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Averaging Method for Functional Differential Equations

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PAR

Mustapha LAKRIB

**MOYENNISATION DANS LES EQUATIONS
DIFFERENTIELLES FONCTIONNELLES**

Soutenue le 20 avril 2002, devant la Commission d'Examen:

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MOYENNISATION DANS LES EQUATIONS
DIFFÉRENTIELLES FONCTIONNELLES

Par
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A la mémoire de mon Père
A ma Mère
A Karima, ma femme
A Ikram, ma fille
A toute ma famille

This thesis is dedicated
To the memory of my Father
To my Mother, my wife Karima,
my daughter Ikram, and to my family

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Introduction

This thesis contains contents of three articles as main topic, however it is not a collection of articles in literal meaning. The articles are:

Chapter 2: “*The method of averaging and functional differential equations with delay*”, International Journal of Mathematics and Mathematical Sciences, Volume 26, Issue 8 (2001), 497-511.

Chapter 3: “*Averaging method for functional differential equations*”, (Submitted for publication).

Chapter 4: “*Time averaging for functional differential equations*”, (Submitted for publication).

They are concerned with the use of the method of averaging to functional differential equations with fast oscillating solutions.

The motivation of this work is based on the following statement: Among authors who interested to the method of averaging for functional differential equations, some of them, as Foduck, Halanay, Hale, Medvedev, Morgunov and Volosov (Chap.1, [1-4] and [7-8]), established results for equations of the form

$$\dot{x}(t) = \varepsilon f(t, x_t)$$

under restrictive conditions on f (almost periodicity, boundedness, Lipschitz, etc...). Others, as Hale and Verduyn (Chap.1, [5]), investigated the more general case

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right). \quad (1)$$

For this, the cited authors gave an extension of the method of averaging to infinite dimensional systems which include functional differential equations. Here, the conditions on f are also restrictive (almost periodicity, boundedness, Lipschitz, etc...).

Our objective, in this thesis, is to make the conditions on f in equation (1) less restrictive and to treat further in each case the equation within its associated natural phase space. All the results are formulated in both classical mathematics and nonstandard analysis. However, they are proved within an axiomatic description of A. Robinson’s *Nonstandard Analysis* (NSA) (Append.A, [16]), namely *Internal Set Theory*, proposed by E. Nelson (Append.A, [13]).

The organization of this thesis is as follows:

In Chapter 1 we recall the basic theory of existence, uniqueness, continuation, and continuous dependence on parameters and initial data. There, we give also a short history on the averaging of functional differential equations.

In Chapter 2 we treat averaging for functional differential equations with delay of the form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t-r)\right). \quad (2)$$

Under smoothness hypotheses that are less restrictive than those ones in classical literature, we show that (2) may be approximated, first on finite time intervals and next for all time, by the averaged functional differential equation

$$\dot{y}(t) = f^o(y(t-r)) \quad (3)$$

where

$$f^o(u) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau. \quad (4)$$

We will emphasize that, in opposition to the approach in [5] cited previously, all the analysis is kept in \mathbf{R}^n .

There, we prove the following results:

Theorem 1 (Theorem 2.2.1) *Assume that f is continuous and bounded on $\mathbf{R}_+ \times \mathbf{R}^n$, that the continuity of $f = f(\tau, u)$ in $u \in \mathbf{R}^n$ is uniform with respect to τ , that for all $u \in \mathbf{R}^n$ there exists a limit (4), and that equation (3) has the uniqueness of the solutions with the prescribed initial data.*

If $y = y(\cdot; 0, \phi)$ is the solution of (3) and J is its maximal interval of existence, then for any $L > 0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (2) with initial value ϕ at 0 is defined at least on $[0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, L]$.

Theorem 2 (Theorem 2.2.2) *Assume that all hypotheses in Theorem 1 hold, and assume that y_e is an uniformly asymptotically stable rest point of (3), and that $\phi \in \mathcal{C}_o$ lies in the basin of attraction of y_e .*

If $y = y(\cdot; 0, \phi)$ is the solution of (3), then for any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (2) with initial value ϕ at 0 is defined for all $t \geq 0$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \geq 0$.

Chapter 3 is fundamental in the sense that there, we are concerned with the use of the method of averaging to functional differential equations of the general form (1). The results discussed in this chapter generalize those ones obtained in Chapter 2. The theorems we prove are the following ones:

Theorem 3 (Theorem 3.2.1) *Assume that f is continuous and bounded on $\mathbf{R}_+ \times \mathcal{C}_o$, that the continuity of $f = f(\tau, u)$ in $u \in \mathcal{C}_o$ is uniform with respect to τ , that for all $u \in \mathcal{C}_o$ there exists a limit (4), and that the averaged equation*

$$\dot{y}(t) = f^o(y_t) \quad (5)$$

has the uniqueness of the solutions with the prescribed initial data.

If $y = y(\cdot; t_0, \phi)$ is the solution of (5) and J is the maximal interval of existence of y , then for any $L > t_0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (1) with initial value ϕ at t_0 is defined at least on $[t_0, L]$ and the inequality $\|x(t) - y(t)\| < \delta$ holds for all $t \in [t_0, L]$.

Theorem 4 (Theorem 3.2.2) *Assume that all hypotheses in Theorem 3 hold, and assume that $f = f(\tau, u)$ is continuously differentiable with respect to u , that y_e is an exponentially stable rest point of (5), and that $\phi \in \mathcal{C}_o$ lies in the domain of exponential stability of y_e .*

If $x = x(\cdot; t_0, \phi)$ is the solution of (1) and $y = y(\cdot; t_0, \phi)$ is the solution of (5), then for any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, x is defined on $[t_0, \infty)$ and the inequality $\|x(t) - y(t)\| < \delta$ holds for all $t \geq t_0$.

In Chapter 4, we propose an alternate result on the averaging of functional differential equations on finite time intervals. For this, we remove the assumption on the boundedness of f and replace the condition on the uniform continuity of $f = f(\tau, u)$ in $u \in \mathcal{C}_o$ with respect to τ , by the (stronger) condition that f is Lipschitz. So, we prove the following result:

Theorem 5 (Theorem 4.2.1) *Assume that f is continuous on $\mathbf{R}_+ \times \mathcal{C}_o$, that f is Lipschitzian; that is, there exists some constant k such that*

$$\|f(\tau, u_1) - f(\tau, u_2)\| \leq k|u_1 - u_2|, \quad \text{for all } \tau \in \mathbf{R}_+ \text{ and } u_1, u_2 \in \mathcal{C}_o,$$

and that for all $u \in \mathcal{C}_o$ there exists a limit (4).

If $x = x(\cdot; t_0, \phi)$ is the solution of (1), $y = y(\cdot; t_0, \phi)$ is the solution of (5) and J is the maximal interval of existence of y , then for any $L > t_0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, x is defined at least on $[t_0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ on $t \in [t_0, L]$.

In Appendix A, we give a tutorial on Internal Set Theory which is an axiomatic approach of the Nonstandard Analysis. The purpose of this appendix is to recall some notions and tools of Nonstandard Analysis which have been either mentioned or used throughout this thesis, namely in Chapters 2-4 and Appendix B.

At the end of this thesis, we have put in Appendices B, C, D and E, some works on other topics, which are either published or in press. These works are:

Appendix B: “*Sur la moyennisation dans les systèmes à plusieurs fréquences*”, Maghreb Mathematical Review (In press).

Appendix C: “*On the validity of the averaging method for all time*”, Maghreb Mathematical Review, Volume 8, No 1 & 2, June & December 1999, 119-128.

Appendix D: “*On solutions of linear difference equations with infinite delay*”, Southwest Journal of Pure and Applied Mathematics, Issue 2 (2000), 52-59.

Appendix E: “*Oscillation of two delays differential equations with positive and negative coefficients*”, Mathematica Pannonica, Volume 12, Issue 2 (2001), 225-234.

Chapter 1

Functional differential equations: Preliminaries and short history on averaging

We recall the basic theory of existence, uniqueness, continuation, and continuous dependence for functional differential equations (see [5] or [7]). In the last section, we review a short history of the existing theory of averaging of functional differential equations.

1.1 Definition of a functional differential equation

Let $r \geq 0$ be a given real number and let $\mathcal{C}([a, b], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[a, b]$ into \mathbb{R}^n with the topology of uniform convergence. If $[a, b] = [-r, 0]$ we let $\mathcal{C}_o = \mathcal{C}([a, b], \mathbb{R}^n)$ and designate the norm of $\phi \in \mathcal{C}_o$ by $|\phi| = \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\}$ where $\|\cdot\|$ is any convenient norm in \mathbb{R}^n . Let $t_0 \in \mathbb{R}$ and $L \geq 0$. If $x \in \mathcal{C}([t_0 - r, t_0 + L], \mathbb{R}^n)$ and $t \in [t_0, t_0 + L]$, then $x_t \in \mathcal{C}_o$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Here $x_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t .

Let D be a subset of $\mathbb{R} \times \mathcal{C}_o$ and $f : D \rightarrow \mathbb{R}^n$ be a given function. Let “ $\dot{\cdot}$ ” represent the right-hand derivative.

Definition 1.1.1 *We say that the relation*

$$\dot{x}(t) = f(t, x_t) \tag{1.1}$$

is a functional differential equation on D .

Definition 1.1.2

1. A function x is said to be a solution of (1.1) on $[t_0 - r, t_0 + L)$ if there are $t_0 \in \mathbb{R}$ and $L > 0$ such that $x \in \mathcal{C}([t_0 - r, t_0 + L), \mathbb{R}^n)$, $(t, x_t) \in D$, and x satisfies (1.1) for $t \in [t_0 - r, t_0 + L)$.
2. For given $t_0 \in \mathbb{R}$ and $\phi \in \mathcal{C}_0$, we say that x is a solution of (1.1) with initial value ϕ at t_0 if there is an $L > 0$ such that x is a solution of (1.1) on $[t_0 - r, t_0 + L)$ and $x_{t_0} = \phi$.

Remark 1.1.1 If x is the unique solution of (1.1) with initial value ϕ at t_0 , we denote it $x = x(\cdot; t_0, \phi)$.

Equation (1.1) is a very general type of equation and includes ordinary differential equations ($r = 0$)

$$\dot{x}(t) = F(t, x(t)),$$

differential difference equations

$$\dot{x}(t) = f(t, x(t), x(t - r_1(t)), \dots, x(t - r_p(t)))$$

with $0 \leq r_i(t) \leq r$, $i = 1, 2, \dots, p$, as well as the integro-differential equation

$$\dot{x}(t) = \int_{-r}^0 g(t, \theta, x(t + \theta)) d\theta.$$

Much more general equations are also included in equation (1.1).

Definition 1.1.3 We say that equation (1.1) is autonomous if $f(t, \phi) = g(\phi)$ where g does not depend on t .

Lemma 1.1.1 If $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}_0$ are given, and $f : D \rightarrow \mathbb{R}^n$ is continuous, then finding a solution of (1.1) with initial value ϕ at t_0 is equivalent to solving the integral equation

$$x(t) = \phi(0) + \int_{t_0}^t f(s, x_s) ds, \quad t \geq t_0; \quad x_{t_0} = \phi. \quad (1.2)$$

1.2 Existence, uniqueness, and continuous dependence

In this section, we recall a basic existence theorem for the initial-value problem of (1.1). Also, a result on continuous dependence and an other on uniqueness, are recalled.

Theorem 1.2.1 (Existence) *Let Ω be an open subset of $\mathbf{R} \times \mathcal{C}_o$ and let $f : \Omega \rightarrow \mathbf{R}^n$ be continuous. If $(t_0, \phi) \in \Omega$, then there is a solution of (1.1) with initial value ϕ at t_0 .*

More generally, if $W \subset \Omega$ is compact and $f : \Omega \rightarrow \mathbf{R}^n$ is continuous, then there are a neighborhood $V \subset \Omega$ of W such that $f|_V$ is bounded, and an $L > 0$ such that for any $(t_0, \phi) \in W$, there is a solution x of (1.1) with initial value ϕ at t_0 that exists on $[t_0 - r, t_0 + L]$.

Theorem 1.2.2 (Continuous dependence) *Let Ω be an open set in $\mathbf{R} \times \mathcal{C}_o$, $(t_0, \phi) \in \Omega$, $f : \Omega \rightarrow \mathbf{R}^n$ be continuous, and $x = x(\cdot; t_0, \phi)$ be a solution of (1.1) with initial value ϕ at t_0 which exists and is unique on $[t_0 - r, b]$. Let $W \subset \Omega$ be the compact set defined by $W = \{(t, x_t) : t \in [t_0, b]\}$ and let V be a neighborhood of W on which f is bounded. If (t^k, ϕ^k, f^k) , $k = 1, 2, \dots$ satisfies $t^k \rightarrow t_0$, $\phi^k \rightarrow \phi$, and $|f^k - f| = \sup_{(t,u) \in V} \|f^k(t, u) - f(t, u)\| \rightarrow 0$ as $k \rightarrow \infty$, then there is a k^0 such that the functional differential equation*

$$\dot{x}(t) = f^k(t, x_t)$$

for $k \geq k^0$ is such that each solution x^k with initial value ϕ^k at t^k exists on $[t^k - r, b]$ and $x^k \rightarrow x$ uniformly on $[t_0 - r, b]$. Since all x^k may not be defined on $[t_0 - r, b]$, $x^k \rightarrow x$ uniformly on $[t_0 - r, b]$ means that for any $\eta > 0$, there is a $k_1 = k_1(\eta)$ such that $x^k(t)$, $k \geq k_1$, is defined on $[t_0 - r + \eta, b]$, and $x^k \rightarrow x$ uniformly on $[t_0 - r + \eta, b]$.

Theorem 1.2.3 (Uniqueness) *Let Ω be an open subset of $\mathbf{R} \times \mathcal{C}_o$ and let $f : \Omega \rightarrow \mathbf{R}^n$ be continuous. Suppose $f = f(t, u)$ is lipschitzian on u in each compact set in Ω . If $(t_0, \phi) \in \Omega$, then there is a unique solution of (1.1) with initial value ϕ at t_0 .*

1.3 Continuation of solutions

Suppose f in (1.1) is continuous and let x be a solution of (1.1) on an interval $[t_0 - r, a)$, $a > t_0$.

Definition 1.3.1 *We say \hat{x} is a continuation of x if there is a $b > a$, such that \hat{x} is defined on $[t_0 - r, b)$, coincides with x on $[t_0 - r, a)$, and satisfies (1.1) on $[t_0 - r, b)$.*

A solution x is noncontinuable if no such continuation exists; that is, the interval $[t_0 - r, a)$ is the maximal interval of existence of the solution x .

The existence of a noncontinuable solution follows from Zorn's lemma. Also, the maximal interval of existence must be open.

Theorem 1.3.1 *Let Ω be an open set in $\mathbf{R} \times \mathcal{C}_o$ and let $f : \Omega \rightarrow \mathbf{R}^n$ be continuous. If x is a noncontinuable solution of (1.1) on $[t_0 - r, b)$, then for any compact set W in Ω , there is a t_W such that $(t, x_t) \notin W$ for $t_W \leq t < b$.*

Corollary 1.3.1 *Let Ω be an open subset of $\mathbf{R} \times \mathcal{C}_o$ and let $f : \Omega \rightarrow \mathbf{R}^n$ be continuous. If x is a noncontinuable solution of (1.1) on $[t_0 - r, b)$ and W is the closure of the set $\{(t, x_t) : t_0 \leq t < b\}$ in $\mathbf{R} \times \mathcal{C}_o$, then W compact implies there is a sequence $\{t_k\}$ of real numbers, $t_k \rightarrow b^-$ as $k \rightarrow \infty$ such that (t_k, x_{t_k}) tends to $\partial\Omega$ as $k \rightarrow \infty$. If $r > 0$, then there is a $\psi \in \mathcal{C}_o$ such that $(b, \psi) \in \partial\Omega$ and $(t, x_t) \rightarrow (b, \psi)$ as $t \rightarrow b^-$.*

1.4 Averaging: A short history

There is a much less extensive literature on the averaging of functional differential equations compared to this one devoted to the averaging of ordinary differential equations.

In the 1960's, authors as V.I. Foduck [1], A. Halanay [2, 3], J.K. Hale [4], G.N. Medvedev [8, 9], V.M. Volosov and B.I. Morgunov [9], introduced the use of the method of averaging to functional differential equations. The most general of the results developed by the cited authors are given in [7]. There, the averaging is discussed for the functional differential equation

$$\dot{x}(t) = \varepsilon f(t, x_t), \quad t \geq t_0; \quad x_{t_0} = \phi \quad (1.3)$$

where $\varepsilon > 0$ is a small parameter.

The associated averaged equation is the ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(\tilde{y}), \quad \tilde{y}(\theta) = y \text{ for } \theta \in [-r, 0] \quad (1.4)$$

where $y(t_0) = \phi(t_0)$ and

$$f^o(u) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

Suitable conditions are given, under which one can compare solutions of (1.3) to those ones of (1.4).

The techniques used in the literature to prove the above assertion follow different approaches. In [7], equation (1.3) is considered as a perturbation of

$$\dot{x}(t) = 0 \cdot x_t,$$

and the solution of (1.3) is decomposed as $x_t = \tilde{I}z(t) + w_t$ where $\tilde{I}(\theta) = I$, the identity, for $\theta \in [-r, 0]$. Then, conditions are derived such that w_t approaches zero faster than any exponential. By use of the invariant manifold theory, it is shown that the flow for (1.3) in any bounded set is equivalent to the flow defined by an ordinary differential equation

$$\dot{z}(t) = \varepsilon g(t, z, \varepsilon), \quad g(t, z, 0) = f(t, \tilde{z}). \quad (1.5)$$

Next, classical averaging procedures of ordinary differential equations are applied to (1.5) to obtain the approximation of solutions of (1.3) by those ones of (1.4).

There is a few works dedicated to the use of the method of averaging to functional differential equations in the general case, that is

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right). \quad (1.6)$$

In [6], the authors introduce an extension of the method of averaging to abstract evolutionary equations in Banach spaces. In particular, they rewrite equation (1.6) as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there.

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Chapter 2

The method of averaging and functional differential equations with delay

This chapter essentially contains the paper [5] “The method of averaging and functional differential equations with delay”, published in International Journal of Mathematics and Mathematical Sciences, Volume 26, Issue 8 (2001), 497-511.

We present a natural extension of the method of averaging to fast oscillating functional differential equations with delay. Unlike the usual approach where the analysis is kept in an infinite dimensional Banach space, our analysis is achieved in \mathbf{R}^n .

2.1 Introduction

An important tool in the rigorous study of differential equations with a small parameter is the method of averaging, which is well known for ordinary differential equations [1, 6, 7, 8] and for functional differential equations with small delay [2, 3, 9]. In both cases, the corresponding averaged equations are ordinary differential equations. However, for fast oscillating functional differential equations with large bounded delay, the method of averaging is not nearly so developed as in the two previous cases. Among recent works devoted to this last case, we will cite the paper of Hale and Verduyn Lunel [4]. There, the authors rewrite a functional differential equation with delay as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there.

In this chapter, we develop an improved theory of averaging for functional differential equations with delay under smoothness hypotheses that are less restrictive than those of [4]. Also all our analysis is kept in \mathbf{R}^n . This is performed in Section 2.2. There, we state closeness of solutions of the averaged and original equations on finite time intervals (Theorem 2.2.1). We also investigate the long time behaviour

of the solutions of the original equation (Theorem 2.2.2). In Section 2.3 we present the nonstandard translates (Theorems 2.3.1 and 2.3.2) in the language of *Internal Set Theory* (See Appendix A) of Theorems 2.2.1 and 2.2.2. We end this section with an external characterization of the uniform asymptotic stability which is the main assumption for the validity of the result of Theorem 2.3.2. Finally, in Section 2.4 we give the proofs of Theorems 2.3.1 and 2.3.2. As ordinary differential equations and functional differential equations with small delay are special cases of functional differential equations with delay, the proofs developed in this section provide alternative proofs to the techniques of averaging on these equations found, for example, in [7, 8].

2.2 The method of averaging

In this section, we present the main results on averaging for functional differential equations with delay.

Suppose $f : \mathbf{R} \times U \rightarrow \mathbf{R}^n$ is a continuous function, where U is an open subset of \mathbf{R}^n . Let $\phi \in \mathcal{C}_o$. Let $\varepsilon > 0$ be a real parameter. Along with the functional differential equation with delay

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t-r)\right), \quad t \geq 0; \quad x_0 = \phi \quad (2.1)$$

we consider the averaged equation

$$\dot{y}(t) = f^o(y(t-r)), \quad t \geq 0; \quad y_0 = \phi, \quad (2.2)$$

where

$$f^o(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, x) d\tau. \quad (2.3)$$

As a first result, we give comparison of solutions of the averaged and original equations on finite time intervals.

Theorem 2.2.1 *Assume that*

- (H0) *The function f is bounded on $\mathbf{R}_+ \times U$.*
- (H1) *The continuity of $f = f(\tau, x)$ in $x \in U$ is uniform with respect to $\tau \in \mathbf{R}_+$.*
- (H2) *For all $x \in U$ there exists a limit (2.3).*
- (H3) *Equation (2.2) has a unique solution.*

Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2) and let $J = [-r, \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. For any $L > 0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (2.1) with initial value ϕ at 0 is defined at least on $[0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, L]$.

Remark 2.2.1 Assume that the initial time $t_0 \neq 0$. Let $y = y(\cdot; t_0, \phi)$ be the solution of (2.2) and let $J = [t_0 - r, t_0 + \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. The conclusions of Theorem 2.2.1 become: For any $L > 0$, $L + t_0 \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (2.1) with initial value ϕ at t_0 is defined at least on $[t_0, t_0 + L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [t_0, t_0 + L]$.

One can also precise the long time behaviour of a solution of (2.1) provided that more is known about the solution of (2.2). To give estimate for all time, we assume that the solution of (2.2) tend toward an equilibrium. Before this, we first recall the concept of *uniform asymptotic stability* of equilibrium points of autonomous functional differential equations with delay.

Consider the autonomous functional differential equation with delay

$$\dot{y}(t) = f^\circ(y(t-r)) \quad (2.4)$$

where $f^\circ : U \rightarrow \mathbb{R}^n$ is a continuous function, U is an open subset of \mathbb{R}^n , and $r > 0$ is a constant.

Since (2.4) is autonomous, the concepts of *asymptotic stability* and *uniform asymptotic stability* of equilibrium points of (2.4) coincide. Then, it is sufficient to deal directly with *uniform notions*.

Definition 2.2.1 The equilibrium point y_e of (2.4) is said to be

1. *Uniformly stable* (in the sense of Liapunov) if for any $\mu > 0$, there $\eta > 0$, there exists $\eta = \eta(\mu) > 0$ with the property that solution y of (2.4) with initial value ϕ at t_0 for which $|\phi - y_e| < \eta$ can be continued for all $t > t_0$ and satisfies $\|y(t) - y_e\| < \mu$.

2. *Uniformly attractive* if there exists $b_0 > 0$ with the respective properties:

a) For all $t_0 \in \mathbb{R}$ any solution y of (2.4) with initial value ϕ at t_0 for which $|\phi - y_e| < b_0$ can be continued for all $t > t_0$.

b) For every $\delta > 0$, there exists $T = T(\delta) > 0$ (T depends on δ but not on t_0) such that $\|y(t) - y_e\| < \delta$ for $t > t_0 + T(\delta)$, i.e., $\lim_{t \rightarrow \infty} y(t) = y_e$ uniformly in $t_0 \in \mathbb{R}$.

3. *Uniformly asymptotically stable* if it is uniformly stable and uniformly attractive.

Remark 2.2.2 The ball \mathcal{B} of center y_e and radius b_0 where the attraction is uniform will be called the *basin of attraction* of y_e .

We now return to the averaged equation (2.2). We assume that y_e is an equilibrium point of (2.2), that is, $f^\circ(y_e) = 0$. As a next result of this section, we prove the validity of the approximation of a solution x of (2.1) by the solution y of (2.2) for all (future) time, under additional conditions about the equilibrium point y_e and the initial value ϕ .

Theorem 2.2.2 *Let the hypotheses (H0) to (H3) of Theorem 2.2.1 be true, and assume that*

(H4) *The point y_e is uniformly asymptotically stable.*

(H5) *The initial value ϕ in (2.2) lies in the basin of attraction of y_e .*

Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2). Then for any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (2.1) with initial value ϕ at 0 is defined for all $t \geq 0$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \geq 0$.

2.3 Nonstandard results

2.3.1 The averaging results

First we give nonstandard formulations of Theorem 2.2.1, Remark 2.2.1, and Theorem 2.2.2. Then, by use of the reduction algorithm (Appendix A, subsection A.2.3), we show that the reduction of Theorem 2.3.1, Remark 2.3.1, and Theorem 2.3.2 are Theorem 2.2.1, Remark 2.2.1 and Theorem 2.2.2, respectively.

Theorem 2.3.1 *Let U be a standard open subset of \mathbb{R}^n . Let standard continuous function. Let $\phi \in \mathcal{C}_o$ be standard. Assume that hypotheses (H0) to (H3) in Theorem 2.2.1 hold. Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2) and let Theorem 2.2.1 hold. Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2) and let $J = [-r, \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > 0$, $L \in J$, any solution x of (2.1) with initial value ϕ at 0 is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

Remark 2.3.1 *Assume that the initial time $t_0 \neq 0$. Let $y = y(\cdot; t_0, \phi)$ be the solution of (2.2) and let $J = [t_0 - r, t_0 + \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. The conclusions of Theorem 2.3.1 become: Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > 0$, $L + t_0 \in J$, any solution x of (2.1) with initial value ϕ at t_0 is defined at least on $[t_0, t_0 + L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [t_0, t_0 + L]$.*

Theorem 2.3.2 *Let U be a standard open subset of \mathbb{R}^n . Let standard continuous function. Let $\phi \in \mathcal{C}_o$ be standard. Let y_e be a standard equilibrium point of (2.2). Assume that hypotheses (H0) to (H5) in Theorem 2.2.2 hold. Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2). Let $y = y(\cdot; 0, \phi)$ be the solution of (2.2). Let $\varepsilon > 0$ be infinitesimal. Then any solution x of (2.1) with initial value ϕ at 0 is defined for all $t \geq 0$ and satisfies $x(t) \simeq y(t)$ for all $t \geq 0$.*

The proofs of Theorems 2.3.1 and 2.3.2 are postponed to Section 4.4. Theorem 2.3.1, Remark 2.3.1, and Theorem 2.3.2 are external statements. We show that the reduction of Theorem 2.3.1 (resp., Remark 2.3.1 and Theorem 2.3.2) is Theorem 2.2.1 (resp., Remark 2.2.1 and Theorem 2.2.2).

Reduction of Theorem 2.3.1 Let B be the formula “If $\delta > 0$ then any solution x of (2.1) is defined at least on $[0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, L]$ ”. To say that “any solution x of (2.1) is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$ ” is the same as saying $\forall^{st} \delta B$. Then Theorem 2.3.1 asserts that

$$\forall \varepsilon (\forall^{st} \eta \varepsilon < \eta \implies \forall^{st} \delta B). \quad (2.5)$$

In this formula L is standard and ε , η and δ range over the strictly positive real numbers. By (A.1) (Appendix A, subsection A.2.3), formula (2.5) is equivalent to

$$\forall \delta \exists^{fin} \eta' \forall \varepsilon (\forall \eta \in \eta' \varepsilon < \eta \implies B). \quad (2.6)$$

For η' a finite set, $\forall \eta \in \eta' \varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min \eta'$, and so formula (2.6) is equivalent to

$$\forall \delta \exists \varepsilon_0 \forall \varepsilon (\varepsilon < \varepsilon_0 \implies B).$$

This shows that for any standard $L > 0$, $L \in J$, the statement of Theorem 2.2.1 holds, thus by transfer (Appendix A, subsection A.2.2), it holds for any $L > 0$, $L \in J$. \square

The reduction of Remark 2.3.1 (resp., Theorem 2.3.2) to Remark 2.2.1 (resp., Theorem 2.2.2) follows almost verbatim the reduction of Theorem 2.3.1 to Theorem 2.2.1 and is not done here.

2.3.2 Uniform asymptotic stability

As the condition (H4) will be used in its external form, we give the external characterizations of the notion of uniform stability and uniform attractivity of the equilibrium point y_e of (2.4), given in Definition 2.2.1.

By transfer (Appendix A, subsection A.2.2), we may assume that f^o , r and y_e are standard.

Lemma 2.3.1 *The equilibrium point y_e of (2.4) is*

1. *Uniformly stable if and only if for all $t_0 \in \mathbf{R}$ any solution y of (2.4) with initial value ϕ at t_0 for which $\phi(t) \simeq y_e$ for $t \in [-r, 0]$, can be continued for all $t > t_0$ and satisfies $y(t) \simeq y_e$.*

2. *Uniformly attractive if and only if it admits a standard basin of attraction, that is, there exists a standard $b_0 > 0$ with the property that for all $t_0 \in \mathbf{R}$ any solution y of (2.4) with initial value ϕ at t_0 for which $|\phi - y_e| < b_0$, ϕ standard, can be continued for all $t > t_0$ and satisfies $y(t) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$.*

Proof. 1. Let A be the formula “ $|\phi - y_e| < \eta$ ” and let B be the formula “Any solution y of (2.4) with initial value ϕ at t_0 can be continued for all $t > t_0$ and

satisfies the inequality $\|y(t) - y_e\| < \mu^n$. The characterization of uniform stability in the lemma is

$$\forall t_0 \forall \phi (\forall^{st} \eta A \implies \forall^{st} \mu B). \quad (2.7)$$

In this formula y , r and y_e are standard parameters and η , μ range over the strictly positive real numbers. By (A.1) (Appendix A, subsection A.2.3), formula (2.7) is equivalent to

$$\forall \mu \exists^{fin} \eta' \forall t_0 \forall \phi (\forall \eta \in \eta' A \implies B). \quad (2.8)$$

For η' a finite set, $\forall \eta \in \eta' A$ is the same as A for $\eta = \min \eta'$, and so formula (2.8) is equivalent to

$$\forall \mu \exists \eta \forall t_0 \forall \phi (A \implies B).$$

This is the usual definition of uniform stability.

2. By transfer (Appendix A, subsection A.2.2), the uniform attractivity of y_e is equivalent to the existence of a standard basin of attraction, that is, b_0 in Remark 2.2.2 is standard. The characterization of standard basin of attraction in the lemma is that for all standard $\phi \in \mathcal{C}_o$ such that $t_0 \in \mathbb{R}$ any solution y of (2.4) with y of (2.4) with initial value ϕ at t_0 can be continued for all $t > t_0$ and satisfies

$$\forall t_0 \forall t (\forall^{st} T \ t - t_0 > T \implies \forall^{st} \delta \|y(t) - y_e\| < \delta). \quad (2.9)$$

In this formula y , ϕ and y_e are standard parameters and T , δ range over the strictly positive real numbers. By (A.1) (Appendix A, subsection A.2.3), formula (2.9) is equivalent to

$$\forall \delta \exists^{fin} T' \forall t_0 \forall t (\forall T \in T' \ t - t_0 > T \implies \|y(t) - y_e\| < \delta). \quad (2.10)$$

For T' a finite set $\forall T \in T' \ t - t_0 > T$ is the same as $t - t_0 > T$ for $T = \max T'$, and so formula (2.10) is equivalent to

$$\forall \delta \exists T \forall t_0 \forall t (t - t_0 > T \implies \|y(t) - y_e\| < \delta).$$

We have shown that for all standard continuous function ϕ in the basin of attraction (and consequently, by transfer (Appendix A, subsection A.2.2), for all continuous function ϕ in the basin of attraction of y_e), for all $t_0 \in \mathbb{R}$ any solution y of (2.4) with initial value ϕ at t_0 can be continued for all $t > t_0$ and satisfies $\lim_{t \rightarrow \infty} y(t) = y_e$, the limit being uniform in t_0 . \square

Assume that (2.4) has the uniqueness of the solutions with prescribed initial values. Let $\phi \in \mathcal{C}_o$. For $t_0 \in \mathbb{R}$, let $y = y(\cdot, t_0, \phi)$ be the solution of (2.4) with initial value ϕ at t_0 . This solution is defined on the interval $I(t_0, \phi) = [t_0 - r, t_0 + \beta)$. It is well known that the function y is continuous with respect to the initial value ϕ . The external formulation of this result is as follows.

Lemma 2.3.2 *Let ϕ and $\phi_0 \in \mathcal{C}_o$, with ϕ_0 standard. If $\phi(t) \simeq \phi_0(t)$ on $[-r, 0]$, then for all $t \in I(t_0, \phi_0)$, $t > t_0$, such that $t - t_0$ is standard, we have $t \in I(t_0, \phi)$ and $y(t; t_0, \phi) \simeq y(t; t_0, \phi_0)$.*

Proof. The reduction of the Lemma 2.3.2 is the usual continuity of solutions with respect to initial values. \square

Lemma 2.3.3 *Assume that (2.4) has the uniqueness of the solutions with prescribed initial values. The equilibrium point y_e of (2.4) is uniformly asymptotically stable if and only if there exists a standard $a > 0$ that for all $t_0 \in \mathbb{R}$ any solution y of with initial value ϕ at t_0 for which $|\phi - y_e| < a$ can be continued for all $t > t_0$ and satisfies $y(t) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$.*

Proof. Assume that y_e is uniformly asymptotically stable. Then it is uniformly attractive, and so it admits a ball \mathcal{B} of center y_e and radius $b_0 > 0$, b_0 standard, as a standard basin of attraction. Let $a > 0$ be standard such that the closure of the ball \mathcal{B}' of center y_e and radius a is included in \mathcal{B} . Let ϕ and $\phi_0 \in \mathcal{C}_o$, ϕ_0 standard, with $\phi(t) \in \mathcal{B}'$ and $\phi_0(t) \in \mathcal{B}$ for all $t \in [-r, 0]$. Assume that $\phi(t) \simeq \phi_0(t)$ for all $t \in [-r, 0]$. For $t_0 \in \mathbb{R}$, let $y = y(\cdot; t_0, \phi)$ and $y_0 = y_0(\cdot; t_0, \phi_0)$ be the solutions of (2.4) with initial values ϕ and ϕ_0 , respectively. By the uniform attractivity of y_e , the solution y_0 is defined for all $t > t_0$ and satisfies $y_0(t) \simeq y_e$ for all $t - t_0 \simeq +\infty$. By Lemma 2.3.2, $y(t) \simeq y_0(t)$ on $[t_0, t_0 + L]$ for all limited $L > 0$. By Robinson's Lemma (Appendix A, section A.4, Lemma A.4.1), there exists $\nu \simeq +\infty$ such that $y(t) \simeq y_0(t)$ on $[t_0, t_0 + \nu]$. Thus $y(t) \simeq y_e$ for all $t \leq t_0 + \nu$, $t - t_0 \simeq +\infty$, and in particular we have $y(t) \simeq y_e$ for all $t \in [t_0 + \nu - r, t_0 + \nu]$. By uniform stability of y_e we have $y(t) \simeq y_e$ for all $t > t_0 + \nu$. Hence $y(t) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$. Conversely, assume y_e satisfies the property in the lemma. By Lemma 2.3.1, 2., the ball \mathcal{B}' is a standard basin of attraction of y_e . Hence y_e is uniformly attractive. Let $\phi \in \mathcal{C}_o$ such that $\phi(t) \simeq y_e$ for all $t \in [-r, 0]$. For $t_0 \in \mathbb{R}$, let that $\phi(t) \simeq y_e$ for all $t \in [-r, 0]$. For $t_0 \in \mathbb{R}$, let By hypothesis we have $y(t) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$, and by Lemma 2.3.2, $y(t) = y(t; t_0, \phi) \simeq y(t; t_0, y_e) = y_e$ for all t such that $t - t_0$ is limited. By Lemma 2.3.1, 1., y_e is uniformly stable. Thus y_e is uniformly asymptotically stable. \square

2.4 Proofs of Theorems 2.3.1 and 2.3.2

2.4.1 Preliminary lemmas

Hereafter we give some results we need for the proof of Theorem 2.3.1. We assume throughout this section that f and ϕ are standard. We suppose also that f satisfies conditions (H0), (H1) and (H2) of Theorem 2.3.1. The conditions (H1) and (H2) will be used in their following external forms

- (H1') $\forall^{st} x \in U \quad \forall x' \in U \quad \forall \tau \in \mathbf{R}_+ : \quad x' \simeq x \implies f(\tau, x') \simeq f(\tau, x).$
(H2') There is a standard function $f^o : U \rightarrow \mathbf{R}^n$ such that

$$\forall^{st} x \in U \quad \forall T \simeq +\infty : \quad f^o(x) \simeq \frac{1}{T} \int_0^T f(\tau, x) d\tau.$$

Lemma 2.4.1 *The function f^o is continuous and we have*

$$f^o(x) \simeq \frac{1}{T} \int_0^T f(\tau, x) d\tau$$

for all x nearstandard in U and all $T \simeq +\infty$.

Proof. ([8, Lemma 4, page 106]). Let $x, {}^o x \in U$ such that ${}^o x$ is standard and $x \simeq {}^o x$. Let $\nu > 0$ be infinitesimal. By condition (H2) there exists $T_0 > 0$ such that, for $T > T_0$

$$\|f^o(x) - \frac{1}{T} \int_0^T f(\tau, x) d\tau\| < \nu.$$

Hence for some $T \simeq +\infty$ we have

$$f^o(x) \simeq \frac{1}{T} \int_0^T f(\tau, x) d\tau.$$

By condition (H1') we have $f(\tau, x) \simeq f(\tau, {}^o x)$ for $\tau \geq 0$. Therefore

$$f^o(x) \simeq \frac{1}{T} \int_0^T f(\tau, {}^o x) d\tau.$$

By condition (H2') we deduce that $f^o(x) \simeq f^o({}^o x)$. Thus f^o is continuous. Moreover for $T \simeq +\infty$ we have

$$f^o(x) \simeq f^o({}^o x) \simeq \frac{1}{T} \int_0^T f(\tau, {}^o x) d\tau \simeq \frac{1}{T} \int_0^T f(\tau, x) d\tau.$$

□

Lemma 2.4.2 *There exists $\mu > 0$ such that whenever $t \geq 0$ is limited and x is nearstandard in U , there is $\alpha > 0$ such that $\mu < \alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x) d\tau \simeq f^o(x).$$

Proof. ([8, Lemma 5, page 107]). Let $t \geq 0$ be limited and let x be nearstandard in U .

i) Suppose t/ε is limited. Let $S > 0$ be unlimited such that $\varepsilon S \simeq 0$. Then

$$\frac{1}{S} \int_{t/\varepsilon}^{t/\varepsilon + S} f(\tau, x) d\tau = \left(1 + \frac{t}{\varepsilon S}\right) \frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau - \frac{1}{S} \int_0^{t/\varepsilon} f(\tau, x) d\tau.$$

By Lemma 2.4.1 we have

$$\frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau \simeq f^o(x).$$

Since

$$\frac{1}{S} \int_0^{t/\varepsilon} f(\tau, x) d\tau \simeq 0 \quad \text{and} \quad \frac{t}{\varepsilon S} \simeq 0$$

we have

$$\frac{1}{S} \int_{t/\varepsilon}^{t/\varepsilon + S} f(\tau, x) d\tau \simeq f^o(x).$$

Then, it suffices to choose $\mu = \varepsilon$ and take $\alpha = \varepsilon S$.

ii) Suppose t/ε is unlimited. Let $S > 0$. We write

$$\begin{aligned} \frac{1}{S} \int_{t/\varepsilon}^{t/\varepsilon + S} f(\tau, x) d\tau &= \frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau \\ &+ \frac{t}{\varepsilon S} \left(\frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau - \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\tau, x) d\tau \right). \end{aligned}$$

By Lemma 2.4.1 we have

$$\frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau \simeq f^o(x) \simeq \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\tau, x) d\tau.$$

Let us denote by

$$\eta(S) = \frac{t}{\varepsilon S} \left(\frac{1}{t/\varepsilon + S} \int_0^{t/\varepsilon + S} f(\tau, x) d\tau - \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\tau, x) d\tau \right).$$

$\eta(S)$ is infinitesimal for all S such that $t/\varepsilon S$ is limited. By permanence (Cauchy principle, see Appendix A, section A.4) this property holds for some S for which $t/\varepsilon S$ is unlimited. S can be chosen so that $S > 1$ and $t/\varepsilon S \simeq +\infty$. Since t is limited we have $\varepsilon S \simeq 0$. Then, it suffices to choose $\mu = \varepsilon$ and take $\alpha = \varepsilon S$. \square

Lemma 2.4.3 *Let $L_1 > 0$ be standard and let \tilde{x} be a function defined on $[-r, L_1]$. We assume that \tilde{x} is continuous on $[-r, 0]$, $\tilde{x}(t)$ is nearstandard in U for all $t \in [0, L_1]$. Then there exist some positive integer ω and some partition $\{t_n : n = 0, \dots, \omega + 1\}$ of $[0, L_1]$ such that $0 = t_0 < \dots < t_n < t_{n+1} < \dots < t_\omega \leq L_1 < t_{\omega+1}$, $t_{n+1} - t_n = \alpha_n \simeq 0$, and*

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_{n+1}/\varepsilon} f(\tau, \tilde{x}(t_n - r)) d\tau \simeq f^o(\tilde{x}(t_n - r)).$$

Proof. Consider the set $A = \{\lambda \in \mathbf{R} / \forall t \in [0, L_1] \exists \alpha \in \mathbf{R} : \mathcal{P}(t, \alpha, \lambda)\}$ where

$$\mathcal{P}(t, \alpha, \lambda) \equiv \mu < \alpha < \lambda \quad \text{and} \quad \left\| \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, \tilde{x}(\tau - r)) d\tau - f^o(\tilde{x}(t - r)) \right\| < \lambda.$$

By Lemma 2.4.2 the set A contains all the standard real numbers $\lambda > 0$. By Lemma A.4.4 (Appendix A) there exists $\lambda_0 \simeq 0$ in A , that is, there exists $0 < \lambda_0 \simeq 0$ such that for all $t \in [0, L_1]$ there exists $\alpha \in \mathbf{R}$ such that $\mathcal{P}(t, \alpha, \lambda_0)$ holds. By the axiom of choice there exists a function $c : [0, L_1] \rightarrow \mathbf{R} : [0, L_1] \rightarrow \mathbf{R}$ such that $c(t) = \alpha$, that is, $\mathcal{P}(t, c(t), \lambda_0)$ holds for all $t \in [0, L_1]$. Since $c(t) > \mu$ for all $t \in [0, L_1]$, the conclusion of the

Lemma 2.4.4 *Let $L_1 > 0$ be standard and let \tilde{x} be a function defined on $[-r, L_1]$. We assume that \tilde{x} is continuous on $[-r, 0]$, $\tilde{x}(t)$ is nearstandard in U for all $t \in [0, L_1]$, and satisfies $\tilde{x}(t) \simeq \tilde{x}(t_n)$ for all $t \in [t_n, t_{n+1}]$ with $0 = t_0 < \dots < t_n < t_{n+1} < \dots < t_\omega \leq L_1 < t_{\omega+1}$ and $t_{n+1} - t_n = \alpha_n \simeq 0$, where the t_n are determined by Lemma 2.4.3. Then*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau \simeq \int_0^t f^o(\tilde{x}(\tau - r)) d\tau \quad \forall t \in [0, L_1].$$

Proof. Let $t \in [0, L_1]$ and let N be a positive integer such that $t_N \leq t < t_{N+1}$. We have

$$\begin{aligned} & \int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau - r)) d\tau \\ &= \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \\ &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \quad (2.11) \\ & \quad + \int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \\ &\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \end{aligned}$$

since

$$\begin{aligned} & \left\| \int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \right\| \\ & \leq \int_{t_N}^t \left(\left\| f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) \right\| + \left\| f^o(\tilde{x}(\tau - r)) \right\| \right) d\tau \\ & \leq 2M(t - t_N) \leq 2M(t_{N+1} - t_N) \leq 2M\alpha \simeq 0 \end{aligned}$$

where $\alpha = \max\{\alpha_n\} \simeq 0$ (see Appendix A, Lemma A.4.2) and M is a bound for f and then for f^o too. M is standard.

On the other hand, it is easy to verify that $\tilde{x}(\tau - r) \simeq \tilde{x}(t_n - r) = cte := \tilde{x}_n$ for $\tau \in [t_n, t_{n+1}]$. By the continuity of f , the condition (H1') and Lemma 2.4.1 (the continuity of f^o) it follows, respectively, that

$$f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) \simeq f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right), \quad f^o(\tilde{x}(\tau - r)) \simeq f^o(\tilde{x}_n),$$

or

$$f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) = f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) + \gamma_n(\tau), \quad f^o(\tilde{x}(\tau - r)) = f^o(\tilde{x}_n) + \delta_n(\tau)$$

with

$$\gamma_n(\tau) \simeq 0 \simeq \delta_n(\tau).$$

Hence, from (2.11) we obtain

$$\begin{aligned} & \int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau - r)) d\tau \\ & \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) + \gamma_n(\tau) + \delta_n(\tau) \right) d\tau, \\ & = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (\gamma_n(\tau) + \delta_n(\tau)) d\tau \\ & \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau + \sum_{n=0}^{N-1} (\gamma_n + \delta_n) \int_{t_n}^{t_{n+1}} d\tau, \\ & \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau + (\gamma + \delta) \sum_{n=0}^{N-1} (t_{n+1} - t_n), \\ & = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau + (\gamma + \delta) \cdot t_N, \end{aligned}$$

with $\gamma_n(\tau) \simeq \delta_n(\tau) \simeq 0$ for $\tau \in [t_n, t_{n+1}]$ and where $\gamma_n + \delta_n = \sup_{\tau \in [t_n, t_{n+1}]} \{\gamma(\tau)\} + \sup_{\tau \in [t_n, t_{n+1}]} \{\delta(\tau)\} \simeq 0$ and $\gamma + \delta = \max\{\gamma_n\} + \max\{\delta_n\} \simeq 0$ (see Appendix A, Lemma A.4.2).

Since L_1 is standard and $t_N \in [0, L_1]$, t_N is limited and then $(\gamma + \delta) \cdot t_N \simeq 0$. Therefore,

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau - r)) d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau$$

By Lemma 2.4.2, we have

$$\begin{aligned}
& \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau \\
&= \int_{t_n}^{t_n + \alpha_n} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) \right) d\tau \\
&= \int_{t_n}^{t_n + \alpha_n} f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) d\tau - \int_{t_n}^{t_n + \alpha_n} f^o(\tilde{x}_n) d\tau \\
&= \int_{t_n}^{t_n + \alpha_n} f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) d\tau - \alpha_n \cdot f^o(\tilde{x}_n) \\
&= \varepsilon \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(s, \tilde{x}_n) ds - \alpha_n \cdot f^o(\tilde{x}_n), \quad \text{where } s = \frac{\tau}{\varepsilon} \\
&= \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(s, \tilde{x}_n) ds - f^o(\tilde{x}_n) \right) \\
&= \alpha_n \cdot \beta_n \quad \text{with } \beta_n \simeq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau - r)) d\tau &\simeq \sum_{n=0}^{N-1} \alpha_n \cdot \beta_n \\
&\simeq \beta \sum_{n=0}^{N-1} \alpha_n, \quad \text{where } \beta = \max\{\beta_n\} \\
&= \beta \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \beta \cdot t_N.
\end{aligned}$$

By Lemma A.4.2 (Appendix A), $\beta \simeq 0$ and then $\beta \cdot t_N \simeq 0$. This implies the lemma. \square

Lemma 2.4.5 *Let $L_1 > 0$ be standard and let x be a solution of (2.1) with initial value ϕ at 0. Assume that $[0, L_1] \subset I$ and $x(t)$ is nearstandard for all $t \in [0, L_1]$. Then x is S -continuous on $[0, L_1]$ and its shadow on $[0, L_1]$ coincides with the solution $y = y(\cdot; 0, \phi)$ of (2.2) on this interval, and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L_1]$.*

Proof. The solution x is given as

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau, \quad \text{for } t \in [0, L_1]. \quad (2.12)$$

As f is bounded on $\mathbf{R}_+ \times U$, it is clear that x is S -continuous on $[0, L_1]$.

Let \tilde{x} be a function which satisfies hypotheses of Lemma 2.4.4 and such that

$$x(t) \simeq \tilde{x}(t), \quad \forall t \in [-r, L_1]. \quad (2.13)$$

Consider now the following equality which is always true

$$\begin{aligned}
& \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau - \int_0^t f^o(x(\tau - r)) d\tau \\
&= \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) - f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) \right) d\tau \\
&+ \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r)) \right) d\tau \\
&+ \int_0^t (f^o(\tilde{x}(\tau - r)) - f^o(x(\tau - r))) d\tau.
\end{aligned} \tag{2.14}$$

As $x(t)$ is nearstandard in U for any $t \in [-r, L_1]$, by the continuity of f , the condition (H1') and (2.13), we have

$$f\left(\frac{t}{\varepsilon}, x(t - r)\right) \simeq f\left(\frac{t}{\varepsilon}, \tilde{x}(t - r)\right)$$

and by Lemma A.4.3 (Appendix A),

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau \simeq \int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau.$$

On the other hand, by Lemma 2.4.1 (the continuity of f^o) and by (2.13), we have

$$f^o(\tilde{x}(t - r)) \simeq f^o(x(t - r))$$

and by Lemma A.4.3 (Appendix A), we obtain

$$\int_0^t f^o(\tilde{x}(\tau - r)) d\tau \simeq \int_0^t f^o(x(\tau - r)) d\tau.$$

By Lemma 2.4.4, for any $t \in [0, L_1]$, we have

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau \simeq \int_0^t f^o(\tilde{x}(\tau - r)) d\tau.$$

Hence, for $t \in [0, L_1]$, (2.14) implies that

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau \simeq \int_0^t f^o(x(\tau - r)) d\tau. \tag{2.15}$$

Using (2.12) and (2.15), we obtain

$$\begin{aligned}
x(t) &= \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau \\
&\simeq \phi(0) + \int_0^t f^o(x(\tau - r)) d\tau.
\end{aligned}$$

Let ${}^o x$ be the shadow of x on $[0, L_1]$. It is easy to see that the function z such that

$$z(t) = \begin{cases} {}^o x(t), & \text{for } t \in [0, L_1] \\ \phi(t), & \text{for } t \in [-r, 0] \end{cases}$$

is a solution of (2.2). The hypothesis (H3) insure that $z = y$ on $[-r, L_1]$. Hence, we have $x(t) \simeq y(t)$ for $t \in [0, L_1]$. \square

2.4.2 Proof of Theorem 2.3.1

Let $L > 0$ be standard in J . Since $\Gamma = y([0, L])$ is a standard compact subset of U , there exists $\rho > 0$, ρ standard, and K , a standard compact neighborhood of Γ included in U , such that $\text{dist}(\Gamma, K) = \inf\{\|y - z\| / y \in \Gamma, z \in \mathbb{R}^n - K\} > \rho$. Let $x : I \rightarrow U$ be a solution of (2.1) with initial value ϕ at 0. Define the set $A = \{L_1 \in I \cap [0, L] / x([0, L_1]) \subset K\}$. A is non empty ($0 \in A$) and bounded above by L . Let L_0 be a lower upper bound of A . There is $L_1 \in A$ such that $L_0 - \varepsilon < L_1 \leq L_0$. Thus $x([0, L_1]) \subset K$. Hence on $[0, L_1]$ the function x is nearstandard in U . By Lemma 2.4.5, we have $x(t) \simeq y(t)$ for $t \in [0, L_1]$. Consider the interval $[0, L_1 + \varepsilon]$. Let $t \in [0, L_1 + \varepsilon]$. As $t - r$ is in $[-r, L_1 + \varepsilon - r] \subset [-r, L_1]$, $x(t - r)$ is defined and so is

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau - r)\right) d\tau.$$

On the other hand, we have, for $t \in [L_1, L_1 + \varepsilon]$, $x(t) \simeq x(L_1) \simeq y(L_1)$ with $y(L_1)$ nearstandard in U . Hence, on $[L_1, L_1 + \varepsilon]$, x is nearstandard in U . Thus, on $[0, L_1 + \varepsilon]$, x is defined and nearstandard in U . By Lemma 2.4.5, we have $x(t) \simeq y(t)$ for $t \in [0, L_1 + \varepsilon]$. Hence $[0, L_1 + \varepsilon] \subset I$ and $x([0, L_1 + \varepsilon]) \subset K$. Suppose $L_1 + \varepsilon \leq L$, then $L_1 + \varepsilon \in A$ which is a contradiction. Thus $L_1 + \varepsilon > L$, that is, we have $x(t) \simeq y(t)$ for all $t \in [0, L] \subset [0, L_1 + \varepsilon]$. \square

2.4.3 Proof of Theorem 2.3.2

By condition (H5) and the uniform attractivity of y_e (see Lemma 2.3.1, 2.), the solution y is defined for all $t > 0$ and satisfies $y(t) \simeq y_e$ for all $t \simeq +\infty$. Let $x : I \rightarrow U$ be a solution of (2.1) with initial value ϕ at 0. By Theorem 2.3.1, for all limited $L > 0$, x is defined on $[0, L]$ and the approximation $x(t) \simeq y(t)$ holds for all $t \in [0, L]$. By Robinson's Lemma (Appendix A, section A.4, Lemma A.4.1), there exists $t_1 \simeq +\infty$ such that $x(t) \simeq y(t)$ on $[0, t_1]$. And then we have

$$x(t) \simeq y(t) \simeq y_e, \quad \forall t \leq t_1, \quad t \simeq +\infty. \quad (2.16)$$

It remains to prove that x is defined for all $t \geq t_1$ and satisfies $x(t) \simeq y(t)$ for all $t > t_1$. Assume that this is false. Then there exists $s > t_1$ such that $x(s) \not\simeq y(s)$, that is

$$\|x(s) - y(s)\| = \frac{2}{3}\kappa \quad (2.17)$$

is appreciable. Since $y(t) \simeq y_e$ for all $t \simeq +\infty$, we have

$$\|x(s) - y_e\| \leq \|x(s) - y(s)\| + \|y(s) - y_e\| \leq \frac{2}{3}\kappa + \frac{\kappa}{3} = \kappa. \quad (2.18)$$

Let \mathcal{B} , the ball of center y_e and radius $b_0 > 0$, b_0 standard, be the basin of attraction of y_e . We can choose s in (2.18) so that the ball \mathcal{B}' of center y_e and radius κ is

included in \mathcal{B} , with $b_0 - \kappa$ appreciable. Let t_2 be the first instant in time such that equality (2.17) holds. Clearly $t_2 > t_1$.

Case 1. $t_{1,2} = t_2 - t_1 \simeq +\infty$.

Redefine in (2.2) initial time $r = t_0$. Let $z_1(\cdot; r, x, t_2)$ denote the solution of (2.2) such that $z_1(t; r, x, t_2) = x(t_2 - t)$ for $t \in [0, r]$. By Theorem 2.3.1 and Remark 2.3.1, for all limited $L > 0$, $z_1(\cdot; r, x, t_2)$ is defined on $[r, r + L]$ and satisfies $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ for $t \in [r, r + L]$. By Robinson's Lemma (Appendix A, section A.4, Lemma A.4.1), there exists $\bar{t}_{1,2} \simeq +\infty$, which one can choose such that $r + \bar{t}_{1,2} \leq t_{1,2}$, with the property that $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ on $[r, r + \bar{t}_{1,2}]$. In particular, $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ on $[\bar{t}_{1,2}, r + \bar{t}_{1,2}] \subset [0, t_{1,2}]$. Since $x(t)$ belongs to \mathcal{B}' for all $t \in [t_1, t_2]$, $x(t_2 - t)$ lies in \mathcal{B}' for all $t \in [0, t_{1,2}]$. This implies that $z_1(t; r, x, t_2)$ is in \mathcal{B} for all $t \in [\bar{t}_{1,2}, r + \bar{t}_{1,2}]$. By the uniform attractivity of y_e (see Lemma 2.3.3), through the transformation $t \mapsto -t$, the solution of (2.2) with initial value $z_1(-t; r, x, t_2)$ for $t \in [-r - \bar{t}_{1,2}, -\bar{t}_{1,2}]$ which coincides with $z_1(\cdot; r, x, t_2)$ (by uniqueness; hypothesis (H3)) is defined for all $t > -\bar{t}_{1,2}$ and satisfies $z_1(-t; r, x, t_2) \simeq y_e$ for $t + \bar{t}_{1,2} \simeq +\infty$. Take $t = 0$, then $x(t_2) \simeq z_1(0; r, x, t_2) \simeq y_e$. Since $y(t_2) \simeq y_e$, this implies that $x(t_2) \simeq y(t_2)$. Which is a contradiction with $\|x(t_2) - y(t_2)\|$ being appreciable.

Case 2. $t_{1,2} = t_2 - t_1$ is limited.

By the continuity of the function $\|x(t) - y(t)\|$, there exists at least $t \in (t_1, t_2)$ such that

$$\|x(t) - y(t)\| = \frac{\kappa}{2}. \quad (2.19)$$

Let t_3 and t_4 , respectively, the first and the last instants in time such that equality (2.19) holds. We have $t_1 < t_3 \leq t_4 < t_2$. It is clear that

$$\|x(t) - y(t)\| < \frac{\kappa}{2}, \quad \forall t \in [t_1, t_3].$$

It is also clear that

$$0 < \frac{\kappa}{2} \leq \|x(t) - y(t)\| \leq \frac{2}{3}\kappa, \quad \forall t \in [t_4, t_2]. \quad (2.20)$$

Redefine in (2.2) initial time $t_1 = t_0$. Let $z(\cdot; t_1, x)$ denote the solution of (2.2) such that $z(t; t_1, x) = x(t)$ for $t \in [t_1 - r, t_1]$. By (2.16) we have

$$z(t; t_1, x) = x(t) \simeq y(t) \simeq y_e, \quad \text{for } t \in [t_1 - r, t_1]. \quad (2.21)$$

Since $t_4 - t_1$ is limited, according to Theorem 2.3.1 and Remark 2.3.1

$$z(t; t_1, x) \simeq x(t), \quad \text{on } [t_1, t_4 + L] = [t_1, t_1 + ((t_4 - t_1) + L)], \quad \forall \text{ limited } L > 0.$$

By Robinson's Lemma (Appendix A, section A.4, Lemma A.4.1), there exists $\omega \simeq +\infty$ such that $z(t; t_1, x) \simeq x(t)$ on $[t_1, t_4 + \omega]$. Thus we have

$$z(t; t_1, x) \simeq x(t), \quad \text{on } [t_1, t_5], \quad \text{where } t_5 = t_4 + \omega. \quad (2.22)$$

By (2.21) and the uniform stability of y_e (see Lemma 2.3.1, 1.) we deduce that

$$z(t; t_1, x) \simeq y(t) (\simeq y_e), \quad \forall t \geq t_1. \quad (2.23)$$

Thus, by (2.22) and (2.23)

$$x(t) \simeq y(t), \quad \forall t \in [t_1, t_5]. \quad (2.24)$$

Therefore $t_5 < t_2$ since $\|x(t_2) - y(t_2)\|$ is appreciable.

Take $t = t_5$. By (2.24) we have $x(t_5) \simeq y(t_5)$. This contradicts (2.20) since $t_5 \in [t_4, t_2]$. \square

2.5 Bibliography

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Chapter 3

Averaging method for functional differential equations

This chapter essentially contains the paper [10] “Averaging method for functional differential equations”, (Submitted for publication).

The method of averaging is extended, in a natural way, to functional differential equations. The first result given here generalizes the corresponding one in [Chap.2, Theorem 2.2.1].

3.1 Introduction

There is a rich literature on the method of averaging for ordinary differential equations (cf. [1, 2, 6, 8, 11, 13, 14] and the references cited therein). The method is also extended to functional differential equations [3, 4, 5, 12, 15, 16] of the form

$$\dot{x}(t) = \varepsilon f(t, x_t) \quad (3.1)$$

where $\varepsilon > 0$ is a small parameter. Under suitable conditions, solutions of (3.1) can be approximated by those ones of the averaged ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(\tilde{y}), \quad \tilde{y}(\theta) = y \quad \text{for } \theta \in [-r, 0] \quad (3.2)$$

where

$$f^o(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau. \quad (3.3)$$

Notice that, if we let $t \mapsto t/\varepsilon$ and $x(t/\varepsilon) = z(t)$, equation (3.1) becomes

$$\dot{z}(t) = f\left(\frac{t}{\varepsilon}, z_{t,\varepsilon}\right) \quad (3.4)$$

with $z_{t,\varepsilon}(\theta) = z(t + \varepsilon\theta)$, $\theta \in [-r, 0]$, which is an equation with a small delay.

The purpose of this chapter is to consider a functional differential equation in the general case, that is

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right) \quad (3.5)$$

and to show that solutions of (3.5) may be approximated by those ones of the averaged equation

$$\dot{y}(t) = f^o(y_t) \quad (3.6)$$

where f^o is given in (3.3). Notice that (3.6) is a functional differential equation and not an ordinary differential equation.

Among authors who studied equation (3.5), we will cite Hale and Verduyn Lunel. In [7], the cited authors introduce an extension of the method of averaging to abstract evolutionary equations in Banach spaces. In particular, they rewrite a functional differential equation as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there. Our approach is different since all the analysis is kept in the associated natural phase space.

This chapter is organized as follows. In Section 3.2, we state closeness of solutions of the original and averaged equations on finite time intervals (Theorem 3.2.1). This result generalizes the corresponding one of [9] and then its proof is directly related to [9] (and [14]). We also investigate the long time behaviour of solutions of the original equation (Theorem 3.2.2). This is done under the assumption that the averaged equation has an exponentially stable equilibrium. For this case, the idea of the proof is the same one used for ordinary differential equations in [13]. In Section 3.3 we present the nonstandard translates (Theorems 3.3.1 and 3.3.2) in the language of *Internal Set Theory* (see Appendix A) of Theorem 3.2.1 and 3.2.2. Finally, in Section 3.4 we first begin with some preliminary lemmas and then give the proofs of Theorems 3.3.1 and 3.3.2.

3.2 Main Results

In this section we state hypotheses and present the main results on averaging for functional differential equations.

First, let us assume that the following conditions are satisfied:

- (H1) The functional f is continuous and bounded on $\mathbf{R}_+ \times \mathcal{C}_o$.
- (H2) The continuity of $f = f(\tau, u)$ in $u \in \mathcal{C}_o$ is uniform with respect to $\tau \in \mathbf{R}_+$.
- (H3) For all $u \in \mathcal{C}_o$ there exists a limit

$$f^o(u) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

- (H4) The averaged equation (3.6) has the uniqueness of the solutions with the prescribed initial conditions.

Remark 3.2.1 *In hypothesis (H4) we anticipate the existence of solutions of (3.6). We will justify this a posteriori. Eendeed, in Lemma 3.4.1 below we will show that f° is continuous so that the existence is guaranteed.*

Under the above assumptions, we will prove a theorem on nearness of the solutions of (3.5) and (3.6) with the same initial conditions.

Theorem 3.2.1 (Averaging on Finite Time Intervals) *Let the hypotheses (H1) to (H4) hold. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbf{R}$. Let $y = y(\cdot; t_0, \phi)$ be the solution of (3.6), and let $J = [t_0 - r, \omega)$, $t_0 < \omega \leq \infty$, be its maximal interval of definition. Then for any $L > t_0$, $L \in J$, and any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, any solution x of (3.5) with initial value ϕ at t_0 is defined at least on $[t_0, L]$ and the inequality $\|x(t) - y(t)\| < \delta$ holds for $t \in [t_0, L]$.*

One can also extend the validity of the averaging technique for all (future) time when the solution of (3.6) lies in the domain of exponential stability of an exponentially stable equilibrium. For this, let us first recall the concept of *exponential stability* of equilibriums.

With the assumption (H4), we suppose that y_e is an equilibrium of (3.6), that is, $f^\circ(y_e) = 0$.

Definition 3.2.1 *y_e is said to be exponentially stable if there exist b, K and $\lambda > 0$ such that, for all $t_0 \in \mathbf{R}$ and all $\phi \in \mathcal{C}_0$, the solution $y = y(\cdot; t_0, \phi)$ of (3.6) for which $|\phi - y_e| < b$, is defined on $[t_0, \infty)$ and the inequality $\|y(t) - y_e\| \leq K e^{-\lambda(t-t_0)} |\phi - y_e|$ holds for all $t \geq t_0$.*

Remark 3.2.2 *The ball \mathcal{B} of center y_e and radius b where the stability is exponential will be called the domain of exponential stability of y_e .*

As a next result of this section, we will prove validity of the approximation of the solutions of (3.5) and (3.6) with the same initial conditions, for all time.

Theorem 3.2.2 (Averaging for All Time) *Let the hypotheses (H1) to (H4) be true. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbf{R}$. Let $y = y(\cdot, t_0, \phi)$ be the solution of (3.6). Assume that*

(H5) *y_e is exponentially stable.*

(H6) *ϕ lies in the domain of exponential stability of y_e .*

Then for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, any solution x of (3.5) with initial value ϕ at t_0 is defined on $[t_0, \infty)$ and the inequality $\|x(t) - y(t)\| < \delta$ holds for all $t \geq t_0$.

3.3 Nonstandard Averaging Results

First we give nonstandard formulations of Theorem 3.2.1 and Theorem 3.2.2. Then, by use of the reduction algorithm, we show that the reduction of Theorem 3.3.1 and Theorem 3.3.2 bellow are Theorem 3.2.1 and Theorem 3.2.2 respectively.

Theorem 3.3.1 *Let $f : \mathbf{R} \times \mathcal{C}_o \rightarrow \mathbf{R}^n$ be standard. Assume that all hypotheses in Theorem 3.2.1 hold. Let $\phi \in \mathcal{C}_o$ and $t_0 \in \mathbf{R}$ be standard. Let $y = y(., t_0, \phi)$ be the solution of (3.6), and let $J = [t_0 - r, \omega)$, $t_0 < \omega \leq \infty$, be its maximal interval of definition. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > t_0$, $L \in J$, any solution x of (3.5) with initial value ϕ at t_0 is defined at least on $[t_0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [t_0, L]$.*

Theorem 3.3.2 *Let $f : \mathbf{R} \times \mathcal{C}_o \rightarrow \mathbf{R}^n$ be standard. Let y_e be a standard equilibrium of (3.6). Assume that all hypotheses in Theorem 3.2.2 hold. Let $\phi \in \mathcal{C}_o$ and $t_0 \in \mathbf{R}$ be standard. Let $y = y(., t_0, \phi)$ be the solution of (3.6). Let $\varepsilon > 0$ be infinitesimal. Then any solution x of (3.5) with initial value ϕ at t_0 is defined on $[t_0, \infty)$ and satisfies $x(t) \simeq y(t)$ for all $t \geq t_0$.*

The proofs of Theorems 3.3.1 and 3.3.2 are postponed to Section 3.4 below. The reduction of Theorems 3.3.1 and 3.3.2 to Theorems 3.2.1 and 3.2.2, respectively, are the same ones as in [Chap.2, subsection 2.3.1].

3.4 Proofs of Theorems 3.3.1 and 3.3.2

3.4.1 Preliminary Lemmas

First Part: Hereafter we are giving some results we need for the proof of Theorem 3.3.1. We assume that $f : \mathbf{R} \times \mathcal{C}_o \rightarrow \mathbf{R}^n$ is standard. Let us give external formulations of conditions (H1), (H2) and (H3) respectively:

$$(H1') \quad \forall^{st} \tau \geq 0 \quad \forall^{st} u \in \mathcal{C}_o \quad \forall \tau' \geq 0 \quad \forall u' \in \mathcal{C}_o:$$

$$\tau' \simeq \tau \text{ and } u' \simeq u \implies f(\tau', u') \simeq f(\tau, u).$$

And, there exists some standard constant M such that

$$\forall^{st} \tau \geq 0 \quad \forall^{st} u \in \mathcal{C}_o : \|f(\tau, u)\| \leq M$$

(and by transfer (Appendix A, subsection A.2.2) the inequality holds for all $\tau \geq 0$ and $u \in \mathcal{C}_o$).

$$(H2') \quad \forall^{st} u \in \mathcal{C}_o \quad \forall u' \in \mathcal{C}_o \quad \forall \tau \geq 0 : \quad u' \simeq u \implies f(\tau, u') \simeq f(\tau, u).$$

$$(H3') \quad \text{There is a standard functional } f^o : \mathcal{C}_o \rightarrow \mathbf{R}^n \text{ such that}$$

$$\forall^{st} u \in \mathcal{C}_o \quad \forall T \simeq +\infty : \quad f^o(u) \simeq \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

Lemma 3.4.1 *The functional f° is continuous and satisfies*

$$f^\circ(u) \simeq \frac{1}{T} \int_0^T f(\tau, u) d\tau$$

for all $u \in \mathcal{C}_o$, u nearstandard, and all $T \simeq +\infty$.

Proof. The proof of Lemma 3.4.1 follows almost verbatim the proof in [Chap.2, Lemma 2.4.1, page 20]. \square

Lemma 3.4.2 *There exists $\mu > 0$ such that whenever $t \geq 0$ is limited and $u \in \mathcal{C}_o$ is nearstandard there exists $\alpha > 0$ such that $\mu < \alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, u) d\tau \simeq f^\circ(u).$$

Proof. The proof of Lemma 3.4.2 follows almost verbatim the proof in [Chap.2, Lemma 2.4.2, page 20]. \square

Lemma 3.4.3 *Let $\phi \in \mathcal{C}_o$ be standard. Let x be a solution of (3.5) with initial value ϕ at 0, let $I = [-r, b)$ be its maximal interval of definition, and let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then x is S-continuous and nearstandard on $[0, L_1]$, and there exist some positive integer N_o and some infinitesimal partition $\{t_n : n = 0, \dots, N_o+1\}$ of $[0, L_1]$ such that $t_0 = 0$, $t_{N_o} < L_1 \leq t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and*

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau \simeq f^\circ(x_{t_n}).$$

Proof. First, as f is bounded on $\mathbf{R}_+ \times \mathcal{C}_o$, it follows that x is S-continuous on $[0, L_1]$. Eendeed, if $t \simeq t'$, with $t, t' \in [0, L_1]$ then $\|x(t) - x(t')\| \leq M|t - t'| \simeq 0$. Furthermore, for $t \in [0, L_1]$, we have

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau$$

where $\phi(0)$ is limited. This implies that x is nearstandard on $[0, L_1]$.

Next, define the set $A = \{\lambda \in \mathbf{R} / \forall t \in [0, L_1] \exists \alpha \in \mathbf{R} : \mathcal{P}(t, \alpha, \lambda)\}$ where

$$\mathcal{P}(t, \alpha, \lambda) \equiv \mu < \alpha < \lambda \quad \text{and} \quad \left\| \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x_t) d\tau - f^\circ(x_t) \right\| < \lambda.$$

By Lemma 3.4.2 the set A contains all the standard real numbers $\lambda > 0$. By Lemma A.4.4 (Appendix A) there exists $\lambda_0 \simeq 0$ in A , that is, there exists $0 < \lambda_0 \simeq 0$ such that for all $t \in [0, L_1]$ there exists $\alpha \in \mathbf{R}$ such that $\mathcal{P}(t, \alpha, \lambda_0)$ holds. By the axiom of choice there exists a function $c : [0, L_1] \rightarrow \mathbf{R}$ such that $c(t) = \alpha$, that is, $\mathcal{P}(t, c(t), \lambda_0)$ holds for all $t \in [0, L_1]$. Since $c(t) > \mu$ for all $t \in [0, L_1]$, the conclusion of the lemma is immediate. \square

Lemma 3.4.4 *Let $\phi \in \mathcal{C}_o$ be standard. Let x be a solution of (3.5) with initial value ϕ at 0, let $I = [-r, b)$ be its maximal interval of definition, and let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then for all $t \in [0, L_1]$*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau \simeq \int_0^t f^o(x_\tau) d\tau.$$

Proof. By Lemma 3.4.3 there exists $\{t_n : n = 0, \dots, N_o + 1\}$ such that $t_0 = 0$, $t_{N_o} < L_1 \leq t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau \simeq f^o(x_{t_n}). \quad (3.7)$$

Let $t \in [0, L_1]$, and let N be a positive integer such that $t_N < t \leq t_{N+1}$. We have

$$\begin{aligned} \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau) d\tau &= \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau \\ &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau \\ &\quad + \int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau \\ &\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau \end{aligned} \quad (3.8)$$

since

$$\begin{aligned} \left\| \int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau \right\| &\leq \int_{t_N}^t \left(\left\| f\left(\frac{\tau}{\varepsilon}, x_\tau\right) \right\| + \left\| f^o(x_\tau) \right\| \right) d\tau \\ &\leq 2M(t - t_N) \leq 2M(t_{N+1} - t_N) \leq 2M\alpha \simeq 0 \end{aligned}$$

where $\alpha = \max\{\alpha_n\} \simeq 0$ (see Appendix A, Lemma A.4.2) and M is a standard bound for f (condition (H1')) and then for f^o too.

By Lemma 3.4.3 we have $x_\tau \simeq x_{t_n}$ for $\tau \in [t_n, t_{n+1}]$ and x_{t_n} is nearstandard, so that by the condition (H2') and Lemma 3.4.1 (the continuity of f^o) it follows respectively that

$$f\left(\frac{\tau}{\varepsilon}, x_\tau\right) = f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) + \gamma_n(\tau)$$

and

$$f^o(x_\tau) = f^o(x_{t_n}) + \delta_n(\tau)$$

with $\gamma_n(\tau) \simeq 0 \simeq \delta_n(\tau)$. Hence, from (3.8) it follows that

$$\begin{aligned} \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau) d\tau &\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) + \eta_n(\tau) \right) d\tau \\ &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) \right) d\tau \\ &\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau) d\tau \end{aligned}$$

where $\eta_n(\tau) = \gamma_n(\tau) + \delta_n(\tau)$, and therefore

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau) d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) \right) d\tau$$

since

$$\left| \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau) d\tau \right| \leq \bar{\eta} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} d\tau = \bar{\eta} \cdot t_N,$$

where $\bar{\eta} = \sup\{\eta_n : 0 \leq n \leq N-1\}$ and $\eta_n = \sup\{\|\eta_n(\tau)\| : t_n \leq \tau \leq t_{n+1}\}$. By Lemma A.4.2 (Appendix A), $\bar{\eta}$ is infinitesimal and so is $\bar{\eta} \cdot t_N$.

Let $0 \leq n \leq N-1$. By means of (3.7) we have

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) \right) d\tau &= \int_{t_n}^{t_n + \alpha_n} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) \right) d\tau \\ &= \int_{t_n}^{t_n + \alpha_n} f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) d\tau - \alpha_n \cdot f^o(x_{t_n}) \\ &= \varepsilon \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau - \alpha_n \cdot f^o(x_{t_n}) \\ &= \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau - f^o(x_{t_n}) \right) \\ &= \alpha_n \cdot \beta_n \quad \text{with } \beta_n \simeq 0. \end{aligned}$$

Therefore

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau) d\tau \simeq \sum_{n=0}^{N-1} \alpha_n \cdot \beta_n \simeq 0$$

since $\left| \sum_{n=0}^{N-1} \alpha_n \cdot \beta_n \right| \leq \bar{\beta} \sum_{n=0}^{N-1} \alpha_n = \bar{\beta} \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \bar{\beta} \cdot t_N$, where $\bar{\beta} = \max\{\|\beta_n\| : 0 \leq n \leq N-1\}$. By Lemma A.4.2 (Appendix A), $\bar{\beta}$ is infinitesimal and so is $\bar{\beta} \cdot t_N$. This completes the proof of Lemma 3.4.4. \square

Lemma 3.4.5 *Let $\phi \in \mathcal{C}_o$ be standard. Let x be a solution of (3.5) with initial value ϕ at 0, let $I = [-r, b)$ be its maximal interval of definition, and let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then the shadow of x on $[0, L_1]$ coincides with the solution $y = y(\cdot; 0, \phi)$ of (3.6) on this interval so that $x(t) \simeq y(t)$ for all $t \in [0, L_1]$.*

Proof. First, by Lemma 3.4.3, x is S-continuous and nearstandard on $[0, L_1]$.

Next, by means of Lemma 3.4.4, for $t \in [0, L_1]$, we have

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau \simeq \phi(0) + \int_0^t f^o(x_\tau) d\tau.$$

If ${}^o x$ is the shadow of x on $[0, L_1]$, it is not difficult to verify that the function z defined as

$$z(t) = \begin{cases} {}^o x(t), & \text{for } t \in [0, L_1] \\ \phi(t), & \text{for } t \in [-r, 0] \end{cases}$$

is a solution of (3.6). By hypothesis (H4) we have $z \equiv y$ on $[-r, L_1]$ so that $x(t) \simeq {}^o x(t) = z(t) = y(t)$ for $t \in [0, L_1]$. \square

Second Part: In this part, we are giving in Lemma 3.4.6 below, the external formulation of an equilibrium exponential stability definition. This result is needed for the proof of Theorem 3.3.2.

Lemma 3.4.6 *The equilibrium y_e of (3.6) is exponentially stable if and only if it admits a standard domain of exponential stability, that is, there exist standard b, K and $\lambda > 0$ such that, for all standard $t_0 \in \mathbf{R}$ and all standard $\phi \in \mathcal{C}_o$, the solution $y = y(\cdot; t_0, \phi)$ of (3.6) for which $|\phi - y_e| < b$, is defined on $[t_0, \infty)$ and the inequality $\|y(t) - y_e\| \leq Ke^{-\lambda(t-t_0)}|\phi - y_e|$ holds for all $t \geq t_0$.*

Proof. The conclusion of the lemma is obtained by successive use of transfer principle (Appendix A, subsection A.2.2). \square

3.4.2 Proof of Theorem 3.3.1

For notation simplicity, let $t_0 = 0$. Let $L > 0$ be standard in J . Let K be a standard tubular neighborhood of diameter ρ around $\Gamma = y([0, L])$. Let I be the maximal interval of definition of x . Define the set $A = \{L_1 \in I \cap [0, L] / x([0, L_1]) \subset K\}$. A is non empty ($0 \in A$) and bounded above by L . Let L_0 be a lower upper bound of A . There is $L_1 \in A$ such that $L_0 - \varepsilon^2 < L_1 \leq L_0$. By continuation, there is L_2, L_2 appreciable, such that x remains defined on $[0, L_1 + \varepsilon L_2]$. Likewise, by continuation, y remains defined in particular on the same interval. By Lemma 3.4.5, we have $x(t) \simeq y(t)$ for $t \in [0, L_1 + \varepsilon L_2]$. Suppose $L_1 + \varepsilon L_2 \leq L$. Then, $[0, L_1 + \varepsilon L_2] \subset I$ and $x([0, L_1 + \varepsilon L_2]) \subset K$, implice that $L_1 + \varepsilon L_2 \in A$, which is a contradiction. Thus $L_1 + \varepsilon L_2 > L$, that is, we have $x(t) \simeq y(t)$ for all $t \in [0, L] \subset [0, L_1 + \varepsilon L_2]$. \square

3.4.3 Proof of Theorem 3.3.2

Let $t_0 = 0$. On $[-r, 0]$ we have $x(t) = y(t) = \phi(t)$ and therefore the conclusion of the theorem holds. By Theorem 3.3.1, the approximation $x(t) \simeq y(t)$ is satisfied for all $t \in [0, L]$, $L > t_0$, L standard. Let $t_1 > t_0$, t_1 standard. t_1 will be chosen convenably later.

Now, for $n = 0, 1, 2, \dots$, let $I_n = [nt_1, (n+1)t_1]$. The collection $\{I_n\}_{n \geq 0}$ is a partition of the positive time axis so that $\mathbf{R}_+ = [0, \infty) = \bigcup_{n \geq 0} I_n$. On each interval

I_n , $n \geq 1$, we define y_n as the solution of (3.6) with initial function $y_n(t) = x(t)$ for $t \in [nt_1 - r, nt_1]$. By Theorem 3.3.1, the approximation $x(t) \simeq y_n(t)$ holds for all $t \in I_n$. From the definition of exponential stability and its properties we have, for

$t \geq nt_1$

$$\|y(t) - y_n(t)\| \leq Ke^{-\lambda(t-nt_1)} \sup_{s \in [nt_1-r, nt_1]} \|y(s) - y_n(s)\| \quad (3.9)$$

where K and λ are positive and standard.

Using the triangle inequality, we have, for $s \in [nt_1 - r, nt_1]$

$$\|y(s) - y_n(s)\| \leq \|y(s) - y_{n-1}(s)\| + \|y_n(s) - y_{n-1}(s)\|. \quad (3.10)$$

However, by Theorem 3.3.1, we have $y_n(s) = x(s) \simeq y_{n-1}(s)$ for all $s \in [nt_1 - r, nt_1]$ and then, by Lemma A.4.2,

$$\max_{n \geq 0} \sup_{s \in [nt_1-r, nt_1]} \|y_n(s) - y_{n-1}(s)\| \leq \alpha \simeq 0.$$

Take $t_1 \geq r$. From (3.9) and (3.10) it follows that, for $t \geq nt_1$

$$\|y(t) - y_n(t)\| \leq Ke^{-\lambda(t-nt_1)} \left(\sup_{s \in [nt_1-r, nt_1]} \|y(s) - y_{n-1}(s)\| + \alpha \right) \quad (3.11)$$

so that

$$\begin{aligned} \sup_{s \in [(n+1)t_1-r, (n+1)t_1]} \|y(s) - y_n(s)\| &\leq K \sup_{s \in [(n+1)t_1-r, (n+1)t_1]} e^{-\lambda(s-nt_1)} \times \\ &\quad \times \left(\sup_{s \in [nt_1-r, nt_1]} \|y(s) - y_{n-1}(s)\| + \alpha \right) \\ &= Ke^{-\lambda(t_1-r)} \left(\sup_{s \in [nt_1-r, nt_1]} \|y(s) - y_{n-1}(s)\| + \alpha \right), \end{aligned}$$

or equivalently

$$|y - y_n|_n \leq Ke^{-\lambda(t_1-r)} (|y - y_{n-1}|_{n-1} + \alpha), \quad n = 1, 2, \dots$$

where

$$|y - y_n|_n := \sup_{s \in [(n+1)t_1-r, (n+1)t_1]} \|y(s) - y_n(s)\|.$$

Suppose $K > 1$ and choose t_1 such that $Ke^{-\lambda(t_1-r)} < 1$. Since $|y - y_0|_0 = 0$ we deduce that

$$|y - y_n|_n \leq \frac{Ke^{-\lambda(t_1-r)}}{1 - Ke^{-\lambda(t_1-r)}} \alpha.$$

Return now to inequality (3.11). For $t \in I_n$, $n \geq 0$, we have

$$\begin{aligned} \|y(t) - y_n(t)\| &\leq Ke^{-\lambda(t-nt_1)} \left(\frac{Ke^{-\lambda(t_1-r)}}{1 - Ke^{-\lambda(t_1-r)}} + 1 \right) \alpha \\ &\leq \frac{K\alpha}{1 - Ke^{-\lambda(t_1-r)}}. \end{aligned}$$

That is, $y(t) \simeq y_n(t)$ on I_n .

Thus, for $t \in I_n$

$$x(t) \simeq y_n(t) \quad \text{and} \quad y(t) \simeq y_n(t) \quad \implies \quad x(t) \simeq y(t).$$

As n is chosen arbitrarily, this completes the proof. \square

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Chapter 4

Time averaging for functional differential equations

This chapter essentially contains the paper [15] “ Time averaging for functional differential equations”, (Submitted for publication).

Comparatively to the result on the method of averaging for functional differential equations given in [Chap.3, Theorem 3.2.1, page 33], here we give an alternate result under different conditions.

4.1 Introduction

The method of averaging is an important tool in the analysis of nonautonomous ordinary differential equations, containing high frequency time oscillations (cf. [2, 3, 6, 11, 13, 17, 18]). The method was extended to many other problems, like ordinary differential equations in Banach spaces (cf. [7]), functional differential equations (cf. [4, 5, 8, 9, 12, 14, 16]), parabolic partial differential equations (cf. [1, 10, 19]), and so forth.

In the present chapter, we will give a basic theorem concerning the method of averaging for functional differential equations of the form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right). \quad (4.1)$$

Section 4.2 contains the conditions required to state and prove our main result as well as the main result itself (Theorem 4.2.1) and its nonstandard translate (Theorem 4.2.2) in the language of *Internal Set Theory* (see Appendix A). The proof of this result is given in Subsection 4.3.2. To simplify the proof, several subsidiary lemmas have been placed in Subsection 4.3.1.

4.2 Conditions and Main Result

The following conditions will be assumed to be true:

- (H1) The functional f is continuous on $\mathbf{R}_+ \times \mathcal{C}_o$.
- (H2) The functional f is Lipschitz, that is, there exists some constant k such that

$$\|f(\tau, u_1) - f(\tau, u_2)\| \leq k|u_1 - u_2|, \quad \text{for all } \tau \geq 0 \text{ and } u_1, u_2 \in \mathcal{C}_o.$$

- (H3) For all $u \in \mathcal{C}_o$ there exists a limit

$$f^o(u) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

Consider the averaged equation

$$\dot{y}(t) = f^o(y_t). \tag{4.2}$$

Under the above assumptions, we will state the main result of this chapter which gives nearness of the solutions x and y of (4.1) and (4.2), respectively, with the same initial conditions.

Theorem 4.2.1 *Let the assumptions (H1), (H2) and (H3) hold true. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbf{R}$. Let $x = x(\cdot; t_0, \phi)$ be the solution of (4.1), let $y = y(\cdot; t_0, \phi)$ be the solution of (4.2), and let J be the maximal interval of definition of y . Then for any $T > t_0$, $T \in J$, and any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(T, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, x is defined at least on $[t_0, T]$ and satisfies $\|x(t) - y(t)\| < \delta$ on $t \in [t_0, T]$.*

Remark 1 *The existence and uniqueness of the solutions of (4.2) will be justified in Lemma 4.3.1 below. Eneed, there we will show that f^o is k -Lipschitz so that existence and uniqueness are guaranteed.*

The nonstandard formulation of Theorem 4.2.1 is as follows:

Theorem 4.2.2 *Let $f : \mathbf{R} \times \mathcal{C}_o \rightarrow \mathbf{R}^n$ be standard. Assume that all assumptions in Theorem 4.2.1 hold true. Let $\phi \in \mathcal{C}_o$ and $t_0 \in \mathbf{R}$ be standard. Let $x = x(\cdot; t_0, \phi)$ be the solution of (4.1), let $y = y(\cdot; t_0, \phi)$ be the solution of (4.2), and let J be the maximal interval of definition of y . Let $\varepsilon > 0$ be infinitesimal. Then for any standard $T > t_0$, $T \in J$, x is defined at least on $[t_0, T]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [t_0, T]$.*

The proof of Theorem 4.2.2 is postponed to next section. The reduction of Theorem 4.2.2 to Theorem 4.2.1 is the same one as in [Chap.2, subsection 2.3.1] and in [Chap.3, section 3.3].

4.3 Proof of Theorem 4.2.2

4.3.1 Preliminary Lemmas

In this subsection we give some results we need for the proof of Theorem 4.2.2. We assume that all assumptions in Theorem 4.2.2 hold true. Let us give external formulations of conditions (H1), (H2) and (H3) respectively:

$$(H1') \quad \forall^{st} \tau \geq 0 \quad \forall^{st} u \in \mathcal{C}_o \quad \forall \tau' \geq 0 \quad \forall u' \in \mathcal{C}_o:$$

$$\tau' \simeq \tau \text{ and } u' \simeq u \implies f(\tau, u') \simeq f(\tau, u).$$

(H2') There is a standard constant k such that

$$\|f(\tau, u_1) - f(\tau, u_2)\| \leq k|u_1 - u_2|, \quad \forall^{st} \tau \geq 0 \quad \forall^{st} u_1, u_2 \in \mathcal{C}_o$$

(and by transfer (Appendix A, subsection A.2.2) the inequality holds for all $\tau \geq 0$ and $u_1, u_2 \in \mathcal{C}_o$).

(H3') There is a standard functional $f^o : \mathcal{C}_o \rightarrow \mathbf{R}^n$ such that

$$\forall^{st} u \in \mathcal{C}_o \quad \forall T \simeq +\infty : \quad f^o(u) \simeq \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

Lemma 4.3.1 *The functional f^o is Lipschitz (with the same constant of Lipschitz as f), and satisfies*

$$f^o(u) \simeq \frac{1}{T} \int_0^T f(\tau, u) d\tau$$

for all $u \in \mathcal{C}_o$, u nearstandard, and all $T \simeq +\infty$.

Proof. First, let u_1 and $u_2 \in \mathcal{C}_o$, with u_1 and u_2 standard. By means of conditions (H2) and (H3), we have

$$\|f^o(u_1) - f^o(u_2)\| \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|f(\tau, u_1) - f(\tau, u_2)\| d\tau \leq k|u_1 - u_2|. \quad (4.3)$$

That is, f^o is k -Lipschitz.

Next, let $u, {}^o u \in \mathcal{C}_o$ such that ${}^o u$ is standard and $u \simeq {}^o u$. By means of (4.3), conditions (H3') and (H2'), respectively, for all $T \simeq +\infty$, we have

$$f^o(u) \simeq f^o({}^o u) \simeq \frac{1}{T} \int_0^T f(\tau, {}^o u) d\tau \simeq \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

□

Lemma 4.3.2 *There exists $\mu > 0$ such that whenever $t \geq 0$ is limited and $u \in \mathcal{C}_o$ is nearstandard there exists $\alpha > 0$ such that $\mu < \alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, u) d\tau \simeq f^o(u).$$

Proof. The proof of Lemma 4.3.2 is the same one as in [Chap.2, Lemma 2.4.2, page 20]. \square

Lemma 4.3.3 *Let $\phi \in \mathcal{C}_o$ be standard. Let $y = y(\cdot; 0, \phi)$ be the solution of (4.2) on J , and let $T_1 > 0$ be standard such that $[0, T_1] \subset J$. Then there exist some positive integer N_o and some infinitesimal partition $\{t_n : n = 0, \dots, N_o + 1\}$ of $[0, T_1]$ such that $t_0 = 0$, $t_{N_o} \leq T_1 < t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and*

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, y_{t_n}) d\tau \simeq f^o(y_{t_n}).$$

Proof. First, notice that $y([-r, T_1])$ is a standard compact subset of \mathbf{R}^n .

Next, define the set $A = \{\lambda \in \mathbf{R} / \forall t \in [0, T_1] \exists \alpha \in \mathbf{R} : \mathcal{P}(t, \alpha, \lambda)\}$ where

$$\mathcal{P}(t, \alpha, \lambda) \equiv \mu < \alpha < \lambda \quad \text{and} \quad \left\| \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, y_t) d\tau - f^o(y_t) \right\| < \lambda.$$

By Lemma 4.3.2 the set A contains all the standard real numbers $\lambda > 0$. By Lemma A.4.4 (Appendix A) there exists $\lambda_0 \simeq 0$ in A , that is, there exists $0 < \lambda_0 \simeq 0$ such that for all $t \in [0, T_1]$ there exists $\alpha \in \mathbf{R}$ such that $\mathcal{P}(t, \alpha, \lambda_0)$ holds. By the axiom of choice there exists a function $c : [0, T_1] \rightarrow \mathbf{R}$ such that $c(t) = \alpha$, that is, $\mathcal{P}(t, c(t), \lambda_0)$ holds for all $t \in [0, T_1]$. Since $c(t) > \mu$ for all $t \in [0, T_1]$, the conclusion of the lemma is immediate. \square

Lemma 4.3.4 *Let $\phi \in \mathcal{C}_o$ be standard. Let $y = y(\cdot; 0, \phi)$ be the solution of (4.2) on J , and let $T_1 > 0$ be standard such that $[0, T_1] \subset J$. Then for all $t \in [0, T_1]$*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau \simeq \int_0^t f^o(y_\tau) d\tau.$$

Proof. Let $f_1(\tau, u) := f(\tau, u) - f^o(u)$, for $\tau \geq 0$ and $u \in \mathcal{C}_o$. The functional f_1 is Lipschitz, that is, there exists some standard constant k_1 ($k_1 = 2k$, where k is the Lipschitz constant of f) such that

$$\|f_1(\tau, u_1) - f_1(\tau, u_2)\| \leq k_1 |u_1 - u_2|, \quad \text{for all } \tau \geq 0 \text{ and } u_1, u_2 \in \mathcal{C}_o.$$

Next, by Lemma 4.3.3 there exists $\{t_n : n = 0, \dots, N_o + 1\}$ such that $t_0 = 0$, $t_{N_o} \leq T_1 < t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f_1(\tau, y_{t_n}) d\tau \simeq 0. \quad (4.4)$$

Let $t \in [0, T_1]$, and let N be a positive integer such that $t_N \leq t < t_{N+1}$. We have

$$\begin{aligned}
& \left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, y_\tau \right) d\tau \right\| \\
\leq & \left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, y_\tau \right) d\tau - \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| + \left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| \\
\leq & \int_{t_N}^t \left\| f_1 \left(\frac{\tau}{\varepsilon}, y_\tau \right) - f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) \right\| d\tau + \left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| \\
\leq & k_1 \int_{t_N}^t |y_\tau| d\tau + \left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\|.
\end{aligned} \tag{4.5}$$

As $y([-r, T_1])$ is a standard compact subset of \mathbf{R}^n , it follows that

$$\int_{t_N}^t |y_\tau| d\tau \simeq 0. \tag{4.6}$$

Let us now estimate the second term in the right hand side of (4.5). For this, consider all the cases.

i) Both t_N/ε and t/ε are limited. In this case, it is clear that

$$\left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| = \varepsilon \left\| \int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s, 0) ds \right\| \simeq 0.$$

ii) Both t_N/ε and t/ε are unlimited. By means of condition (H3'), we have

$$\begin{aligned}
\left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| &= \varepsilon \left\| \int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s, 0) ds \right\| \\
&\leq t_N \left\| \frac{1}{t_N/\varepsilon} \int_0^{t_N/\varepsilon} f_1(s, 0) ds \right\| + t \left\| \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f_1(s, 0) ds \right\| \simeq 0.
\end{aligned}$$

iii) t_N/ε is limited and t/ε is unlimited. This case is a combination of cases i) and ii). We write

$$\begin{aligned}
\left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| &= \varepsilon \left\| \int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s, 0) ds \right\| \\
&\leq \varepsilon \left\| \int_0^{t_N/\varepsilon} f_1(s, 0) ds \right\| + t \left\| \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f_1(s, 0) ds \right\| \simeq 0.
\end{aligned}$$

Thus, we have

$$\left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, 0 \right) d\tau \right\| \simeq 0. \tag{4.7}$$

Therefore, from (4.5) and by means of (4.6) and (4.7), we obtain that

$$\left\| \int_{t_N}^t f_1 \left(\frac{\tau}{\varepsilon}, y_\tau \right) d\tau \right\| \simeq 0$$

so that

$$\begin{aligned}
\int_0^t f\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau - \int_0^t f^o(y_\tau) d\tau &= \int_0^t f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau \\
&\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) \right) d\tau \\
&\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau.
\end{aligned} \tag{4.8}$$

As f^o is bounded on $y([-r, T_1])$ by some standard positive constant, it is not difficult to verify that $y_\tau \simeq y_{t_n}$ for $\tau \in [t_n, t_{n+1}]$, $n = 0, \dots, N$, and then by Lemma A.4.2 (Appendix A), we have

$$\sup_{0 \leq n \leq N-1} \sup_{\tau \in [t_n, t_{n+1}]} \{|y_\tau - y_{t_n}|\} \simeq 0$$

and so is

$$k_1 \cdot \sup_{0 \leq n \leq N-1} \sup_{\tau \in [t_n, t_{n+1}]} \{|y_\tau - y_{t_n}|\} \cdot t_N,$$

so that

$$\begin{aligned}
&\left\| \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau \right\| \\
&\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) \right\| d\tau \\
&\leq k_1 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} |y_\tau - y_{t_n}| d\tau \\
&\leq k_1 \cdot \sup_{0 \leq n \leq N-1} \sup_{\tau \in [t_n, t_{n+1}]} \{|y_\tau - y_{t_n}|\} \cdot t_N \simeq 0.
\end{aligned}$$

Therefore, from (4.8) and by means of (4.4), it follows that

$$\begin{aligned}
\int_0^t f\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau - \int_0^t f^o(y_\tau) d\tau &\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_n + \alpha_n} f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau \\
&= \varepsilon \sum_{n=0}^{N-1} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f_1(\tau, y_{t_n}) d\tau \\
&= \sum_{n=0}^{N-1} \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f_1(\tau, y_{t_n}) d\tau \right) \\
&= \sum_{n=0}^{N-1} \alpha_n \beta_n \simeq 0
\end{aligned}$$

since $\left\| \sum_{n=0}^{N-1} \alpha_n \cdot \beta_n \right\| \leq \bar{\beta} \sum_{n=0}^{N-1} \alpha_n = \bar{\beta} \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \bar{\beta} \cdot t_N$, where $\bar{\beta} = \max\{\|\beta_n\| : 0 \leq n \leq N-1\}$. By Lemma A.4.2 (Appendix A), $\bar{\beta}$ is infinitesimal and so is $\bar{\beta} \cdot t_N$. This completes the proof of Lemma 4.3.4. \square

Lemma 4.3.5 *Let $\phi \in \mathcal{C}_o$ be standard. Let $x = x(\cdot; 0, \phi)$ be the solution of (4.1) on I , and $y = y(\cdot; 0, \phi)$ be the solution of (4.2) on J . Let $T_1 > 0$ be standard such that $[0, T_1] \subset I \cap J$. Then $x(t) \simeq y(t)$ for all $t \in [0, T_1]$.*

Proof. For $t \in [0, T_1]$ we have

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau \quad (4.9)$$

$$y(t) = \phi(0) + \int_0^t f^o(y_\tau) d\tau. \quad (4.10)$$

Substraction of (4.9) and (4.10) gives

$$\begin{aligned} & \|x(t) - y(t)\| \\ & \leq \left\| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f\left(\frac{\tau}{\varepsilon}, y_\tau\right) \right) d\tau \right\| + \left\| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^o(y_\tau) \right) d\tau \right\| \\ & \leq \int_0^t \left\| f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f\left(\frac{\tau}{\varepsilon}, y_\tau\right) \right\| d\tau + \left\| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^o(y_\tau) \right) d\tau \right\| \\ & \leq k \int_0^t |x_\tau - y_\tau| d\tau + \left\| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^o(y_\tau) \right) d\tau \right\|. \end{aligned} \quad (4.11)$$

Since, for $\tau \in [0, t]$, $|x_\tau - y_\tau| \leq \sup_{s \in [0, \tau]} \|x(s) - y(s)\|$, it follows from (4.11) that

$$\|x(t) - y(t)\| \leq k \int_0^t \sup_{s \in [0, \tau]} \|x(s) - y(s)\| d\tau + \left\| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^o(y_\tau) \right) d\tau \right\|. \quad (4.12)$$

The first term of the right hand side of (4.12) is increasing, and therefore

$$\begin{aligned} & \sup_{\tau \in [0, t]} \|x(\tau) - y(\tau)\| \\ & \leq k \int_0^t \sup_{s \in [0, \tau]} \|x(s) - y(s)\| d\tau + \sup_{\tau \in [0, t]} \left\| \int_0^\tau \left(f\left(\frac{s}{\varepsilon}, y_s\right) - f^o(y_s) \right) ds \right\|. \end{aligned}$$

By Gronwall's Lemma, this implies that

$$\sup_{\tau \in [0, t]} \|x(\tau) - y(\tau)\| \leq e^{kt} \sup_{\tau \in [0, t]} \left\| \int_0^\tau \left(f\left(\frac{s}{\varepsilon}, y_s\right) - f^o(y_s) \right) ds \right\|$$

and by means of Lemma 4.3.4 and Lemma A.4.2 (Appendix A), respectively, we conclude that $x(t) \simeq y(t)$, which finishes the proof. \square

4.3.2 Proof of Theorem 4.2.2

The proof of Theorem 4.2.2 is the same one as in [Chap.3, section 3.4.2, page 39].
□

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Appendix A

Internal Set Theory: A Tutorial

“... there are good reasons to believe that nonstandard analysis, in some version or other, will be the analysis of the future.”
Kurt Gödel

Very little is needed to be known of the theoretical details of Nonstandard Analysis which is a modern theory of infinitesimals as introduced by A. Robinson [16] in order to apply it well. The purpose of this appendix is to communicate the common background necessary to the understanding of Chapters 2-4 and Appendix B of this thesis.

A.1 Brief history

Early in the 20th century the logician Thoralf Skolem discovered there must exist what he called “strange” models for arithmetic. Later, in the 1960’s, Abraham Robinson gave a precise meaning to the word “strange”:

“In the fall of 1960 it occurred to me that the concepts and methods of contemporary mathematical logic are capable of providing a suitable framework for the development of the Differential Calculus by means of infinitely small and infinitely large numbers. [This framework] was, in part, inspired by the so-called Non-standard models of arithmetic whose existence was first pointed out by T. Skolem [19]”

Robinson built an alternate model to the standard real number system that contained, in addition to the standard reals, infinitesimals and their reciprocals, infinitely large numbers. For years others had experimented with the use of infinitesimals, but no one had ever given them rigorous credibility. Probably the most famous application of infinitesimals occurred 300 years earlier when Leibnitz used them in his discovery of the differential calculus, and was promptly ridiculed in spite of the obvious correctness of his results. He and others who followed tried, in vain, to justify their use, eventually grounding calculus instead in a more complicated theory of limits. Robinson’s discovery showed, at last, Leibnitz’s intuition was right. In his

nonstandard model we can not only reason without contradiction, we can infer with impunity regarding the usual (standard) real numbers.

A.2 Presentation of Internal Set Theory

Internal Set Theory (IST) is an axiomatic description of *Nonstandard Analysis* (NSA) proposed by Nelson [13]. It is an extension of classical axiomatic set theory (Zermelo-Frankel set theory with axiom of choice: ZFC). The only nonlogical symbol of ZFC is \in . Here ZFC is completed by the use of a new undefined monadic predicate symbol st (read standard), and three axioms to operate on it, Transfer, Idealization and Standardization. A number (and more generally a set) is either standard or not.

A.2.1 To be internal or external

Recall that any *mathematical formula* is built of constants, variables, quantifiers and connectives, displayed in some coherent manner. A *constant* is an object such as 0, 1, 10, e , π , \emptyset , \mathbf{N} , \mathbf{R} , that has been defined once for all and to which some notation has been assigned; this terminology is somewhat misleading as functions like \cos , \ln , $+$, \times are also logical constants, as are predicates such as \leq , \in and now st . Objects like f or ε that, formally, are *variables*, but that one treats like constants, as though they were objects fixed for all, are called *parameters*.

We call *internal*, the formulas of IST without any occurrence of the predicate st in them; otherwise, we call them *external*. Thus internal formulas are the formulas of ZFC.

A.2.2 The axioms

Here are the rules as to how to manipulate the predicate “standard”. There are three of them: transfer, idealization and standardization.

A.2.2.1 Transfer

Principle A.2.1 *Let F be an internal formula. Then*

$$\forall^{st} t_1, t_2, \dots, t_l [\forall^{st} x F(x, t_1, \dots, t_l)] \implies \forall x F(x, t_1, \dots, t_l),$$

or equivalently

$$\forall^{st} t_1, t_2, \dots, t_l [\exists x F(x, t_1, \dots, t_l)] \implies \exists^{st} x F(x, t_1, \dots, t_l).$$

The axiom of transfer implies that all explicitly defined sets of classical mathematics, such as \emptyset , 1 , 2 , π , e , \cos , \ln , ..., \mathbf{N} , \mathbf{R} , ..., $[0, 1]^{\mathbf{R}}$, $\mathcal{C}^{\infty}[0, 1]$, ... are standard. Indeed, $\exists!x F(x) \implies \exists^{st}x F(x)$.

An important consequence of this axiom is that *any standard function has standard values at standard points*.

A.2.2.2 Idealization

Principle A.2.2 *Let B be an internal formula. Then*

$$[\forall^{st}Y, Y \text{ finite} \implies \exists x \forall y \in Y B(x, y)] \iff [\exists x \forall^{st}y B(x, y)]$$

One of this axiom important consequences is: *there exists a finite set that contains all standard objects*. As a consequence, all infinite sets have nonstandard elements.

Here are some useful examples of the consequences of the axiom of idealization:

- The relation $B(x, y) \equiv (x \in \mathbf{N}) \ \& \ (y \in \mathbf{N}) \implies (x \geq y)$ leads to the existence of infinitely large integers.
- The relation $B(x, y) \equiv (x \in E) \ \& \ (x \neq y)$ leads to the fact that any set E is standard and finite if and only if it has only standard elements. A consequence is that any limited integer (i.e. less than some standard integer) is itself standard.

As useful consequences we have:

1. A real number x is *infinitesimal*, denoted by $x \simeq 0$, if its absolute value $|x|$ is smaller than any standard strictly positive real number.
2. A real number x is *limited* if its absolute value $|x|$ is smaller than some standard real number.
3. A real number x is *unlimited*, denoted by $x \simeq \pm\infty$, if it is not limited.
4. A real number x is *appreciable* if it is neither unlimited nor infinitesimal.
5. Two real numbers x and y are *infinitely close*, denoted by $x \simeq y$, if their difference $x - y$ is infinitesimal.

A.2.2.3 Standardization

Principle A.2.3 *Let F be any formula (internal or external) of IST. Then*

$$\forall^{st}\mathcal{E} \exists^{st}\mathcal{S}_F \forall^{st}x [x \in \mathcal{S}_F \iff x \in \mathcal{E} \ \& \ F(x)]$$

In words: within a standard reference set \mathcal{E} , the standard elements of \mathcal{E} satisfying an arbitrary property $F(x)$ define a standard subset \mathcal{S}_F of \mathcal{E} . As a consequence of the transfer axiom, this set \mathcal{S}_F is unique. It is called the *standardized* of $\{st(x) \mid F(x)\}$. We write

$$\mathcal{S}_F := {}^S\{x \in \mathcal{E} \mid F(x)\}.$$

An important consequence is that *any limited real number x possess a shadow*. It is the standard real number denoted by ${}^o x$ such that ${}^o x \simeq x$.

A.2.2.4 Consistency of IST

Recall that the axioms of IST are all axioms of ZFC, restricted to internal formulas (in other words, IST is an extension of ZFC), plus three others which govern the use of the new predicate. Thus *all theorems of ZFC remain valid* in IST. IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC.

IST is a relatively consistent extension of ZFC, that is, IST is consistent with ZFC.

A.2.3 Reduction algorithm

Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is an algorithm (a well-known *reduction algorithm*) to reduce any external formula $F(x_1, \dots, x_n)$ of IST without other free variables than x_1, \dots, x_n , to an internal formula $F'(x_1, \dots, x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \iff F'$ for all standard values of the free variables. In other words, any result which may be formalized within IST by a formula $F(x_1, \dots, x_n)$ is equivalent to the classical property $F'(x_1, \dots, x_n)$, provided the parameters x_1, \dots, x_n are restricted to standard values.

We give the reduction of the frequently occurring formula $\forall x (\forall^{st} y A \implies \forall^{st} z B)$ where A and B are internal formulas

$$\forall x (\forall^{st} y A \implies \forall^{st} z B) \equiv \forall z \exists^{fin} y' \forall x (\forall y \in y' A \implies B). \quad (\text{A.1})$$

A.3 Shadows and S-properties

Definitions. For x and y in a standard metric space E , the notation $x \simeq y$ means that the distance from x to y is infinitesimal. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *nearstandard* in E and the standard point x_0 is called the *standard part* (or *the shadow*) of x (it is unique) and is also denoted by ${}^o x$. The *halo* of x , denoted by $\text{hal}(x)$, is the set, usually external, of all y such that $x \simeq y$.

For any subset X of E , a point $x \in E$ is called *nearstandard in X* if there exists a standard point $x_0 \in X$ such that $x \simeq x_0$. We recall that, if X is standard, X is open if and only if it contains the halo of all its standard elements. The *shadow* of a subset X of E , denoted by oX , is the unique standard set whose standard elements are precisely those whose halo intersects X .

Let E and F be standard metric spaces, and g be an internal function defined on $\mathcal{D}(g) \subset E$ and with values in F . g is called *S-continuous* at $x_0 \in \mathcal{D}(g)$ if for all $x \simeq x_0$, $g(x) \simeq g(x_0)$ holds, S-continuous in $E \times F$ if it is S-continuous at each point $x \in \mathcal{D}(g)$ such that $(x, g(x))$ is nearstandard in $E \times F$.

For g standard, the continuity and the S-continuity in $\mathcal{D}(g) \times \mathcal{B}(g)$, where $\mathcal{B}(g)$ is a target of g , coincide.

The shadow in $E \times F$ of the graph of a S-continuous function g is the graph of a standard continuous function g_0 , called the shadow of g , and denoted by og .

A.4 Permanence principles

In ZFC in principle all sets are defined using the only non logical symbol \in . In IST there is also the possibility to define collections with the non logical symbol *st*. Those collections which fall outside the range of ZFC are called *external sets*. External sets are often easily recognized: mostly some elementary classical property fails to hold. For instance, the set of infinitesimal real numbers $hal(0)$ must be external, for it constitutes a bounded subset of \mathbb{R} without lower upper bound.

It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is remarked that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a large domain. In Nonstandard Analysis, statements which affirm that the validity of a property exceeds the domain where it was established in direct way are called *permanence principles*.

Many permanence results used in Nonstandard Analysis are based upon the self evident statement:

Principle A.4.1 (Cauchy principle) *No external set is internal.*

It has the following frequently used application.

Lemma A.4.1 (Robinson's Lemma) *If r is an internal real function such that $r(t) \simeq 0$ for all limited $t \geq 0$, then there exists $\nu \simeq +\infty$ such that $r(t) \simeq 0$ for all $t \in [0, \nu]$.*

Proof. Indeed, $\{l \in \mathbb{R} / l \geq 1, l \text{ limited}\} \subset \{l \in \mathbb{R} / \forall t \in [0, l] |r(t)| < 1/l\}$. The first set is external and the second set is internal. By the Cauchy principle the inclusion is strict. \square

As useful consequences of Cauchy's principle which will be used throughout this thesis we have:

Lemma A.4.2 *Let I be an internal set and $h : I \rightarrow \mathbf{R}$ be an internal function such that $h(t) \simeq 0$ for all $t \in I$. Then $\sup_{t \in I} \{h(t)\} \simeq 0$.*

Proof. We have $\{l \in \mathbf{R}_+ / l \notin \text{hal}(0)\} \subset \{l \in \mathbf{R} / \forall t \in I |h(t)| < l\}$. The first set is external otherwise $\text{hal}(0)$ would be internal, and the second set is internal. By the Cauchy principle the inclusion is strict. \square

Lemma A.4.3 (Corollary of Lemma A.4.2) *Let $a < b$, $b - a$ limited, and let $g, \tilde{g} : [a, b] \rightarrow \mathbf{R}^n$ be internal integrable functions such that $g(t) \simeq \tilde{g}(t)$ for all $t \in [a, b]$. Then*

$$\int_a^b g(t) dt \simeq \int_a^b \tilde{g}(t) dt.$$

Lemma A.4.4 *If $\mathcal{P}(\cdot)$ is an internal property such that $\mathcal{P}(\lambda)$ holds for all appreciable real numbers $\lambda > 0$, then there exists $0 < \lambda_0 \simeq 0$ such that $\mathcal{P}(\lambda_0)$ holds.*

We conclude this appendix with the following remark.

Remark A.4.1 *The use of Nonstandard Analysis in perturbation theory of differential equations goes back to the seventies with the Reebian school. It gave birth to the nonstandard perturbation theory of differential equations which has become today a well-established tool in asymptotic theory (see the special five-digits classification 34E18 of the 2000 Mathematics Subject Classification). To have an idea of the rich literature on the subject, the reader is referred to general references below, and the references therein.*

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Appendix B

Sur la moyennisation dans les systèmes à plusieurs fréquences

Par Mustapha LAKRIB¹

Abstract. In this paper, the behaviour of solutions of differential equations of the form $\dot{x} = \varepsilon X(x, \phi)$, $\dot{\phi} = \Omega(x) + \mathcal{O}(\varepsilon)$, is studied using the method of averaging. This method consists in replacing X by \bar{X} , its spatial average, and considering instead the equation $\dot{y} = \varepsilon \bar{X}(y)$. For most initial conditions $x - y$ remains small, but for others, resonance between $\Omega_1(x), \dots, \Omega_m(x)$ is significant and $x - y$ can become large. Rather than to consider a near identity change of variables (which is classical approach), we exploit through the local study of the slow variable, the proximity of the temporal and spatial averages of X , to show that $x - y$ remains small as long as x evolves into the $\sqrt{\varepsilon}$ -nonresonant domain. Terminology in formulation and proofs are the one of the Nonstandard Analysis.

(1991) *A.M.S Subject Classification Codes.* 34C29, 34E10, 03H05.

Keywords and Phrases. Averaging, regular perturbations, nonstandard analysis.

B.1 Introduction

On considère un système d'équations différentielles de la forme

$$\begin{cases} \dot{x} = \varepsilon X(x, \phi) \\ \dot{\phi} = \Omega(x) + \mathcal{O}(\varepsilon) \end{cases} \quad (\text{B.1})$$

où $(\cdot) = d/d\tau$, $\tau \in \mathbf{R}$, $x \in \mathbf{R}^n$, $\phi \in \mathbf{T}^m = \mathbf{R}^m/2\pi\mathbf{Z}^m$ le tore de dimension m , et $\varepsilon > 0$ un petit paramètre réel. On précisera la régularité du second membre

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de (B.1). Le système (B.1) est composé d'une variable lente à n composantes: x_1, x_2, \dots, x_n et d'une variable rapide à m composantes: $\phi_1, \phi_2, \dots, \phi_m$. La méthode de moyennisation consiste à remplacer ce système par celui (plus simple) obtenu en moyennisant X par rapport aux variables de phase

$$\dot{y} = \varepsilon \bar{X}(y) \tag{B.2}$$

où la fonction moyenne \bar{X} est définie par

$$\bar{X}(y) = \frac{\int_{\mathbf{T}^m} X(y, \phi) d\phi}{\int_{\mathbf{T}^m} d\phi} = \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} X(y, \phi) d\phi_1 \dots d\phi_m.$$

Pourvu que les trajectoires du système (B.1) soient uniformément réparties sur la surface du tore \mathbf{T}^m , il est raisonnable de remplacer X par sa moyenne sur \mathbf{T}^m et espérer approximer les composantes lentes des solutions du système original par les solutions du système moyennisé. Cependant si les fréquences sont résonantes, i.e., $\Omega_1, \Omega_2, \dots, \Omega_m$ sont rationnellement dépendantes, la répartition du mouvement n'est plus uniforme (le mouvement non perturbé est partout dense dans un tore de dimension inférieure) et le mouvement moyennisé n'est plus nécessairement proche du mouvement réel. Pour les systèmes génériques, les fréquences sont non résonantes pour presque toutes les valeurs de x . Il est donc raisonnable d'espérer que, pour la plupart des conditions initiales, la méthode de moyennisation apporte une description correcte de l'évolution de la variable lente du véritable mouvement. Pour $m = 2$, Arnold [2] et Neishtadt [10] montrent que c'est effectivement le cas, le premier en supposant que la variation du rapport des fréquences le long des trajectoires du système original est strictement monotone; le second impose une condition qui force les trajectoires du système moyennisé à ne visiter les résonances qu'au plus une fois, de manière transversale et à vitesse finie. Des résultats généraux ($m \geq 3$) sont établis par Anosov [1] et Kasuga [8] sous l'hypothèse d'indépendance des fréquences au sens que le rang de la dérivée des fréquences, par rapport à la variable lente, est égale au nombre des composantes de la variable rapide. Une estimation de la mesure de l'ensemble des conditions initiales qui doivent être exclues dans ce cas (mais aussi dans le cas où $m = 2$) a été obtenue par Neishtadt [11] (voir [10] pour $m = 2$).

La démonstration des résultats de moyennisation repose sur le principe suivant [3, 4, 9]: On cherche à définir une nouvelle variable lente $z = x + \varepsilon u(x, \phi)$, proche de la variable d'origine, dont l'évolution soit décrite par un système proche du système moyennisé

$$\frac{dz}{d\tau} = \varepsilon \bar{X}(z) + \text{termes d'ordre plus élevé en } \varepsilon.$$

Le résultat recherché suivra alors de l'approximation de x par z d'une part et de celle de z par y d'autre part, sur un intervalle de temps d'ordre $1/\varepsilon$.

On vérifie que la fonction u doit satisfaire l'équation

$$\Omega(x) \cdot \frac{\partial u}{\partial \phi} = -\tilde{X}(x, \phi) \quad (\text{B.3})$$

où \tilde{X} désigne la partie purement oscillante de X ; soit pour les coefficients de Fourier u_k et X_k de u et X ($k \in \mathbf{Z}^m$):

$$i(k, \Omega(x))u_k(x) = -X_k(x),$$

équation dans laquelle apparaissent les petits dénominateurs $(k, \Omega(x))$. La procédure est alors la suivante:

i) En raison de la décroissance rapide des coefficients de Fourier de X (X étant supposée suffisamment régulière), on peut tronquer les séries de Fourier à l'ordre $N = N(\varepsilon)$ et remplacer dans (B.3) \tilde{X} par \tilde{X}_N , la somme partielle d'ordre N de la série de Fourier \tilde{X} ; ce qui revient à dire qu'il est inutile de tenir compte des résonances qui peuvent se produire entre des harmoniques d'ordres très élevés.

ii) On résoud exactement l'équation ainsi obtenue sur le domaine non résonant, c'est à dire l'ensemble des x de l'espace des variables lentes tels que les dénominateurs ne soient pas trop petits. Et l'on traite séparément les traversées de zones résonantes, complémentaires de cet ensemble. Pour la plupart des conditions initiales, les zones résonantes sont franchies assez rapidement pour que l'écart entre $x(\tau)$ et $y(\tau)$ ne croisse outre mesure lors de ces franchissements. Les autres conditions initiales, qui correspondent aux trajectoires capturées par les résonances ou errant le long des surfaces résonantes, en passant de l'une à l'autre, constituent l'ensemble de mesure petite pour lequel la méthode de moyennisation ne s'applique pas. Notons que dans sa simplicité, ce qui précède constitue le noyau de preuves plus complexes.

Dans l'étude ci-après, moyennant les méthodes non standard [5, 6, 7], on se propose d'apporter une justification autre que celle décrite plus haut (considérée comme classique) de la méthode de moyennisation, en dehors des zones de résonance.

Le travail est organisé de la manière suivante: A la Section B.2 on rappelle les notions de moyennes temporelle et spatiale, puis on donne une condition nécessaire et suffisante pour que celles-là soient infiniment proches pour les fonctions qui nous intéressent. A la Section B.3, on définit la notion de domaine $\sqrt{\varepsilon}$ -non résonant sur lequel on montre l'approximation de la composante lente de la solution du système original par la solution du système moyennisé. L'approche n'est plus basée sur un changement de variables proche de l'identité. A travers l'étude locale de la variable lente, la démonstration, basée sur la méthode de stroboscopie [6, 12, 13], exploite le fait que dans un tel domaine, les moyennes temporelles le long du flot associé au système non perturbé ($\varepsilon = 0$) et les moyennes spatiales sont infiniment proches.

B.2 Moyennes temporelle et spatiale

Soit $\Omega = (\Omega_1, \dots, \Omega_m) \in \mathbf{R}^m$, Ω fixé, et soit le système d'équations différentielles ordinaires défini sur le tore \mathbf{T}^m , $m \in \mathbf{N}^*$, par

$$\dot{\phi} = \Omega. \quad (\text{B.4})$$

Au système (B.4) est associé le flot $\{h^\tau\}_{\tau \in \mathbf{R}}$ défini par $h^\tau(\phi) = \phi + \Omega \cdot \tau$, $\tau \in \mathbf{R}$ et $\phi \in \mathbf{T}^m$.

Soit $X : \mathbf{T}^m \rightarrow \mathbf{R}^m$ une fonction intégrable.

Définition B.2.1 *On appelle moyenne temporelle de la fonction X le long du flot $\{h^\tau\}_{\tau \in \mathbf{R}}$, la fonction X^* définie par*

$$\begin{aligned} X^*(\phi) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(h^\tau(\phi)) d\tau \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(\phi + \Omega \cdot \tau) d\tau, \quad \phi \in \mathbf{T}^m. \end{aligned}$$

Abus de langage: Sans risque de confusion, nous conviendrons d'appeler moyenne temporelle de X la quantité

$$\begin{aligned} X^*(\phi, T) &= \frac{1}{T} \int_0^T X(h^\tau(\phi)) d\tau \\ &= \frac{1}{T} \int_0^T X(\phi + \Omega \cdot \tau) d\tau, \quad \phi \in \mathbf{T}^m, T \in \mathbf{R}_+. \end{aligned}$$

Définition B.2.2 *On appelle moyenne spatiale de la fonction X sur le tore \mathbf{T}^m , le vecteur scalaire \bar{X} défini par*

$$\bar{X} = \frac{1}{(2\pi)^m} \int_{\mathbf{T}^m} X(\phi) d\phi.$$

Si on suppose que X admet un développement en série de Fourier, X s'écrit: $X(\phi) = \bar{X} + \tilde{X}(\phi)$, $\phi \in \mathbf{T}^m$, où \bar{X} est la moyenne spatiale de X et \tilde{X} la partie oscillante de X . On a

$$\tilde{X}(\phi) = \sum_{k \neq 0} X_k e^{i(k, \phi)} \quad (\text{B.5})$$

où $k \in \mathbf{Z}^m$ et les X_k sont les coefficients de Fourier de X .

Nous allons commencer par examiner l'approximation des moyennes temporelle et spatiale des éléments de la bases $\{e^{i(k, \phi)}\}_{k \in \mathbf{Z}^m}$ (le long du flot $\{h^\tau\}_\tau$), puis moyennant (B.5) nous traiterons le cas de la fonction X .

Le résultat du lemme ci-après est une caractérisation de l'approximation (au sens d'infiniment proche) des moyennes temporelle et spatiale des monômes trigonométriques homogènes $e^{i(k, \phi)}$, $k \in \mathbf{Z}^m$, $k \neq 0$.

Lemme B.2.1 Soit $p(\phi) = e^{i(k,\phi)}$, $k \in \mathbf{Z}^m$, $k \neq 0$. Les deux propositions suivantes sont équivalentes:

$$P_1: \quad \forall \lim l \in \mathbf{R} : |(k, \Omega)| \geq l \cdot \sqrt{\varepsilon}$$

$$P_2: \quad \exists T_0 \in \mathbf{R}, T_0 \simeq +\infty, \sqrt{\varepsilon}T_0 \simeq 0 : \\ \forall T \in \mathbf{R}, \forall \phi \in \mathbf{T}^m (T \geq T_0 \implies p^*(\phi, T) \simeq 0 (= \bar{p}))$$

Preuve du Lemme B.2.1

i) Condition nécessaire:

k s'écrit: $k = (k_1, \dots, k_j, \dots, k_m)$. k étant non nul, il existe j tel que $k_j \neq 0$. On a d'une part

$$\int_0^{2\pi} e^{i(k_1\phi_1 + \dots + k_j\phi_j + \dots + k_m\phi_m)} d\phi_j = \frac{e^{i(k_1\phi_1 + \dots + k_j\phi_j + \dots + k_m\phi_m)}}{ik_j} \Big|_0^{2\pi} = 0$$

et donc

$$\begin{aligned} \bar{p} &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} e^{i(k_1\phi_1 + \dots + k_j\phi_j + \dots + k_m\phi_m)} d\phi_1 \dots d\phi_j \dots d\phi_m \\ &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{e^{i(k_1\phi_1 + \dots + k_j\phi_j + \dots + k_m\phi_m)}}{ik_j} \Big|_0^{2\pi} d\phi_1 \dots \widehat{d\phi_j} \dots d\phi_m \\ &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} 0 \cdot d\phi_1 \dots \widehat{d\phi_j} \dots d\phi_m = 0. \end{aligned}$$

D'autre part, pour $\phi \in \mathbf{T}^m$ et $T \in \mathbf{R}$

$$\begin{aligned} p^*(\phi, T) &= \frac{1}{T} \int_0^T p(\phi + \Omega \cdot \tau) d\tau = \frac{1}{T} \int_0^T e^{i(k, \phi + \Omega \cdot \tau)} d\tau \\ &= \frac{e^{i(k, \phi)}}{Ti(k, \Omega)} \left[e^{i(k, \Omega)\tau} \right]_0^T = \frac{e^{i(k, \phi)}}{Ti(k, \Omega)} \left[e^{i(k, \Omega)T} - 1 \right]. \end{aligned} \quad (\text{B.6})$$

D'où $|p^*(\phi, T)| \leq \frac{2}{T|(k, \Omega)|}$. Or, par hypothèse, on a: $\forall \lim l \in \mathbf{R} : |(k, \Omega)| \geq l\sqrt{\varepsilon}$, par permanence (Principe de Cauchy, voir [5, 6]) cette propriété reste vraie jusqu'à un certain $\nu \in \mathbf{R}$, $\nu \simeq +\infty$ et $\sqrt{\varepsilon}\nu \simeq 0$. Ainsi

$$\exists \nu \in \mathbf{R}, \nu \simeq +\infty, \sqrt{\varepsilon}\nu \simeq 0, \text{ tel que } : |(k, \Omega)| \geq \nu\sqrt{\varepsilon}.$$

Posons $T_0 = \frac{1}{\sqrt{\varepsilon\nu}}$. On a

$$T_0 = \frac{1}{\sqrt{\varepsilon\nu}} \sqrt{\nu} \simeq +\infty \quad \text{et} \quad \sqrt{\varepsilon}T_0 = \frac{1}{\sqrt{\nu}} \simeq 0,$$

et pour $T \geq T_0$, on a

$$|p^*(\phi, T)| \leq \frac{2}{T|(k, \Omega)|} \leq \frac{2}{T_0|(k, \Omega)|} \leq \frac{2}{\frac{1}{\sqrt{\varepsilon\nu}}\sqrt{\varepsilon\nu}} = \frac{2}{\sqrt{\nu}} \simeq 0.$$

Ce qui termine la preuve de la nécessité de P_2 pour P_1 .

ii) Condition suffisante:

On montre que la contraposée correspondante est vraie. On suppose alors l'existence d'un réel limité l tel que: $|(k, \Omega)| \leq l\sqrt{\varepsilon}$. On peut même supposer que: $|(k, \Omega)| = l\sqrt{\varepsilon}$. On supposera encore que $(k, \Omega) \geq 0$. Soit $T_0 \in \mathbf{R}$, $T_0 \simeq +\infty$ et $\sqrt{\varepsilon}T_0 \simeq 0$. Deux cas sont possibles:

1. $l = 0$:

Pour $T \geq T_0$ et $\phi \in \mathbf{T}^m$, on a

$$\begin{aligned} p^*(\phi, T) &= \frac{1}{T} \int_0^T e^{i(k, \phi + \Omega \cdot \tau)} d\tau = \frac{e^{i(k, \phi)}}{T} \int_0^T e^{i(k, \Omega)\tau} d\tau \\ &= e^{i(k, \phi)} \quad \text{car } e^{i(k, \Omega)\tau} = e^0 = 1; \forall \tau \in [0, T] \end{aligned}$$

et donc pour $T = T_0$ et $\phi = 0$, on a $p^*(0, T_0) = 1 \not\simeq 0 (= \bar{p})$. Ainsi

$$(\forall T_0 \in \mathbf{R} : T_0 \simeq +\infty \text{ et } \sqrt{\varepsilon}T_0 \simeq 0), \exists T \geq T_0, \exists \phi \in \mathbf{T}^m : p^*(\phi, T) \not\simeq 0 (= \bar{p}).$$

2. $l \neq 0$:

Pour $T \geq T_0$ et $\phi \in \mathbf{T}^m$, on a

$$p^*(\phi, T) = \frac{1}{T} \int_0^T e^{i(k, \phi + \Omega \cdot \tau)} d\tau = e^{i(k, \phi)} \frac{1}{T} \int_0^T e^{i(k, \Omega)\tau} d\tau = e^{i(k, \phi)} \frac{e^{i(k, \Omega)T} - 1}{i(k, \Omega)T}.$$

Comme l est limité et $\sqrt{\varepsilon}T_0 \simeq 0$, alors $(k, \Omega)T_0 = l\sqrt{\varepsilon}T_0 \simeq 0$ et $\frac{e^{i(k, \Omega)T_0} - 1}{i(k, \Omega)T_0} \simeq 1$, et donc $p^*(\phi, T_0) \simeq e^{i(k, \phi)}$. Il suffit alors de prendre $\phi = 0$ pour avoir $p^*(0, T_0) \simeq e^{i(k, 0)} = 1 \not\simeq 0 (= \bar{p})$. Ce qui termine la preuve du lemme B.2.1. \square

Remarque B.2.1 *Le lemme B.2.1 apporte une estimation du temps requis pour que les moyennes temporelle et spatiale de $e^{i(k, \phi)}$ (le long du flot $\{h^\tau\}_\tau$) soient proches à χ près, puisque de (B.6) on a*

$$|p^*(\phi, T)| \leq \frac{2}{T|(k, \Omega)|}$$

et donc

$$T_0 = \frac{2}{\chi|(k, \Omega)|}.$$

Ce temps est inversement proportionnel à la distance $|(k, \Omega)|$ de la résonance de numéro k . D'où, si on entoure la résonance par une zone résonante de "largeur" δ définie par $|(k, \Omega)| \leq \delta$, en dehors de cette zone les moyennes temporelle et spatiale de la fonction $e^{i(k, \phi)}$ deviennent proches à χ près après un temps $T_0 = 2/\delta\chi$. (Dans notre cas, $\delta = \sqrt{\varepsilon\nu}$, $\chi = 2/\sqrt{\nu}$ et $T_0 = 1/\sqrt{\varepsilon\nu}$).

Reprenons la fonction $X : \mathbf{T}^m \longrightarrow \mathbf{R}^m$. Pour $\phi \in \mathbf{T}^m$, $X(\phi) = \bar{X} + \tilde{X}(\phi)$ où \bar{X} est la moyenne spatiale de X et \tilde{X} est la partie oscillante de X . En utilisant le résultat du lemme B.2.1., on se propose ci-après de montrer que pour $\phi \in \mathbf{T}^m$ et $T \geq T_0$, où T_0 est un réel que l'on déterminera, on a

$$\tilde{X}^*(\phi, T) = \frac{1}{T} \int_0^T \tilde{X}(\phi + \Omega \cdot \tau) d\tau \simeq 0$$

Ce qui a pour conséquence

$$X^*(\phi, T) = \bar{X} + \tilde{X}^*(\phi, T) \simeq \bar{X}$$

i.e.; les moyennes temporelle et spatiale de X sont infiniment proches.

Pour se faire, décomposons \tilde{X} de sorte que $\tilde{X}(\phi) = \tilde{X}_N(\phi) + R_N(\phi)$ où

$$\tilde{X}_N(\phi) = \sum_{1 \leq |k| \leq N} X_k e^{i(k, \phi)} \quad \text{et} \quad R_N(\phi) = \sum_{|k| > N} X_k e^{i(k, \phi)}.$$

$k \in \mathbf{Z}^m$; X_k les coefficients de Fourier de X et N un entier positif que l'on déterminera. On choisit alors N de sorte que R_N (et delà R_N^*) soit petit, puis on montre que \tilde{X}_N^* est petit aussi, et donc \tilde{X}^* sera de même.

Soient $\phi \in \mathbf{T}^m$ et $T \in \mathbf{R}$. Nous avons,

$$\begin{aligned} \tilde{X}_N^*(\phi, T) &= \frac{1}{T} \int_0^T \tilde{X}_N(\phi + \Omega \cdot \tau) d\tau = \frac{1}{T} \int_0^T \sum_{1 \leq |k| \leq N} X_k e^{i(k, \phi + \Omega \cdot \tau)} d\tau \\ &= \sum_{1 \leq |k| \leq N} X_k \frac{1}{T} \int_0^T e^{i(k, \phi + \Omega \cdot \tau)} d\tau. \end{aligned}$$

Si on suppose que X est analytique, les X_k sont à décroissance exponentielle de sorte que $\|X_k\| \leq c_1 \exp(-c_2 |k|)$. On suppose par ailleurs que l'hypothèse suivante est réalisée: $\forall \lim l \in \mathbf{R}, \forall k \in \mathbf{Z}^m, k \neq 0 : |(k, \Omega)| \geq l \cdot \sqrt{\varepsilon}$. Le principe de Cauchy (voir [5, 6]) permet alors d'affirmer l'existence de $\nu \in \mathbf{R}$, $\nu \simeq +\infty$ et $\sqrt{\varepsilon\nu} \simeq 0$ tel qu'on ait:

$$\forall k \in \mathbf{Z}^m, k \neq 0 : |(k, \Omega)| \geq \nu \sqrt{\varepsilon}.$$

Alors pour $T \geq T_0 = 1/\sqrt{\varepsilon\nu}$, nous avons (voir i) de la preuve du lemme B.2.1)

$$\|\tilde{X}_N^*(\phi, T)\| \leq 2 \sum_{1 \leq |k| \leq N} \frac{\|X_k\|}{T |(k, \Omega)|} \leq 2 \sum_{1 \leq |k| \leq N} \frac{c_1 e^{-c_2 |k|}}{\sqrt{\nu}} \leq 2c_1 \sum_{1 \leq |k| \leq N} \frac{1}{\sqrt{\nu}}.$$

Comme le nombre de $k \in \mathbf{Z}^m$ tels que $|k| \leq N$ n'excède pas $2^m N^m$, alors

$$\|\tilde{X}_N^*(\phi, T)\| \leq 2^{m+1} c_1 \frac{N^m}{\sqrt{\nu}}.$$

De l'analyticit  de X , on d duit que pour $\mu \in (0, 1/2)$, la norme de R_N est major e par μ pourvu que $N \geq N(\mu) = [c \log(1/\mu)]$ (voir [9], Appendice 1), donc pour $\mu = 1/\sqrt{\nu}$ et $N = [c \log \sqrt{\nu}]$, on a

$$\|\tilde{X}_N^*(\phi, T)\| \leq 2^{m+1} \frac{(c \log \sqrt{\nu})^m}{\sqrt{\nu}} \simeq 0,$$

et del 

$$\|\tilde{X}^*(\phi, T)\| \leq \|\tilde{X}_N^*(\phi, T)\| + \|R_N^*(\phi, T)\| \leq 2^{m+1} \frac{(c \log \sqrt{\nu})^m}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}} \simeq 0,$$

puisque

$$\|R_N^*(\phi, T)\| \leq \frac{1}{T} \int_0^T \|R_N(\phi + \Omega \cdot \tau)\| d\tau \leq \frac{1}{T} \int_0^T \frac{1}{\sqrt{\nu}} d\tau = \frac{1}{\sqrt{\nu}}.$$

A ce stade du raisonnement, en utilisant la lin arit  de l'int gration, une argumentation analogue   celle de ii) de la preuve du lemme B.2.1, permet de d duire le r sultat suivant:

Lemme B.2.2 *Les deux propositions suivantes sont  quivalentes:*

$$P_1: \quad \forall \text{lim} l \in \mathbf{R}, \forall k \in \mathbf{Z}^m, k \neq 0: |(k, \Omega)| \geq l \cdot \sqrt{\varepsilon}$$

$$P_2: \quad \exists T_0 \in \mathbf{R}, T_0 \simeq +\infty, \sqrt{\varepsilon} T_0 \simeq 0:$$

$$\forall T \in \mathbf{R}, \forall \phi \in \mathbf{T}^m (T \geq T_0 \implies X^*(\phi, T) \simeq \bar{X})$$

Remarque B.2.2 *C'est pour simplifier les calculs que nous avons suppos  X analytique. Ce qui pr c de reste vraie pour X de classe C^{2m+1} puisque dans ce cas aussi la suite des coefficients de Fourier de X est d croissante (bien que la d croissance ne soit pas exponentielle) de sorte que pour $\mu > 0$, la norme de R_N soit major e par μ d s lors que $N = N(\mu) = [1/\mu^{2m}]$ (voir [9], Appendice 1).*

B.3 Moyennisation et $\sqrt{\varepsilon}$ -non r sonance

Consid rons le syst me d' quations diff rentielles ordinaires

$$\begin{cases} \frac{dx}{d\tau} &= \varepsilon X(x, \phi) \\ \frac{d\phi}{d\tau} &= \Omega(x) + \mathcal{O}(\varepsilon), \end{cases} \quad (\text{B.7})$$

o  $x \in \mathbf{B}$, \mathbf{B}  tant un compact standard de \mathbf{R}^n , et $\phi \in \mathbf{T}^m$. On suppose que

i) X et Ω sont standard de classe \mathbf{C}^1 .

ii) X est analytique en ϕ .

Remarque B.3.1 *L'hypothèse ii) peut être affaiblie en supposant X de classe C^{2m+1} (voir remarque B.2.2).*

Puisque nous nous intéressons au comportement des solutions du système original et du système moyennisé pour des intervalles du temps τ d'ordre $1/\varepsilon$, il est plus commode d'introduire un changement dans l'échelle du temps en posant $t = \varepsilon\tau$ (temps lent).

Ainsi, par rapport au nouveau temps, le système (B.7) s'écrit (en gardant la même notation pour le système obtenu)

$$\begin{cases} \frac{dx}{dt} &= X(x, \phi) \\ \frac{d\phi}{dt} &= \frac{\Omega(x)}{\varepsilon} + \mathcal{O}(1). \end{cases} \quad (\text{B.8})$$

Soit $(x(t), \phi(t))$ sa solution de condition initiale $(x(0), \phi(0))$.

Soit \bar{X} la moyenne sur une période (par rapport à la variable angulaire ϕ) de X , définie par

$$\bar{X}(y) = \frac{1}{(2\pi)^m} \int_{\mathbf{T}^m} X(y, \phi) d\phi, \quad y \in \mathbf{B}.$$

Notons par $y(t)$, la solution, issue de $y(0)$, du système moyennisé

$$\frac{dy}{dt} = \bar{X}(y). \quad (\text{B.9})$$

Soit $J = [0, \omega)$, $0 < \omega \leq +\infty$, le demi-intervalle positif maximal d'existence de y , et soit b limité dans \mathbf{R}_+ , $b < \omega$, tel que $y([0, b])$ soit non infiniment proche de la frontière de \mathbf{B} .

B.3.1 Domaine $\sqrt{\varepsilon}$ -non résonant

Définition B.3.1 *L'ensemble $\mathbf{D} \subset \mathbf{R}^n$ défini par*

$$\mathbf{D} = \{x \in \mathbf{B} / \forall^{lim} l \in \mathbf{R}, \forall k \in \mathbf{Z}^m, k \neq 0 : |(k, \Omega(x))| \geq l\sqrt{\varepsilon}\}$$

est appelé domaine $\sqrt{\varepsilon}$ -non résonant.

Proposition B.3.1 *Soit $x \in \mathbf{R}^n$, alors*

$$x \in \mathbf{D} \Leftrightarrow \begin{cases} (\exists T_0 \in \mathbf{R}, T_0 \simeq +\infty, \sqrt{\varepsilon}T_0 \simeq 0) : \forall T \in \mathbf{R}, \forall \phi \in \mathbf{T}^m, \\ T \geq T_0 \implies X^*(x; \phi, T) \simeq \bar{X}(x) \end{cases}$$

où

$$X^*(x; \phi, T) = \frac{1}{T} \int_0^T X(x, \phi + \Omega(x) \cdot \tau) d\tau.$$

Preuve. Cette proposition est une conséquence du lemme B.2.2. □

B.3.2 Résultat principal

Le théorème ci-après permet l'approximation (au sens d'infiniment proche) de la composante lente $x(t)$ de la solution $(x(t), \phi(t))$ de (B.8) par la solution $y(t)$ de (B.9) tant que l'évolution de $x(t)$ se fait dans le domaine $\sqrt{\varepsilon}$ -non résonant.

Théorème B.3.1 *Avec les hypothèses i) et ii), on suppose que les conditions initiales de $x(t)$ et $y(t)$ sont standard, avec $x(0) = y(0) \in \mathbf{D}$. Alors*

$$\forall t \in [0, b], \quad x(t) \simeq y(t), \quad \text{tant que } x(t) \text{ reste dans } \mathbf{D}.$$

Remarque B.3.2 *Pour démontrer le théorème B.3.1, on utilise la méthode de stroboscopie [6, 12, 13]. Cela consiste à regarder une fonction non standard, oscillant autour d'une valeur moyenne, en des instants infiniment proches, à des endroits privilégiés pour ne retenir que la variation moyenne de cette fonction.*

Méthode de stroboscopie [6, 12, 13] Soit \mathbf{B} un compact standard de \mathbb{R}^n , I un intervalle de \mathbb{R} , $\bar{X} : \mathbf{B} \rightarrow \mathbb{R}^n$ une fonction standard continue et $x : I \rightarrow \mathbb{R}^n$ avec $0 \in I$ et $x(0)$ standard. On suppose que

i) Il existe $\mu > 0$ tel qu'à chaque fois que $(t_0, x(t_0)) \in I \times \mathbb{R}^n$ est limité, il existe $t'_0 \in \mathbb{R}$ avec $t'_0 - t_0 > \mu$, tel que la fonction x vérifie la propriété stroboscopique

$$t'_0 \simeq t_0, \quad \forall s \in [t_0, t'_0] \quad x(s) \simeq x(t_0), \quad \text{et} \quad \frac{x(t'_0) - x(t_0)}{t'_0 - t_0} \simeq \bar{X}(x(t_0)).$$

t_0 et t'_0 s'appellent instants successifs d'observation de la stroboscopie.

ii) Le problème à valeur initiale

$$\frac{dy}{dt} = \bar{X}(y), \quad y(0) = x(0)$$

possède une solution unique. Soit $y(t)$ cette solution.

Théorème B.3.2 (Lemme de stroboscopie) ([6] p.148, [12] p.11, [13] p.97) *On suppose que les hypothèses i) et ii) sont réalisées. Soit $J = [0, \omega)$, $0 < \omega \leq +\infty$, le demi intervalle positif maximal d'existence de y , et soit b un réel limité, $b < \omega$, tel que $y([0, b])$ soit non infiniment proche de la frontière de \mathbf{B} . Alors, la fonction $x(t)$ est définie au moins sur $[0, b]$ et vérifie $x(t) \simeq y(t)$, pour tout $t \in [0, b]$.*

Preuve du Théorème B.3.1

On montre que la composante $x(t)$ de la solution $(x(t), \phi(t))$ du système (B.8) vérifie les hypothèses du théorème B.3.2. Ainsi, on suppose connu un instant d'observation

de la stroboscopie, soit t_0 cet instant, et on détermine l'instant suivant d'observation t'_0 . Posons $x_0 = x(t_0)$ et $\phi_0 = \phi(t_0)$. De l'hypothèse $x_0 \in \mathbf{D}$, on a

$$\begin{cases} (\exists T_0, \quad T_0 \simeq +\infty, \quad \sqrt{\varepsilon}.T_0 \simeq 0) : \forall T \in \mathbf{R}, \quad \forall \phi \in \mathbf{T}^m \\ T \geq T_0 \quad \implies \quad X^*(x_0; \phi, T) \simeq \bar{X}. \end{cases} \quad (\text{B.10})$$

Sous le changement de variables

$$T = \frac{t - t_0}{\varepsilon}, \quad \mathbf{y}(T) = \frac{x(t_0 + \varepsilon T) - x_0}{\varepsilon}, \quad \text{avec } \varphi(T) = \phi(t_0 + \varepsilon T) - \phi_0$$

le système (B.8) devient

$$\begin{cases} \frac{d\mathbf{y}}{dT} = X(x_0 + \varepsilon\mathbf{y}(T), \phi_0 + \varphi(T)) & ; \quad \mathbf{y}(0) = 0 \\ \frac{d\varphi}{dT} = \Omega(x_0 + \varepsilon\mathbf{y}(T)) + \varepsilon\mathcal{O}(1) & ; \quad \varphi(0) = 0. \end{cases} \quad (\text{B.11})$$

Considérons le système déduit de (B.11) pour $\varepsilon = 0$

$$\begin{cases} \frac{d\bar{\mathbf{y}}}{dT} = X(x_0, \phi_0 + \bar{\varphi}(T)) & ; \quad \bar{\mathbf{y}}(0) = 0 \\ \frac{d\bar{\varphi}}{dT} = \Omega(x_0) & ; \quad \bar{\varphi}(0) = 0. \end{cases} \quad (\text{B.12})$$

On vérifie que:

Lemme B.3.1 *Si $(\mathbf{y}(T), \varphi(T))$ et $(\bar{\mathbf{y}}(T), \bar{\varphi}(T))$ sont les solutions respectives de (B.11) et (B.12), alors pour tout $T \in \mathbf{R}$, $0 < T \leq T_0$, où T_0 est défini par (B.10), on a*

$$\begin{cases} \varphi(T) \simeq \bar{\varphi}(T) = \Omega(x_0) \cdot T \\ \frac{\mathbf{y}(T)}{T} \simeq \frac{\bar{\mathbf{y}}(T)}{T} = \frac{1}{T} \int_0^T X(x_0, \phi_0 + \Omega(x_0) \cdot \tau) d\tau. \end{cases} \quad (\text{B.13})$$

La seconde approximation de (B.13) permet alors de définir le second instant d'observation de la stroboscopie en posant $t'_0 = t_0 + \varepsilon.T_0$. D'où

$$\begin{aligned} \frac{x(t'_0) - x(t_0)}{t'_0 - t_0} &= \frac{\varepsilon.\mathbf{y}(T_0)}{\varepsilon.T_0} = \frac{\mathbf{y}(T_0)}{T_0} \simeq \frac{1}{T_0} \int_0^{T_0} X(x_0, \phi_0 + \Omega(x_0) \cdot \tau) d\tau \\ &= X^*(x_0; \phi_0, T_0) \simeq \bar{X}(x_0). \end{aligned}$$

Puisque $t'_0 - t_0 = \varepsilon T_0 > \varepsilon$ et que $x(t) - x(t_0) = \varepsilon\mathbf{y}(T) \simeq 0$ pour tout $t \in [t_0, t'_0]$ (voir (B.14) ci après), on a montré qu'il existe $\mu = \varepsilon$ tel qu'à chaque fois que $(t_0, x(t_0))$ est limité, il existe $t'_0 \in \mathbf{R}$ avec $t'_0 - t_0 > \mu$, tel que la fonction x vérifie la propriété stroboscopique. Du théorème B.3.2 découle alors le résultat du théorème B.3.1. \square

Preuve du Lemme B.3.1

Soit $T \in \mathbf{R}$, $0 < T \leq T_0$.

• Considérons $\|\varphi(T) - \bar{\varphi}(T)\|$. On a

$$\begin{aligned}
\|\varphi(T) - \bar{\varphi}(T)\| &\leq \int_0^T \left\| \frac{d\varphi}{dT}(s) - \frac{d\bar{\varphi}}{dT}(s) \right\| ds \\
&\leq \int_0^T \|\Omega(x_0 + \varepsilon \mathbf{y}(s)) + \varepsilon \mathcal{O}(1) - \Omega(x_0)\| ds \\
&\leq \int_0^T \|\Omega(x_0 + \varepsilon \mathbf{y}(s)) - \Omega(x_0)\| ds + \varepsilon \int_0^T \|\mathcal{O}(1)\| ds \\
&\leq L_1 \int_0^T \|\varepsilon \mathbf{y}(s)\| ds + \varepsilon \int_0^T \|\mathcal{O}(1)\| ds
\end{aligned}$$

où $L_1 = \sup_{x \in \mathbf{B}} \left\| \frac{d\Omega}{dx}(x) \right\|$. Or, pour $T \in \mathbf{R}$, $0 \leq T \leq T_0$, on a

$$\|\varepsilon \mathbf{y}(T)\| = \|x(t_0 + \varepsilon T) - x(t_0)\| \leq \varepsilon T L_0 \quad (\text{B.14})$$

où $L_0 = \sup_{\mathbf{B} \times \mathbf{R}^m} \|X(x, \phi)\|$. Alors,

$$\begin{aligned}
\|\varphi(T) - \bar{\varphi}(T)\| &\leq L_1 L_0 \varepsilon \int_0^T s ds + \varepsilon T L_2 \\
&\leq \varepsilon L_1 L_0 \frac{T^2}{2} + \varepsilon T L_2 \\
&\leq L_1 L_0 \frac{(\sqrt{\varepsilon} T_0)^2}{2} + \sqrt{\varepsilon} (\sqrt{\varepsilon} T_0) L_2 \simeq 0
\end{aligned} \quad (\text{B.15})$$

où L_2 est tel que $\sup_{\mathbf{B} \times \mathbf{R}^m} \|\mathcal{O}(1)\| \leq L_2$. Les L_i étant tous limités dans \mathbf{R}_+ .

Ainsi $\|\varphi(T) - \bar{\varphi}(T)\| \simeq 0$ et donc $\varphi(T) \simeq \bar{\varphi}(T) = \Omega(x_0) \cdot T$.

• Pour $\left\| \frac{\mathbf{y}(T)}{T} - \frac{\bar{\mathbf{y}}(T)}{T} \right\|$, on a

$$\begin{aligned}
\left\| \frac{\mathbf{y}(T)}{T} - \frac{\bar{\mathbf{y}}(T)}{T} \right\| &\leq \frac{1}{T} \int_0^T \left\| \frac{d\mathbf{y}(s)}{dT} - \frac{d\bar{\mathbf{y}}(s)}{dT} \right\| ds \\
&\leq \frac{1}{T} \int_0^T \|X(x_0 + \varepsilon\mathbf{y}(s), \phi_0 + \varphi(s)) - X(x_0, \phi_0 + \bar{\varphi}(s))\| ds \\
&\leq L_3 \frac{1}{T} \int_0^T \|\varepsilon\mathbf{y}(s)\| ds + L_4 \frac{1}{T} \int_0^T \|\varphi(s) - \bar{\varphi}(s)\| ds \\
&\leq L_3 L_0 \varepsilon \frac{T}{2} + L_4 \frac{1}{T} \int_0^T (L_0 L_1 \varepsilon \frac{s^2}{2} + \varepsilon s L_2) ds \quad (\text{voir (B.14) et (B.15)}) \\
&\leq L_3 L_0 \varepsilon \frac{T}{2} + L_4 L_0 L_1 \varepsilon \frac{T^2}{6} + L_4 L_2 \varepsilon \frac{T}{2} \\
&\leq L_3 L_0 \sqrt{\varepsilon} \frac{\sqrt{\varepsilon} T_0}{2} + L_4 L_0 L_1 \frac{(\sqrt{\varepsilon} T_0)^2}{6} + L_4 L_2 \sqrt{\varepsilon} \frac{\sqrt{\varepsilon} T_0}{2} \simeq 0
\end{aligned}$$

où $L_3 = \sup_{\mathbf{B} \times \mathbb{R}^m} \left\| \frac{\partial X}{\partial x}(x, \phi) \right\|$ et $L_4 = \sup_{\mathbf{B} \times \mathbb{R}^m} \left\| \frac{\partial X}{\partial \phi}(x, \phi) \right\|$, avec les L_i limités dans \mathbb{R}_+ .

Ainsi

$$\left\| \frac{\mathbf{y}(T)}{T} - \frac{\bar{\mathbf{y}}(T)}{T} \right\| \simeq 0 \quad \text{et donc} \quad \frac{\mathbf{y}(T)}{T} \simeq \frac{\bar{\mathbf{y}}(T)}{T} = \frac{1}{T} \int_0^T X(x_0, \phi_0 + \Omega(x_0) \cdot \tau) d\tau.$$

Ce qui termine la preuve du lemme B.3.1. \square

B.4 Bibliographie

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Appendix C

On the validity of the averaging method for all time

By Mustapha LAKRIB¹

Abstract. We present the k -th order averaging theory for periodic systems. Then we show how to obtain $\mathcal{O}(\varepsilon^k)$ -approximations to the solutions of the differential equation $\dot{x} = \varepsilon f(t, x)$ in such a way that they are uniformly valid for all time under the condition that the first order averaged equation has an exponentially attracting rest point.

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C.1 Introduction

As is well-known, many perturbation problems in nonlinear oscillations, and some in celestial mechanics, can be put into the form

$$\dot{x} = \varepsilon f(t, x) \tag{C.1}$$

where $\varepsilon > 0$ is a small parameter, $t \in \mathbf{R}_+$, $x \in \mathbf{R}^n$ and f is periodic in t (with period 2π , for convenience). Approximate solutions of (C.1) can be constructed using the method of averaging which gives asymptotic estimates by comparing the solutions of (C.1) with those of a derived k -th order averaged equation, where k is a chosen integer. Since the publication of the now classic book by Bogoliubov and Mitropolosky [2] the literature on averaging has grown immensely. The interested reader is referred to [12] for more details and bibliographical information. The aim of this work is to show how to obtain $\mathcal{O}(\varepsilon^k)$ -approximations to the solutions of (C.1)

¹*The references of this paper are:*

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by the k -th order averaging method in such a way that they are uniformly valid for all time. The results are as follows. The k -th order averaging approximation is normally valid for a time-scale $\mathcal{O}(1/\varepsilon)$ [3, 4, 7, 9], but becomes valid for all (future) time, in the Sanchez-Palencia case [5, 10, 11, 12].

The organization of this paper is as follows. Some preliminaries are collected in Section C.2. In Section C.3 we give exposition of the k -th order averaging method, and Section C.4 treats the Sanchez-Palencia case; that is, the case where the first order averaged equation admits an exponentially attracting rest point.

C.2 Preliminaries

In this section we collect some definitions and lemmas in the form that will be used later.

We assume that all functions are smooth (infinitely differentiable). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such a function, and let $D^m f(x)$ denote the m -th derivative of f at $x \in \mathbb{R}^n$. $D^m f(x)$ is a m -multilinear map on \mathbb{R}^n for each $x \in \mathbb{R}^n$. For $\xi_i \in \mathbb{R}^n$, $i = 1, \dots, m$, and $x \in \mathbb{R}^n$, the value of $D^m f(x)$ at (ξ_1, \dots, ξ_m) is given by

$$D^m f(x)\xi_1 \cdots \xi_m = \sum \frac{\partial^m f(x)}{\partial x^{(j_1)} \cdots \partial x^{(j_m)}} \xi_1^{(j_1)} \cdots \xi_m^{(j_m)} \in \mathbb{R}^n,$$

the summation ranging over $j_1, \dots, j_m = 1, \dots, n$. By definition $D^0 f(x) = f(x)$. For $\xi \in \mathbb{R}^n$, $\xi^{(j)}$ denotes the j -th component of ξ while ξ^m is an abbreviation for $\xi \cdots \xi$ (m times).

In this notation, Taylor's formula [4, 9] for a smooth function f and $x_0 \in \mathbb{R}^n$, can be written in the form

$$f(x) = f(x_0) + \cdots + \frac{1}{(m-1)!} D^{m-1} f(x_0) h^{m-1} + \rho(x_0)$$

where

$$\rho(x_0) = \frac{1}{(m-1)!} \int_0^1 (1-y)^{m-1} D^m f(x_0 + yh) h^m dy,$$

with $h = x - x_0$ and $x \in \mathbb{R}^n$.

In this paper $f(t, x)$ denotes a mapping of $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $D_x^m f(t, x)$ is used to denote the m -th partial derivative of f with respect to x at $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. The above formulas holds with $f(x)$ and $D^m f(x)$ replaced by $f(t, x)$ and $D_x^m f(t, x)$ respectively. If f is 2π -periodic in t , its mean value with respect to t will be designated as f^o and is defined, for each $x \in \mathbb{R}^n$, by

$$f^o(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t, x) dt.$$

The oscillating part $f - f^o$ of f will be denoted by \tilde{f} .

The Besjes inequality is used to estimate various functions on expanding intervals of time with length $\mathcal{O}(1/\varepsilon)$. The version below differs from the original proof given in [1] and [4], in some details, uses much stronger hypotheses, and may be easily proved by the method used in [8].

Lemma C.2.1 *Let $g(t, z)$ be a smooth function with values in \mathbb{R}^n , defined for $t \in \mathbb{R}$ and $z \in \mathbb{R}^n$, which is 2π -periodic in t with zero mean, i.e., $g^o \equiv 0$. Suppose that $y(t)$ is a smooth function which is defined and remains bounded for t in an interval $I(\varepsilon) = [0, b(\varepsilon)]$ for some positive function $b(\varepsilon)$, and is slowly varying in the sense that*

$$\left\| \frac{dy}{dt}(t) \right\| \leq c\varepsilon. \quad (\text{C.2})$$

Then there exist constants M and N such that

$$\left\| \int_0^t g(s, y(s)) ds \right\| \leq M + N\varepsilon t \quad (\text{C.3})$$

for $t \in I(\varepsilon)$.

Remark 2 *We will refer to the integrand of (C.3) as a mean-free periodic function with slow input. Notice that (C.2) guarantees that the “input” y will remain bounded for at least on a time-scale $\mathcal{O}(1/\varepsilon)$, so we could always take $I(\varepsilon)$ to be of this order. However, the theorem remains valid for longer than this if y remain bounded for longer.*

We formulate the familiar Gronwall inequality in a slightly different form which make it adapted to our study.

Lemma C.2.2 *Suppose that for $t \in [0, L/\varepsilon]$*

$$\psi(t) \leq \varepsilon c_1 \int_0^t \psi(s) ds + c_2 \varepsilon^k$$

with $\psi(t)$ a continuous function, $\psi(t) \geq 0$ for $t \in [0, L/\varepsilon]$, $\psi(0) = 0$ and c_1, c_2 some constants. Then there exists a constant $c > 0$ such that

$$\psi(t) \leq c\varepsilon^k$$

for $t \in [0, L/\varepsilon]$.

C.3 The k -th order averaging expansion

Here we present a treatment of the k -th order averaging theory for periodic systems of the form

$$\dot{x} = \varepsilon f(t, x), \quad x(0) = \alpha \quad (\text{C.4})$$

where f is 2π -periodic in t and $\alpha \in \mathbf{R}^n$. The k -th order approximation to the solution of (C.4) by the method of averaging is formally defined for $k \geq 1$ as

$$x_k(t) = \hat{x}_k(t) + \sum_{i=1}^{k-1} \varepsilon^i u_i(t, \hat{x}_k(t)) \quad (\text{C.5})$$

where $\hat{x}_k(t)$ is the solution of the autonomous system

$$\dot{\hat{x}} = \varepsilon h_1^o(\hat{x}) + \cdots + \varepsilon^k h_k^o(\hat{x}), \quad \hat{x}(0) = \alpha \quad (\text{C.6})$$

We will suppose \hat{x}_k to be defined and bounded in intervals of time with length $\mathcal{O}(1/\varepsilon)$. The functions h_i^o and u_i are defined by a recursive process which may be described as follows.

Beginning with

$$h_1(t, y) = f(t, y),$$

we may construct recursively, for $i = 1, \dots, k$, the sequence of functions $h_1, u_1, h_2, u_2, \dots, h_k, u_k$, defined by

$$\frac{\partial u_i}{\partial t}(t, y) = \tilde{h}_i(t, y), \quad u_i^o(y) = v_i(y) \quad (\text{C.7})$$

and

$$h_i(t, y) = \sum_{j=1}^{i-1} \left[\frac{1}{j!} D_x^j f(t, y) \sum_{i_1 + \dots + i_j = i-1} u_{i_1} \cdots u_{i_j}(t, y) - D_x u_j(t, y) h_{i-j}^o(y) \right]. \quad (\text{C.8})$$

Here h_i^o, u_i^o are mean values with respect to t of h_i and u_i respectively, and \tilde{h}_i is the oscillating part of h_i . These equations are obtained by substituting the expressions (C.5) and (C.6) into the differential equation (C.4), expanding the function $f(t, y)$ in a Taylor series about the point $x = \hat{x}_k$ and equating the coefficients of like powers of ε , the v_i being chosen to make the functions u_i 2π -periodic in t with $u_i(0, y) = 0$ for all y . The reason for the name k -th order averaging is now apparent since the h_i^o 's are determined by a time average.

It remains to establish the error of the approximation. Let

$$R_k(t) = x(t) - x_k(t) \quad (\text{C.9})$$

where $x(t)$ is the exact solution of (C.4). We will show that $R_k = \mathcal{O}(\varepsilon^k)$ on a time-scale $\mathcal{O}(1/\varepsilon)$.

Theorem C.3.1 *There exist positive constants c , L and ε_0 such that*

$$\|R_k(t)\| \leq c\varepsilon^k$$

for $t \in [0, L/\varepsilon]$ and $\varepsilon \in [0, \varepsilon_0]$.

Proof. From (C.4), (C.5), (C.6), (C.7) and (C.8), computations lead to

$$\begin{aligned} \frac{dR_k}{dt} &= \frac{dx}{dt} - \frac{dx_k}{dt} \\ &= \varepsilon [f(t, x(t)) - f(t, x_k(t))] + \varepsilon^k g_k(t, \hat{x}_k(t)) + \varepsilon^{k+1} l_k(t, \hat{x}_k(t)) \end{aligned} \quad (\text{C.10})$$

where

$$g_k(t, \hat{x}_k) = -\tilde{h}_k(t, \hat{x}_k),$$

$$l_k(t, \hat{x}_k) = \sum_{i=0}^{k-2} \varepsilon^i \sum_{l=i+1}^{k-1} D_x u_l(t, \hat{x}_k) h_{k+1+i-l}^o(\hat{x}_k) - \sum_{i=1}^k \rho_i(t, \hat{x}_k)$$

and $\rho_i(t, \hat{x}_k)$ is the remainder obtained by expanding $f(t, x_k)$ about the point \hat{x}_k . The estimate $l_k(t, \hat{x}_k) = \mathcal{O}(1)$ holds as long as \hat{x}_k remain in compact subsets; but we know from the hypothesis under \hat{x}_k that this is so on a time-scale $\mathcal{O}(1/\varepsilon)$, and therefore the estimate holds on this interval. Integrate (C.10) and take the norm lead to

$$\begin{aligned} \|R_k(t)\| &= \|x(t) - x_k(t)\| \\ &\leq \varepsilon \int_0^t \|f(s, x(s)) - f(s, x_k(s))\| ds + \varepsilon^k \left\| \int_0^t g_k(s, \hat{x}_k(s)) ds \right\| + \mathcal{O}(\varepsilon^k) \end{aligned}$$

Since g_k is a mean-free periodic function with slow input, the Besjes inequality (Lemma C.2.1) applies, and the norm of this integral is bounded by some $M + N\varepsilon t \leq B_0$ on a time-scale $\mathcal{O}(1/\varepsilon)$. The term $\mathcal{O}(\varepsilon^k)$ is bounded by some $\varepsilon^k B_1$, on the same interval, and the first term can be bounded using a Lipschitz constant. Therefore

$$\|R_k(t)\| \leq \varepsilon L \int_0^t \|R_k(s)\| ds + B\varepsilon^k,$$

and the proof is concluded with Gronwall's inequality (Lemma C.2.2). \square

C.4 The k-th order averaging expansion in the contracting case

In this section we will show that the averaging method can be extended beyond time intervals with length $\mathcal{O}(1/\varepsilon)$. This is true under the Sanchez-Palencia condition. The approximation is then uniformly valid for $t \geq 0$. First, we state this condition and

show that it is equivalent to certain contraction hypothesis on the linear variational equation. Then we prove validity of the averaging method for all (future) time under this hypothesis.

Let $\hat{x}_1(\tau)$ be a solution (with $\tau = \varepsilon t$), of

$$\frac{d\hat{x}_1}{d\tau} = f^o(\hat{x}_1), \quad \hat{x}_1(0) = \alpha. \quad (\text{C.11})$$

Suppose that $f^o(0) = 0$ and that the matrix $A = f_x^o(0)$ has its eigenvalues in the left half-plane, so that $x = 0$ is an asymptotically stable rest point (if this statement holds for $x = x_c$ with $f^o(x_c) = 0$, we translate this rest point to the origin). Suppose furthermore that α lies in the basin of attraction of $x = 0$, so that $\hat{x}_1(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$. These hypotheses are the conditions of the Sanchez-Palencia theorem.

We now express these hypotheses in an alternate form. Consider the linear variational equation of (C.11) along its solution, namely

$$\frac{dz}{d\tau} = f_x^o(\hat{x}_1(\tau))z = A(\tau)z. \quad (\text{C.12})$$

If the Sanchez-Palencia condition holds, then the matrix $A(\tau)$ in (C.12) satisfies $A(\tau) \rightarrow A$ as $\tau \rightarrow +\infty$, and as (see [6]) the matrix A has eigenvalues with all real parts negative, there exist constants $a > 0$ and $b > 0$ such that the fundamental matrix solution $\Phi(\tau)$ of (C.12) satisfies

$$\|\Phi(\tau_2)\Phi^{-1}(\tau_1)\| \leq ae^{-b(\tau_2-\tau_1)} \quad (\text{C.13})$$

for $0 \leq \tau_1 \leq \tau_2$; in other words, the trivial solution $z = 0$ of (C.12) is exponentially asymptotically stable. Conversely, (C.13) implies that the Sanchez-Palencia condition holds. To see this, note that $z = d\hat{x}_1/d\tau$ is a solution of (C.12), and therefore

$$\frac{d\hat{x}_1}{d\tau} = \Phi(\tau)\Phi^{-1}(0)\frac{d\hat{x}_1}{d\tau}(0).$$

Integrating and using (C.13) gives

$$\|\hat{x}_1(\tau_2) - \hat{x}_1(\tau_1)\| \leq \frac{a}{b} \left(e^{-b\tau_1} - e^{-b\tau_2} \right).$$

From this and the Cauchy criterion, it follows that $\hat{x}_1(\tau)$ approaches an exponentially asymptotically stable rest point of (C.11).

Now we will prove that under hypothesis (C.13), the k -th order averaging is valid for all time. We rewrite the differential equation (C.10) satisfied by the remainder in a way which exhibits it as a perturbation of (C.12), and use (C.13) as an aid in estimating the solution via the Besjes inequality.

Theorem C.4.1 *Let $x_k(t)$ be the k -th order approximation of the solution of (C.4) by the method of averaging, defined by (C.5), and let $R_k(t)$ be its error, given by*

(C.9). Assume that the contracting hypothesis (C.13) is satisfied. Then there exist positive constants c and ε_0 such that

$$\|R_k(t)\| \leq c\varepsilon^k$$

for $t \geq 0$ and $\varepsilon \in [0, \varepsilon_0]$.

Proof. We take up again the differential equation (C.10) satisfied by R_k . If we adopt the expedient of writing $\mathcal{O}(\tilde{\varepsilon}^j)$ for a term which is ε^j times a mean-free function with slow input, (C.10) can be written

$$\frac{dR_k}{dt} = \varepsilon [f(t, x(t)) - f(t, x_k(t))] + \mathcal{O}(\tilde{\varepsilon}^k) + \mathcal{O}(\varepsilon^{k+1}) \quad (\text{C.14})$$

In the discussion after (C.10) it was pointed out that the $\mathcal{O}(\varepsilon^{k+1})$ estimate holds as long as \hat{x}_k remain bounded. In the general case of section 3 this remains bounded on a time-scale $\mathcal{O}(1/\varepsilon)$. Here we will suppose that \hat{x}_k remains bounded for all $t \geq 0$. Therefore (C.14) holds for $t \geq 0$.

Next we use induction on k to show that $R_k = \mathcal{O}(\varepsilon^k)$ for $t \geq 0$. For $k = 1$, it is simplest to work directly from the definition of R_1 rather than from (C.14), although the same conclusions follow either way. Differentiating $R_1 = x - x_1 = x - \hat{x}_1$ gives

$$\begin{aligned} \frac{dR_1}{dt} &= \varepsilon f(t, x) - \varepsilon f^o(\hat{x}_1) \\ &= \varepsilon [f^o(x) - f^o(\hat{x}_1)] + \varepsilon \tilde{f}(t, x) \\ &= \varepsilon [f^o(\hat{x}_1 + R_1) - f^o(\hat{x}_1)] + \varepsilon \tilde{f}(t, x) \\ &= \varepsilon f_x^o(\hat{x}_1) R_1 + \mathcal{O}(\varepsilon \|R_1\|^2) + \mathcal{O}(\tilde{\varepsilon}) \\ &= \varepsilon A(\varepsilon t) R_1 + \mathcal{O}(\varepsilon \|R_1\|^2) + \mathcal{O}(\tilde{\varepsilon}). \end{aligned} \quad (\text{C.15})$$

Here we have expanded f^o in a Taylor expansion in R_1 , and identified $\tilde{f}(t, x(t))$ as a mean-free function with a slow input. For the solution R_1 of (C.15), we may write the integral equation

$$R_1(t) = \int_0^t \Phi(\varepsilon t) \Phi^{-1}(\varepsilon s) [O(\varepsilon \|R_1\|^2) + O(\tilde{\varepsilon})] ds.$$

Using c as the constant to defining the first "big-oh" estimate, restoring the function \tilde{f} in the second, and using (C.13) we have

$$\begin{aligned} \|R_1(t)\| &\leq c\varepsilon \int_0^t a e^{-\varepsilon b(t-s)} \|R_1(s)\|^2 ds \\ &\quad + \varepsilon \|\Phi(\varepsilon t)\| \cdot \left\| \int_0^t \Phi^{-1}(\varepsilon s) \tilde{f}(s, x(s)) ds \right\|. \end{aligned} \quad (\text{C.16})$$

The function $g(t, z_0, z_1) = \Phi^{-1}(z_0)\tilde{f}(t, z_1)$ is mean-free in t for fixed $z_0 \in \mathbf{R}$ and $z_1 \in \mathbf{R}^n$. Therefore the Besjes lemma (Lemma C.2.1) applies, and using (C.13) again (with $\tau_2 = \varepsilon t$ and $\tau_1 = 0$), the last term of (C.16) is bounded by $\varepsilon a e^{-\varepsilon b t}(M + N\varepsilon t)$ for some M and N , which is $\mathcal{O}(\varepsilon)$ for all time. (Its maximum is $\varepsilon(aN/be)e^{bM/N}$). Let $\delta(t)$ denote the maximum of $\|R_1(t)\|$ for $s \in [0, t]$. Then (C.16) and these calculations imply that for some β ,

$$\delta(t) \leq c\varepsilon \int_0^t a e^{-\varepsilon b(t-s)} \delta^2(s) ds + \beta\varepsilon \quad (\text{C.17})$$

for all $t \geq 0$.

We will show that (C.17) implies $\delta(t) = \mathcal{O}(\varepsilon)$ for $t \geq 0$, which will complete proof of the theorem in the leading order case $k = 1$. Since $\delta(0) = 0$, there exists an initial interval $I(\varepsilon)$ of time for which $\delta(t) < \sqrt{\varepsilon}$, that is, $\delta^2(t) < \varepsilon$. Using this on the right hand side of (C.17) and evaluating the integral, we find a constant K such that $\delta(t) < K\varepsilon$ for $t \in I(\varepsilon)$; that is, $\delta < K\varepsilon$ for as long as $\delta < \sqrt{\varepsilon}$, or in other words, δ must exceed $\sqrt{\varepsilon}$ before it can exceed $K\varepsilon$. Choosing $\varepsilon_0 < 1/K^2$ guarantees that for $\varepsilon \in [0, \varepsilon_0]$, $K\varepsilon < \sqrt{\varepsilon}$, so that escape beyond $K\varepsilon$ is impossible, and that the proof is finished.

Now we turn to the inductive step. Assuming that $R_{k-1} = \mathcal{O}(\varepsilon^{k-1})$ it follows (by elementary calculus) that $R_k = \mathcal{O}(\varepsilon^{k-1})$; we wish to show that $R_k = \mathcal{O}(\varepsilon^k)$. We can write

$$R_k = \varepsilon^{k-1} S$$

with $S(t) = \mathcal{O}(1)$ for $t \geq 0$, and we wish to show that $S = \mathcal{O}(\varepsilon)$. From (C.14), after separating f into mean and mean-free parts, we have

$$\begin{aligned} \frac{dR_k}{dt} &= \varepsilon^{k-1} \frac{dS}{dt} \\ &= \varepsilon[f^o(x) - f^o(x_k)] + \varepsilon[\tilde{f}(t, x) - \tilde{f}(t, x_k)] + \mathcal{O}(\tilde{\varepsilon}^k) + \mathcal{O}(\varepsilon^{k+1}). \end{aligned} \quad (\text{C.18})$$

The terms in \tilde{f} are mean-free functions with slow inputs x and x_k , respectively. Also, using a Lipschitz constant for \tilde{f} we see (from the induction hypothesis and the initial factor ε) that the expression is actually of order ε^k ; therefore it can be assimilated to the term $\mathcal{O}(\tilde{\varepsilon}^k)$. On the other hand, we have

$$\begin{aligned} \varepsilon[f^o(x) - f^o(x_k)] &= \varepsilon[f^o(x_k + \varepsilon^{k-1}S) - f^o(x_k)] \\ &= \varepsilon^k f_x^o(x_k)S + \mathcal{O}(\varepsilon(\varepsilon^{k-1}\|S\|)^2) \\ &= \varepsilon^k f_x^o(x_k)S + \mathcal{O}(\varepsilon^{k+1}) \\ &= \varepsilon^k f_x^o(\hat{x}_1)S + \mathcal{O}(\varepsilon^{k+1}) \\ &= \varepsilon^k A(\varepsilon t)S + \mathcal{O}(\varepsilon^{k+1}). \end{aligned}$$

Here we have firstly expanded f^o in a Taylor expansion in $\varepsilon^{k-1}S$, (higher order terms in S have been pushed to order at least ε^{k+1} , and these terms are uniformly of that order for $t \geq 0$, because S is known, by the inductive hypothesis, to be bounded). As $\hat{x}_k = \hat{x}_1 + \mathcal{O}(\varepsilon)$, we have $x_k = \hat{x}_1 + \mathcal{O}(\varepsilon)$, and then we have secondly expanded f_x^o in a Taylor expansion in $\mathcal{O}(\varepsilon)$. Now (C.18) becomes

$$\frac{dS}{dt} = \varepsilon A(\varepsilon t)S + \mathcal{O}(\tilde{\varepsilon}) + \mathcal{O}(\varepsilon^2).$$

Expressing it as an integral equation, taking norms, and using the Besjes inequality on the $\mathcal{O}(\tilde{\varepsilon})$ term, immediatly shows that $S = \mathcal{O}(\varepsilon)$. \square

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Appendix D

On solutions of linear difference equations with infinite delay

By **Mustapha LAKRIB**¹

Abstract. The difference equation with infinite delay $(DE)_\varepsilon x_{n+1} = Ax_n + \varepsilon \sum_{k=-\infty}^n R_{n-k}x_k$ is considered. Sufficient conditions for the existence of a unique solution are proved. An $O(\varepsilon)$ -approximation of this solution for $-N \leq n \leq N$, $N > 0$, by the solution of the equation $(DE)_0$ is obtained.

(1991) *A.M.S Subject Classification Codes.* 45D, 34C

Keywords and Phrases. Difference equation, Infinite delay, Approximation

D.1 Introduction

In [1], Ryabov considers a problem concerning the existence of so-called two-sided solutions of linear integrodifferential equations of the form

$$\frac{dx}{dt} = Ax + \varepsilon \int_{-\infty}^t R(t-s)x(s)ds, \quad (\text{D.1})$$

where ε is a positive small parameter, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, A is a constant $m \times m$ matrix and R is a matrix of continuous functions satisfying the inequality

$$\|R(t)\| \leq c \frac{e^{-\gamma t}}{t^{1-\alpha}},$$

where c, γ, α are positive constants and $0 < \alpha < 1$. It is proven there that if $\min\{\operatorname{Re}(\lambda_i) : 1 \leq i \leq m\} > -\gamma$, where $\lambda_1, \dots, \lambda_m$ are eigenvalues of A , then there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any $x_0 \in \mathbb{R}^m$ the equation (D.1) has

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a unique solution x_ε (so-called two-sided solution) defined on the entire axis t for every given initial condition $x_\varepsilon(0) = x_0$ and satisfying $\lim_{\varepsilon \rightarrow 0} \|x_\varepsilon - x\|_L = 0$ for any $L > 0$, where $\|x_\varepsilon - x\|_L = \sup\{\|x_\varepsilon(t) - x(t)\| : -L \leq t \leq L\}$, $x(t) = e^{At}x_0$.

The equation (D.1) has the discretization

$$x_{n+1} = Ax_n + \varepsilon(R_0x_n + R_1x_{n-1} + \cdots + R_kx_{n-k} + \cdots), \quad (\text{D.2})$$

which is a difference equation with infinite delay. Our purpose in this work is to study (D.2) in a Banach space E , with $A, R_k \in \mathcal{L}(E)$, the space of continuous linear mappings from E into E ($k = 0, 1, \dots$), A is invertible and $A^{-1} \in \mathcal{L}(E)$. Under sufficient conditions, we shall prove the existence of solutions of (D.2) determined uniquely by the initial value and defined for all integers. These solutions are approximated, to order ε , by those one of the equation obtained from (D.2) for $\varepsilon = 0$. This is performed in Section D.2 below.

D.2 Main results

The existence, uniqueness and approximation results are stated in the following theorem.

Theorem D.2.1 *Suppose that the following conditions are satisfied*

$$\|A^{-1}\| < 1, \quad \|R_0\| \leq 1, \quad \|R_n\| \leq \frac{\gamma^{-n}}{n^{1-\alpha}}, \quad n = 1, 2, \dots, \quad (\text{D.3})$$

where γ, α are constants, $e \leq \gamma$, $0 < \alpha < 1$.

Then there exist a positive constant ε_0 and a unique operator $B \in \mathcal{L}(E)$, such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any $x_0 \in E$, the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A + \varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$ is the unique solution of (D.2) satisfying the condition $x_0(\varepsilon) = x_0$ with $x \in F := \{y = \{y_n\}_{n=-\infty}^{\infty} : y_n \in E, \sup\{|y_n| : -\infty < n \leq 0\} < \infty\}$, where $|\cdot|$ is the norm on E . Moreover, for any positive integer N , there exists a positive constant C (depending on ε_0, x_0 and N) such that $\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon$.

Proof. Let us seek the solution of (D.2) in the form $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A + \varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$, where $B \in \mathcal{L}(E)$ is such that $AB = BA$ and $A + \varepsilon B$ is invertible. The matrix B satisfy the equation

$$B = P_\varepsilon(B) := R_0 + R_1(A + \varepsilon B)^{-1} + \cdots + R_k(A + \varepsilon B)^{-k} + \cdots. \quad (\text{D.4})$$

Consider the mapping $P_\varepsilon : \mathcal{D} \rightarrow \mathcal{L}(E)$, defined by (D.4), where $\mathcal{D} = \{B \in \mathcal{L}(E) : AB = BA \text{ and } \|B\| \leq 1\}$, $\|B\| := \sup\{|Bx| : |x| \leq 1\}$. If $B_1, B_2 \in \mathcal{D}$ then

$$\begin{aligned} & \|P_\varepsilon(B_1) - P_\varepsilon(B_2)\| \\ &= \|R_1[(A + \varepsilon B_1)^{-1} - (A + \varepsilon B_2)^{-1}] + R_2[(A + \varepsilon B_1)^{-2} \\ & \quad - (A + \varepsilon B_2)^{-2}] + \cdots + R_k[(A + \varepsilon B_1)^{-k} - (A + \varepsilon B_2)^{-k}] + \cdots\|. \end{aligned}$$

Since $(A + \varepsilon B_i) = A(I + \varepsilon A^{-1} B_i)$ and $AB_i = B_i A$, $i = 1, 2$ where I is the unit matrix, we have

$$\begin{aligned} & \|P_\varepsilon(B_1) - P_\varepsilon(B_2)\| \\ & \leq \|R_1\| \|A^{-1}\| \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \\ & \quad + \|R_2\| \|A^{-1}\|^2 \|(I + \varepsilon A^{-1} B_1)^{-2} - (I + \varepsilon A^{-1} B_2)^{-2}\| + \dots \\ & \quad + \|R_k\| \|A^{-1}\|^k \|(I + \varepsilon A^{-1} B_1)^{-k} - (I + \varepsilon A^{-1} B_2)^{-k}\| + \dots \end{aligned} \quad (\text{D.5})$$

We have the following estimate

$$\begin{aligned} & \|(I + \varepsilon A^{-1} B_1)^{-k} - (I + \varepsilon A^{-1} B_2)^{-k}\| \\ & = \|[(I + \varepsilon A^{-1} B_1)^{-1}]^k - [(I + \varepsilon A^{-1} B_2)^{-1}]^k\| \\ & \leq \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \|(I + \varepsilon A^{-1} B_1)^{-1}\|^{k-1} \\ & \quad + \|(I + \varepsilon A^{-1} B_1)^{-1}\|^{k-2} \|(I + \varepsilon A^{-1} B_2)^{-1} - (I + \varepsilon A^{-1} B_1)^{-1}\| \|(I + \varepsilon A^{-1} B_1)^{-1}\|^{k-2} \\ & \quad + \dots + \|(I + \varepsilon A^{-1} B_2)^{-1}\|^{k-1} \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \\ & \leq \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \{ \|(I + \varepsilon A^{-1} B_1)^{-1}\|^{k-1} \\ & \quad + \|(I + \varepsilon A^{-1} B_1)^{-1}\|^{k-2} \|(I + \varepsilon A^{-1} B_2)^{-1}\| + \dots + \|(I + \varepsilon A^{-1} B_2)^{-1}\|^{k-1} \}. \end{aligned}$$

As for $B_1, B_2 \in \mathcal{D}$ and $\varepsilon \in (0, \varepsilon_1]$, where $\varepsilon_1 < 1/\|A^{-1}\|$, we have

$$\begin{aligned} & \|(I + \varepsilon A^{-1} B_i)^{-1}\| \\ & = \|I - \varepsilon(A^{-1} B_i) + \varepsilon^2(A^{-1} B_i)^2 - \dots\| \\ & \leq 1 + \varepsilon\|A^{-1}\| + \varepsilon^2\|A^{-1}\|^2 + \dots \\ & = \frac{1}{1 - \varepsilon\|A^{-1}\|}, \end{aligned}$$

we obtain the inequality

$$\begin{aligned} & \|(I + \varepsilon A^{-1} B_1)^{-k} - (I + \varepsilon A^{-1} B_2)^{-k}\| \\ & \leq \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \frac{k}{(1 - \varepsilon\|A^{-1}\|)^{k-1}}. \end{aligned} \quad (\text{D.6})$$

Thus, for all $B_1, B_2 \in \mathcal{D}$ and $\varepsilon \in (0, \varepsilon_1]$, from (D.5) we obtain the estimate

$$\begin{aligned} & \|P_\varepsilon(B_1) - P_\varepsilon(B_2)\| \\ & \leq \{ \|R_1\| \|A^{-1}\| + \|R_2\| \|A^{-1}\|^2 \frac{2}{1 - \varepsilon\|A^{-1}\|} + \dots \\ & \quad + \|R_k\| \|A^{-1}\|^k \frac{k}{(1 - \varepsilon\|A^{-1}\|)^{k-1}} + \dots \} \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \\ & \leq S_1 \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\|, \end{aligned} \quad (\text{D.7})$$

where $S_1 = (1 - \varepsilon\|A^{-1}\|) \sum_{k=1}^{\infty} k \|R_k\| \frac{\|A^{-1}\|^k}{(1 - \varepsilon\|A^{-1}\|)^k}$. As $\|A^{-1}\| < \gamma$ then $\gamma^{-1} < 1/\|A^{-1}\|$. Choose ε_2 sufficiently small such that $\gamma^{-1} + \varepsilon < 1/\|A^{-1}\|$ for $\varepsilon \in (0, \varepsilon_2]$. We obtain

$$\frac{\gamma^{-1}\|A^{-1}\|}{1 - \varepsilon\|A^{-1}\|} < 1 \quad \text{for } \varepsilon \in (0, \varepsilon_2]. \quad (\text{D.8})$$

The conditions (D.3) and (D.8) insure the convergence of the series S_1 .

The following estimate is needed

$$\begin{aligned}
& \|(I + \varepsilon A^{-1} B_1)^{-1} - (I + \varepsilon A^{-1} B_2)^{-1}\| \\
&= \|(I - \varepsilon(A^{-1} B_1) + \varepsilon^2(A^{-1} B_1)^2 - \dots) \\
&\quad - (I - \varepsilon(A^{-1} B_2) + \varepsilon^2(A^{-1} B_2)^2 - \dots)\| \\
&\leq \varepsilon \|A^{-1} B_1 - A^{-1} B_2\| + \varepsilon^2 \|(A^{-1} B_1)^2 - (A^{-1} B_2)^2\| + \dots \\
&\quad + \varepsilon^k \|(A^{-1} B_1)^k - (A^{-1} B_2)^k\| + \dots.
\end{aligned} \tag{D.9}$$

The mean value theorem yields

$$\begin{aligned}
& \|(A^{-1} B_1)^k - (A^{-1} B_2)^k\| \\
&\leq \|A^{-1}\|^k \|B_1^k - B_2^k\| \\
&\leq \|A^{-1}\|^k \sup\{\|k[(1 - \theta)B_1 + \theta B_2]^{k-1}\| : 0 \leq \theta \leq 1\} \|B_1 - B_2\| \\
&\leq k \|A^{-1}\|^k \|B_1 - B_2\|.
\end{aligned} \tag{D.10}$$

From (D.7), (D.9) and (D.10) it follows that for all $B_1, B_2 \in \mathcal{D}$ and $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\}]$, we have

$$\|P_\varepsilon(B_1) - P_\varepsilon(B_2)\| \leq \varepsilon \|A^{-1}\| S_1 S_2 \|B_1 - B_2\|, \tag{D.11}$$

where $S_2 = \sum_{k=0}^{\infty} (k+1) \varepsilon^k \|A^{-1}\|^k$ is a convergent series. Therefore the inequality

(D.11) implies that if $\varepsilon \in (0, \varepsilon_3]$, where $\varepsilon_3 < \frac{1}{\|A^{-1}\| S_1 S_2}$, then the mapping P_ε is a contraction on \mathcal{D} and thus it has a unique fixed point in \mathcal{D} . So if we set $x_n(\varepsilon) = (A + \varepsilon B)^n x_0$, where B is the unique fixed point of P_ε , the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{\infty}$ is a solution of (D.2) satisfying the condition $x_0(\varepsilon) = x_0$. As $\|A^{-1}\| < 1$, choose ε_4 sufficiently small, such that $\|(A + \varepsilon B)^{-p}\| \leq \|(A + \varepsilon B)^{-1}\|^p < 1$ for all positive integer p and $\varepsilon \in (0, \varepsilon_4]$. This implies that $\sup\{|x_n(\varepsilon)| : -\infty < n \leq 0\} \leq |x_0| < \infty$, i.e. $x = \{x_n(\varepsilon)\}_{n=-\infty}^{\infty} \in F$.

We shall prove the uniqueness of the solution in F of (D.2).

Consider the space

$$\tilde{F} := \{x = \{x_n\}_{n=-\infty}^0 : x_n \in E, \sup\{|x_n| : -\infty < n \leq 0\} < \infty\}$$

which is a Banach space with the norm $\|x\| := \sup\{|x_n| : -\infty < n \leq 0\}$.

Let $x_n = A^n y_n$, $-\infty < n \leq 0$. Then by substituting into (D.2), we obtain

$$y_{n+1} = y_n + \varepsilon A^{-(n+1)} (R_0 A^n y_n + R_1 A^{n-1} y_{n-1} + \dots + R_k A^{n-k} y_{n-k} + \dots).$$

Observe that

$$\begin{aligned}
y_0 - y_{-1} &= \varepsilon (R_0 A^{-1} y_{-1} + R_1 A^{-2} y_{-2} + \dots + R_k A^{-1-k} y_{-1-k} + \dots), \\
y_{-1} - y_{-2} &= \varepsilon A (R_0 A^{-2} y_{-2} + R_1 A^{-3} y_{-3} + \dots + R_k A^{-2-k} y_{-2-k} + \dots), \\
&\vdots \\
y_{-(p-1)} - y_{-p} & \text{ots} \\
&\quad + \varepsilon (R_0 A^{-1} y_{-1} + R_1 A^{-2} y_{-2} + \dots + R_k A^{-1-k} y_{-1-k} + \dots).
\end{aligned}$$

A return to the initial variable gives

$$\begin{aligned} A^{-p}x_0 - x_{-p} &= \varepsilon A^{-1}(R_0x_{-p} + R_1x_{-p-1} + \cdots + R_kx_{-p-k} + \cdots) \\ &\quad + \cdots + \varepsilon A^{-p+1}(R_0x_{-2} + R_1x_{-3} + \cdots + R_kx_{-2-k} + \cdots) \\ &\quad + \varepsilon A^{-p}(R_0x_{-1} + R_1x_{-2} + \cdots + R_kx_{-1-k} + \cdots). \end{aligned}$$

Define the mapping

$$Q_\varepsilon : \tilde{F} \rightarrow \{x = \{x_n\}_{n=-\infty}^0, x_n \in E\},$$

by

$$\begin{aligned} (Q_\varepsilon x)_{-p} &= \varepsilon A^{-1}(R_0x_{-p} + R_1x_{-p-1} + \cdots + R_kx_{-p-k} + \cdots) + \cdots \\ &\quad + \varepsilon A^{-p}(R_0x_{-1} + R_1x_{-2} + \cdots + R_kx_{-1-k} + \cdots), \end{aligned}$$

for all positive integer p .

If $x = \{x_n\}_{n=-\infty}^0 \in \tilde{F}$, we have for all positive integer p the estimate

$$\begin{aligned} |(Q_\varepsilon x)_{-p}| &\leq \varepsilon \|x\| \{ \|A^{-1}\| (\|R_0\| + \|R_1\| + \cdots + \|R_k\| + \cdots) + \cdots \\ &\quad + \|A^{-1}\|^p (\|R_0\| + \|R_1\| + \cdots + \|R_k\| + \cdots) \} \\ &\leq \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} (1 + \gamma^{-1} + \cdots + \frac{\gamma^{-k}}{k^{1-\alpha}} + \cdots) \\ &\leq \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} \int_0^\infty e^{-t} t^{\alpha-1} dt = \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} \Gamma(\alpha), \end{aligned}$$

where $\Gamma(\alpha)$ is Euler's Gamma-function. Thus, for $x \in \tilde{F}$, we have $Q_\varepsilon x \in \tilde{F}$. Since Q_ε is linear, it satisfies

$$\|Q_\varepsilon x_1 - Q_\varepsilon x_2\| = \|Q_\varepsilon(x_1 - x_2)\| \leq \frac{\varepsilon \Gamma(\alpha)}{1 - \|A^{-1}\|} \|x_1 - x_2\|$$

for all $x_1, x_2 \in \tilde{F}$. This implies that if $\varepsilon \in (0, \varepsilon_5]$, where $\varepsilon_5 < \frac{1 - \|A^{-1}\|}{\Gamma(\alpha)}$, then the mapping Q_ε is a contraction on \tilde{F} and thus has a unique fixed point $x^* \in \tilde{F}$. Since Q_ε is linear, we conclude that $x^* = 0$.

Let $x = \{x_n\}_{n=-\infty}^\infty, y = \{y_n\}_{n=-\infty}^\infty \in F$ be two solutions of (D.2), satisfying the condition $y_0 = x_0$ and let $x^* = \{y_n - x_n\}_{n=-\infty}^0$. The sequence x^* is in \tilde{F} and is the fixed point of Q_ε and therefore $x^* = 0$, i.e. $y_n = x_n$, for all $n \leq 0$. This prove the uniqueness of the solution of (D.2) for all $n \leq 0$.

On the other hand, from (D.2) and for $n > 0$ we have

$$\begin{aligned} &|x_n - y_n| \\ &= \varepsilon \left| \sum_{i=0}^{n-1} A^{n-i-1} [R_0(x_i - y_i) + R_1(x_{i-1} - y_{i-1}) + \cdots + R_k(x_{i-k} - y_{i-k}) + \cdots] \right|. \end{aligned}$$

As $y_k = x_k$ for all integer $k \leq 0$, the above equality gives

$$\begin{aligned} & |x_n - y_n| \\ & \leq \varepsilon \{ \|A\|^{n-1} |x_0 - y_0| + \|A\|^{n-2} [|x_1 - y_1| + \gamma^{-1} |x_0 - y_0|] \\ & + \cdots + [|x_{n-1} - y_{n-1}| + \gamma^{-1} |x_{n-2} - y_{n-2}| + \cdots + \frac{\gamma^{-(n-2)}}{(n-2)^{1-\alpha}} |x_0 - y_0|] \}. \end{aligned}$$

From this inequality, it follows that

$$|x_1 - y_1| \leq \varepsilon |x_0 - y_0| = 0,$$

$$|x_2 - y_2| \leq \varepsilon \{ \|A\| [|x_0 - y_0| + |x_1 - y_1|] \} = 0,$$

etc. Therefore, by induction we have that $y_n = x_n$ for all $n > 0$.

Consequently, the solution in F of (D.2) is uniquely defined by the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A + \varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$.

Finally, we shall prove that for any positive integer N , there exists a positive constant C , depending on ε_0 , x_0 and N , where $\varepsilon_0 = \min\{\varepsilon_i : 1 \leq i \leq 5\}$, such that $\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon$.

Let n be an integer such that $0 \leq n < N$. By use of the binomial formula, we deduce the following estimate

$$\begin{aligned} |x_n(\varepsilon) - A^n x_0| & \leq |(A + \varepsilon B)^n - A^n| |x_0| \\ & \leq \| (A + \varepsilon B)^n - A^n \| |x_0| \\ & \leq \varepsilon |x_0| \{ C_n^{n-1} \|A\|^{n-1} \|B\| + \varepsilon C_n^{n-2} \|A\|^{n-2} \|B\|^2 + \cdots + \varepsilon^{n-1} \} \\ & \leq c_1 \varepsilon, \end{aligned}$$

where

$$c_1 = c_1(\varepsilon_0, x_0, N) = |x_0| \{ C_N^{N-1} \|A\|^{N-1} + \varepsilon_0 C_N^{N-2} \|A\|^{N-2} + \cdots + \varepsilon_0^{N-1} \},$$

and

$$C_n^p = \frac{n!}{p!(n-p)!} \quad \text{for } 0 \leq p \leq n \leq N.$$

For $-N \leq n < 0$, we set $n = -k$. Then we have

$$\begin{aligned} |x_n(\varepsilon) - A^n x_0| & \leq |[(A + \varepsilon B)^n - A^n] x_0| \\ & \leq \| (A + \varepsilon B)^n - A^n \| |x_0| \\ & = |x_0| \| (A + \varepsilon B)^{-k} - A^{-k} \| \\ & = |x_0| \| [(A + \varepsilon B)^{-1}]^k - [A^{-1}]^k \| \\ & \leq |x_0| \| (A + \varepsilon B)^{-1} - A^{-1} \| \{ [(A + \varepsilon B)^{-1}]^{k-1} \\ & \quad + [(A + \varepsilon B)^{-1}]^{k-2} A^{-1} + \cdots + [A^{-1}]^{k-1} \} \\ & \leq |x_0| \| A^{-1} \| \| (I + \varepsilon A^{-1} B)^{-1} - I \| \{ \| (A + \varepsilon B)^{-1} \|^{k-1} \\ & \quad + \| A^{-1} \| \| (A + \varepsilon B)^{-1} \|^{k-2} + \cdots + \| A^{-1} \|^{k-1} \}. \end{aligned} \tag{D.12}$$

As we have seen above, $\|(A + \varepsilon B)^{-1}\|^k < 1$, for all positive integer k and $\varepsilon \in (0, \varepsilon_0] \subset (0, \varepsilon_4]$. On the other hand, we have

$$\begin{aligned} \|(I + \varepsilon A^{-1}B)^{-1} - I\| &= \|(I - (\varepsilon A^{-1}B) + (\varepsilon A^{-1}B)^2 - \dots) - I\| \\ &\leq \varepsilon \|A^{-1}\| \{1 + \varepsilon \|A^{-1}\| + \varepsilon^2 \|A^{-1}\|^2 + \dots\} \\ &\leq \varepsilon \frac{\|A^{-1}\|}{1 - \varepsilon \|A^{-1}\|}. \end{aligned}$$

Therefore, from (D.12) we obtain

$$\begin{aligned} |x_n(\varepsilon) - A^n x_0| &\leq \varepsilon |x_0| \frac{\|A^{-1}\|^2}{1 - \varepsilon \|A^{-1}\|} \{1 + \|A^{-1}\| + \dots + \|A^{-1}\|^{k-1}\} \\ &\leq \varepsilon |x_0| \frac{\|A^{-1}\|^2}{1 - \varepsilon \|A^{-1}\|} \cdot \frac{1}{1 - \|A^{-1}\|} \\ &\leq c_2 \varepsilon, \end{aligned}$$

where

$$c_2 = c_2(\varepsilon_0, x_0) = |x_0| \frac{\|A^{-1}\|^2}{1 - \varepsilon_0 \|A^{-1}\|} \cdot \frac{1}{1 - \|A^{-1}\|}.$$

Hence, the following inequality holds for $C = \min\{c_1, c_2\}$

$$\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon.$$

This completes the proof of the theorem. \square

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Appendix E

Oscillation of two delays differential equations with positive and negative coefficients

By Mustapha LAKRIB¹

Abstract. Sufficient conditions for the oscillation of all the solutions of the first order delay differential equation with positive and negative coefficients

$$\dot{x}(t) + p(t)x(t - \tau) - q(t)x(t - \sigma) = 0$$

are given.

MSC 2000. 34K15, 34C10

Keywords. Oscillation, Delay differential equations

E.1 Introduction

The oscillation theory of delay differential equations has been mostly developed during the past few years. This is motivated by the many applications of delay differential equations in physics, biology, ecology and physiology. We refer, for example, to [5, 9, 11, 13, 15, 27, 30, 32] and to the references cited therein.

The purpose of this paper is to consider the delay differential equation

$$\dot{x}(t) + p(t)x(t - \tau) - q(t)x(t - \sigma) = 0, \tag{E.1}$$

where p and $q \in \mathcal{C}([t_0, +\infty), \mathbf{R}^+)$ and $\tau, \sigma \in \mathbf{R}^+$.

By a solution of equation (E.1) on $[t_0, +\infty)$, where $t_0 \geq 0$, we mean a continuous function defined on $[t_0 - \max\{\tau, \sigma\}, +\infty)$, which is a differentiable function

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x on $[t_0, +\infty)$ and satisfies equation (E.1) for all $t \geq t_0$. Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of the delay differential equation with positive coefficient

$$\dot{x}(t) + p(t)x(\tau(t)) = 0 \quad (\text{E.2})$$

was undertaken by Mishkis. In 1950 [31], he proved that every solution of the equation (E.2) oscillates if

$$\limsup_{t \rightarrow +\infty} [t - \tau(t)] < +\infty, \quad \liminf_{t \rightarrow +\infty} [t - \tau(t)] \cdot \liminf_{t \rightarrow +\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [25] proved that the same conclusion holds if

$$\limsup_{t \rightarrow +\infty} \int_{\tau(t)}^t p(s) ds > 1. \quad (\text{E.3})$$

In 1979, Ladas [24] and in 1982, Koplatadze and Chanturiya [18] improved (E.3) to

$$\liminf_{t \rightarrow +\infty} \int_{\tau(t)}^t p(s) ds > \frac{1}{e}. \quad (\text{E.4})$$

Concerning the constant $1/e$ in (E.4), it is to be noted that if the inequality

$$\int_{\tau(t)}^t p(s) ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [18], the equation (E.2) has a non-oscillatory solution.

How to fill the gap between the conditions (E.3) and (E.4) when the limit

$$\lim_{t \rightarrow +\infty} \int_{\tau(t)}^t p(s) ds$$

does not exist, is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [10] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible non-oscillatory solutions x of the equation (E.2). Their result says that all the solutions of the equation (E.2) are oscillatory if $0 < m \leq 1/e$ and

$$M > 1 - \frac{m^2}{4} \quad (\text{E.5})$$

where

$$m = \liminf_{t \rightarrow +\infty} \int_{t-\tau}^t p(s) ds \quad \text{and} \quad M = \limsup_{t \rightarrow +\infty} \int_{t-\tau}^t p(s) ds.$$

Since then, several authors obtained better results by improving the upper bound for $x(\tau(t))/x(t)$. Among them, we can cite Chao [4], Yu and Wang [33] and Yu, Wang, Zhang and Qian [34].

In 1990, Elbert and Stavroulakis [8] and in 1991, Kwong [23], using different techniques, improved condition (E.5) in the case where $0 < m \leq 1/e$ to the conditions

$$M > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2$$

and

$$M > \frac{\ln \lambda_1 + 1}{\lambda_1},$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$.

We also mention that in the case where

$$\int_{\tau(t)}^t p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \int_{\tau(t)}^t p(s)ds = \frac{1}{e},$$

this problem has been studied in 1995 by Elbert and Stavroulakis [8], Kozakiewicz [19] and Li [28], and in 1996 by Li [29] and by Domshlak and Stavroulakis [6].

In 1998, Domshlak and Stavroulakis [7] and in 1999, Jaros and Stavroulakis [17] established sufficient conditions for the oscillation of all solutions of the equation (E.1) in the critical state that the corresponding limiting equation admits a non-oscillatory solution.

Among several other works devoted to the study of oscillatory properties of delay differential equations, we can cite the papers by Agwo [1], Arino and Györi [2], Arino, Ladas and Sficas [3], Gopalsamy [12], Gopalsamy and Ladas [14], Györi and Ladas [15], Györi, Ladas and Pakula [16], Kulenovic and Ladas [20, 21], Kulenovic, Ladas and Meimaridou [22] and Ladas and Stavroulakis [26].

In this paper, we provide new sufficient conditions for the oscillation of all the solutions of the equation (E.1) by means of the generalized characteristic equation.

E.2 The main result

We first give some results needed in the proof of our main theorem (Theorem E.2.1).

Lemma E.2.1 *Let $x \in \mathbb{R}$ and $r > 0$. Then*

$$re^{rx} \geq rx + \ln er. \tag{E.6}$$

Proof. Write the right hand side of (E.6) as

$$\begin{aligned} rx + \ln er &= \ln e^{rx} + \ln er \\ &= \ln e + \ln re^{rx} \\ &= 1 + \ln re^{rx}. \end{aligned}$$

So, inequality (E.6) becomes

$$re^{rx} \geq 1 + \ln re^{rx}$$

or

$$re^{rx} - \ln re^{rx} \geq 1. \quad (\text{E.7})$$

Let $z = re^{rx}$ and consider the function $f(z) = z - \ln z$. In terms of z and $f(z)$, inequality (E.7) reads

$$f(z) \geq 1.$$

Note that $f(1) = 1$ and $\frac{d}{dz}f(z) = \frac{z-1}{z}$, which implies that f admits 1 as a minimum and then $f(z) \geq 1$ for all $z > 0$. \square

Lemma E.2.2 ([15]) *Suppose that $x \in \mathcal{C}([t_0, +\infty), \mathbb{R})$ satisfies the inequality*

$$x(t) \leq c + \max_{t-\tau \leq s \leq t} x(s) \quad \text{for } t \geq t_0,$$

where $c \leq 0$, $\tau \in \mathbb{R}^+$ and $t_0 \in \mathbb{R}$. Then x cannot be a nonnegative function.

Lemma E.2.3 ([15]) *Let $p_i \in \mathcal{C}([t_0, +\infty), \mathbb{R}^+)$ and $\tau_i \in \mathbb{R}^+$, $i = 1, \dots, n$. The differential inequality*

$$\dot{x}(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i) \leq 0, \quad t \geq t_0$$

has an eventually positive solution if and only if the differential equation

$$\dot{y}(t) + \sum_{i=1}^n p_i(t)y(t - \tau_i) = 0, \quad t \geq t_0$$

has an eventually positive solution.

Consider the delay differential equation

$$\dot{y}(t) + a(t)y(t - \tau) = 0, \quad t \geq t_0, \quad (\text{E.8})$$

and let

$$m = \liminf_{t \rightarrow +\infty} \int_{t-\tau}^t a(s)ds \quad \text{and} \quad M = \limsup_{t \rightarrow +\infty} \int_{t-\tau}^t a(s)ds.$$

Lemma E.2.4 ([17]) *Suppose that $m > 0$ and the equation (E.8) has an eventually positive solution y . Then $m \leq 1/e$ and*

$$M \leq c_1 = \frac{1 + \ln \lambda_1}{\lambda_1} - L,$$

where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$ and $L = \liminf_{t \rightarrow +\infty} \frac{y(t)}{y(t - \tau)}$.

Lemma E.2.5 ([33]) *Let $0 < m \leq 1/e$ and let y be an eventually positive solution of the equation (E.8). Then*

$$\limsup_{t \rightarrow +\infty} \frac{y(t - \tau)}{y(t)} \leq c_2 = \frac{2}{1 - m - \sqrt{1 - 2m - m^2}}.$$

Lemma E.2.6 *Consider the delay differential equation (E.1) and assume that the following conditions hold:*

$$(H1) : \quad p, q \in \mathcal{C}([t_0, +\infty), \mathbb{R}^+), \tau, \sigma \in \mathbb{R}^+ \text{ and } \tau \geq \sigma,$$

$$(H2) : \quad p(t) \geq q(t + \sigma - \tau) \text{ for } t \geq t_0 + \tau - \sigma,$$

$$(H3) : \quad \int_{t-\tau}^{t-\sigma} q(s + \sigma) ds \leq 1 \text{ for } t \geq t_0 + \tau,$$

Let x be an eventually positive solution of the equation (E.1) and X the function defined by

$$X(t) = x(t) - \int_{t-\tau}^{t-\sigma} q(s + \sigma)x(s)ds, \quad t \geq t_0 + \tau - \sigma. \quad (\text{E.9})$$

Then X is decreasing and positive.

Proof. By differentiation, (E.9) gives

$$\dot{X}(t) = \dot{x}(t) - q(t)x(t - \sigma) + q(t + \sigma - \tau)x(t - \tau).$$

By (E.1) we see that

$$\dot{X}(t) = -[p(t) - q(t + \sigma - \tau)]x(t - \tau). \quad (\text{E.10})$$

Because that x is positive for $t \geq t_1 - \tau$, where $t_1 \geq t_0 + \tau$, and from hypothesis (H2), we conclude that

$$\dot{X}(t) < 0 \quad \text{for} \quad t \geq t_1 + \tau, \quad (\text{E.11})$$

which implies that X is decreasing on $[t_1 + \tau, +\infty)$.

Now, by contradiction we prove that X is positive. Suppose that there exists a $t_2 \geq t_1$ such that $X(t_2) \leq 0$. Inequality (E.11) implies that there exists a $t_3 \geq t_2$ such that $X(t) \leq X(t_3) \leq 0$ for $t \geq t_3$. From (E.9) it follows that for $t \geq t_3$ we have

$$\begin{aligned} x(t) &= X(t) + \int_{t-\tau}^{t-\sigma} q(s + \sigma)x(s)ds \\ &\leq X(t_3) + \int_{t-\tau}^{t-\sigma} q(s + \sigma)x(s)ds \\ &\leq X(t_3) + \left(\max_{t-\tau \leq s \leq t-\sigma} x(s) \right) \int_{t-\tau}^{t-\sigma} q(s + \sigma)ds. \end{aligned}$$

Hypothesis (H3) yields

$$x(t) \leq X(t_3) + \max_{t-\tau \leq s \leq t-\sigma} x(s) \quad \text{for } t \geq t_3.$$

By Lemma E.2.2, x cannot be nonnegative on $[t_3, +\infty)$, which is a contradiction to the assumptions of our lemma. This completes the proof. \square

Our main result about the oscillation of all solutions of the equation (E.1) is embodied in the following:

Theorem E.2.1 *Let the hypotheses (H1) to (H3) of Lemma E.2.6 are true. Let $a(t) = p(t) - q(t + \sigma - \tau)$ and assume that*

$$(H4) : \liminf_{t \rightarrow +\infty} \int_{t-\tau}^t a(s) ds > 0$$

$$(H5) : \limsup_{t \rightarrow +\infty} \int_{\bar{t}_0}^t a(s) \ln \left(e \int_s^{s+\tau} a(u) du \right) ds = +\infty, \text{ for some } \bar{t}_0.$$

Then every solution of the equation (E.1) is oscillatory.

Proof. For the sake of contradiction, we assume that (E.1) has a positive solution x . (For the case that (E.1) has a negative solution \bar{x} , we simply let $x = -\bar{x}$). By Lemma E.2.6, the function X defined by (E.9) is positive. Also by (E.10) we have

$$\dot{X}(t) + [p(t) - q(t + \sigma - \tau)]x(t - \tau) = 0.$$

Since $0 < X(t) \leq x(t)$ and $p(t) \geq q(t + \sigma - \tau)$ (see (H2)), X satisfies

$$\dot{X}(t) + [p(t) - q(t + \sigma - \tau)]X(t - \tau) \leq 0.$$

By Lemma E.2.3, the delay differential equation

$$\dot{y}(t) + [p(t) - q(t + \sigma - \tau)]y(t - \tau) = 0 \tag{E.12}$$

has a positive solution. Let y be such a solution. Note that y is decreasing for sufficiently large t . Dividing both sides of equation (E.12) by $y(t)$, we have

$$\frac{\dot{y}(t)}{y(t)} + [p(t) - q(t + \sigma - \tau)] \frac{y(t - \tau)}{y(t)} = 0.$$

Integrating both sides of this equation from $t - \tau$ to t , for sufficiently large t , we have

$$\ln \frac{y(t)}{y(t - \tau)} + \int_{t-\tau}^t [p(s) - q(s + \sigma - \tau)] \frac{y(s - \tau)}{y(s)} ds = 0.$$

Let $W(t) = \frac{y(t-\tau)}{y(t)}$. By the last equation, we have

$$\int_{t-\tau}^t [p(s) - q(s + \sigma - \tau)]W(s)ds = \ln W(t),$$

or

$$W(t) = \exp \left(\int_{t-\tau}^t a(s)W(s)ds \right),$$

where $a(t) = p(t) - q(t + \sigma - \tau)$. Multiplying the factor $a(t)$ in both sides of this equation, the function $\alpha(t) = a(t)W(t)$ satisfies the generalized characteristic equation

$$\alpha(t) = a(t) \exp \left(\int_{t-\tau}^t \alpha(s)ds \right).$$

Let $r(t) = \int_{t-\tau}^t a(s)ds$ and $x(t) = \frac{1}{r(t)} \int_{t-\tau}^t \alpha(s)ds$. The equation above is equivalent to

$$r(t)\alpha(t) = a(t)r(t)e^{r(t)x(t)}.$$

By Lemma E.2.1 we get

$$r(t)\alpha(t) \geq a(t) [r(t)x(t) + \ln er(t)],$$

which is equivalent to

$$r(t)\alpha(t) \geq a(t) \left[\int_{t-\tau}^t \alpha(s)ds + \ln er(t) \right]$$

or

$$\left(\int_{t-\tau}^t a(s)ds \right) \alpha(t) - a(t) \int_{t-\tau}^t \alpha(s)ds \geq a(t) \ln \left(e \int_{t-\tau}^t a(s)ds \right).$$

Integrating both sides of this inequality from \bar{t}_0 to T , for $T > \bar{t}_0$, \bar{t}_0 sufficiently large, we have

$$\begin{aligned} & \int_{\bar{t}_0}^T \left(\int_{t-\tau}^t a(s)ds \right) \alpha(t)dt - \int_{\bar{t}_0}^T a(t) \int_{t-\tau}^t \alpha(s)dsdt \\ & \geq \int_{\bar{t}_0}^T a(t) \ln \left(e \int_{t-\tau}^t a(s)ds \right) dt. \end{aligned} \tag{E.13}$$

By interchanging the order of integration, we find that

$$\begin{aligned} & \int_{\bar{t}_0}^T a(t) \left(\int_{t-\tau}^t \alpha(s)ds \right) dt \geq \int_{\bar{t}_0}^{T-\tau} \left(\int_s^{s+\tau} a(t)\alpha(s)dt \right) ds \\ & = \int_{\bar{t}_0}^{T-\tau} \alpha(s) \left(\int_s^{s+\tau} a(t)dt \right) ds = \int_{\bar{t}_0}^{T-\tau} \alpha(t) \left(\int_t^{t+\tau} a(s)ds \right) dt. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\bar{t}_0}^T \alpha(t) \left(\int_{t-\tau}^t a(s)ds \right) dt - \int_{\bar{t}_0}^{T-\tau} \alpha(t) \left(\int_t^{t+\tau} a(s)ds \right) dt \\ & \geq \int_{\bar{t}_0}^T \left(\int_{t-\tau}^t a(s)ds \right) \alpha(t)dt - \int_{\bar{t}_0}^T a(t) \int_{t-\tau}^t \alpha(s)dsdt \end{aligned} \tag{E.14}$$

and therefore, from (E.13) and (E.14), it follows that

$$\int_{T-\tau}^T \alpha(t) \left(\int_{t-\tau}^t a(s) ds \right) dt \geq \int_{\bar{t}_0}^T a(t) \ln \left(e \int_{t-\tau}^t a(s) ds \right) dt.$$

Taking “limsup” on both sides of this inequality, we have

$$\begin{aligned} M \cdot \limsup_{t \rightarrow +\infty} \int_{t-\tau}^t \alpha(s) ds &\geq \limsup_{t \rightarrow +\infty} \int_{t-\tau}^t \alpha(s) \left(\int_{s-\tau}^s a(u) du \right) ds \\ &\geq \limsup_{t \rightarrow +\infty} \int_{\bar{t}_0}^t a(s) \ln \left(e \int_{s-\tau}^s a(u) du \right) ds, \end{aligned} \quad (\text{E.15})$$

where $M = \limsup_{t \rightarrow +\infty} \int_{t-\tau}^t a(s) ds$. Since $\alpha(t) = a(t)W(t) = -\frac{\dot{y}(t)}{y(t)}$, inequality (E.15) gives

$$M \cdot \limsup_{t \rightarrow +\infty} \ln \frac{y(t-\tau)}{y(t)} \geq \limsup_{t \rightarrow +\infty} \int_{\bar{t}_0}^t a(s) \ln \left(e \int_{s-\tau}^s a(u) du \right) ds.$$

Using Lemma E.2.4 (M is finite) and hypothesis (H5), we deduce that

$$\limsup_{t \rightarrow +\infty} \frac{y(t-\tau)}{y(t)} = +\infty.$$

In view of Lemma E.2.5, we have a contradiction. Thus the result of Theorem E.2.1 holds. \square

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