allows to exhibit, to construct the object in question, for if mathematics is resulting from experience, any definition (or proof) should allow the mathematician to decide, in each case, whether or not a certain given concept satisfy the said definition. These demands guide the elaborations of the algorithms. Thus, for Kronecker, concepts are the results of computations and the concrete expression of the operations made is more important than the result obtained.

The idea, then, is to replace the negative or fractional numbers by congruences. The computation, the concrete algebraic expression completely takes the place of an enlarged, faulty concept of number. For example, for the introduction of negative numbers, Kronecker proposes the following solution.

³⁵ *“Diese allgemeine Theorie gestattet alle der eigentlichen Arithmetik fremden Begriffe, den der negativen, der gebrochenen, der reellen und der imaginären algebraischen Zahlen, auszuscheiden.”*

³⁶ See p. 309 and the quotation by Vlăduț: “Kronecker’s views are quintessentially expressed in two ideas. The first is the rejection of actual infinity as mathematical reality. The second is also prohibitive in nature and can best be expressed by his well-known dictum “God Himself made the integers – everything else is the work of men”. This leads to the prohibition against using mathematical notions and methods which cannot be reduced to the arithmetic of integers. Despite their prohibitive form, these statements carry a powerful positive charge. (...) The desire to argue “arithmetically” influenced Kronecker’s works, many of which were based on the method of computation of some mathematical objects.” ([Vlăduț, 1991], 11-12)

Consider $(a - b) + b = a$. In this expression, there is no subtraction actually carried out, the negative factor “-1” can be replaced by an indeterminate x . One can thus write: $(a + xb) + b = a$, in which one looks at x as being a numerical magnitude given by the equation $x + 1 = 0$. Considered in terms of congruences, $x + 1$ is everywhere equal to 0 can be written as $x + 1 \equiv 0 \pmod{x + 1}$. Thus, $(a + xb) + b = a$ can be written as $(a + xb) + b \equiv a \pmod{x + 1}$. Kronecker uses the example $7 - 9 = 3 - 5$ which is changed into $7 + 9x \equiv 3 + 5x \pmod{x + 1}$, in which $7 - 9 = 3 - 5$ is a fictive equation, and $7 + 9x \equiv 3 + 5x \pmod{x + 1}$ is an actual equality. Moreover, according to Kronecker, one “gains in content” by doing so, because, here, the algebraic expression (congruence) is valid for any positive integer x : the division of $7 + 9x$ by $x + 1$ gives the same remainder as the division of $3 + 5x$ by $x + 1$.³⁷

The idea for the definition of rational numbers is similar: rather than using a factor $\frac{1}{m}$, Kronecker uses an indeterminate x_m verifying that

$$m \cdot \frac{x}{m} = a \iff (ax_m)m = a$$

in which x_m is such that $mx_m - 1 \equiv 0 \pmod{mx_m - 1}$, hence

$$(ax_m)m \equiv a \pmod{mx_m - 1}.$$

To properly define the fractional numbers, it is necessary to define the rules of computations, which are given by replacing the three equalities

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}$$

$$\frac{a}{m} \cdot \frac{b}{n} = \frac{ab}{mn}$$

$$\frac{a}{m} : \frac{b}{n} = \frac{an}{bm}$$

with the three congruences³⁸

$$ax_m + bx_n \equiv (an + bm)x_mx_n \pmod{mx_m - 1, nx_n - 1, mnx_mn - 1}$$

$$ax_m bx_n \equiv abx_{mn} \pmod{mx_m - 1, nx_n - 1, mnx_mn - 1}$$

$$ax_m x_{bx_n} \equiv anx_{bm} \pmod{mx_m - 1, nx_n - 1, bmx_{bm}x_{bx_n} - 1}.$$

For irrational numbers, the issue is more delicate. Indeed, from Kronecker’s viewpoint, irrationals not only do not have an arithmetical existence, they are not authorized in the realm of arithmetic at all. This is

³⁷Note that this idea (and the following ones) are very peculiar to Kronecker. Gauss did not exclude negative numbers, and the mathematician Hermann Schubert who wrote a textbook (*System der Arithmetik und Algebra*) referenced by Kronecker as having served as an inspiration for his own work (*ibid.*, footnote, 262) did not use congruences.

³⁸ $a \equiv b \pmod{m, n} \iff a - b = km + ln$

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so especially because there is no constructive procedure involving a finite number of arithmetical operations from natural numbers that would be able to define irrational numbers.

In the case of algebraic irrational magnitudes, Kronecker proposes a method to isolate the roots of polynomials, and it is the only case for which the introduction of irrational magnitudes is considered as anything else than superfluous. This can fully be handled without introducing, in the algebraic expressions, anything else than natural numbers.

I have shown the fact that the introduction and application of algebraic numbers is dispensable in all the cases where the isolation of the reciprocal conjugates is not necessary, in a previous work [*Ein Fundamentalsatz der allgemeinen Arithmetik*]. The fact that this isolation can itself happen without the introduction of new concepts, and that it is only if it happens in this way that it clearly puts to the fore the essence of the matter, it is what we are going to explain here, in the same way as I have been used to doing it for ten years in my university lectures, and we will also give with this the “more precise analysis of the concept of real roots of algebraic equations.”³⁹ (*ibid.*, 262-263)

Kronecker's project of a “general arithmetic”, whose very first elements were given above, are, for him, nothing else but “a gathering of results obtained by giving integral values to the indeterminates”. Because of this, the results of the general arithmetic are to be considered as part of the usual number theory, and “all the results of the deepest mathematical research must eventually be expressed in these simple forms using properties of the integers”. The definitions of the negative and fractional magnitudes as algebraic congruences are given as an “appropriate and clear way to express and represent the numbers”.

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Dedekind states, in the introduction of *Stetigkeit*, that while he “fully agrees” with the “content” (*Inhalt*) of Heine's definition⁴⁰ of the irrational numbers,

³⁹ “Dass die Einführung und Verwendung der algebraischen Zahlen überall da entbehrlich ist, wo nicht die Isolirung der unter einander conjugirten erfordert wird, habe ich in einem früheren Aufsätze gezeigt [*Ein Fundamentalsatz der allgemeinen Arithmetik*]; dass diese Isolirung selbst aber auch ohne Einführung neuer Begriffe geschehen kann und nur dann, wenn sie so geschieht, das Wesen der Sache klar hervortreten lässt, soll hier in derselben Weise, wie ich es seit zehn Jahren in meinen Universitätsvorlesungen zu thun pflege, dargelegt und damit zugleich jene ‘genauere Analyse des Begriffs der reellen Wurzeln algebraischer Gleichungen’ gegeben werden.”

⁴⁰Dedekind only published *Stetigkeit* after receiving Heine's paper in which Cantor's definition is used. Before that, Dedekind hesitated about publishing his definition because

as it “could hardly be otherwise”, he considers his own “presentation (...) to be simpler in form and to bring out the vital point more clearly.”⁴¹ In 1872, Dedekind had not read Cantor’s or Weierstrass’s theories, but he wrote in 1888, in the preface of *Zahlen*, that he considered both theories to be “perfectly rigorous”, his own theory being essentially “simpler, smoother” (*ruhiger*).⁴²

7.2.1 A rigorous and arithmetical foundation for continuity

It is when teaching differential calculus in 1858 in Zürich, that Dedekind came to the realization that an arithmetical and rigorous ground was needed for real numbers.

I felt more keenly than ever before the lack of a truly scientific foundation for arithmetic.⁴³ ([Dedekind, 1872], 767)

At first, Dedekind was tempted to find “refuge” in “geometrical evidences”, but while geometrical intuition can be useful in teaching and can indeed play an important heuristic role, for “no one will deny that this form of introduction into the differential calculus can make no claim to being scientific”⁴⁴ (*ibid.*, 767).

Dedekind wrote in the introduction to *Stetigkeit* that the “feeling of dissatisfaction” was so “overpowering” in 1858, that he decided to find a way to find “a purely arithmetical and perfectly rigorous foundation (*Begründung*) for the principles of infinitesimal analysis.” The theorem that particularly arose Dedekind’s dissatisfaction about the foundation of infinitesimal analysis is that of the existence of a limit for “every magnitude which grows continually, but not beyond all limits.” Indeed, as it was the case for Cantor and Weierstrass, Dedekind wished to have a rigorous definition of the

“the presentation is not altogether easy, and (...) the theory itself is not very fruitful” ([Dedekind, 1872], 767).

⁴¹ “*Dem Wesen nach stimme ich zwar vollständig mit dem Inhalte dieser Schrift überein, wie es ja nicht anders sein kann, aber ich will freimütig gestehen, daß meine Darstellung mir der Form nach einfacher zu sein und den eigentlichen Kernpunkt präziser hervorzuheben scheint.*”

⁴² Like Cantor’s definition of irrational numbers, Dedekind’s cuts are explained in the works on the arithmetization of mathematics cited at the beginning of the chapter. On Dedekind more specifically, one can refer to [Dugac, 1976b], [Sinaceur and Dedekind, 2008], and [Ferreirós, 2008]. I will, here, stay close to Dedekind’s text. In particular, I will not consider more philosophical issues such as the axiomatic and / or structuralist character of Dedekind’s approach, as one can find in [Sieg and Schlimm, 2005], [Reck, 2003], [Reck, 2009] and [Yap, 2009]. My aim, here, is rather to highlight the specific status of arithmetic for Dedekind and its relations with Dedekind’s concept of number.

⁴³ “*Ich fühlte dabei empfindlicher als jemals früher den Mangel einer wirklich wissenschaftlichen Begründung der Arithmetik*”

⁴⁴ “*daß diese Art der Einführung in die Differentialrechnung keinen Anspruch auf Wissenschaftlichkeit machen kann, wird wohl niemand leugnen.*”

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quantities in question, in the cases in which it is not a rational number. By providing a rigorous definition, Dedekind could, in the same move, give a proof of the existence of such irrational limit. Indeed, as Dedekind wrote to Lipschitz about his definition of irrational numbers, “nothing is more dangerous in mathematics than to admit existence without proof.” His 1872 booklet provides a proof that the existence of irrational numbers is well founded. It is particularly important because “a definition of this continuity is nowhere given” even though differential calculus is developed using “continuous quantities”. The aforementioned theorem appears to Dedekind to be suited to serve as a foundation for Analysis.

For Dedekind, it is by a proper and purely arithmetical definition of irrational numbers and continuity, that one can obtain a satisfactory ground for this important theorem and thereby for Analysis:

a more careful investigation convinced me that this theorem [of the convergence of a bounded monotonic sequence], or any one equivalent to it, can be regarded as a more or less sufficient foundation for infinitesimal analysis. It only remained to discover its true origin in the elements of arithmetic, and thereby to secure a real definition of the essence of continuity.⁴⁵ (*ibid.*, 767)

The definition of real numbers should be “arithmetical” in the sense of Weierstrass’s arithmetization: it should be developed solely on the basis of rational numbers. But there is an additional criterion, for Dedekind: the definition must neither use external tools such as geometrical notions, nor unclear ideas such as that of “extensive magnitude”.

[T]he way in which the irrational numbers are usually introduced is based directly upon the concept of extensive magnitudes – which itself is nowhere carefully defined – and number is explained as the result of measuring such a magnitude by another of the same kind. Instead of this I demand that arithmetic shall develop out of itself.⁴⁶ (*ibid.*, 771)

Here, the “development of the arithmetic of rational numbers” is supposed to be known and well founded. Dedekind explains shortly how he considers that arithmetic was developed, so far:

⁴⁵ “eine genauere Untersuchung überzeugte mich, daß dieser oder auch jeder mit ihm äquivalente Satz gewissermaßen als ein hinreichendes Fundament für die Infinitesimalanalysis angesehen werden kann. Es kam nur noch darauf an, seinen eigentlichen Ursprung in den Elementen der Arithmetik zu entdecken und hiermit zugleich eine wirkliche Definition von dem Wesen der Stetigkeit zu gewinnen.”

⁴⁶ “Die bisher übliche Einführung der irrationalen Zahlen knüpft nämlich geradezu an den Begriff der extensiven Größen an – welcher aber selbst nirgends streng definiert wird – und erklärt die Zahl als das Resultat der Messung einer solchen Größe durch eine zweite gleichartige. Statt dessen fordere ich, da die Arithmetik sich aus sich selbst heraus entwickeln soll.”

I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetical act, that of counting, and counting itself is nothing other than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding. (...) Addition is the result of bringing together into a single act an arbitrary number of repetitions of the simplest act; multiplication arises from it in a similar way. While these two operations can always be carried out, the inverse operations, subtraction and division, are admissible only with restrictions. (...) [T]his limitation in performing the indirect operations has in each case been the real motive for a new creative act; thus negative and fractional numbers have been created by the human mind, and in the system of all rational numbers an instrument of infinitely greater perfection has been gained.⁴⁷ (*ibid.*, 768)

The standpoint adopted, here, presents strong similarities with the ideas exposed in the 1854 *Habilitationsvortrag*, although some aspects of the overall conception have been refined. In particular, for Dedekind in 1872 and later, the arithmetical operations are defined as the result of the operation applied to the operands, i.e. one defines the addition of a and b as $a + b = c$, rather than *just* the operation itself, as a formal tool.⁴⁸

It seems, even though Dedekind does not make this precise point explicit, that such a definition of the operations provides a better justification for the *creation* of new numbers to satisfy the demand that operations be applicable without restrictions. Indeed, if one considers, for example, the subtraction of a and b as defined by $a - b = d$, such a d does not exist if $a < b$ and one creates the negative numbers so that the subtraction can be universally

⁴⁷ “*Ich sehe die ganze Arithmetik als eine notwendige oder wenigstens natürliche Folge des einfachsten arithmetischen Aktes, des Zählens, an, und das zählen selbst ist nichts anderes als die sukzessive Schöpfung der unendlichen Reihe der positiven ganzen Zahlen, in welcher jedes Individuum durch das unmittelbar vorhergehende definiert wird; der einfachste Akt ist der Übergang von einem schon erschaffenen Individuum zu dem darauffolgenden neu zu erschaffenden. (...) Die Addition ist die Zusammenfassung einer beliebigen Wiederholung des obigen einfachsten Aktes zu einem einzigen Akte, und aus ihr entspringt auf dieselbe Weise die Multiplikation. Während diese beiden Operationen stets ausführbar sind, zeigen die umgekehrten Operationen, die Subtraktion und Division, nur eine beschränkte Zulässigkeit. (...) genug, gerade diese Beschränktheit in der Ausführbarkeit der indirekten Operationen ist jedesmal die eigentliche Ursache eines neuen Schöpfungsaktes geworden; so sind die negativen und gebrochenen Zahlen durch den menschlichen Geist erschaffen, und es ist in dem System aller rationalen Zahlen ein Instrument von unendlich viel größerer Vollkommenheit gewonnen.*”

⁴⁸It is a point explained in a letter to Weber, in 1878: “It does not please me that, here, one defines the operation more than the result of the operation. I prefer when, for example, the sum is defined as a number completely determined by the two summands, rather than defining the addition, and this already with the rational numbers.” (Letter from 19 November 1878, in [Dedekind, 1932] III, 486)

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valid. Note that the “limitation” of the applicability of inverse operations is a *motive* for the creation of new numbers: the actual creation itself is done with a proper rigorous definition.

7.2.2 The incomplete domain of rational numbers

To provide a purely arithmetical definition of the irrational numbers, Dedekind starts by considering the field R of rational numbers and the fact that it is totally ordered:

For our immediate purpose, however, another property [than forming a field] of the system R is still more important; it may be expressed by saying that the system R forms a well-ordered (*wohlgeordnetes*) domain of one dimension extending to infinity on two opposite sides.⁴⁹ ([Dedekind, 1872], 768)

Dedekind thus bases his investigations on the fact that the rational numbers form a linear (“one dimension”) infinite total order. The use of “expressions borrowed from geometric ideas” allows to understand what is meant by these properties but not to provide an appropriate definition. This is an additional reason, for Dedekind, to provide a clear and precise arithmetical definition of the linear continuum. Dedekind thus makes completely explicit the idea of “well-ordered domain of one dimension”, introducing the idea that R is dense (but Dedekind does not name this property, it will be done by Weber in the *Lehrbuch*).

The order is defined by considering what it means for two numbers a and b to be unequal:

Two rational numbers a, b are different just in case the difference $a - b$ has either a positive or negative value. In the former case a is said to be greater than b , and b less than a ; this is also indicated by the symbols $a > b$, $b < a$. Because $b - a$ has a positive value in the latter case, it follows that $b > a$, and $a < b$.⁵⁰ (*ibid.*, 769)

The relation $>$ is transitive and verifies “if a, c are two different numbers, there are infinitely many different numbers b lying between a and c .” Moreover the rational numbers verify that every number generates what is called, later in the essay, a “cut”:

⁴⁹ “Für unseren nächsten Zweck ist aber noch wichtiger eine andere Eigenschaft des Systems R , welche man dahin aussprechen kann, daß das System R ein wohlgeordnetes, nach zwei entgegengesetzten Seiten hin unendliches Gebiet von einer Dimension bildet.”

⁵⁰ “Die Verschiedenheit zweier rationaler Zahlen a, b zeigt sich darin, daß die Differenz $a - b$ entweder einen positiven oder einen negativen Wert hat. Im ersten Falle heist a größer als b , b kleiner als a , was auch durch die Zeichen $a > b$, $b < a$ angedeutet wird. Da im zweiten Falle $b - a$ einen positiven Wert hat, so ist $b > a$, $a < b$.”

If a is any definite number, then all numbers of the system R fall into two classes, A_1 and A_2 , each of which contains infinitely many individuals; the first class A_1 comprises all numbers a_1 that are $< a$, the second class A_2 comprises all numbers a_2 that are $> a$; the number a itself may be assigned at pleasure to the first or second class, and it is then respectively the greatest number of the first class or the least of the second. In either case the decomposition of the system R into the two classes A_1, A_2 is such that every number of the first class A_1 is less than every number of the second class A_2 .⁵¹ (*ibid.*, 769)

So as to highlight the essential property of continuity, as opposed to the density of the system of rational numbers, Dedekind uses an analogy with the straight line. The straight line, here, only has a heuristic role: the properties given, Dedekind says, “recall the corresponding relations of position between the points of a straight line.”

Following this thread, Dedekind emphasizes what he considers to be the “essence of continuity”, which he wants to be able to “follow up arithmetically”, putting to the fore the fact that rational numbers are insufficient to characterize continuity. It is well-known, at least since the Greek mathematicians, that there are “infinitely many lengths which are incommensurable with the unit of length,” which allows to state that

[t]he straight line L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals.⁵² (*ibid.*, 770)

This is the occasion for Dedekind to state clearly that while geometry can *inspire* extensions of the number concept, as it is the case with incommensurable lengths, it is not acceptable to take it as foundation, as means of definition, for the new numbers, which are new *arithmetical* elements and ought to be defined as such, by purely arithmetical means.

The comparison between the domain R of rational numbers and the straight line allows to find “in the former a gappiness, incompleteness, discontinuity”. Having highlighted that one of the common properties of R and the straight line is that every point divides the domain / the line in two distinct ordered parts, Dedekind finds “the essence of continuity” in the “converse” property:

⁵¹ “Ist a eine bestimmte Zahl, so zerfallen alle Zahlen des Systems R in zwei Klassen, A_1 und A_2 , deren jede unendlich viele Individuen enthält; die erste Klasse A_1 umfaßt alle Zahlen a_1 , welche $< a$ sind, die zweite Klasse A_2 umfaßt alle a_2 , welche $> a$ sind; die Zahl a selbst kann nach Belieben der ersten oder der zweiten Klasse zugeteilt werden, und sie ist dann entsprechend die größte Zahl der ersten oder die kleinste Zahl der zweiten Klasse. In jedem Falle ist die Zerlegung des Systems R in die beiden Klassen A_1, A_2 von der Art, da jede Zahl der ersten Klasse A_1 kleiner als jede Zahl der zweiten Klasse A_2 ist.”

⁵² “Die Gerade L ist unendlich viel reicher an Punktindividuen, als das Gebiet R der rationalen Zahlen an Zahlindividuen.”

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If all points of the straight line fall into two classes such that every point of the first class (*Klasse*) lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.⁵³ (*ibid.*, 771)

It is important to underline that continuity is *not* a necessary or intrinsic property of the straight line, or even of space, for Dedekind. In fact, continuity is a property that mathematicians attribute to the line: “to the straight line we ascribe absence of gaps, completeness, continuity.” To assume that the straight line is continuous is an axiom for Dedekind:

The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we think continuity into the line.⁵⁴ (*ibid.*, 772)

Note that Dedekind is not stating that ‘the continuity of the straight line’ is an axiom. Rather, the fact that one admits the continuity of the line is an axiom which confers the continuity to the line: the mathematician, in thought, makes the line complete, continuous. Continuity, for Dedekind, is a property “conceived and not perceived” as Sinaceur explains:

This principle of continuity “corresponds” to the intuitive geometric representation of the straight line that most analysts used to have, but Dedekind gives it the form of an explicit statement, precisely because he refuses this conception of an intuitive continuity. (...) The mathematical continuum is conceived and not perceived; one does not *see* the continuity of the line, one *thinks* the continuity, one *defines* it by a principle or a theorem, and in mathematics, there is, in all rigor, nothing but principles and theorems.⁵⁵ ([Sinaceur and Dedekind, 2008], 37)

In fact, Dedekind explains that not only is space not necessarily continuous, but even Euclidean geometry does not require the continuity of space in its

⁵³ “Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, daß jeder Punkt der ersten Klasse links von jedem Punkte der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt.”

⁵⁴ “Die Annahme dieser Eigenschaft der Linie ist nichts als ein Axiom, durch welches wir erst der Linie ihre Stetigkeit zuerkennen, durch welches wir die Stetigkeit in die Linie hineindenken.”

⁵⁵ “Ce principe de continuité ‘correspond’ à la représentation géométrique intuitive de la droite qu’avaient la plupart des analystes, mais Dedekind lui donne la forme d’un énoncé explicite précisément parce qu’il récusé cette conception d’une continuité intuitive. (...) Le continu mathématique est conçu, et non perçu ; on ne voit pas la continuité d’une ligne, on la pense, on la définit par un principe ou un théorème, et en mathématiques il n’y a, en toute rigueur, rien que des principes et des théorèmes.”

premises to be valid. Nor do the axioms of Euclidean geometry imply continuity.⁵⁶ Dedekind discussed questions related to Euclid's *Elements* in detail in the correspondence with Lipschitz. It is always possible, if needed, to "fill in" discontinuities in thought, that is to create, by a rigorous definition, the elements that allow to make the discontinuous domain or space continuous – in the case of R , it will be the irrational numbers. Here, the 'principle of continuity' is taken as one of the laws that will allow to define the continuity for real numbers, for it reveals the discontinuities in R and the need to introduce new elements to obtain a complete, continuous domain.

7.2.3 Definition of irrational numbers

The rigorous arithmetical definition of irrational numbers is made in four steps. First, Dedekind defines the notion of cut:

Every rational number a separates the system R into two classes A_1, A_2 such that every number a_1 of the first class A_1 is less than every number a_2 of the second class A_2 , the number a is either the greatest number of the class A_1 , or the least number of the class A_2 . If now any separation of the system R into two classes A_1, A_2 is given which possesses only this characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2 , then for brevity we shall call such a separation a *cut* (*Schnitt*) and we shall designate it by (A_1, A_2) .⁵⁷ (op.cit., 772)

Dedekind says that "every rational number a produces" the cut (A_1, A_2) .⁵⁸ It is easy to prove that there is an infinity of cuts which are not produced by rational numbers which amounts to proving the incompleteness of R . To prove this, it suffices to consider a positive integer D such that it is not the square of any other integer. Then, there exists a certain λ such that

$$\lambda^2 < D < (\lambda + 1)^2.$$

One defines a cut (A_1, A_2) for which each number a_2 in A_2 has a square which is greater than D and each number a_1 in A_1 has a square $\leq D$.

⁵⁶In fact, continuity only became a subject of concern with the development of infinitesimal calculus, in the 17th-18th centuries.

⁵⁷"jede rationale Zahl a eine Zerlegung des Systems R in zwei Klassen A_1, A_2 von der Art hervorbringt, daß jede Zahl a_1 der ersten Klasse A_1 kleiner ist als jede Zahl a_2 der zweiten Klasse A_2 ; die Zahl a ist entweder die größte Zahl der Klasse A_1 , oder die kleinste Zahl der Klasse A_2 . Ist nun irgendeine Einteilung des Systems R in zwei Klassen A_1, A_2 gegeben, welche nur die charakteristische Eigenschaft besitzt, daß jede Zahl a_1 in A_1 kleiner ist als jede Zahl a_2 in A_2 , so wollen wir der Kürze halber eine solche Einteilung einen Schnitt nennen und mit (A_1, A_2) bezeichnen."

⁵⁸The cut produced by the greatest number in A_1 and that produced by the lowest number in A_2 are not, here, considered essentially different.

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Then, by showing that there is no rational number q such that $q^2 = D$, which is a elementary number theoretical proof, one can see that there is neither a greatest number in A_1 nor a lowest number in A_2 . Hence, there doesn't exist any rational number generating the cut (A_1, A_2) .

Having proved that there are gaps in R , Dedekind proposes the following:

whenever we have a cut (A_1, A_2) produced by no rational number, we create a new number, an irrational number a , which we regard as completely defined by this cut (A_1, A_2) ; we shall say that the number a corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal if and only if they correspond to essentially different cuts.⁵⁹ (*ibid.*, 773)

First, note that Dedekind is not stating that he is *inventing* new numbers which did not exist before, he does not “claim to invent (...) some number which wouldn't have been, already, grasped more or less clearly in the mind of every mathematician” (Letter to Lipschitz, in [Dedekind, 1932] III, 475). Rather, as he explains to Lipschitz:

The global tendency of my writing, which I believe has been expressed clearly in the introduction and the §3, is rather simply to use the generally known phenomena of the cut to prove (which has never been done, to my knowledge) that on the sole ground of the arithmetic of rational numbers, thus without the resort of the rather obscure and complicated concept of magnitude, the irrational number can be defined in one strike, and that, which is the most important, in their completeness (continuity) which is sufficient and in the same time indispensable for an absolutely rigorous scientific construction of the arithmetic of the real numbers.⁶⁰ (*ibid.*, 476)

⁵⁹ “Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so erschaffen wir eine neue, eine irrationale Zahl a , welche wir als durch diesen Schnitt (A_1, A_2) vollständig definiert ansehen; wir werden sagen, daß die Zahl a diesem Schnitt entspricht, oder daß sie diesen Schnitt hervorbringt. Es entspricht also von jetzt ab jedem bestimmten Schnitt eine and nur eine bestimmte rationale oder irrationale Zahl und wir sehen zwei Zahlen stets und nur dann als verschieden oder ungleich an, wenn sie wesentlich verschiedenen Schnitten entsprechen.”

⁶⁰ “Die ganze Tendenz meiner Schrift, die ich in der Einleitung und in §3 deutlich bezeichnet zu haben glaube, geht vielmehr lediglich darauf hinaus, mit Benutzung der allgemein bekannten Schnitt-Erscheinung nachzuweisen (was meines Wissens noch nirgends geschehen war), daß auf der alleinigen Grundlage der Arithmetik der rationalen Zahlen, also ohne jede Zuziehung des ziemlich dunkelen und complicirten Größen-Begriffes, die irrationalen Zahlen mit einem Schlage definirt werden können, und zwar, was das Wichtig-

The cuts, thus, identify the gaps in R that need to be filled to obtain a continuous domain. This incites the mathematician to *create*, in thought, new numbers. What Dedekind *invents* is a new tool, the concept of cut, that allows to define the *system* of real numbers, since the definition given by Dedekind, here, yields all the irrational numbers in one move and provides the concept of ordered (as we will see later) field of real numbers.

Irrational numbers should thus not be *identified* with cuts, but only considered as *corresponding* to cuts, a mathematical concept which offers the means for the definition/creation of irrational numbers.⁶¹ This point has been often contentious for many commentators for whom Dedekind is, in fact, assuming the existence of the irrational number he is creating. Most notably Russell wrote, in *Introduction to Mathematical Philosophy*:

From the habit of being influenced by spatial imagination, people have supposed that series must have limits in cases where it seems odd if they do not. Thus, perceiving that there was no rational limit to the ratios whose square is less than 2, they allowed themselves to “postulate” an irrational limit, which was to fill the Dedekind gap. Dedekind, in the above-mentioned work, set up the axiom that the gap must always be filled, i.e. that every section must have a boundary. (...) But there are an infinite number of series for which it is not verified.

The method of “postulating” what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil. ([Russell, 1903], 71)

Several decades later, Dugac wrote that while Dedekind first “states clearly that the irrational number a is defined by the cut (A_1, A_2) ”, the status of cuts corresponding to no rational numbers in the system R is less clear.

Indeed, the rational number exists before generating the cut, and the manner of forming the cuts from the elements in \mathbb{Q} fully justifies the use of the word “generate”. But we believe that when the cut (A_1, A_2) does not correspond to any rational number, the word “generate” seems to presuppose (against Dedekind’s insisting conception that new numbers are “created”) a certain

ste ist, in derjenigen Vollständigkeit (Stetigkeit), welche für einen absolut strengen, wissenschaftlichen Aufbau der Arithmetik der reellen Zahlen ausreichend und zugleich unentbehrlich ist.”

⁶¹For this reason, the definitions given by Cantor and Weierstrass are as valid as Dedekind’s: the only difference is in the “simplicity” of the means of definition.

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a priori existence of a , while in fact, it is a which is “generated” by the cut.⁶² ([Dugac, 1976b], 43)

The point that Dugac seems to be overlooking, here, is that the cuts provide a method in two steps: first, one constructs the cut, which is a mathematical tool that will be *used for* the creation, then one properly speaking creates the irrational number determined by the cut. The cut is an *instrument* at the mathematician's disposal to create the new numbers. The cut is a mathematical object, designed by the mathematician, and is not itself creating the numbers: cuts do not have a creative power, unlike the mathematician who, according to Dedekind, has godlike powers of creation on these matters. And this creation is, for Dedekind, justified by the fact that there exists, as it was rigorously proved, an infinity of cuts corresponding to no rational number. According to Boniface:

Dedekind would probably say that the cut was built (as the field of rational numbers was built), while the corresponding irrational number was created (...). [T]he number created thus come to “complete” the set of rational numbers and are “entirely determined” by the cut which served for their creation.⁶³ ([Boniface, 2004], 25)

To these remarks, let us answer in two steps. First, with precisions about the creation of new numbers *per se*, so as to clarify the idea of “creation” for Dedekind, and in a second step by considering more specifically the creation of irrational numbers.

The creation of an irrational number is and can only be made from the already defined domain of rational numbers, and it is in this extent that the cut is the instrument for the creation of the irrational numbers. Built from the rational numbers, the cuts allows to put to the fore the necessity to create new numbers to fill in the gaps, the discontinuities of R . If the irrational number “corresponds” to the cut, it is because there is a one-to-one correspondence between essentially different cuts and the domain of rational numbers completed with the irrational numbers.

The creation is, thus, of the same nature as the creation of negative and rational numbers: it is done only on the basis of already defined concepts.

⁶² “*En effet, le nombre rationnel existe avant d’engendrer la coupure et la façon de former les coupures, à partir des éléments de \mathbb{Q} , justifie pleinement l’emploi du mot ‘engendrer’. Mais, nous croyons, que lorsque la coupure $(A_1; A_2)$ ne correspond pas à un nombre rationnel, alors le mot ‘engendrer’ semble présupposer (contrairement à la conception insistante de Dedekind affirmant la ‘création’ de nouveaux nombres) une certaine existence a priori de a , tandis qu’en réalité, c’est a qui est ‘engendré’ par la coupure.*”

⁶³ “*Dedekind dirait sans doute que la coupure a été construite (comme a a été construit le corps des nombres rationnels), alors que le nombre irrationnel qui lui correspond a été créé (...). [L]es nombres créés viennent donc “compléter” l’ensemble des nombres rationnels et sont “entièrement déterminés” par la coupure qui a servi à leur création.*”

It is not an *ex nihilo* creation, rather it is intrinsically related to already defined domains of numbers, and it is done with tools that do not use “foreign elements”. The idea of a constrained creation was already present in Dedekind’s *Habilitationsvortrag*, albeit without any explanation as to how this creation was supposed to be done.⁶⁴ Here (and as it will also be the case for natural numbers), the creation is synonymous to the (rigorous) definition of the new numbers by means of a new mathematical concept built on “earlier” well-defined arithmetical concepts.

In 1888, when Weber raised the question of the identification between numbers and cuts, Dedekind answered:

you say that the irrational number is nothing other than the cut itself, while I prefer to create something *new* (different from the cut) that corresponds to the cut and of which I say that it brings forth, creates the cut. We have the right to ascribe such a creative power to ourselves; and moreover, because of the identity of nature (*Gleichartigkeit*) of all numbers, it is more expedient to proceed in this way.⁶⁵ (Letter to Weber, 24 January 1888, transl. slightly altered in [Ewald, 2005], 835)

Dedekind explains that if one were to say that the numbers are “identical with the cut” they produce, then one would talk about numbers and ascribe to them “such attributes that would sound in the highest degree peculiar were they to be applied to the numbers themselves”. Thus, identifying the numbers to the cut would imply to attribute to numbers properties that are not arithmetical.⁶⁶ This would imply differences of nature between the different systems of numbers, whereas Dedekind carefully, since the initial concept of natural number, defined each extension of the concept of number on the basis of the previously established systems and this without introduction of “foreign elements” so as to preserve their “identity of nature”. For Dedekind, such an extension allows to widen the number concept without changing its nature – which explains why he calls the new objects created irrational *numbers* and not irrational *magnitudes* as Cantor, Weierstrass and

⁶⁴The essential difference between the *Habilitationsvortrag* and *Stetigkeit* is that in this case, rather than the demand that an operation be uniformly applicable, one demands that all cuts be generated by a number.

⁶⁵*Du sagst, die Irrationalzahl sei überhaupt Nichts anderes als der Schnitt selbst, während ich es vorziehe, etwas neues (vom Schnitte Verschiedenes) zu erschaffen, was dem Schnitte entspricht, und wovon ich sage, da es den Schnitt hervorbringe, erzeuge. Wir haben das Recht, uns eine solche Schöpfungskraft zuzusprechen, und außerdem ist es der Gleichartigkeit aller Zahlen wegen viel zweckmäßiger, so zu verfahren.*

⁶⁶The argument is the same concerning the identification of natural numbers with cardinal numbers construed as the “class”, i.e. ‘the system of all finite systems that [correspond one-to-one] to each other, but “something new (corresponding to this class) which the mind creates”.

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many other did.⁶⁷

In defining the irrational numbers, Dedekind is introducing new arithmetical elements, and his definition satisfies the requirements, which I have already mentioned, that (1) arithmetic be “kept free from intermixture with foreign elements” and “the definition or creation of irrational numbers ought to be based on phenomena one can already observe clearly in the domain R of rational numbers”; (2) the definition must allow to “engender simultaneously” all numbers (rather than several definitions as roots of equations, as logarithms, ...); and (3) the definition must allow “a perfectly clear definition of the calculations (addition, etc.) one needs to make on the new numbers” ([Dedekind, 1876-1877], 58, footnote, transl. slightly altered), which is done for real numbers using only cuts, later in *Stetigkeit*.

7.2.4 Order and operations for real numbers

The second step for a complete definition of irrational numbers is to define an order for the *all* the real numbers, i.e. for irrational and rational numbers alike. This is done by studying relations between cuts. Let α and β be two real numbers corresponding to the cuts (A_1, A_2) and (B_1, B_2) . A cut being a partition of the domain of real numbers in two disjoint sets, it suffices to know, for example, A_1 to fully know the cut itself. Hence, to be able to state the equality of the two real numbers α and β , it suffices to know if A_1 and B_1 are identical, i.e. every number in A_1 is a number in B_1 and every number in B_1 is a number in A_1 , hence (A_2 and B_2 are identical as well and) $\alpha = \beta$.

If the classes A_1 and B_1 are not identical, then there are two possibilities:

- there exists at least one number a'_1 which is in A_1 but not in B_1 , and is thus in B_2 , i.e. there exists a number in B_2 such that $a'_1 = b'_2$;
- there exists at least one number b'_1 which is in B_1 but not in A_1 , and is thus in A_2 , i.e. there exists a number in A_2 such that $b'_1 = a'_2$.

If there is only one such number a'_1 (resp. b'_1), then it is easy to show that $\alpha = a'_1 = b'_2 = \beta$ (resp. $\alpha = b'_1 = a'_2 = \beta$). If there is more than one such number, then the cuts (A_1, A_2) and (B_1, B_2) are “essentially different”. In the first case, A_1 and B_1 are such that there are more elements in A_1 than in B_1 and hence, $\alpha > \beta$, while in the second case, A_1 and B_1 are such that there are less elements in A_1 than in B_1 and hence, $\alpha < \beta$. It is not possible to have any other case than the one presented.

The careful re-definition of the order in the real numbers is important for Dedekind, insofar as it is crucial to justify the choice of terminology and

⁶⁷Note that Weber seems to have been convinced by Dedekind's argument, since he followed exactly Dedekind's definition of irrational numbers, in his *Lehrbuch*, including not identifying irrationals with cuts.

symbols. Indeed, by doing so one makes sure to avoid “[allowing] oneself to make inadmissible transfers from one domain to the other” motivated, even with the “best intention to be honest”, by “a hasty choice of expressions borrowed from other notions that have already been developed” (*ibid.*, 775). It was already the case, as we saw, with ideal numbers and the analogy with rational number theory, which taken for granted led to “hasty conclusions and incomplete proofs”. The transfer of expressions, notions, methods must, thus, be motivated and justified, and it should not be taken as pretext to assume the validity of the properties, as we saw for ideal theory.

The order established for the real numbers allows to show that the “system of real numbers” denoted by \mathfrak{R} is a “completely ordered unidimensional domain”, i.e. that its elements verify the properties on p. 411. More importantly, \mathfrak{R} also “possesses continuity”, that is:

If the system \mathfrak{R} of all real numbers divides into two classes $\mathfrak{A}_1, \mathfrak{A}_2$ such that every number α_1 of the class \mathfrak{A}_1 is less than every number α_2 of the class \mathfrak{A}_2 then there exists one and only one number α by which this separation is produced.⁶⁸ ([Dedekind, 1872], 776)

All these properties are theorems proved only using cuts. Thus \mathfrak{R} is a completely order linear continuous domain and Dedekind thus provided a perfectly rigorous and purely arithmetical characterization of continuity.

Since the transfer of expressions and properties should not be made without proper definition (and proofs, when necessary), the definition of the arithmetical operations is the fourth and last crucial point in the definition of the real numbers. Again, everything is done by means of cuts. As mentioned at the beginning of the paragraph, operations should be defined by the factors on which they act. Hence, to define the addition of two real numbers α corresponding to the cut (A_1, A_2) and β corresponding to the cut (B_1, B_2) in R (and not in \mathfrak{R} , for that would be assuming the validity of the transfer of the operations) means to be able to characterize the cut (C_1, C_2) corresponding to the “result γ of the computation”. The definition of the cut (C_1, C_2) must be done on the sole basis of the cuts (A_1, A_2) and (B_1, B_2) . For this, Dedekind proposes that a rational number c be “put into the class C_1 ” if there exists a_1 in A_1 and b_1 in B_1 such that $a_1 + b_1 \geq c$. This generates a partition of the domain in two distinct classes, and “evidently forms a cut”. This cut is produced by the result of $\alpha + \beta$: any number c_1 in C_1 is such that $c_1 \leq a_1 + b_1 \leq \alpha + \beta$ and one can prove that any number c_2

⁶⁸ “Zerfällt das System \mathfrak{R} aller reellen Zahlen in zwei Klassen $\mathfrak{A}_1, \mathfrak{A}_2$ von der Art, daß jede Zahl α_1 der Klasse \mathfrak{A}_1 kleiner ist als jede Zahl α_2 der Klasse \mathfrak{A}_2 , so existiert eine und nur eine Zahl α , durch welche diese Zerlegung hervorgebracht wird.”

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in C_2 is such that $c_2 \geq \alpha + \beta$. Indeed, suppose that there is a number c_2 in C_2 such that $c_2 < \alpha + \beta$. One could write $c_1 + p = \alpha + \beta$ with p a positive rational number, i.e.

$$c_2 = \left(\alpha - \frac{1}{2}p\right) + \left(\beta - \frac{1}{2}p\right)$$

but then, since $\alpha - \frac{1}{2}p$ is in A_1 and $\beta - \frac{1}{2}p$ is in B_1 , that would contradict $c_2 < \alpha + \beta$. Thus, any number c_2 in C_2 is such that $c_2 \geq \alpha + \beta$, which means that (C_1, C_2) is indeed produced by $\alpha + \beta$.

Thus we shall not violate the definition which holds in the arithmetic of rational numbers if we always understand the sum $\alpha + \beta$ of any two real numbers α, β to be that number γ by which the cut (C_1, C_2) is produced. Further, if only one of the two numbers α, β is rational, e.g., α , it is easy to see that it makes no difference to the sum $\gamma = \alpha + \beta$ whether the number α is put into the class A_1 or into the class A_2 .⁶⁹ (*ibid.*, 777)

It is perfectly possible, Dedekind tells us, to define all the other operations of elementary arithmetic: subtraction, multiplication, quotient, but also powers, roots, logarithms, and “in this way we arrive at real proofs of theorems (as, e.g., $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$), which to the best of my knowledge have never been established before.”⁷⁰

This statement was a contentious point for Lipschitz, who affirmed that Euclid's *Elements* V, 5. already contained “the necessary principles for the proof of these theorems”.⁷¹ The letters exchanged with Lipschitz gave Dedekind the opportunity to explain many of his ideas, and I have used his answers to Lipschitz several times. Regarding this particular point, Dedekind carefully explained how his ideas were distinct from what can be found in Euclid's *Elements*.⁷² An important difference between what is done in the *Elements* and Dedekind's definition of the irrational is that there is, in Euclid, no consideration of a “concept of a *continuous* domain

⁶⁹ “Man verstößt daher nicht gegen die in der Arithmetik der rationalen Zahlen geltende Definition, wenn man in allen Fällen unter der Summe $\alpha + \beta$ von zwei beliebigen reellen Zahlen α, β diejenige Zahl γ , versteht, durch welche der Schnitt (C_1, C_2) hervorgebracht wird. Ist ferner nur eine der beiden Zahlen α, β , z.B. α rational, so überzeugt man sich leicht, daß es keinen Einfluß auf die Summe $\gamma = \alpha + \beta$ hat, ob man die Zahl in die Klasse A_1 oder in die Klasse A_2 aufnimmt.”

⁷⁰The letters exchanged with Lipschitz tell us that Dedekind did not know of Weierstrass's work on irrational numbers, in 1872.

⁷¹The consideration of Dedekind's cuts as similar to Eudoxus's method of exhaustion can also be found in [Weyl, 1946], 4-5.

⁷²As Sinaceur underlines, Dedekind “proposes a very subtle and absolutely original analysis of the theory of proportions of the books V and X of Euclid's *Elements*”. See [Sinaceur and Dedekind, 2008], pp. 258-260.

of magnitudes". If one were to consider a domain of magnitudes with incommensurable magnitudes but without the principle of continuity given by Dedekind, this domain would still be "incomplete" (i.e. discontinuous),

and the generally valid definitions of the arithmetical operations is therefore downright impossible, because in such incomplete domains of numbers, the sum, difference, etc., of two actually existing numbers may not exist in the same domain of numbers.⁷³ (Letter to Lipschitz, 10 June 1876, in [Lipschitz and al. 1986], 68)

The possibility to give a generally valid definition of the operations is a core argument for Dedekind's approach. If one is ready to give up the demand that the operations defined are "general", i.e. valid for all and any numbers of the domains, then it suffices to state that "as the product $\sqrt{2} \cdot \sqrt{3}$, I mean the number $\sqrt{6}$, hence $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, qed!" (*ibid.*, 68). If this is a "conceivable" way of treating operations, it is not, however "advisable". In particular, it would imply that the operations would need to be defined every time they are applied to new numbers.

In *Stetigkeit*, Dedekind does not give the definitions of the other operations of arithmetic for brevity's sake. To shorten these definitions, Dedekind proposes to consider the "concept of interval", that is,

a system A of rational numbers possessing the following characteristic property: if a and a' are numbers of the system A , then all rational numbers lying between a and a' are contained in A .⁷⁴ ([Dedekind, 1872], 777)

The concept of interval allows to introduce a notion of supremum and infimum and to give a theorem that "the arithmetical operations possess a certain continuity":

If the number λ is the result of an operation performed on the numbers $\alpha, \beta, \gamma, \dots$ and λ lies within the interval L , then intervals A, B, C, \dots can be found which contain the numbers $\alpha, \beta, \gamma, \dots$ and such that the result of the same operation in which the numbers $\alpha, \beta, \gamma, \dots$ are replaced by arbitrary numbers of the in-

⁷³ "und es sind schon deshalb allgemeingültige Definitionen der arithmetischen Operationen geradezu unmöglich, weil in solchen lückenhaften Zahlen-Gebieten die aus zwei wirklich darin existirenden Zahlen abzuleitende Summe, Differenz u.s.w. in demselben Zahlen-Gebiete vielleicht nicht existirt."

⁷⁴ "eines Intervalls, d.h. eines Systems A von rationalen Zahlen, welches folgende charakteristische Eigenschaft besitzt: sind a und a' Zahlen des Systems A , so sind auch alle zwischen a und a' liegenden rationalen Zahlen in A enthalten."

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tervals A, B, C, \dots is always a number lying within the interval L .⁷⁵ (*ibid.*, 777)

The theorem given by Dedekind, here, is according to Sinaceur “of an exceptional generality”. Dedekind, however, finds it too “clumsy” (*Schwerfälligkeit*). This “clumsiness” incites Dedekind to turn towards the “ideas of variable magnitudes, functions, and limiting values,” which are “an aid to expression” (*um der Sprache zu Hilfe*). This shows that Dedekind would rather design new notions that are more practical, easier to use in mathematics, rather than systematically bring back all definitions to basic principles. It was also the case with the definition adopted for finite fields, a “concession” made to allow a smoother development of algebraic number theory.

These developments make it possible for Dedekind to provide a purely arithmetical proof of the theorem: “If a magnitude x grows continually but not beyond all limits it approaches a limiting value”, which was considered as a good ground for infinitesimal analysis.

Dedekind shows that this result is equivalent to the principle of continuity: it ceases to be valid “as soon as we assume a single real number not to be contained” in the domain \mathfrak{R} of the real numbers, and if this theorem is valid so is the theorem stating the continuity of \mathfrak{R} . This is shown by Dedekind using the method of analysis and synthesis of the problem: the analysis shows that the principle of continuity is necessary for the theorem to be true, and the synthesis shows that it is also a sufficient condition for a proof of the theorem. Other theorems can be proved as well, such as the convergence of Cauchy sequences.

We see, thus, how with the definition of real numbers, their order, the proof of the continuity of the system of real numbers and the elementary arithmetical operations, Dedekind was able to give a purely arithmetical and perfectly rigorous definition of the linear continuum of the real numbers. I will come back to the issue of rigor later in this chapter.

The distinction between dense and continuous domains is an essential point for Dedekind's approach (and which was, of course, also noted by Cantor). The terminology was coined by Weber in his *Lehrbuch*:

An ordered set which has the property that between any two elements, one can always find again another element is called *dense*.⁷⁶ ([Weber, 1895-96] I, 4)

⁷⁵ “Ist die Zahl λ das Resultat einer mit den Zahlen $\alpha, \beta, \gamma, \dots$ angestellten Rechnung, und liegt sie innerhalb des Intervalls L , so lassen sich Intervalle A, B, C, \dots angeben, innerhalb deren die Zahlen $\alpha, \beta, \gamma, \dots$ liegen, und von der Art, daß das Resultat derselben Rechnung, in welcher die Zahlen $\alpha, \beta, \gamma, \dots$ durch beliebige Zahlen der Intervalle A, B, C, \dots ersetzt werden, jedesmal eine innerhalb des Intervalls L liegende Zahl wird.”

⁷⁶ “Eine geordnete Menge von der Eigenschaft, dass zwischen je zwei Elementen immer

Weber explains that one can form a dense set of numbers by forming the rational numbers as pairs of natural numbers. He, then, introduces the notion of cut, and states that

When any cut in a dense set is produced by a determinate element μ , the set is called *continuous*.⁷⁷ (*ibid.*, 6)

Rational numbers are characterized by Weber as a dense but not continuous set of numbers, and the real numbers as a continuous set of numbers.

Dedekind's viewpoint was completely adopted by Weber, despite the latter's initial reservations about the "creation" of irrational numbers. The definition of reals by means of cuts can also be found in Ulisse Dini's 1878 treatise on functions (*Fondamenti per la teorica della funzioni di variabili reali*), who initiated the interest in the foundation of analysis in Italy. This interest is taken up by Peano who developed a more Weierstrassian viewpoint. Cuts were also used by Camille Jordan in the 1893 edition of his textbook on Analysis. Despite its non-constructive aspect and the difficulties of using cuts in effective computations, the methods of cuts was thus widely adopted.

We saw that for Klein, arithmetization was essentially a process of rigorization of the foundation of Analysis, in particular of the real numbers. The approach earned its name of "arithmetization" from the clearly displayed demand that the definition of irrational numbers be made on the basis of the rational numbers, and in particular not in a geometrical context.

Although Klein's description does point out some important specificities of the changes happening during the 19th century, it seems to present two drawbacks: the first is the negation of important differences in the ideas and mathematics of the 'arithmetizers'. The second is, as pointed out by Jahnke and Otte, the distinction between foundation and development of mathematics.⁷⁸

Does this separation between development and foundation really apply to the mathematics of the 19th century, which did show the marks of a historically unprecedented productivity?
([Jahnke and Otte, 1981], 23)

From their viewpoint, the arithmetization of mathematics is not restricted to the foundation of Analysis, but is a "matter of (...) reshaping and reformulating mathematics as a whole." For them, at the turn of 19th century, arithmetic became "the language of algebra resp. of the entire mathematics,

noch andere Elemente gefunden werden, heisst dicht."

⁷⁷ "Wenn jeder Schnitt in einer dichten Menge durch ein bestimmtes Element μ erzeugt wird, so heisst die Menge stetig."

⁷⁸ Jahnke and Otte also underline the question of the relation between foundation and application of mathematics, a point that seems out of the scope of my inquiry.

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by means of which, and in which, all mathematical facts must ultimately be expressible”, a process that culminated with Hilbert’s program.

In the following paragraph, I will inquire into the idea that “arithmetization” played a predominant role in the development of Dedekind’s mathematics, as the strategy elaborated by Dedekind for his most important works – both foundational and non-foundational. This will allow me to propose an explanation for the importance of arithmetic observed in the previous chapters.

Note that while following Jahnke and Otte’s initial proposition, I will not be following the idea that the concept number had become a “symbol”, for this would not fit Dedekind’s idea of numbers. An appropriate explanation of Dedekind’s arithmetization will thus have to be made in two parts: first, in the next paragraph, I will highlight the characteristics of his approach, and in a second step, I will relate them to his concept of number. This second point will take place in the last chapter of the dissertation.

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If the arithmetization that can be identified in Dedekind’s works is not supported by a concept of number changed into a pure “symbol”, then the process on which arithmetization relies still needs to be made explicit.⁷⁹ In which way is what Dedekind does an expression of mathematics in arithmetical forms? Why should arithmetic be used rather than other approaches? What kind of demands does the arithmetization respond to?

One of the core motivations of the arithmetization of the “foundation of Analysis” proposed by Dedekind in 1872 is the demand that “Arithmetic is developed out of itself.” And indeed, we saw that, for Dedekind, a properly done extension of a certain domain of numbers yields new *numbers*, and not magnitudes. This implies that from the extension of the number concept follows an extension of arithmetic, whose “first object” are the natural numbers.

In the following, I will propose a characterization of Dedekind’s “arithmetization” susceptible to encompass the definition of the irrational numbers as well as Dedekind’s resort to arithmetical methods in algebraic function and algebraic number theory. With the elements presented in previous chapters, we know that the development of arithmetical methods is particularly important for Dedekind’s algebraic number theory (and subsequently for his algebraic function theory), and became particularly noticeable in the rewriting of algebraic number theory in 1876-77 in a form that appears to be more

⁷⁹Once this is done, I will turn to the concept of natural number, so as to highlight the foundation of arithmetic which supports Dedekind’s arithmetization.

‘arithmetical’.

When using the phrase “arithmetization”, I refer to the attempts to develop theories on the basis of elementary arithmetic so as to fulfill epistemological requirements. I investigate the strategy developed to designate the set of concepts and methods used to provide a new rigorous and uniform ground and/or expression to certain theories.

Contrary to Klein and to most of the later commentators, I will consider that the “putting into arithmetical form” is an important, if not essential, part of the process of arithmetization. Moreover, I will look at arithmetization in a somewhat broader way, and will not reduce it to considerations on the definition of real numbers. Hence, like Janke and Otte, I will speak of “arithmetization of mathematics” rather than “arithmetization of analysis”. In fact, when introduced by Kronecker and Klein, “to arithmetize” aimed at all (pure) mathematics. Kronecker’s ‘program’ was a reduction of all pure mathematics to arithmetic. Klein had a very broad view of arithmetization as being a rigorization program running through mathematics in the 19th century. In 1897, in his *Theory of Algebraic Number Fields*, Hilbert wrote “the modern development of pure mathematics takes place chiefly under the sign of number” (quoted in Petri and Schappacher 2007, 366), and mentions the arithmetization of function theory by Dedekind and by Weierstrass, as well as the arithmetization of geometry. Hilbert’s notion of arithmetization “touches algebra, analysis, and geometry alike” according to Petri and Schappacher (*ibid.*, 367).

In what I will present, there is more to arithmetization than the mere rigid logic argued for by Klein. Indeed, as shown by the variety of examples given by Klein himself, “arithmetization” takes on several meanings, each based on a different conception of arithmetic. In Dedekind’s works, the use of arithmetic does, indeed, provide a more rigorous and more general ‘foundation’ or ‘definition’ to continuity and the real numbers and aims at avoiding the use of intuition, but his approach seems more systematic and more ambitious than solely the increase of rigid rigor – in fact, rigor appears as a part of the creation process of mathematicians.

7.3.1 Extension of arithmetic?

We saw that it is clearly stated by Dedekind that arithmetic is the “Science of Numbers”, whose first object are the natural numbers (see Sec. 2.6.1). For him, an “arithmetical” theory should imply that “the consideration of the realm of numbers (the absolute constants) would form the main foundation” ([Edwards, Neumann, and Purkert, 1982], 54). Arithmetic, described as an immediate consequence of the “act of counting”, is also, with the introduction of the arithmetical operations, a science of immeasurable depth and width, for Dedekind. As he explains, in *Zum Zahlbegriff*,

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Arithmetic, whose sole subject matter is [the number] concept, is already by now a science of immeasurable extension, and there is no doubt that no limits are set to its further developments. (*Zum Zahlbegriff*, in [Dugac, 1976b], 315)

As we saw, that particular characterization of arithmetic does not keep Dedekind from considering extensions of the systems of numbers as broadenings of the number concept itself. Indeed, the element of each new system created are called *numbers* and treated as arithmetical elements, in particular with the redefinition of the elementary operations for each new system. Dedekind's contemporaries, on the other hand, be they engaged in a generalization of the number concept like Cantor or committed to recognizing only natural integers as actual numbers like Kronecker, did not write about irrational numbers, but about irrational *magnitudes*. Dedekind's viewpoint on the number concept and the enlargement of arithmetic is singular, on this matter.

Irrationals are considered as numbers in relation to the fact that the process of enlargement of the domains of numbers preserves the arithmetical nature of their elements, as I mentioned earlier. Inasmuch as no “foreign elements” are introduced, that the extension is made without using non-arithmetical methods, and with a consistent definition of the elementary operations for the new numbers, then the objects obtained are numbers. For Dedekind, that *Stetigkeit* is as much about arithmetic being given a “truly scientific” foundation and satisfying the demand “that arithmetic shall develop out of itself” ([Dedekind, 1872], 767, 770), as it is about providing “a purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal Analysis”. It is important to emphasize that the definition of irrationals is not arithmetical in order for them to enter the realm of arithmetic, but *because* they are part of arithmetic and, as such, require an arithmetical definition.⁸⁰

A possible additional reason for Dedekind to consider irrationals as numbers properly speaking, is their use in number theory by Dirichlet. Indeed, Dedekind is explicitly erasing the long lasting distinction between “discrete” and “continuous” magnitudes. Recall that Gauss excluded the irrationals from the scope of his *Disquisitiones*. Dirichlet, in 1837, introduced the use of infinite series and analytical methods in number theory, on the basis of which he was able to prove his theorem on arithmetic progressions.

I was only able to establish [that any arithmetical progression whose first term and common ration are integers without common divisors contains infinitely many prime numbers] by relying on properties of a class on infinite series, which have a lot

⁸⁰Whereas Kronecker admits in arithmetic magnitudes that are not natural numbers only insofar as he can provide an acceptable arithmetical definition for them.

in common with the ones used by Euler in the chapter XV of his *Introduction to Infinitesimal Analysis*.⁸¹ ([Lejeune-Dirichlet, 1838], 359)

Dirichlet, convinced that he had found a very fruitful new approach to number theoretical problems, pursued this road further and presented some of the core ideas and results obtained in two articles in 1838 and 1839: “*Sur l’usage des séries infinies dans la théorie des nombres*” and “*Recherches sur diverses applications de l’analyse infinitésimale à la théorie des nombres*” (repr. in [Lejeune-Dirichlet, 1889-1897]). In these articles, he explains that through the researches on the arithmetical progression theorem, he came to understand that “the consideration of this kind of series constitutes a very fruitful method of indeterminate analysis, which can be applied to various questions” (*ibid.*, 360). He thus called the attention to the links between infinitesimal analysis and higher arithmetic, which had been overlooked by Gauss. This new perspective on the relations between analysis and arithmetic allowed him to “consider a great number of questions related to numbers with a entirely new viewpoint”, and to “attach them to the principles of infinitesimal analysis” and the properties of infinite series and products (*ibid.*, 413). Not only did the introduction of analytical methods open new questions and perspectives in number theory, which Dirichlet details in his two papers, it also gave him the possibility to simplify greatly some of the proofs in the theory of binary quadratic forms:

The Analysis that led us to the complete solution of this interesting question will give us, in the same time and, so to speak, on the way, new and very simple proofs of several beautiful theorems due to M. Gauss, but that this illustrious mathematician only established by very complicated considerations, in the second part of the 5th section of his *Disquisitiones arithmeticae*.⁸² (*ibid.*, 413-414)

The possibility to give this new shorter, simpler treatment of some of Gauss’s results is, for Dirichlet, a great advantage of his method. His hope, he says, is that his work can “contribute to the progress of science by establishing new basis” and by making more elementary some “beautiful and important

⁸¹ “Je suis parvenu à établir [que toute progression arithmétique dont le premier terme et la raison sont des entiers sans diviseur commun, renferme une infinité de nombres premiers], en m’appuyant sur les propriétés d’une classe de séries infinies, qui ont beaucoup d’analogie avec celles qu’Euler considère dans le chap. XV. de son *Introd. à l’Anal. de l’inf.* (...)”

⁸² “L’analyse qui nous conduira à la solution complète de cette question intéressante, nous fournira en même temps et pour ainsi dire, chemin faisant, des démonstrations nouvelles et très simples de plusieurs beaux théorèmes dus à M. Gauss, mais que cet illustre géomètre n’avait établis qu’au moyen de considérations très compliquées dans la seconde partie de la 5e section de ses *Disquisitiones arithmeticae*.”

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theories” which had hitherto been accessible to a small number of mathematicians able to follow “long sequences of computations” (*ibid.*).

Dirichlet’s inauguration of analytical methods in number theory is, thus, presented as the possibility to establish a new ground for the investigations of number theory, allowing to bypass difficulties. By this, an extension of arithmetic is made in the consideration of methods of infinitesimal analysis, as legitimate tools for arithmetic. In particular, then, for Dirichlet, continuity and “surd” magnitudes (both algebraic and transcendental) are integrated into arithmetic.

The acceptance of irrationals as numbers and genuine objects of arithmetic comes as an inheritance from Dirichlet and a continuation of his works. Indeed, Dedekind did learn number theory with Dirichlet, and also shared his views about the importance for a method to be able to shorten complicated computations and tedious reasonings.⁸³

It is with this conception of arithmetic as the *science* of numbers, a non-rigid theory susceptible to be widened by the extension of its objects, that Dedekind can introduce continuity in arithmetic – and with this conception that he approaches number theory. His number theoretical works appear to be based on the idea of arithmetic as being an activity of the human understanding subjected to changes, progresses, innovations. As such, arithmetic can be remodeled so as to ease proofs and allow more developments. On these grounds, Dedekind develops, for number theory, new methods in which arithmetical concepts are applied to objects which are not, properly speaking, numbers. As we saw in the previous chapters, Dedekind extends the scope of arithmetic by introducing the consideration of ideals as arithmetical elements which are yet not numbers but systems of numbers.

The introduction of arithmetical operations for objects which are not numbers is first made as a way to reinterpret relationships between concepts, such as groups. The efficiency of this approach, particularly striking in algebraic number theory, seems to have incited Dedekind to pursue this road further and develop an arithmetical approach to other theories, such as algebraic function theory in which not only are number theoretical methods transferred, but an arithmetical ground for the Riemann surface is also given.

With this move, Dedekind extends again the scope of arithmetic by widening the possibility of applications of the arithmetical operations. Indeed, as we saw, ideals are introduced as arithmetical elements. Moreover, in the methods derived from ideal theory, such as module theory or the polygons in Riemannian function theory, the central role played by divisibility

⁸³Note that this specific inheritance is made possible by Dedekind’s general standpoint about mathematics, as a science whose greatest developments are made by the introduction of new concepts. Kronecker, although another of Dirichlet’s followers, considered irrationals as geometrical objects and certainly not admissible as objects of arithmetic – especially not transcendental irrationals.

gives the methods an arithmetical character.

As it has been mentioned several times and is particularly explicit in *Stetigkeit*, one of the core motivations of Dedekind's works is to reach a greater rigor and uniformity in the definition of certain notions, or more widely in the presentation of a theory. In the following, I will inquire into the tight relations between the desire of rigor guiding Dedekind's research and the resort to arithmetical methodology, which should allow me to explain the recurrence of arithmetical methods in Dedekind works, which was put to the fore in the previous chapters.

7.3.2 Reduction to earlier concepts and rigor

The arithmetical methodology in use in Dedekind's algebraic number and algebraic function theories, as well as for the "arithmetization" of continuity in 1872, are not merely the reformulation of already known theories into an arithmetical form. They are the introduction of new concepts, new methods, best-suited for Dedekind's goal, and to which an arithmetical form is given.

According to Dedekind, mathematicians can and should introduce new concepts to stimulate the development of mathematics:

I see nothing meritorious (...) in (...) insisting on the use and recognition of no other than natural numbers. On the contrary, the greatest and most fruitful advances in mathematics and other sciences have preferably been made by the creation and introduction of new concepts, after the frequent recurrence of complex phenomena, which could not be mastered by the old notions, compelled to this.⁸⁴ ([Dedekind, 1888]., transl. slightly altered, 792)

New domains of numbers are, for Dedekind, an example of such a "creation and introduction of new concepts". In the meantime, definitions such as the definition of real numbers should always be done by reduction to "earlier concepts":

the step-by-step extension of the number-concept is (...) to be carried out – the creation of zero, of the negative, rational, irrational, and complex numbers – always by a reduction to earlier concepts, and indeed without any introduction of foreign conceptions (such as for example that of measurable magnitudes),

⁸⁴ "[I]ch erblicke keineswegs etwas Verdienstliches darin (...) keine anderen als die natürlichen Zahlen benutzen und anerkennen zu wollen. Im Gegenteil, die größten und fruchtbarsten Fortschritte in der Mathematik und anderen Wissenschaften sind vorzugsweise durch die Schöpfung and Einführung neuer Begriffe gemacht, nachdem die häufige Wiederkehr zusammengesetzter Erscheinungen, welche von den alten Begriffen nur mühselig beherrscht werden, dazu gedrängt hat."

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which according to my view can attain perfect clearness only through the science of numbers.⁸⁵ (*ibid.*, 792)

The key to satisfy these two requirements, then, seems to be to introduce new concepts based on the previously rigorously defined (“earlier”) concepts, and to use these “earlier” concepts as tools.

7.3.2.1 Rigor and “Dedekind’s principle”

To characterize rigor more precisely, one can follow a proposition by Michael Detlefsen, in his paper “Dedekind Against Intuition. Rigor, Scope and the Motives of His Logicism” ([Detlefsen, 2012]), in which Detlefsen proposes to take the very first sentence of Dedekind’s *Was sind und was sollen die Zahlen?* as a criterion of rigor:

In science nothing capable of proof ought to be believed without proof.⁸⁶ ([Dedekind, 1888], 790)

Detlefsen proposes to interpret this statement as guiding what it is acceptable to believe, what justification can be considered as valid for a scientific proposition. He calls it “Dedekind’s principle” and states that Dedekind’s view is that

[t]he only proper, and the best possible scientific justification of a provable proposition is a proof. More accurately, it is a proof in which all the basic premisses are unprovable. ([Detlefsen, 2012], 210)

From this viewpoint, then, Dedekind’s principle gives us a standard for rigor in mathematics, according to which a proof is rigorous, and therefore acceptable in science, only if it relies on premisses which are believed to be unprovable.

“Dedekind’s principle”, Detlefsen suggests, should be understood as guiding what is acceptable to *believe* in science, which justifications should be considered as acceptable for a scientific proposition. The standard of rigor given by “Dedekind’s principle” demands that any proof, in order to be considered as an acceptable (rigorous) one, ought to rely on premisses which

⁸⁵ “[D]ie schrittweise Erweiterung des Zahlbegriffes, die Schöpfung der Null, der negativen, gebrochenen, irrationalen und komplexen Zahlen stets durch Zurückführung auf die früheren Begriffe herzustellen ist, und zwar ohne jede Einmischung fremdartiger Vorstellungen (wie z. B. der der meßbaren Größen), die nach meiner Auffassung erst durch die Zahlenwissenschaft zu vollständiger Klarheit erhoben werden können.”

⁸⁶ “Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.”

have been believed to be unprovable. This principle, Detlefsen tells us, underlies *Zahlen* and guides a certain Dedekindian logicism.⁸⁷

The idea of the “Dedekind’s principle” appears very consistent with the definition of the natural numbers, which I will study in the next chapter. Indeed, this definition is presented by Dedekind as based on the “Laws of Pure Thought” which were considered as being the only non-provable truths, in the 19th century. Considering this, it is easy to understand how Dedekind’s essay on natural numbers meets this criteria of rigor: the building of the sequence of natural integers, and the arithmetic following from it, are founded on the laws of thought (via systems and applications). It is thus guaranteed that the bases of the reasoning are unprovable premisses (see Section 8.1.1). Hence, following this logical principle of rigor, one *defines* the natural numbers by means of general concepts representing the operations of thought.

Note that there seems to be an ambiguity, here, between definition and proof. The principle of rigor could be stated as a demand of definition, in which it would mean to try to avoid “illicit hypotheses of existence”: one verifies, or proves, the hypothesis of existence by effectively defining the notion whose existence is supposed by a systematic reduction to earlier (already defined) concepts. To avoid gaps, circularities or even impossibilities in demonstrations, one has to never admit something that can be defined, without a definition.⁸⁸ This tension between definition and proof seems to run through Dedekind’s works. Indeed, he often states that a more rigorous definition (of the irrational numbers, of ideals, of the Riemann surface) is essential for more rigorous proofs, for a more rigorous development of the theory investigated.

In *Zahlen*, on which Detlefsen’s reading of Dedekind’s ideal of rigor is based, in order to provide the desired rigorous definition of natural numbers, Dedekind decomposes the reasoning at the roots of the creation of natural numbers. The essay presents itself as a “long series of simple inferences” which, while it can seem tedious and useless, corresponds exactly, from Dedekind’s viewpoint, to the way in which our mind is working, to our “stepped understanding” (*Treppen-Verstand*). This idea, also mentioned in a letter to Cantor, reflects the precept stated earlier: to systematically reduce definitions and proofs to “earlier concepts”, so that each step of the reasoning is known and controlled.

However, in practice, when working with more elaborated concepts, it is not necessary, and not always fruitful, to systematically go back to the first

⁸⁷For Detlefsen, the standard of rigor attached to “Dedekind’s principle” is a “commitment to an objective standard for determining when proof has been pursued to this point [of unprovable premisses]” (*ibid.*, 211). Understood this way, logicism largely represents a norm of properly scientific reasoning, in particular in mathematics, rather than a view concerning the nature of mathematical objects and truths.

⁸⁸I will come back to this question in Sec. 8.1.2.

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unprovable premisses. Dedekind stated very clearly, in the introduction of *Zahlen*, that a systematic reduction to natural numbers is a too cumbersome and tedious work to be fruitful. Likewise, in *Stetigkeit*, Dedekind states that to express all the properties of arithmetic operations by means of cuts would be lengthy considerations. Recall that it would rely on a theorem (see p. 423) of such a “repulsive clumsiness” (*abschreckende Schwerfälligkeit*) that it “convinces us” that one should introduce an “aid to expression” (*etwas (...) um der Sprache zu Hilfe*, here, the notions of variable magnitudes, functions, and limiting values). According to Dedekind, from then on, “it would be best to base the definitions of even the simplest arithmetical operations upon these ideas”⁸⁹ ([Dedekind, 1872], 777).

One should, therefore, contextualize Dedekind’s principle: it is required to define the natural numbers on the ground of unprovable premisses, because one is defining the first object of the simplest science. In further developments of mathematics, one should take advantage of the definitions already given and “reduce the long chains of inferences” by taking the previous, already well-defined, concepts as a basis.

This reduction to “earlier concepts”, which I mentioned above, can play this role, and indeed appears to be an essential part of what Dedekind considers a rigorous method in mathematics. This allows a reduction to “simpler” truths, which are already known, already proven. This way, one is assured to have a theory whose concepts are founded on well-defined notions, in which each step, each inference is controlled, that is, in which one is assured that proofs and definitions present no gaps, no unverified hypothesis of existence, no circularities... Every step is suitably justified and every step is fully and clearly understood. One is thus assured to have a definition allowing complete proofs: nothing is admitted without a proof.

The practice of rigor appears to be closely related to Dedekind’s principle and articulated with the idea that everything that can be reduced to simpler truths should effectively be so. Be it for extending the number concept or to define / prove in more elaborate mathematical works, the reduction to earlier concepts should not be taken as a reductionism, but as proposing that what is already well-defined in earlier conceptual fields can be used as tools for future developments. This way, one can consider the rigorous definition (or proof) obtained as given, and secure future developments in mathematics. By doing so, one assures the consistency of the concept defined, secures the inferences and controls perfectly each of them: each step is justified and clearly understood. One assures a definition allowing gapless / complete proofs: nothing susceptible of proof is admitted without proof.

This is a requirement that should be met even if it calls for long technical preliminaries, even if the sequence of inferences to get to the right definition

⁸⁹ “... zwar wird es das Zweckmäßigste sein, schon die Definitionen der einfachsten arithmetischen Operationen auf diese Begriffe zu gründen. “

is long and, as Dedekind himself stated, “apparently artificial”. Indeed, once this long sequence of inferences is over, one can fix the definition and allow a surer development of the theory investigated. In Dedekind’s algebraic number and algebraic function theories, it seems that it is possible to do so with rational arithmetic operations, with “the simplest principles of arithmetic”, making arithmetic a tool for rigor.

7.3.3 A quest for rigor?

Dedekind’s insistence on rigor is highlighted in the requirement to actually make explicit every step of definitions and proofs which seem, to the trained mathematician, as immediately clear and evidently true as the possibility to consistently extend the order of natural number to negative numbers.

In 1888, Dedekind gives a definition of the natural numbers in terms of mappings and systems. His definition is thus based on earlier notions and even directly related the Laws of pure thought, assuring that arithmetic is ultimately well founded. I will suggest in the following that, in fact, such requirements are guiding many of his most important mathematical researches. It is in relation of such requirements that Dedekind elaborates methods and definitions answering the demand to build new notions and methods of proof on the basis of “earlier concepts” – in particular, using arithmetic. Dedekind is thus able to provide definitions whose rigor is assured because they are only elaborated using already rigorously and uniformly defined notions. By doing so, a more rigorous ground for the theory is obtained. This new definition can be taken as a solid soil on which further developments of the theory can be based in such a way that one needs not to come back to the grounding notions. Arithmetization plays the central role in this approach.

A definition is considered the best possible, by Dedekind, if it satisfies not only the criterion of rigor and avoids relying on *Darstellungsformen*, but also can be seen to be uniformly valid for all cases concerned. This possibility depends largely on the tools used to set up the definition and the theory deriving from it. By using “earlier” and more general concepts, such as the notion of “System”, one avoids the introduction of “foreign” elements, risks of circularities, gaps in the proofs, or illicit hypotheses of existence. By making sure that the definition is rigorous (in Dedekind’s sense) and uniform (a unique valid for all the cases, which Dedekind will often refer to as “general”), one assures that the rest of the theory will be developed as rigorously and uniformly as the definition is.

In 1872, in the preface to *Stetigkeit*, Dedekind notices that “Arithmetic lacks a truly scientific foundation”. To establish such a properly scientific foundation, it is not authorized to rely on theorems which haven’t been proved in all generality, on illicit hypotheses of existence or again to resort to intuition – which is well-known to be the first motive of arithmetization.

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Dedekind's endeavors are, thus, directed towards finding a "purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal calculus". He considers this to have been done in a satisfying way once the definition by means of cuts is given. Cuts only necessitate the supposition of the existence of rational numbers and make it possible to define all irrationals in one move. In addition, it also makes it possible to prove that rational numbers form a continuous domain – an indispensable part, if one is to rigorously ground the arithmetic of irrational numbers.

Besides, for Dedekind, the demands stated about the definitions are, by extension, a guarantee for the rigor and uniformity of the proofs. Indeed, there is implicitly in Dedekind's works, the idea that if one is able to provide the right definition (a rigorous, uniform definition), then the proofs should follow without too much pain. For Dedekind, theorems "result immediately" ([Dedekind, 1854a]), laws are "derived from the sole conditions" ([Dedekind, 1888]) of the definition, proofs "follow immediately from the definitions" ([Dedekind, 1872]).⁹⁰ A rigorous definition allows, in particular, to give rigorous proofs to statements which lacked such proofs. The well-known example of the proof that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ on the basis of a general definition of operations was indeed, for Dedekind, an argument for the efficiency of his definition of the irrational numbers.

In fact, the positive consequences of securing the basic definitions have, from Dedekind's viewpoint, a wider scope than the sole possibility to have rigorous proofs in the theory studied: it also allows to assure future developments in mathematics in a larger way. For example, an appropriate definition of the irrational numbers is essential because

[t]he problem is to indicate a precise characteristic (*Merkmal*) of continuity that can serve as the basis for valid deductions.⁹¹
([Dedekind, 1872], 771)

And a precise characteristic of continuity is essential for infinitesimal Analysis as well as for a fair number of other parts of mathematics. In *Zahlen*, Dedekind explains that he wishes that his "attempt to establish the science of numbers upon a uniform foundation" will incite mathematicians "to reduce the long series of inferences to more moderate and attractive proportions" ([Dedekind, 1888], 792) – a point put to the fore in *Algebraischen Funktionen* as well.

Rigor requirements and insistence on finding the appropriate definition appear in Dedekind's works in algebraic number and algebraic function the-

⁹⁰Dedekind's attention to the deductive structure of a theory, notably expressed in this idea, leads him to adopt a position in which one can distinguish the premisses of Hilbert's axiomatic approach.

⁹¹"es kommt darauf an, ein präzises Merkmal der Stetigkeit anzugeben, welches als Basis für wirkliche Deduktionen gebraucht werden kann."

ories as well, and this conception of rigor is not restricted to foundational areas. In addition, the particular kind of benefit for proofs that I just mentioned is often brought out by Dedekind. But how would a principle such as “Dedekind’s principle” and the demand that definitions be done using earlier notions be used in a non-foundational inquiry? Let us recapitulate the approaches adopted by Dedekind in algebraic number and function theory, so as to highlight the way in which these works appear to answer the demand of rigor – this shall allow me to characterize it more in the next paragraph.

In algebraic number theory, Dedekind wants to “establish the general laws of divisibility” governing algebraic integers. The whole enterprise is guided by the possibility to find the right generalization of Kummer’s concept of ideal, and to generalize the results already obtained by him for cyclotomic integers. Dedekind, thus, wants to be able to answer the question that naturally arises from Kummer’s promising results: are the laws of divisibility generally valid, that is, valid for “all the numerical domains of the most general kind” (i.e. the domains of algebraic integers)? In order to be able to obtain the “general and exceptionless theory”, Dedekind had to give up Kummer’s “formal” approach and find a suitable concept for the generalization. Indeed, as we saw, Kummer’s approach had many drawbacks – from not meeting Dedekind’s standards to erroneous proofs.

For Dedekind, the aim could only be considered reached on the basis of a definition of ideal numbers and their multiplication “exact and common to all ideal numbers”. But because the definition of ideal numbers requires so much reliance on *Darstellungsformen* and is so difficult to generalize, Dedekind rather introduces the concept of ideal defined by two necessary and sufficient conditions of closure by arithmetical operations. The concept of ideal is freed from any consideration of particular factors and from taking into account the individual nature of the numbers studied – or, as Dedekind puts it, ideals are invariant and independent. This definition allows to define arithmetical operations, which will be proved to verify the same properties and laws as rational arithmetic does. It is only on the basis of systems of algebraic integers and arithmetical operations that the general validity of the divisibility laws is therefore proved – on purely arithmetical grounds, without introducing foreign elements or indeterminate variables and in one move for any algebraic number.

For algebraic function theory, Dedekind (and Weber) is following the same path. The relative mathematical vagueness in which Riemann left his readers did not escape Dedekind, who confessed to Weber that he would only truly master Riemann’s works once he would be able to “overcome in [his] way, with the rigor that is customary in number theory, a whole series of obscurities.” It is, as we saw, what Dedekind and Weber propose to do in the 1882 paper, in which a clearer more general foundation is given to Riemann’s

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function theory. Unlike the unsatisfying previous treatments which were using theorems whose truth was admitted on the basis of their geometrical evidence, or were forced to distinguish between cases or even restricting their inquiry to particular kinds of functions, Dedekind and Weber are able to obtain a “more reliable basis for the fundamental notions, as well as for a general and exceptionless treatment of the theory” by grounding their investigation on ideal theory.

The long, technical paper that we have considered in detail at the beginning of this dissertation is the path that led to the said surer basis, and a definition that Dedekind and Weber feel entitled to present as a perfectly precise, rigorous and general foundation for Riemann surfaces. To get there, they use the methods introduced by Dedekind in algebraic number theory, adapt them to function theory, and develop a deeply arithmetical approach for Riemann surfaces, introducing notions such as the greatest common divisors of complexes of points.

The definition of a point given by Dedekind and Weber, as a correspondence between the functions of the field studied and the numerical constants, and the subsequent definition and study of the concept of Riemann surface are thus only involving well defined notions, do not resort to intuition, vague notions, do not make *a priori* hypotheses on the singularities of functions. Moreover, the new definition of the Riemann surface and the new conceptual apparatus developed allow to give the first rigorous proof of the Riemann-Roch theorem and (for Dedekind) promises great developments in Riemannian function theory. In particular, if integration theory and topological considerations had to be excluded from Dedekind and Weber’s works, they should benefit from the foundations established in the paper. Indeed, the “completely precise and rigorous definition of the point of a Riemann surface” that they are able to obtain “can also serve as a basis for the investigation of continuity and related questions” ([Dedekind and Weber, 1882], 42-43).

The definition of irrational numbers was presented as providing a rigorous basis for deductions related to the linear continuum, and the definition of the Riemann surface guarantees a stable basis for the mathematicians wishing to develop Riemann’s theory further and investigate “continuity and related questions”.

A comparable statement is made for algebraic number theory, in 1877. Indeed, for Dedekind, to have a (rigorous and uniform) proof of the general validity of the laws of divisibility of algebraic numbers is of “extremely practical value” for it gives a ground for larger developments (in modular arithmetic, quadratic forms, etc) and “[t]he certainty that these general laws [of divisibility] really exist greatly facilitates the discovery and proof of special phenomena in a given field” ([Dedekind, 1876-1877], transl. slightly altered, 138).

From this summary and from the texts that we have studied in the earlier chapters, one can see that an essential aim of Dedekind's research is to assure the basis of theories so as to support and even carry the development of mathematics. With this emphasis on designing a perfectly rigorous and uniform definition, Dedekind elaborates a strategy that can answer a demand for rigor both ways: everything used in the definition has to be proved,⁹² and everything following on from the definition ought to be (made) provable. One can see that Dedekind does not spare his endeavors to elaborate concepts such as ideals, which are part of an "independent" and "invariant" theory and in such acquire a certain generality for the theory's validity is indifferent to the individual nature of the elements. For each theory investigated, Dedekind adapts his approach and conceives appropriate grounding concepts: cuts for irrational numbers, modules and ideals for algebraic number theory, ideals and points for Riemann surfaces. These grounding concepts are such that they can serve the same purpose of uniformity for the theory developed, the same attempt to provide what Dedekind calls a "general and exceptionless theory". This is, I will argue, the core of Dedekind's strategy of arithmetization.⁹³

7.3.4 A strategy of arithmetization in Dedekind's works?

Note that in the following, one should understand as "arithmetical" what Dedekind counts as arithmetical. In this, his approach is different from Kronecker's and, more generally, might not be seen as "putting into an arithmetical form" such by other mathematicians. The point, here, is not a systematic comparison with other approaches which would take us too far from the point. The following reflection does not either have any pretension to be anything beyond an interpretation of the methods developed by Dedekind, which would avoid to use other categories than his own.

To describe and understand Dedekind's approach as an "arithmetization", here, allows to provide a reading that is consistent from a global perspective, to have a comprehensive explanation of the unity of Dedekind's works. But importantly, it also allows for local explanations of Dedekind's works, that is, to explain the particular methods developed for treating a certain subject. Again, Dedekind's arithmetization is not a reductionist program, it is not answering the desire to unite all mathematics under one

⁹²At least, insofar as the definition is such that going back to the first unprovable premisses is more laborious than fruitful and / or is not necessary.

⁹³The foundation of Dedekind's strategy of arithmetization, on the other hand, are the natural numbers, the definition of which I will consider in the next chapter. In this case, Dedekind also elaborates the definition using "earlier" grounding concepts, mappings and systems. Yet, those concepts trace back the notions and theorems to elements which are not capable of being proved and suggest that the natural numbers hold a specific position in Dedekind's conception of mathematics.

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same arithmetical flag. Rather, it puts to the fore the idea that arithmetic furnishes a powerful set of methods and concepts for the development of rigorous and fruitful theories – be it for the enlargement of domains of numbers or for the definition of the Riemann surface. As such, “arithmetization” is adapted to the context in which it is used, and provides an explanation of the persisting use of arithmetic in Dedekind’s works.

As a method for rigorous definition of systems of numbers, arithmetization is at the heart of Dedekind’s approach. It is not, however, driven exclusively by a desire to rigorize the foundation of Analysis. Rather, it is a systematic approach in which the definition of new systems of numbers is done by relying exclusively on the previous systems and involving only arithmetical tools. For Dedekind, this approach alone can allow for a truly clear understanding of the successive extensions of the numbers concepts, by highlighting the roots and process of the generalization of the number concept, and by providing appropriate definition of the new numbers (and their operations). The resort to arithmetical methods allows to assure that no “foreign element” is introduced in the reasoning, for the extension of the systems of numbers is made “by reduction to earlier concepts, and indeed without any introduction of foreign conceptions” ([Dedekind, 1888], 792).

Arguments of uniformity of the approach, reduction to earlier concepts, conservation of the arithmetical nature of the concepts, are put forward in algebraic number theory as well. To answer these requirements, the whole theory is developed on the basis of a generalization of divisibility. And again, the same kind of arguments appear in algebraic function theory which, developed on the ground of ideal theory and arithmetical operations, is said to “stay in its own sphere”. The theory is, thus, developed by designing new tools using conceptually earlier and well-known notions. According to Dedekind, it is clearer, more precise, and each step of the definitions and proofs, each inference are controlled. It is, in addition to proposing a uniform treatment of algebraic functions, a perfectly rigorous theory.

Another point relating algebraic function theory to arithmetization is the arithmetical treatment given to the point of a Riemann surface, a concept fundamentally related to continuity. As Schappacher pointed out, the definition of the point of a Riemann surface and that of a point of the linear continuum in *Stetigkeit* present “remarkable similarities”:

Both definitions are remarkably similar; both try to conceptualize the intuition of what a concrete point does for you. On the real line, fixing a point can tell you, can it not, where to cut the line in two, and in Dedekind’s analysis, the idea of continuity is precisely that every cut in the line is also afforded by a point. ([Schappacher, 2010], 3262)

To demand a definition of the notion of point that completely bans intuition, and restriction imposed to use already defined concepts, are directly responsible for choosing to define irrational by means of cuts or points as evaluation morphisms. The treatment given for the linear continuum in 1872 is an arithmetization, as we saw, and, as Schappacher insists

the definition of a point on an algebraic Riemann surface given in [*Algebraischen Funktionen*] plainly relies on an arithmetization of the theory of algebraic Riemann surfaces because Dedekind and Weber prepare this definition by a good fifty pages “of a purely formal nature” which carry over to rings of (entire) algebraic functions most of the apparatus that Dedekind had developed earlier for the theory of algebraic integers in number fields. It is this theory which then allowed them, among many other things, to quickly deduce that every point – in the sense of evaluation mapping – corresponds to a prime ideal of the coordinate ring. (*ibid.*, 3263)

Recall that Dedekind considered that space is not necessarily continuous, and that the space on which the mathematicians are working is not (necessarily) a reflection of the real space. And while Dedekind had his own justifications for this belief, it is good to remember that a lot of his ideas were shaped in close relation to Riemann’s approach. Riemann detached the study of space from the consideration of metrics with his concept of *Mannigfaltigkeit* and was well aware of the possibility to develop spherical geometry. Dedekind also saw the development of non-Euclidean geometries, with the works of Bolyai, Lobatchevski and Beltrami. Moreover, unlike many of his contemporaries and in particular Gauss or Kronecker, Dedekind does not consider that geometry is to be excluded from pure mathematics and does not take it to be an empirical science. Geometrical theories should, thus, be treated with the same methods, and of course with the same rigor, as other parts of mathematics.

As underlined by Schappacher ([Schappacher, 2010]), the 1885 *Jahrbuch über die Fortschritte der Mathematik* classifies Dedekind and Weber’s paper as generalities on function theory, while Dedekind’s 1882 work on the discriminant of an algebraic number field is classed as algebra in the chapter on the theory of forms – while Dedekind himself seemed to consider it as pertaining to number theory.

Neither Weber nor Dedekind assessed, at any point, that Riemann’s function theory should be seen as an “arithmetical” subject – wherea in *Stetigkeit*, the “arithmetization” was prompted by the fact that irrationals are numbers and their development is thus part of the “science of numbers”. The development of an arithmetical methodology to present a more rigorous and uniform theory of algebraic functions seems to adopt arithmetic as a

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tool, and put it at the service of the foundation and development of function theory. Arithmetization, here, appears to be a rewriting of certain theories so as to be more faithful to certain epistemological values. However it does not inflect a modification of the nature of the objects in the investigated theory, nor a change of disciplinary area of the theory. Arithmetization is not developed in order to make a theory enter the realm of arithmetic, but because it is the best way, according to Dedekind, to provide a rigorous, uniform treatment of the theory investigated.

Dedekind's rewritings to obtain rigorous theories are done using a combination of set-theoretic notions and rational arithmetic. It was particularly striking in *Algebraischen Funktionen*, in which this strategy is used all along the paper. Arithmetization, construed in this way, seems to point at a possible way of satisfying the conception of rigor stated above, and is in fact largely exploited by Dedekind. It seems that, for him, what could satisfy this demand is to exhibit a construction based on the algebraic concepts of field and ideals and on the operations of rational arithmetic, so as to build a rigorous basis and set up an arsenal to put at the service of certain theories. The arithmetical methodology developed by Dedekind appears to be answering the demand to give a logically rigorous ground to theories. By these choices of methods – the exclusive use of arithmetical operations – and by its motives – to provide a logically rigorous foundation – Dedekind's approach is typically an arithmetization but largely oversteps the scope usually attributed to 'arithmetization'.

Of this, again, ideal theory, as a means of arithmetization, is a particularly good example: the new concept of ideal is a system of already existing elements verifying simple conditions of closure by rational operations, and whose theory is developed relying essentially on the definition of a divisibility relation between ideals and the set up of a "higher level" of arithmetic. This theory offers the possibility to establish an algebraico-arithmetic arsenal involving relatively simple notions, like divisibility, and thanks to which it is possible to give, for example, a new definition of the Riemann surface (for algebraic functions of one complex variable) that is more rigorous, more general and resting largely on the "simplest principles of arithmetic". The arithmetical definition possible through the use of ideals seems to grant, for Dedekind, the possibility to satisfy the essential criterion for rigor: everything that can be defined or proved is effectively defined or proved by reduction to earlier concepts by virtue of arithmetical tools.

It is clear here, and in algebraic function theory, the advantage that Dedekind could have seen in using ideals and arithmetic: rather than engaging himself in long developments requiring laborious computations, Dedekind can deploy an extremely elaborated and fruitful theory whose core are the "simplest principles of arithmetic" and whose generality and rigorous definition is assured. And it will especially be so once Dedekind is able to

provide a rigorous and uniform definition of natural numbers. Note that while Dedekind's book on natural numbers was only published in 1888, a first sketch of the theory, essentially similar to the version published, was written between 1872 and 1878 – a point mentioned by Dedekind in the first preface to *Zahlen* and confirmed by manuscripts. This corresponds to the period during which the second version of algebraic number theory was written, as well as *Algebraischen Funktionen*.

Hence, it appears that Dedekind develops a strategy of putting into arithmetical forms, of arithmetization, which promotes what could be called an 'abstract' arithmetic and allows the introduction of new concepts: the operations of rational arithmetic are used to invent new concepts considered as suited to founding mathematical theories. Arithmetic operations appear, here, to play an active role in the development of mathematical knowledge.

Born in the lectures on Galois theory given in Göttingen in 1856-58, and fully exploited in algebraic number theory, the approach developed by Dedekind articulates the use of infinite systems taken as wholes, following the example of Gauss's notion of class, and the set up of arithmetical operations for these systems. This approach allows to conciliate the demands stated by Dedekind.

First, arithmetic possesses an intrinsic rigor, especially since for Dedekind it is completely independent from any intuition. This is a particularly palpable element in Dedekind's foundational works and very explicit in Dedekind's writings. A role of intuition in arithmetic had been rejected explicitly by mathematicians such as Gauss, who saw numbers as pure creations of the understanding. Arithmetic, based on clearly defined concepts, proceeds by simple inferences resorting exclusively to logic. I will come back to the definition of numbers and arithmetic, and their 'logical' nature in the next chapter.

Secondly, arithmetic also allows to successfully operate the reduction to earlier concepts, without necessarily going all the way up to unprovable premisses – which would be more tedious than efficient, the definition of numbers and operations allowing to not systematically go back to the first premisses. Moreover, it also provides efficient and rigorous methods of proof, for as we saw, Dedekind does not simply state definitions in an arithmetical form, he designs them so that methods of proof can be developed that also follow the arithmetical approach – a point particularly clear in the second part of *Algebraischen Funktionen*.

Finally, and it is not the least advantage of arithmetization, the use of arithmetic combines the requirement of rigor with the freedom to introduce new concepts claimed by Dedekind. The invention of new concept was, for Dedekind, one of the most important sources of development of mathematics. Arithmetic, as we saw in *Algebraischen Funktionen*, can be developed as an 'abstract' theory inasmuch as one can define arithmetical operations for

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objects that are not numbers and, in this, it offers the possibility to introduce new concepts, such as fields or ideals. These new concepts open new areas of research in mathematics and are also extremely fruitful as tools in other parts of mathematics – for binary quadratic forms or algebraic functions.

7.3.5 Conclusion. What arithmetic for Dedekind’s arithmetization?

Dedekind, engaged in a quest for rigorous mathematics, finds a way to answer his requisites by elaborating arithmetical frameworks and methods, using and adapting notions of arithmetic to design new concepts. Arithmetic, from the science of numbers, the “simplest science”, becomes a powerful tool at the service of the mathematicians.

Rigor, as a demand to never believe in what can be proven without a proof or to never use any notion without having defined it in such a way that it only relies on well-defined earlier concepts, guides the elaboration of new concepts and new methods, for Dedekind. From his viewpoint, it is essential for the development of mathematics, to secure the definitions of the concepts, by assuring the rigor by reduction to earlier concepts, but also by assuring the width of validity of the said definition.

As Dedekind’s works on extending the domains of numbers and his careful explanation of the “*Gedankengang*” towards ideal theory show, the analysis of the old theory and the problem(s) to be solved is an important part of Dedekind’s writings.⁹⁴ The analysis enables to identify the key properties and indicates suitable methods for the definition and the proofs. The theory of algebraic integers is a good example of this, since having identified the notion of algebraic integer was a necessary step for the possibility to develop ideal theory in fields of algebraic numbers, and prove the general validity of the laws of arithmetic. Not only is Dedekind able to prove this result, but he does it using essentially the “simplest principles of arithmetic”, putting to the fore the preeminence of the role played by arithmetic in the development of more elaborated theories.

The use of arithmetic to build up definitions and methods of proof meeting Dedekind’s demands, as we saw, can be observed in many of his works. Arithmetization, understood widely as the recourse to arithmetical methods and concepts to obtain better, more rigorous concepts and proofs, appears as a description fitting Dedekind’s approach on a global scale. More importantly, because it implies that arithmetic provides epistemic tools for the development of mathematics and comes as an answer to localized desires of providing a more rigorous theory, the strategy of arithmetization can only be clearly understood together with the idea that it has to be adapted to each theory investigated.

⁹⁴[Sieg and Morris, 2014] relate this to the influence of Lotze.

Chapter 7 - Arithmetic, arithmetization, extension of the number concept

In the above paragraphs, I have underlined how arithmetic is taken advantage of. Arithmetic is made into an epistemic tool so as to take advantage of its benefits and constitute a conceptual arsenal for Dedekind to exploit in his mathematical works. A question that still needs some consideration is the following: while we can understand how arithmetical methods answer Dedekind's requirements, is arithmetic invested of a special rigor by default, or is there some part of Dedekind's ideas on numbers and their science that allows for the development of this arithmetical methods? Conversely, could the approach developed in Dedekind's mathematical works have had an impact on his researches on how to define natural numbers?

Chapter 8

The natural number concept in perspective

In the previous chapters, we saw that throughout Dedekind's works, arithmetic is omnipresent and plays an active role in the development of mathematics. Extended for the sake of giving more rigorous definitions, arithmetic is used by Dedekind as a tool, as a means to elaborate rigorous and uniform definitions and methods of proof, elaborating a strategy of arithmetization which runs through his works. This suggests that arithmetic operations came to be considered as offering the possibility to set up a conceptual arsenal providing the best way to answer Dedekind's epistemological and mathematical concerns. In order to truly understand Dedekind's arithmetization and his conception of arithmetic, the "science of numbers", the link with his conception of number, which begins to appear in *Stetigkeit*, should be made more explicit.

By considering the definition of natural numbers and the foundation of arithmetic, which were published relatively late in Dedekind's career, I wish to investigate further the possibility that Dedekind's practice of mathematics influenced his ideas on numbers and arithmetic. I am not, here, suggesting that the mathematical works are integrally responsible for Dedekind's ideas, but that it affected, left a mark, on Dedekind's ideas, that there is a web of internal influences inside Dedekind's corpus, responsible for a significant part of the peculiar conceptions he developed. Indeed, the persistence with which he uses the same approaches, the rewritings of his works and his tendency to refer to his works as guided by the same methodological stances, suggest a net of – more or less strong, and more or less explicit – influences inherent to the Dedekindian corpus.

Note that I am not stating that Dedekind's mathematics developed in autarky, for as we saw the influence of other mathematicians can be felt in his mathematical works as well. The influences of Gauss, Kummer and

Dirichlet on his number theory are pregnant, and we saw that Dedekind regularly works with typical methods of 19th century mathematics. His relationship with Riemann, his works and correspondence with Weber and with Frobenius all influenced his mathematics in ways I have mentioned before. The “arithmetization” that I have underlined is openly embedded in a tradition going back to Gauss and, in fact, in some ways typical of 19th century mathematics in Germany. Yet, it is also a singular approach in some of the standpoints adopted: to transfer the research to a “higher” viewpoint, the consideration of actual infinities, the definition of ‘abstract’ arithmetical relationships. . . In the same way, his ideas on natural numbers were part of rather widely spread conceptions: numbers as creations of the mind, a rather classical conception of logic, the status and role attributed to the laws of thought. . . and highly singular, in taking mappings and sets at grounding concepts, defining a general concept of simply infinite system conceptual prior to the definition of natural numbers. . . Rather, I am suggesting that Dedekind’s ideas on numbers, on arithmetic, on rigor even were shaped with his writings, by his practice of mathematics, as much as they shaped them.

Indeed, the chronology and the time spent looking for an appropriate definition for the concept of natural number suggest that his definition of numbers bears the mark of his use of arithmetical operations for ideals, for ideals of functions, for classes, for polygons, for classes of polygons, and so on.

In this chapter, which will be organized in four sections, I will bring to the fore the ways in which Dedekind’s mathematical practice and his concept of numbers seem to have affected each other. The chapter has two core aims: to highlight links between arithmetization and the definition of natural numbers, and to argue against a reading of *Zahlen* isolated from the other parts of Dedekind’s works.¹

For this, I will give a presentation of Dedekind’s 1888 *Was sind und was sollen die Zahlen?* displaying the conceptual apparatus involved in defining the natural numbers and highlight some of the most important properties of Dedekind’s definition, so that the definition of natural numbers given by Dedekind is as clear as I have tried to make his other works. Starting with the concepts of system and mapping, which Dedekind takes as grounding concepts for the natural numbers, I will underline how the foundation chosen answers in the same time the need for a definition of the natural numbers

¹ *Zahlen* has been largely commented, in historical works such [Dugac, 1976b] and in recent works in philosophy of mathematics such as [Sieg and Schlimm, 2005], [Potter, 2002], [Reck, 2009], [Reck, 2003]. One can also refer to [Belna, 1996], [Ferreirós, 2008], [Sinaceur and Dedekind, 2008]. For a comparison with Frege, one can refer to [Tait, 1997]. Like for *Stetigkeit*, my aim in this chapter will not be to consider the philosophical questions raised by these authors, nor to propose a systematic comparison of Dedekind’s concept of numbers with other concurrent conceptions.

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following Dedekind's standard of rigor, and a desire to justify the idea that arithmetic and numbers are created by the mind and even participate in human thinking.

I would like to defend the idea that the core aim of his 1888 essay was the *definition* of natural numbers. This implies that *Zahlen* was not the start a program of reformulation of all (elementary) mathematics in terms of sets and mappings. In particular, despite the similarities of their results, Dedekind's and Peano's works on natural numbers are based on very different conceptions and even have different aims. A short inquiry into Peano's ideas shall allow to make this clear – and at the same time, to emphasize my previous point, by contrast.

I will use these ideas and the previous observations on a Dedekindian arithmetization to consider the question of arithmetic as an epistemic tool, as used in the works studied in the previous parts. I will suggest a deeper link between the definition of natural numbers given in *Zahlen* and the use of arithmetic as epistemic tools.

Whenever relevant, during this chapter, I will consider links between *Zahlen* and Dedekind's previous works. Some of these links have been previously hinted at in the literature, in particular as regards the use of sets and mappings. I will consider these in particular in order to highlight the possibility that certain aspects of *Zahlen* serve as a retrospective justification or foundation of the methods employed in previous works.

8.1 Defining the natural numbers

Natural numbers were, in the 19th century, considered to be given (e.g. by Peano, Ohm or Cantor), or as symbols arising from the act of counting (by Schröder), or as given by experience (e.g. by Helmholtz). Weber, in his *Lehrbuch der Algebra*, considers the natural numbers as “already known”, despite the interest shown in letters for *Zahlen*. It seems that a definition of the natural numbers, that is, a definition that produces a new concept, did not appear as a requirement or as a fruitful mathematical work to many mathematicians. In fact, as we saw, it was even seen by some, in particular by Kronecker, as a mistake to try to define natural numbers.

As I pointed out before, the desire to provide more uniform and rigorous grounds for theories is a driving force in Dedekind's major works: in the 1882 paper on Riemann function theory, in the reformulation of Kummer's ideal numbers, in *Stetigkeit*, and even in the rewritings of his own works. . . The importance of providing the “right” definition has been pointed out several times by Dedekind, not only to answer epistemological requirements, but also because of the crucial role played by definitions in setting up the deductive structure of the theory.² This is, in particular, implied in

²This is a point noted by Ferreirós as well: “The new definitions must offer a solid

Dedekind's requirement that a definition of new arithmetical elements provide the means to define the operations in a consistent way and to elaborate methods of proof (for the relevant theorems) without having to import any external notion. In the case of *Zahlen*, this point is particularly crucial since the arithmetical operations are defined for the first time. Since operations are taken to be responsible for the "inexhaustible wealth" of arithmetical laws, as stated in *Stetigkeit*, it is essential for further developments of arithmetic that operations be well-founded. Hence, a definition of numbers and the arithmetical operations is, for Dedekind, a truly rigorous foundation to the "simplest" and "first" science.

I will argue that *Zahlen* should be considered in this perspective. By doing so, it is possible to put *Zahlen* in a more global perspective in Dedekind's works, and understand its significance in relation to Dedekind's other (mathematical) works.

The earlier comparison with Kronecker's concept of number and arithmetization (which aims at reducing the whole of mathematics to natural numbers) and a comparison with Peano's work (which aims at deriving large parts of mathematics from the symbolic rendition of the axioms of natural numbers) which I will propose in Sec. 8.3, underline that Dedekind's work on natural numbers is directed essentially towards providing a definition of the sequence of natural numbers – a point noticed by Peano himself (see p. 509). Dedekind himself stressed both his non-reductionist views and the focus of his book, in his introduction:

In accordance with the purpose of this memoir I restrict myself to the consideration of the series of so-called natural numbers. (...) [I]t appears as something self-evident and not new that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers (...)

foundation for the deductive structure of the whole theory: they ought to enable a sound definition of operations on the new 'elements', and the proof of all relevant theorems" ([Ferreirós, 2008], 103). Ferreirós comes back several times on Dedekind's attention to the deductive structure of *Zahlen* and of mathematical theories more generally. For example, he states that "a set-theoretical definition of ideals implied a strict reliance on sets and set-theoretical relations throughout the theory", considering that it is the reason for Dedekind's rewritings of his algebraic number theory. However, providing a purely set-theoretic presentation of the theory of algebraic numbers, was never explicitly stated as the essential aim of Dedekind's works: the increased uses of systems, here modules and ideals, was motivated by the desire to give the "great difficulty" of the theory (the proof of the reciprocity of multiplication and divisibility of ideals) a sounder basis. The *goal* was a theory that would follow more closely arithmetical lines of development, and it resulted in a greater importance for sets – but always conceived as wholes and studied through arithmetical reformulations of their relations. In fact, a purely set-theoretical version of algebraic number theory, one which would be cleaned up of the arithmetical treatment we observed, would be one that attribute to numbers properties of sets, properties that are not numerical and spoil the unity of nature of numbers.

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[b]ut I see nothing meritorious (...) in actually performing this wearisome circumlocution and insisting on the use and recognition of no other than natural numbers.³ ([Dedekind, 1888], 792)

It is important to understand the aim and consequently the limits of *Zahlen*: by the end of the book, the definition of natural numbers is assured to be rigorous and complete so that the numbers themselves can be used, which was Dedekind's goal. In particular, there is no reason, for Dedekind, to try to reformulate all propositions of arithmetic in terms of systems and mappings, nor to follow a road similar to Peano's, for his aim was (simply) to provide a definition of the natural numbers and the elementary operations. A reduction of mathematics to systems and mappings would not be more fruitful or less of a wearisome circumlocution than a reduction to natural numbers. As Dedekind pointed out himself, the extension of the number concept to negative, fractional, irrational numbers is out of the scope of the inquiry of his book.

It is also important in relation to Dedekind's conception of rigor and for a better understanding of the relation between *Zahlen* and the other parts of Dedekind's corpus. This last point shall allow me to suggest that Dedekind's mathematical works, and in particular the use of arithmetic as a tool to develop more rigorous theories, left a mark on his ideas about how natural numbers and the operations of arithmetic should be defined. Conversely, a better understanding of Dedekind's definition of numbers and arithmetic should give the last piece of explanation concerning Dedekind's strategy of arithmetization.

One of the essential characteristics of numbers, for Dedekind, is the fact that they are creations of the mind, a relatively common idea in the 19th century, and a point for which the definition of natural numbers should be able to account. The definition given by Dedekind, in 1888, is based on the laws of pure thought, and answers several demands. I will propose, in the following, to start by considering the laws of thought in their context and in Dedekind's book, and in a second step to turn to the links between the definition of natural numbers and Dedekind's notion of rigor, as well as between arithmetic, human thinking and the uses of arithmetic as an epistemic tool. With these ideas at hand, I will propose to consider Dedekind's definition in more details.

³ "Dem Zwecke dieser Schrift gemäß beschränke ich mich auf die Betrachtung der Reihe der sogenannten natürlichen Zahlen. (...) [E]rscheint es als etwas Selbstverständliches und durchaus nicht Neues, da jeder auch noch so fern liegende Satz der Algebra und höheren Analysis sich als ein Satz über die natürlichen Zahlen aussprechen läßt, (...) aber ich erblicke keineswegs etwas Verdienstliches darin (...) diese mühselige Umschreibung wirklich vornehmen und keine anderen als die natürlichen Zahlen benutzen und anerkennen zu wollen."

8.1.1 Laws and operations of pure thought

Dedekind's famous *Was Sind und Was Sollen die Zahlen?* is the result of many years of reflection on the number concept. Manuscripts from 1872 to 1878 and letters to Heinrich Weber attest that Dedekind worked on foundations of the number concept for fifteen years. But Dedekind had doubts about publishing this work, as he explains to Weber, and from 1879⁴ to 1887, the project seems to have been on hold.

When, in 1890, Dedekind had to answer criticism about *Zahlen* from Hans Kieferstein, he went on, in a famous exchange of letters, to explain "how [his] essay came to be written." The first part of his researches for finding the right foundation for \mathbb{N} consisted in an analysis of the sequence of natural numbers aiming at the identification of the essential properties of the system of natural numbers:

How did my essay come to be written? Certainly not in one move; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. What are the mutually independent fundamental properties of the sequence N , that is, those properties that are not derivable from one another but from which all others follow?⁵ (Letter to Kieferstein, 27 February 1890, transl. slightly altered in [Heijenoort, 1967], 99-100)

This was followed by the efforts to figure out how these essential properties can be expressed in a perfectly general way, that is, independently from the individual nature of numbers. In Dedekind's words:

how should we divest these properties of their specifically arithmetic content so that they are subsumed under general concepts and under activities of the understanding *without* which no thinking is possible at all but with which a foundation is provided for the reliability and completeness of proofs as well as for the construction of definitions of concepts free of contradictions?⁶ (*ibid.*, translation slightly altered, 100)

⁴Dedekind mentions the essay in the 1879 edition of Dirichlet's *Vorlesungen*, as we saw earlier.

⁵"*Wie ist meine Schrift entstanden ? Gewiss nicht in einem Zuge, sondern sie ist eine nach langer Arbeit aufgebaute Synthese, die sich auf eine vorausgehende Analyse der Reihe der natürlichen Zahlen stützt, so wie diese sich, gewissermassen erfahrungsmässig, unserer Betrachtung darbietet. Welches sind die von einander unabhängigen Grundeigenschaften dieser Reihe N , d.h. diejenigen Eigenschaften, welche sich nicht auseinander ableiten lassen, aus denen aber alle anderen folgen.*"

⁶"*wie muss man diese Eigenschaften ihres spezifisch arithmetischen Characters entkleiden, der Art, dass sie sich allgemeinen Begriffen und solchen Tätigkeiten des Verstandes*

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By “divest these properties [of the sequence of natural numbers] of their specifically arithmetical content”, Dedekind seems to be suggesting that one should be able to define the sequence of natural numbers independently of it being the system of the objects we use for counting and for arithmetical activities linked to the four rational operations. Then, the definition given for natural numbers should provide, in the same move, a rigorous mathematical definition of the “act of counting” and subsequently, of the arithmetical operations. In fact, this demand leads to the attempt to define arithmetic by means of “earlier concepts”, and to attain a conceptual stage that would be ‘earlier’ than arithmetic. The ‘set-theoretic’ construction he gives in *Zahlen*, which I will consider below in detail, strongly suggests that Dedekind was trying to separate what (from his point of view) constitutes a proper definition for the natural numbers – that is, what captures their characteristic properties (what we would call the structure) and gives a proper rigorous construction of them – from what is built from the numbers, what derives from the definition and should be part of further studies of their properties, and is thus specifically *arithmetic* (in particular, the four fundamental operations of arithmetic and their properties and laws).

8.1.1.1 Laws and operations of pure thought as foundation

Since Dedekind’s aim is to establish the “science of numbers” independently from any representation of time and space, or again on purely logical grounds,⁷ it is important that he is able to provide a construction of the “science of numbers” that will allow “to investigate precisely our notions of space and time by bringing them into relation with this number-domain created in our mind”⁸ ([Dedekind, 1888], transl. slightly altered, 791). Natural numbers originate in the mind, which requires that they be treated independently from “the notions or intuitions of space and time”. Numbers, Dedekind tells us, should rather be considered “as flowing immediately from the pure laws of thought.”

Dedekind’s deep analysis of the sequence of natural numbers allowed him to highlight that two basic notions are essential to it: systems and mappings (*Abbildung*). *Abbildungen*, which require the idea of systems, are describing

unterordnen, ohne welche überhaupt kein Denken möglich ist, mit welchen aber auch die Grundlage gegeben ist für die Sicherheit und Vollständigkeit der Beweise, wie für die Bildung widerspruchsfreier Begriffserklärungen?”

⁷For Dedekind, logic is the means for rigor. Rather than a logicist project, Dedekind’s objective seems to be that of providing perfectly rigorous grounds for natural numbers, that is, to give them a perfectly rigorous definition. I will come back to that point at the end of this chapter.

⁸“*sind wir erst in den Stand gesetzt, unsere Vorstellungen von Raum und Zeit genau zu untersuchen, indem wir dieselben auf dieses in unserem Geiste geschaffene Zahlen-Reich beziehen.*”

the capacity of the mind to relate things in a system to things in another system (i.e., to map systems onto each other). Dedekind explains that

[i]f we scrutinize closely what is done in counting a set or number (*Anzahl*) of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and otherwise absolutely indispensable foundation, as I have already affirmed in an announcement of this paper, the whole science of numbers must, in my opinion, be established.⁹ (*ibid.*, transl. slightly altered, 791)

For Dedekind, the concept of number and the way in which it is used are so deeply embodied in human thinking, that counting is an “activity of thought” comparable to that of reading:

[R]eading [for an accomplished reader] always remains a more or less complete repetition of the individual steps which the beginner has to take in his wearisome spelling-out; but a very small number of these steps, and therefore a very small effort or exertion of the mind, is sufficient to enable the practised reader to recognize the correct, true word.¹⁰ (*ibid.*, 792)

It is because mathematicians are so used to these “simple inferences” which constitute the basis of the “laws of numbers”, that they are sometimes mistaking them for intuitive or immediate. In fact, arithmetical reasoning is so deeply embodied in the human thinking since “the time of birth” that a large part of it has become an unconscious act:

So from the time of birth, we are constantly and in an ever increasing measure led to relate things to things and thus to exercise that faculty of the mind on which also depends the creation of numbers; this exercise goes on continually, though without definite purpose, in our earliest years and with the accompanying formation of judgements and chains of reasoning, we acquire

⁹“*Verfolgt man genau, was wir bei dem Zählen der Menge oder Anzahl von Dingen tun, so wird man auf die Betrachtung der Fähigkeit des Geistes geführt, Dinge auf Dinge zu beziehen, einem Dinge ein Ding entsprechen zu lassen, oder ein Ding durch ein Ding abzubilden, ohne welche Fähigkeit überhaupt kein Denken möglich ist. Auf dieser einzigen, auch sonst ganz unentbehrlichen Grundlage muß nach meiner Ansicht, wie ich auch schon bei einer Ankündigung der vorliegenden Schrift ausgesprochen habe, die gesamte Wissenschaft der Zahlen errichtet werden.*”

¹⁰“*dieses Lesen bleibt immer eine mehr oder weniger vollständige Wiederholung der einzelnen Schritte, welche der Anfänger bei dem mühseligen Buchstabieren auszuführen hat; ein sehr kleiner Teil derselben, und deshalb eine sehr kleine Arbeit oder Anstrengung des Geistes reicht aber für den geübten Leser schon aus, um das richtige, wahre Wort zu erkennen*”

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a store of properly arithmetical truths to which our first teachers later refer as to something simple, self-evident, and given in the inner intuition.¹¹ (*ibid.*, transl. slightly altered, 791-792)

The foundation of the “science of numbers” on the laws and operations of pure thought allows Dedekind to provide a rigorous justification of the idea that numbers are creations of the human mind – controlled by the laws of pure thought:

I consider [the number-concept] as flowing immediately from the laws of pure thought. My answer to the problems propounded in the title of this writing is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things.¹² (*ibid.*, 791-792)

8.1.1.2 Numbers as creations of the mind

The idea that numbers are creations of the mind was fairly common, among German mathematicians, in the 19th century. Gauss notably had strong views about arithmetic, the “queen of mathematics”. A well-known idea is that of numbers as creations of the human mind:

It is my deepest conviction that the theory of space has a completely different position in our *a priori* knowledge than does the pure theory of quantity. Our knowledge of the former utterly lacks the complete conviction of necessity (and also of absolute truth) that belongs to the latter; we must in humility grant that, if number is merely the product of our mind, space also possesses a reality outside our mind, and that we cannot entirely prescribe its laws *a priori*.¹³ (Letter to Bessel, 9 April 1830, quoted and

¹¹“*So sind wir auch schon von unserer Geburt an beständig und in immer steigendem Maße veranlaßt, Dinge auf Dinge zu beziehen und damit diejenige Fähigkeit des Geistes zu üben, auf welcher auch die Schöpfung der Zahlen beruht; durch diese schon in unsere ersten Lebensjahre fallende unablässige, wenn auch absichtslose Übung und die damit verbundene Bildung von Urteilen und Schlußreihen erwerben wir uns auch einen Schatz von eigentlich arithmetischen Wahrheiten, auf welche später unsere ersten Lehrer sich wie auf etwas Einfaches, Selbstverständliches, in der inneren Anschauung Gegebenes berufen[.]*”

¹²“*ich [den Zahlbegriff] für einen unmittelbaren Ausfluß der reinen Denkgesetze halte. Meine Hauptantwort auf die im Titel dieser Schrift gestellte Frage lautet: die Zahlen sind freie Schöpfungen des menschlichen Geistes, sie dienen als ein Mittel, um die Verschiedenheit der Dinge leichter und schärfer aufzufassen.*”

¹³“*Nach meiner innigsten Überzeugung hat die Raumlehre in unserm Wissen a priori eine ganz andere Stellung wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Nothwendigkeit (also auch von ihrer absoluten Wahrheit) ab, die der letztern eigen ist; wir müssen in Demuth zugeben, dass, wenn die Zahl bloss unseres Geistes Product ist, der Raum auch ausser unserm Geiste eine Realität hat, der wir a priori ihre Gesetze nicht vollständig vorschreiben können.*”

translated in [Ferreirós, 2007], 238)

Note that this statement comes in discussions about geometry which, for Gauss, requires the use of empirical intuition. Indeed, in his view, logical principles themselves in geometry “only bloom sterile flowers, if the fructifying, living intuition of the object does not act everywhere”. Geometry is not an *a priori* science and thus is not part of “pure mathematics.” Gauss explicitly opposes geometry to arithmetic, which stands purely *a priori*.¹⁴ This distinction does not appear in Dedekind’s writings. For Dedekind, all concepts of mathematics are created by the mind, and geometry is not an exception to the rule.

Martin Ohm, who formed the project to reduce all mathematics to the natural numbers, adopted a broadly formal approach related to that of late 18th century combinatorialism. He considered the natural numbers as given and as the only objects with a real existence, the remaining parts of mathematics being a sort of theory of numerical signs, reconstructed by manipulation of formulae. Interestingly, Ohm combined a formalist approach with the consideration of symbols as having primarily a mental existence, as Ferreirós explains ([Ferreirós, 2008], 12). For example, a formal power series is, for Ohm, a function of degree infinitely high and “therefore, an entire function that is never really representable, but only lives in the idea within ourselves” (quoted and translated in [Ferreirós, 2008], 12). Furthermore, operations rather than numbers are considered by Ohm as essential in calculus, and operations are “actions of the understanding”:

In the most diverse phenomena of the calculus (of arithmetic, algebra, analysis, etc.) the author sees, not properties of quantities, but properties of the operations, that is to say, actions of the understanding (...). It turns out that one only calculates with “forms,” that is, with symbolized operations, actions of the understanding that have been suggested (...) by the consideration of the abstract whole numbers.¹⁵ (Quoted and translated in [Ferreirós, 2008], 13)

Ferreirós explains that the general symbolical rules, in Ohm’s writings, “represent mental actions performed on mentally existing forms or symbols” (*ibid.*, 12). For Ohm, the symbolical operations are suggested by the consideration of the natural numbers, a statement he makes in reference to what will later be called the “principle of permanence of formal laws.”

Dedekind’s notion of operation was that of an action of thought on the operands and opposed to a formal or symbolical one. This allowed him

¹⁴See [Ferreirós, 2008], §I. 2.

¹⁵“In den verschiedensten Erscheinungen des Kalküls (der Arithmetik, Algebra, Analysis, u.u.) erblickt der Verf. nicht Eigenschaften der Grössen, sondern Eigenschaften der Operationen, d.h. Akten des Verstandes ...”

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to develop arithmetical methods in non-arithmetical settings: by defining operations as relations between given objects, Dedekind was able to extend the operations of arithmetic in a way that enabled the development of non-formal arithmetical approaches. The idea that the individual nature of objects or elements of a system should not enter into account in the investigation should not be taken as a move towards formal mathematics, since, as we saw, Dedekind expresses no interest in developing the “formal” theory of modules or ideals.

8.1.1.3 Some context about the laws of thought

The second point in need of background, here, are the laws of pure thought. In the German speaking world, debates about the laws of thought and their nature were numerous.¹⁶ It is not the place, here, to go over these questions, but it is good to know that these ideas were relatively widespread, if only to see that Dedekind’s mathematical innovations are integrated in a rather traditional philosophical setting. It was then common,¹⁷ in England as well as in Germany, to broadly equate “logic” and the “laws of thought” – each author making its own refinements – following Kant:

Logic is a science of reason not only as to mere form but as to matter; a science a priori of the necessary laws of thought, not in regard to particular objects, however, but to all objects in general — hence a science of the correct use of the understanding and of reason in general, not subjectively, however, i.e., not according to empirical (psychological) principles for how the understanding does think, but objectively, i.e. according to a priori principles for how it ought to think. ([Kant, 2012], 531)

Albeit not thoroughly Kantian, the epistemological framework in which Dedekind is thinking does present strong Kantian characteristics, especially on two points: the creativity of the mind and the kind of “objectivity” possessed by mathematical concepts as created by the activity of the understanding.¹⁸ His understanding of logic as the laws of thought should, then, be understood in this context. Sinaceur underlined that if *mathematically*,

¹⁶See [Kusch, 1995].

¹⁷But there are exceptions, notably Bolzano refuses to attribute any central role to the laws of thought, because even if they are certainly true

one can by no means say that they contain “the ground of all truth in thinking, and are sufficient to determine the truth and correctness of all our thoughts.” For they contain, as far as I can see, neither the objective, nor even some subjective, ground for the deduction of any truth worth talking about, let alone of all truths (WL §45).

¹⁸For a transcendental reading of Dedekind’s ideas on numbers, see [McCarty, 1995].

Dedekind is strongly opposed to Kantian ideas or to strict constructivism, he seems to adopt “from an *epistemological* viewpoint, a revised Kantian constructivism” ([Sinaceur and Dedekind, 2008], 123).

Logic as strictly related to the laws of thought is an idea that is developed and investigated in detail by William Hamilton in his *Logic*. The links between logic and the laws of thought are explicitly stated: “Logic is the Science of the Laws of Thought as Thought” ([Hamilton, 1863], 9). He explains later that this statement means that “Logic is the science of the necessary forms of thought” (*ibid.*, 16). Hamilton, quoted and studied by Detlefsen in [Detlefsen, 2012], states that “Thought as the knowledge of one thing in relation to another, is a mediate and complex cognition” ([Hamilton, 1863], 36).

What are the laws of thought, for Hamilton, then? They are the object of “Pure logic”¹⁹ and they are directing, even norming, our thinking²⁰ The laws of thought (“or logical necessity”) should not be mistaken for physical laws, such as the law of gravitation, they are rather

general precepts which we are able certainly to violate, but which if we do not obey, our whole process of thinking is suicidal, or absolutely null. These laws are, consequently, the primary conditions of the possibility of valid thought, and as the whole of Pure Logic is only an articulate development of the various modes in which they are applied, their consideration in general constitutes the first chapter in an orderly system of the science. (*ibid.*, 39)

Unlike Dedekind, Hamilton explicitly states four laws of thought:

The Fundamental Laws of Thought or the conditions of the thinkable, as commonly received, are four: 1. The Law of Identity; 2. The Law of Contradiction; 3. The Law of Exclusion or of Excluded Middle; and, 4. The Law of Reason and Consequent, or of Sufficient Reason” (*ibid.*, 39-40).

According to Detlefsen, these four laws were “commonly regarded as laws of pure thought” ([Detlefsen, 2012]). Writers gave lists similar to Hamilton’s, sometimes excluding the principle of sufficient reason, or including more

¹⁹“Pure Logic considers the laws of thought proper, as contained a priori in the nature of pure intelligence itself.” (*ibid.*, 29)

²⁰“Pure Logic may, I think, best be distributed upon the following principles. We may think; and we may think well. On the one hand, the conditions of thinking do not involve the conditions of thinking well; but the conditions of thinking well involve the conditions of thinking. Logic, therefore, as the science of thought, must necessarily consider the conditions of the possibility of thought. On the other hand, the end of thought is not merely to think, but to think well; therefore, as the end of a science must be conformed to the end of its object-matter, Logic, as the science of thought, must display not only the laws of possible, but the laws of perfect thinking.” (*ibid.*, 31)

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laws. The “laws of pure thought” were a relatively widespread idea, in the 19th century, and can also be found in writers closer to Dedekind.

The laws of thought conceived as normative laws for the process of thinking are important in Lotze’s writings. Laws of thought are for him “the necessary condition of any thought being true.” In *Outlines of Logic*, he explains that

they are those subjective modes of the connection of our thoughts which are necessary to us, if we are by thinking to know the objective truth. ([Lotze, 1882], 6)

In *Mikrokosmos*, Lotze explains that the laws of thought “arise from the organization of our mind as innate modes of its activity without having any original relation to the nature of the objects with which they are destined to deal” ([Lotze, 1856], Book VIII, 322). For Lotze, thought acts on ideas not as a simple distinction of what is true or false, but in such a way that they become subjected to laws, or *capable* of being true or false. One can, then, come to apprehend truth with the operations of thought. Laws of thought can serve to test the accuracy of a reasoning:

we have certain laws of thought by which, given certain valid premises, logical conclusions derived from them are distinguished from illogical. ([Lotze, 1843], 450)

Note that for Lotze, thinking, is “an activity of relating”, a point that we find in Hamilton’s and Dedekind’s thinkings as well.²¹

8.1.1.4 Operations of pure thought and arithmetic, for Dedekind

The interesting point, here, is that Dedekind both adopts Gauss’s ‘intellectualism’ and places logic, as the laws of thought, at the foundation of *all* mathematics. This point makes a fundamental difference between him and

²¹One should also mention Frege’s particular position on the matter. Summed up by Kusch, Frege’s main point is a differentiation between “prescriptive” and “descriptive” laws. The prescriptive laws might be called “laws of thought” by some authors since “they legislate how one ought to think”, but only the “set of psychological [descriptive laws]” should properly be labelled “laws of thought” ([Kusch, 1995], 32). Frege’s anti-psychologism leads him to reject the idea of laws of thought as the basis of logic. His take on the Laws of thought is significantly different from Dedekind’s. Frege’s accusations of psychologism to Dedekind are well-known, as is the fact that the Kantian standpoint of Dedekind’s conception of logic bypasses these accusations. In any case, Dedekind did not read Frege’s works until after the publication of *Zahlen*. Among significant authors who Dedekind had not read by the time he wrote *Zahlen*, one should also mention Hermann Grassmann, and his *Lehrbuch der Arithmetik für höhere Lehranstalten* (1861). We know by a letter of Dedekind to Lipschitz, that, in 1876, Dedekind still hadn’t read Grassmann’s *Lehrbuch*.

the authors mentioned. Indeed, the conjunction of these two ideas gives rise to a peculiar conception of mathematics in which *all* mathematics (geometry included and with no part of it being “given”) is produced by the mind, on the basis of and under the control of the laws of thought (or logic), but without being a formal science.

Another important, and most interesting difference with previous authors, here, is that Dedekind does not merely state that arithmetic flows from the laws of thought, that it is a product of the human understanding, he provides a mathematical framework to support his claim. Indeed, the concepts chosen by Dedekind as grounds for arithmetic are mathematical conceptualisations of operations of the mind, giving ground to the idea that arithmetic is the “science of numbers”, an activity of the understanding.²² Moreover, with this foundation, Dedekind gives a philosophical and mathematical ground to the idea that arithmetic, and all of mathematics, are an activity of the understanding – a point to which I will come back.

The first operation is collecting things with a common property – that is forming sets of elements satisfying a same condition:

It very frequently happens that different things, a, b, c, \dots , apprehended for some reason from a common point of view, can be associated in the mind, and we then say that they form a system S ; we call the things a, b, c, \dots elements of the system S , they are contained in S ; conversely, S consists of these elements. Such a system S (a collection, a manifold, a totality) as an object of our thought is likewise a thing; S is completely determined if for every thing, it is determined whether it is an element of S or not.²³ ([Dedekind, 1888], transl. slightly altered, 797)

A “thing” is defined as an “object of thought” – note that Dedekind defines and studies the *things* and not the objects of thought. With the notion of system, Dedekind wants to provide a rigorous mathematical theory of any object (or so to speak, of generic objects) of thought.²⁴

Besides, the notion of system, here, is not strictly similar to our modern notion of set, insofar as modern set-theory considers sets from an extensional viewpoint, as an aggregate of elements. For Ferreirós, “Dedekind’s conception of set can be regarded as a typical example of the old ‘naïve’

²²Sinaceur stresses this point in her introductory note to *Zahlen*, p. 106.

²³“*Es kommt sehr häufig vor, daß verschiedene Dinge a, b, c, \dots aus irgendeiner Veranlassung unter einem gemeinsamen Gesichtspunkte aufgefaßt, im Geiste zusammengestellt werden, und man sagt dann, daß sie ein System S bilden; man nennt die Dinge a, b, c, \dots die Elemente des Systems S , sie sind enthalten in S ; umgekehrt besteht S aus diesen Elementen. Ein solches System S (oder ein Inbegriff, eine Mannigfaltigkeit, eine Gesamtheit) ist als Gegenstand unseres Denkens ebenfalls ein Ding; es ist vollständig bestimmt, wenn von jedem Ding bestimmt ist, ob es Element von S ist oder nicht.*”

²⁴See Sinaceur’s introductory notes in [Sinaceur and Dedekind, 2008].

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approach, although he avoided associating too closely sets and concepts” ([Ferreirós, 2008], 226). If there is, at the end of his definition of system, a statement resembling the principle of extensionality, Dedekind “preferred to leave undetermined what kind of ‘common point of view’ may determine the set” (*ibid.*, 228). In fact, as Sinaceur explains, Dedekind’s systems are

first the correlate of a *point of view* defining, *in thought*, the assembling, the unity of a multiplicity of elements, the goal of [*Zahlen*] being precisely to give a mathematical content to this thought, to this unificatory *concept*.²⁵ ([Sinaceur and Dedekind, 2008], 153, footnote 1, my translation)

Hence, Dedekind’s systems are a mathematical conceptualisation of the action, made in thought, of collecting things (possibly infinitely many of them) sharing a common property. In one of the drafts of *Zahlen*, Dedekind wrote:

The things with a common property (...) should be treated as a new thing, called system.

Dedekind told Bernstein that he considered sets “as a closed bag, which contains things entirely determined but that one would not see and about which one wouldn’t know anything else than that they are existent and determined.”²⁶

The use of a single letter to designate a set of elements, used by Dedekind since his lectures on Galois theory, emphasizes the idea that systems are themselves objects of thought “and likewise a thing”. The possibility to define arithmetical relationships between ideals (or groups, fields, etc.) relies essentially on this idea, as does the possibility to consider classes of ideals, or to study systems of systems (see for example p. 318).

Dedekind, here, is laying the first stone of a general theory of systems, essential premisses for his definition of natural numbers, but which also allows a retrospective foundation of his previous uses of (infinite) systems, in algebraic number and function theories and for the definition of real numbers. The importance of “systems” in Dedekind’s mathematics is well-known, and we have seen in the previous chapters how it played an essential role in bypassing and even sometimes erasing computations and avoiding to base

²⁵ “Le ‘système’ de Dedekind est d’abord le corrélat d’un point de vue définissant en pensée l’assemblage, l’unité d’une multiplicité d’éléments, l’objet de l’ouvrage étant précisément de donner un contenu mathématique à cette pensée, à ce concept unificateur.”

²⁶ “[E]r stelle sich eine Menge vor wie einen geschlossenen Sack, der ganz bestimmte Dinge enthalte, die man aber nicht sähe, und von denen man nichts wisse, außer daß sie vorhanden und bestimmt seien.” (Quoted by E. Noether in [Dedekind, 1932] III, 449) Bernstein also recalls that, some time later, Cantor “straightened his colossal stature, made a grandiose gesture with upraised arms and said, staring into space, ‘I picture a set as an abyss.’”

researches on *Darstellungsformen*. The possibility to provide a general theory of systems and to establish their essential properties for any element of indeterminate (and thus indifferent) nature seems to show that the use of set-theoretic methods is perfectly well founded. It suggests that the validity of Dedekind's 'set-theoretic' methods does not depend on the nature of the elements.

In particular, the many details and proofs about union, intersection, inclusion of systems seem to give a theoretic ground to their use for groups, modules and ideals. Indeed, the arithmetical operations and notions defined for these concepts (e.g., inclusion as a division relation) are reformulations of the set-theoretic relationships, and it would be naïve to believe that the fact that it is possible to set up an arithmetical framework for these investigations is sufficient to assure a rigorous foundation for infinite systems. By giving a rigorous theory of *any* system, Dedekind provides a solid theoretical ground to the intensive use of systems made in his mathematical works. Notably, he had defined the intersection of several systems as their least common multiple without giving any algorithm or even any indication for exhibiting explicitly the LCM of two or more systems. In *Zahlen*, he explains in a footnote the validity and legitimacy of non-constructive definitions:

How this determination [of the system S] is brought about, and whether we know a way of deciding upon it, is entirely indifferent for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances. I mention this expressly because Kronecker not long ago (Crelle's Journal, Vol. 99, pp. 334-336) has endeavoured to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified; but there seems to be no call to enter upon this matter with more detail until the distinguished mathematician shall have published his reasons for the necessity or merely the expediency of these limitations.²⁷ ([Dedekind, 1888], 797)

Once the notion of system defined, Dedekind gives a series of properties and further definitions for the concept of system. He defines, for a system A , the property of being a part of a system S :²⁸

²⁷ "Auf welche Weise diese Bestimmtheit zustande kommt, und ob wir einen Weg kennen, um hierüber zu entscheiden, ist für alles Folgende gänzlich gleichgültig; die zu entwickelnden allgemeinen Gesetze hängen davon gar nicht ab, sie gelten unter allen Umständen. Ich erwähne dies ausdrücklich, weil Herr Kronecker vor kurzem (im Band 99 des Journals für Mathematik, S. 334 bis 336) der freien Begriffsbildung in der Mathematik gewisse Beschränkungen hat auferlegen wollen, die ich nicht als berechtigt anerkenne; näher hierauf einzugehen, erscheint aber erst dann geboten, wenn der ausgezeichnete Mathematiker seine Gründe für die Notwendigkeit oder auch nur die Zweckmäßigkeit dieser Beschränkungen veröffentlicht haben wird."

²⁸In a manuscript entitled "Dangers of the theory of systems" (French transl. in [Sinaceur

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A system A is said to be part of a system S when every element of A is also an element of S . Since this relation between a system A and a system S will occur continually in what follows, we shall express it briefly by the symbol $A3S$.²⁹

Dedekind then gives properties of the inclusion, the definition of union (denoted by $\mathfrak{M}(A, B, C \dots)$) and intersection (noted by $\mathfrak{G}(A, B, C \dots)$) of several systems $A, B, C \dots$, and the essential related results.

The growing importance of systems in Dedekind's writings was underlined by Ferreirós as resulting from "the influence of Riemann and Hamilton, but above all in connection with his own experiences in the fields of algebra and number theory":

Having been accustomed to traditional logic since the *Gymnasium*, it seemed clear to him that a set or class can be regarded as the extension of a concept, and that a theory of sets ought to be a part of logic. Dedekind's algebraic work, and his analysis of arithmetic, led him to focus on a second notion, that of mapping, which presupposed the idea of set. But he could convince himself that it is also a logical notion. ([Ferreirós, 2008], 243)

Mappings conceptualize the second operation of the mind. They are also linked to Dedekind's previous works (and explicitly so) and consist in relating things to things, namely to map a set onto another:

By a mapping (*Abbildung*) φ of a system S we understand a law according to which, to every determinate element s of S , there belongs a determinate thing which is called the image (*Bild*) of s and which is denoted by $\varphi(s)$; we say also that $\varphi(s)$ corresponds to the element s , that $\varphi(s)$ results or is produced from s by the mapping φ .³⁰ ([Dedekind, 1888], 799)

and Dedekind, 2008], 317-320) written in the late 1890s but whose main idea was mentioned in a letter to Weber in 1888, Dedekind underlines the risks inherent to a confusion between "being an element of" and "being a part of" a system. It is particularly dangerous in the case of a system with only one element, and Dedekind stresses the necessity to make a clear distinction between the two ideas.

²⁹ "Ein System A heißt Teil eines Systems S , wenn jedes Element von A auch Element von S ist. Da diese Beziehung zwischen einem System A und einem System S im folgenden immer wieder zur Sprache kommen wird, so wollen wir dieselbe zur Abkürzung durch das Zeichen $A3S$ ausdrücken." (*ibid.*, 797)

I will keep Dedekind's notation rather than using the modern \subset .

³⁰ "Unter einer Abbildung φ eines Systems S wird ein Gesetz verstanden, nach welchem zu jedem bestimmten Element s von S ein bestimmtes Ding gehört, welches das Bild von s heißt und mit $\varphi(s)$ bezeichnet wird; wir sagen auch, daß $\varphi(s)$ dem Element s entspricht, daß $\varphi(s)$ durch die Abbildung φ aus s entsteht oder erzeugt wird."

The notion of *Abbildung* was fairly often used in geometrical contexts, for example for mapping a surface onto another, by Gauss and Riemann notably. We saw (p. 364) that, in Dedekind’s case, the notion of *Abbildung* appears in a published work for the first time in the 1879 version of algebraic number theory, in the context of the definition of conjugate fields.³¹ Dedekind explicitly refers to this paragraph, in *Zahlen*. Recall that in 1879, Dedekind acknowledged the importance of the concept of *Abbildung*, presented here as a “general substitution”. From “substitutions” conceived more generally as transforming a system of numbers into another, Dedekind elaborates and defines a more general notion, that of *Abbildung*, which he considers as essential for mathematics, and even for thinking. In 1879, he explains in a programmatic footnote:

Upon this faculty of the mind of comparing a thing ω with a thing ω' , or relating ω with ω' , or making ω' correspond to ω , without which no thinking at all is possible, is also based the entire science of numbers, as I shall try to show in a different place.³² ([Lejeune-Dirichlet, 1879], 470, transl. slightly altered in [Ferreirós, 2008], 89)

As it was the case for systems, Dedekind’s notion of mapping comes from his works in number theory (i.e., Galois theory and algebraic integer theory³³). Systems and mappings are central in Dedekind’s approach for they are elementary, fundamental notions which can be fruitfully used in many different theories. In particular, they provide a basis for the definition of natural numbers. The fact that these notions are developed for *any* object of thought shows the possibility to provide a rigorous theoretical ground for systems and mappings, that allows them to be universally (in all areas of mathematics) usable.³⁴

Dedekind gives several properties of mappings, such as for example:

If $A3B$, then $A'3B'$.³⁵ ([Dedekind, 1888], 800)

And the definition of the composition of mappings:

³¹See p. 364 and see [Sieg and Schlimm, 2014] for more details about Dedekind’s concept of *Abbildung*.

³²“*Auf dieser Fähigkeit des Geistes, ein Ding ω mit einem Ding ω' zu vergleichen, oder ω auf ω' zu beziehen, oder dem ω ein ω' entsprechen zu lassen, ohne welche ein Denken überhaupt nicht möglich ist, beruht, wie ich an einem anderen Orte nachzuweisen versuchen werde, auch die gesammte Wissenschaft der Zahlen.*”

³³It is possible that the definition of the point of a Riemann surface strengthened Dedekind’s ideas on mappings, but “*Abbildungen*” appear in the drafts for *Zahlen* before the writing of *Algebraische Funktionen*.

³⁴For example, Dedekind’s proof that the propositions of Euclid’s *Elements* do not require space to be continuous is made by exhibiting a one-to-one correspondence between points and algebraic integers which form a discontinuous domain.

³⁵“*Ist $A3B$, so is $A'3B'$.*”

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If ϕ is a mapping of a system S , and ψ a mapping of the image $S' = \phi(S)$, there always results a mapping θ of S , compounded out of ϕ and ψ , which consists in this: to every element s of S there corresponds the image

$$\theta(s) = \psi(s') = \psi(\phi(s))$$

where again we have put $\phi(s) = s'$. This mapping θ can be denoted briefly by the symbol $\psi.\phi$ or $\psi\phi$, and the image $\theta(s)$ by $\psi\phi(s)$ where the order of the symbols ψ, ϕ is to be considered, since in general the symbol $\phi\psi$ has no interpretation and actually has a meaning only when $\psi(S')\exists S$.³⁶ (*ibid.*, 800-801)

Dedekind also defines “similar mappings”, which are injective mappings (and in fact, since Dedekind’s mappings are defined from S into their image, similar mappings are bijective) and inverse mappings:

A mapping φ of a system S is said to be similar or distinct, when to different elements a, b of the system S there always correspond different images $a' = \varphi(a), b' = \varphi(b)$. Since, in this case, if $s' = t'$ then $s = t$, every element of the $S' = \varphi(S)$ is the image s' of single, perfectly determinate element s of the system S , and we can therefore set over against the mapping φ of S an inverse mapping of the system S' , to be denoted by $\bar{\varphi}$, which consists in this: to every element s' of S' there corresponds the image $\bar{\varphi}(s') = s$; obviously this mapping is also similar.³⁷ (*ibid.*, 801)

The discovery of the paradoxes of set theory published in Russell’s *Principles of Mathematics* in 1903, which affect certain parts of Dedekind’s essay, led Dedekind to refuse the third re-edition of *Zahlen* for eight years. In fact, the paradox linked to the consideration of the totality of all systems, used for the infamous Theorem 66 (see p. 484), was signaled to Dedekind by Cantor,

³⁶ “Ist ϕ eine Abbildung eines Systems S , und ψ eine Abbildung des Bildes $S' = \phi(S)$, so entspringt hieraus immer eine aus ϕ und ψ zusammengesetzte Abbildung θ von S , welche darin besteht, daß jedem Elemente s von S das Bild $\theta(s) = \psi(s') = \psi(\phi(s))$ entspricht, wo wieder $\phi(s) = s'$ gesetzt ist. Diese Abbildung θ kann kurz durch das Symbol $\psi.\phi$ oder $\psi\phi$, das Bild $\theta(s)$ durch $\psi\phi(s)$ bezeichnet werden, wobei auf die Stellung der Zeichen ψ, ϕ wohl zu achten ist, weil das Zeichen $\phi\psi$ im allgemeinen bedeutungslos ist und nur dann einen Sinn hat, wenn $\psi(S')\exists S$ ist.”

³⁷ “Eine Abbildung φ eines Systems S heißt ähnlich (oder deutlich), wenn verschiedenen Elementen a, b des Systems S stets verschiedene Bilder $a' = \varphi(a), b' = \varphi(b)$ entsprechen. Da in diesem Falle umgekehrt aus $s' = t'$ stets $s = t$ folgt, so ist jedes Element des Systems $S' = \varphi(S)$ das Bild s' von einem einzigen, vollständig bestimmten Elemente s des Systems S , und man kann daher der Abbildung φ von S eine umgekehrte, etwa mit $\bar{\varphi}$ zu bezeichnende Abbildung des Systems S' gegenüber stellen, welche darin besteht, daß jedem Elemente s' von S' das Bild $\bar{\varphi}(s') = s$ entspricht, und offenbar ebenfalls ähnlich ist.”

in 1896-1897. For Dedekind, this was the cause of doubts about “the complete rationality of human thought”, according to Bernstein’s testimony.³⁸ Dedekind, while acknowledging that this was “a fatal blow to his mistakes”, agreed to the third re-edition of *Zahlen* in 1911 for he still believed that

a rigorous investigation of the creativity (*Schöpferkraft*) of the mind to create from determinate elements a new determinate, their system, that is necessarily different from each of these elements, will certainly lead to an unobjectionable formulation of the foundations of my work.³⁹ (Preface to the third edition, transl. slightly altered, 796)

And in fact, as Zermelo himself noticed, it is clear that the mathematical core of *Zahlen* is not affected if one takes away Theorem 66 and replaces it by Zermelo’s axiom of infinity.

It is Dedekind’s principle of rigor, that led him to attempt to prove the existence of infinite systems. This proof is a bright example of *Zahlen*’s motto: “nothing capable of proof ought to be believed without proof”, but it is far from being its only illustration in *Zahlen*.

8.1.2 Defining numbers and rigor, one last time

Dedekind’s book, whose title is a question about the nature and the role of numbers, is opened by a sentence whose link with a foundation of natural numbers does not seem *a priori* crystal-clear:

In science nothing capable of proof ought to be believed without proof. (*ibid.*, 790)

We saw, in Sec. 7.3.2, that Detlefsen proposes an interpretation of this idea, which he calls “Dedekind’s principle”, as being a logical principle guiding what kind of statements can be accepted and believed as scientific propositions. It tells us that the “only proper, and the best possible scientific justification of a provable proposition is a proof (...) in which all the basic premises are unprovable” ([Detlefsen, 2012], 210).

This standard of rigor demanding that any proof rely on premisses which have been acknowledged as unprovable, according to Detlefsen, is at the core of *Zahlen* and a certain Dedekindian logicism. Detlefsen’s argument is to be related to the idea, commonly accepted in the 19th century, that the “laws of pure thought”, on which *Zahlen* is built, would be the only unprovable

³⁸See E. Noether’s note in [Dedekind, 1932] III, 449, in which she reproduces the letter from Bernstein which we mentioned earlier.

³⁹“*ich glaube, daß eine strenge Untersuchung der Schöpferkraft des Geistes, aus bestimmten Elementen ein neues Bestimmtes, ihr System zu erschaffen, das notwendig von jedem dieser Elemente verschieden ist, gewiß dazu führen wird, die Grundlagen meiner Schrift einwandfrei zu gestalten.*”

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(logical) truths (see [Detlefsen, 2012], 208-210). Following “Dedekind’s principle”, then, being unprovable truths makes of the laws of thought perfect candidates to be the ultimate premisses and answer the requirement of rigor:

This being so, [the laws of thought] provided a similarly objective terminus for the pursuit of rigor. According to Dedekind’s Principle, a genuinely rigorous or scientific mathematics required that proof be pursued to the fullest extent logically possible – that is, to the point where every proposition susceptible of proof had been proved. (ibid., 211)

It is then clear that (and how) Dedekind’s essay answers the requirement of rigor demanding that nothing capable of being proved be accepted without a proof: the construction of the sequence of natural numbers and the arithmetic deduced from it are indeed based on the laws of thought, ensuring that one relies on unprovable premisses. It is to answer the principle of rigor that nothing should be left unproven (if it is provable), that Dedekind *defines* the sequence of natural numbers with the general concepts of system and mappings representing operations of thought. The laws of pure thought, thus, provide a rigorous ground – the only acceptable logical ground – for the definition of numbers, the objects of the “simplest science”, insofar as they are the only unprovable logical truths.

“Dedekind’s principle” appears as a driving force, in *Zahlen*, to *define* the sequence of natural number on the basis of the operations of thought. I have underlined earlier a certain ambiguity between proof and definition, if one considers *Zahlen* as being essentially an application of “Dedekind’s principle”. Indeed, the content of *Zahlen* is an answer to the question asked in its title, it tells us what the numbers are and what they are for. But then, how is “Dedekind’s principle” inserted in the book, in this perspective? Is Dedekind proving or defining?

Two possible readings can easily be dismissed. First, I believe one can rule out a logicist interpretation solely based on the rejection of intuition in arithmetic. If Dedekind does openly refuse intuition, he does not, as I have mentioned, propose a foundationalist or reductionist project comparable to (orthodox) logicism. The logic, as presented in the principle considered by Detlefsen to be at the basis of a Dedekindian ‘logicism’ is essentially a tool to guide the reasonings. Secondly, one can exclude the possibility that “Dedekind’s principle” only refers to some of the demonstrations in the book, such as the proof of the validity of complete induction, for it would miss the core of the book: the definition of the sequence of natural numbers, and the first elements of arithmetic (operations, cardinal numbers. . .).

The ambiguity between “definition” and “proof”, here, seems to be solved by the understanding that definitions work as a proof of existence. Indeed, as in previous works, it is to avoid these “illicit hypotheses of existence”, that

Dedekind's principle comes into play: one verifies (or proves) the assumption of existence by effectively defining the notion whose existence is supposed, by a systematic reduction to earlier concepts. This is exactly what is done in *Zahlen*, to get over the tendency to consider natural numbers as "something simple, evident, given by inner intuition".⁴⁰ So as to avoid faulty proofs, one should not accept any notion that can be defined without actually defining it.

The writing of *Zahlen* was impelled by the same kind of demand that motivated his writing of *Stetigkeit* or *Algebraische Funktionen*: to provide a rigorous, uniform definition to a certain concept. Dedekind explains, in the introduction to *Zahlen*, that he wants to establish the science of numbers on a "uniform basis", to give it a general foundation. He adds that he wishes that this could help the other mathematicians to reduce the length and complexity of their own "sequences of inferences" – since the science of numbers is (finally) well grounded. Indeed, Dedekind's book unfolds and proves these long chains of inferences from which results the concept of number; it shows that numbers are indeed pure creations of the mind, and gives rigorous definitions and demonstrations to all the fundamental notions properties of the natural numbers.

In 1888, the definition of the natural numbers on the basis of *the operations of thought* and the long and technical preliminary developments are described as representing "the long series of simple inferences corresponding to our step-by-step understanding" (*ibid.*). So as to give a definition of natural numbers, Dedekind dissects and gives a rigorous logical expression of the "chains of reasoning on which the laws of numbers depend" (*ibid.*). The amount of technical developments needed for the definition of simply infinite systems shows how important unfolding the "*Gedankenprozeß*" is to assure not only the rigor of the definition. And it is also essential for a better understanding of what constitutes the essential properties of the concept – in particular, here, through this analysis, natural numbers are identified as grounded on the concept of *Abbildung*. It is important to remember that there can be, within definitions themselves, elements that need to be proven. Thus, to unfold the "*Gedankenprozeß*" for the definition can allow to have a deeper and more precise understanding of the concept investigated.

In *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen*, a manuscript dedicated to the "extension on the concept of number on the basis of sequence of natural numbers" written after 1888, Dedekind argues that

[a]lthough any mathematician knows in advance that the plan will be completely successful, one must however not renounce to

⁴⁰Sinaceur, in her introductory note to her translation of *Zahlen* points, as well, the quasi-equivalence between to prove and to define, in *Zahlen's* first sentence.

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expose in detail at least the essential of the process of thought (*Gedankenprozeß*) [which leads to this success].

This was written about the possibility to extend the order of natural numbers to negative numbers. It also applies perfectly to *Zahlen*, in which “the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend.” For Dedekind, the apparent obviousness of some of the demonstrations are “a convincing proof that their possession (or belief in them) is never given by inner intuition but is always gained only by a (more or less complete) repetition of the individual inferences” ([Dedekind, 1888], 791). The “*Gedankenprozeß*” highlights the importance of an appropriate definition of the numbers following the steps of the formation of numbers in our understanding.

This conviction is expressed in relation to the necessity to assure our clear understanding of the concept of numbers, which is all the more crucial as it allows to recognize that some notions which seem simple are, in fact, complex notions:

Thus it happens that many very complicated notions (such as, for example, that of the number (*Anzahl*) of things) are erroneously regarded as simple.⁴¹ (*ibid.*, 792)

In *Zahlen*, Dedekind’s rigor is pushed further than before, bringing to the fore that by false assumptions of simplicity, one takes the risk of overlooking implicit hypotheses of existence, which may very well be illicit hypotheses or lacking a rigorous definition. For Dedekind, a consistent definition is a proof of existence, and for this reason, it should never be neglected. A clear example of such illicit hypotheses of existence can be found in the treatment of irrational numbers and continuity, for which if one were to try to “build arithmetic on the concept of ratios of magnitudes”, then

in this way of founding Arithmetic, the completeness of the concept of number depends uniquely on the completeness of the concept of magnitude; but since the continuity (continuous completeness) of real numbers is necessary to build Arithmetic scientifically, it is essential to *know* exactly at the beginning the degree of completeness of magnitudes, because, in mathematics, nothing is more dangerous than to admit existence without sufficient proof.⁴² ([Dedekind, 1932] III, 77-78, my translation,

⁴¹ “und so kommt es, daß manche, eigentlich sehr zusammengesetzte Begriffe (wie z B. der der Anzahl von Dingen) fälschlich für einfach gelten.”

⁴² “da vielmehr die Vollständigkeit des Zahlbegriffs bei dieser Begründung der Arithmetik lediglich von der Vollständigkeit des Grössen-Begriffs abhängt, und da die stetige Vollständigkeit der reellen Zahlen für den wissenschaftlichen Aufbau der Arithmetik unentbehrlich ist, so ist unerlässlich von vorneherein genau zu wissen, wie vollständig das Gebiet der Grössen ist, weil Nichts in der Mathematik gefährlicher ist, als ohne genügenden Beweis Existenzen anzunehmen.”

original emphasis)

This kind of improper assumptions, for natural numbers as for other parts of mathematics, may lead to incomplete, if not faulty, characterizations of concepts and demonstrations. In the case of natural numbers, a possible equivalent of such an assumption would be the definition of numbers as aggregates of units.

Arithmetic itself is said to be the “simplest” science:

(...) the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. ([Dedekind, 1888], 790)

This is so because arithmetic is the science of objects which are defined (and the propositions pertaining to them proved) with the most elementary, most fundamental concepts of logic and mathematics. I have argued that rigor, in Dedekind’s works, is articulated with the idea that anything, in mathematics, that can be expressed by means of “simpler” truths should effectively be so, even if it implies that one has to go through a long sequence of logical and “apparently artificial” inferences. Indeed, it is then possible to fix the definition so as to allow a more certain and easier development of mathematics. *Zahlen* is entirely dedicated to this goal:

[M]ay other mathematicians be led with [the following pages, as an attempt to establish the science of numbers upon a uniform foundation] to reduce the long series of inferences to more moderate and attractive proportions.⁴³ (*ibid.*, transl. altered, 792)

Because numbers are at the basis of mathematics – and even the most efficient and fruitful of all the creations of the human mind – it is all the more crucial to be assured of the rigor and consistency of their definition. Moreover, insofar as numbers are the result of a process of thought repeated since “the time of birth”, the definition should be able to make this thought process explicit, to make explicit the more general and more fundamental (simpler) concepts constituting the sequence of natural numbers. Natural numbers, then, are in need of a rigorous definition by means of earlier concepts.

How did Dedekind proceed to do so? As Dedekind explains, in the letter to Keferstein, he sought to identify

⁴³ “mögen die andere Mathematiker dazu [die folgenden Blätter als ein Versuch, die Wissenschaft der Zahlen auf einheitlicher Grundlage zu errichten] anregen, die langen Reihen von Schlüssen auf ein bescheideneres, angenehmeres Maß zurückzuführen.”

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What are the mutually independent fundamental properties of the sequence N , that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under activities of the understanding *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions?⁴⁴ (transl. van Heijenoort, 99-100)

The properties “that are not derivable from one another but from which all others follow,” which Dedekind seeks to subsume “under more general notions and under activities of the understanding” are these properties identified by analyzing the “sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration.” This results in a definition that we recognize as “axiomatic” according to standards inherited from Hilbert – but “axiom” is a term that Dedekind does not use. The “completeness of proofs” alluded to in the letter is (again) the demand to avoid all illicit (implicit) hypotheses, and reflects the very first sentence of *Zahlen*. By this analysis, Dedekind is able to highlight which properties are relevant to the number concept, and which are not.

A point that should also result clearly from the definition is that numbers are products of human understanding, that they “flow” from the “laws of thought”. Dedekind conceives the definition of natural numbers as a mathematical conceptualization of the “*Gedankenprozeß*”. The sequence of natural numbers is thus defined on the ground of systems and mappings, mathematical expressions of the operations of thought, thereby linking arithmetic directly to the laws of thought, the only unprovable premisses. By this reduction, one assures the coherence of the concept of number: the elements of the theory of systems and the theory of mappings assure that the inferences of the definition are secured and completely mastered, every step can be suitably justified and clearly understood. The definition given produces all the natural numbers in one move. It gives Dedekind the means to define the operations and to prove the validity of the proof and definition by induction on an integer n . The definition, thus, allows complete proofs: nothing provable is admitted without a proof.

⁴⁴ “*Welches sind die von einander unabhängigen Grundeigenschaften dieser Reihe N , d.h. diejenigen Eigenschaften, welche sich nicht auseinander ableiten lassen, aus denen aber alle anderen folgen. Und wie muss man diese Eigenschaften ihres spezifisch arithmetischen Characters entkleiden, der Art, dass sie sich allgemeinen Begriffen und solchen Tätigkeiten des Verstandes unterordnen, ohne welche überhaupt kein Denken möglich ist, mit welchen aber auch die Grundlage gegeben ist für die Sicherheit und Vollständigkeit der Beweise, wie für die Bildung widerspruchsfreier Begriffserklärungen?*”

The process of definition in *Zahlen* is a bright illustration of Dedekind's demand that the mathematicians be careful in assuming simplicity and in making hypotheses of existence. Rigor, in the set up of *Zahlen*, is pushed further than before – and probably as far as it can be (from Dedekind's viewpoint) – by defining natural numbers on the basis of the mathematical definition of the fundamental operations of the mind, and thus by reaching unprovable premisses.

8.1.3 Arithmetic and the act of thinking

We saw that in the introduction of *Zahlen*, Dedekind compared the “individual inferences” leading to the formation of the number concept to “the action which an accomplished reader performs in reading”. Arithmetic appears to be not only a creation of the human mind, but deeply embedded in our thinking. This, for Dedekind, constitutes the explanation for the tendency to mistakenly consider numbers as being “something simple, self-evident and given in the inner consciousness”.

It is also the idea expressed in the quote inspired by Gauss (‘s arithmetization of Plutarch) that Dedekind famously placed at the head of *Zahlen*:

ἼΑεὶ ὁ ἄνθρωπος ἀριθμητίζει.

Man always arithmetizes.

By no less than replacing God by man, Dedekind asserts two key elements of his conception of mathematics: that mathematics, as a science, is an activity of the human mind, and that arithmetic is intimately and constitutively linked to the nature of human thought. These two points are intrinsically linked. We saw, in Sec. 3.1.1, that Dedekind considered science as an activity of the understanding, the “endeavors to fathom the truth”. Hence, it belongs to the mathematician to invent, to design the appropriate concepts to “fathom the truth” and develop mathematics.

The second point is explicitly stated at several places in *Zahlen* or in a manuscript written after 1888 and which I have quoted before:

Of all the auxiliary means (*Hilfsmitteln*), that the human mind has yet created to ease its life, i.e., the work in which the thought consists, none is as effective and far-reaching and as inextricably connected to its innermost Nature, as the concept of number. Arithmetic, whose sole subject matter is this concept, is already by now a science of immeasurable extension, and there is no doubt that no limits are set to its further developments. Equally limitless is the area of its applications, since every thinking man, even when he is not distinctly aware of it, is a number-man, an arithmetician. (*Zum Zahlbegriff*, in [Dugac, 1976b], 315, my translation)

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Arithmetic is, thus, attributed a constitutive role in the act of thinking itself and placed at the core of all human thought – no matter how much the thinking man is aware of the arithmetical nature of his thinking. It is a creation of the human mind so as to “ease” its work, a means to develop in all rigor the human reasonings. As such, numbers are said to be one of the most powerful concepts created by the human mind.

In [Sinaceur, Forthcoming], Sinaceur suggests an intimate link between arithmetic and the structure of thought, which would attribute a prominent role to arithmetic in thinking. After a careful analysis of Dedekind’s conception of logic, in order to differentiate it from Frege’s and argue against a Dedekindian logicism, she states that

For Dedekind, what matters are not the numbers themselves but their structure. Arithmetic is fundamental not only because numbers are applied everywhere, but because we can, following the arithmetical laws, calculate with things which are not numbers. What matters is not what can be said of numbers in themselves, but as satisfying the four conditions brought to the light (*Z*, 71, 73). And it is why we can say that arithmetic is a formal structure of our experience. The “logic of the mind” is arithmetic taken generally. As Dedekind writes, “every thinking man, even if he is not clearly aware of that, is an arithmetic-man, an arithmetician” ([Dugac, 1976b], 315), for thinking is representing a thing by a thing, relating a thing with a thing. One may understand the famous phrase “arithmetic is a part of logic” as meaning that arithmetic affords also a rational (logical) norm of thinking. ([Sinaceur, Forthcoming])

In her view, arithmetic is for Dedekind deeply involved in the mathematician’s endeavors to extend mathematical knowledge. And indeed, for Dedekind arithmetic holds a very peculiar status not only in mathematics but in human understanding more generally. For example, to have a precise, rigorous understanding of arithmetic appears also to be a necessary condition for any accurate investigation “our notions of space and time”

It is only through the purely logical process of building up the science of numbers and by thus conquering the continuous number-domain, that we are enabled accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind.⁴⁵ ([Dedekind, 1888], 791)

⁴⁵ “Durch den rein logischen Aufbau der Zahlen-Wissenschaft und durch das in ihr gewonnene stetige Zahlen-Reich sind wir erst in den Stand gesetzt, unsere Vorstellungen von Raum und Zeit genau zu untersuchen, indem wir dieselben auf dieses in unserem Geiste geschaffene Zahlen-Reich beziehen.”

Note that Dedekind *builds up* the science of numbers and “by thus” conquers continuity. Dedekind, then, is stating that if one wants to investigate our notions of space and time with any clarity, it has to be brought into relation with the number concept. It is a point that Dedekind stated before giving his definition of the natural numbers, he wrote in *Stetigkeit*, in 1872:

The chain of [the natural] numbers already forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws which one obtains by introducing the four fundamental operations of arithmetic.⁴⁶
([Dedekind, 1872], 768)

However, it has to be brought into relation with the appropriate notion of number: it is the continuous domain of the real numbers that will allow for a better grasp of our notion of space.⁴⁷

I would like to defend an idea slightly different from the one advanced by Sinaceur, who relates arithmetic to the structure of thought. I will propose that the fundamental position of arithmetic in the human thought gives them the role of an epistemic tool, which we observed in Dedekind’s works.

Numbers and arithmetic can, as the quotes above suggest, be seen as “auxiliary means” and used to help the mathematician. We saw with the idea of a Dedekindian arithmetization that they can be used as tools to build new concepts, to develop theories as we saw in the case of Riemannian function theory. They can be *epistemic tools* used in the process of the development of mathematical knowledge, due to their “inexhaustible wealth of remarkable laws” but also, I will argue, to the place attributed to them in the human mind by Dedekind.

The writing of *Zahlen* highlights and justifies this idea at the same way that it claims and justifies numbers being creations of the mind, by defining the sequence of natural numbers *and* the operations and basic methods of proof and of definition⁴⁸ on the basis of the operations of thought, and with

⁴⁶“Die Kette dieser Zahlen bildet an sich schon ein überaus nützliches Hilfsmittel für den menschlichen Geist, und sie bietet einen unerschöpflichen Reichtum an merkwürdigen Gesetzen dar, zu welchen man durch die Einführung der vier arithmetischen Grundoperationen gelangt.”

⁴⁷Note also, in addition to being fruitful as a way to assure (a Dedekindian) rigor, the unfolding of the “*Gedankenprozeß*” can help to reveal that some things which were hitherto considered “inconceivable” can, in fact, be proved to be true, and enter the realm of the conceivable. In particular, it was the case of the analysis of the concept of continuity, mentioned by Dedekind in the introduction of *Zahlen*. Indeed, the analysis of the concept of continuity allowed Dedekind to point out that continuity is not a necessary property of our (Euclidean) space, for all propositions of Euclidean geometry can be proved in the everywhere discontinuous space of the algebraic numbers. His research for a rigorous and general definition of continuity allowed him not only to identify how to define irrational numbers, but also to enlarge our conception of space.

⁴⁸We will see that Dedekind proves the validity of the proof and definition by induction.

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a conceptual arsenal involving sets and mappings alone.

To define the natural numbers, Dedekind, carefully unfolds each step of the long process of defining the ordered structure of natural numbers, the “long series of simple inferences corresponding to our step-by-step understanding, [and] the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend” (*ibid.*, 791). He dissects the “*Gedankenprozeß*” through which the human mind is going to produce numbers and formulates it in rigorous logical terms with the concepts of system and mappings. It is by this unfolding of the intricate relationship of arithmetic to thought, that Dedekind is justified in saying that the numbers “flow directly from the laws of pure thought”, that the human mind is continually – albeit often unconsciously – doing arithmetic. He highlights that thought and arithmetic are following the same step-by-step process rooted in the two fundamental operations of the mind and builds up the whole edifice of arithmetic on the “operations of thought”.

Before developing the idea of numbers as tools produced by the mind to “ease its life”, I will expose the details of the definition of natural numbers.

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In the following, in order to support the claims above and to have a sound basis for inquiring into the relations between Dedekind’s definition of natural numbers and the “arithmetization”, I will present Dedekind’s definition of natural numbers. The presentation of *Zahlen* that I will propose will make use of very little references to secondary literature. My reading does not aim at challenging or even commenting upon previous interpretations of *Zahlen*. My reading has been informed by the many commentaries available,⁴⁹ but it relies essentially on Dedekind’s texts themselves.

I will use the text of *Zahlen* to highlight how it is answering a demand for a definition of natural numbers, and to clarify how the strategy deployed to give this definition meets requirements of rigor, yields an idea of arithmetic of a great generality, and allows to define the concept of number in such a way that it gives rise to a conception of arithmetic that can account for its uses in previous works.

The ‘logical’ approach advocated by Dedekind, if applied systematically, helps to gain clarity, because it allows to understand the chains of inferences lying behind notions we considered as evident. Logic is an instrument for the thought, a means of rigor, related to the Laws of thought and close to a

⁴⁹In particular [Dugac, 1976b], [Belna, 1996], [Tait, 1997], [Ferreirós, 2008], [Potter, 2002], [Reck, 2009], [Reck, 2003], [Sieg and Schlimm, 2005], [Schlimm, 2005], [Sinaceur and Dedekind, 2008], [Sinaceur, Forthcoming], [Detlefsen, 2012].

theory of inference. As Sinaceur underlines, Dedekind is not interested by “logical issues in and for themselves,” but by the possibility to instigate more “rigorous logic” into the foundation of mathematical theory, leading him to unfold the “series of inferences” on which concepts such as natural numbers rely. Logic is related to the “*Treppen-Verstand*” and, for Dedekind, to provide a “logical” foundation or definition of a concept means to provide a foundation which makes every step possible, from the ground to the actual definition of the concept (see [Sinaceur, Forthcoming]). Dedekind’s logic has to be put into relation with the fact that thought is an activity, a creative activity: logic is guiding the thought. In that sense, as we will see, arithmetic is a part of logic (whose scope of applicability is considerably wider than arithmetic’s). And in that sense, *Zahlen* is a logical project: it aims to show that numbers are creations of the mind which can be defined by more general means, that they are not given (in any sense possible: an inner intuition, a realm of ideas, God, . . .), and that one can give a purely logical and perfectly rigorous ground to arithmetic. Logic must be the only means with which one *constructs* arithmetic. Once arithmetic is defined, mathematicians can use arithmetic itself.

Dedekind was not motivated either, unlike Frege or Peano, by the possibility to provide a foundation from which one could deduce the whole of mathematics. He himself mentions in the introduction of *Zahlen* the restricted area covered by his essay.⁵⁰ The possibility to extend “step-by-step” the domains of numbers was already highlighted in *Stetigkeit* and Dedekind does not consider that what he wrote before *Zahlen* should be reconsidered in this light.

8.2.1 The “simply infinite systems”

The definition of the sequence of natural numbers requires a certain amount of technical developments and is given approximately in the middle of Dedekind’s book. The first step is to introduce the notion of chain which will give a rigorous mathematical definition of the “act of counting”. With chains alone, Dedekind can prove, before having defined the sequence of natural numbers, the theorem of complete induction. The next step is to define finite and infinite sets. From there, it becomes possible for Dedekind to give his definition of the system of natural numbers, as an instance of what he calls “simply infinite systems”. The definitions of an order on the natural numbers, of the arithmetical operations and of the notion of cardinal numbers come after this definition.

⁵⁰“In accordance with the purpose of this memoir I restrict myself to the consideration of the sequence of so-called natural numbers.” ([Dedekind, 1888], 792)

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8.2.1.1 Dedekind's previous attempts to define the natural numbers

Dedekind's first researches on the sequence of natural numbers, which he himself qualified of "naïve"⁵¹, proposed a definition of natural numbers using a Leibnizian flavored definition by induction:

[First version] Act of creation $1; 1 + 1 = 2; 2 + 1 = 3; 3 + 1 = 4 \dots$ numbers (ordinal).⁵²

[Second version] Creation of the numbers: $1; 1 + 1 = 2; 2 + 1 = 3; 3 + 1 = 4 \dots$ from each number a the following number $a+1$ is formed by the act $+1$. Therefore, everything by complete induction.⁵³

Dedekind also touches on the possibility to define the successor operation $+1$ by a mapping φ : $1, \varphi(1) = 2, \varphi(2) = 3, \varphi(3) = 4, \dots$. This gives him the possibility to distinguish between successor and addition:

$$a + \varphi(b) = \varphi(a + b) \text{ and } a + 1 = \varphi(a).$$

The idea is not further developed in this early manuscript. It is however an important step, in which the "act of counting" is seen as a mapping producing the sequence of natural numbers, initially independent of addition, which will be defined as satisfying certain recurrence conditions. The distinction between the successor operation and the addition is important for further developments of Dedekind's conception of number – "the beginning of a quite dramatic conceptual shift", according to Sieg and Schlimm. By distinguishing the generating act of numbers from the operations of arithmetic, Dedekind also allows for the consideration of arithmetic operations as determined by the *operands* and not as formal operations.

The first draft of *Zahlen*, reproduced in [Dugac, 1976b] (293-309), spreads over a period of six years between 1872 and 1878 and presents a certain number of similarities with the ideas published in 1888.⁵⁴ It marks a significant break with the ideas about numbers in the *Habilitationsvortrag* and in

⁵¹The title of the first manuscript is "*Versuch einer Analyse des Zahlbegriffs vom naiven Standpunkte aus*" (Cod. Ms. Dedekind III, 4). It is dated, in [Sieg and Schlimm, 2005], from dated from 1872/1878. Some copies of the manuscripts were communicated to me by Sean Walsh, who I would like to thank warmly. One can also find a retranscription of some of these manuscripts in [Schlimm, 2000].

⁵²"*Schöpfungsakt* $1; 1 + 1 = 2; 2 + 1 = 3; 3 + 1 = 4 \dots$ "

⁵³"*Erschaffung der Zahlen*: $1; 1 + 1 = 2; 2 + 1 = 3; 3 + 1 = 4 \dots$ aus jeder Zahl a wird durch den Act $+1$ die folgende Zahl $a + 1$ gebildet. – Deshalb Alles durch vollständige Induction."

⁵⁴[Sieg and Schlimm, 2005] provides great explanations on its content and traces some changes in Dedekind's approach to the definition of systems of numbers. My commentary in the following paragraph will rely on their work, yet I will not consider the "axiomatic" aspect of Dedekind's approach and will have some details on points which were out of the scope of inquiry of the authors.

Stetigkeit by identifying mappings as the core of what “counting” consists in. The definition of an infinite system and the notion of chain (which we will consider below) as a justification for induction and a means of producing the sequence of natural numbers are first introduced in the 1872/78 draft. The notion of chain is refined throughout the manuscript, which contains three versions (or three “layers”, as put by Sieg and Schlimm, since several versions are written next to each other on the same page) of Dedekind’s researches, between 1872 and 1878. At each “layer”, one can notice subtle changes in his ideas which gradually place the notion of *Abbildung* at the center of the investigation, adopting the viewpoint announced in 1879.

In the first draft, Dedekind still considers the systems to play the central role, since for the characterization of chains, he uses the ideas of “mappable” (*abbildbar*), “similarly mappable” (*deutlich abbildbar*), “image” (*Bild*) related to systems. As Sieg and Schlimm explain, this “match[es] 1871 (Section I of §159 in Supplement X) as well as 1872 in terminology and outlook”: the definition of conjugate fields in 1871 is given in terms of numbers $\varphi(a)$ corresponding to numbers a in the field A , rather than in terms of mappings from one field onto another.

The second attempt, which is the longest of the three, introduces the notion of *Abbildung* and states that mappings are necessary for the foundation of arithmetic:

If one accurately tracks what we are doing when we count a set or a number of things, one is necessarily led to the concept of correspondence or mapping.

The concepts of system, of mapping, which shall be introduced in the following in order to ground the concept of number, cardinal number, remain indispensable for arithmetic even if one wants to assume the concept of cardinal number as being immediately evident (“inner intuition”).⁵⁵ (Quoted and translated in [Sieg and Schlimm, 2005], 141)

Note that Dedekind considers that the concept of number is the cardinal number (*Anzahl*), here, whereas he wrote to Weber after the publication of *Zahlen* that he took the ordinal number to be the true concept of number. A possible explanation for this change is the fact that the definition of numbers by means of mappings and chains requires to define the ordered sequence

⁵⁵ “Verfolgt man genau, was wir beim Abzählen der Menge oder Anzahl von Dingen thun, so wird man nothwendig auf den Begriff der Correspondenz oder Abbildung geführt. Die Begriffe des Systems, der Abbildung, welche im Folgenden eingeführt werden, um den Begriff der Zahl, der Anzahl zu begründen, bleiben auch dann für die Arithmetik unentbehrlich, selbst wenn man den Begriff der Anzahl als unmittelbar evident (‘innere Anschauung’) voraussetzen wollte.”

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formed by ordinal numbers. Only after that can the cardinal number be introduced as “number of things” in a system.

The main properties of N , such as induction or the definition of the operations, are, however, completely independent of the cardinal numbers.

Dedekind, in this second “layer” calls a chain a “group” (*Gruppe*):

G is called a *group* (relating to this mapping $[\varphi]$) when $G' [= \varphi(G)]$ is a part of G .⁵⁶ (In [Dugac, 1976b], 296)

Dedekind develops his ideas all the way up to finite cardinals. He also considers the proof of the “correctness of the method of proof” by induction on natural numbers, which is one the core elements of *Zahlen*:

The proof of the correctness of the method of proof from n to $n+1$ is correct; in contrast, the proof (completeness) of the definition of concepts by the method from n to $n + 1$ is not yet sufficient at this point; the existence (consistent) of the concept remains in doubt. This will become possible only by injectivity, by the consideration of the system $[n]!!!!!!$ Foundation.⁵⁷ (Quoted and translated in [Sieg and Schlimm, 2005], 143)

Note the distinction between the “proof” and the “definition” by induction, which Dedekind will develop in *Zahlen*.

Dedekind also defines “the sum of two numbers” using both induction and mappings, that is, as satisfying the two conditions $a + 1 = \varphi(a)$ and $a + \varphi(n) = \varphi(a + n)$. In *Zahlen*, these are properties which characterize addition, but not the definition properly speaking.⁵⁸

The third layer of the manuscript is a “very polished version of [part of the previous] considerations” as Sieg and Schlimm notice. Some of the points are very close to the ideas presented in 1888. In particular, it completely adopts the standpoint according to which the mappings are the core of the investigation, rather than the systems themselves. The sequence of natural numbers is characterized in the following way:

Characteristic of the system Z . There is a similar mapping of Z – if T is a part of Z , then the image of T is denoted by T' – which has the following property.

⁵⁶ “ G heisst (mit bezug auf diese Abbildung $[\varphi]$) a Gruppe, wenn $G' [= \varphi(G)]$ Theil von G ist.”

⁵⁷ “Der Beweis der Richtigkeit der Beweismethode von n auf $n + 1$ ist richtig; dagegen ist der Beweis (Vollständigkeit) der Begriffserklärung durch die Methode von n auf $n + 1$ an dieser Stelle noch nicht genügend; die Existenz (widerspruchsfrei) des Begriffs bleibt zweifelhaft. Dies wird erst möglich durch die Deutlichkeit, durch die Betrachtung des Systems $[n]!!!!!!$ Fundament.”

⁵⁸ The definition of addition is made using the “Theorem of the definition by induction” which proved that one can define a “function” by induction. Addition is a special case of such function. In modern terms, the definition given by Dedekind is recursive.

- I. Z' is a part of Z .
- II. There is a number (i.e., a thing contained in Z), which is not contained in Z' . This number shall be called “one” and is denoted by 1.
- III. A number chain (i.e., each part T of Z , whose image T' is a part of T) that contains the number 1 is identical with Z .⁵⁹ (*ibid.*, transl. slightly altered, 144)

To give mappings the central role in the theory is a significant move towards Dedekind’s final definition of the natural numbers. It suggests that Dedekind went from the idea of the sequence of natural numbers as a domain whose order could be represented by a certain mapping (a chain) to the idea that the chain is the essential concept which generates the ordered sequence. As we will see below, Dedekind in 1888 defines the system of numbers as satisfying the condition that there exists a mapping which “makes of N the chain of an element [1] not contained” in the said chain. The main difference between the 1888 definition and that given in the first draft is the fact that the generality of the standpoint adopted in *Zahlen*. Indeed, *Zahlen* is overall more general, in the sense that Dedekind defines a concept of simply infinite system whose elements are of indeterminate nature, and considers the natural numbers to be a particular instance of this concept.⁶⁰

Dedekind left this work unfinished until 1887. The two last drafts of *Zahlen*, written in 1887, are similar to the published version. There is one notable difference in the 1887 draft, which is the absence of “proof” of the existence of an infinite system. It is likely that Dedekind’s reading of Bolzano’s *Paradoxien des Unendlichen*, sent to him by Cantor, inspired the writing of this proof.

8.2.1.2 Chains

The paragraph in which the notion of chain is defined aims at studying the mappings of a system S into itself, i.e., such that $\varphi(S) = S$. Consider such a function $\varphi : S \rightarrow S$. A chain is a subset K such that $\varphi(K) \subset K$.⁶¹

⁵⁹ “Charakteristik des Systems Z . Es giebt eine deutliche Abbildung von Z - ist T ein Theil von Z , so soll das Bild von T mit T' bezeichnet werden, welche folgende Eigenschaft besitzt.

I. Z' ist Theil von Z .

II. Es giebt eine Zahl (d.h. ein in Z enthaltenes Ding), welche nicht in Z' enthalten ist. Diese Zahl soll ‘Eins’ heissen und mit 1 bezeichnet werden.

III. Eine Zahlkette (d.h. jeder Theil T von Z , dessen Bild T' ein Theil von T ist), welche die Zahl 1 enthält, ist identisch mit Z .”

⁶⁰Note also that *Zahlen* contains considerably more developments.

⁶¹Dedekind often uses the notation K' for $\varphi(K)$. For clarity purposes, I will often keep the notation $\varphi(K)$.

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K is called a chain (*Kette*)⁶² when $K'3K$. We remark expressly that this name does not in itself belong to the part K of the system S , but is given only with respect to the particular mapping φ ; with reference to another mapping of the system S into itself, K can very well not be a chain.⁶³ ([Dedekind, 1888], 803)

In contemporary terminology, one would say that a chain is the minimal closure of a set K in a set S containing K under a function φ on S (where “minimal” is related to the general notion of intersection)⁶⁴. Dedekind’s remark following the definition is important: being a chain depends on the *Abbildung* considered, it is not a property of the set K itself. There are several interesting properties of chains proved by Dedekind in the following pages such as:

- S is a chain.
- If K is a chain, $\varphi(K)$ is a chain.
- If A, B, C, \dots are chains, then (A, B, C, \dots) and $\mathfrak{G}(A, B, C, \dots)$ are chains.
- If K is a chain and $A \subset K$, then $\varphi(A) \subset K$.

For what will follow, an important notion is the intersection of all the chains in S containing a certain subset A of S , and denoted by A_0 . It is itself a chain and is called “the chain of the system A or briefly the chain of A ” (*ibid.*, 804). Once again, the definition is dependent of the choice of φ :

This definition too is strictly related to the fundamental determinate mapping φ of the system S into itself, and if later, for the sake of clearness, it is necessary, we shall at pleasure use the symbol $\varphi_0(A)$ instead of A_0 and likewise designate the chain of A corresponding to another mapping ω_0 by $\omega_0(A)$.⁶⁵ (*ibid.*, 804)

And it is possible to take for A a singleton $\{a\}$ and form its chain. Three results are said to “completely characterize” the chain A_0 :

⁶²The term “Kette” was rather largely spread in number theory (e.g., *Kettenbrüche* for continuous fractions) and can be found in various authors’s works, such as Gauss or Dirichlet. Throughout Dirichlet’s *Vorlesungen*, “Kette” is used to designate (finite or infinite) sequence of numbers defined recursively, and it was the case in the previous editions. It was also used to designate the sequence of natural numbers in *Stetigkeit* (see p. 472).

⁶³“ K heißt eine Kette, wenn $K'3K$ ist. Wir bemerken ausdrücklich, da dieser Name dem Teile K des Systems S nicht etwa an sich zukommt, sondern nur in Beziehung auf die bestimmte Abbildung φ erteilt wird; in bezug auf eine andere Abbildung des Systems S in sich selbst kann K sehr wohl keine Kette sein.”

⁶⁴I borrow this remark to [Reck, 2012], but changed the notation according to the previous note.

⁶⁵“Auch diese Erklärung bezieht sich durchaus auf die zugrunde liegende bestimmte Abbildung φ des Systems S in sich selbst, und wenn es später der Deutlichkeit wegen nötig wird, so wollen wir statt A_0 lieber das Zeichen $\varphi_0(A)$ setzen, und ebenso werden wir die einer anderen Abbildung ω_0 entsprechende Kette von A mit $\omega_0(A)$ bezeichnen.”

- A is a subset of A_0 ;
- The image of A_0 by φ is included in A_0 ;
- If K is a chain and $A \subset K$, then $A_0 \subset K$.

Furthermore, if A is itself a chain, then $A_0 = A$. We also have that $\varphi(A_0) = (\varphi(A))_0$, that is,

the image of the chain of A is at the same time the chain of the image of A . Hence we can designate this system in short by A'_0 [i.e., $\varphi(A)_0$] and at pleasure call it the *chain-image* or *image-chain* [*das Kettenbild oder die Bildkette*] of A . With the clearer notation given [above] the theorem might be expressed

$$\varphi(\varphi_0(A)) = \varphi_0(\varphi(A)).^{66}$$

Dedekind does not explain why the notion of chain is important, he does not motivate the long technical preliminaries necessary to obtain a rigorous definition of the natural numbers – just as the exposition of module theory was not motivated in 1871. As underlined by Ferreirós, this omission probably made it more difficult for his contemporaries to “appreciate [the concept of chain] adequately”.⁶⁷ Explanations were only given in the correspondence with Keferstein.

Again, Dedekind proceeds by analysis and synthesis. The need for a notion of “similar mapping” was identified through the analysis of the sequence of natural numbers. This mapping serves to define the notion of successor. It has to be injective since distinct numbers always have distinct successors.⁶⁸ In order to characterize correctly the sequence of natural numbers, one has to make sure that the properties (closure by successor, unicity of the successor, existence of a first element) do not, in fact, define a system S which would contain the natural numbers n and another set T of additional elements t which are not natural numbers. These “aliens intruders” (which would be non-standard elements) must be avoided. Dedekind explains to

⁶⁶ “*das Bild der Kette von A ist zugleich die Kette des Bildes von A. Man kann daher dieses System kurz durch A'_0 bezeichnen und nach Belieben das Kettenbild oder die Bildkette von A nennen. Nach der deutlicheren in 44 angegebenen Bezeichnung würde der Satz durch $\varphi(\varphi_0(A)) = \varphi_0(\varphi(A))$ zu sein.*”

⁶⁷Ferreirós writes: “It would have been very important to explain the role of chains in connection with the definition of numbers, a topic that appears only in the letter to Keferstein. And the reception of Dedekind’s work would probably have been better had he explained that results such as the Cantor-Bernstein theorem were consequences of chain theory. As he presented the matter, only a few mathematicians (especially Schröder and Zermelo, but surprisingly not Cantor) were able to appreciate his contribution adequately.” ([Ferreirós, 2008], 230) The reference to the Cantor-Bernstein theorem could only have been made in the Preface to the third edition in 1911, since the theorem was given by Cantor in 1895 and proved by Dedekind in 1897 (Dedekind himself never published this proof).

⁶⁸These elements were already present in Dedekind’s first drafts.

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Keferstein that it is necessary to solve this problem by purely logical means (that is, not by arithmetical means, but not by any external means either) “without presupposing any arithmetic knowledge”, in order to avoid a vicious circle in the definition. So as to “give an unambiguous conceptual foundation to the distinction between the elements n and the elements t ,” one must therefore do it

[s]olely through consideration of the chains (articles 37 and 44 of [*Zahlen*]), and yet, by means of these, completely!⁶⁹ (Letter to Keferstein, transl. van Heijenoort slightly altered, 101)

From there, Dedekind can prove what he calls the “theorem of complete induction”, a general version of the well-known mathematical induction, whose aim is to prove that a chain A_0 belongs to a certain system:

Theorem of complete induction. To show that the chain A_0 is part of any system Σ – be this latter part of S or not – it is sufficient to show,

ρ . that A_3S , and

σ . that the image of every common element of A_0 and Σ is likewise an element of Σ .⁷⁰ ([Dedekind, 1888], 805)

What this theorem tells us, as Dedekind explains, is that proving that the chain A_0 is included in a certain system Σ amounts to proving that all the elements of A_0 have a certain property P , such that all the elements of Σ satisfy P . In order to demonstrate this, one has to prove that

- (1) A , the “initial element”, is in T , i.e., that all the elements of A have the property P ;
- (2) for any n in A_0 which satisfies the property P , its image $\varphi(n)$ satisfies the property as well.

Expressed in this manner, it is evident that “the form of demonstration known by the name of complete induction (the inference from n to $n + 1$)” (*op. cit.*, 805) is just a special case of the theorem mentioned above. Indeed, the “theorem of complete induction” is given before the definition of the

⁶⁹ “Ganz allein durch die Betrachtung der Ketten (37, 44 meiner Schrift), durch diese aber auch vollständig!”

⁷⁰ “Satz der vollständigen Induktion. Um zu beweisen, daß die Kette A_0 Teil irgendeines Systems Σ ist – mag letzteres Teil von S sein oder nicht – genügt es zu zeigen,

ρ . daß A_3D , und

σ . daß das Bild jedes gemeinsamen Elementes von A_0 und Σ ebenfalls Element von Σ ist.”

sequence of natural numbers, and is thus independent of the mode of inference well known for natural numbers. This theorem of complete induction is, as Sinaceur explains, a “logical derivation from the abstract set-theoretic concepts, of the arithmetic process of complete induction” ([Sinaceur and Dedekind, 2008], 116) and presents itself as a general form of induction.

To have this theorem in perfectly general form allows for a “scientific foundation” of induction. And if one wishes, as Dedekind does, to unfold step by step each inference of the definition of numbers and to recognize which properties are *arithmetic* i.e., dependent of having a system of numbers, it is essential to be able to point out a theorem of complete induction which does not depend on natural numbers. With the proof based on the concept of chain, as it is here, and not on numbers themselves, Dedekind assures the independence of “complete induction” from the very nature of numbers, and avoids a circular reasoning when giving the definition by induction. Moreover, for Dedekind, this theorem is one of the central points of the book, for it is essential to assure that one has a “sufficient method of proof” for theorems valid for any natural numbers, as he wrote to Keferstein. It shows that

the form of argument known as complete induction (or the inference from n to $n + 1$) is really conclusive (59), (60), (80), and that therefore the definition by induction (or recursion) is determinate and consistent (126).⁷¹ (*ibid.*, 791)

8.2.1.3 Finite and infinite systems

The next crucial point for Dedekind’s book is the definition of infinite and finite systems, one of the most striking passages of the book – and even more so since Dedekind’s contemporaries still had reticences towards actual infinities. It has been analyzed and commented in depth by previous authors, and I will not spend much time on it.

Dedekind’s “logical proof of existence” (Letter to Keferstein, 101) of an infinite system appears as an essential point in relation to his insistence that “nothing in mathematics is more dangerous than to accept existence without sufficient proof”. Importantly, Dedekind changes the order of deduction usually adopted by authors who accepted the infinite. Cantor, for example, considered natural numbers as given and with them, had a presupposed notion of infinite set.⁷² For Dedekind, the infinite systems are primary and serve as a basis for the definition of integral numbers. This implies that if

⁷¹ “daß die unter dem Namen der vollständigen Induktion (oder des Schlusses von n auf $n + 1$) bekannte Beweisart wirklich beweiskräftig (59, 60, 80), and daß auch die Definition durch Induktion (oder Rekursion) bestimmt und widerspruchsfrei ist (126).”

⁷²In fact, Cantor only started to give definition of the finite and the infinite after the publication of *Zahlen*.

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one is to prove that such a system exists, it must be done independently of the system of natural numbers itself (hence the “purely logical proof”). Not only is this point essential for the definition of natural numbers, which are a particular kind of infinite system, but also seems to come as a retrospective justification of his many uses of infinite systems in previous mathematical works.

By giving the conceptual priority to infinite systems, Dedekind made an important move which reverses the traditional order of explanation: infinite systems, abstract notions subjected to heated debates, became the fundamental notion with which one should define the familiar (and concrete) companions that are the natural integers. Following Dedekind’s insistence about the importance to give rigorous definitions of the grounding concepts and to never make illicit hypotheses of existence, it becomes indispensable to provide a definition of the infinite and, if possible, to prove the existence of such systems.

Dedekind thus gives a rigorous definition of finite and infinite systems from which one should be able to (rigorously) develop the “science of numbers”:

A system S is said to be *infinite* when it is similar to a proper part of itself; otherwise S is said to be a *finite* system.⁷³ (*ibid.*, 806)

This definition is now called “Dedekind infinite”.

It is the first time that such a definition of infinite systems is given. Although the property of infinite sets taken as definition by Dedekind had been noticed by Cantor and Bolzano, it was not considered as the essential property of infinite systems that should be taken as a definition, as mentioned by Dedekind in the preface of the second edition of *Zahlen* (1893):

[b]ut neither of these authors made the attempt to use this property for the definition of the infinite and upon this foundation to establish with rigorous logic the science of numbers.⁷⁴ (*ibid.*, 795)

This point is another instance of how Dedekind carefully identified the “characteristic properties”⁷⁵ to be taken as definition. Note that, mathematicians

⁷³ “Ein System S heißt unendlich, wenn es einem echten Teile seiner selbst ähnlich ist (32); im entgegengesetzten Falle heißt S ein endliches System.”

⁷⁴ “Aber keiner der genannten Schriftsteller hat den Versuch gemacht, diese Eigenschaft zur Definition des Unendlichen zu erheben and auf dieser Grundlage die Wissenschaft von den Zahlen streng logisch aufzubauen.”

⁷⁵ The shift from previous approaches is significant: this property of infinite systems had been considered as a “paradox” by many authors, the best known being Galileo who gave it his name.

who did accept actual infinite, such as Riemann, tend to think that it could not be defined directly.⁷⁶

The definition of infinite systems is followed by the infamous Theorem 66, in which Dedekind proves the existence on infinite systems using “[his] own realm of thoughts, i.e., the totality S of all things”. Dedekind’s reasoning is likely inspired by Bolzano’s in the *Paradoxien des Unendlichen*, which was sent to Dedekind by Cantor shortly before the last draft of *Zahlen*. Commented and criticized at length, this proof should be considered in its context. Dugac considers that this proof is unsuited to Dedekind’s mathematics, and that the proof, inspired by Bolzano, leaves its reader “perplexed” and does not seem mathematical, but rather “of philosophy and even of psychology!”. In fact, Dedekind’s proof does fit well in Dedekind’s approach, and in particular his rigor requisites: “nothing susceptible of proof ought to be accepted without a proof”.⁷⁷ In the second letter to Keferstein (27 February 1880), Dedekind states clearly his intentions:

Does such a [simply infinite] system exist at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay).⁷⁸ (transl. van Heijenoort 1967, 101)

Dedekind’s and Bolzano’s proofs, as noticed by Ferreirós, bring to the fore “the need to establish the existence of infinite sets as an explicit proposition within set theory” ([Ferreirós, 2008], 234). Moreover, considering that, for Dedekind, systems and mappings are objects of thought, operations of the understanding, and that he wants to make sure that an infinite system exists *in our realm of ideas*, the form of the proof of existence itself is not surprising.⁷⁹ For M. Potter as well, it is “not surprising that Dedekind’s proof of the existence of an infinite system has the same apparently psychological slant” considering his “his understanding of systems as creations of our minds” ([Potter, 2002], 100).

In addition, insofar as “there is nothing, in mathematics, more dangerous than to admit existences, without sufficient proof,” for infinite systems as well, it is essential to provide a rigorous consistent definition of the notion

⁷⁶See [Ferreirós, 2008], §II, 4.2.

⁷⁷[Detlefsen, 2014] proposes to see it as a proof of consistency.

⁷⁸“[E]xistiert überhaupt ein solches System in unserer Gedankenwelt? Ohne den logischen Existenzbeweis würde es immer zweifelhaft bleiben, ob nicht der Begriff eines solchen Systems vielleicht innere Widersprüche enthält. Daher die Notwendigkeit solcher Beweise (66, 72 meiner Schrift).”

⁷⁹And, as mentioned before, the Kantian background of Dedekind’s theory dismisses accusations of psychologism.

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whose existence is supposed, so as to prove that the hypothesis of existence is valid.

8.2.1.4 Simply infinite systems and the system of natural numbers

With the notion of chain and the proof that an infinite system does exist, Dedekind can now define the notion of simply infinite system:

A system N is said to be simply infinite when there exists a similar mapping φ of N into itself such that N appears as the chain of an element not contained in $\varphi(N)$. We call this element, which we shall denote in what follows by the symbol 1, the base-element of N and say the simply infinite system N is ordered (*geordnet*) by this mapping φ . If we retain the earlier convenient symbols for images and chains (§4) then the essence of a simply infinite system N consists in the existence of a mapping φ of N and an element 1 which satisfy the following conditions $\alpha, \beta, \gamma, \delta$:

- α . $N' \ni 3N$ [i.e., $\varphi(N) \subset N$]
- β . $N = 1_0$ [i.e., N is the chain of 1]
- γ . The element 1 is not contained in $N' [= \varphi(N)]$
- δ . The mapping φ is similar.⁸⁰ (*Zahlen*, 807)

The condition β . ensures that N is such that it satisfies the requirement that there are no “alien intruders” (i.e., nonstandard elements).⁸¹

These four conditions define *all* simply infinite systems, since the concept defined is that of systems composed of uninterpreted elements. To define the sequence of natural numbers requires one more step. As is well known, Dedekind defines the natural numbers as a *creation* resulting from a process

⁸⁰ “Ein System N heißt einfach unendlich, wenn es eine solche ähnliche Abbildung φ von N in sich selbst gibt, daß N als Kette (44) eines Elementes erscheint, welches nicht in $\varphi(N)$ enthalten ist. Wir nennen dies Element, das wir im folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und sagen zugleich, das einfach unendliche System N sei durch diese Abbildung φ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§ 4), so besteht mithin das Wesen eines einfach unendlichen Systems N in der Existenz einer Abbildung φ von N und eines Elementes 1, die den folgenden Bedingungen $\alpha, \beta, \gamma, \delta$, genügen:

- α . $N' \ni 3N$
- β . $N = 1_0$
- γ . Das Element 1 ist nicht in N' enthalten.
- δ . Die Abbildung φ ist ähnlich.”

⁸¹A similar definition can be found in Peano’s works. It also inspired Hilbert in *Über die Grundlagen der Logik und der Arithmetik*.

of abstraction from the simply infinite systems. More exactly, the sequence of natural numbers is the “abstract type” of the simply infinite systems, as he wrote to Keferstein:

If in the consideration of a simply infinite system N ordered by a mapping φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another (...), then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers* (...). With reference to this liberation of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind.⁸² (*ibid.*, 809)

There is, here, an important difference with the 1878 draft. In 1878, Dedekind only stated these properties as characterizing the sequence of natural numbers, whereas in 1888, he defines a general concept of simply infinite system from which he creates the sequence of natural numbers.⁸³

The creative act of abstraction giving the sequence of natural numbers is described by Dedekind as a transformation of the elements n of the system N into new elements \mathbf{n} (the numbers), and N is itself transformed into an abstract system \mathbf{N} . Because this transformation is a similar mapping, we can allow ourselves the convenience “to speak of the numbers as of the original elements of the system N ” (in the 1887 draft of *Zahlen*, quoted in [Sieg and Schlimm, 2005]). As soon as the definition of the numbers is given, Dedekind works with what we could call a ‘model’ of the simply infinite ordered system – a point whose legitimacy he will *prove* a little further down in the book, by proving (what we would call) the ‘categoricity’ of his version of second order arithmetic. For Dedekind, this means that one can work with the natural numbers, the familiar numbers, without loss of generality and of course without risk of “illicit hypothesis of existence”. From there, one can define the operations of arithmetic and eventually study the arithmetical properties with the ‘model’, with the “familiar numbers” – which are the basis from which the number concept will be extended.

The abstraction Dedekind is alluding to has been described by M. Potter as “gestur[ing] towards two competing metaphors, of forgetfulness and of creation, which correspond to different ways of regarding abstraction” ([Potter, 2002], 103):

⁸²“Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt (...), so heißen diese Elemente natürliche Zahlen oder Ordinalzahlen oder auch schlechthin Zahlen (...). In Rücksicht auf diese Befreiung der Elemente von jedem anderen Inhalt (Abstraktion) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen.”

⁸³The considerations about the proof and definition by induction are subjected to the same change of scope, between 1878 and 1888.

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The metaphor of forgetfulness suggests that we can by inattention to its special features make one triangle (or simply infinite system) go proxy for them all. The metaphor of creation suggests that once we have at least one triangle (or simply infinite system) we can create a new one with the special feature that it has no special features: no more is true of it than is true of every triangle (or simply infinite system). (*ibid.*, 103)

W. Tait considers that Dedekind's abstraction is a case of "logical abstraction"⁸⁴ and raises the following question: "instead of abstracting the simply infinite set of number from an already given simply infinite set M , why did Dedekind not simply take the system of numbers to be this latter system?" ([Tait, 2005], 224) For Tait, the answer, which Dedekind did not give, is the same as that given to Weber's (and others) questions regarding the definition of real numbers by means of cuts: the irrational numbers are not identified to the cut, for it would give them properties (that of the cuts) which have nothing to do with their nature of numbers. Indeed, Dedekind explains that if one were to identify numbers with cuts or with infinite sets (classes), then

one will say many things about the class (e.g., that it is a system of infinitely many elements, namely, of all similar systems) that one would apply to the number only with the greatest reluctance; does anybody think, or won't he gladly forget, that the number four is a system of infinitely many elements? (But that the number four is the child of the number three and the mother of the number five is something that nobody will forget.)⁸⁵ (Letter to Weber, 24 January 1888, transl. in [Ewald, 2005], 835)

Rather, one says that they are defined by (or with) the cut, created by the mind to fill the 'voids' highlighted by the cuts. In the same way that Dedekind does not identify the natural numbers with the concept of simply infinite system: he creates the natural numbers from it.

⁸⁴Logical abstraction is characterized by Tait as follows: "In any case, what seems to me to be essential to this kind of abstraction is this: the propositions about the abstract objects translate into propositions about the things from which they are abstracted and, in particular, the truth of the former is founded upon the truth of the latter. So the abstraction in question has a strong claim to the title logical abstraction: the sense of a proposition about the abstract domain is given in terms of the sense of the corresponding proposition about the (relatively) concrete domain." ([Tait, 2005], 223)

⁸⁵"*man wird vieles von der Classe sagen (z.B. daß sie ein System von unendlich vielen Elementen, nämlich allen ähnlichen Systemen ist), was man der Zahl selbst doch gewiß höchst ungern (als Schwergewicht) anhängen würde; denkt irgend Jemand daran, oder wird er es nicht gern bald vergessen, daß die Zahl vier ein System von unendlich vielen Elementen ist? (Daß aber die Zahl 4 das Kind der Zahl 3 und die Mutter der Zahl 5 ist, wird Jedem stets gegenwärtig bleiben).*"

This shows that Dedekind is not trying to re-define numbers in a way that changes their nature or their meaning – and indeed, he states that his “answer to the problems propounded in the title (...) [is that] numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things.” To not identify numbers with the means of the definition and creation (simply infinite systems, pairs, congruences, cuts) allows to preserve the “identity of nature” of numbers.

However, it seems that there is a subtle difference between the two definitions: Dedekind’s cuts are a *tool* to bring to the fore the discontinuity of the domain of rational numbers, a tool which allows to identify the need of the creation of new numbers to complete the domain. It seems that if Dedekind had wanted to adopt an approach strictly comparable to that in *Stetigkeit*, he would have stopped at the definition of the sequence of natural numbers as given in the 1872/78 draft: the concept of chain allows to create the natural numbers. In *Zahlen*, Dedekind is looking for a definition based on unprovable premisses, exhibiting the roots of natural numbers in unprovable logical laws. On this basis, Dedekind defines *simply infinite systems*, a general concept of which one takes an instance, created by abstraction.

In addition, the act of abstraction is justified by the demand that numbers be produced by an act of creation of the mind, as Dedekind explains in a 1877 draft of *Zahlen*:

By this abstraction, the originally given elements n of N are turned into new elements n , namely into numbers (and N itself is consequently also turned into a new abstract system \mathcal{N}). Thus, one is justified in saying that the numbers owe their existence to an act of free creation of the mind. For our mode of expression, however, it is more convenient to speak of the numbers as of the original elements of the system N and to disregard the transition from N to \mathcal{N} , which itself is a similar (*deutlich*) mapping. Thereby, as one can convince oneself using the theorems regarding definition by recursion, nothing essential is changed, nor is anything obtained subreptitiously in illegitimate ways.⁸⁶ (Quoted and translated in [Sieg and Schlimm, 2005], 152)

⁸⁶ “Da durch diese Abstraction die ursprünglich vorliegenden Elemente n von N (und folglich auch N selbst in ein neues abstraktes System \mathcal{N}) in neue Elemente n , nämlich in Zahlen umgewandelt sind, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsacte des Geistes verdanken. Für die Ausdrucksweise ist es aber bequemer, von den Zahlen wie von den ursprünglichen Elementen des Systems N zu sprechen, und den Übergang von N zu \mathcal{N} , welcher selbst eine deutliche Abbildung ist, außer Acht zu lassen, wodurch, wie man sich mit Hilfe der Sätze über Definition durch Recursion (...) überzeugt, nichts Wesentliches geändert, auch Nichts auf unerlaubte Weise erschlichen wird.”

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Abstraction, here, is a mathematical process, “an active constituent of the creative power of thought,” as Sinaceur underlines. It is an integral part of the mathematician’s practice, and even, as it has been argued, a logical procedure. By “abstracting” the system of natural numbers from the simply infinite systems, Dedekind isolates one particular system, that he designates as the natural numbers and on the basis of which he will develop his work. The system of natural numbers, here, is an instance – the particular system having this very special property to have absolutely no particular property – of a more general (uninterpreted) system. It is in reference to this peculiar system that Dedekind will talk about “numbers”, in the rest of the essay. However, this will not imply any loss of generality, for not only is this ‘instantiation’ an abstraction, but Dedekind will prove that all simply infinite systems are isomorphic to each other. Thus, everything Dedekind defines and/or proves about the “abstract type” N is still valid for any other simply infinite system.

The last step is to state what *arithmetic* is. Arithmetic is said to be the science treating of the general laws governing the structure defined above and derived from the set of conditions defining a simply infinite system:

The relations or laws which are derived entirely from the conditions [defining the notion of simply infinite system] and therefore are always the same in all ordered simply infinite systems, whatever fortuitous names may happen to be given to the individual elements form the first object of the *science of numbers* or *arithmetic*.⁸⁷ (*op. cit.*, transl. slightly altered, 809)

Arithmetic and its laws, then, do not depend on the choice of a simply infinite system. The arithmetic which Dedekind is defining, here, is thus general, in the sense that it is the propositions, theorems, and laws for any element of unspecified nature satisfying the four conditions α , β , γ , δ – hence, in particular, for the natural numbers.

The notion of chain gives an order for the elements of the simply infinite systems, that is, it defines the notion of succession in natural numbers, the “act of counting”. The only thing left, in order to be able to state and prove the propositions of arithmetic – inequalities of numbers, operations, etc. – is to prove the validity of a “*method of proof* sufficient to establish, in full generality, propositions that are supposed to hold for all numbers n ” (Letter to Keferstein, original emphasis). Based on the general complete induction

⁸⁷“Die Beziehungen oder Gesetze, welche ganz allein aus den Bedingungen α , β , γ , δ in 71 abgeleitet werden und deshalb in allen geordneten einfach unendlichen Systemen immer dieselben sind, wie auch die den einzelnen Elementen zufällig gegebenen Namen lauten mögen, bilden den nächsten Gegenstand der Wissenschaft von den Zahlen oder der Arithmetik.”

given earlier, Dedekind states the “complete induction” theorem for natural numbers:

Theorem of complete induction (inference from n to $n + 1$). In order to show that a theorem holds for all numbers n of a chain m_0 , it is sufficient to show,

- ρ . that it holds for $n = m$, and
- σ . that from the validity of the theorem for a number n of the chain m_0 its validity for the following number n' always follows.⁸⁸ (*ibid.*, 809)

Note that the definition by induction has not been given, yet, and it won't, in fact, be given until three sections later.

8.2.2 Properties of natural numbers and operations

Once the system natural numbers N is defined, a fair amount of its properties are still to be defined and proved. The numbers which Dedekind just defined are solely the structure of the ordinal numbers. In order to be able to know more and do more with this concept, one needs to define and prove in more details certain of its properties:

- (1) an order relationship ($<$) for the elements of N (§7)
- (2) the distinction between finite and infinite subsets of N (§8)
- (3) the definition by induction (§9)
- (4) the similarity (in the technical sense defined by Dedekind) of all simply infinite systems (§10)
- (5) arithmetical operations (§§11-13)
- (6) cardinal numbers (§14)

It is important to note that everything defined and proved by Dedekind is defined and proved using solely mappings, chains and complete induction. The definitions and proofs, then, do not use anything whose definition has not been done (and rigorously so), or has not been done with tools intern to the theory (hence, they are not foreign to the theory). Moreover, the

⁸⁸ “Satz der vollständigen Induktion (Schluß von n auf n'). Um zu beweisen, daß ein Satz für alle Zahlen *neiner* Kette m_0 gilt, genügt es zu zeigen,

ρ . daß er für $n = m$ gilt, und

σ . da aus der Gültigkeit des Satzes für eine Zahl n der Kette m stets seine Gültigkeit auch für die folgende Zahl n' folgt.”

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exclusive use of notions elaborated within the theory allows to preserve the generality of the investigations that was granted by taking systems and mappings as grounding concepts. As we will see, to consider the system of natural numbers N as an instance of simply infinite systems does not lead to a loss of generality, insofar as it can be shown that all simply infinite systems are isomorphic to N , and thus to each other.

I will consider, for brevity's sake, only (4) the proof that all simply infinite systems are isomorphic to each other and (5) the definition of arithmetical operations.

8.2.2.1 The “categoricity theorem”

Dedekind defined, earlier in the book, the notion of similarity (i.e., one-to-one correspondence) between systems and the notion of “class”:

[We can therefore separate all systems into classes by putting into a determinate class all and only those systems Q, R, S, \dots that are similar to a determinate system R , called the representative of the class; according to [the theorem 33⁸⁹] the class is not changed by taking as representative any other system belonging to it.⁹⁰ (*ibid.*, 802)

The idea of the §10 is to prove that the simply infinite systems form a class. This is done in two steps: first, prove that all simply infinite systems are similar to each other; then, prove that any system similar to a simply infinite system is itself a simply infinite system.

The first step is expressed by Dedekind in the following way:

[Theorem 132] All simply infinite systems are similar to the number-series N and consequently by [the theorem 33] to one another.⁹¹ (*ibid.*, 821)

This theorem, proved by complete induction, allows to show that the results proven for N are valid for *any* simply infinite system. Indeed, since “being similar to” is a transitive relation, the choice of a representative does not change anything. This result shows that, as in the case of congruences, to isolate a “representative” (here N) does not imply a loss of generality. What Dedekind develops, from there, is thus a sort of general arithmetic,

⁸⁹“33. Theorem. If R, S are similar systems, then every system Q similar to R is also similar to S .”

⁹⁰“Man kann daher alle Systeme in Klassen einteilen, indem man in eine bestimmte Klasse alle und nur die Systeme $Q, R, S \dots$ aufnimmt, welche einem bestimmten System R , dem Repräsentanten der Klasse, ähnlich sind; nach dem vorhergehenden Satze 33 ändert sich die Klasse nicht, wenn irgendein anderes ihr angehöriges System S als Repräsentant gewählt wird.”

⁹¹“Alle einfach unendlichen Systeme sind der Zahlenreihe N und folglich (nach 33) auch einander ähnlich.”

an arithmetic of uninterpreted elements – “shadowy forms” in Dedekind’s words.

The second step to show that the simply infinite systems form a class, consists in showing that not only are all simply infinite systems similar to each other, but any system similar to a simply infinite system is itself simply infinite.⁹² With these two results, Dedekind is able to prove what is now called the categoricity of (his version of) second order arithmetic. As explained, for example, by Awodey and Reck:

Dedekind does not yet work with a completely general notion of isomorphism, nor does he use the term “categorical”.⁹³ Nevertheless, these two theorems [132 and 133] (and their proofs) show that he basically knows his characterization is categorical. ([Awodey and Reck, 2002], 7)

Dedekind considers that a direct consequence of the “categoricity theorem” is what we would call the “semantic completeness of his axiomatization,”⁹⁴ that is, every result proved for a simply infinite system N can be translated and be valid in any other simply infinite M :

[Remark 134] By the two preceding theorems (132), (133) all simply infinite systems form a class (...) At the same time, with reference to [the definition of the simply infinite systems and of the natural numbers] it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N ordered by the mapping φ (and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement φ) possesses perfectly general validity for every other simply infinite system Ω ordered by a mapping θ and its elements v , and that the passage from N to Ω (e.g., the translation of an arithmetical theorem from one language into another) is effected by the mapping ψ considered in (132), (133).⁹⁵ ([Dedekind, 1888], 823)

⁹²[Theorem 133] Every system which is similar to a simply infinite system and therefore by (132), (33) to the number-sequence N , is simply infinite.” (*ibid.*, 823)

⁹³The idea of a categorical theory (relatively to a certain semantic) is that there exists an isomorphism between any two models of the theory. These notions were unknown to Dedekind.

⁹⁴See [Awodey and Reck, 2002]. Here, “semantic completeness” is to be understood as: For all sentences φ of all models M , N of a theory, if M verifies φ , then N verifies φ .

⁹⁵“Zufolge der beiden vorhergehenden Sätze 132, 133 bilden alle einfach unendlichen Systeme eine Klasse im Sinne von 34. Zugleich leuchtet mit Rücksicht auf 71, 73 ein, daß jeder Satz über die Zahlen, d.h. über die Elemente n des durch die Abbildung φ geordneten einfach unendlichen Systems N , und zwar jeder solche Satz, in welchem von der besonderen Beschaffenheit der Elemente n gänzlich abgesehen wird und nur von solchen Begriffen

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It should be underlined that the results stated in the theorems 132 and 133 are, for Dedekind the proof that all propositions of arithmetic are of “perfectly general validity” (*ibid.*, 823), that is, valid for *all* simply infinite systems (counter to a model theoretic interpretation of the result).⁹⁶ The Remark 134, then, allows Dedekind to highlight the fact that it is rigorously proven that elementary arithmetic is generally defined (i.e., valid for all relevant cases) and that the “creation” of natural numbers did not entail any loss of generality. Dedekind himself mentions (*ibid.*), the possibility of transferring or translating (*übertragen*) the propositions and theorems from one system to another – a point he seems to consider as self-evident. This further allows to “justify” the definition of numbers, as Dedekind explains. Consider a system Ω of elements v ordered by a mapping θ , and a mapping ψ which makes the passage from the natural numbers N (ordered by φ) to Ω , i.e., ψ changes every element n of N into an element v of Ω , i.e., into $v = \psi(n)$, then

[t]his element v can be called the n th element of Ω and accordingly the number n is itself the n th number of the number-sequence N . The same significance which the mapping φ possesses for the laws in the domain N , insofar as every element n is followed by a determinate element $\varphi(n) = n'$ is found, after the change effected by ψ , to belong to the mapping θ for the same laws in the domain Ω , insofar as the element $v = \psi(n)$ arising from the change of n is followed by the element $\theta(v) = \psi(n')$ arising from the change of n' . We are therefore justified in saying that φ is changed by ψ into θ , which is symbolically expressed by $\theta = \psi\varphi\bar{\psi}$, $\varphi = \bar{\psi}\theta\psi$.⁹⁷ (*ibid.*, 823)

die Rede ist, die aus der Anordnung φ entspringen, ganz allgemeine Gültigkeit auch für jedes andere durch eine Abbildung θ geordnete einfach unendliche System Ω und dessen Elemente v besitzt, und da die Übertragung von N auf Ω (z. B. auch die Übersetzung eines arithmetischen Satzes aus einer Sprache in eine andere) durch die in 132, 133 betrachtete Abbildung ψ .”

⁹⁶Tait, following his reflections about logical abstraction proposes the following interpretation for the “categoricity theorem” : “[I]t would seem that logical abstraction, as it is described here, does play a role, not in proofs, but in that it fixes grammar, the domain of meaningful propositions, concerning the objects in question, and so determines the appropriate subject matter of proofs. For example, proving the categoricity of the axioms of simply ordered sets fixes the sense of all propositions in the pure theory of numbers, but it would not do so if numbers were sets, since the sense of $0 \in 1$ is not fixed.” ([Tait, 2005], 224)

⁹⁷“Dieses Element v kann man das n -te Element von Ω nennen, und hiernach ist die Zahl n selbst die n -te Zahl der Zahlenreihe N . Dieselbe Bedeutung, welche die Abbildung φ für die Gesetze im Gebiete N besitzt, insofern jedem Elemente n ein bestimmtes Element $\varphi(n) = n'$ folgt, kommt nach der durch $\psi(n)$ bewirkten Verwandlung der Abbildung θ zu für dieselben Gesetze im Gebiete Ω , insofern dem durch Verwandlung von n entstandenen Elemente $v = \psi(n)$ das durch Verwandlung von n' entstandene Element $\theta(v) = \psi(n')$ folgt; man kann daher mit Recht sagen, da. durch ψ in θ verwandelt wird, was sich

Hence, the “transfer” or “translation” (*übertragen*) of one system into another preserves the order structure: if n is transformed in v , then the successor of n is transformed into the successor of v . This, for Dedekind “fully justifies” the “definition of the notion of numbers” given as an abstraction in which one considers only the relations between numbers established by “the ordering mapping φ ”. And indeed, shortly after the publication of *Zahlen*, Dedekind writes to Weber that he

still regard[s] the ordinal number [i.e., the abstract elements of the ordered simply-infinite system] and not the cardinal number (*Anzahl*) as the original number-concept.⁹⁸

8.2.2.2 Operations of arithmetic

We saw when considering the definition of ideals and of irrational numbers that, for Dedekind, one of the major requirements in defining arithmetical elements (and in particular numbers) is the possibility to give “a perfectly clear definition of the calculations (addition, etc.) one needs to make on the new numbers” ([Dedekind, 1876-1877], Note, 58). He repeats this statement in the letter to Keferstein, mentioning it as one the requirements for the certitude that the definition of the sequence of natural numbers is “completely characterized”:

Finally, is it possible also to set up the definitions of numerical operations consistently for all numbers n ? Yes!⁹⁹ (Letter to Keferstein, transl. van Heijenoort, 101)

It is the “Theorem of the definition by induction” (Theorem 126), which allows, in *Zahlen*, to define the arithmetic operations by means of “earlier concepts”. I will unfold Dedekind ‘set-theoretical’ definition, so as to highlight this last point. It shall also allow me to point out how Dedekind’s definition of the operation does indeed define a new number, the result of the operation, and not a formal operation.

The definition by induction allows to give definitions valid for all numbers n . The abstract statement is the following. Consider any system Ω , θ a mapping (injective or not) in Ω , and ω a determinate element of Ω . Then, there exists *one and only one* mapping ψ of N such that

Theorem 126
I. $\psi(N) \subset \Omega$

symbolisch durch $\theta = \psi\varphi\bar{\psi}$, $\varphi = \bar{\psi}\theta\psi$ ausdrückt.”

⁹⁸ “Außerdem muß ich Dir gestehen, daß ich bis jetzt immer noch die Ordinalzahl, nicht die Cardinalzahl (*Anzahl*) als den ursprünglichen Zahlbegriff ansehe.”

⁹⁹ “Endlich: ist es auch möglich, die Definitionen für Zahlen Operationen widerspruchsfrei für alle Zahlen n aufzustellen? Ja!”

8.2 Was Sind und Was Sollen die Zahlen?

- II. $\psi(1) = \omega$
- III. $\psi(\varphi(n)) = \theta(\psi(n))$ where n designates any number.

ψ is a mapping which associates an element of Ω to any number n . The condition II is the initialization of the induction. The third condition is what we would call a recursion condition on ψ : the image by ψ of the successor of n is the image by a certain mapping θ of Ω of $\psi(n)$. The definition by induction is different from the “Theorem of complete induction”, since it was true of *any* system S and mapping φ , while the definition by induction is restricted to the ordered system of the natural numbers.

To define addition, we consider a mapping ψ from N into N satisfying the following conditions:

- I. $\psi(N) \subset N$
- II. $\psi(1) = 1$
- III. $\psi(\varphi(n)) = \varphi(\psi(n))$ where n designates any number.

The mapping ψ exists and is unique, since it is the mapping defined in the Theorem 126 stated above (with $\theta = \varphi$). Here, ψ corresponds to the “identity” mapping. In fact, ψ is completely determined by the choice of $\psi(1) = \omega$. Dedekind explains that since the choice of the initial element completely determines the result of ψ , if one chooses ψ such that $\psi(1) = \varphi(m)$ with m in N , then $\psi(n)$ corresponds to a certain number k which one denotes by $m + n$ and called “the addition, or the sum, of the numbers m and n .”¹⁰⁰ For $\psi(1) = 1$, $\psi(n) = n$ the image of the successor is the successor of the image, and for $\psi(1) = m$, $\psi(n) = n + m$.¹⁰¹ Addition is completely determined by the three conditions:

- I. $\psi(N) \subset N$
- II. $m + 1 = \varphi(m)$
- III. $m + \varphi(n) = \varphi(m + n)$

One can recognize, here, the conditions stated by Dedekind in the manuscripts. However, it is important to underline that these are properties and not the definition *per se*. The distinction was absent in the manuscripts and illustrates well Dedekind’s wish that the operations be defined as the production of a new number.

Dedekind gives the usual results on addition and the successor operation, and on the addition of any two natural numbers (again, all proofs are made by induction).

¹⁰⁰Indeed, one has $\psi(1) = m + 1$ and $\psi(n + 1) = \psi(n) + 1$. Then, $\psi(2) = \psi(1 + 1) = \psi(1) + 1 = m + 1 + 1 = m + 2$. Hence, $\psi(n) = \psi((n - 1) + 1) = \psi(n - 1) + 1 = m + (n - 1) + 1 = m + n$.

¹⁰¹The definition of addition as the iteration of the successor operation is “an easier way to define addition”, but more difficult to define *rigorously* according to Dedekind.

Since the determination of the mapping ψ depends entirely on the choice of ω , the initial element of the recursion, Dedekind has defined, as he explains, “an infinite system of new mappings of the number-sequence N into itself.” To define multiplication, he proposes to use these mappings to “produce new mappings ψ of N ”. For this, one takes $\Omega = N$ and $\theta(n) = m + n$. Thus, $\psi(N) \subset N$. To determine ψ , one has to choose ω (in N). Dedekind takes $\omega = m$, then $\psi(n)$ is a number k which is denoted by $m.n$ or $m \times n$ or mn , and “[we] call this number the product arising from the number m by multiplication by the number n , or, for short, the product of the numbers m, n ” (*ibid.*, 826). The product is completely determined by the conditions:

- I. $\psi(N) \subset N$
- II. $m.\varphi(n) = m.n + m$
- III. $m.1 = m$

Here, again, Dedekind gives the usual results on multiplication and the successor operation, and on the multiplication of any two natural numbers, with proofs by induction. For the exponentiation of numbers, the idea is similar: one takes $\omega = a$, $\theta(n) = an = na$, and obtains $\psi(n) = a^n$ and the usual properties.

Note that the recursion condition on the definitions of these three operations seems to allow for a mathematical expression of Dedekind’s statement, in the 1854 *Habilitationsvortrag* that addition is the collection “into a single act of the multiply-repeated performance of” the successor operation (which is the “act of counting” in mathematical clothes) and the multiplication is formed from addition “in a similar manner, and from multiplication that of exponentiation” ([Dedekind, 1854a], 757).

8.2.2.3 Definition of the negative numbers

The inverse operations of subtraction and division are not given, here. The manuscript “*Die Erweiterung des Zahlbegriff aus Grund den Reihe der natürlischen Zahlen*” ([Dedekind, \approx 1890], transl. in French in [Sinaceur and Dedekind, 2008], 333-330), gives the definition of subtraction, and extends the system of natural numbers into the system of all integral numbers. In this text, Dedekind proposes a definition of the subtraction and the negative numbers on the basis of the definition of addition given in *Zahlen*, and using congruences to introduce negative numbers.¹⁰² Unlike Kronecker, however, Dedekind does not introduce indeterminate variables and uses the congruences to *create* new numbers.

For this, Dedekind explicitly relies on the definition of natural numbers and the arithmetic operations given in *Zahlen*, and defines the subtraction

¹⁰²Sieg and Schlimm consider that Dedekind provides “models” for the laws exhibited in the manuscripts commented in the previous paragraph.

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as a difference: for α_1, α_2 natural numbers such that $\alpha_1 < \alpha_2$, there “always exists one and only one (natural) number n ” such that

$$n + \alpha_1 = \alpha_2$$

n is called “the difference of the numbers α_1 and α_2 and denoted by $\alpha_2 - \alpha_1$.” However, if $\alpha_1 \geq \alpha_2$, n does not “have any meaning” and to “give it the meaning of a number demands an extension of the number concept.” This extension is carried over by introducing the (usual) notion of congruence: pairs of numbers are said to be congruent if their difference is a same number n . If one designates, for short, a pair of numbers α_1, α_2 by α and a pair of numbers β_1, β_2 by β , then $\alpha \equiv \beta$ means that $\alpha_1 - \alpha_2 = \beta_1 - \beta_2$. That is, that there exists a certain number n such that

$$n + \alpha_1 = \alpha_2 \text{ and } n + \beta_1 = \beta_2.$$

This, however, only has a meaning if $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, and Dedekind proposes to enlarge the notion of congruence so that it can be valid for any number and satisfy

$$\alpha_1 + \beta_2 = \alpha_2 + \beta_1$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2$. This allows to see that one can distribute all “pairs of numbers α ” into classes denoted by (α) according to whether they are congruent or non-congruent to α .

Dedekind designates by P the system of all classes and proposes to obtain a “distinct (*deutlich*) view” of the system, “i.e., a view of all the elements (α) contained in it such that they are different”. For this, Dedekind distinguishes positive classes, i.e., classes such that for a representative α , one has $\alpha_1 < \alpha_2$. The class (α) is thus “completely characterized” by the number $n = \alpha_2 - \alpha_1$, which Dedekind calls the “character” (*Charakter*) of the class. There is, thus, a one-to-one correspondence between the elements of the sequence N of natural numbers and the positive classes. So as to be able to “extend this mapping into a [one-to-one] mapping of the system P ”, Dedekind proposes to “*extend the concept of number by creating new numbers*, i.e., new individuals which will serve as characters (images [*Bilder*]) for the non-positive classes of the system P ”. Hence, each non-positive class defines a new number, which are distinguished according to the classes to which they correspond.

The case in which a class (α) contains “a pair whose numbers α_1, α_2 are identical” defines the “null class” and is denoted by 0.

If a class contains a pair of numbers such that $\alpha_1 > \alpha_2$, then any pair contained in this class must satisfy this equality. In this case, then, “there is one and only one (old) natural number n satisfying the condition”

$$n + \alpha_2 = \alpha_1.$$

If β belongs to the same class, one has $n + \beta_2 = \beta_1$. If m runs through the natural numbers, the class is formed by all the pairs μ with $\mu_1 = n + m$ and

$\mu_2 = m$. One calls the class (α) “*negative* or *opposite* to the positive class characterized above by the number n .” So as to fully characterize this class, Dedekind states, one has to “create” a “new number”. This number “will be called a *negative* number” and is “temporarily designated by \bar{n} .” By doing so, all classes of P are “exhausted” and the new numbers form a domain M which contains the positive numbers n , the negative numbers \bar{n} and 0. In order to characterize more precisely the domain M , Dedekind defines an order for its elements, by extending the “natural order” given for the natural number by a mapping φ (which we will consider in more details in the next chapter), so as to adapt the concepts of successor and predecessor to all the numbers in M . This will allow him to give the definition of the addition for the numbers of the domain M .

This, Dedekind writes, ought to be methodically developed, with each step appropriately defined and each property proved, even though “every mathematician knows in advance that the plan will be completely successful.”

The intricate relationships between operations, their laws and the numbers, which was noticeable in the 1854 *Habilitationsvortrag*, disappears in later works. The definition of operations “as a number completely determined by the summands” highlights the idea that arithmetic operations are an act of the understanding which, from two numbers, produces a third one, which is clearly brought to the fore by the definition in terms of mappings. Dedekind’s insistence on defining the numbers instead of the operations also seems to be linked to his requirement to always work with objects which are actually defined rather than answering formal conditions, as it was the case, for example, with Kummer’s ideal numbers.

As I have mentioned, Dedekind is well aware of the importance of the role played by definitions in shaping a theory and its development, which led him, as we saw in the first part of this dissertation, to dedicate an entire paper to providing a general and precise definition of the Riemann surface for further developments. In this case, the definition of arithmetical operations is crucially important, insofar as the whole of arithmetic relies on it. The definition ought to satisfy two essential conditions: on one hand, it should be given uniformly for all numbers in N and only by means of previously defined notions (i.e., numbers and mappings) so as to not presuppose any of their properties; on the other hand, the definition should allow to define the inverse operations and develop arithmetic further.

Now that the content of Dedekind’s book is clear, and before proposing an interpretation of Dedekind’s conception of arithmetic in relation to his mathematical work, I will propose to contrast Dedekind’s approach with Peano’s. This shall allow to make the specificities of *Zahlen* more visible.

8.3 Peano vs.(?) Dedekind

8.3 Peano vs.(?) Dedekind

In this paragraph, I will propose to compare Dedekind's approach in *Zahlen* to Giuseppe Peano's works on natural numbers. There was a fair number of works published, in the 19th century, about the foundation of arithmetic by authors such as Ohm, Graßmann, Schröder, Helmholtz, Kronecker (whose ideas we considered in Sec. 7.1.3), Frege and to a certain extent Cantor and Weierstrass. Dedekind explains that the publication of Helmholtz's and of Kronecker's works prompted him to publish his own work.¹⁰³

Peano's and Dedekind's works on natural numbers both started from a desire to instill rigor in arithmetic, based on a logical analysis of the sequence of natural numbers, and Peano's axioms (published in 1889) are, as is well known, equivalent to the properties given in Dedekind's definition of a simply infinite system. However, their approaches present some fundamental differences, that I will try to highlight, and which bring to the fore the specificity of Dedekind's approach.

8.3.1 Peano's axioms

Peano's famous axioms were published in 1889, in *Arithmetices principia: nova methodo exposita* ("The principles of arithmetic, presented by a new method", translated by H. Kennedy in [Peano, 1973]¹⁰⁴) and are presented as the result of a logical analysis of the sequence of natural numbers. Before considering Peano's ideas in more details, let me state the axioms. I give them partly in natural language and with modern symbols, for clarity's sake. Peano's investigations are greatly led by the desire to identify the right symbols and notations, and to explain each symbol used in his 1889 essay would take too long.

The sign N means *number (positive integer)*; 1 means unity; $a + 1$ means the successor of a , or a plus 1; and $=$ means is equal to (this must be considered as a new sign, although it has the appearance of a sign of logic).

1. $1 \in N$
2. $a \in N \rightarrow a = a$
3. $a, b \in N$, if $a = b$ then $b = a$

¹⁰³He also mentions Schröder's *Lehrbuch der Arithmetik und Algebra*, published in 1878, which he considers to be a "very valuable" work.

¹⁰⁴The page references to this article are to Kennedy's translation. In the literature, a thorough historical comparison of Peano's and Dedekind's approaches (beyond the similarity of the axioms) has not, as far as I am aware of, been proposed. One is more likely to find comparisons of Dedekind's definition with Frege's works, than with Peano's. For my presentation Peano's works, I rely on Kennedy's works and [Segre, 1994] – especially for the translations.

4. $a, b, c \in N$, if $a = b$ and $b = c$ then $a = c$
5. If $a = b$ and $b \in N$, then $a \in N$
6. $a \in N \rightarrow a + 1 \in N$
7. If $a, b \in N$ and $b \in N$, then $a = b \rightarrow a + 1 = b + 1$
8. There exists no $a \in N$ such that $a + 1 = 1$
9. The induction principle: If $k \in K$ is such that $1 \in k$ and for any $x \in k$, $x + 1 \in k$, then $N \supset k$. ([Peano, 1973], 113)

Recall that the induction is not included in Dedekind's definition, it is proved later in the book. Moreover, Dedekind's notion of chain allows to reduce considerably the number of properties to state.

Following Russell's observations on his axioms, Peano does mention that the propositions given "express the necessary and sufficient conditions that the entities of a system can be put into one-to-one correspondence with the series of natural numbers." In the *Formulario*, in 1898, Peano explains that

These primitive propositions (...) suffice to deduce all the properties of the numbers that we shall meet in the sequel. There is, however, an infinity of systems which satisfy the five primitive propositions (...). All systems which satisfy the five primitive propositions are in one-to-one correspondence with the natural numbers. The natural numbers are what one obtains by abstraction from all these systems; in other words, the natural numbers are the system which has all the properties and only those properties listed in the five primitive propositions. (Quoted and translated in [Kennedy, 1963], 263)

This apparently strengthens the similarity with Dedekind's definition in 1888, who, unlike Russell¹⁰⁵ did not consider it to be problematic. However, Peano did not think of Dedekind's work as actually similar to his, despite their similar results. As we will see, Peano provides an axiomatization of an existing concept, in his work "number is not defined, but the fundamental properties of it are given", while "Dedekind defines number, and calls number precisely that which satisfies the conditions stated above" (*Sul concetto di numero*, quoted and translated in [Kennedy, 1963], 263).

¹⁰⁵Russell, in *The Principles of Mathematics*, compares Dedekind's and Peano's expositions of the axioms for the natural numbers and begins with the observation that

Dedekind proves mathematical induction, while Peano regards it as an axiom. This gives Dedekind an apparent superiority, which must be examined. ([Russell, 1903], 25)

Russell, in fact, considers Peano's approach to be more acceptable.

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8.3.2 Peano's conception of logic

Peano was one of the actors of the wide movement of rigorization of analysis in the second half of the 19th century, and exhibited the first space-filling curve. After working on differential calculus (which he taught in Turin) and geometry, he turned to foundations of arithmetic.

For the foundation of mathematics, the first question that should be considered, he says, is “Can the idea of number be defined, using simpler ideas?” and for example, is it possible to deduce properties such as the commutativity of addition “from simpler properties”. To this, Peano answered:

To these questions may be given different answers by various authors, since simplicity can be diversely understood. For my part, the answer to the first is that number (positive integer) cannot be defined (seeing that the ideas of order, succession, aggregate, etc., are just as complex as that of number). The answer to the second has been affirmative. (Quoted and transl. in [Kennedy, 2006], 54)

For Peano, the attempts to give a definition on the basis of simpler terms do not constitute a productive research program, because simplicity is a very subjective notion, which changes from one person to another. Evidence is, as well, much too subjective to be taken as a criterion for the exactness of a reasoning.¹⁰⁶ Peano, an error hunter in infinitesimal calculus and the inventor of the first space-filling curve, is well aware of the difficulties embedded in epistemological values such as evidence or simplicity. Moreover, Peano continues, such criteria are particularly difficult to handle in a foundational research, since the objects investigated in this kind of work “may appear to many as almost self-evident”:

[Evidence as a criterion] is particularly unsatisfactory in our research, which refers to propositions to which we are *very accustomed* and which may appear to many as almost self-evident. But this question can be given an entirely satisfactory solution. In fact, reducing the propositions, as is done here, to formulas analogous to algebraic equations and then examining the usual proofs, we discover that these consist in transformations of propositions and groups of propositions, having a high degree of analogy with the transformations of simultaneous equations. These transformations, or logical identities, of which we

¹⁰⁶“For this reason, several mathematicians, Descartes among them, declared that evidence is the only criterion to recognize the exactness of a reasoning. But this principle is unsatisfactory. A demonstration may be more or less evident; it may be evident to one person, but doubtful to another; and it may have happened to anyone to find unsatisfactory proofs which were already regarded as exact.” (Quoted and translated in [Segre, 1994], 79-80.)

make constant use in our arguments, can be stated and studied.
(Quoted and transl. in [Segre, 1994], 279-280)

If Dedekind and Peano share the idea that one should not consider certain concepts and ideas as self-evident for the sole reason that they are familiar, they adopt a very different approach of the problem. Peano does not intend to *define* the natural numbers with his axioms, but to highlight their fundamental properties by giving a logical, symbolical expression of them. In this, his position is fundamentally different from Dedekind's who did not consider that there was such things as indefinable mathematical concepts.¹⁰⁷ Moreover, and this is an obvious point of divergence, for Peano the logical analysis consists in being able to put the basic properties into the right symbolism.

To give his axioms, Peano, thus, analyzes the arithmetical reasonings and ideas, since it is, for him, the role of logic in mathematics:

Mathematical logic itself studies the properties of operations and logical relationships, which it indicates by symbols. (...) Some of the principles of this science can be found in the general Logic (see Aristotle). Its true founder is Leibniz, who stated the principal properties of the ideas now denoted by the signs \cap , \cup , $-$, \supset , \wedge .¹⁰⁸ ([Peano, 1901], iii)

First, logic should be used to identify the “fundamental ideas” which compose the more complex ideas and reasonings, as well as the *relations* between ideas which are themselves “logical identities”, such as for example the notion of successor in the sequence of natural numbers (and its role to define the elementary operations). In a second step, logic should serve to express these ideas in terms of “conventional signs”. The desirable outcome of a logical analysis is the representation of the “simplest ideas” in terms of “conventional signs” and the possibility to combine the signs to express complex ideas and relations.

The same analysis can and should be conducted for logic itself. Peano writes that he “was fortunate enough to arrive at a complete analysis of the ideas of logic, reducing them to a quite limited number, which are expressed by the symbols ϵ , \supset , $=$, \cup , \cap , $-$, \wedge .” Note that for Peano the reduction of

¹⁰⁷The laws of thought are unprovable, but in order to use them as “ultimate premisses”, one has to introduce the concepts of system and mappings which are, as we saw, perfectly definable.

¹⁰⁸“*La Logique mathématique à son tour étudie les propriétés des opérations et des relations logiques, qu'elle indique par des symboles. (...) Quelques principes de cette science se rencontrent dans la Logique générale (voir Aristote). Son vrai fondateur est Leibniz, qui a énoncé les principales propriétés des idées représentées maintenant par les signes \cap , \cup , $-$, \supset , \wedge .*”

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the number of signs indicates, a deeper analysis.¹⁰⁹

Logic appears, then, to be a formal and symbolic instrument which allows a better understanding of mathematical reasonings, and gives tools to establish objectively rigorous grounds. For example, if one wants to give an appropriate characterization of addition, one should not – as it was usually done in schools – give a definition using vague terms such as:

‘Addition is that operation by which two or more numbers are summed (united),’ which expresses to add by means of the synonym to sum or to unite. ([Peano, 1973], 222)

This does not allow for a logical construction of the science, a correct logical derivation of complex notions and theorems from primitive notions, for which Peano advocates. The logical analysis, which Peano presents as the first step towards a rigorous logical presentation of theories, should enable the mathematician to solve this problem, and to give actually adequate definitions. Rather, addition should be defined on the basis of the axioms given in 1889 using the successor function and basic properties such as:

[Definition] [I]f a and b are numbers, and $(a + b) + 1$ has meaning (that is, if $a + b$ is a number), but $a + (b + 1)$ has not yet been defined, then $a + (b + 1)$ indicates the number that follows $a + b$.
(...)

19. $a, b \in N \rightarrow a + b \in N$.

20. def[inition]. $a + b + c = (a + b) + c$.

21. $a, b, c \in N \rightarrow a + b + c \in N$.

22. $a, b, c \in N$ and $a = b$ then $a + c = b + c$.

(in [Peano, 1973], 114-116)

For Peano, having expressed the primitive ideas (successor, induction, ...) by means of symbols, here, opens up the possibility of a completely precise characterization of the complex notions such as addition, and the other operations... and eventually, all analysis – a project in which Dedekind was not interested. According to Peano, “symbols are all that may be imagined to be most rigorous and, at the same time, simple, clear, and easy for the pupils.” And it assures the rigor and objectivity of the development of the theory (the definition of composite notions, the expression of theorems, the

¹⁰⁹He states in his 1895 review of Frege’s *Grundgesetze der Arithmetik* that his own ideography having fewer signs than Frege’s indicates that his analysis is deeper. In fact, when reviewing Frege’s *Grundgesetze*, Peano does not address foundational issues, yet crucial for Frege’s book, but concentrates on the *ideography* itself, considering it as the most important element of the book, or even that it is what makes the book important. Peano showed, throughout his works, a tendency to avoid addressing philosophical and even foundational considerations.

development of proofs),¹¹⁰ especially inasmuch as logic can be developed in analogy with algebra.

I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs. The signs pertain either to logic or to arithmetic (. . .) With this notation every proposition assumes the form and precision equations enjoy in algebra, and from the propositions so written others may be deduced, by a process which resembles the solution of algebraic equations. That is the chief reason for writing this paper. ([Peano, 1889], [Peano, 1973], 101-102)

Symbols, thus, are used by Peano to express the “primitive ideas” identified by the logical analysis. While the choice of symbols themselves, indicated by considerations of simplicity, has a part of arbitrariness, the logical process of derivation is completely objective and independent of the symbols themselves. Hence, the arbitrariness is not a problem *per se*. Furthermore, from Peano’s point of view, it should be possible to translate any definition or proposition from one ideography to another.¹¹¹

While Dedekind did not comment on Peano’s works, it is clear, from many statements we already saw, that the importance given to notations and symbols is a point on which he disagrees. Notations or formal approaches in mathematics are not considered, by Dedekind, to provide any clarity, any help, in fact they are hiding the essential characteristics of concepts and could only allow for a partial or biased understanding. In addition, we saw that Dedekind’s conception of logic is that of a guide for rigorous reasonings, which also goes against Peano’s conception.

8.3.3 Peano on numbers

After the publication, in 1889, of “*The principles of arithmetic, presented by a new method*”, Peano regularly came back to the question of defining the numbers – or, at least, as we shall see, of identifying and expressing the right “primitive notions” from which one can logically deduce the properties of the set of natural numbers. In *Sul concetto di numero, Nota I*, he exposes again his axioms, and gives more details about his approach.

¹¹⁰Peano considers himself a follower of Leibniz, and seems to think that, to a certain extent, his *Formulario* is Leibniz’s dream of a “general Characteristic” come true.

¹¹¹Peano repeatedly asked Frege to translate his *Begriffsschrift* into his own notations, showing that he did not grasp entirely the important differences between their approaches. Frege, on his part, “expresses doubts that [Peano’s] ideography can serve to do more than express propositions” (Peano quoting Frege’s review of the *Formulario*). To which Peano answers that “in the works cited above [the ideography’s] importance as a means of reasoning is evident.”

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The first numbers presented, with which we form all the others, are the positive integers. And the first question is: Can we define one, number, sum of two numbers? ([Peano, 1891], 84, translated in [Kennedy, 1974])

One can try to “clarify” the notion of number with “[t]he common, Euclidean, definition of number, ‘number is the collection of several units.’” However this characterization “is not satisfactory as a definition.” It is acceptable as a clarification, because history and pedagogy prove that it works very well:

Indeed, very young children use the words one, two, three, etc. They later adopt the word number, and only much later does the word collection appear in their vocabulary. Indeed, philology teaches that these words appear in this same order in the development of the Indo-European languages. Hence, from the practical side, the question appears to me to be settled. (*ibid.*, 84)

To this, he adds that, from this practical point of view, “there is no *need* for the teacher to give any definition of number” (my emphasis). Indeed, the idea is clear enough for pupils and “any definition would only have the effect of confusing them.”¹¹² However, the “theoretical side” offers a different perspective:

From the theoretical side, to decide the question of the definition of number, one should be told first what ideas he may use. Here we suppose known only the ideas represented by the signs *cap* (and), \cup (or), $-$ (not), ϵ (is), etc., which have been treated in the preceding note. Therefore, number *cannot be defined*, since it is evident that however these words are combined among themselves, we can never have an expression equivalent to number. (*ibid.*, 84-85 my emphasis)

Peano, then, very explicitly states that his axioms *do not define numbers*. This does not mean that natural numbers are out of the scope of a logical analysis or exposition. Indeed, although number cannot be defined, it is still possible to logically express the fundamental (primitive) properties “from which the many other well known properties of the numbers are derived.” The concepts that are not definable are the following:

¹¹²Dedekind’s position is, here, different again. While Dedekind would probably not recommend to treat the definition of natural numbers as given in *Zahlen* in elementary schools, we know that for the definition of real numbers, he considers that it is better for the students to have the rigorous general definition in order to understand the notion they are working with.

[the concept] of number N , of one 1, and of successor of a number a , which we indicate for the moment by $a+$. These concepts may not be obtained by deduction; it is necessary to obtain them by induction (abstraction). (*ibid.*, 85)

Note that the successor is not designed by “ $a + 1$ ”, Peano intended this to make clear that the “single sign, +” designate “the fundamental operation ‘successor of.’” The successor operation is therefore at the foundation of the natural numbers, as it is in Dedekind’s definition of the set of natural numbers. For Peano, it will be essential for the definition of the operations of arithmetic, but it is itself undefinable. However, the distinction between the successor operation and the definition of addition is not made by Peano, as it is by Dedekind.

8.3.4 Definitions according to Peano

Peano holds a particular view on definition, which he exposes relatively clearly in several places, and notably in the last paragraphs of *Notations de logique mathématique* in 1894. After a long exposition of the notations used in the *Formulario*, he proceeds to explain the specific status of definitions. The definition of a concept or an object is equivalent to the introduction of a new sign to designate the concept. Peano begins by stating that it is *not necessary* to define every new notion as long as it is expressible as a reasonably simple combination of words:

The usefulness of definitions is well known. But it should be remarked that they may not be necessary to rigor. One can always, in place of Np [prime number], write what the sign represents. (...) In the *Formulario*, one should introduce definitions, and hence new signs, only when this definition yields a notable simplification. If the idea expressed by a word from the ordinary language can be expressed by a group of symbols simple enough, then it is better to always write this group rather than to represent it by a [new] sign.¹¹³ ([Peano, 1894], §41)

The definition, for example, of a prime number must be done in terms of symbols expressing the property that p is a number only divisible by 1 and by itself.

¹¹³ “L’utilité des définitions est bien connue. Mais il faut remarquer que, à la rigueur, elles ne sont pas nécessaires. On peut toujours au lieu de Np [nombre premier] écrire ce que représente ce signe. (...) Dans le *Formulaire*, il convient d’introduire des définitions, et en conséquence des signes nouveaux, seulement lorsque cette définition porte une notable simplification. Si l’idée exprimée par un mot dans le langage ordinaire est exprimable par un groupe de symboles assez simples, il est mieux d’écrire toujours ce groupe que de le représenter par un signe.”

8.3 Peano vs.(?) Dedekind

The importance of symbolism, for Peano, leads him to adopt a relatively restrictive notion of definition: to be acceptable, a definition must be reducible to an equality. In its simplest form, Peano writes in *Notations mathématiques* a definition is

$$x = a \text{ Def.}$$

where x is a sign which does not yet have a definition, a is a group of signs which have a known sense, and we agree to write the single sign x in place of the group a . This convention is expressed by writing the sign $=$ between x and a and Def. at the end of the line.¹¹⁴ (*ibid.*, §36)

As it should be clear, not every definition can be as simple as the case described above. One could want to define groups of signs in which new signs appear or a group of signs which each have a signification but such that they don't if they are put together. Then

[t]he definition follows an hypothesis h and has the form:

$$h \rightarrow x = a \quad \text{Def.}^{115}$$

(*ibid.*, §37)

An important concern for Peano is the fact that several sentences can define the same idea.¹¹⁶ Therefore, one should fix the meaning with symbols.

More significantly as we saw with the axioms for the natural numbers, for Peano not every idea is definable:

One cannot define everything. This well known proposition follows from what we said. To define a sign x , one must be able to compose by the known signs, a sign a such that one has $x = a$. Hence, one must already know a few signs.¹¹⁷ (*ibid.*)

The primitive ideas (such as numbers) are not definable. Undefinable notions are identified as those which wouldn't be simplified if expressed as a

¹¹⁴ $x = a$ Def. où x est un signe qui n'a pas encore de signification, a est un groupe de signes ayant une signification connue ; et nous convenons d'écrire le signe simple x au lieu du groupe a . Cette convention est exprimée en écrivant le signe $=$ entre x et a et Def. à la fin de la "

¹¹⁵ "[l]a définition suit une hypothèse h et a la forme : $h \rightarrow x = a$ Def."

¹¹⁶ For example: "Instead of saying 'A sphere of center O and radius r cuts a straight line whose distance to O is lesser than r in two points', one can say 'On a straight line, whose distance to O is smaller than r , there are two points whose distance to O is r '" (*ibid.*)

¹¹⁷ "On ne peut pas tout définir. Cette proposition, bien connue, résulte aussi de ce que vous avons dit. Pour définir un signe x , il faut pouvoir composer par les signes connus un signe a tel que l'on ait $x = a$. Donc, il faut déjà connaître quelques signes."

combination of other – tentatively simpler – ideas (the successor operation, for example). In fact, for Peano, the good question is not whether one can define something, but whether one can define something by means of other (simpler) objects:

The question that appears so often “can we define the object x ?” is not correctly put. One can ask it under the form “can we define x by means of the objects a, b, c ?”, “can we form, with the signs a, b, c a group of signs equal to x ?” In practice, the answer to that question only means “can we define the object x by means of simpler ideas?” and there is arbitrariness in the evaluation of simplicity.¹¹⁸ (*ibid.*, §42)

The logical process of analysis leads to express complex ideas as a composition of simple(r) ones. There exists, Peano writes, ideas that can’t be further simplified and these are the “primitive ideas of the science” which can be obtained only through “experience or induction”. Mathematics does not depart from this principle, and “the study of the primitive ideas” has been done, for example, in arithmetic, enabling us to know that “one can define everything by means of the primitive ideas represented by the signs $N, 1, +$.” Peano notes that the sign $+$ is a primitive idea only when it designates the successor operation – it is, then, the primitive sign which expresses the primitive idea of (mathematical) induction.¹¹⁹

8.3.5 What Peano thought Dedekind was doing

Peano recognized Dedekind’s priority in giving the “axioms”. He affirmed the independence of his work and that his booklet was being printed when he read Dedekind’s essay. He would have, then, only drawn “moral support for his conviction of the independence of the axioms.” Dedekind and Peano seem to meet at some points, in particular as regards the emphasis that one is mistaken in believing that certain complex notions are simple, or the idea that only a deep analysis enabling to identify the basic notions and followed by a logical exposition can provide a clear and rigorous ground necessary for a better development of mathematics. However, we saw important differences in their works.

In addition to the divergences noted above, I would like to underline that the *aims* of their work are significantly different. Dedekind does not consider

¹¹⁸ “La question qui se présente tant de fois “peut-on définir l’objet x ?” à la rigueur n’est pas bien posée. On la peut mettre sous la forme “peut-on définir x au moyen des objets a, b, c ?”, “peut-on former avec les signes a, b, c un groupe de signes égal à x ?” Dans la pratique, la question posée signifie seulement “peut-on définir l’objet x au moyen d’idées plus simples ?” et il y a de l’arbitraire dans l’évaluation de la simplicité.”

¹¹⁹ There is, however, in the choice of the “primitive ideas”, a certain arbitrariness “for, if one define d by means of a, b, c and one defines c by means of a, b, d , then one can take as primitive ideas either a, b, c or a, b, d .”

8.4 Mathematics, arithmetization and number concept

numbers or the successor operation as primitive undefinable notions which only need to be rendered more explicit. On the contrary, his whole book is devoted to the definition of these notions. This difference is pointed out by Peano himself, in *Sul concetto di numero*:

Between the preceding and what Dedekind says, there is an apparent contradiction that should immediately be pointed out. Here, number is not defined, but its principal properties are stated. Instead, Dedekind defines number as precisely that which satisfies the preceding conditions. Evidently, the two coincide.¹²⁰ ([Peano, 1891], 88, transl. in [Kennedy, 1974])

In this very clear and moderate statement, Peano, who tended to stay away from philosophy, seems to have perfectly grasped Dedekind's intention.

Dedekind and Peano approach a different problem. While Dedekind intends to *define* the natural numbers, Peano explicitly does not, on the contrary, he assumes that they exist, develops a formal system and begins the transition towards modern *logic* – whereas Dedekind begins the transition towards set theory. Moreover, not only does Peano assume the existence of numbers for his axiomatization, he even considers that numbers *cannot* be defined, that they are a primitive concept to which his work is giving a mere clarification. Peano wants to be able to logically derive all analysis from the axioms, a problem that Dedekind's does not consider, nor does he consider the question of formal logic.

Hence, as mentioned earlier, Peano presents the principal properties of the sequence of natural numbers, and shows how logic can help mathematics, by developing logic as a (formal) theory. Dedekind, however, did not have any of these considerations, he did not present a system of formal rules and did not seem interested in deducing Analysis, or even number theory from the notions introduced in *Zahlen*. From Dedekind's viewpoint, it seems rather that once a concept is created, through a rigorous definition, for example the natural numbers, one can use this concept without going through the “long series of inferences” that preceded its definition.

8.4 Mathematics, arithmetization and number concept

Considering the recurrent use of arithmetical operations in Dedekind's works, and in particular their essential importance in ideal theory, it seems that with his definition of numbers and arithmetic, Dedekind is also providing (retrospectively) a rigorous ground for the arithmetical methods developed in his works through a rigorous and general foundation to arithmetic.

The importance of definition in Dedekind's mathematics is made particularly significant through his requirements of rigor. It becomes all the

¹²⁰By “evidently the two coincide”, Peano refers to the independence of the axioms.

more crucial to secure the definition of natural numbers that would provide a rigorous foundation of arithmetic, considering the role played by arithmetic in Dedekind's mathematics. The 'genetic' component of Dedekind's approach in extending the concepts of numbers and his insistence on preserving the "identity of nature" of numbers emphasizes the importance of the foundation of natural numbers.

I will suggest that the definition provided by Dedekind allows for a justification of the use of arithmetic as a tool for producing new knowledge in mathematics. And, conversely, these uses of arithmetic had an indicative role (*mutatis mutandis*) in the definition of natural numbers and the foundation of arithmetic.

8.4.1 A need for a definition of natural numbers?

The continued interest of Dedekind for the number concept and his extensive use of arithmetic suggest that Dedekind's mathematical works are highly dependent of the foundation of arithmetic. To build the theory on "earlier concepts" emphasizes the need to consolidate the basis. In this case, the importance to be able to provide a definition of the natural numbers based on primitive unprovable notions is particularly stressed.

It is not uncommon, in mathematics, that results are proved before an adequate rigorous definition of the fundamental concepts can be given. From differential calculus or the continuity of functions to complex numbers, many examples can be given. The relative mathematical imprecision of the definition of Riemann's function theory did not keep mathematicians from developing his works, and we saw that Dedekind and Weber's 1882 paper was written essentially to correct this insufficiency. To secure the basic definitions of a theory, be it natural numbers or Riemann surfaces, constitutes, for Dedekind, a very crucial part of the rigorization of a theory. In the light of previous mathematical developments, from the extension of the domains of numbers to the arithmetic of ideals, and as far as the natural numbers are not considered as "given" or "evident" a definition of the sequence natural numbers appears indispensable.

The extension of the number concept is one of the main driving forces of mathematics – in fact for Dedekind, as we saw with the *Habilitationsvortrag*, the extension of concepts and methods is one of the main driving forces in mathematics, as it was already the case for Gauss. We saw that the extensions of the domains of numbers are conducted by Dedekind by building on previous content, adopting an approach close to the genetic approach (yet underlaid by an axiomatic analysis of the concepts to be defined). This implies that each extension of the domain(s) of numbers requires that the restricted domain be rigorously defined. As a consequence, the foundation

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of natural numbers is crucial for Dedekind, in order to assure the consistency of the extensions of the number concept.

Indeed, we saw on p. 496, that for Dedekind, the negative integers are created by defining pairs or congruences of natural numbers in order to allow the unconditional applicability of subtraction and thus form the domain of all integers; the rational numbers are created by defining pairs of integers to allow the unconditional applicability of division and thus allow to form again a wider domain; the irrational numbers are created by means of cuts on the basis of the domain of rational numbers alone; the complex numbers are defined as pairs by Hamilton.

For algebraic numbers, the failure of the uniqueness of the prime decomposition is a motive for defining a wider notion of integer, so as to regain the “simple laws” of the old theory. The theory of algebraic numbers relies entirely on an extension of the concept of integer, algebraic integers, which is the “broadest generalization of the concept of integer” because it covers all known notions of integers (positive, negative, Gaussian, cyclotomic, etc.). In the definition of an algebraic integer, one can see that rational integers play an essential use by their role as coefficients of the polynomial.

Since the whole edifice is built one floor after the other, and not with an intrinsic axiomatic definition for each system of numbers, a problem in the definition of natural numbers could put at risk all domains of numbers up to complex numbers.

Articulated with Dedekind’s conception of rigor, then, if nothing that can be proved (or defined) should be accepted without appropriate proof (or definition), there is no reason for natural numbers to escape this rule. At least, it requires a further examination. And indeed, as I mentioned above, the analysis of the sequence of inferences highlights the fact that natural numbers are all but indefinable or immediate notions.

The second reason is linked to the question of “arithmetization”, as I presented it in Sec. 7.3. Dedekind’s arithmetization is then expressed through the demand of going back to the “simplest principles of arithmetic”, even if the presentation ends up demanding long technical preliminaries. Arithmetization plays a role in satisfying Dedekind’s suggestion that new concepts should be built on “earlier concepts” and simpler notions.

While he never used, unlike Kronecker, the phrase “arithmetization”, Dedekind does mention such a development of the number concept and stresses its advantages. As I have underlined, to bring certain notions into relation with the number concept or with methods of number theory is said to provide the same increase in clarity, rigor and even precision. Dedekind expresses this fact several times, and notably to Weber in relation to gaining a true and complete understanding of Riemann’s works. Arithmetization thus allows for a rigorous and clear presentation and understanding of certain theories, as attested by the theory of algebraic functions and the 1877

rewriting of algebraic number theory – in this last case, it even “highlights” the “main difficulty” (which was not mentioned as a “difficulty” in 1871).

I observed, in the earlier parts of this dissertation, that by analyzing where and how arithmetic is used in Dedekind’s mathematical texts, one can notice that arithmetic presents itself with the very particular ability to be applicable in various domains whose elements are not necessarily numbers. It is then possible to extend the operations of arithmetic into several “levels”, using arithmetic as a tool to develop mathematics. In several mathematical papers, as we saw, Dedekind exploits the possibility to develop arithmetic for elements that are not numbers, and laying the ground for more developments – for ideals of numbers, ideals of functions, polygons. . . Dedekind’s mathematics relies heavily on arithmetic, and if Dedekind is to pursue his own principle of rigor to its full extent, an appropriate foundation of arithmetic is required. Arithmetic is said many times to be inherently related to natural numbers, hence the roots of the many extensions of divisibility present in Dedekind’s works should be found in the natural numbers. Recall that Kronecker’s theory of algebraic magnitudes is said to not be arithmetical because, unlike what one “should suspect”, the “consideration of the realm of numbers (the absolute constants)” does not “form the main foundation” – at least, according to Dedekind.

8.4.2 Arithmetic operations as epistemic tools

The idea that numbers and arithmetic can be a guide for the thought, which can be related to the possibility to define arithmetic as “flowing from the laws of thought”, should also be related to Dedekind’s use of arithmetic in his mathematical works. Indeed, if the foundation of arithmetic is to be acceptable as a foundation for Dedekind’s many uses of arithmetic in mathematics, it should lay bases general enough so that arithmetic can be understood as transferable, or translatable into different settings. It seems that the possibility to define arithmetic on the basis of the general laws of and as an emanation of the mind can act as a justification of the preeminence of arithmetical methods in Dedekind’s works.

The particular way in which Dedekind chose to define the natural numbers seems to contain a justification for his uses of arithmetic as an epistemic tool. The use of arithmetic operations as epistemic tools, and the generality it suggests for arithmetic, are significant in Dedekind’s most elaborate (mathematical) works and sustained by many applications and results. The use of arithmetic appears as a shining illustration of a more fundamental claim about the natural numbers: that they are logical entities, defined by the general concepts of sets and mappings and directly flowing from the laws of pure thought.

This suggestion relates to my attempts to place Dedekind’s definition of natural numbers in perspective of the other works in his mathematical

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corpus and to understand how his number concept is articulated with his mathematical practice, how its definition bears the mark of previous mathematical researches, and what effect it can have had on later works.

Dedekind adopts a rather Kantian position regarding the limited, finite and sequential character of our understanding. A consequence of this is that he considers the development of mathematics, beginning with the extension of the concept of number, to be made step-by-step. Moreover, it should always be done with a systematic resort to “earlier concepts” if one is to assure the rigor of the reasoning. I proposed that the use of arithmetical operations, notions and methods of proof in his mathematical works, even in non-numerical contexts, was related to the desire to meet this certain standard of rigor.

But recall that for ideals, the definition of a new divisibility relationship is introduced as a way to make the development of the theory *easier*: the recourse to arithmetic, even if it requires to define new operations for new objects and / or long preliminaries, is a way to bypass the difficulties such as the long tedious computations that Dedekind dislikes so much. It is a way to *simplify* the inferences by reducing the complicated computations to very simple ones – manipulations of arithmetical operations similar to elementary number theory. The changes inflected to the theory between 1871 and 1876, which led to a theory of ideals following, almost to the letter, the same development as rational number theory, was also justified by a desire to *simplify and clarify* the theory. As Dedekind tells us, “if the theory has not been shortened it has at least been simplified a little” ([Dedekind, 1876-1877], 119). In the same way, in 1882, Dedekind and Weber’s use of arithmetical notions openly aimed at a more rigorous and *clearer* theory, which they obtained adopting a strongly arithmetical approach with several layers of arithmetical notions – an approach which was not perpetuated by Weber.

An additional example, not seen yet, is a paper on ideal classes, *Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers*, which Dedekind published in 1877. The paper is dedicated to the study of “all the ideals of the order \mathfrak{o}' ”, which Dedekind promised to study in [Dedekind, 1876-1877] (see p. 356). Dedekind introduces notations for the divisibility, GCD and LCM of modules which are more than strongly reminiscent of elementary arithmetic. For two modules \mathfrak{a} and \mathfrak{b} , the module constituted by all the numbers of the form $\alpha + \beta$ with α in \mathfrak{a} and β in \mathfrak{b} is denoted by $\mathfrak{a} + \mathfrak{b}$ and called “the sum”. The “sum” can be seen to also be the greatest common divisor of the modules \mathfrak{a} and \mathfrak{b} . The sum / GCD satisfies common properties of addition, such as commutativity and associativity. It also satisfies that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ (idempotency) and if \mathfrak{d} divides \mathfrak{m} , $\mathfrak{m} + \mathfrak{d} = \mathfrak{d}$. The least common multiple of an arbitrary number of modules, i.e., the numbers which belong to every module, is denoted by “the symbol

$\mathfrak{a} - \mathfrak{b}$ " ([Dedekind, 1877], 121). It satisfies idempotency, commutativity and associativity. The new notation introduced allows Dedekind to study properties of modules (and, later in the paper, of ideals) and to state general equalities such as:

$$(\mathfrak{a} + \mathfrak{b}) - (\mathfrak{a} + \mathfrak{c}) = \mathfrak{a} + (\mathfrak{b} - (\mathfrak{a} + \mathfrak{c}))$$

$$(\mathfrak{a} - \mathfrak{b}) + (\mathfrak{a} - \mathfrak{c}) = \mathfrak{a} - (\mathfrak{b} + (\mathfrak{a} - \mathfrak{c})).$$

These properties will play a crucial role in the 1894 version of algebraic number theory.¹²¹

Hence, the key role played by elementary arithmetic operations in Dedekind's mathematics strongly suggests that the operations of arithmetic, tightly linked to thought itself, are given a very particular and active role in the development of mathematics. With the possibility to be used or applied in other areas of mathematics, arithmetic appears to be a powerful tool at the mathematician's disposal. It allows to elaborate a strategy of arithmetization which reduces the chains of inferences and the length of computations by transposing the study of arithmetic to systems of elements (what Dedekind calls a "higher level"). Arithmetization allows to develop concepts and methods answering the criterion for rigor proposed in Chapter 7. By involving developing new methods based on an arithmetic of a "higher level", by rewriting certain theories with arithmetical tools, Dedekind elaborates what he considers to be a simpler, clearer approach to the various theories involved – e.g. for algebraic functions. This shows the active role played by arithmetic in the rational activity of producing new knowledge.

For Dedekind, as we saw, mathematics is an activity of the thinking human. Arithmetic, in texts such as *Zum Zahlbegriff* and in the way in which the natural numbers are defined, is presented as possessing an essential role in the act of thinking itself: arithmetical thinking appears to be a fundamental act in thinking generally, whether one is aware of it or not. Dedekind highlighted how thought and arithmetic are following the same step-by-step process rooted in the two fundamental operations of the mind: collecting things with a common property, and relating things to things "without which no thinking is possible". By the unfolding of the "*Gedankenprozeß*" from which *Zahlen* proceeds, Dedekind justifies his belief that numbers are immediately created by the mind and their science is so deeply embedded in our thinking that every thinking man is an arithmetician. Arithmetic is then said to be the most fundamental and simple science, and *at the same time* a science with "limitless applications". Numbers (and

¹²¹This notation and the properties proved for modules on this basis will be of great importance in the 1894 version of ideal theory, as well as for later works in which Dedekind develops the notion of "Dualgruppe", which correspond to today's notion of lattice ([Dedekind, 1897], [Dedekind, 1900], see also Schlimm, 2011).

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therefore arithmetic) are one of the simplest and most fruitful, most efficient of all the “auxiliary means” created by the mind: as a part of logic, arithmetic is capable of providing a help for thought, a tool to assist the thought – an “auxiliary means that the human mind has yet created to ease its life” as put in *Zum Zahlbegriff*.¹²²

With arithmetic, Dedekind has a theory with a double advantage. More developed than the logical notions (systems and mappings), it has an “inexhaustible wealth” of laws and still is elementary and rigorously defined for it to guide the thought. It can even allow to eliminate difficulties in the development of theories. It appears to be a particularly appropriate tool for extending mathematical knowledge.

In addition, arithmetic appears to present itself with a wide scope of possible applications – as has been known for a long time. Unlike the demand to recognize only natural numbers, which, Dedekind states very clearly, would not be fruitful, arithmetic used in this way can help to master the new “complex phenomena”. Within Dedekind’s mathematical works, it appears that arithmetic possesses a certain generality which allows for arithmetic to be used ‘transversally’ in several different (not necessarily arithmetical) theories, and to be used to develop new concepts. This allows to combine two apparently exclusive claims: a mathematical and epistemic primacy of arithmetic, and the benefits of inventing new concepts for the development of mathematics.

Arithmetic, as a part of logic and as intimately linked to thought, appears to be a powerful tool at the mathematician’s disposal because it can offer large possibilities of applications. Through arithmetization, it allows to instill a greater rigor in the theory. The possibility to “transfer” arithmetic to other frameworks (on which I will come back shortly) suggests that arithmetic possesses a certain proper generality. Furthermore, the 1882 theory of algebraic functions is said to “stay in its own sphere”, which in turn suggests that setting up higher levels in which arithmetical notions are defined for systems of objects (e.g., ideals or polygons) and studying their properties in an arithmetical form do not “mar the purity” of the theory. Arithmetic as part of logic, as an epistemic tool, then, seems to present itself with a certain generality in the sense of an ontological neutrality, just as the notions of system and mapping do.

Recall that the four fundamental operations of arithmetic are themselves defined in terms of mappings, a few paragraphs after the definition of the sequence of natural numbers. The definition of the operations is thus as general as the previous developments of the book. It appears, from this

¹²²Sinaceur’s suggestion that arithmetic being a part of logic could be understood as arithmetic providing a “rational (logical) norm of thinking” also follows the idea of logic as a help to conduct reasoning rigorously.

definition, that their properties (i.e., the laws of rational arithmetic) should be given the same level of generality as the generality inherent to the system of natural numbers: the highest one, that of the laws of pure thought. The laws of thought, whose improvability was underlined, following Detlefsen, as a requisite to serve as premisses for the definition of natural numbers, are also of the utmost generality. As logical operations, the operations of thought expressed as ‘set-theoretical’ concepts have the highest level of generality because they are applicable, objectively, to any and all thoughts. The operations of thought have a universal generality, thus, and it seems to be an important point for them to be at the ground of arithmetic, for when involving concepts or laws of a lesser generality in a definition, one risks to unduly impose restrictions to the said definition. The operations of *arithmetic*, through the definition given by Dedekind, thus acquire an analogous degree of generality and are applicable to any object of thought.

The whole edifice of arithmetic being built up on the “operations of thought” seems to not only be a proof of arithmetic’s fundamentality in human thinking, but also to suggest that the laws and propositions of arithmetic are, for Dedekind, of the same level of generality as the laws of pure thought. The operations themselves being defined in terms of mappings, the laws of elementary arithmetic shall be given the same level of generality: the highest one, that of the laws of pure thought.

On the other hand, arithmetic possesses the generality exhibited by the ‘categoricity’ theorem. Indeed, as I underlined, this theorem is seen by Dedekind as the proof that all propositions of arithmetic are of “perfectly general validity” (*ibid.*, 823) for *all* simply infinite systems. It is then proven that elementary arithmetic is general. Dedekind himself mentioned, in the Remark 134 (*ibid.*, 823), the possibility of transferring (*übertragen*) the propositions and theorems from one system to another.

Note that “*übertragen*” is the same word used by Dedekind for the transfer of ideal theory from algebraic number theory to algebraic function theory:

the right approach is indicated by the methods applied with great success in number theory which are derived from Kummer’s creation of ideal numbers, and which can be transferred to the theory of functions.¹²³ ([Dedekind and Weber, 1882], transl. modified, 41)

as well as for the transfer of divisibility properties from functions to ideals:

Such a divisor does not indeed exist in the general case, but when the theorems about the rational functions are not related

¹²³ “*wiesen die mit bestem Erfolge in der Zahlentheorie angewandten Methoden, die sich an Kummers Schöpfung der idealen Zahlen anschließen, und der Übertragung auf die Theorie der Funktionen fähig sind, auf den richtigen Weg.*”

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to the divisor itself but rather to the system of functions divisible by it, they can be completely transferred to general algebraic functions.¹²⁴ (*ibid.*, transl. slightly altered, 42)

It is also used, for example, when studying quadratic forms and “transferring” the problems and methods to ideals. In *Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers*, Dedekind explains that the problem can be solved by transferring the

With the help of these principles, which I must here consider as already known, one can determine the relation of the number of classes for different orders, from the decomposable forms of arbitrary degree or the corresponding problem transferred to classes of ideals.¹²⁵ ([Dedekind, 1877], 110)

Later in the paper, he also explains that the “transfer” to ideals of some of the “principles” (*Übertragung der Prinzipien*, *ibid.*, 118) which Dirichlet introduced for the proof of the theorem on arithmetic progressions allows to obtain the results that we saw on p. 362.

The remark following this “categoricity” result states that any theorem of arithmetic – any theorem “regarding the elements n of the simply infinite system N ordered by the mapping φ (and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement φ)” ([Dedekind, 1888], 823) – is perfectly general. That is, it “possesses perfectly general validity for every other [than N] simply infinite system” (*ibid.*) ordered by a certain mapping θ . Dedekind explicitly mentions the possibility of transferring the theorems from one system to another, giving the outlines of how the order (successor relation) is preserved. This, he writes, “fully justifies” the definition of the numbers as the “abstract type” of the concept simply infinite systems. And arithmetic is characterized as the science having for first object “the relations or laws which are derived entirely from the conditions [defining the simply infinite systems] and therefore are always the same in all ordered simply infinite systems, *whatever names may happen to be given to the individual elements (compare 134)*” (*ibid.*, 809, my emphasis). In this claim already appears the idea that what constitutes arithmetic are general laws valid in any simply infinite system – independent of any instantiation.

¹²⁴ “Ein solcher Teiler existiert zwar im allgemeinen Falle nicht, wenn man aber die bezüglichen Sätze über rationale Funktionen nicht an den Teiler selbst, sondern an das System der durch denselben teilbaren Funktionen knüpft, so gestatten sie eine vollkommene Übertragung auf die Allgemeinen algebraischen Funktionen.”

¹²⁵ “Mit Hilfe dieser Prinzipien, welche ich hier als bekannt voraussetzen muß, läßt sich nun das auf die zerlegbaren Formen von beliebigem Grade oder auf die entsprechenden Ideal-Klassen übertragene Problem, das Verhältnis der Klassen- Ahnzahlen für verschiedene Ordnungen zu bestimmen. . . ”

Dedekind doesn't seem to interpret his categoricity result as implying that there exists exactly one model up to isomorphism for natural numbers, as second order logic would. Admittedly, the categoricity allows that one can work with the system of natural numbers since there is no loss of generality. Yet, Dedekind's immediate jump to the possibility of transferring theorems from one simply infinite system to another (the semantic completeness, noted by Reck and Awodey) is suggesting that he saw this result as establishing arithmetic as "perfectly general", in the sense that it can be applied and the validity of arithmetic properties can be proven in any set in a one-to-one correspondence with the natural numbers.

The "categoricity" theorem implies that one can consider the "class of all simply infinite systems", all equivalent to each other and for which one can take a representative without loss of generality, as for congruence classes and for the many other equivalence classes that Dedekind came upon in his works.

For someone familiar with the notion of equinumerosity ("similarity" in *Zahlen*) of sets, the "categoricity" result seems to hold certain possibilities of applications. For example, Dedekind gave a proof of the one-to-one correspondence between algebraic numbers and natural numbers to Cantor in their correspondence. The letters from Dedekind to Cantor of this period do not seem to be available, but according to what Cantor wrote, the proof of the existence of a one-to-one correspondence between natural numbers and algebraic numbers given by Dedekind was

approximately the same as the way in which I prove the statement in my previous letter [i.e., the set of natural numbers can be mapped one-to-one with the more general set $(a_{n_1}, a_{n_2}, \dots, a_{n_v})$, in which n_1, n_2, \dots, n_v are positive integers in any (unlimited) number]. I take $n_1^2 + n_2^2 + \dots + n_v^1 = \aleph$ and order accordingly the elements.

Isn't it good and convenient, in and for itself, as it was emphasized by your remark, that it is possible to talk about the n th algebraic number, so that each appears once in the sequence?¹²⁶ (Letter from 2 December 1873, in [Cantor and Dedekind, 1937], 13)

We saw that the existence of biunivoque correspondences does justify the transfer of arithmetical properties from algebraic numbers or functions to ideals or from ideals to polygons. Moreover, recall that Dedekind justified

¹²⁶ "ungefähr derselbe, wie ich meine Behauptung im vorigen Briefe erhärte. Ich nehme $n_1^2 + n_2^2 + \dots + n_v^1 = \aleph$ und ordne darnach die Elemente. Ist es nicht an und für sich gut und bequem, dass man, wie von Ihnen bezeichnend hervorgehoben wird, von der n ten algebraischen Zahl reden kann, so dass jede einmal an die Reihe kommt?"

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the transfer of the study of divisibility from algebraic integers to ideals because of the “complete coincidence” of the relations of divisibility between numbers and ideals. In algebraic function theory, the transfer of all arithmetical laws and properties from ideals to polygons was justified by the existence of a one-to-one correspondence between ideals and points, but without much of a proof of the validity of the said laws.

As a ground, a justification of the methods used in his works, then, *Zahlen* seems to show a certain impact of the previous mathematical works on its writing. In addition, the slow development of Dedekind’s ideas on natural numbers and the uses made of arithmetic while Dedekind was reflecting on defining natural numbers suggest that the practice of mathematics affected the definition of natural numbers, in that it suggested the possibility to define a more general concept.

8.4.3 Could the uses of arithmetic have influenced the definition of natural numbers?

In the reflection on natural numbers, started around 1872, one can see that while Dedekind identified the “act of counting” as establishing a correspondence between elements, his ideas only gradually grew towards the general idea of simply infinite system. One can distinguish, in the first draft from 1872-1878 quoted above, the evolutions of Dedekind’s ideas, as I have mentioned, from considering the *system* as central, together with ideas such as “mappable” or “similarly mappable”, to putting the *Abbildungen* themselves as the core of the study. The definition of the sequence of natural numbers, thus, is slowly detached from the consideration of the system itself as holding the key to the definition of the ordered sequence, to embrace the idea of mappings as the fundamental concept. On the other hand, the definitions given in this draft are far from the level of generality proposed in *Zahlen*, not only because the “categoricity” theorem is absent, but also because the idea that one could define a general concept of ordered sequence of which our familiar concept of natural numbers would be an instance does not appear. The definition given in the second “layer” of the drafts, between 1872 and 1878, defines numbers as elements of the chain of 1. The chain itself, in 1872/78 is defined in the following way: if a system S is mapped onto a system S' different from S , then there exists an element in S which is not in S' . This element is “designed by the symbol 1 and named ‘one’” (in [Dugac, 1976b], 299)

Each thing contained in (1) [the chain of 1] is called a *number*.¹²⁷
(*ibid.*, 300)

¹²⁷ “Jedes in (1) enthaltene Ding heisse eine Zahl.”

In the last (hence from 1878) “layer” of draft, Dedekind gives “characterization of N ” (*Charakteristik des Systems Z*) with the properties that will be used to define the simply infinite system:

Characteristic of the system Z . There is a similar mapping of Z (...) which has the following properties.

I. Z' is a part of Z .

II. There is a number (i.e., a thing contained in Z), which is not contained in Z' . This number shall be called “one” and is denoted by 1.

III. A number chain (i.e., each part T of Z , whose image T' is a part of T) that contains the number 1 is identical with Z . (*ibid.*, transl. slightly altered, 144)

Here, we see that Dedekind introduces the notion of numbers before the defining properties, they seem to not avoid being more symbols satisfying properties than creations of the mind properly speaking. In addition, it is clear that the concept defined by Dedekind, here, is not supposed to be the “general” concept of “simply infinite system”. The 1872/78 manuscript shows part of the “protracted labor” through which Dedekind went for the “analysis of the sequence of natural numbers just as it presents itself”. Despite the great generality to which Dedekind arrived in the concepts of system and mapping, and the increasing generality of the characterization of the sequence of natural numbers, Dedekind defines *numbers* and not elements of indifferent nature satisfying certain properties, such that one of the infinite possibilities of such systems is the sequence of natural numbers.

Note that prior to his researches on the sequence of natural numbers, Dedekind wrote and published his first important works in number theory (in particular [Dedekind, 1871]) and *Stetigkeit*, that during his struggles to find a satisfying characterization of natural numbers, he wrote the second version of ideal theory, and that 1878 marks the beginning of Dedekind and Weber collaboration on algebraic function theory. While Dedekind did announce the imminent publication of (what was supposed to be) *Zahlen* in 1879, as we know, the book was only published almost ten years later.

To be clear, there are links drawn explicitly by Dedekind himself, underlining the inner unity of his works, and that I have mentioned, but there aren't many explicit statements from Dedekind regarding the influence of mathematical practice on his conception of arithmetic. Thus, it is difficult to do more than pointing out textual coincidences such as the ones above. However, our study of the texts does suggest that the possibility to develop arithmetical methods of proofs in several different domains might have modulated Dedekind's conception of arithmetic, and through this, of number.

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Neither in Dedekind's statements about arithmetic nor in his uses of arithmetical methods, can statements about the generality of arithmetic be found in a way similar to what is in *Zahlen*. In fact, it seems that this idea appears progressively through a process of enlargement of Dedekind's conception of arithmetic, suggested or even enabled by the uses made in his mathematical works. We saw that from one step to another, Dedekind went from considering only the extensions of domains of numbers in 1854 to Galois theory and higher congruences developed in a "rigorous analogy" with rational arithmetic in 1857. Later, he introduced the concept of ideal as an extension of Kummer's ideal numbers and started to develop an extended notion of divisibility, already outlined in his lectures on Galois theory, which was expanded to a greater extent in a rewriting of algebraic number theory and transferred to algebraic function theory to be used in investigations of the Riemann surface itself. The crescendo, thus, is palpable.

This sort of "generality" of arithmetic, the possibility to transfer arithmetic in different frameworks, is central in Dedekind's mathematical works, and is sustained by many applications and results in Dedekind's writings on algebraic numbers and algebraic functions. And we saw to which extent Dedekind exploits the possibility to develop arithmetic for elements that are not numbers, and to prove the validity of the laws of rational arithmetic for those objects, as a consequence of the definition of arithmetical (rational) operations and independently of the individual nature of the elements of the domains under consideration. Dedekind seems to be putting forward an 'abstract' arithmetic by developing divisibility theories in his most prominent mathematical works, and exhibiting the validity of the laws of (rational) arithmetic in more general settings (fields of algebraic numbers) or in theories that seem to be out of the scope of the "science of numbers" (algebraic functions of one complex variable). The general (or abstract) character of this arithmetic is obtained by the study of particular 'models' of arithmetic, following Dedekind's tendency to develop his most general ideas in specific frameworks (such as the development of the notion of *Abbildung* from permutations of numbers to general mappings via the definition of conjugated fields and of the point of a Riemann surface), to build on content.

As I have mentioned at the beginning of this chapter, I am not arguing that Dedekind's practice bears all the responsibilities of Dedekind's ideas on numbers, for it would amount to denying a large portion of what was explained above – which I do not, of course, wish to do. Rather, I am suggesting that considering the recurrence of certain methods, the long and repeated endeavors to develop theories in an ever more "satisfying" way, and the net of auto-references shown by Dedekind,¹²⁸ it seems that his works

¹²⁸Dedekind, in fact, rarely considers that other mathematicians could have found a better method of definition or proof. Even in the case of Weierstrass's and Cantor's definitions of real numbers, if Dedekind does recognize that they are as rigorous and correct as his own definition, he insists that his approach is easier, and less heavy.

and ideas have nourished each other. This does not imply that all works influence each other, nor that they all should answer to the same standards (a rigorous theory of algebraic numbers does not require to unfold completely the “*Gedankenprozeß*” as in *Zahlen*).

8.4.4 Some closing remarks

The guiding idea for this last chapter was to consider Dedekind’s definition of the natural numbers and its relation to the strategy of arithmetization highlighted in the previous chapters. Indeed, the strategy of arithmetization brought to the fore in Chapter 7, in which to “arithmetize” is taken in the sense of “putting into an arithmetical form”, requires a particular understanding of what numbers and arithmetic are and can do.

It was argued by Otte and Jahnke that, more broadly, the development of a purely symbolical or formal concept of number was a crucial premisses for the development of the movement of arithmetization of mathematics in the 19th century. The number concept was indeed slowly detached from strong ontological commitments, eroded by the many interactions between fields and disciplines in the first half of the 19th century. But it cannot be said to be formal or symbolical in Dedekind’s works. If the hypothesis of the existence of a link between the number concept and arithmetization, in Dedekind’s works, is to be taken seriously, then it seems that it is located elsewhere.

To consider this issue, I proposed an analysis of *Zahlen* as a project of *definition* of the natural numbers. Arithmetization was related, in Chapter 7 to Dedekind’s standard of rigor – to never consider anything provable or definable by means of “earlier concepts” without providing an acceptable proof and/or a rigorous definition for it – and the application of Dedekind’s ideal of rigor to arithmetic itself arises naturally.

I argued that *Zahlen* was essentially designed to provide a definition of the sequence of natural numbers that would answer Dedekind’s requisites, and especially his conception of rigor. The definition given in 1888 is uniform and for the definition of arithmetic operations as well as for the set up of methods of proof (whose validity has been rigorously proved). It also answers Dedekind’s principle of rigor: the definition and all the proofs involved in the process are grounded on “earlier concepts”, each inference is carefully verified, and in fact in this particular case, it even is grounded on “unprovable premisses” for numbers are the simplest objects in mathematics.

Moreover, the definition provided by Dedekind allows to (rigorously) justify the statement according to which numbers are creations of the mind and Dedekind’s repeated assertions on numbers and arithmetic being some of the most powerful tools created by the mind. Through this, it seems that Dedekind also gives a justification for the primary importance of arithmetical methods in his works. Indeed, the recurring resort to arithmetic as

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an epistemic tool in the set up of theories and the genetic approach to the extension of the number systems stressed the need for a definition of natural numbers and the operations of arithmetic to assure the overall consistency of Dedekind's works. But there seems to be another way in which Dedekind's mathematical works relate to *Zahlen*, namely the gradual extension of arithmetic through its uses in ideal theory, function theory, etc. This last point is particularly important in relation to the general definition provided in 1888 and the 'categoricity' theorem.

It is important to underline that the play of influences between arithmetic in practice and the foundation of arithmetic is mutual, reciprocal. On one hand, the possibility to do arithmetic and prove the validity of the laws of arithmetic in several different domains highlighted and might have provided a source for the generality of arithmetic. On the other hand, it is supported by a more fundamental claim about the natural numbers: that they are logical entities defined by the general concepts of sets and mappings and directly flowing from the laws of pure thought. In fact, the retrospective justification that I have mentioned can be considered as two-folded: the definition of numbers given in *Zahlen* gives a rigorous ground to Dedekind's methods, and the said methods provide a shining illustration – if not a proof – that numbers are, indeed, creations of the mind providing one of the most powerful “auxiliary means” created by the mind.

We have seen that Dedekind's mathematics was carried in great part by the uses of arithmetical methods in “higher levels”, leading to what is often identified as “conceptual” mathematics. From the definition of ideals to the study of “divisibility laws of polygons” as a ground for studying Riemann surfaces, Dedekind used arithmetic as a tool, as a guide, to develop new concepts, involving it actively in the production of new knowledge. Rather than a superficial advantage of re-using familiar terminologies, it seems that in using arithmetic to build new concepts and to develop rigorous foundations for the theory, Dedekind is taking advantage of arithmetic being an “auxiliary means”, of being embedded in thought and as such susceptible to participate in the development of the “endeavors to fathom the truth”.

In Dedekind's works, the idea of a general, abstract arithmetic arose slowly, from building on the content of arithmetized theories, from the slow continued extension of arithmetic through its applications in different mathematical frameworks. Between what it is possible to do with arithmetical operations, and the definition of natural numbers and the arithmetical operations, Dedekind builds several levels of arithmetic, taking advantage of the many layers of systems and arithmetical methods possible. Through this, mathematical practice and foundational researches seem to inspire and nurture each other, bringing to the fore what Dedekind thought to be an intricate relationship between the human understanding and the arithmetical operations. Arithmetic, for Dedekind, is an activity of the human mind, presented as an integral part of the human thinking. It is part of logic which,

from the laws of numbers, can become an aid, a guide for the thought.

8.5 Epilogue, the 1894 version of algebraic number theory

In order to be complete, at least to a certain extent, an argument according to which mathematical works have influenced each other should have turned towards the works Dedekind did after his book on natural numbers, which I unfortunately had to leave out of my dissertation. To be clear, the definition of natural numbers was not some sort of ultimate goal for Dedekind to reach, and it was certainly not my intention to suggest so. In the same way that the gradual extension of arithmetic seems to have left marks on Dedekind's ideas about arithmetic and numbers, the development in *Zahlen* of a powerful and very general conceptual apparatus which yielded a perfectly rigorous concept of number opened new possibilities – if only about the uses of systems and mappings. However, the other side of the play of influences, namely how the development of the set-theoretical concepts and the acknowledgement of their potential fruitfulness in the works written after 1888, would demand a study taking into account a lot more than this dissertation (at this point) could allow. In particular, since after the third edition of Dirichlet's *Vorlesungen* in 1879, Dedekind and Weber's paper on algebraic function theory was published, and Kronecker published his own theory on algebraic magnitudes which Dedekind studied closely. If Dedekind's ideas grew very confined and more opposed to Kronecker's than anything else until the late 1880s, it was less the case in the later years of his life. Before the 1890s, there was barely any work published using ideal theory or fields (and even less modules, which were even overlooked later by Hilbert in his *Zahlbericht*) that was not written by Dedekind himself. The reception of Dedekind started to be more substantial, and so were his exchanges with the mathematical community. As a consequence, the changes noticeable in Dedekind's later works should be investigated in a broader light, involving more sociological considerations, and an analysis of his exchanges with Frobenius, Weber and of the reception of his works by the young generation of mathematicians. This work could very well fill up a second dissertation.

Let me, however, say a few words about Dedekind's later works from a purely intra-Dedekind viewpoint. Dedekind's approach continued to change and move towards more 'abstract' methods, yet it still exhibited arithmetical features.

Dedekind published a new version of his algebraic number theory in 1894, in the fourth re-edition of Dirichlet's *Vorlesungen*. One of the main motivations of the new version of the theory of algebraic number is, again, to untie in a more satisfactory manner the "main difficulty" of the theory, namely the proof of the reciprocity of multiplication and divisibility of ideals,

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and to provide a satisfying ground for it. Ideal theory was thus still seen as an arithmetical theory for which it was crucial that the notion of divisibility as inclusion be proved equivalent to the arithmetical notion of divisibility. The aspect which left Dedekind unsatisfied, in the 1877/79 version of the theory is the fact that the means to prove this statement were only given towards the very end of the theory, by successively removing hypotheses and introducing the concept of *Ordnung*. It seems that, in the second version of the theory, Dedekind had not yet uncovered a satisfying way to found the theory, so that the “difficulty” be solved earlier by the introduction of the “*Ordnung*”. There are no clear explanations about that point, in 1877/79. However, in 1894 *Ordnungen* are introduced much earlier in the theory, allowing to develop a large part of module theory in which to transfer the difficulties. Module theory, then, offers a possibility to be used to develop the preliminary basis allowing ideal theory to be developed more easily.

The new version of Dedekind’s XIth Supplement, in Dirichlet’s *Vorlesungen*, is considerably longer than the previous ones (more than 200 pages). It provides a large amount of developments on the theory of fields and Galois theory (§§160-167, 41 pages) and concentrates a great part of preliminary researches in module theory (§§168-175, 47 pages). Ideal theory and the divisibility laws of algebraic numbers take about 30 pages, a third of which is a presentation of the failure of unique factorization of algebraic integers, and the passage from Kummer’s ideal numbers to ideals. The rest of the Supplement gives further investigations of algebraic number theory and the possible application of ideal and module theory for the composition of form, the division of the circle, etc., as in the previous versions. It is, thus, a great illustration of how Dedekind preferred to spend more time on preliminary developments so that the core of the theory – here, the divisibility of ideals – would be easier to develop, as well as of the idea that the proof of the laws of divisibility is a just step, albeit an essential one, in the development of number theory.

Despite the many additional developments that Dedekind provides in this new version of the XIth Supplement, the aim of the work itself has not changed: the theory is designed to prove the general validity of the divisibility laws in fields of algebraic numbers. The changes inflected to the new version are yet another instance of Dedekind’s tendency to completely reshape concepts, and even theories, to have them fit more closely his epistemological and methodological requisites.

Dedekind mentioned in the previous version of his theory that a more advanced study of the theory of fields could lead to developments of Galois theory. However, he never actually gave any of these developments because he considered that it would take him too far. Here, Dedekind’s reservations about developing field theory, and in particular the investigation of permutations of fields, were raised. There seems to be several possible reasons, for

this change. First, it should be underlined that Dedekind dedicated several works to groups, group determinants and group character during the 1880s, which he hadn't done before. He did not published most of these works, but discussed them in his correspondence with Frobenius. In fact, as noticed by Hawkins, Dedekind's investigations on groups were mostly related to questions raised by his researches on number theory, such as the conditions for the subfields of the field (extension) considered for the theory of algebraic numbers to all be normal ([Hawkins, 1978], 78). The second reason is the increasing interest for group theory that one can notice in the late 19th century, with works by Netto, Sylow, Lie, Frobenius, or again Weber.

Dedekind's 1894 Xth Supplement makes many references to *Zahlen*, and in particular to the concept of *Abbildung*, which is used as a basis for the "permutations" of the field. Dedekind, however, is very explicit on the fact that he is considering only a special kind of *Abbildung* and that for his purpose, it is clearer to restrict the attention to permutations.

The 1894 theory of algebraic integers does not adopt the same approach strictly analogous to rational number theory as the previous version did. Nevertheless, it still exhibits signs of the essential importance of arithmetic for Dedekind's practice. In particular, Dedekind, despite giving a greater importance to *Abbildungen*, does not give up his arithmetical terminology, neither for the 'set-theoretic' relations between fields (and modules and ideals), nor for the investigation of the properties of permutations themselves. And in fact, arithmetic is again used as a guide: arithmetical investigations are transferred to mappings and indicate which questions to consider. For example, Dedekind considers a notion of common divisor for permutations and naturally turns to the notion of common multiple which is, in fact, not transposable as such. Dedekind thus introduces a new notion: "compatible (harmonious) or incompatible" permutations ([Dedekind, 1894a], 465) according to whether the permutations have a common multiple or not. These elements are interesting in relation with *Zahlen*, in which Dedekind brought to the fore the constitutive role of mappings and systems in arithmetic. Not only are mappings and systems concepts "without which no thinking at all is possible", but Dedekind showed how they are the core fundamental concepts to define the sequence of natural numbers as well as the arithmetic operations. Here, in rewriting once again his theory of algebraic numbers, having identified mappings as an essential concept for developing mathematics, he changes the focus again and studies fields with mappings. And yet, he repeats the same move of defining arithmetical notions for their study.

In this version of Dedekind's algebraic number theory, the researches on modules are considerably expanded in comparison with previous versions. He introduces a symbol for divisibility between modules: $\mathfrak{m} > \mathfrak{d}$ for \mathfrak{m} divisible by \mathfrak{d} . He defines further arithmetical operations for modules using

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the ideas introduced in [Dedekind, 1877] (which I mentioned on p. 514). Dedekind also defines a notion of quotient of module¹²⁹ and introduces the *Ordnung* of a module \mathfrak{a} as the quotient of \mathfrak{a} with itself. The concept of *Ordnung*, here, is not defined as a system closed by certain properties of closure – although these properties will be shown to be necessary and sufficient conditions. It is not introduced as a way to reframe the theory, as it was in 1877. Rather, it is a particular kind of module and will be used to ease the development of ideal theory. Just as in 1877, the concept of *Ordnung* will play a crucial role in proving the reciprocity of multiplication and divisibility of ideals and it becomes clear in the Supplement that it eases certain crucial aspects of ideal theory. While it is introduced earlier in the theory and does play a significant role in its development, the *Ordnung* still does not play the role usually attributed to rings in modern algebra. Nevertheless, throughout the rewritings of his algebraic number theory, Dedekind gives more importance to the concept of *Ordnung* and seems to notice the importance of *Ordnung* in structuring the theory of algebraic numbers.

In the paragraphs on ideal theory, the previous developments in module theory are used to present and prove properties of ideals and their divisibility, as well as to give new properties of ideals. What was considered as the “great difficulty” of the theory in 1877 is relatively easy to prove, thanks to previous investigations on properties of modules. Dedekind stresses, once again, how important it is to be able to prove the “correlation (*Zusammenhang*) between divisibility of ideals and their multiplication”. Here, the “greatest difficulty” to overcome in the “first foundation of ideal theory” was resolved at the very beginning with module theory. Dedekind, once again, studies in depth the divisibility of ideals and uses the previous developments to extend his investigation of the divisibility properties in ideal theory. The theorems are still stated with the arithmetical terminology and there is no doubt that Dedekind is investigating divisibility laws. Note that many of the proofs keep a form close to the arithmetical form they had in the previous version.¹³⁰

Thus, reciprocity of divisibility and multiplication of ideals is still seen as such an important theorem for the theory that to be able to prove it in a satisfying way justifies complete rewritings of the theory. In 1895, in a response to Hurwitz’s *Zur Theorie der Ideale*, Dedekind explains the epistemological motivations of his approach in ideal theory, as well as how he was able to formulate a new satisfying presentation of his theory of algebraic numbers.

¹²⁹For two ideals \mathfrak{a} and \mathfrak{b} , $\frac{\mathfrak{a}}{\mathfrak{b}}$ is the collection of all numbers ν such that $\mathfrak{a}\nu > \mathfrak{b}$.

¹³⁰The proof of the unique factorization theorem follows the same general idea as the proof given in the 1877 version (or in rational number theory), but is shortened by the use of the idea of a “chain” of divisors which corresponds only to a sequence of ideals ordered by divisibility. It is the idea that will lead to the “chain condition” for ideals.

The role played by module theory is highlighted by Dedekind's explanations of his quest for a foundation of ideal theory that would avoid relegating the proof of a crucial theorem towards the end of the theory. Dedekind exposes his "Prague theorem"¹³¹ and how he was led to understand that it was not a suitable foundation for his theory of ideals. This way to "shorten" ideal theory, for Dedekind, was "disrupting the uniform character of the theory" (*den einheitlichen Charakter der Theorie zu stören*, [Dedekind, 1894b], 54). Dedekind considers that the "intrusion of functions of variable" is "marring the purity"¹³² ([Dedekind, 1894b], 55) of the theory (as it was the case with Kronecker's *Grundzüge*, for Dedekind). In addition, involving in number theory the consideration of such explicit algebraic expressions goes against "the decision for the internal in contrast to the external": it gives too much weight to the algebraic expression, while the focus should be on numbers and divisibility. The solution, for Dedekind, lied in considerations about the divisibility of modules (with the equality given on p. 514). Dedekind, then, hides away the computations once again, by finding a way to transfer the investigations to the "higher level" of modules. The arithmetical symbolism introduced to support the arithmetical methodology allows to develop further the investigation of module theory and to obtain general results such as the equality quoted above and the ones mentioned in the previous paragraph.

These considerations on modules and the developments given to module theory led Dedekind to develop the notions of *Dualgruppe*, equivalent of our modern concept of lattice, on which Dedekind published two papers in 1897 and 1900 ([Dedekind, 1897], [Dedekind, 1900]) which have been studied by Dirk Schlimm in [Schlimm, 2011]. These late works show how the reflections motivated by number theoretical manipulations became increasingly general, abstract and even formal at some points.

¹³¹ "When the coefficients of a product of two functions A, B are all [algebraic] integers, then each product of a coefficient of A and a coefficient of B is an [algebraic] integer." ([Dedekind, 1892], 32)

¹³² "... durch die Einmischung der Funktionen von Variablen die Reinheit der Theorie nach meiner Ansicht getrübt wird ..."

General conclusion

9.1 Between mathematical explorations and foundational investigations

In this dissertation, I have suggested that Dedekind's mathematical practice relies in an essential way on arithmetic used and construed in a specific way, to the point that his foundational works bear the mark of the mathematical practice.

This conclusion does not stem from an attempt to thoroughly reconsider all understandings and interpretations of Dedekind's concept of number. Rather, it is to be related with the desire to deepen our understanding of Dedekind's practice by stepping away from the sole idea of a "conceptual approach". Moreover, it is embedded in the wish to approach Dedekind's works from a global viewpoint, to consider his works as an articulated whole. The foundational and the mathematical 'sides' of Dedekind's writings have usually been treated separately, and one of the aims of this dissertation was to provide an account of Dedekind's mathematics that would be able to make sense of his mathematical practice as well as of the more philosophical parts of his works. That is, to bring to the fore the idea that mathematical practice and epistemological conceptions are not, in Dedekind's mathematics, independent from each other, and that the latter do not (necessarily) precede the first. I have thus proposed to combine a rigorous historical approach with a close attention to the way in which epistemological values shaped Dedekind's works. I aimed at inquiring into the shaping of his methods and ideas by a close study of a selection of texts, to highlight significant recurring elements of practice. This brings to the fore a key role played by "arithmetic" in Dedekind's works. The mark(s) left by the mathematical practice on Dedekind's foundational researches are thus modulations in the ideas about arithmetical methods, and the epistemological properties attributed to arithmetic and numbers.

By suggesting that arithmetic operations used as epistemic tools in the mathematical works, had an impact on the conception of arithmetic and, through this, of natural numbers, I wished to underline (1) that Dedekind's conception of arithmetic is intrinsically related to his concept of number,

and (2) that *Zahlen*, its aim and its scope, can only truly be understood in relation to Dedekind's practice of mathematics. Through a chronological approach of Dedekind's works, proposing in a sense a micro-historical account of Dedekind's practice of mathematics and conception of arithmetic, it can thus be seen that Dedekind's particular methods should be considered as linked to his foundational investigations and conversely. Indeed, if it is always difficult to provide a *proof* in this kind of historical inquiries, the long genesis of *Zahlen* which resulted in a concept of the sequence of numbers in which generality is intertwined with the idea of numbers as creations of the mind, suggest the foundational works could be better understood if read as integral part of his works.

9.1.1 Mathematical works and arithmetic

A core aim of this dissertation was to propose a precise positive characterization of Dedekind's practice, rather than to assess it in opposition to non-conceptual approaches. The idea of a "conceptual approach", widespread in the historiography, seems to provide incomplete informations about Dedekind's mathematics. It is, thus, not inaccurate properly speaking, but it seems elusive on many aspects of Dedekind's practice. Indeed, to assess the "conceptual" aspect of Dedekind's approach focuses all the attention on the elaboration of the fundamental concepts (e.g., ideals for algebraic number theory) without investigating further the new practices developed in relation with these new concepts. Yet, an attentive reading of Dedekind's writings brings out certain habits of demonstration, certain specific elements of practice, providing a more detailed understanding of Dedekind's approach.

So as to provide this positive characterization, I proposed to start by analyzing in more details the 1882 article co-written with Heinrich Weber, *Theorie der algebraischen Funktionen einer Veränderlichen*, which was little studied in the literature and in which these specific elements of practice are particularly pregnant and striking. A singular paper in Dedekind's corpus by its subject, the 1882 paper transfers typically Dedekindian methods to a different framework, thus shedding light on their specificities. This paper proposes a rewriting of Riemann's function theory based on the transfer to algebraic functions of methods elaborated by Dedekind for the theory of algebraic numbers. It answers a desire to provide a foundation to (a part of) Riemannian function theory that would be uniform and as rigorous as number theory can be.¹ After the writing of their paper, Dedekind wrote to Weber to thank him and underline the joy brought to him by their work:

¹"I am not the profound expert on Riemann's work that you take me to be. I certainly know those works, and I *believe* in them, but I do not master them, and I will not master them until having overcome in my way, with the rigor that is customary in number theory, a whole series of obscurities." (Letter to Weber, November 1874, Quoted and translated in [Ferreirós, 2008], 78)

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I take this opportunity to thank you again sincerely for the entire, almost two years long, work [on algebraic functions theory], which gave you infinite trouble, and of which taking part brought me the greatest joy and an important enrichment of knowledge. It is a truly special feeling, to be joined by the exploration of the truth, what Pascal in his first letter to Fermat so admirably phrased: “Car je voudrais désormais vous ouvrir mon cœur, s’il se pouvait, tant j’ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris.” Often, had I to think at this place, through the progresses of our work, which though after many oscillations still has assumed the character of intrinsic necessity. It shall bring me also much joy, if the thing finds some success...² (Letter to Weber, 30 October 1880, in [Dedekind, 1932] III, 488, my translation)

The study of Dedekind and Weber’s rewriting of the basic concepts of Riemann’s theory of functions highlighted that arithmetical notions are deeply involved in the development of the new rigorous, simpler and uniform basis of the theory.

The arithmetical aspect of their work is not only related to the fact that Dedekind and Weber found their work on the concepts of field, module and ideal which they transfer from Dedekind’s number theoretical works. The core of the “arithmetical” approach proposed by Dedekind and Weber, here, is the definition of new arithmetical operations, relationships between objects reinterpreted into arithmetical terms. Such arithmetical terminology and the design of arithmetical notions are introduced throughout all of Dedekind and Weber’s paper. This implies, in particular, the consideration of the laws of divisibility of complexes of points of the surface and the demonstration of theorems of Riemannian function theory, such as the Riemann-Roch theorem, using notions of greatest common divisor and least common multiple of polygons, equivalence classes, etc., derived from their divisibility laws. It thus appears, in this paper, that the demand of a more rigorous, clearer, more precise, and more uniform treatment of Riemannian function theory is fulfilled by taking an arithmetical path, by deploying

²“Ich benutze die Gelegenheit, um Dir nochmals meinen innigsten Dank für die ganze, beinahe zweijährige Arbeit zu sagen, von der Du so unendlich viel Mühe gehabt hast, und an der Theil zu nehmen mir die größte Freude und eine bedeutende Bereicherung an Wissen gebracht hat; es ist ein ganz besonderes schönes Gefühl, sich so bei der Erforschung der Wahrheit zu begegnen, was Pascal in seinem ersten Brief an Fermat so trefflich ausdrückt: “Car je voudrais désormais vous ouvrir mon cœur, s’il se pouvait, tant j’ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris.” Oft habe ich an diese Stelle denken müssen bei den Fortschritten unserer Arbeit, die nach mancherlei Oscillationen doch immer mehr den Charakter innerer Nothwendigkeit angenommen haben. Es soll mich nun auch herzlich freuen, wenn die Sache einigen Beifall finden wird, worauf ich aber vorläufig nicht allzu sehr baue, weil die langweiligen Moduln gewiß manchen zurückschrecken werden. . . .”

an arithmetical strategy. Although a definition of the point of a Riemann surface without any resort to geometrical intuition was one of the aims of Dedekind and Weber's paper, the systematicity with which they use arithmetical notions suggest that it plays a more significant role. Thus, the approach to Dedekind's corpus by a different text, "through the side door", allowed to show that one needed to pay more attention to arithmetic in Dedekind's practice. It suggests several questions about the properties and characteristics of arithmetic, about its status and about the relation between this use of arithmetic and the conception of number.

So as to inquire into the actual role and status of arithmetic and broaden the understanding of the status held by arithmetic in Dedekind's mathematics, I proposed a more comprehensive inquiry into Dedekind's writings under the form of a chronological study of a selection of Dedekind's most important texts, with the risk of going over already well known works. It seemed, however, essential to study Dedekind's algebraic number theory, not in the hope to add to the understanding of the mathematics involved in it, but so as to mark the gradual changes in his approach between 1871 and 1879. The study of how Dedekind's works were reshaped by the desire to stick always closer to certain epistemological requisites and the comparison of the changes showed the gradual increase of importance of arithmetic for Dedekind's approach. It showed that the development of methods using arithmetic operations for objects that are not numbers is a strategy to which Dedekind resorts repeatedly and more and more often. Thus, a specific use of arithmetic is brought to the fore, whose development can be traced from writing to writing, suggesting that the practice of mathematics itself could have affected the conception of arithmetic, a point considered in two times: by the investigation of the practice and by an analysis in relation to the foundational texts.

The investigation of Dedekind's practice, from his very first works, to the transfer of the methods of his algebraic number theory to algebraic functions, allowed to suggest that his use of arithmetic evolved through his works. Indeed, in Dedekind's *Inauguraldissertation* and *Habilitationsvortrag*, considerations about the development of mathematics and about "arithmetic" do not show any particular interest for the kind of problems and methods, related to divisibility, that Dedekind's develop in his later works. Dedekind's description of the development of arithmetic as the successive extensions of the concept of number, from the "act of counting" to complex numbers (and even Hamilton's quaternions) is led by considerations about the validity of basic laws of number analog to a principle of permanence.

Dedekind stated, in 1877, that divisibility is at the ground of arithmetic, the science of numbers. The fact that arithmetic is a science, and thus for Dedekind an activity of the human mind rather than a rigid theory, suggests

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that arithmetic can change, evolve with the changes and extensions of the number concept, so as to preserve the validity of the laws governing numbers. And indeed, to each extension of the number domains corresponds an extension of the arithmetical operations, and consequently of scope of validity of the laws deriving from them. Divisibility concerns arise when one studies in more details the properties of the extensions of the concept of *integer*. For algebraic integers, the “most general concept of integer”, it becomes necessary to broaden the notion of divisibility if one is to preserve the validity of essential properties of numbers: the unique decomposition in prime factors. For Dedekind, this is done by setting up a “higher level” and considering the divisibility of systems of numbers, which is proved to correspond to the divisibility of numbers themselves.

Dedekind’s first two works in number theory were a lecture on Galois theory given during the winter semesters of 1856-57 and 1857-58 and an article on higher congruences, in which he aimed at developing an approach “rigorously tied to an analogy with elementary number theory”. In these two works, Dedekind introduces for the first time strategies and uses of arithmetic which will be developed in a greater extent and be dominant in his mathematical works from 1871 to 1882. Dedekind transfers the study of (algebraic) relationships between numbers to what he will call later a “higher level” in which he is guided by a “rigorous analogy” with elementary arithmetic.

Indeed, in Dedekind’s most renowned work, the theory of algebraic integers, the definition of a “higher level” in which to transfer the study of the properties of numbers is essential. Ideals, infinite systems of numbers defined by conditions of closure by arithmetical operations. Insofar as they are extensions of Kummer’s ideal numbers, ideals appear as fundamentally linked to arithmetic. And, indeed, they are studied as arithmetical elements for which arithmetical operations are defined. With the definition of ideals, modules and even of fields, Dedekind introduces a new notion of divisibility (“to divide is to contain”) which allows to study the properties of divisibility of *numbers* through the considerably simpler divisibility of ideals.

The 1871 algebraic integer theory yields the assurance that the divisibility behavior of ideals is, indeed, the same as rational numbers. In 1876, it is the desire to provide a clearer, simpler and better founded theory of algebraic integers, which incites Dedekind to propose a rewriting of his own work. In the second version of algebraic integers said to rely on an “improved form” of ideal theory, Dedekind substitutes a generalization of rational divisibility for the divisibility tests on which the 1871 Xth Supplement relied in great part. By this move, the arithmetical operations defined between ideals become the essential tool to study them, and ideal theory is developed in a perfect analogy with rational number theory, emphasizing the arithmetical nature of ideals. In this version, the study of the divisibility properties is

transferred to the “higher level”, in which the laws of rational arithmetic can be found again – and in a similar form. Ideals, thus, go from an extension of ideal numbers to an extension of divisibility itself.

The 1877 version of Dedekind’s algebraic number theory was particularly interesting for this dissertation because it is the version transferred to algebraic function theory. In 1877, it seems that the possibility to develop the theory by involving only notions and tools extending rational arithmetic plays a crucial role in providing a simpler, clearer foundation to the theory. It would be tempting to attribute this increase of simplicity and clarity to the sole familiarity of the arithmetical methods. Tempting, but too easy (especially since we know that it was not Dedekind’s last version of algebraic number theory). The 1877 version, by relying only on divisibility properties of sets of numbers, does not rely on anything “extrinsic” to the theory, it does not involve indeterminate variables nor long computations. Rather, it replaces all complicated divisibility tests by very simple computations similar to rational arithmetic in the “higher level”. The same will be valid when the transfer to algebraic functions is done in 1882: rational arithmetic, here, appears to provide a model to follow. Through the set up of new operations, it thus appears to give tools that can actually be used to develop powerful mathematics.

By shifting the focus from the “conceptual” aspect of his practice to the role and status of arithmetic in Dedekind’s works, important aspects of his approach are highlighted. Dedekind seems to suggest, by his practice itself, that to give oneself the possibility of using only the “simplest principles of arithmetic” justifies abstract and technical detours, so as to be able to obtain an expression of the theory in terms of these “simplest principles”. For that purpose, arithmetic operations are performed at several levels, as if there were several layers of arithmetic(s), as is particularly clear in the 1882 paper. The almost invariable recourse to arithmetic to build up a theory, in Dedekind’s works, suggests that a very peculiar role is given to them in the development of mathematics. It seems, and I believe *Algebraische Funktionen* is a good example of that point, that for Dedekind the operations of arithmetic are used as tools to develop certain theories in more suitable ways, with more rigorous and uniform definitions and methods of proof. Yet, once this has been said, there is still a lot to clarify. In particular why should arithmetic be more suited than other approaches? Is the sole possibility to have an intrinsic approach sufficient to support the extent of the arithmetization of Dedekind’s approach – especially for algebraic functions? What is the ideal of rigor supporting Dedekind’s demands? And is arithmetic better suited to satisfy it? After all, the emphasis on elementary arithmetic was a rather common idea in the second half of the 19th century. In particular, it was of the greatest importance for Kronecker. To qualify Dedekind’s approach as being centered on arithmetic cannot, thus, be the

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end of the analysis.

9.1.2 On characterizing arithmetic

Let us come back to the essential question of “arithmetic”, which is a key point. If Dedekind’s “conceptual approach” is more precisely characterized as “arithmetical”, it is only insofar as this statement is supplemented by a characterization of what it means to be “arithmetic”.

Indeed, “arithmetic” is also the core of Kronecker’s approach. However, as is well known and as I have evoked in the dissertation, their approaches are opposed in many aspects. If adopting an “arithmetical” approach is essential for their practices, they both have very different conceptions of what the ‘correct’ practice of arithmetic is. This point was emphasized by Schappacher in [Schappacher, 2010]. Schappacher underlines the similarity in Dedekind’s and Kronecker’s works of the appeal to arithmetic “as a model approach to the theory of algebraic functions”, the statements about generality, simplicity, the “superiority of the arithmetic point of view” ([Schappacher, 2010], 3266-3267) which are invoked by both authors.

Ferreirós has suggested that “arithmetic”, in the 19th century, broadly meant “all pure mathematics” ([Ferreirós, 2007]). It is true that there was an emphasis on arithmetic among (German speaking) mathematicians in the 19th century which yields this larger picture. Yet, by adopting this broad view, it becomes difficult to distinguish between Kronecker’s and Dedekind’s approaches. With a more detailed investigation, such as the one I conducted in this dissertation, it becomes possible to better characterize what opposed them, despite their looking for the same things. Indeed, what is arithmetical for Kronecker is *not* arithmetical for Dedekind. In the same way, Dedekind and Kronecker both considered their approach more concrete than the other’s. Dedekind worked only with systems of actually existing numbers and thought that this was *concrete* in opposition to the abstraction of introducing formal considerations (e.g., using indeterminate variables). Kronecker had the opposite position: he considered Dedekind’s approach as too abstract, and his own methods more concrete since they were constructive, finitist methods.

If one is to have a finely-shaded understanding of the mathematical practices of mathematicians such as Dedekind and Kronecker, it is thus essential to acknowledge that there are several conceptions and practices of arithmetic just as there are several conceptions and practices of concreteness / abstraction or several conceptions and practices of generality (as highlighted by the works in [Chemla, Chorlay, and Rabouin, Forthcoming] mentioned in the introduction of this dissertation). In Brechenmacher’s study ([Brechenmacher, Forthcoming]), several conceptions of generality are highlighted and, in fact, it is explicit among the actors that there is a conflict between them regarding what is the “right way” to develop a general approach. The differences can

be understood once it is clear that mathematicians are interested in different aspects and characteristics of generality. In the same way, arithmetical approaches claimed by the actors rely on different features and epistemic benefits attributed to arithmetic. By this parallel, I do not wish to suggest that arithmetic is an epistemological value as generality is. However, arithmetic does have such a crucial importance, for these actors, that it is considered a “superior” approach. In the same time, arithmetical approaches are opposed depending on the authors. This implies that the understanding of what is characteristic of arithmetic, the conception of arithmetic multifolded among the actors, played a key role in their practice.³

9.1.3 Arithmetic and foundational researches

To inquire into Dedekind’s conception of arithmetic in his foundational works was the second moment of my analysis, which was done in two steps. First, I proposed to identify the epistemic benefits which could, for Dedekind, explain the recurring resort to arithmetic. Secondly, I tried to provide an epistemologically sound reading of Dedekind’s conception of arithmetic, justifying these epistemic benefits, on the basis of foundational works. Here, again, it implied to investigate well-known works, to walk a path walked by many historians and philosophers. Admittedly, the mathematical content of Dedekind’s works did not need much clearing and I often walked around philosophical issues. This was less because of a desire to avoid them, than because they constitute localized inquiries which would have made me lose the focus on arithmetic as embedded in a larger body of practices. A deeper understanding of the ‘structuralist’ or axiomatic approach in his foundational works would certainly refine the perception of these works and how they influenced further philosophical and foundational thinkers. I don’t believe this aspect needs to be contradictory with what was presented in this dissertation.

Putting aside, for one moment, the fact that the introduction of a divisibility notion for ideals makes things “easier”, simpler, the use of arithmetic to provide more rigorous theories is of course strongly reminiscent of the movement of “arithmetization of analysis” which Klein described in a famous 1895 talk. My analysis of Dedekind’s works suggest that both rigorization and the putting into an arithmetical form are of the greatest importance – albeit with an idea of what “arithmetical form” means significantly different from Kronecker’s. On that matter, an important moment of Dedekind’s mathematical works is, of course, his definition of the real numbers by means of cuts, in 1872. A typical example of arithmetization, the definition of real numbers was, for Dedekind, directly related to a desire

³For Kronecker, see [Boniface, 2005], [Kronecker, 1891], [Smadja, 2002], [Vlăduț, 1991].

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to instigate more rigor at the foundation of analysis.

Following a suggestion by M. Detlefsen, who identified Dedekind's ideal of rigor in the first sentence of *Zahlen*, "nothing susceptible of proof ought to be accepted without a proof", I proposed that Dedekind's conception of rigor is linked to the demand that one should use "earlier concepts", notions which have effectively been defined. The typical example of notions which should *not* be used is that of "magnitude" for the definition of irrational numbers. This idea should, however, be understood in relation to advocating for the introduction of new concepts. Dedekind, in the works we have studied, exhibited the possibility to design new concepts using the "simplest principles of arithmetic". He provided arithmetical rewriting of the theories. To use arithmetical notions, to ground reasonings on rational arithmetic and the sort of extensions of rational arithmetic embodied by ideal theory, and, by this, to give an arithmetical form to arguments, appear for Dedekind as a strategy enabling the satisfaction of these two requirements. This strategy corresponds to an arithmetization by its motives and by the methods used.

Thus, arithmetization for Dedekind allowed to satisfy rigor requisites without doubling them with a reduction to natural numbers and without denying the mathematician's creative abilities. However, to be effective, such an arithmetization requires to be able to extend arithmetic to non-numerical objects, to develop arithmetic as an 'abstract' theory.

Stetigkeit, which was written between the first two versions of Dedekind's theory of algebraic integers, is also important because it sheds light on his ideas about the extensions of the number concept. Indeed, Dedekind's 1872 essay brings to the fore the idea that extensions of the number domains produce new *numbers* and not "magnitudes", at least as long as the extension of the number domains are made with intrinsic means and in such a way that the "identity of numbers" is preserved (i.e., without attributing to non-numerical properties to numbers by identifying them with infinite classes such as cuts, for example). This suggests strongly that the "science of numbers" has a scope of application that can be widened, providing that the extension is consistently done. The possibility to define ideals as arithmetical elements, then, appears as a crucial argument for a strategy of arithmetization deployed in Dedekind's mathematics.

No matter how often it is used as a means for rigor, for Dedekind, arithmetic is still fundamentally conceived as the "science of numbers" throughout his career. Considering the preeminence of arithmetic, and the development of a certain strategy of arithmetization in Dedekind's works, the links between arithmetization and his concept of number become all the more important to clarify.

To be consistent with Dedekind's rigor principles, it appears essential to provide a rigorous foundation to arithmetic and to be able to provide it as relying only on "earlier concepts". It is what is done in *Zahlen*, a "synthesis" written after a deep analysis of the number concept, a "protracted labor"

that can be traced up to fifteen years before its final version and publication. The definition of natural numbers and the elementary operations of arithmetic given in *Zahlen* yields a foundation for arithmetic which is perfectly rigorous. It is also perfectly general in the sense that elementary properties and notions of arithmetic are defined by means of systems and mappings and, thus, so that everyone of them is also valid for “all ordered simply infinite systems”. The foundation of arithmetic, “flowing directly from the laws of thought”, is given the same level of generality as the laws of thought, and even more so since it is proven that everything valid for one instance of the simply infinite systems is valid for and can be transferred to any simply infinite system.

Dedekind presents natural numbers as a concept deeply embedded in the human understanding and as creations of the human mind. By providing the rigorous, logical, and general definition of natural numbers given in 1888, arithmetic appears to be intrinsically linked with thought itself, to be a part of logic guiding the reasoning. As such, it is perfectly justified as a tool to guide the development of certain mathematical theories. In fact, Dedekind expresses several times the idea that numbers are “an exceedingly useful instrument for the human mind” ([Dedekind, 1872]). After 1888, Dedekind writes that numbers are “auxiliary means” created by the mind – in fact, the most “far-reaching and (...) inextricably connected to [the understanding’s] innermost Nature” of all auxiliary means (*Zum Zahlbegriff*). Moreover, he continues, arithmetic is “already by now a science of immeasurable extension”, which can still be further developed and possesses an “[e]qually limitless (...) area of applications, since every thinking man, even when he is not distinctly aware of it, is a number-man, an arithmetician.” Thus,

For Dedekind, numbers and arithmetical operations defined for them can be considered and used as “auxiliary means”, as helps for rigorous reasonings. They can, in fact, as is clearly illustrated in *Algebraische Funktionen*, be used as epistemic tools involved in the production of new mathematical knowledge, in the design of new concepts and new methods of proof, and/or to strengthen the foundation of certain theories.

9.2 Further on up the road

9.2.1 Investigating the differences deeper

There seems to be an additional step left to make in the understanding of the various conceptions of arithmetic. Indeed, it is somewhat intriguing that Dedekind and Kronecker, who were both deeply influenced by Dirichlet, Gauss, and Kummer, were working in the same field of research, and whose works were often shaped by the same epistemological values (albeit not necessarily conceived in the same way), developed approaches so opposed. There is one important difference which is the philosophy classes

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Dedekind followed with Lotze in Göttingen. Rather than locating the source of their differences in the “conceptual” approach – which, recall, is seen as led by Dirichlet, who had a great influence on Kronecker – it seems that this specific philosophical component of Dedekind’s education. Where Kronecker wished to treat mathematics as a natural science, to “describe simply and completely” phenomena, Dedekind’s approach seems to present certain Lotzean aspects. A forthcoming paper by W. Sieg and R. Morris ([Sieg and Morris, 2014]), which proposes a deepened investigation into Dedekind’s structuralism and the “creation of concepts”, considers this question in more detail.⁴ They link Dedekind’s insistence on the finiteness of understanding in [Dedekind, 1854a] with some of Lotze’s remarks in his *Logik*. More significantly, they exhibit elements of Lotze’s philosophy which shed light on Dedekind’s ideas about “*Begriffsbildung*”, characteristic properties (*Merkmale*) and creation “by abstraction”. According to them, Lotzean abstraction is “at work” in Dedekind’s creation of the natural numbers – which incidentally also explains Dedekind’s insistence on the need of a proof of the existence of an infinite system. For them, Dedekind’s “structuralism” is “located (...) within the logic of his time, in particular, that of Lotze and Schröder.”

We know that Dedekind followed Lotze’s classes in Göttingen, and the web of influences between Herbart, Lotze, Riemann and Dedekind does make the influence of Lotze on Dedekind’s thinking likely. And indeed, there is a lot reminiscent of Lotze’s philosophy, in Dedekind’s conception of logic and of the role of the thought. This being said, although Dedekind seems to adopt a Lotzean idea of abstraction, there seems to be reasons to be careful in considering the Lotzean heritage of Dedekind. For example, we know how thought, logic and mathematics are intrinsically related for Dedekind. While they are as well for Lotze, he conceived mathematical logic as close to Leibniz’s “*lingua characteristica*“ (or at least, what was known of it at the time) and more generally of logic as intrinsically linked with language ([Sullivan, 2014]).

While this would have been out of the scope of my inquiry, an approach from the history of philosophy which would inquire into Dedekind’s reception of Lotze and how it can be articulated with and integrated into his conception of mathematics, would probably provide additional insights on Dedekind’s ideas by giving elements of philosophical context.⁵ As suggested by Sieg and Morris’s paper, these elements of philosophical context should allow to understand better what is often called Dedekind’s structuralism.

Dedekind’s structuralism is related to his adoption of an axiomatic approach. As it was made clear in [Sieg and Schlimm, 2005], Dedekind’s ap-

⁴[Gandon, 2006] suggested this possible direction of research, but he did not pursue it.

⁵In addition, considering works such as [Heis, 2013], it seems that this could provide the ground for a different approach of the relations (or absence thereof) between Dedekind’s and Frege’s thinking.

proach shows elements of both the axiomatic and the genetic approaches. To be sure that the coexistence of these two views does not, as Sieg and Schlimm affirm, entail any conflict, it seems important to clarify the peculiar brand of axiomatic adopted by Dedekind, and in particular to which extent it was *only* a matter of logical organization of the definition and theory.⁶

9.2.2 About rewriting Riemann, again

An analysis of the rewriting of Riemann's function theory by Dedekind and Weber, which I have considered essentially from the viewpoint of Dedekind's mathematics, would only be complete when it will conciliate two aspects that I had to omit, for lack of time. On one hand, the role played by Weber in the elaboration of the theory is important to clarify. On this point, we hopefully will know more when the correspondence between Dedekind and Weber will be published. On the other hand, the letters between Dedekind and Weber also contain a key element for the second aspect of the rewriting: the editorial work made by Dedekind and Weber on Riemann's works. Indeed, a considerable part of Riemann's *Gesammelte Werke* is constituted by unpublished works and manuscripts. The letters exchanged for the edition of Riemann's *Gesammelte Werke* show us that Dedekind and Weber did a great amount of reconstruction and edition of Riemann's manuscripts. It would be interesting to be able to combine a reading of their editorial work described in the letters with a comparison of Riemann's manuscripts and Riemann's *Gesammelte Werke*. Hopefully, this would allow a better insight at how Dedekind and Weber assimilated Riemann's works. And it could help us understand how much the edited works of Riemann are marked by the editorial work made by Dedekind and Weber.

Moreover, this would constitute a first step towards focusing on Weber's side of the story, which I unfortunately had to neglect, pursuing reluctantly the tendency to look at Weber as a secondary character. If only for his essential role in the circulation of the now so widely adopted Dedekindian concepts, he would deserve more attention in future historiographical works. In fact, as a mediator between different approaches in mathematics and through his many articles, his teaching, correspondence and collaborations, and of course his *Lehrbuch der Algebra*, Weber was certainly not, in his time, a secondary character.

⁶The tension between axiomatic and genetic approach seems solvable if one considers that Dedekind was only adopting axiomatic as a logical organization of the definition and theory, which corresponds to what Detlefsen called "descriptive axiomatization" in [Detlefsen, 2013]. Yet, Detlefsen underlined in [Detlefsen, 2014] that Dedekind's insistence on proving the existence of infinite systems in *Zahlen* can be interpreted as a desire to give a proof of consistency of his theory. This suggests that Dedekind might have had an understanding of his theory related to an "abstract science" which challenges the coexistence of genetic and axiomatic approaches.

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Résumé

En 1882, Richard Dedekind et Heinrich Weber proposent une re-définition algébraico-arithmétique de la notion de surface de Riemann utilisant les concepts et méthodes introduits par Dedekind en théorie des nombres algébriques. Dans un effort pour regarder au-delà de l'idée d'une "approche conceptuelle", ce travail se propose d'identifier les éléments de pratique propres à Dedekind, partant de l'article co-écrit avec Weber. Nous mettons en avant l'idée selon laquelle, dans les travaux de Dedekind, l'arithmétique peut jouer un rôle actif et essentiel pour l'élaboration de connaissances mathématiques. Pour cela, nous proposons l'étude, dans la pratique mathématique, de la conception de l'arithmétique chez Dedekind, de la place donnée à et du rôle joué par les notions arithmétiques, et des possibles évolutions de ces idées dans les travaux de Dedekind. Cette étude est faite par l'examen serré d'une sélection de textes. Dans un premier temps, sont étudiés les premiers travaux de Dedekind, son *Habilitationsvortrag* en 1854 et ses premières recherches en théorie des nombres. Suite à cela, nous proposons une comparaison des deux premières versions de la théorie des nombres algébriques publiée par Dedekind en 1871 et 1877. Enfin, ayant mis en évidence le rôle central de l'arithmétique, pour les mathématiques dedekindiennes, nous nous tournons vers les travaux fondationnels de Dedekind, afin d'explicitier la spécificité de sa conception en élucidant, à travers ses travaux sur la définition des nombres, ce qui donne à l'arithmétique cette place de choix et les liens avec la définition des entiers naturels donnée dans le fameux *Was sind und was sollen die Zahlen?* en 1888.

Mots clefs : Richard Dedekind, Arithmétique, Théorie des nombres, Théorie des fonctions algébriques, Histoire des mathématiques au 19e siècle.

Abstract

In 1882, Richard Dedekind and Heinrich Weber offer an arithmetico-algebraic re-definition of the Riemann surface, using concepts and methods introduced by Dedekind in algebraic number theory. In an attempt to investigate Dedekind's works beyond the mere idea of a "conceptual approach", this work proposes to identify the elements of practice specific to Dedekind, starting from the paper co-written with Weber. I put forward the idea that in Dedekind's works, arithmetic can play an essential and active role in the elaboration of mathematical knowledge. For this, I propose to study, in Dedekind's mathematical practice, the conception of arithmetic, the place and role of arithmetical notions and the possible evolutions in Dedekind's ideas about arithmetic. This study is based on a careful analysis of a selection of Dedekind's texts. For this, I study Dedekind's early works, his 1854 *Habilitationsvortrag* and his first works in number theory. Then, I propose a comparison between the first two versions of Dedekind's theory of algebraic numbers published in 1871 and 1877. Finally, I turn to Dedekind's foundational works, to make explicit the specificities of his conception and elucidate, through the study of his works on the definition of numbers, what gives arithmetic its pride of place and the link with the definition of natural numbers given in *Was sind und was sollen die Zahlen?* in 1888.

Keywords: Richard Dedekind, Arithmetic, Number theory, Algebraic function theory, History of mathematics in the 19th century.

Corrigenda*

- Page 51, a quote from Riemann's 1851 doctoral dissertation is called "a transparent allusion to Weierstrass's works." This is incorrect since Weierstrass had not published any work on the subject in 1851. Riemann is more likely referring to Cauchy (see [Laugwitz, 2009], 104 and [Neuenschwander, 1981]).
- Page 64, footnote 40, "je n'ai pas changer d'avis" should be "je n'ai pas changé d'avis".
- Page 81, a more adequate presentation of Gauss's *Disquisitiones Arithmeticae* is required. In particular, the beginning of the first paragraph should be modified as follow:
Gauss's *Disquisitiones Arithmeticae* has often been presented as marking the birth of number theory as a discipline in its own right in Germany. However, as [Goldstein and Schappacher, 2007a] show, the situation is more complicated and requires a more finely-shaded analysis, for the developments following Gauss's works tend to challenge the boundaries between disciplines. In this very succinct presentation of Gauss's *Disquisitiones Arithmeticae*, I attempt to point out the various characteristics of "number theory", "arithmetic" and "higher arithmetic" for Gauss and his successors (e.g. Dirichlet, Jacobi, ...). Here, I will not be looking at disciplines *per se*, but about what actors would qualify as "arithmetical" – a delicate point, as the thorough investigation in [Goldstein and Schappacher, 2007a] suggests.
- Page 104, last paragraph, "Particularly well-know" should be "Particularly well-known".
- Pages 258-259, "modular equations" should be "congruences".
- Page 278, the quote from [Edwards, 1980] is incomplete and should be:

A cyclotomic integer $f(\alpha)$ is said to be divisible by p if $f(\alpha)\psi(\alpha)$ is divisible by 47. ([Edwards, 1980], 325)

*I would like to thank Norbert Schappacher, Hourya Sinaceur and Jean-Jacques Szczeciniarz for pointing out these mistakes.

- Page 293, footnote 49, “in next paragraph” should be “in the next paragraph”. And “congruent with respect with respect to α ” should be “congruent with respect to α ”.
- Page 370, footnote 56, “un nombrea” should be “un nombre α ”.
- Page 373, missing reference: “for Dieudonné...” refers to [Dieudonné, 1974].
- Page 392, footnote 12, “Méry’s was” should be “Méry was”.
- Page 421, quote, “ $\gamma = \alpha + \beta$ ” should be “ $\gamma = \alpha + \beta$ ”.
- Page 422, Dedekind’s quotation in the third paragraph should be “as the product $\sqrt{2}.\sqrt{3}$, I mean the number $\sqrt{6}$, hence $\sqrt{2}.\sqrt{3} = \sqrt{6}$, qed!”.
- Page 539, the last sentence of the second paragraph should be reformulated as follow:
 Lotze, on the other hand, although he considered such relations between thought, mathematics and logic, conceived mathematical logic as close to Leibniz’s “*lingua characteristica*” (or at least, what was known of it at the time) and more generally of logic as intrinsically linked with language ([Sullivan, 2014]).

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The reference for Hourya Benis Sinaceur’s paper on Dedekind’s logicism, referred to in the dissertation as [Sinaceur, forthcoming], is:

Sinaceur, Hourya Benis (forthcoming). “Is Dedekind a logicist? Why does such a question arise?”. In *Functions and Generality of Logic: Reflections on Dedekind’s and Frege’s Logicisms*. Ed. by M. Panza, G. Sandu, H. Benis Sinaceur. Springer.