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Maxwell and Yang-Mills Equations on Curved Black Hole Space-Times

Sari Ghanem

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Équations de Maxwell et de Yang-Mills sur des Espaces-Temps Courbes avec un Trou Noir

Thèse de Doctorat
Présentée à l'École Doctorale
de
Sciences Mathématiques de Paris Centre
pour obtenir le grade de
Docteur en Mathématiques
de
l'Université Paris Diderot - Paris 7
Institut de Mathématiques de Jussieu

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Maxwell and Yang-Mills Equations on Curved Black Hole Space-Times

A Dissertation
Presented to the Faculty of the Graduate School
of
Sciences Mathématiques de Paris Centre
in Candidacy for the Degree of
Doctor of Philosophy
of
Université Paris Diderot - Paris 7
Institut de Mathématiques de Jussieu

by
Sari Ghanem

Dissertation Directors: Frédéric Hélein Université Paris 7
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November 2014

Résumé en Français

Équations de Maxwell et de Yang-Mills sur des Espaces-Temps Courbes avec un Trou Noir

Sari Ghanem

2014

Dans cette thèse, nous menons une étude systématique de régularité des champs de Maxwell et de Yang-Mills sur des espaces-temps courbes et en présence d'un trou noir. Dans le premier chapitre, nous écrivons la preuve de la non-explosion de la courbure de Yang-Mills sur des espaces-temps courbes quelconques, fixes, en utilisant la paramétrix de Klainerman-Rodnianski combinée avec des inégalités de type Grönwall appropriées. Alors que l'argument de Chruściel-Shatah nécessite un contrôle de deux dérivées de la courbure de Yang-Mills, nous pouvons en sortir en contrôlant uniquement une seule dérivée, et écrire une nouvelle preuve indépendante de tout choix de jauge. Dans le chapitre qui suit, nous étudions les équations de Maxwell dans le domaine extérieur du trou noir de Schwarzschild. Nous montrons que si nous supposons que les composantes du milieu des solutions non-stationnaires des équations de Maxwell vérifient une certaine estimée de type Morawetz sur une région compacte dans l'espace autour de la surface piégée, alors nous pouvons prouver des propriétés de décroissance uniforme pour les composantes des champs de Maxwell dans tout l'extérieur du trou noir de Schwarzschild, y compris des points sur l'horizon, en faisant uniquement recours à des inégalités de Sobolev combinées avec des estimées d'énergie en utilisant directement les équations de Maxwell. Cette preuve ne passe pas par l'équation d'onde scalaire, n'a pas besoin de séparer les composantes du milieu, et serait alors utile pour le cas des champs de Yang-Mills où la séparation ne peut pas se produire. Le dernier chapitre est une ouverture sur des problèmes différents en équations aux dérivées partielles.

Abstract

Maxwell and Yang-Mills Equations on Curved Black Hole Space-Times

Sari Ghanem

2014

In this thesis, we take a systematic study of global regularity of the Maxwell equations and of the Yang-Mills equations on curved black hole space-times. In the first chapter, we write the proof of the non-blow up of the Yang-Mills curvature on arbitrary curved space-times using the Klainerman-Rodnianski parametrix combined with suitable Grönwall type inequalities. While the Chruściel-Shatah argument requires a control on two derivatives of the Yang-Mills curvature, we can get away by controlling only one derivative instead, and write a new gauge independent proof on arbitrary, fixed, sufficiently smooth, globally hyperbolic, curved 4-dimensional Lorentzian manifolds. In its sequel, we study the Maxwell equations in the domain of outer-communication of the Schwarzschild black hole. We show that if we assume that the middle components of the non-stationary solutions of the Maxwell equations verify a Morawetz type estimate supported on a compact region in space around the trapped surface, then we can prove uniform decay properties for the components of the Maxwell fields in the entire exterior of the Schwarzschild black hole, including the event horizon, by making only use of Sobolev inequalities combined with energy estimates using the Maxwell equations directly. This proof does not pass through the scalar wave equation on the Schwarzschild black hole, does not need to separate the middle components for the Maxwell fields, and would then be in particular useful for the non-abelian case of the Yang-Mills equations where the separation of the middle components cannot occur. The last chapter is an opening to different problems in partial differential equations.

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Chapter 1

Présentation de la Thèse en Français

Les théories des champs de jauge, comme les équations de Maxwell et les équations de Yang-Mills, se posent dans des théories physiques importantes pour décrire l'électromagnétisme et les interactions faibles et fortes, et sont dans une certaine mesure mathématiquement liées aux équations d'Einstein dans le vide en Relativité Générale. En effet, en utilisant le formalisme de Cartan les équations d'Einstein dans le vide peuvent être écrites sous la forme des équations de Yang-Mills, sauf pour le fait que la géométrie de fond fait partie de la solution inconnue du problème d'évolution dans la Relativité Générale, alors que dans la théorie de Yang-Mills, on peut fixer la géométrie de fond comme étant un espace-temps donné.

Dans un article classique, [EM1]-[EM2], Eardley et Moncrief ont prouvé l'existence globale des solutions des équations de Yang-Mills sur l'espace-temps de Minkowski à 4 dimensions. L'objectif de cette thèse est d'étendre leur résultat de régularité globale sur des espace-temps courbes, comme le trou noir de Schwarzschild.

Dans le premier chapitre, nous écrivons la preuve de la non-explosion des champs de Yang-Mills sur un espace-temps courbe fixe quelconque. Pour cela, nous faisons usage

d'une formulation hyperbolique du problème contrairement au cas abélien des équations de Maxwell. Dans un article récent, [KR1], Klainerman et Rodnianski ont fourni une formule de représentation covariante pour l'équation des ondes sur des espaces-temps courbes quelconques, lisses, globalement hyperboliques, dans laquelle les termes intégrales sont pris sur le cône de lumière passé. Comme les auteurs l'ont souligné, leur paramétrix peut être immédiatement adaptée pour des dérivées covariantes de jauge; c'est parce que le produit scalaire sur l'algèbre de Lie $\langle \cdot, \cdot \rangle$ est invariant par l'action adjointe. Ils l'utilisent pour donner une nouvelle preuve indépendante de tout choix de jauge du résultat d'Eardley-Moncrief dans [EM2] concernant la non-explosion des champs de Yang-Mills, dont le seul ingrédient est la conservation de l'énergie. Comme les auteurs l'ont mentionné, on peut généraliser leur preuve de la non-explosion de la courbure de Yang-Mills sur l'espace-temps plat de Minkowski à des espaces-temps courbes quelconques lisses, globalement hyperboliques, sous l'hypothèse qu'il existe un champ de vecteur unitaire de type temps dont le tenseur de déformation est fini, comme il l'a été supposé dans les travaux antérieurs de Chruściel et de Shatah, voir [CS]. Dans cette section, nous fournissons du matériel standard, mais pas aussi clairement souligné dans la littérature, qui consiste à écrire la preuve de la non-explosion des champs de Yang-Mills sur des espaces-temps courbes quelconques, sous les hypothèses mentionnées ci-dessus, en utilisant la paramétrix de Klainerman-Rodnianski combinée avec des inégalités de type Grönwall appropriées. Alors que l'argument de Chruściel-Shatah nécessite un contrôle simultané de la norme L_{loc}^∞ et de la norme H_{loc}^2 de la courbure de Yang-Mills, nous pouvons en sortir par le seul contrôle de la norme H_{loc}^1 à la place. Cependant, se débarrasser de tout contrôle sur le gradient de la courbure de Yang-Mills, comme c'est le cas dans la preuve sur l'espace-temps de Minkowski dans [KR1], a été tenté par l'auteur, et nous n'avons pas pu le faire. Comme l'espace-temps étendu du trou noir de Schwarzschild est un cas particulier d'une variété Lorentzienne lisse, globalement hyperbolique, à 4 dimensions avec la présence d'un champ de vecteur de type temps $\frac{\partial}{\partial t}$ (voir l'annexe pour l'espace-temps de Schwarzschild prolongé) dont le champ de vecteur unitaire admet un tenseur de déformation fini, nous obtenons que les solutions locales des équations

de Yang-Mills n'explorent pas en t' à l'écart de l'hypersurface $r = 0$, comme on peut encore appliquer le lemme de Grönwall dans le passé de chaque point, et son intersection avec le futur d'une hypersurface de Cauchy prescrit par $t' = \text{constante}$, pour obtenir une borne sur le flux d'énergie le long du cône de lumière passé.

L'objectif du deuxième chapitre est de prouver une décroissance uniforme des champs de Maxwell dans le domaine extérieur du trou noir de Schwarzschild. A l'extérieur du trou noir de Schwarzschild $\frac{\partial}{\partial t}$ est un champ de vecteur de type temps et Killing. Appliquant le théorème de divergence à $T_{t\mu}$, où T est le tenseur d'énergie-impulsion de Maxwell, entre deux hypersurfaces de type espace prescrites par $t = \text{constante}$ conduit à une conservation de l'énergie qui est au niveau d'une intégrale sur l'espace de T_{tt} . Dans ce cas, cela donne une énergie positive conservée au niveau de la norme L^2 des champs de Maxwell. Cependant, près de l'horizon, une difficulté est que nous perdons le contrôle sur la norme L^2 de certaines composantes des champs en raison de la présence du facteur $(1 - \frac{2m}{r})$ dans l'expression intégrale de la conservation de l'énergie, qui s'annule à l'horizon. Afin de prouver une décroissance uniforme des solutions des équations de Maxwell dans l'extérieur, nous aimerions majorer l'énergie conforme obtenue à partir du champ de vecteur de Morawetz. Pour ce faire, nous supposons que les composantes du milieu des solutions non-stationnaires vérifient une estimée de type Morawetz, c'est à dire qu'une intégrale sur l'espace-temps des composantes du milieu, appuyée sur une zone délimitée dans l'espace autour de la surface piégée, $r = 3m$, est bornée par l'énergie sans les composantes du milieu - ce qui serait le cas si l'on obtient cette estimation en utilisant un champ de vecteur de type espace après l'élimination dans l'argument les solutions stationnaires des équations de Maxwell à l'extérieur du trou noir de Schwarzschild, qui sont les soi-disant les solutions de Coulomb. Nous prouvons alors qu'une telle intégrale sur l'espace-temps décroît dans le temps, et nous obtenons une borne supérieure sur l'énergie conforme par une quantité, qui dépend des données initiales, qui n'est pas finie pour les solutions de Coulomb. Cela sera utilisé pour prouver des propriétés de décroissance des composantes des champs de Maxwell loin de l'horizon. Près de l'horizon, nous parvenons à contourner la difficulté en faisant usage

d'un champ de vecteur de type horizon pour obtenir une décroissance uniforme pour quatre composantes dans l'ensemble de la région extérieure, y compris des points à l'horizon.

Le dernier chapitre est une ouverture sur des problèmes différents dans les équations aux dérivées partielles. Le problème de régularité des solutions de l'équation quasi-géostrophique critique avec des données initiales quelconques lisses a été suggéré par S. Klainerman dans [Kl2] comme l'un des problèmes les plus difficiles dans les équations aux dérivées partielles du XXI^{ème} siècle. Dans un papier élégant, [KNV], Kiselev, Nazarov, et Voldberg, ont prouvé la régularité globale des solutions de l'équation quasi-géostrophique dissipative critique à 2 dimensions avec des données initiales périodiques et lisses. Cette ouverture est une remarque qui en utilisant une adaptation de la technique inventée par A. Kiselev, F. Nazarov, et A. Voldberg, avec un argument de scaling modifié, nous pouvons prouver la régularité globale des solutions de l'équation quasi-géostrophique dissipative critique à 2 dimensions avec force lisse et périodique, sous l'hypothèse que la donnée initiale est lisse et périodique, et la force est bornée dans l'espace and dans le temps, et α -Hölder continu dans l'espace, avec $\alpha > 0$.

Remarque. L'ensemble de la thèse est écrite d'une manière détaillée, où nous détaillons tous les calculs, et en cas de besoin, nous montrons du matériel standard pour rendre ce travail autonome. Pour être complet, nous détaillons du matériel bien connu dans l'annexe.

Les trois parties majeures de la thèse sont présentées dans ce qui suit :

1.1 L'Existence Globale des Champs de Yang-Mills sur des Espaces-Temps Courbes

Dans ce chapitre, nous étudions les équations de Yang-Mills sur un espace-temps courbe donné quelconque. Pour cela, nous allons utiliser une formulation hyperbolique

du problème. En effet, tandis que les équations de Yang-Mills disent que la courbure de Yang-Mills est à divergence nulle sur la géométrie de fond, on peut obtenir une formulation hyperbolique en prenant la divergence covariante de l'identité de Bianchi. Cela conduit à une équation d'onde covariante tensorielle sur la courbure de Yang-Mills avec un terme non-linéaire. C'est exactement l'étude de ce terme non-linéaire qui permet de répondre à la question de l'existence locale, et de l'existence globale des solutions des équations. Dans cette formulation, la donnée initiale se compose du potentiel de Yang-Mills, qui est une 1-forme à valeur dans l'algèbre de Lie, et le champ électrique (vaguement parlant, il s'agit de la dérivée temporelle du potentiel) sur une hypersurface de Cauchy Σ donnée de type espace. Les données initiales doivent vérifier elles-mêmes les équations de Yang-Mills, ce qui est que la divergence covariante du champ électrique s'annule. On cherche une courbure de Yang-Mills qui satisfait les équations de Yang-Mills de sorte qu'une fois réstreint sur cette hypersurface Σ la courbure de Yang-Mills correspondrait à celle donnée par les champs électriques et les potentiels prescrits.

Eardley et Moncrief ont prouvé l'existence globale des solutions des équations de Yang-Mills sur l'espace-temps de Minkowski à 4 dimensions en prouvant un résultat d'existence locale et en fournissant des estimations ponctuelles sur la courbure, [EM1]-[EM2]. Leur approche dépendait de la solution fondamentale de l'équation des ondes sur un espace-temps plat, et l'utilisation de la condition de jauge Cronström, qui présente l'avantage remarquable d'exprimer directement le potentiel en fonction de la courbure en termes d'une intégrale, pour estimer le terme non-linéaire. Plus tard, ce résultat a été étendu par Chruściel et Shatah, [CS], à des espace-temps courbes en utilisant la même approche, en faisant usage de la paramétrix de Friedlander pour l'équation des ondes dans des domaines causals dans des espaces-temps courbes, [Fried], ainsi que la condition de jauge Cronström. Dans un article récent, [KR1], Klainerman et Rodnianski ont construit une paramétrix pour l'équation d'onde qui leur a permis de donner une nouvelle preuve indépendante de tout choix de jauge du résultat d'Eardley-Moncrief dans [EM2] concernant la non-explosion des champs de Yang-Mills dans le contexte de l'espace-temps de Minkowski.

L'approche de Klainerman-Rodnianski repose sur leur dérivation d'une formule de représentation covariante pour l'équation des ondes sur des espaces-temps courbes quelconques lisses, globalement hyperboliques, dans laquelle les termes intégraux s'appuient sur le cône de lumière passé. Comme les auteurs l'ont souligné, leur paramétrix peut être immédiatement adaptée pour des dérivées covariantes de jauge; c'est parce que le produit scalaire sur l'algèbre de Lie $\langle \cdot, \cdot \rangle$ est invariant par l'action adjointe. Ils l'ont utilisée pour donner une nouvelle preuve indépendante de tout choix de jauge du résultat d'Eardley-Moncrief [EM2], dont le seul ingrédient est la conservation de l'énergie. Comme les auteurs l'ont mentionné, on peut généraliser leur preuve de la non-explosion de la courbure de Yang-Mills sur l'espace-temps plat de Minkowski à des espaces-temps courbes quelconques lisses, globalement hyperboliques, sous l'hypothèse qu'il existe un champ de vecteur unitaire $\frac{\partial}{\partial t}$ de type temps dont le tenseur de déformation est fini, comme il l'a été supposé dans les travaux antérieurs de Chruściel et de Shatah, [CS].

Dans ce chapitre, nous fournissons du matériel standard, mais pas aussi clairement souligné dans la littérature, qui consiste à écrire la preuve de la non-explosion de la courbure de Yang-Mills pour des solutions locales sur des espaces-temps courbes quelconques en utilisant le paramétrix de Klainerman-Rodnianski combinée avec des inégalités appropriées de type Grönwall. Alors que l'argument de Chruściel-Shatah nécessite un contrôle simultané de la norme L_{loc}^∞ et de la norme H_{loc}^2 de la courbure de Yang-Mills, nous pouvons en sortir en contrôlant uniquement la norme H_{loc}^1 . Cependant, nous avons été incapables de se débarrasser de tout contrôle sur le gradient de la courbure de Yang-Mills, comme il est le cas dans la preuve sur l'espace-temps de Minkowski dans [KR1]. Ainsi, cela donne une nouvelle preuve indépendante de tout choix de jauge et améliore le résultat du Chruściel-Shatah, [CS], d'existence globale des champs de Yang-Mills sur des variétés Lorentziennes courbes quelconques, suffisamment lisses, globalement hyperboliques, à 4 dimensions.

1.2 Sur la Décroissance Uniforme des Champs de Maxwell sur des Espaces-Temps Courbes avec un Trou Noir

Dans ce chapitre, nous étudions les équations de Maxwell sur le trou noir de Schwarzschild. Dans un article récent, [DR1]-[DR2], Dafermos et Rodnianski ont prouvé la décroissance des solutions de l'équation d'onde scalaire libre $\square_{\mathbf{g}}\phi = 0$ à l'extérieur du trou noir de Schwarzschild, jusqu'à des points sur l'horizon. Nous ne savons pas comment faire marcher ces méthodes pour $\square_{\mathbf{g}}\phi = c\cdot\phi$ ou pour $\square_{\mathbf{g}}\phi = \phi^2$ en utilisant le tenseur d'énergie-impulsion de l'équation des ondes. Ainsi, cela exclut la possibilité d'utiliser ces méthodes pour les équations de Maxwell dans une formulation hyperbolique où les termes sources seraient $(\square_{\mathbf{g}}F)_{\mu\nu}$, où $F_{\mu\nu}$ est le champ de Maxwell. Dans un article récent, [Bl], Blue a prouvé une décroissance pour les champs de Maxwell dans l'extérieur sur l'espace-temps de Schwarzschild. La preuve de Blue a nécessité une étude d'une équation d'onde sur l'espace-temps de Schwarzschild pour les composantes dites du milieu, qui peuvent être séparées des autres composantes dans le cas abélien des équations de Maxwell. Cette approche a été ensuite étendue par Andersson et Bleu à la métrique de Kerr, [AB]. Toutefois, dans le cas non-linéaire des équations de Yang-Mills, on ne peut dissocier les composantes du milieu des autres. Pourtant, il semble difficile de généraliser les résultats de Dafermos et Rodnianski pour l'équation d'onde scalaire libre aux équations de Maxwell en utilisant directement le tenseur d'énergie-impulsion de Maxwell, sans se référer à l'équation d'onde scalaire, combiné avec des inégalités de Sobolev appropriées. Une étape clé pour atteindre cet objectif serait de majorer l'énergie conforme sans séparation des composantes du milieu des champs de Maxwell. Cela donnerait une nouvelle preuve indépendante et améliorerait le résultat de Blue, et serait en particulier utile pour le cas non commutatif des équations de Yang-Mills où cette séparation ne peut pas se produire. J'ai essayé de faire cela dans ma thèse dans le but de prouver une borne uniforme pour les champs de Yang-Mills à l'extérieur du trou noir de Schwarzschild et la métrique de Kerr. Cependant, comme Klainerman m'a fait

remarquer plus tard, beaucoup de gens ont essayé d'obtenir une preuve plus conceptuelle de décroissance pour les équations de Maxwell, sans passer par l'équation d'onde scalaire, et y parvenir serait d'une grande portée. Même si je n'étais pas en mesure de résoudre cela encore, dans ce chapitre, je vous écris une preuve de décroissance pour les champs de Maxwell sur l'extérieur du trou noir de Schwarzschild, directement, sans séparation des composantes du milieu et sans se référer à l'équation d'onde, en supposant que nous avons une estimatée de type Morawetz au niveau de la 0-dérivée sur une région compacte dans l'espace. Pour expliquer, rappelons-le (voir l'annexe) qu'à l'extérieur, la métrique de Schwarzschild peut s'écrire ainsi:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

Si nous définissons,

$$\begin{aligned} r^* &= r + 2m \log(r - 2m) \\ v &= t + r^* \\ w &= t - r^* \end{aligned}$$

nous aurons alors,

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{r}\right)dvdw + r^2d\sigma^2 \\ &= -\frac{\left(1 - \frac{2m}{r}\right)}{2}dv \otimes dw - \frac{\left(1 - \frac{2m}{r}\right)}{2}dw \otimes dv + r^2d\sigma^2 \end{aligned}$$

Soient,

$$\frac{\hat{\partial}}{\partial w} = \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\partial}{\partial w} \tag{1.1}$$

$$\frac{\hat{\partial}}{\partial v} = \frac{\partial}{\partial v} \tag{1.2}$$

$$\frac{\hat{\partial}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \quad (1.3)$$

$$\frac{\hat{\partial}}{\partial \phi} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (1.4)$$

et à un point de l'espace-temps de Schwarzschild, soient e_1, e_2 une base normalisée de \mathbf{S}^2 , qui vérifie pour tout $A, B \in \{1, 2\}$,

$$\mathbf{g}(e_A, \frac{\hat{\partial}}{\partial w}) = \mathbf{g}(e_A, \frac{\hat{\partial}}{\partial v}) = 0 \quad (1.5)$$

$$\mathbf{g}(e_A, e_B) = \delta_{AB} \quad (1.6)$$

Dans ce contexte, par "en supposant une estimatée de Morawetz", nous entendons exactement que pour r_0, R_0 comme dans la preuve de (4.55), et pour tout $t_i = (1.1)^i t_0$, où t_0 est un nombre réel positif et i est un nombre naturel, si nous définissons:

$$J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2]. dr^* d\sigma^2 dt \quad (1.7)$$

alors, nous supposons que les solutions non-stationnaires vérifient

$$\begin{aligned} & J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \\ & \lesssim |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1})| + \sum_{j=1}^k (|\hat{E}_{\mathcal{L}_{K_j} F}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\mathcal{L}_{K_j} F}^{(\frac{\partial}{\partial t})}(t_{i+1})|) \end{aligned} \quad (1.8)$$

où $K_j, j \in \{1, \dots, k\}$, sont des champs de vecteurs de type Killing, et où

$$\begin{aligned} \hat{E}_F^{(\frac{\partial}{\partial t})}(t) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} [\frac{1}{r^2(1-\frac{2m}{r})} |F_{t\theta}|^2 + \frac{1}{r^2(1-\frac{2m}{r}) \sin^2 \theta} |F_{t\phi}|^2 \\ &+ \frac{1}{r^2(1-\frac{2m}{r})} |F_{r^*\theta}|^2 + \frac{1}{r^2(1-\frac{2m}{r}) \sin^2 \theta} |F_{r^*\phi}|^2] r^2(1-\frac{2m}{r}) d\sigma^2 dr^* \end{aligned} \quad (1.9)$$

qui est l'énergie sans les composantes du milieu $F_{tr^*}, F_{\theta\phi}$. Les seules solutions stationnaires

des équations de Maxwell à l'extérieur du trou noir de Schwarzschild, sont les soi-disant solutions de Coulomb. Par conséquent, l'hypothèse ci-dessus est supposée pour les solutions non-Coulomb. Pour simplicité, nous écrivons notre preuve avec $k = 3$ et $K_j = \Omega_j$, où Ω_j , $j \in \{1, 2, 3\}$, est une base des opérateurs de moment angulaire, bien que la preuve marche encore avec un produit quelconque des dérivées de Killing sur le côté droit de (4.8), en ajustant en conséquence les quantités dans le théorème qui dépendent des données initiales.

Nous allons montrer que si les composantes du milieu des solutions non-stationnaires vérifient une estimation de type Morawetz au niveau de la zéroième dérivée, (4.8), alors nous pouvons prouver des propriétés de décroissance uniforme des solutions des équations de Maxwell dans l'ensemble du domaine extérieur du trou noir de Schwarzschild, y compris l'horizon, en faisant usage des inégalités de Sobolev appropriées combinées avec des estimations d'énergie en utilisant le tenseur d'énergie-impulsion des champs de Maxwell. Nous ne faisons aucune utilisation de la décomposition en harmoniques sphériques. Nous commençons par une hypersurface de Cauchy prescrite par $t = t_0$ où la donnée initiale doit vérifier certaines conditions de régularité (il n'y a pas de condition d'annulation pour F sur la sphère de bifurcation). Loin de l'horizon (dans la région $r \geq R > 2m$, pour un R fixe), nous allons montrer que

$$\begin{aligned} |F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\leq \frac{C}{(1 + |v|)} \\ |F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\leq \frac{C}{(1 + |w|)} \end{aligned}$$

pour tout $\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$, où $F_{\mu\nu}$ est le champ de Maxwell. Près de l'horizon, et dans toute la région l'extérieure $r \geq 2m$, jusqu'à des points sur l'horizon, nous allons prouver que

$$\begin{aligned} |F_{\hat{v}\hat{w}}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}}, & |F_{e_1 e_2}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}} \\ |F_{\hat{v}e_a}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}}, & |\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_a}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}} \end{aligned}$$

Pour expliquer vraiment bien :

Si l'on essaie de généraliser la preuve de Dafermos et Rodnianski pour l'équation d'onde scalaire libre aux équations de Maxwell, sur l'espace-temps de Schwarzschild, en utilisant le tenseur d'énergie-impulsion des champs de Maxwell à la place, certaines difficultés qui apparaissent sont :

1. Pour majorer l'énergie conforme des champs de Maxwell et de leurs dérivés dans la direction des champs de vecteurs de Killing, il faut contrôler une intégrale sur l'espace-temps près de la surface piégée $r = 3m$, qui implique les composantes de soi-disant du milieu de champ de Maxwell (voir (4.44)). En fait, les termes dans l'intégrale sur l'espace-temps $J^{(K)}$, obtenue en appliquant le théorème de divergence sur le champ de vecteur de Morawetz $K = -w^2 \frac{\partial}{\partial w} - v^2 \frac{\partial}{\partial v}$ contracté avec le tenseur d'énergie-impulsion des champs de Maxwell, sont non-négatifs dans une région $r_0 \leq r \leq R_0$ qui contient $r = 3m$. Il semble que cette intégrale ne peut pas être contrôlée à l'aide de ces méthodes en raison de la présence des autres composantes avec le "mauvais" signe dans l'intégrale sur l'espace-temps générée à partir d'un champ de vecteur de contrôle de type espace (voir (4.49)). En effet, en utilisant le théorème de divergence avec le champ de vecteur $G = f(r^*) \frac{\partial}{\partial r^*}$, où f est une fonction bornée, nous obtiendrons une intégrale sur l'espace-temps et une intégrale sur le bord. Cependant, l'intégrale sur l'espace-temps obtenue à partir de G a des termes qui n'apparaissent pas dans $J^{(K)}$ qui entrent avec le mauvais signe, et donc il ne peut pas être rendue positive. Nous allons montrer que si nous dépassons cela, voir hypothèse (4.8), nous pouvons alors écrire une preuve indépendante de tout choix de jauge de la décroissance uniforme des champs de Maxwell à l'extérieur du trou noir de Schwarzschild jusqu'à des points à l'horizon, en utilisant les équations de Maxwell directement.
2. Il faut construire un nouveau champ qui vérifie les équations de Maxwell et les identités de Bianchi, qui coïncide avec le champ d'origine dans une région et s'annule identiquement en dehors d'une autre région spécifique (voir la preuve de (4.58)).

Dans le cas de l'équation d'onde $\square_{\mathbf{g}}\phi = 0$, on peut multiplier les données initiales dans le problème de Cauchy par une fonction de coupure, et examiner l'évolution de ces données pour obtenir une solution qui vérifie l'équation d'onde et les propriétés énoncées précédemment. Dans le cas des équations de Maxwell et de Yang-Mills, si l'on multiplie les données initiales par une fonction de coupure alors les équations de contraintes ne seraient plus satisfaites. Il semble à première vue que l'on ne peut pas obtenir un nouveau champ qui vérifie les propriétés nécessaires. Alors que cela est vrai si l'on voulait le faire pour toutes les composantes, néanmoins, on peut le faire pour toutes les composantes à l'exception des composantes F_{rt} et $F_{\theta\phi}$, où la multiplication doit être au niveau de la dérivée spatiale des composantes. Par conséquent, on peut construire un nouveau champ qui peut être fait pour coïncider avec F dans une certaine région, et s'annule en dehors de l'autre sauf pour ces deux dernières composantes. Le peu de bonnes nouvelles, c'est que les calculs montrent que ces "mauvais" termes ne figurent pas dans les intégrales de bord générées par le théorème de divergence appliqué à un champ de vecteurs du type espace contracté avec le tenseur d'énergie-impulsion (voir (4.52)). Ainsi, dans notre hypothèse (4.8), nous supposons qu'une certaine intégrale sur l'espace-temps autour de la surface piégée $r = 3m$ des composantes du milieu, peut être majorée par l'énergie sans les composantes du milieu, ce qui serait le cas si cette estimation a été obtenue en contrôlant l'intégrale sur espace-temps en termes des intégrales de bord générées à partir des champs de vecteurs de type espace multipliés par une fonction bornée, comme montré dans estimation (4.53). Ceci est crucial pour établir (4.58) qui est l'estimation principale pour majorer l'énergie conforme dans (4.74).

3. Pour prouver une décroissance pour une énergie généralisée qui permettrait de contrôler la norme L^2 des champs de Maxwell près de l'horizon, on est confronté à une situation où il semble crucial de contrôler une intégrale sur l'espace-temps prise sur une zone bornée dans l'espace près de l'horizon, qui contient toutes les composantes.

Nous surmontons ce par notre hypothèse (4.8); elle peut également être utilisée pour majorer l'intégrale sur l'espace-temps contenant les autres composantes, comme dans (4.104).

En outre, en plus de ce qui précède :

4. Contrairement au cas de l'équation d'onde, les flux d'une énergie généralisée qui contrôle la norme L^2 des champs de Maxwell près de l'horizon ne contiennent pas la totalité de leurs composantes. Sur des hypersurfaces $v = \text{constante}$ il ne contient que $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}\hat{\theta}}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}\hat{\phi}}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{v}\hat{w}}$ et $\sqrt{1 - \frac{2m}{r}} F_{\hat{\theta}\hat{\phi}}$, et sur des hypersurfaces $w = \text{constante}$, il ne contient que les composantes $F_{\hat{v}\hat{w}}$, $F_{\hat{\theta}\hat{\phi}}$, $F_{\hat{v}\hat{\theta}}$ et $F_{\hat{v}\hat{\phi}}$. En outre, en utilisant les inégalités de Sobolev près de l'horizon, puisque $\frac{\partial}{\partial v}$ et $\frac{\partial}{\partial w}$ ne sont pas des champs de vecteurs de Killing cela rajouterait une difficulté supplémentaire, tandis que dans le cas de l'équation des ondes, les carrés de ces dérivés apparaissent dans les flux qui sont facilement contrôlés. Nous contournons ces problèmes en utilisant des inégalités de type Sobolev appropriées pour chaque composante, combinées avec les identités de Bianchi et les équations de champ, de manière à ce que les dérivés dans les directions de $\frac{\partial}{\partial v}$ et $\frac{\partial}{\partial w}$ peuvent être contrôlées par des dérivés de Killing, ∂_t et ∂_{Ω_j} , des composantes qui apparaissent dans les flux pour lesquels nous aurions prouvé une décroissance.

Dans le cas des équations de Yang-Mills il y a l'obstacle supplémentaire que les équations sont non-linéaires. Si l'on obtient une identité d'énergie pour les champs de Yang-Mills, on ne peut pas écrire directement la même identité d'énergie pour les dérivés des champs dans la direction des champs de vecteurs de Killing, par opposition au cas des champs de Maxwell, en raison de la non-linéarité des équations.

1.3 Ouverture sur des Problèmes Différents en EDP

Le problème de régularité des solutions de l'équation quasi-géostrophique critique avec des données initiales quelconques lisses a été suggéré par S. Klainerman dans [Kl2] comme l'un des problèmes les plus difficiles dans les équations aux dérivées partielles du XXIème siècle. Dans un papier élégant, [KNV], Kiselev, Nazarov, et Voldberg, ont prouvé la régularité globale des solutions de l'équation quasi-géostrophique dissipative critique à 2 dimensions avec des données initiales périodiques et lisses. Ce chapitre est une remarque qui en utilisant une adaptation de la technique inventée par A. Kiselev, F. Nazarov, et A. Voldberg, avec un argument de scaling modifié, nous pouvons prouver que le gradient des solutions locales de l'équation quasi-géostrophique dissipative critique à 2 dimensions avec force lisse et périodique, n'explose pas dans le temps, sous l'hypothèse que la donnée initiale est lisse et périodique, et la force est bornée dans l'espace and dans le temps, et α -Hölder continu dans l'espace, avec $\alpha > 0$. Cela permet de montrer que les solutions locales peuvent être étendues globalement dans le temps.

Chapter 2

Introduction

Gauge field theories, such as the Maxwell equations and the Yang-Mills equations, arise in important physical theories to describe electromagnetism and the weak and strong interactions, and are to some extent mathematically related to the Einstein vacuum equations in General Relativity. Indeed, using Cartan formalism the Einstein vacuum equations can be written as the Yang-Mills equations except to the fact that the background geometry is part of the unknown solution of the evolution problem in General Relativity, while in Yang-Mills theory one can fix the background to be a given space-time.

In a classical paper, [EM1]-[EM2], Eardley and Moncrief proved global existence of solutions of the Yang-Mills equations in the 4-dimensional Minkowski background. The aim of this thesis is to extend their global regularity result to curved backgrounds, such as the Schwarzschild black hole.

In the first chapter, we write the proof of the non-blow up of the Yang-Mills curvature on arbitrary fixed curved space-time. For this, we do make use of a hyperbolic formulation of the problem as opposed to the abelian case of the Maxwell equations. In a recent paper, [KR1], Klainerman and Rodnianski provided a covariant representation formula for the wave equation on arbitrary, smooth, globally hyperbolic, curved space-times, in which the integral terms are supported on the past null cone. As the authors pointed out, their

parametrix can be immediately adapted to gauge covariant derivatives; this is because the scalar product on the Lie algebra $\langle \cdot, \cdot \rangle$ is Ad-invariant. They use it to give a new gauge independent proof of the Eardley-Moncrief result in [EM2] concerning the non-blow up of the Yang-Mills fields, of which the only ingredient is the conservation of energy. As the authors mentioned, one can generalize their proof of the non-blow up of the Yang-Mills curvature on the flat Minkowski space-time to arbitrary smooth, globally hyperbolic, curved space-times under the assumption that there exists a unit timelike vector field of which the deformation tensor is finite, as it has been assumed in previous work by Chruściel and Shatah, see [CS]. In this section, we provide standard material, but not so clearly pointed out in literature, that consists in writing the proof of the non-blow up of Yang-Mills fields on arbitrary curved space-times, under the assumptions mentioned previously, using the Klainerman-Rodnianski parametrix combined with suitable Grönwall type inequalities. While the Chruściel-Shatah argument, [CS], requires a simultaneous control of the L_{loc}^∞ and the H_{loc}^2 norms of the Yang-Mills curvature, we can get away by controlling only the H_{loc}^1 norm instead. However, getting rid of any control on the gradient of the Yang-Mills curvature, as it is the case in the proof on Minkowski space-time in [KR1], was attempted by the author, and we were unable to do so. As the extended Schwarzschild black hole is a special case of a smooth, globally hyperbolic, 4-dimensional Lorentzian manifold with the presence of a timelike vector field $\frac{\partial}{\partial t'}$ (see Appendix for the extended Schwarzschild space-time) of which the normalized vector field has a deformation tensor that is finite, we get that the Yang-Mills curvature of local solutions of the Yang-Mills equations will not blow up in t' away from the hypersurface $r = 0$, as one can still apply Grönwall lemma in the past of each point, and its intersection with the future of a Cauchy hypersurface prescribed by $t' = \text{constant}$, to obtain a bound on the energy flux along the past null cone.

The goal of the second chapter is to prove uniform decay of the Maxwell fields in the domain of outer-communication of the Schwarzschild black hole. In the exterior of the Schwarzschild black hole $\frac{\partial}{\partial t}$ is timelike Killing vector field. Applying the divergence

theorem to $T_{t\mu}$, where T is the Maxwell energy momentum tensor, between two space-like hypersurfaces prescribed by $t = \text{constant}$ leads to a conservation of energy that is at the level of a space integral of T_{tt} . In this case, this gives a conserved positive energy that is at the level of the L^2 norm of the Maxwell fields. However, near the event horizon, a difficulty is that we lose control on the L^2 norm of some components of the fields due to the presence of the factor $(1 - \frac{2m}{r})$ in the integral expression of the conserved energy, which happens to vanish at the event horizon. In order to prove uniform decay for solutions of the Maxwell equations in the exterior, we would like to bound the conformal energy obtained from the Morawetz vector field. To achieve this, we assume that the middle components of the non-stationary solutions verify a Morawetz type estimate, i.e. that the space-time integral of the middle components, supported on a bounded region in space around the trapped surface, $r = 3m$, is bounded by the energy without the middle components - which would be the case if one obtained this estimate using space like vector fields after eliminating in the argument the stationary solutions to the Maxwell equations on the exterior of the Schwarzschild black hole, which are the so-called Coulomb solutions. We would then prove that such a space-time integral would be decaying in time, and we will get a uniform bound on the conformal energy by certain quantity, dependant on the initial data, that is not finite for the Coulomb solutions. This will be used to prove decay properties for components of the Maxwell fields away from the horizon. Near the horizon, we manage to get around the difficulty by making use of a horizon type vector field to obtain uniform decay for four components in the entire exterior region, including points on the event horizon.

The last chapter is an opening to different problems in partial differential equations. The problem of breakdown of solutions of the critical quasi-geostrophic equation with arbitrary smooth initial data was suggested by S. Klainerman in [Kl2] as one of the most challenging problems in partial differential equations of the twenty-first century. In an elegant paper, [KNV], Kiselev, Nazarov, and Voldberg, proved global well-posedness of the critical 2-D dissipative quasi-geostrophic equation with smooth periodic initial data. This opening is a remark that by using an adaptation of the technique invented by A. Kiselev, F.

Nazarov, and A. Voldberg, with a modified scaling argument, we can prove global no blow-up of the gradient of solutions to the critical 2-D dissipative quasi-geostrophic equation with periodic force, under the assumption that the initial data is smooth and periodic, and the force is bounded in space and time, and α -Hölder continuous in space, $\alpha > 0$.

Remark 2.0.1. The whole thesis is written in an expository way, where we detail all the calculations, and when needed we show standard material to make this work self-contained. For completeness, we detail well known material in the Appendix.

2.1 The Field Equations

In this section we present the Yang-Mills curvature, and we derive the Yang-Mills equations from the Yang-Mills Lagrangian. We will also show the Bianchi identities.

2.1.1 The Yang-Mills curvature

Let (M, \mathbf{g}) be a four dimensional globally hyperbolic Lorentzian manifold. Let G be a compact Lie group, and \mathcal{G} its Lie algebra such that it has a faithful real matrix representation $\{\theta_a\}$. Let $\langle \cdot, \cdot \rangle$ be a positive definite Ad-invariant scalar product on \mathcal{G} . The Yang Mills potential can be regarded locally as a \mathcal{G} -valued one form A on M , say

$$A = A_\alpha^{(a)} \theta_a dx^\alpha = A_\alpha dx^\alpha$$

in a given system of coordinates. The gauge covariant derivative of a \mathcal{G} -valued tensor Ψ is defined as

$$\mathbf{D}_\alpha^{(A)} \Psi = \nabla_\alpha \Psi + [A_\alpha, \Psi] \tag{2.1}$$

where ∇_α is the space-time covariant derivative of Levi-Civita on (M, \mathbf{g}) , and $\nabla_\alpha \Psi$ is the tensorial covariant derivative of Ψ , that is

$$\begin{aligned}
 (\nabla_\alpha \Psi)(X, Y, Z, \dots) &= \partial_\alpha(\Psi(X, Y, Z, \dots)) - \Psi(\nabla_\alpha X, Y, Z, \dots) \\
 &\quad - \Psi(X, \nabla_\alpha Y, Z, \dots) - \Psi(X, Y, \nabla_\alpha Z, \dots) \\
 &\quad - \dots - \dots
 \end{aligned} \tag{2.2}$$

The tensorial second order derivative is defined as

$$\begin{aligned}
 (\nabla_\beta \nabla_\alpha \Psi)(X, Y, Z, \dots) &= \partial_\beta[(\nabla_\alpha \Psi)(X, Y, Z, \dots)] - (\nabla_{\nabla_\beta e_\alpha} \Psi)(X, Y, Z, \dots) \\
 &\quad - (\nabla_\alpha \Psi)(\nabla_\beta X, Y, Z, \dots) - (\nabla_\alpha \Psi)(X, \nabla_\beta Y, Z, \dots) \\
 &\quad - \dots - \dots
 \end{aligned} \tag{2.3}$$

By letting

$$\begin{aligned}
 (\nabla_\beta (\nabla_\alpha \Psi))(X, Y, Z, \dots) &= \partial_\beta[(\nabla_\alpha \Psi)(X, Y, Z, \dots)] - (\nabla_\alpha \Psi)(\nabla_\beta X, Y, Z, \dots) \\
 &\quad - (\nabla_\alpha \Psi)(X, \nabla_\beta Y, Z, \dots) - \dots
 \end{aligned}$$

We can then write

$$(\nabla_\beta \nabla_\alpha \Psi)(X, Y, Z, \dots) = (\nabla_\beta (\nabla_\alpha \Psi))(X, Y, Z, \dots) - (\nabla_{\nabla_\beta e_\alpha} \Psi)(X, Y, Z, \dots) \tag{2.4}$$

The Yang-Mills curvature is a \mathcal{G} -valued two form

$$F = F_{\alpha\beta}^{(a)} \theta_a dx^\alpha \wedge dx^\beta = F_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

obtained by commuting in a system of coordinates two gauge covariant derivatives of a \mathcal{G} -valued tensor Ψ , where the tensorial second order gauge derivative of Ψ is defined by

$$\mathbf{D}_\alpha^{(A)} \mathbf{D}_\beta^{(A)} \Psi = \mathbf{D}_{\alpha\beta}^2 \Psi = \mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \Psi) - \mathbf{D}_{\nabla_\alpha e_\beta}^{(A)} \Psi \quad (2.5)$$

$$\begin{aligned} \mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \Psi) &= \nabla_\alpha (\mathbf{D}_\beta^{(A)} \Psi) + [A_\alpha, \mathbf{D}_\beta^{(A)} \Psi] \\ &= \nabla_\alpha (\nabla_\beta \Psi + [A_\beta, \Psi]) + [A_\alpha, \nabla_\beta \Psi + [A_\beta, \Psi]] \\ &= \nabla_\alpha (\nabla_\beta \Psi) + [\partial_\alpha A_\beta, \Psi] + [A_\beta, \nabla_\alpha \Psi] + [A_\alpha, \nabla_\beta \Psi] + [A_\alpha, [A_\beta, \Psi]] \end{aligned}$$

Then,

$$\begin{aligned} &\mathbf{D}_\alpha^{(A)} \mathbf{D}_\beta^{(A)} \Psi - \mathbf{D}_\beta^{(A)} \mathbf{D}_\alpha^{(A)} \Psi \\ &= \mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \Psi) - \mathbf{D}_\beta^{(A)} (\mathbf{D}_\alpha^{(A)} \Psi) - (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)} \Psi) + (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)} \Psi) \\ &= \mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \Psi) - \mathbf{D}_\beta^{(A)} (\mathbf{D}_\alpha^{(A)} \Psi) + (\mathbf{D}_{\nabla_\beta e_\alpha - \nabla_\alpha e_\beta}^{(A)} \Psi) \\ &= \nabla_\alpha (\nabla_\beta \Psi) + [\partial_\alpha A_\beta, \Psi] + [A_\beta, \nabla_\alpha \Psi] + [A_\alpha, \nabla_\beta \Psi] + [A_\alpha, [A_\beta, \Psi]] \\ &\quad - \nabla_\beta (\nabla_\alpha \Psi) - [\partial_\beta A_\alpha, \Psi] - [A_\alpha, \nabla_\beta \Psi] - [A_\beta, \nabla_\alpha \Psi] - [A_\beta, [A_\alpha, \Psi]] \\ &\quad + \nabla_{(\nabla_\beta e_\alpha - \nabla_\alpha e_\beta)} \Psi + [A(\nabla_\beta e_\alpha - \nabla_\alpha e_\beta), \Psi] \\ &= \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta} \Psi_{\dots\gamma\dots} + [\partial_\alpha A_\beta, \Psi] + [A_\alpha, [A_\beta, \Psi]] - [\partial_\beta A_\alpha, \Psi] - [A_\alpha, \nabla_\beta \Psi] \\ &\quad - [A_\beta, [A_\alpha, \Psi]] + [A(\nabla_\beta e_\alpha - \nabla_\alpha e_\beta), \Psi] \\ &= \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta} \Psi_{\dots\gamma\dots} + [\partial_\alpha A_\beta - \partial_\beta A_\alpha + A(\nabla_\beta e_\alpha - \nabla_\alpha e_\beta) + [A_\alpha, A_\beta], \Psi] \\ &\quad \text{(where we used Jacobi's identity)} \end{aligned} \quad (2.6)$$

$$\begin{aligned} &= \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta} \Psi_{\dots\gamma\dots} + [\nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta], \Psi] \\ &= \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta} \Psi_{\dots\gamma\dots} + [F_{\alpha\beta}, \Psi] \end{aligned} \quad (2.7)$$

where $\Psi = \Psi_{a_1 a_2 \dots a_i \dots}$, and γ is at the i^{th} place, and $F_{\alpha\beta}$ is a 2-tensor given by,

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta] \quad (2.8)$$

2.1.2 The Yang-Mills equations

The Yang-Mills Lagrangian is given by

$$L = -\frac{1}{4} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle$$

A compact variation $(F(s), U)$, where U is any compact set of M , can be written in terms of a compact variation $(A(s), U)$ of a gauge potential in the following manner:

$$\dot{F}_{\alpha\beta} = \frac{d}{ds} F_{\alpha\beta}(s)|_{s=0} = \nabla_\alpha \dot{A}_\beta - \nabla_\beta \dot{A}_\alpha + [\dot{A}_\alpha, A_\beta] + [A_\alpha, \dot{A}_\beta]$$

where

$$\dot{A} = \frac{d}{ds} A(s)|_{s=0}$$

The action principle gives

$$\begin{aligned} \int_U \frac{d}{ds} L(s)|_{s=0} &= \int_U -\frac{1}{2} \langle \dot{F}_{\alpha\beta}, F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} = 0 \\ &= -\frac{1}{2} \int_U \langle \nabla_\alpha \dot{A}_\beta - \nabla_\beta \dot{A}_\alpha + [\dot{A}_\alpha, A_\beta] + [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} \\ &= -\frac{1}{2} \int_U \langle \nabla_\alpha \dot{A}_\beta, F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} + \frac{1}{2} \int_U \langle \nabla_\beta \dot{A}_\alpha, F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} \\ &\quad - \frac{1}{2} \int_U \langle [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} - \frac{1}{2} \int_U \langle [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} \\ &= - \int_U \langle \nabla_\alpha \dot{A}_\beta, F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} - \int_U \langle [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} \\ &\quad \text{(where we have used the anti-symmetry of } F) \\ &= - \int_U \langle \dot{A}_\beta, \nabla_\alpha F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} - \int_U \langle [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} \end{aligned}$$

(where we used the divergence theorem, and the boundary terms are zero since F has

compact support).

On the other hand

$$-\int_U \langle [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}} = \int_U \langle [\dot{A}_\beta, A_\alpha], F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}}$$

(by anti-symmetry of F)

$$= \int_U \langle \dot{A}_\beta, [A_\alpha, F^{\alpha\beta}] \rangle_{\mathbf{g}} dv_{\mathbf{g}}$$

because $\langle \cdot, \cdot \rangle$ is Ad-invariant. This yields to

$$0 = \int_U \langle \dot{A}_\beta, \nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] \rangle_{\mathbf{g}} dv_{\mathbf{g}} = \int_U \langle \dot{A}_\beta, \mathbf{D}_\alpha^{(A)} F^{\alpha\beta} \rangle_{\mathbf{g}} dv_{\mathbf{g}}$$

So the covariant divergence of the curvature is zero

$$\mathbf{D}_\alpha^{(A)} F^{\alpha\beta} = 0 \tag{2.9}$$

On the other hand, computing

$$\begin{aligned} & \mathbf{D}_\alpha^{(A)} F_{\mu\nu} + \mathbf{D}_\mu^{(A)} F_{\nu\alpha} + \mathbf{D}_\nu^{(A)} F_{\alpha\mu} \\ &= \nabla_\alpha F_{\mu\nu} + [A_\alpha, F_{\mu\nu}] + \nabla_\mu F_{\nu\alpha} + [A_\mu, F_{\nu\alpha}] + \nabla_\nu F_{\alpha\mu} + [A_\nu, F_{\alpha\mu}] \\ &= \nabla_\alpha (\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]) + [A_\alpha, \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]] \\ & \quad + \nabla_\mu (\nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]) + [A_\mu, \nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]] \\ & \quad + \nabla_\nu (\nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]) + [A_\nu, \nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]] \\ &= \nabla_\alpha \nabla_\mu A_\nu - \nabla_\alpha \nabla_\nu A_\mu + [\nabla_\alpha A_\mu, A_\nu] + [A_\mu, \nabla_\alpha A_\nu] \\ & \quad + [A_\alpha, \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]] + \nabla_\mu \nabla_\nu A_\alpha - \nabla_\mu \nabla_\alpha A_\nu \\ & \quad + [\nabla_\mu A_\nu, A_\alpha] + [A_\nu, \nabla_\mu A_\alpha] + [A_\mu, \nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]] \\ & \quad + \nabla_\nu \nabla_\alpha A_\mu - \nabla_\nu \nabla_\mu A_\alpha + [\nabla_\nu A_\alpha, A_\mu] \\ & \quad + [A_\alpha, \nabla_\nu A_\mu] + [A_\nu, \nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]] \end{aligned}$$

(where $\nabla_\alpha \nabla_\mu A = \nabla_\alpha (\nabla_\mu A) - \nabla_{\nabla_\alpha e_\mu} A$ is the tensorial covariant derivative of A)

$$\begin{aligned}
&= R_\nu^\gamma{}_{\alpha\mu} A_\gamma + R_\alpha^\gamma{}_{\mu\nu} A_\gamma + R_\mu^\gamma{}_{\nu\alpha} A_\gamma + [A_\alpha, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\alpha]] \\
&\quad + [A_\nu, [A_\alpha, A_\mu]] \\
&= -(R^\gamma{}_{\nu\alpha\mu} + R^\gamma{}_{\alpha\mu\nu} + R^\gamma{}_{\mu\nu\alpha}) A_\gamma + [A_\alpha, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\alpha]] \\
&\quad + [A_\nu, [A_\alpha, A_\mu]] \\
&= 0
\end{aligned}$$

by Bianchi identity and symmetry of the curvature tensor.

So we have,

$$\mathbf{D}_\alpha^{(A)} F_{\mu\nu} + \mathbf{D}_\mu^{(A)} F_{\nu\alpha} + \mathbf{D}_\nu^{(A)} F_{\alpha\mu} = 0 \quad (2.10)$$

The equations (2.9) and (2.10) form the Yang-Mills equations. The Maxwell equations correspond to the abelian case where $[,] = 0$, and therefore $\mathbf{D}^{(A)} = \nabla$.

The Cauchy problem for the Yang-Mills equations formulates as the following: given a Cauchy hypersurface Σ in M , and a \mathcal{G} -valued one form A_μ on Σ , and a \mathcal{G} -valued one form E_i on Σ satisfying $\mathbf{D}_i^{(A)} E^i = 0$, we are looking for a \mathcal{G} -valued two form $F_{\mu\nu}$ satisfying the Yang-Mills equations such that once $F_{\mu\nu}$ restricted on M we have $F_{0i} = E_i$, and such that $F_{\mu\nu}$ corresponds to the curvature derived from the Yang-Mills potential A_μ (i.e. $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta]$).

2.2 Motivation

Our motivation for the systematic study of gauge field theories such as the Yang-Mills equations is to have insights into the Einstein vacuum equations in General Relativity. Indeed, using Cartan formalism the Einstein vacuum equations can be viewed as to some

extent mathematically related to the Yang-Mills equations. General Relativity postulates that the space-time is a 4-dimensional Lorentzian manifold (M, \mathbf{g}) , that satisfies the Einstein vacuum equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci curvature, i.e. $R_{\mu\nu} = R^\gamma{}_{\mu\gamma\nu}$, and where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor associated to the metric \mathbf{g} , defined by,

$$\begin{aligned} R(X, Y, U, V) &= \mathbf{g}(X, [\nabla_U \nabla_V - \nabla_V \nabla_U]Y) \\ &= \mathbf{g}(X, \nabla_U(\nabla_V Y) - \nabla_V(\nabla_U Y) - \nabla_{[U, V]}Y) \end{aligned}$$

where X, Y, U, V are vectorfields in the tangent bundle of M . We will see that using Cartan formalism one can write the Riemann tensor as a Yang-Mills curvature.

2.2.1 Cartan formalism

At a point p of the space-time, one can choose a normal frame, which means a frame such that $\mathbf{g}(e_\alpha, e_\beta)(p) = \text{diag}(-1, 1, \dots, 1)$, and $\frac{\partial}{\partial \sigma} \mathbf{g}(e_\alpha, e_\beta)(p) = 0$, i.e. the first partial derivatives of the metric at p vanish. Cartan formalism consists in defining the connection 1-form,

$$(A)_{\alpha\beta}(X) = \mathbf{g}(\nabla_X e_\beta, e_\alpha) \tag{2.11}$$

where ∇ is the Levi-Civita connection. Thus, since $A = A_\mu dx^\mu$, we can write,

$$(A_\mu)_{\alpha\beta} = (A)_{\alpha\beta}\left(\frac{\partial}{\partial \mu}\right) = \mathbf{g}(\nabla_\mu e_\beta, e_\alpha)$$

Lemma 2.2.2. *If we define the Lie bracket $[A_\mu, A_\nu]$ as*

$$([A_\mu, A_\nu])_{\alpha\beta} = (A_\mu)_\alpha{}^\lambda (A_\nu)_{\lambda\beta} - (A_\nu)_\alpha{}^\lambda (A_\mu)_{\lambda\beta} \tag{2.12}$$

and,

$$(F_{\mu\nu})_{\alpha\beta} = (\nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}])_{\alpha\beta} \quad (2.13)$$

with A defined as in (2.11), then, we have

$$R_{\alpha\beta\mu\nu} = (F_{\mu\nu})_{\alpha\beta} \quad (2.14)$$

Proof.

Computing,

$$\begin{aligned} R(e_{\alpha}, e_{\beta}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) &= \mathbf{g}(e_{\alpha}, (\nabla_{\frac{\partial}{\partial\mu}}\nabla_{\frac{\partial}{\partial\nu}} - \nabla_{\frac{\partial}{\partial\nu}}\nabla_{\frac{\partial}{\partial\mu}} - \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]})e_{\beta}) \\ &= \mathbf{g}(e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}\nabla_{\frac{\partial}{\partial\nu}}e_{\beta} - \nabla_{\frac{\partial}{\partial\nu}}\nabla_{\frac{\partial}{\partial\mu}}e_{\beta} - \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]}e_{\beta}) \\ &= \frac{\partial}{\partial\mu}\mathbf{g}(e_{\alpha}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) \\ &\quad - [\frac{\partial}{\partial\nu}\mathbf{g}(e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta})] \\ &\quad - \mathbf{g}(e_{\alpha}, \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]}e_{\beta}) \\ &= \frac{\partial}{\partial\mu}\mathbf{g}(e_{\alpha}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) - \frac{\partial}{\partial\nu}\mathbf{g}(e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) \\ &\quad + \mathbf{g}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) \\ &\quad ([\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}] = 0 \text{ since they are coordinate vectorfields}) \\ &= \frac{\partial}{\partial\mu}(A_{\nu})_{\alpha\beta} - \frac{\partial}{\partial\nu}(A_{\mu})_{\alpha\beta} + \mathbf{g}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) \\ &= \frac{\partial}{\partial\mu}(A_{\nu})_{\alpha\beta} - \frac{\partial}{\partial\nu}(A_{\mu})_{\alpha\beta} + \mathbf{g}(e^{\lambda}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha})e_{\lambda}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) - \mathbf{g}(e^{\lambda}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha})e_{\lambda}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) \\ &= \frac{\partial}{\partial\mu}(A_{\nu})_{\alpha\beta} - \frac{\partial}{\partial\nu}(A_{\mu})_{\alpha\beta} + e^{\lambda}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha})\mathbf{g}(e_{\lambda}, \nabla_{\frac{\partial}{\partial\mu}}e_{\beta}) - e^{\lambda}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha})\mathbf{g}(e_{\lambda}, \nabla_{\frac{\partial}{\partial\nu}}e_{\beta}) \\ &= \frac{\partial}{\partial\mu}(A_{\nu})_{\alpha\beta} - \frac{\partial}{\partial\nu}(A_{\mu})_{\alpha\beta} + e^{\lambda}(\nabla_{\frac{\partial}{\partial\nu}}e_{\alpha})(A_{\mu})_{\lambda\beta} - e^{\lambda}(\nabla_{\frac{\partial}{\partial\mu}}e_{\alpha})(A_{\nu})_{\lambda\beta} \end{aligned}$$

Computing at the point p , using the normal frame,

$$(A_{\mu})^{\lambda}{}_{\alpha} = \mathbf{g}^{\gamma\lambda}(A_{\mu})_{\gamma\alpha} = \mathbf{g}^{\lambda\lambda}(A_{\mu})_{\lambda\alpha} = \mathbf{g}(e_{\lambda}, e_{\lambda})^{-1}(A_{\mu})_{\lambda\alpha} = \mathbf{g}(e_{\lambda}, e_{\lambda})^{-1}\mathbf{g}(\nabla_{\mu}e_{\alpha}, e_{\lambda})$$

Since the metric is compatible, we have $\nabla \mathbf{g} = 0$, and thus,

$$\begin{aligned}\nabla_{\mu} \mathbf{g}(e_{\alpha}, e_{\lambda}) &= \frac{\partial}{\partial \mu} \mathbf{g}(e_{\alpha}, e_{\lambda}) - \mathbf{g}(\nabla_{\mu} e_{\alpha}, e_{\lambda}) - \mathbf{g}(e_{\alpha}, \nabla_{\mu} e_{\lambda}) \\ &= -\mathbf{g}(\nabla_{\mu} e_{\alpha}, e_{\lambda}) - \mathbf{g}(e_{\alpha}, \nabla_{\mu} e_{\lambda}) = 0\end{aligned}$$

Therefore,

$$\mathbf{g}(\nabla_{\mu} e_{\alpha}, e_{\lambda}) = -\mathbf{g}(e_{\alpha}, \nabla_{\mu} e_{\lambda})$$

and thus, the matrix A is anti-symmetric, ie. $(A_{\mu})_{\alpha\beta} = (A_{\mu})_{\beta\alpha}$.

We have,

$$\nabla_{\frac{\partial}{\partial \mu}} e_{\alpha} = e^{\lambda} (\nabla_{\frac{\partial}{\partial \mu}} e_{\alpha}) e_{\lambda}$$

Thus,

$$\mathbf{g}(e_{\lambda}, \nabla_{\frac{\partial}{\partial \mu}} e_{\alpha}) = \mathbf{g}(e_{\lambda}, e^{\lambda'} (\nabla_{\frac{\partial}{\partial \mu}} e_{\alpha}) e_{\lambda'}) = \mathbf{g}(e_{\lambda}, e_{\lambda}) e^{\lambda} (\nabla_{\frac{\partial}{\partial \mu}} e_{\alpha})$$

Consequently,

$$e^{\lambda} (\nabla_{\frac{\partial}{\partial \mu}} e_{\alpha}) = \mathbf{g}(e_{\lambda}, e_{\lambda})^{-1} \mathbf{g}(e_{\lambda}, \nabla_{\frac{\partial}{\partial \mu}} e_{\alpha}) = (A_{\mu})^{\lambda}{}_{\alpha}$$

and,

$$e^{\lambda} (\nabla_{\frac{\partial}{\partial \nu}} e_{\alpha}) = \mathbf{g}(e_{\lambda}, e_{\lambda})^{-1} \mathbf{g}(e_{\lambda}, \nabla_{\frac{\partial}{\partial \nu}} e_{\alpha}) = (A_{\nu})^{\lambda}{}_{\alpha}$$

Therefore,

$$\begin{aligned}
R(e_\alpha, e_\beta, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}) &= \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} + (A_\nu)^\lambda{}_\alpha(A_\mu)_{\lambda\beta} - (A_\mu)^\lambda{}_\alpha(A_\nu)_{\lambda\beta} \\
&= \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} - (A_\nu)_\alpha{}^\lambda(A_\mu)_{\lambda\beta} + (A_\mu)_\alpha{}^\lambda(A_\nu)_{\lambda\beta}
\end{aligned}$$

(by anti-symmetry of the matrix A).

We define the Lie bracket $[A_\mu, A_\nu]$ as

$$([A_\mu, A_\nu])_{\alpha\beta} = (A_\mu)_\alpha{}^\lambda(A_\nu)_{\lambda\beta} - (A_\nu)_\alpha{}^\lambda(A_\mu)_{\lambda\beta}$$

Consequently, we get,

$$R_{\alpha\beta\mu\nu} = \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} + ([A_\mu, A_\nu])_{\alpha\beta}, \quad (2.15)$$

We have by definition,

$$\nabla_\mu A_\nu = \partial_\mu(A_\nu) - (A)(\nabla_\mu \frac{\partial}{\partial \nu})$$

and,

$$\nabla_\nu A_\mu = \partial_\nu(A_\mu) - (A)(\nabla_\nu \frac{\partial}{\partial \mu})$$

Thus,

$$\begin{aligned}
\nabla_\mu A_\nu - \nabla_\nu A_\mu &= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)(\nabla_\nu \frac{\partial}{\partial \mu}) - (A)(\nabla_\mu \frac{\partial}{\partial \nu}) \\
&= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)(\nabla_\nu \frac{\partial}{\partial \mu} - \nabla_\mu \frac{\partial}{\partial \nu}) \\
&= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)([\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \mu}])
\end{aligned}$$

(because the metric is symmetric)

$$= \partial_\mu(A_\nu) - \partial_\nu(A_\mu)$$

(since $\frac{\partial}{\partial\mu}$, and $\frac{\partial}{\partial\nu}$ are coordinate vectorfields, therefore they commute).

As a result,

$$R_{\alpha\beta\mu\nu} = (\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu])_{\alpha\beta} \quad (2.16)$$

Since the curvature tensor of the connection A is,

$$(F_{\mu\nu})_{\alpha\beta} = (\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu])_{\alpha\beta} \quad (2.17)$$

We can write,

$$R_{\alpha\beta\mu\nu} = (F_{\mu\nu})_{\alpha\beta} \quad (2.18)$$

□

2.2.3 The Einstein equations in a Yang-Mills form

The following is a well known proposition, of which we sketch the proof.

Proposition 2.2.4. *Let (M, \mathbf{g}) be a 4-dimensional Lorentzian manifold that is Ricci flat, i.e. $R_{\mu\nu} = 0$. Then, we have $(\mathbf{D}^{(A)\mu} F_{\mu\nu})_{\alpha\beta} = 0$, where A and F are defined as in (2.11) and (2.13).*

Proof.

Lemma 2.2.5. *We have*

$$\nabla_\sigma R_{\alpha\beta\mu\nu} = (\mathbf{D}_\sigma^{(A)} F_{\mu\nu})_{\alpha\beta}$$

Proof.

Computing,

$$\begin{aligned} \nabla_\sigma R_{\alpha\beta\mu\nu} &= \frac{\partial}{\partial\sigma} R(e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(\nabla_\sigma e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, \nabla_\sigma e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) \\ &\quad - R(e_\alpha, e_\beta, \nabla_\sigma \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \nabla_\sigma \frac{\partial}{\partial\nu}) \\ &= \frac{\partial}{\partial\sigma} R(e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, e_\beta, \nabla_\sigma \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \nabla_\sigma \frac{\partial}{\partial\nu}) \\ &\quad - R(\nabla_\sigma e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, \nabla_\sigma e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) \\ &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - R(\nabla_\sigma e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, \nabla_\sigma e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) \\ &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - R(e^\lambda (\nabla_\sigma e_\alpha) e_\lambda, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - R(e_\alpha, e^\lambda (\nabla_\sigma e_\beta) e_\lambda, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) \\ &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - e^\lambda (\nabla_\sigma e_\alpha) R(e_\lambda, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - e^\lambda (\nabla_\sigma e_\beta) R(e_\alpha, e_\lambda, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) \\ &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - (A_\sigma)^\lambda{}_\alpha (F_{\mu\nu})_{\lambda\beta} - (A_\sigma)^\lambda{}_\beta (F_{\mu\nu})_{\alpha\lambda} \\ &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} + (A_\sigma)_\alpha{}^\lambda (F_{\mu\nu})_{\lambda\beta} - (F_{\mu\nu})_{\alpha\lambda} (A_\sigma)^\lambda{}_\beta \end{aligned}$$

(using the anti-symmetry of A). Thus,

$$\begin{aligned} \nabla_\sigma R_{\alpha\beta\mu\nu} &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} + ([A_\sigma, F_{\mu\nu}])_{\alpha\beta} \\ &= (\mathbf{D}_\sigma^{(A)} F_{\mu\nu})_{\alpha\beta} \end{aligned} \tag{2.19}$$

□

Computing,

$$\begin{aligned} (\mathbf{D}^{(A)\mu} F_{\mu\nu})_{\alpha\beta} &= \nabla^\mu R_{\alpha\beta\mu\nu} = \mathbf{g}^{\mu\sigma} \nabla_\sigma R_{\alpha\beta\mu\nu} \\ &= \nabla_\sigma (\mathbf{g}^{\mu\sigma} R_{\alpha\beta\mu\nu}) \end{aligned}$$

(because the metric is compatible)

$$= \nabla_{\sigma} R_{\alpha\beta}{}^{\sigma}{}_{\nu} = \nabla_{\sigma} R^{\sigma}{}_{\nu\alpha\beta}$$

(using the symmetry of the Riemann tensor)

$$= -\nabla_{\alpha} R^{\sigma}{}_{\nu\beta\sigma} - \nabla_{\beta} R^{\sigma}{}_{\nu\sigma\alpha}$$

(where we have used another symmetry of the Riemann tensor)

$$= \nabla_{\alpha} R^{\sigma}{}_{\nu\sigma\beta} - \nabla_{\beta} R^{\sigma}{}_{\nu\sigma\alpha} = \nabla_{\alpha} R_{\nu\beta} - \nabla_{\beta} R_{\nu\alpha} = 0$$

(since the Einstein vacuum equations say that $R_{\mu\gamma} = 0 = R^{\sigma}{}_{\mu\sigma\gamma}$). We get,

$$(\mathbf{D}^{(A)\mu} F_{\mu\nu})_{\alpha\beta} = 0 \tag{2.20}$$

□

The second Bianchi identity for the Riemann tensor,

$$\begin{aligned} 0 &= \nabla_{\alpha} R_{\gamma\sigma\mu\nu} + \nabla_{\mu} R_{\gamma\sigma\nu\alpha} + \nabla_{\nu} R_{\gamma\sigma\alpha\mu} \\ &= (\mathbf{D}_{\alpha}^{(A)} F_{\mu\nu} + \mathbf{D}_{\mu}^{(A)} F_{\nu\alpha} + \mathbf{D}_{\nu}^{(A)} F_{\alpha\mu})_{\gamma\sigma} \end{aligned} \tag{2.21}$$

which is the Bianchi identity for the Yang-Mills fields. The equations above (2.20) and (2.21) are the Yang-Mills equations except to the fact that the background geometry (M, \mathbf{g}) is part of the unknown that we are looking for while trying to solve the Einstein vacuum equations.

This analogy between the Einstein equations in General Relativity and the Yang-Mills equations has been pursued by V. Moncrief in [M], by developing an integral representation formula for the curvature tensor in General Relativity, and then independently by I.

Rodnianski and S. Klainerman in [KR1] and [KR3], partly as a desire to adapt the Eardley-Moncrief argument [EM1]-[EM2] to General Relativity. The analogy was exploited by I. Rodnianski, S. Klainerman, and J. Szeftel, in [KRS] in their proof of the Bounded L^2 Curvature Conjecture in General Relativity.

Chapter 3

The Global Existence of Yang-Mills Fields on Curved Space-Times

3.1 Introduction

In this chapter, we study the Yang-Mills equations on an arbitrary given curved space-time. For this, we will make use of a hyperbolic formulation of the problem. Indeed, while the Yang-Mills equations say that the Yang-Mills curvature is divergence free on the background geometry, one can obtain a hyperbolic formulation by taking the covariant divergence of the Bianchi identity. This leads to a tensorial covariant wave equation on the Yang-Mills curvature with a non-linear term. It is exactly the study of this non-linear term that permits one to answer the question of local well-posedness, and global well-posedness of the equations. In this formulation, the initial data consists of the Yang-Mills potential, that is a one form valued in the Lie algebra, and the electric field (loosely speaking the time derivative of the potential) on a given spacelike Cauchy hypersurface Σ . The initial data set has to verify itself the Yang-Mills equations, that is the covariant divergence of

the electric field vanishes. One looks for a Yang-Mills curvature that satisfies the Yang-Mills equations such that once restricted on this hypersurface Σ the Yang-Mills curvature corresponds to that given by the prescribed potential and electric field.

Eardley and Moncrief proved global existence of solutions of the Yang-Mills equations in the 4-dimensional Minkowski background by proving a local existence result and providing pointwise estimates on the curvature, [EM1]-[EM2]. Their approach depended on the fundamental solution of the wave equation on flat space-time, and the use of the Cronström gauge condition, that has the remarkable advantage of expressing the potential as a function of the curvature directly in terms of an integral, to estimate the non-linear term. Later on, this result was extended by Chruściel and Shatah, [CS], to curved space-times using the same approach, by making use of the Friedlander parametrix for the wave equation in causal domains in curved space-times, [Fried], and the Cronström gauge condition as well. In a recent paper, [KR1], Klainerman and Rodnianski constructed a parametrix for the wave equation which permitted them to give a new gauge independent proof of the Eardley-Moncrief result in [EM2] concerning the non-blow up of the Yang-Mills fields in a Minkowski background.

The Klainerman-Rodnianski's approach relies on their derivation of a covariant representation formula for the wave equation on arbitrary, smooth, globally hyperbolic, curved space-times, in which the integral terms are supported on the past null cone. As the authors pointed out, their parametrix can be immediately adapted to gauge covariant derivatives; this is because the scalar product on the Lie algebra $\langle \cdot, \cdot \rangle$ is Ad-invariant. They used it to give a new gauge independent proof of the Eardley-Moncrief result [EM2], of which the only ingredient is the conservation of the energy. As the authors mentioned, one can generalize their proof of the non-blow up of Yang-Mills fields on the flat Minkowski space-time to arbitrary smooth, globally hyperbolic, curved space-times under the assumption that there exists a timelike vector field $\frac{\partial}{\partial t}$ of which the deformation tensor is finite, as it has been assumed in previous work by Chruściel and Shatah, [CS].

In this chapter, we provide standard material, but not so clearly pointed out in litera-

ture, that consists in writing the proof of the non-blow up of the Yang-Mills curvature on arbitrary curved space-times by using the Klainerman-Rodnianski parametrix combined with suitable Grönwall type inequalities. While the Chruściel-Shatah argument requires a simultaneous control of the L_{loc}^∞ and the H_{loc}^2 norms of the Yang-Mills curvature, we can get away by controlling only the H_{loc}^1 norm instead. However, we were unable to get rid of any control on the gradient of the Yang-Mills curvature, as it is the case in the proof on Minkowski space-time in [KR1]. Hence, this provides a new gauge independent proof of the control on the L_{loc}^∞ norm of the Yang-Mills curvature and improves the Chruściel-Shatah's result, [CS], for the global existence of Yang-Mills fields on sufficiently smooth, globally hyperbolic, curved 4-dimensional Lorentzian manifolds.

3.1.1 The statement

More precisely, we will prove the following theorem,

Theorem 3.1.2. *Let (M, \mathbf{g}) be a curved 4-dimensional Lorentzian manifold. We assume that \mathbf{g} is sufficiently smooth, and M is globally hyperbolic. We know by then that there exist a smooth vector field $\frac{\partial}{\partial t}$ such that M is foliated by Cauchy hypersurfaces Σ_t . Let \hat{t} be a unit timelike vector field orthogonal to Σ_t .*

Let $\{\hat{t}, n, e_1, e_2\}$ be an orthonormal frame where \hat{t} is as above. We assume there exists $C_{loc}(t) \in L_{loc}^1$, such that for all $\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_1, e_2\}$, the components of the deformation tensor $\pi^{\hat{\mu}\hat{\nu}}(\hat{t}) = \frac{1}{2}[\nabla^{\hat{\mu}}\hat{\nu} + \nabla^{\hat{\nu}}\hat{\mu}]$ verify,

$$|\pi^{\hat{\mu}\hat{\nu}}(\hat{t})|_{L_{loc}^\infty(\Sigma_t)} \leq C_{loc}(t) \quad (3.1)$$

where Σ_t are the $t = \text{constant}$ hypersurfaces.

Let $F_{\hat{\mu}\hat{\nu}}$ be the components of the Yang-Mills fields in the frame $\{\hat{t}, n, e_1, e_2\}$, defined as the anti-symmetric 2-tensor solution of the Cauchy problem of the Yang-Mills equations $\mathbf{D}_\alpha^{(A)} F^{\alpha\beta} = 0$, where the initial data prescribed on the Cauchy hypersurface $\Sigma_{t=t_0}$ verifies

the Yang-Mills constraint equations,

$$\mathbf{D}^{(A)\beta} F_{t\beta}(t = t_0) = (\mathbf{D}^{(A)\beta} F_{\nu\beta})\left(\frac{\partial}{\partial t}\right)^\nu(t = t_0) = 0$$

Let,

$$E_F^{\hat{t}}(t = t_0) = \int_{q \in \Sigma_{t=t_0}} \frac{1}{2} [|F_{\hat{t}n}|^2 + |F_{\hat{t}a}|^2 + |F_{\hat{t}b}|^2 + |F_{na}|^2 + |F_{nb}|^2 + |F_{ab}|^2](q) \cdot dV_\Sigma(q)$$

and,

$$E_{\mathbf{D}^{(A)F}}^{\hat{t}}(t = t_0) = \int_{q \in \Sigma_{t=t_0}} \frac{1}{2} [|\mathbf{D}^{(A)} F_{\hat{t}n}|^2 + |\mathbf{D}^{(A)} F_{\hat{t}a}|^2 + |\mathbf{D}^{(A)} F_{\hat{t}b}|^2 + |\mathbf{D}^{(A)} F_{na}|^2 + |\mathbf{D}^{(A)} F_{nb}|^2 + |\mathbf{D}^{(A)} F_{ab}|^2](q) \cdot dV_\Sigma(q)$$

where,

$$|\mathbf{D}^{(A)} F_{\hat{\mu}\hat{\nu}}|^2 = |\mathbf{D}_{\hat{t}}^{(A)} F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}_n^{(A)} F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}_{e_a}^{(A)} F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}_{e_b}^{(A)} F_{\hat{\mu}\hat{\nu}}|^2$$

and where $|\cdot|$ is the norm associated to the Ad-invariant scalar product $\langle \cdot, \cdot \rangle$.

If

$$E_F^{\hat{t}}(t = t_0) < \infty$$

and,

$$E_{\mathbf{D}^{(A)F}}^{\hat{t}}(t = t_0) < \infty$$

then, the Yang-Mills curvature $F_{\hat{\mu}\hat{\nu}}$ of local solutions defined for all $t \in [t_0, T)$ will not blow-up in time t , i.e. at each point q of the space-time at time $t_q = T$, we have

$$\lim_{p \rightarrow q, t_p < T} |F_{\hat{\mu}\hat{\nu}}|(p) < \infty,$$

also,

$$\lim_{t \rightarrow T, t < T} E_F^{\hat{t}}(t) < \infty,$$

and

$$\lim_{t \rightarrow T, t < T} E_{\mathbf{D}^{(A)}F}^{\hat{t}}(t) < \infty.$$

3.1.3 A hyperbolic formulation for the Yang-Mills equations

It is known that the Yang-Mills fields can be shown to satisfy a tensorial hyperbolic wave equation with sources, on the background geometry. To see this, we start by taking the covariant divergence of (2.10), we obtain:

$$\mathbf{D}^{(A)\alpha} \mathbf{D}^{(A)}_{\alpha} F_{\mu\nu} + \mathbf{D}^{(A)\alpha} \mathbf{D}^{(A)}_{\mu} F_{\nu\alpha} + \mathbf{D}^{(A)\alpha} \mathbf{D}^{(A)}_{\nu} F_{\alpha\mu} = 0$$

We have:

$$\begin{aligned} \mathbf{D}^{(A)\alpha} \mathbf{D}^{(A)}_{\mu} F_{\nu\alpha} &= \mathbf{D}^{(A)}_{\mu} \mathbf{D}^{(A)\alpha} F_{\nu\alpha} + \nabla^{\alpha} \nabla_{\mu} F_{\nu\alpha} - \nabla_{\mu} \nabla^{\alpha} F_{\nu\alpha} + [F^{\alpha}_{\mu}, F_{\nu\alpha}] \\ &= \mathbf{D}^{(A)}_{\mu} \mathbf{D}^{(A)\alpha} F_{\nu\alpha} + R_{\nu}^{\gamma\alpha} F_{\gamma\alpha} + R_{\alpha}^{\gamma\alpha} F_{\nu\gamma} + [F^{\alpha}_{\mu}, F_{\nu\alpha}] \\ &= \mathbf{D}^{(A)}_{\mu} (\mathbf{D}^{(A)\alpha} F_{\nu\alpha}) - \mathbf{D}^{(A)\nabla_{\nu} e_{\alpha}} F_{\nu\alpha} + R_{\nu}^{\gamma\alpha} F_{\gamma\alpha} + R_{\alpha}^{\gamma\alpha} F_{\nu\gamma} \\ &\quad + [F^{\alpha}_{\mu}, F_{\nu\alpha}] \\ &= \partial_{\mu} (\mathbf{D}^{(A)\alpha} F_{\nu\alpha}) + [A_{\mu}, \mathbf{D}^{(A)\alpha} F_{\nu\alpha}] - \mathbf{D}^{(A)\alpha} F(\nabla_{\mu} e_{\nu}, e_{\alpha}) - \mathbf{D}^{(A)\alpha} F(e_{\nu}, \nabla_{\mu} e_{\alpha}) \\ &\quad - \mathbf{D}^{(A)\nabla_{\nu} e_{\alpha}} F_{\nu\alpha} + R_{\nu}^{\gamma\alpha} F_{\gamma\alpha} + R_{\alpha}^{\gamma\alpha} F_{\nu\gamma} + [F^{\alpha}_{\mu}, F_{\nu\alpha}] \\ &= 0 - \mathbf{D}^{(A)\alpha} F(e_{\nu}, \nabla_{\mu} e_{\alpha}) - \mathbf{D}^{(A)\nabla_{\nu} e_{\alpha}} F_{\nu\alpha} + R_{\nu}^{\gamma\alpha} F_{\gamma\alpha} + R_{\alpha}^{\gamma\alpha} F_{\nu\gamma} + [F^{\alpha}_{\mu}, F_{\nu\alpha}] \end{aligned}$$

(by equation (2.9)). By choosing a normal frame at each point in space-time, i.e. a frame where $\mathbf{g}(e_{\alpha}, e_{\beta}) = \text{diag}(-1, 1, \dots, 1)$ and $\nabla_{\alpha} e_{\beta} = 0$ at that point, to compute the

contractions $\mathbf{D}^{(A)\alpha}F(e_\nu, \nabla_\mu e_\alpha)$ and $\mathbf{D}^{(A)\nabla_\nu e_\alpha}F_{\nu\alpha}$, we get that they vanish. So

$$\begin{aligned}\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}_\mu F_{\nu\alpha} &= R_{\nu\gamma\alpha\mu}F^{\gamma\alpha} + R_{\alpha\gamma}{}^\alpha{}_\mu F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}] \\ &= R_{\nu\gamma\alpha\mu}F^{\gamma\alpha} + R_{\gamma\mu}F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}] \\ &= R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} + R_{\mu\gamma}F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}]\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}_\nu F_{\alpha\mu} &= \mathbf{D}^{(A)}_\nu\mathbf{D}^{(A)\alpha}F_{\alpha\mu} + \nabla^\alpha\nabla_\nu F_{\alpha\mu} - \nabla_\nu\nabla^\alpha F_{\alpha\mu} + [F^\alpha{}_\nu, F_{\alpha\mu}] \\ &= 0 + R_{\alpha}{}^\gamma{}_\nu F_{\gamma\mu} + R_{\mu}{}^\gamma{}_\nu F_{\alpha\gamma} + [F^\alpha{}_\nu, F_{\alpha\mu}] \\ &\quad \text{(by equation (2.9), and computing the contractions in a normal frame)} \\ &= R_{\alpha\gamma}{}^\alpha{}_\nu F^\gamma{}_\mu + R_{\mu\gamma\alpha\nu}F^{\alpha\gamma} + [F_{\alpha\nu}, F^\alpha{}_\mu] \\ &= R_{\gamma\nu}F^\gamma{}_\mu + R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} + [F^\alpha{}_\mu, F_{\nu\alpha}]\end{aligned}$$

(where we have used the anti-symmetry of F)

$$= R_{\nu\gamma}F^\gamma{}_\mu + R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} + [F^\alpha{}_\mu, F_{\nu\alpha}]$$

As we have

$$\mathbf{D}^{(A)}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)}_\nu F_{\alpha\mu} = 0$$

we get

$$\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}_\alpha F_{\mu\nu} + 2R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} + R_{\mu\gamma}F_\nu{}^\gamma + R_{\nu\gamma}F^\gamma{}_\mu + 2[F^\alpha{}_\mu, F_{\nu\alpha}] = 0$$

We obtain:

$$\square_{\mathbf{g}}^{(A)} F_{\mu\nu} = \mathbf{D}^{(A)\alpha} \mathbf{D}^{(A)}_{\alpha} F_{\mu\nu} = -2R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} - R_{\mu\gamma} F_{\nu}{}^{\gamma} - R_{\nu\gamma} F^{\gamma}{}_{\mu} - 2[F^{\alpha}{}_{\mu}, F_{\nu\alpha}] \quad (3.2)$$

Due to the equation (3.2), the held belief is that the Yang-Mills equations are hyperbolic in nature.

3.1.4 Strategy of the proof

As we showed, see (3.2), the Yang-Mills fields satisfy a non-linear hyperbolic differential equation on the background geometry. Since the scalar product on the Lie algebra \langle , \rangle is Ad-invariant, the Klainerman-Rodnianski parametrix can be immediately generalized (see Appendix) to gauge covariant derivatives to give a representation formula for solutions of $(\square_{\mathbf{g}}^{(A)} F)_{\mu\nu} = S_{\mu\nu}$, where $S_{\mu\nu}$ is a source tensor, and hence it can be used for the Yang-Mills fields, see (6.103).

We would like to bound all the terms in the representation formula in a way that we could use Grönwall lemma to deduce that the L^{∞} norm of F will stay finite (see (3.66)). For this we need a parameter in which the extension of local solutions can make sense; this would be the vector field $\frac{\partial}{\partial t}$.

The main advantage of the parametrix is that all its integral terms are supported on the past null cone. Naively, one can hope that those can be bounded by the flux of the energy generated from \hat{t} . Thus, if one can bound the energy flux along the null cones, the proof might go through. To bound the energy flux, one needs, as in [CS], to assume that the deformation tensor of a unit timelike vector field has its integral in t finite on bounded

domains:

$$|\pi(\hat{t})|_{L^\infty(\Sigma_t)} \leq C_{loc}(t) \in L^1_{loc}$$

Using the divergence theorem on the energy-momentum tensor contracted with \hat{t} will lead to an inequality on the energy. The assumption on the deformation tensor above can show using Grönwall lemma that the local energy will stay finite, see (3.22). Using this and the assumption on the deformation tensor again, one can show that the space-time integral generated from the divergence theorem will stay finite in t , see (3.23), from which one can deduce the finiteness of the flux (3.24).

The integral terms supported on the past null cone in the Klainerman-Rodnianski parametrix involve a term that is a generalization of the fundamental solution of the wave equation on flat space-time to curved space-times. This is $\lambda_{\mu\nu}$ that is a two tensor solution of a transport equation along the null cone given by (6.8) and (6.9). Using the transport equation, one can prove that the L^∞ norm of $s\lambda_{\mu\nu}$, where s is the geodesic parameter for a null vector field L normal to the null cone used to define the transport equation for $\lambda_{\mu\nu}$, will be bounded by the initial data for $\lambda_{\mu\nu}$, see (3.25). Yet, since we would want to apply Grönwall lemma, the terms which contain $\lambda_{\mu\nu}$ and $F_{\mu\nu}$ can be bounded as in (3.2.18), by controlling $s\lambda_{\mu\nu}$, see (3.25), and $s^{-1}F_{\mu\nu}$, see (3.67). The terms which contain $\lambda_{\mu\nu}$ and $[F, F]$ can be bounded as in (3.2.20) by using the finiteness of the energy flux, see (3.70).

However, a major difference with the situation on Minkowski space, is in the way to deal with the term which contains $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ and F , where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the induced Laplacian on the 2-sphere prescribed by $s = constant$ defined by (6.49). In Minkowski space, one can control directly $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ as shown by Rodnianski and Klainerman in [KR1], because one can close a system of transport equations along the null cone. On curved space-times, we are unable to close such a system, consequently, we will use the divergence theorem on \mathbb{S}^2 , see (3.75), so as to bring the problem to controlling $\mathbf{D}_a^{(A)}\lambda$ and $\mathbf{D}_a^{(A)}F$, where these are the derivatives tangential to the 2-sphere prescribed by $s = constant$.

To control $\mathbf{D}_a^{(A)}\lambda$ we will follow [KR3], see (3.2.12). Since the area element on the 2-spheres is at the level of s^2 , see (6.59), one would want to control the L^2 norm of $s\mathbf{D}_a^{(A)}\lambda$ on the null cone, with respect to the measure $dsd\sigma^2$, where $d\sigma^2$ is the usual volume form on \mathbb{S}^2 . One could try to use the fundamental theorem of calculus directly to control the L^2 norm on \mathbb{S}^2 then integrate in s , yet by doing so, we would find ourselves confronted to controlling near the vertex p ($s = 0$) a quantity of the type $(\frac{1}{s} - tr\chi)$, where χ is the null second fundamental form of the null hypersurfaces. This quantity cannot be controlled even in the 4-dimensional Minkowski space where $tr\chi = \frac{2}{s}$. To change the factor in front of $\frac{1}{s}$ from 1 to 2, one would need to apply the fundamental theorem of calculus to control the L^2 norm on the null cone of $s^2\mathbf{D}_a^{(A)}\lambda$ instead of $s\mathbf{D}_a^{(A)}\lambda$, see (3.38). Since it is the L^2 norm, this means that one would have to consider $s^4|\mathbf{D}_a^{(A)}\lambda|^2$ instead of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$ for applying the fundamental theorem of calculus. However, since what we want to control is the integral on the null cone of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$, which is bigger than that of $s^4|\mathbf{D}_a^{(A)}\lambda|^2$, near $s = 0$, we would need to lower the power on s , for this one can actually control the integral on \mathbb{S}^2 of $s^{-1}s^4|\mathbf{D}_a^{(A)}\lambda|^2$ by applying the fundamental theorem of calculus to $s^4|\mathbf{D}_a^{(A)}\lambda|^2$ as described above, see (3.39) and (3.40). This would allow then to control the integral on \mathbb{S}^2 of $s^3|\mathbf{D}_a^{(A)}\lambda|^2$ in a way that one could then get an estimate on the L^1 norm on \mathbb{S}^2 for $s^2|\mathbf{D}_a^{(A)}\lambda|^2$, see (3.50), which permits one to apply the L^2 maximum principle to control the integral on the null cone of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$ near the vertex p ($s = 0$), see (3.52). Away from the vertex $s = 0$ the integral is clearly finite and hence, this would give the desired control.

In order to control the L^2 norm of $\mathbf{D}_a^{(A)}F$ on the null cone, we will use the energy momentum tensor of the wave equation T_1 after contracting the free indices of the Yang-Mills fields with respect to a Riemannian metric h , as in [CS], see (3.55). Since it is a full contraction, we can compute it by choosing a normal frame, i.e. a frame where the Christoffel symbols vanish at that point, and hence we can get the derivatives inside the scalar product as covariant derivatives (and also as gauge covariant derivatives using the fact that the scalar product is Ad-invariant) instead of partial derivatives. Since it is the

energy momentum tensor for the wave equation, the boundary term supported on the null cone obtained after contracting T_1 with the normalized timelike vector field, $\frac{\partial}{\partial t}$, and applying the divergence theorem in a region inside the null cone, is at the level of the L^2 norm of $\mathbf{D}_a^{(A)}F$ and $\mathbf{D}_L^{(A)}F$, see (3.59), and thus it controls the L^2 norm of $\mathbf{D}_a^{(A)}F$. We know by then, from the divergence theorem, that this can be controlled by a quantity that is at the level of a space integral of $T_1^{\hat{t}\hat{t}}$ on the initial spacelike hypersurface and in addition a spacetime integral of $|\mathbf{D}^{(A)}F|(|\mathbf{D}^{(A)}F| + |F| + |F|^2)$, see (3.62), where $|F|$ and $|F|^2$ arise from the sources of the tensorial gauge hyperbolic wave equation verified by F , and $|\mathbf{D}^{(A)}F|$ in the parenthesis is due to the fact that the deformation tensor of $\frac{\partial}{\partial t}$, as well as the covariant derivative of h , do not vanish. As we wish to get rid of the gradient of F , so as to have a control that involves an integral or a double integral of the square of the L^∞ norm of F , see (3.65), we recall that the divergence theorem that we applied previously also permits one to control the space integral of $T_1^{\hat{t}\hat{t}}$ on the spacelike hypersurface, that is at the level of the L^2 norm of the $\mathbf{D}^{(A)}F$, by the same quantity that controls the boundary term on the null cone, (3.62). This allows one to use Grönwall lemma, after using $a.b \lesssim a^2 + b^2$, and the conservation of the energy that is at the level of the L^2 norm of F , to control the L^2 norm of $\mathbf{D}^{(A)}F$ on the spacelike hypersurfaces by the desired quantity, see (3.63). Injecting this in the previous control on the L^2 norm of $\mathbf{D}_a^{(A)}F$, (3.62), and using again $a.b \lesssim a^2 + b^2$ and the conservation of the energy, leads to the desired control (3.65).

Now, the parametrix (6.103) permits us to control the value of the Yang-Mills fields contracted with an arbitrary tensor, at a point q in space-time, by the estimates mentioned above. We would want to establish a Grönwall type inequality in t on the L^∞ norm of F on Σ_t^p , the spacelike hypersurfaces prescribed by $t = \text{constant}$ in the past of a point p , so as to deduce the finiteness of the fields at the point p . To obtain this, we take the supremum on $q \in \Sigma_t^p$ in the inequality described above, i.e. after using the parametrix and the above estimates, see (3.66). This can be used to show that $\|F\|_{L^\infty(\Sigma_t^p)}$ verifies a generalized Grönwall type inequality (3.78) to which Pachpatte in [Pach], proved a result that ensures that the solutions will stay finite. A local existence result would give that so-

lutions of the Yang-Mills equations will either blow up in finite time, or they will be defined globally in time if a certain "energy" norm does not blow up. The non-blow up result for the Yang-Mills curvature that we have established can be used to show that this suitable energy for a local solution does not blow up, and therefore prove that the local solution of the Yang-Mills equations can be extended globally in time t , under the assumptions of theorem (3.1.2).

3.2 The Proof of Global Existence of Yang-Mills Fields on Arbitrary, Sufficiently Smooth, Globally Hyperbolic, Curved Lorentzian Manifolds

3.2.1 Energy estimates

Consider the energy momentum tensor:

$$T_{\mu\nu} = \langle F_{\mu\beta}, F_{\nu}^{\beta} \rangle - \frac{1}{4} \mathbf{g}_{\mu\nu} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle \quad (3.3)$$

We will write $\langle F_{\alpha\beta}, F^{\alpha\beta} \rangle$ as $F_{\alpha\beta} \cdot F^{\alpha\beta}$ to lighten the notation.

Taking the covariant divergence of $T_{\mu\nu}$ we obtain:

$$\begin{aligned} \nabla^{\nu} T_{\mu\nu} &= \nabla^{\nu} (F_{\mu\beta} \cdot F_{\nu}^{\beta} - \frac{1}{4} \mathbf{g}_{\mu\nu} F_{\alpha\beta} \cdot F^{\alpha\beta}) \\ &= (\mathbf{D}^{(A)\nu} F_{\mu\beta}) \cdot F_{\nu}^{\beta} + F_{\mu\beta} \cdot \mathbf{D}^{(A)\nu} F_{\nu}^{\beta} - \frac{1}{4} \mathbf{g}_{\mu\nu} \mathbf{D}^{(A)\nu} F_{\alpha\beta} \cdot F^{\alpha\beta} \\ &\quad - \frac{1}{4} \mathbf{g}_{\mu\nu} F_{\alpha\beta} \cdot \mathbf{D}^{(A)\nu} F^{\alpha\beta} \end{aligned}$$

(where we used the fact that the metric is Killing, i.e. $\nabla \mathbf{g} = 0$, and that $\langle \cdot, \cdot \rangle$ is Ad-invariant)

$$= (\mathbf{D}^{(A)\nu} F_{\mu\beta}) \cdot F_{\nu}^{\beta} - \frac{1}{2} \mathbf{g}_{\mu\nu} \mathbf{D}^{(A)\nu} F_{\alpha\beta} \cdot F^{\alpha\beta}$$

(we used the field equations)

$$\begin{aligned}
&= (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} - \frac{1}{2} (\mathbf{D}_\mu^{(A)} F_{\alpha\beta}) \cdot F^{\alpha\beta} \\
&= (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} + \frac{1}{2} (\mathbf{D}_\alpha^{(A)} F_{\beta\mu}) \cdot F^{\alpha\beta} + \frac{1}{2} (\mathbf{D}_\beta^{(A)} F_{\mu\alpha}) \cdot F^{\alpha\beta}
\end{aligned}$$

(using the Bianchi identities)

$$\begin{aligned}
&= (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} - \frac{1}{2} (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} + \frac{1}{2} (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\beta\alpha} \\
&= (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} - \frac{1}{2} (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta} - \frac{1}{2} (\mathbf{D}_\alpha^{(A)} F_{\mu\beta}) \cdot F^{\alpha\beta}
\end{aligned}$$

(where we used the anti-symmetry of F in the last two equalities)

$$= 0 \tag{3.4}$$

Considering a vector field V^ν we let

$$J_\mu(V) = V^\nu T_{\mu\nu} = T_{\mu V}$$

We have,

$$\nabla^\mu J_\mu(V) = \partial^\mu T_{\mu V} - T(\nabla^\mu e_\mu, V) = \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V)$$

Hence,

$$\begin{aligned}
\nabla^\mu J_\mu(V) &= \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V) \\
&= V^\nu (\nabla^\mu T_{\mu\nu}) + (\nabla^\mu V^\nu) T_{\mu\nu} \\
&= (\nabla^\mu V^\nu) T_{\mu\nu}
\end{aligned}$$

(since T is divergenceless)

Therefore,

$$\nabla^\mu J_\mu(V) = \pi^{\mu\nu} T_{\mu\nu}$$

(where we used the symmetry of $T_{\mu\nu}$), where $\pi^{\mu\nu}(V)$ is the deformation tensor that is,

$$\pi^{\mu\nu}(V) = \frac{1}{2}(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \quad (3.5)$$

Applying the divergence theorem on $J_\mu(V)$ in a region B bounded to the past by a spacelike hypersurface Σ_1 and to the future by a spacelike hypersurface Σ_2 , and by a null hypersurface N , we obtain:

$$\int_B \pi^{\mu\nu}(V) T_{\mu\nu} dV_B = \int_{\Sigma_1} J_\mu(V) w^\mu dV_{\Sigma_1} - \int_{\Sigma_2} J_\mu(V) w^\mu dV_{\Sigma_2} - \int_N J_\mu(V) w_N^\mu dV_N \quad (3.6)$$

where w^μ are the unit normal to the hypersurfaces Σ , w_N^μ is any null generator of N , dV_Σ are the induced volume forms and dV_N is defined such that the divergence theorem applies.

Taking $V = \hat{t}$, where \hat{t} is a unit timelike vector field.

Taking $B = \Sigma_+ \cap J^-(p)$, we get:

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B = \int_{\Sigma \cap J^-(p)} J_\mu(\hat{t}) w^\mu dV_{\Sigma_+} - \int_{N^-(p) \cap \Sigma_+} J_\mu(\hat{t}) w_N^\mu dV_N$$

where w is a unit timelike vector field normal to Σ . If $\Sigma = \Sigma_t$ then $w = \hat{t}$ where

$$\mathbf{g}(\hat{t}, \hat{t}) = -1 \quad (3.7)$$

Definition 3.2.2. Define the energy $E_t^{\hat{t}}$ by,

$$E_t^{\hat{t}} = \int_{\Sigma_t} J_\mu(\hat{t}) \hat{t}^\mu dV_{\Sigma_t} \quad (3.8)$$

And define the flux $F^{\hat{t}}(N^-(p) \cap \Sigma_+)$ by,

$$F^{\hat{t}}(N^-(p) \cap \Sigma_+) = - \int_{N^-(p) \cap \Sigma_+} J_\mu(\hat{t}) w_{N^-(p)}^\mu dV_{N^-(p)} \quad (3.9)$$

We get:

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B = -E_{t=0}^{\hat{t}}(\Sigma \cap J^-(p)) + F^{\hat{t}}(N^-(p) \cap \Sigma_+) \quad (3.10)$$

Definition 3.2.3. We define $\{L, \underline{L}, e_1, e_2\}$ a null frame as in (6.3), (6.4), (6.5), and (6.6), in the following manner:

We define L as in (6.1.1), and we define \underline{L} as:

$$\underline{L} = -\mathbf{g}(L, \hat{t})^{-1} (2\hat{t} + \mathbf{g}(L, \hat{t})^{-1} L) \quad (3.11)$$

Define e_i , $i \in \{1, 2\}$, such that,

$$\mathbf{g}(e_i, e_j) = \delta_{ij} \quad (3.12)$$

$$\mathbf{g}(L, e_i) = \mathbf{g}(\underline{L}, e_i) = 0 \quad (3.13)$$

We verify that

$$\begin{aligned}
\mathbf{g}(\underline{L}, \underline{L}) &= \mathbf{g}_{L\hat{t}}^{-2} \mathbf{g}(2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1} L, 2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1} L) = 4\mathbf{g}_{L\hat{t}}^{-2} \mathbf{g}(\hat{t}, \hat{t}) + 4\mathbf{g}_{L\hat{t}}^{-2} \mathbf{g}_{L\hat{t}}^{-1} \mathbf{g}(\hat{t}, L) \\
&= -4\mathbf{g}_{L\hat{t}}^{-2} + 4\mathbf{g}_{L\hat{t}}^{-2} \\
&= 0
\end{aligned} \tag{3.14}$$

and,

$$\begin{aligned}
\mathbf{g}(L, \underline{L}) &= -\mathbf{g}_{L\hat{t}}^{-1} \mathbf{g}(L, 2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1} L) = -2\mathbf{g}_{L\hat{t}}^{-1} \mathbf{g}(L, \hat{t}) \\
&= -2
\end{aligned} \tag{3.15}$$

We have

$$\hat{t} = -\frac{(\mathbf{g}_{L\hat{t}}^{-1} L + \mathbf{g}_{L\hat{t}} \underline{L})}{2} \tag{3.16}$$

Computing explicitly $F^{\hat{t}}(N^-(p) \cap \Sigma_+)$

$$\begin{aligned}
&J_\mu(\hat{t})L^\mu \\
&= T_{\mu\hat{t}}L^\mu = T_{L\hat{t}} = F_{L\beta} \cdot F_{\hat{t}}^\beta - \frac{1}{4} \mathbf{g}_{L\hat{t}} F_{\alpha\beta} \cdot F^{\alpha\beta} \\
&= F_{LL} \cdot F_{\hat{t}}^L + F_{Le_a} \cdot F_{\hat{t}}^{e_a} + F_{Le_b} \cdot F_{\hat{t}}^{e_b} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{LL} \cdot F^{LL} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{La} \cdot F^{La} \\
&\quad - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{Lb} \cdot F^{Lb} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{La} \cdot F^{La} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{Lb} \cdot F^{Lb} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{ab} \cdot F^{ab} \\
&= -\frac{1}{2} F_{LL} \cdot F_{\hat{t}L} + F_{Le_a} \cdot F_{\hat{t}e_a} + F_{Le_b} \cdot F_{\hat{t}e_b} + \frac{1}{8} \mathbf{g}_{L\hat{t}} F_{LL} \cdot F_{LL} + \frac{1}{4} \mathbf{g}_{L\hat{t}} F_{La} \cdot F_{La} \\
&\quad + \frac{1}{4} \mathbf{g}_{L\hat{t}} F_{Lb} \cdot F_{Lb} + \frac{1}{4} \mathbf{g}_{L\hat{t}} F_{La} \cdot F_{La} + \frac{1}{4} \mathbf{g}_{L\hat{t}} F_{Lb} \cdot F_{Lb} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{ab} \cdot F_{ab} \\
&= -\frac{1}{2} F_{LL} \cdot F_{\hat{t}L} + F_{Le_a} \cdot F_{\hat{t}e_a} + F_{Le_b} \cdot F_{\hat{t}e_b} + \frac{1}{8} \mathbf{g}_{L\hat{t}} F_{LL} \cdot F_{LL} + \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{La} \cdot F_{La} \\
&\quad + \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{Lb} \cdot F_{Lb} - \frac{1}{2} \mathbf{g}_{L\hat{t}} F_{ab} \cdot F_{ab}
\end{aligned}$$

We have,

$$\begin{aligned}
-\frac{1}{2}F_{LL}\cdot F_{\hat{t}L} &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\mathbf{g}(L, \hat{t})F_{LL}\cdot F_{LL} = -\frac{1}{4}\mathbf{g}_{L\hat{t}}F_{LL}\cdot F_{LL} \\
F_{Lea}\cdot F_{\hat{t}ea} + F_{Leb}\cdot F_{\hat{t}eb} &= -\frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Lea}\cdot F_{Lea} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lea}\cdot F_{Lea} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Leb}\cdot F_{Leb} \\
&\quad - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Leb}\cdot F_{Leb}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&J_{\mu}(\hat{t})L^{\mu} \\
&= -\frac{1}{4}\mathbf{g}_{L\hat{t}}F_{LL}\cdot F_{LL} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Lea}\cdot F_{Lea} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lea}\cdot F_{Lea} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Leb}\cdot F_{Leb} \\
&\quad - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Leb}\cdot F_{Leb} + \frac{1}{8}\mathbf{g}_{L\hat{t}}F_{LL}\cdot F_{LL} + \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{La}\cdot F_{La} + \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lb}\cdot F_{Lb} \\
&\quad - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}\cdot F_{ab} \\
&= -\frac{1}{8}\mathbf{g}_{L\hat{t}}F_{LL}\cdot F_{LL} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Lea}\cdot F_{Lea} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Leb}\cdot F_{Leb} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}\cdot F_{ab}
\end{aligned}$$

Thus,

$$\begin{aligned}
&F^{\hat{t}}(N^-(p) \cap \Sigma_+) \\
&= \int_{N^-(p) \cap \Sigma_+} \left(\frac{1}{8}\mathbf{g}_{L\hat{t}}|F_{LL}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}|F_{Lea}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}|F_{Leb}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}|F_{ab}|^2 \right) (q)
\end{aligned} \tag{3.17}$$

where $|\cdot|$ is the norm deduced from $\langle \cdot, \cdot \rangle$.

Computing $E_t^{\hat{t}}$

$$E_t^{\hat{t}} = \int_{\Sigma_t} J_{\mu}(\hat{t})\hat{t}^{\mu} dV_{\Sigma_t} = \int_{\Sigma_t} T_{\hat{t}\hat{t}} dV_{\Sigma_t} \tag{3.18}$$

$$T_{\hat{t}\hat{t}} = F_{\hat{t}\beta}\cdot F_{\hat{t}}^{\beta} - \frac{1}{4}\mathbf{g}_{\hat{t}\hat{t}}F_{\alpha\beta}\cdot F^{\alpha\beta}$$

Choosing an orthonormal frame $\{\hat{t}, n, e_1, e_2\}$

$$\begin{aligned}
T_{\hat{t}\hat{t}} &= F_{\hat{t}n} \cdot F_{\hat{t}n} + F_{\hat{t}a} \cdot F_{\hat{t}a} + F_{\hat{t}b} \cdot F_{\hat{t}b} + \frac{1}{2} F_{\hat{t}n} \cdot F^{\hat{t}n} + \frac{1}{2} F_{\hat{t}a} \cdot F^{\hat{t}a} \\
&\quad + \frac{1}{2} F_{\hat{t}b} \cdot F^{\hat{t}b} + \frac{1}{2} F_{na} \cdot F^{na} + \frac{1}{2} F_{nb} \cdot F^{nb} + \frac{1}{2} F_{ab} \cdot F^{ab} \\
&= F_{\hat{t}n} \cdot F_{\hat{t}n} + F_{\hat{t}a} \cdot F_{\hat{t}a} + F_{\hat{t}b} \cdot F_{\hat{t}b} - \frac{1}{2} F_{\hat{t}n} \cdot F_{\hat{t}n} - \frac{1}{2} F_{\hat{t}a} \cdot F_{\hat{t}a} - \frac{1}{2} F_{\hat{t}b} \cdot F_{\hat{t}b} \\
&\quad + \frac{1}{2} F_{na} \cdot F_{na} + \frac{1}{2} F_{nb} \cdot F_{nb} + \frac{1}{2} F_{ab} \cdot F_{ab} \\
&= \frac{1}{2} F_{\hat{t}n} \cdot F_{\hat{t}n} + \frac{1}{2} F_{\hat{t}a} \cdot F_{\hat{t}a} + \frac{1}{2} F_{\hat{t}b} \cdot F_{\hat{t}b} + \frac{1}{2} F_{na} \cdot F_{na} + \frac{1}{2} F_{nb} \cdot F_{nb} + \frac{1}{2} F_{ab} \cdot F_{ab} \\
&= \frac{1}{2} [|F_{\hat{t}n}|^2 + |F_{\hat{t}a}|^2 + |F_{\hat{t}b}|^2 + |F_{na}|^2 + |F_{nb}|^2 + |F_{ab}|^2](q) \geq 0
\end{aligned}$$

Thus,

$$E_{t=t_0}^{\hat{t}}(\Sigma \cap J^-(p)) \leq E_{t=t_0}^{\hat{t}} \quad (3.19)$$

Finiteness of the flux from finite initial energy

Let Σ_t^- be the past of Σ_t .

We also have,

$$\begin{aligned}
&\int_{\Sigma_+ \cap \Sigma_t^- \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B \\
&\lesssim \sum_{\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_a, e_b\}} \int_{\Sigma_+ \cap \Sigma_t^- \cap J^-(p)} |\pi^{\hat{\mu}\hat{\nu}}(\hat{t})| \langle F_{\hat{\mu}\beta}, F_{\hat{\nu}}{}^\beta \rangle - \frac{1}{4} \mathbf{g}^{\hat{\mu}\hat{\nu}} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle |dV_B \\
&\quad \text{(by computing the contraction in the basis } \{\hat{t}, n, e_1, e_2\}) \\
&\lesssim \sum_{\alpha, \beta, \hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_a, e_b\}} \int_{t_0}^t |\pi^{\hat{\mu}\hat{\nu}}(\hat{t})|_{L^\infty_{\Sigma_t^- \cap J^-(p)}} \int_{\Sigma_t^- \cap J^-(p)} (|F_{\hat{\mu}\beta}|^2 + |F_{\hat{\nu}}{}^\beta|^2 \\
&\quad + |F_{\alpha\beta}|^2 + |F^{\alpha\beta}|^2) \cdot dV_{\Sigma_t^-} \quad (3.20)
\end{aligned}$$

(by using $a \cdot b \lesssim a^2 + b^2$).

As in [CS] we assume that the deformation tensor of \hat{t} is finite. More precisely, we assume that for all $\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_1, e_2\}$, the components of the deformation tensor, $\pi^{\hat{\mu}\hat{\nu}}(\hat{t}) = \frac{1}{2}[\nabla^{\hat{\mu}}\hat{t}^{\hat{\nu}} + \nabla^{\hat{\nu}}\hat{t}^{\hat{\mu}}]$, verify,

$$|\pi^{\hat{\mu}\hat{\nu}}(\hat{t})|_{L_{loc}^\infty(\Sigma_t)} \leq C_{loc}(t) \quad (3.21)$$

where $C_{loc}(t) \in L_{loc}^1$.

Applying the divergence theorem again in the future of Σ and the past of $\Sigma_t \cap J^-(p)$, we get:

$$\begin{aligned} E_t^{\hat{t}}(\Sigma_t \cap J^-(p)) &\lesssim \int_{\Sigma_+ \cap \Sigma_{\bar{t}}^- \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B + E_{t=t_0}^{\hat{t}}(\Sigma_{t_0} \cap J^-(p)) \\ &\lesssim E_{t=t_0}^{\hat{t}} + \int_{t=t_0}^t C(\bar{t}) E_{\bar{t}}^{\hat{t}}(\Sigma_{\bar{t}} \cap J^-(p)) d\bar{t} \end{aligned} \quad (3.22)$$

(where we used (3.20), (3.1), and the positivity of the energy (3.19)).

Using Grönwall lemma, we get that $E_t^{\hat{t}}(\Sigma_t \cap J^-(p))$ is finite and continuous in t , and therefore

$$\begin{aligned} \int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B &\lesssim \int_{\hat{t}=t_0}^t c(\bar{t}) E_{\bar{t}}^{\hat{t}}(\Sigma_{\bar{t}} \cap J^-(p)) d\bar{t} \\ &\lesssim \int_{t_0}^t c(\bar{t}) d\bar{t} \\ &\lesssim c(t_p) \end{aligned} \quad (3.23)$$

and therefore

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\hat{t}) T_{\mu\nu} dV_B + E_{t=t_0}^{\hat{t}}(\Sigma \cap J^-(p)) \lesssim c(t_p) E_{t=t_0}^{\hat{t}}$$

Hence,

$$F^{\hat{t}}(N^-(p) \cap \Sigma_+) \lesssim c(t_p)E_{t=t_0}^{\hat{t}} \quad (3.24)$$

This finiteness of the flux will play a key role in the proof.

3.2.4 Definitions and notations

Definition 3.2.5. We define $\lambda_{\alpha\beta}$ as in (6.8) and (6.9), by fixing at p a \mathcal{G} -valued anti-symmetric 2-tensor \mathbf{J}_p , and defining $\lambda_{\alpha\beta}$ as the unique 2-tensor field along $N^-(p)$, the boundary of the causal past of p , that verifies the linear transport equation:

$$\begin{aligned} \mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\lambda_{\alpha\beta} &= 0 \\ (s\lambda_{\alpha\beta})(p) &= \mathbf{J}_{\alpha\beta}(p) \end{aligned}$$

where s is the affine parameter on $N^-(p)$ defined as in (6.1.1), and χ is the null second fundamental form of $N^-(p)$ defined as in (6.2), and $tr\chi$ defined as in (6.1.3).

Definition 3.2.6. We define positive definite Riemannian metric as in (6.52), in the following manner:

$$h(e_\alpha, e_\beta) = \mathbf{g}(e_\alpha, e_\beta) + 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t})$$

where

$$\mathbf{g}(\hat{t}, \hat{t}) = -1$$

Definition 3.2.7. We define the spacelike foliation Σ_t of the space-time M by considering $t = \text{constant}$ hypersurfaces as in (3.1.2).

Definition 3.2.8. For any \mathcal{G} -valued 2-tensor K , we define

$$|K|^2 = h_{\alpha\mu}h_{\beta\nu}|K^{\mu\nu}| \cdot |K^{\alpha\beta}|$$

and

$$|K|_{L^\infty} = \| |(h_{\alpha\mu}h_{\beta\nu}|K^{\mu\nu}| \cdot |K^{\alpha\beta}|)^{\frac{1}{2}} \|_{L^\infty} = \| (|K|^2)^{\frac{1}{2}} \|_{L^\infty}$$

We recall (6.55), that for any two \mathcal{G} -valued tensors K and G , we have

$$| \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle | \lesssim (|K|^2)^{\frac{1}{2}} \cdot (|G|^2)^{\frac{1}{2}}$$

Notations

We denote by $N_\tau^-(p)$ the portion of $N^-(p)$ to the past of Σ_{t_p} and to the future of $\Sigma_{t_p-\tau}$.

t^* and \bar{t} are values of t , where \hat{t} is a timelike vector field verifying (3.1).

\bar{s} and \hat{s} are values of s .

We denote by s_τ , the largest value of s on $N_\tau^-(p)$.

We let $\Sigma_t^p = \Sigma_t \cap J^-(p)$.

3.2.9 Estimates for $s\lambda_{\alpha\beta}$

Proposition 3.2.10. *We have,*

$$\sup_{N_\tau^-(p)} |s\lambda| \leq C(p, \tau) |J| \tag{3.25}$$

Proof.

We proved in (6.1.27) that,

$$\sup_{0 \leq \bar{s} \leq s} |\bar{s}\lambda|^2 \leq C(p, s)|J|^2 \quad (3.26)$$

Hence,

$$\sup_{N_\tau^-(p)} |s\lambda|^2 \leq C(p, s_\tau)|J|^2 \quad (3.27)$$

(where in this last inequality s_τ is the largest value of s on $N_\tau^-(p)$).

In view of (6.64),

$$s = t_p - t + O(t_p - t) \quad (3.28)$$

we get,

$$\sup_{N_\tau^-(p)} |s\lambda| \leq C(p, \tau)|J| \quad (3.29)$$

□

Proposition 3.2.11.

$$\|\lambda\|_{L^2(N_\tau^-(p))} \leq (\tau)^{\frac{1}{2}} C(p, \tau)|J| \quad (3.30)$$

Proof.

We have,

$$\begin{aligned} \int_{\mathbb{S}^2} |s\lambda|^2(u=0, s, \omega) da_s &\lesssim \int_{\mathbb{S}^2} C(p, s_\tau)^2 |J|^2 da_s \\ &\quad \text{(by (3.27))} \\ &\leq C(p, s_\tau)^2 |J|^2 \int_{\mathbb{S}^2} 1 da_s \\ &\lesssim C(p, s_\tau)^2 |J|^2 s^2 \end{aligned}$$

Thus,

$$\int_{\mathbb{S}^2} |\lambda|^2(u=0, s, \omega) da_s \leq C(p, s_\tau)^2 |J|^2$$

$$\|\lambda_{\alpha\beta}\|_{L^2(N_\tau^-(p))}^2 = \int_0^{s_\tau^*} \int_{\mathbb{S}^2} |\lambda|^2(u=0, s, \omega) da_s ds$$

(where s_τ^* is the largest value of s for a fixed ω such that $(u=0, s, \omega) \in N_\tau^-(p)$)

$$\lesssim C(p, s_\tau)^2 |J|^2 \int_0^{s_\tau} 1 ds \leq s_\tau C(p, s_\tau)^2 |J|^2$$

Thus,

$$\|\lambda\|_{L^2(N_\tau^-(p))} \leq (s_\tau)^{\frac{1}{2}} C(p, s_\tau) |J|$$

Therefore, since $t_p - t = s + o(s)$,

$$\|\lambda\|_{L^2(N_\tau^-(p))} \leq (\tau)^{\frac{1}{2}} C(p, \tau) |J| \tag{3.31}$$

□

3.2.12 Estimates for $\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_\tau^-(p))}$

Definition 3.2.13. Let

$$\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2} \text{tr} \chi \delta_{ab} \tag{3.32}$$

and let ζ_a , $a \in \{1, 2\}$, be defined as in (6.35):

$$\zeta_a = \frac{1}{2} \mathbf{g}(\nabla_a L, \underline{L}) \tag{3.33}$$

Lemma 3.2.14.

$$\begin{aligned}
\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} &= -tr\chi\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta} - \frac{1}{2}\zeta_a tr\chi\lambda_{\alpha\beta} \\
&\quad + [F_{La}, \lambda_{\alpha\beta}] + R_{\alpha}{}^{\gamma}{}_{La}\lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{La}\lambda_{\alpha\gamma}
\end{aligned} \tag{3.34}$$

Proof.

We have,

$$\begin{aligned}
\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} &= \mathbf{D}_a^{(A)}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_{\alpha}{}^{\gamma}{}_{La}\lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{La}\lambda_{\alpha\gamma} \\
&= \mathbf{D}_a^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_a L}^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_{\alpha}{}^{\gamma}{}_{La}\lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{La}\lambda_{\alpha\gamma} \\
&= \mathbf{D}_a^{(A)}\left(-\frac{tr\chi}{2}\lambda_{\alpha\beta}\right) - \mathbf{D}_{\nabla_a L}^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_{\alpha}{}^{\gamma}{}_{La}\lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{La}\lambda_{\alpha\gamma}
\end{aligned}$$

on $N_{\tau}^{-}(p)$.

We remind that for any vectorfield X , we have

$$X = -\frac{1}{2}\mathbf{g}(X, \underline{L})L - \frac{1}{2}\mathbf{g}(X, L)\underline{L} + \mathbf{g}(X, e_a)e_a$$

Taking $X = \nabla_a L$, we get

$$\nabla_a L = -\frac{1}{2}\mathbf{g}(\nabla_a L, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_a L, L)\underline{L} + \mathbf{g}(\nabla_a L, e_b)e_b$$

we get,

$$\nabla_a L = \mathbf{g}(\nabla_a L, e_b)e_b - \zeta_a L = \chi_{ab}e_b - \zeta_a L \tag{3.35}$$

Thus,

$$\begin{aligned}
\mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} &= -\frac{1}{2}(\nabla_a \text{tr} \chi) \lambda_{\alpha\beta} - \frac{1}{2} \text{tr} \chi (\mathbf{D}_a^{(A)} \lambda_{\alpha\beta}) + \zeta_a \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} \\
&\quad - \chi_{ab} \mathbf{D}_b^{(A)} \lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_{\alpha}^{\gamma} L_a \lambda_{\gamma\beta} + R_{\beta}^{\gamma} L_a \lambda_{\alpha\gamma} \quad (3.36)
\end{aligned}$$

Since $\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2} \text{tr} \chi \delta_{ab}$, and since we have $\mathbf{D}_L^{(A)} \lambda_{\alpha\beta} = -\frac{1}{2} \text{tr} \chi \lambda_{\alpha\beta}$ on $N_{\tau}^{-}(p)$, we get,

$$\begin{aligned}
\mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} &= -\frac{1}{2}(\nabla_a \text{tr} \chi) \lambda_{\alpha\beta} - \frac{1}{2} \text{tr} \chi (\mathbf{D}_a^{(A)} \lambda_{\alpha\beta}) - \frac{1}{2} \zeta_a \text{tr} \chi \lambda_{\alpha\beta} \\
&\quad - \hat{\chi}_{ab} \mathbf{D}_b^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} \text{tr} \chi \delta_{ab} (\mathbf{D}_b^{(A)} \lambda_{\alpha\beta}) + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_{\alpha}^{\gamma} L_a \lambda_{\gamma\beta} + R_{\beta}^{\gamma} L_a \lambda_{\alpha\gamma} \\
&= -\text{tr} \chi \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} - \hat{\chi}_{ab} \mathbf{D}_b^{(A)} \lambda_{\alpha\beta} - \frac{1}{2}(\nabla_a \text{tr} \chi) \lambda_{\alpha\beta} - \frac{1}{2} \zeta_a \text{tr} \chi \lambda_{\alpha\beta} \\
&\quad + [F_{La}, \lambda_{\alpha\beta}] + R_{\alpha}^{\gamma} L_a \lambda_{\gamma\beta} + R_{\beta}^{\gamma} L_a \lambda_{\alpha\gamma}
\end{aligned}$$

□

We want to control $\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_{\tau}^{-}(p))}$, where we are summing over $a \in \{1, 2\}$, by abuse of notation. Following [KR3], let's compute,

$$\bar{s} \int_{\mathbb{S}^2} \bar{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2 d\sigma^2 = \int_{\mathbb{S}^2} \bar{s}^{-1} \bar{s}^4 |\mathbf{D}_a^{(A)} \lambda|^2 d\sigma^2 = \int_{\mathbb{S}^2} |\bar{s}^{-1} \int_0^{\bar{s}} \nabla_L |s^2 \mathbf{D}_a^{(A)} \lambda|^2 ds| d\sigma^2$$

Lemma 3.2.15. *Let Ψ be a \mathcal{G} -valued tensor, we have,*

$$|\nabla_{\sigma} |\Psi|^2|(p) \leq C(p) [|\mathbf{D}_{\sigma}^{(A)} \Psi| \cdot |\Psi| + |\Psi|^2] \quad (3.37)$$

where $C(p)$ depends on the space-time geometry on the point p .

Proof.

$$\begin{aligned}\nabla_\sigma |\Psi|^2 &= \nabla_\sigma (h_{\alpha\mu} h_{\beta\nu} |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}|) = \nabla_\sigma (h_{\alpha\mu} h_{\beta\nu}) \cdot |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}| \\ &\quad + h_{\alpha\mu} h_{\beta\nu} \cdot \nabla_\sigma (|\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}|)\end{aligned}$$

Therefore,

$$\begin{aligned}|\nabla_\sigma |\Psi|^2| &\leq |(\nabla_\sigma h_{\alpha\mu}) h_{\beta\nu}| \cdot |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}| + |h_{\alpha\mu} (\nabla_\sigma h_{\beta\nu})| \cdot |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}| \\ &\quad + |h_{\alpha\mu} h_{\beta\nu}| \cdot (|\mathbf{D}_\sigma^{(A)} \Psi^{\mu\nu}| + |\Psi(\nabla_\sigma e^\mu, e^\nu)| + |\Psi(e^\mu, \nabla_\sigma e^\nu)|) \cdot |\Psi^{\alpha\beta}| \\ &\quad + |h_{\alpha\mu} h_{\beta\nu}| \cdot |\Psi^{\mu\nu}| \cdot (|\mathbf{D}_\sigma^{(A)} \Psi^{\alpha\beta}| + |\Psi(\nabla_\sigma e^\alpha, e^\beta)| + |\Psi(e^\alpha, \nabla_\sigma e^\beta)|)\end{aligned}$$

(due to (6.66)).

Choosing a normal frame (where the Christoffel symbols vanish at that point) to consider the contactions, using (6.67), and the fact that the metric is smooth, we get,

$$|\nabla_\sigma |\Psi|^2|(p) \leq C(p) [h_{\alpha\mu} h_{\beta\nu} |\mathbf{D}_\sigma^{(A)} \Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}| + h_{\alpha\mu} h_{\beta\nu} |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}|]$$

Using Cauchy-Schwarz, we obtain the desired estimate.

□

We can compute,

$$\begin{aligned}\mathbf{D}_L^{(A)} (s^2 \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}) &= 2s \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + s^2 \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} \\ &= s^2 (2s^{-1} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta})\end{aligned}$$

Thus, using (3.37), we get

$$|\nabla_L |s^2 \mathbf{D}_a^{(A)} \lambda|^2 \lesssim |s^2(2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda)| \cdot |s^2 \mathbf{D}_a^{(A)} \lambda| + |s^2 \mathbf{D}_a^{(A)} \lambda|^2 \quad (3.38)$$

Therefore,

$$\begin{aligned} & \bar{s} \bar{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2 \\ \lesssim & \bar{s}^{-1} |\bar{s}^2 \mathbf{D}_a^{(A)} \lambda|^2 \\ \lesssim & \bar{s}^{-1} \int_0^{\bar{s}} |\nabla_L |\bar{s}^2 \mathbf{D}_a^{(A)} \lambda|^2| ds \quad (3.39) \\ \lesssim & \bar{s}^{-1} \int_0^{\bar{s}} [|s^2(2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda)| \cdot |s^2 \mathbf{D}_a^{(A)} \lambda| + |s^2 \mathbf{D}_a^{(A)} \lambda|^2] ds \\ \lesssim & \bar{s}^{-1} \int_0^{\bar{s}} \sup_{s \in [0, \bar{s}]} s^4 |\mathbf{D}_a^{(A)} \lambda|^2 ds \\ & + \bar{s}^{-1} \int_0^{\bar{s}} \epsilon^{-\frac{1}{2}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| \cdot \epsilon^{\frac{1}{2}} s^{\frac{3}{2}} |\mathbf{D}_a^{(A)} \lambda| ds \\ \lesssim & \bar{s}^{-1} \sup_{s \in [0, \bar{s}]} (s) \int_0^{\bar{s}} \sup_{s \in [0, \bar{s}]} (s^3 |\mathbf{D}_a^{(A)} \lambda|^2) ds + \bar{s}^{-1} \epsilon \int_0^{\bar{s}} \sup_{s \in [0, \bar{s}]} (s^3 |\mathbf{D}_a^{(A)} \lambda|^2) ds \\ & + \frac{1}{\epsilon} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 \\ \lesssim & \bar{s} \sup_{s \in [0, \bar{s}]} (s^3 |\mathbf{D}_a^{(A)} \lambda|^2) + \epsilon \sup_{s \in [0, \bar{s}]} (s^3 |\mathbf{D}_a^{(A)} \lambda|^2) \\ & + \frac{1}{\epsilon} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 \quad (3.40) \end{aligned}$$

Taking the supremum in this last inequality on $\bar{s} \in [0, \hat{s}]$, where \bar{s} and \hat{s} are values of s , we get,

$$\begin{aligned} & \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2(0, \bar{s}, \omega) d\sigma^2 \\ \lesssim & (\hat{s} + \epsilon) \sup_{\bar{s} \in [0, \hat{s}]} \sup_{s \in [0, \bar{s}]} (s^3 |\mathbf{D}_a^{(A)} \lambda|^2) \\ & + \sup_{\bar{s} \in [0, \hat{s}]} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 \end{aligned}$$

Choosing \hat{s} and ϵ small enough depending on the space-time geometry on p , we obtain:

$$\begin{aligned} & \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2(0, \bar{s}, \omega) \\ & \lesssim \sup_{\bar{s} \in [0, \hat{s}]} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 \end{aligned}$$

Integrating on \mathbb{S}^2 , we get

$$\begin{aligned} & \int_{\mathbb{S}^2} \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2(0, \bar{s}, \omega) d\sigma^2 \\ & \lesssim \int_{\mathbb{S}^2} \sup_{\bar{s} \in [0, \hat{s}]} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 d\sigma^2 \end{aligned} \quad (3.41)$$

We want to control

$$\begin{aligned} & \sup_{\bar{s} \in [0, \hat{s}]} \int_{\mathbb{S}^2} [\bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds]^2 d\sigma^2 \\ & = \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds \right\|_{L_\omega^2}^2 \end{aligned}$$

where the L_ω^p denotes the L^p norm on $s = \text{constant}$, with the canonical induced volume form $d\sigma^2$ induced on \mathbb{S}^2 .

$$\begin{aligned} & \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds \right\|_{L_\omega^2}^2 \\ & = \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda - \text{tr} \chi \mathbf{D}_a^{(A)} \lambda + \text{tr} \chi \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds \right\|_{L_\omega^2}^2 \\ & = \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} - \text{tr} \chi \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} - \hat{\chi}_{ab} \mathbf{D}_b^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} (\nabla_a \text{tr} \chi) \lambda_{\alpha\beta} \right. \\ & \quad \left. - \frac{1}{2} \zeta_a \text{tr} \chi \lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_\alpha^\gamma L_a \lambda_{\gamma\beta} + R_\beta^\gamma L_a \lambda_{\alpha\gamma}] ds \right\|_{L_\omega^2}^2 \end{aligned}$$

where we used (3.34), and where $|\cdot|_h$ means that we consider a full contraction with respect to the metric h , in the indices α, β . Hence,

$$\begin{aligned}
& \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda + \mathbf{D}_L^{(A)} \mathbf{D}_a^{(A)} \lambda| ds \right\|_{L^2_{\omega}}^2 \\
& \lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda - \text{tr} \chi \mathbf{D}_a^{(A)} \lambda - \hat{\chi}_{ab} \mathbf{D}_b^{(A)} \lambda| ds \right\|_{L^2_{\omega}}^2 \\
& \quad + \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} \left| -\frac{1}{2} (\nabla_a \text{tr} \chi) \lambda \right| ds \right\|_{L^2_{\omega}}^2 \\
& \quad + \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} \left| -\frac{1}{2} \zeta_a \text{tr} \chi \lambda \right| ds \right\|_{L^2_{\omega}}^2 \\
& \quad + \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |[F_{La}, \lambda_{\alpha\beta}]|_h ds \right\|_{L^2_{\omega}}^2 \\
& \quad + \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |R_{\alpha}{}^{\gamma}{}_{La} \lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{La} \lambda_{\alpha\gamma}|_h ds \right\|_{L^2_{\omega}}^2 \\
& = I_1 + I_2 + I_3 + I_4 + I_5 \tag{3.42}
\end{aligned}$$

where I_i , $i \in \{1, \dots, 5\}$, are defined in order.

Estimating I_1

$$\begin{aligned}
I_1 & = \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |2s^{-1} \mathbf{D}_a^{(A)} \lambda - \text{tr} \chi \mathbf{D}_a^{(A)} \lambda - \hat{\chi}_{ab} \mathbf{D}_b^{(A)} \lambda| ds \right\|_{L^2_{\omega}}^2 \\
& \lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \sup_{s \in [0, \bar{s}]} |s^{\frac{5}{2}} \mathbf{D}_a^{(A)} \lambda| \int_0^{\bar{s}} |2s^{-1} - \text{tr} \chi - \hat{\chi}_{ab}| ds \right\|_{L^2_{\omega}}^2
\end{aligned}$$

(we remind that we were summing over a with abuse of notation)

$$\begin{aligned}
&\lesssim \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \int_0^{\bar{s}} |2s^{-1} - \text{tr}\chi - \hat{\chi}_{ab}| ds \right\|_{L_\omega^\infty}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \int_0^{\bar{s}} 1 \cdot |2s^{-1} - \text{tr}\chi - \hat{\chi}_{ab}| ds \right\|_{L_\omega^\infty}^2 \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \int_0^{\bar{s}} |2s^{-1} - \text{tr}\chi - \hat{\chi}_{ab}|^2 ds \right\|_{L_\omega^\infty} \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \int_0^{\bar{s}} |2s^{-1} - \text{tr}\chi - \hat{\chi}_{ab}|^2 ds \right\|_{L_\omega^\infty} \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \right\|_{L_\omega^\infty} \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \int_0^{\bar{s}} [|2s^{-1} - \text{tr}\chi|^2 + |\hat{\chi}_{ab}|^2] ds \right\|_{L_\omega^\infty} \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \right\|_{L_\omega^\infty} \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2
\end{aligned}$$

We know that

$$|2s^{-1} - \text{tr}\chi| = O(s^2) \quad (3.43)$$

and

$$\int_0^{\bar{s}} |\hat{\chi}_{ab}|^2 ds \lesssim 1 \quad (3.44)$$

(see proposition 3.1 in [Wang]).

We get,

$$I_1 \lesssim \hat{s} \left\| \sup_{s \in [0, \hat{s}]} (s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda) \right\|_{L_\omega^2}^2 \quad (3.45)$$

Estimating I_2

$$\begin{aligned}
I_2 &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} \left| -\frac{1}{2} (\nabla_a \text{tr} \chi) \lambda \right| ds \right\|_{L^2_\omega}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \lambda \right\|_{L^\infty}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{3}{2}} \left| -\frac{1}{2} (\nabla_a \text{tr} \chi) \right| ds \right\|_{L^2_\omega}^2 \\
&\lesssim |J|^2 \cdot \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \left| \bar{s} \nabla_a \text{tr} \chi \right| \int_0^{\bar{s}} s^{\frac{1}{2}} ds \right\|_{L^2_\omega}^2 \\
&\lesssim |J|^2 \cdot \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \bar{s}^{\frac{3}{2}} \right\|^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \nabla_a \text{tr} \chi \right\|_{L^2_\omega}^2 \\
&\lesssim \hat{s} \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \nabla_a \text{tr} \chi \right\|_{L^2_\omega}^2
\end{aligned}$$

We have

$$\left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \nabla_a \text{tr} \chi \right\|_{L^2_\omega}^2 \lesssim 1$$

(see proposition 3.2 in [Wang]).

We get,

$$I_2 \lesssim \hat{s} \tag{3.46}$$

Estimating I_3

$$\begin{aligned}
I_3 &= \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} \left| -\frac{1}{2} \zeta_a \text{tr} \chi \lambda \right| ds \right\|_{L^2_\omega}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \lambda \right\|_{L^\infty}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{3}{2}} \left| -\frac{1}{2} \zeta_a \text{tr} \chi \right| ds \right\|_{L^2_\omega}^2 \\
&\lesssim |J|^2 \cdot \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s} \text{tr} \chi \right\|_{L^\infty}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{1}{2}} |\zeta_a| ds \right\|_{L^2_\omega}^2 \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \left(\int_0^{\bar{s}} s ds \right)^{\frac{1}{2}} \left(\int_0^{\bar{s}} |\zeta_a|^2 ds \right)^{\frac{1}{2}} \right\|_{L^2_\omega}^2 \\
&\quad \text{(from (3.43))} \\
&\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \bar{s} \right\|_{L^\infty}^2 \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2
\end{aligned}$$

$$I_3 \lesssim \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \quad (3.47)$$

Estimating I_4

$$\begin{aligned} I_4 &= \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |[F_{La}, \lambda_{\alpha\beta}]|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} s^{-1} |[F_{La}, s\lambda_{\alpha\beta}]|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} |\bar{s}\lambda|_h \right\|_{L_\omega^\infty}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \bar{s}^{\frac{1}{2}} \int_0^{\bar{s}} 1 \cdot s |F_{La}| ds \right\|_{L_\omega^2}^2 \\ &\lesssim |J|^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-\frac{1}{2}} \bar{s}^{\frac{1}{2}} \left(\int_0^{\bar{s}} s^2 |F_{La}|^2 ds \right)^{\frac{1}{2}} \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \left(\int_0^{\bar{s}} s^2 |F_{La}|^2 ds \right)^{\frac{1}{2}} \right\|_{L_\omega^2}^2 \end{aligned}$$

$$I_4 \lesssim \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \quad (3.48)$$

Estimating I_5

$$\begin{aligned} I_5 &= \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |R_\alpha^\gamma{}_{La} \lambda_{\gamma\beta} + R_\beta^\gamma{}_{La} \lambda_{\alpha\gamma}|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{5}{2}} |h^{\sigma\gamma}| \cdot |R_{\alpha\sigma La} \lambda_{\gamma\beta} + R_{\beta\sigma La} \lambda_{\alpha\gamma}|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} |\bar{s}\lambda|_h \right\|_{L_\omega^\infty}^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \int_0^{\bar{s}} s^{\frac{3}{2}} |R_{\alpha\beta La}|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim |J|^2 \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-1} \bar{s}^{\frac{1}{2}} \int_0^{\bar{s}} s |R_{\alpha\beta La}|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-\frac{1}{2}} \int_0^{\bar{s}} 1 \cdot s |R_{\alpha\beta La}|_h ds \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^{-\frac{1}{2}} \bar{s}^{\frac{1}{2}} \left[\int_0^{\bar{s}} s^2 |R_{\alpha\beta La}|_h^2 ds \right]^{\frac{1}{2}} \right\|_{L_\omega^2}^2 \\ &\lesssim \left\| \sup_{\bar{s} \in [0, \hat{s}]} \left[\int_0^{\bar{s}} s^2 |R_{\alpha\beta La}|_h ds \right]^{\frac{1}{2}} \right\|_{L_\omega^2}^2 \end{aligned}$$

$$I_5 \lesssim \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \quad (3.49)$$

Estimating $\|\mathbf{D}_a^{(A)}\lambda\|_{L^2(N_\tau^-(p))}$

Injecting (3.45), (3.46), (3.47), (3.48), (3.49) in (3.42), and then in (3.41), we obtain:

$$\begin{aligned} & \int_{\mathbb{S}^2} \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)}\lambda|^2(0, \bar{s}, \omega) d\sigma^2 \\ & \lesssim \hat{s} \left\| \sup_{s \in [0, \hat{s}]} |s^{\frac{3}{2}} \mathbf{D}_a^{(A)}\lambda|_h \right\|_{L_w^2}^2 + \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \\ & \quad + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \end{aligned}$$

There exists $C(p)$ (constant depending on p) such that for $\hat{s} \lesssim C(p)$, we have:

$$\begin{aligned} & \int_{\mathbb{S}^2} \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)}\lambda|^2(0, \bar{s}, \omega) d\sigma^2 \\ & \lesssim \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \\ & \quad + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \end{aligned}$$

For $\hat{s} \lesssim C(p)$, we have,

$$\begin{aligned} & \hat{s} \int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)}\lambda|^2(0, \hat{s}, \omega) d\sigma^2 \\ & \lesssim \int_{\mathbb{S}^2} \sup_{\bar{s} \in [0, \hat{s}]} \bar{s}^3 |\mathbf{D}_a^{(A)}\lambda|^2(0, \bar{s}, \omega) d\sigma^2 \\ & \lesssim \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2(0, \hat{s}, \omega) d\sigma^2 \\
& \lesssim 1 + \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \\
& \quad + \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2
\end{aligned} \tag{3.50}$$

Integrating, we obtain

$$\begin{aligned}
& \int_0^{C(p)} \int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2(0, \hat{s}, \omega) d\sigma^2 d\hat{s} \\
& \lesssim \int_0^{C(p)} 1 d\hat{s} + \int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \right] d\hat{s} \\
& \quad + \int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \right] d\hat{s} + \int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \right] d\hat{s} \\
& \lesssim 1 + \left(\int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} + \left(\int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \\
& \quad + \left(\int_0^{C(p)} \left[\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}}
\end{aligned} \tag{3.51}$$

Using the L^2 maximum principle, and letting $t(C(p))$ be the value of t for which $s(t) = C(p)$, we get in view of (6.59):

$$\begin{aligned}
& \| \mathbf{D}_a^{(A)} \lambda \|_{L^2(N_{t(C(p))}^-(p))} \\
& \lesssim 1 + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \\
& \quad + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{\hat{s}} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \\
& \lesssim 1 + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{s_\tau} |\zeta_a|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{s_\tau} s^2 |F_{La}|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \\
& \quad + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{s_\tau} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \\
& \lesssim 1 + \left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{s_\tau} |\zeta_a|^2 ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} + [C(p)(F^{\hat{t}}(N_\tau^-(p)))^2]^{\frac{1}{2}}
\end{aligned} \tag{3.52}$$

(since the metric is smooth).

We have,

$$\left(\int_0^{C(p)} \left[\int_{\mathbb{S}^2} \int_0^{s_\tau} (|\zeta_a|^2) ds d\sigma^2 \right]^2 d\hat{s} \right)^{\frac{1}{2}} \lesssim 1 \quad (3.53)$$

(see proposition 3.1 in the Appendix of [Wang]) .

Thus,

$$\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_{t(C(p))}^-(p))} \lesssim 1$$

Since the metric is smooth, $\lambda_{\alpha\beta}$ is smooth away from $s = 0$; we finally obtain:

$$\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_\tau^-(p))} \lesssim 1 \quad (3.54)$$

3.2.16 Estimates for $\|\mathbf{D}_a^{(A)} F\|_{L^2(N_\tau^-(p))}$

We want to control $\|\mathbf{D}_a^{(A)} F\|_{L^2(N_\tau^-(p))}$. For this, as in [CS], we take the energy momentum tensor for the wave equation, after considering a full contraction with respect to the Riemannian metric h , and define the 2-tensor:

$$T_1^{\alpha\beta} = h^{\mu\nu} h^{\rho\sigma} [\langle \mathbf{D}^{(A)\alpha} F_{\mu\rho}, \mathbf{D}^{(A)\beta} F_{\nu\sigma} \rangle - \frac{1}{2} \mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_\lambda^{(A)} F_{\nu\sigma} \rangle] \quad (3.55)$$

Let $\hat{t}_\beta = (\hat{t})_\beta$

We have

$$T_1^{\alpha\beta}\hat{t}_\beta = h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[\langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle]$$

We would like to compute $\nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta)$. Since it is a full contraction, we can compute it by choosing a normal frame, i.e. a frame where the Christoffel symbols vanish at that point, and hence we can get the derivatives inside the scalar product as covariant derivatives and also as gauge covariant derivatives using the fact that the scalar product is Ad-invariant, instead of partial derivatives. We obtain

$$\begin{aligned} & \nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta) \\ = & \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[\langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \\ & + h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[\langle \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\ & - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \end{aligned}$$

We have

$$-\frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle = -\frac{1}{2} \langle \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle$$

Computing

$$h^{\mu\nu}h^{\rho\sigma} \langle \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle = h^{\rho\sigma}h^{\mu\nu} \langle \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\nu\sigma}, \mathbf{D}_\lambda^{(A)}F_{\mu\rho} \rangle$$

We get

$$\begin{aligned}
& \nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta) \\
= & \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[\langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \\
& + h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[\langle \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\
& - \langle \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\nu\sigma}, \mathbf{D}_\lambda^{(A)}F_{\mu\rho} \rangle] \\
= & \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[\langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \\
& + h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[\langle \square_{\mathbf{g}}^{(A)}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\
& + \langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\beta}F_{\nu\sigma} - \mathbf{D}^{(A)\beta}\mathbf{D}_\alpha^{(A)}F_{\nu\sigma} \rangle] \\
= & \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[\langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\alpha\beta} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \\
& + h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[-2 \langle R_{\gamma\mu\rho\alpha}F^{\alpha\gamma}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\
& - \langle R_{\mu\gamma}F_\rho{}^\gamma, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle - \langle R_{\rho\gamma}F^\gamma{}_\mu, \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\
& - 2 \langle [F^\alpha{}_\mu, F_{\rho\alpha}], \mathbf{D}^{(A)\beta}F_{\nu\sigma} \rangle \\
& + \langle \mathbf{D}^{(A)\alpha}F_{\mu\rho}, [F_\alpha{}^\beta, F_{\nu\sigma}] + R_{\nu}{}^\gamma{}_\alpha{}^\beta F_{\gamma\sigma} + R_\sigma{}^\gamma{}_\alpha{}^\beta F_{\nu\gamma} \rangle]
\end{aligned}$$

(where we used (3.2)).

From (3.11), we have

$$\hat{t} = -\frac{1}{2}(\mathbf{g}(L, \hat{t})\underline{L} + \mathbf{g}(L, \hat{t})^{-1}L)$$

we get,

$$T_1^{\hat{t}L} = -\frac{1}{2}\mathbf{g}(L, \hat{t})T^{LL} - \frac{1}{2}\mathbf{g}(L, \hat{t})^{-1}T^{LL} \quad (3.56)$$

Computing

$$\begin{aligned}
& \langle \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_{\lambda}^{(A)} F_{\nu\sigma} \rangle \\
&= \langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}_L^{(A)} F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}_{\underline{L}}^{(A)} F_{\nu\sigma} \rangle \\
&\quad + \langle \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} \rangle \\
&= \mathbf{g}_{LL} \langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}} F_{\nu\sigma} \rangle + \mathbf{g}_{L\underline{L}} \langle \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} \rangle \\
&\quad + \langle \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} \rangle \\
&= -4 \langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}} F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} \rangle
\end{aligned}$$

(using (3.15)).

We get,

$$\begin{aligned}
& T_1^{LL} \\
&= h^{\mu\nu} h^{\rho\sigma} [\langle \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} \rangle - \frac{1}{2} \mathbf{g}^{\underline{L}L} \langle \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_{\lambda}^{(A)} F_{\nu\sigma} \rangle] \\
&= h^{\mu\nu} h^{\rho\sigma} [\langle \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} \rangle + \frac{1}{4} (-4 \langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}} F_{\nu\sigma} \rangle \\
&\quad + \langle \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} \rangle)] \\
&= \frac{1}{4} h^{\mu\nu} h^{\rho\sigma} \langle \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} \rangle
\end{aligned} \tag{3.57}$$

and we have,

$$\begin{aligned}
T_1^{LL} &= h^{\mu\nu} h^{\rho\sigma} [\langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} \rangle - \frac{1}{2} \mathbf{g}^{LL} \langle \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_{\lambda}^{(A)} F_{\nu\sigma} \rangle] \\
&= h^{\mu\nu} h^{\rho\sigma} \langle \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} \rangle
\end{aligned} \tag{3.58}$$

Injecting (3.57) and (3.58) in (3.56), we obtain

$$\begin{aligned}
T_1^{\hat{t}L} &= -\frac{1}{2}\mathbf{g}(L, \hat{t})T^{\underline{L}L} - \frac{1}{2}\mathbf{g}(L, \hat{t})^{-1}T^{LL} \\
&= -\frac{1}{2}\left[\frac{1}{4}\mathbf{g}(L, \hat{t})h^{\mu\nu}h^{\rho\sigma} \langle \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}_a^{(A)}F_{\nu\sigma} \rangle \right. \\
&\quad \left. + \mathbf{g}(L, \hat{t})^{-1}h^{\mu\nu}h^{\rho\sigma} \langle \mathbf{D}^{(A)L}F_{\mu\rho}, \mathbf{D}^{(A)L}F_{\nu\sigma} \rangle\right] \\
&= -\frac{1}{2}\left[\frac{1}{4}\mathbf{g}(L, \hat{t})|\mathbf{D}^{(A)a}F|^2 + \mathbf{g}(L, \hat{t})^{-1}|\mathbf{D}^{(A)L}F|^2\right] \tag{3.59}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
T_1^{\hat{t}\hat{t}} &= h^{\mu\nu}h^{\rho\sigma}[\langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}}F_{\nu\sigma} \rangle - \frac{1}{2}\mathbf{g}^{\hat{t}\hat{t}} \langle \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} \rangle] \\
&= h^{\mu\nu}h^{\rho\sigma}[\langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}}F_{\nu\sigma} \rangle + \frac{1}{2}(\langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}_{\hat{t}}^{(A)}F_{\nu\sigma} \rangle \\
&\quad + \langle \mathbf{D}^{(A)\hat{n}}F_{\mu\rho}, \mathbf{D}_{\hat{n}}^{(A)}F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}_a^{(A)}F_{\nu\sigma} \rangle)] \\
&= h^{\mu\nu}h^{\rho\sigma}[\langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}}F_{\nu\sigma} \rangle - \frac{1}{2} \langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}}F_{\nu\sigma} \rangle \\
&\quad + \frac{1}{2} \langle \mathbf{D}^{(A)\hat{n}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{n}}F_{\nu\sigma} \rangle + \frac{1}{2} \langle \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}^{(A)a}F_{\nu\sigma} \rangle)] \\
&= \frac{1}{2}h^{\mu\nu}h^{\rho\sigma}[\langle \mathbf{D}^{(A)\hat{t}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}}F_{\nu\sigma} \rangle + \langle \mathbf{D}^{(A)\hat{n}}F_{\mu\rho}, \mathbf{D}^{(A)\hat{n}}F_{\nu\sigma} \rangle \\
&\quad + \langle \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}^{(A)a}F_{\nu\sigma} \rangle] \\
&= \frac{1}{2}[|\mathbf{D}^{(A)\hat{t}}F|^2 + |\mathbf{D}^{(A)\hat{n}}F|^2 + |\mathbf{D}^{(A)a}F|^2] \tag{3.60}
\end{aligned}$$

Denoting by $N_{t_p-\tau, t}^-(p)$ the portion of $N^-(p)$ that is to the future of $\Sigma_{t_p-\tau}$ and to the past of Σ_t . Denoting the gradient of F by $\mathbf{D}^{(A)}F$, and defining,

$$|\mathbf{D}^{(A)}F|^2 = h^{\alpha\beta}h^{\mu\gamma}h^{\nu\sigma} \langle \mathbf{D}_\alpha^{(A)}F_{\gamma\sigma}, \mathbf{D}_\beta^{(A)}F_{\mu\nu} \rangle \tag{3.61}$$

Applying the divergence theorem to $T_1^{\alpha\beta}\hat{t}_\beta$ in $J^-(p) \cap \Sigma_{t_p-\tau}^+ \cap \Sigma_t^-$, using (3.56) and the fact that the metric is sufficiently smooth so that $\nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)$ is finite, using (3.59),

(3.60), and applying Cauchy-Schwarz, we obtain:

$$\begin{aligned}
& \| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_t \cap J^-(p))}^2 + \| \mathbf{D}_a^{(A)} F \|_{L^2(N_{t_p-\tau, t}^-(p))}^2 \\
& \lesssim \| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_{t_p-\tau} \cap J^-(p))}^2 \\
& \quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F | (| \mathbf{D}^{(A)} F | + | F | + | F |^2) dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t}
\end{aligned} \tag{3.62}$$

We get,

$$\begin{aligned}
& \int_{\Sigma_t \cap J^-(p)} | \mathbf{D}^{(A)} F |^2 dV ol_{\Sigma_t \cap J^-(p)} + \| \mathbf{D}_a^{(A)} F \|_{L^2(N_{t_p-\tau, t}^-(p))}^2 \\
& \lesssim \| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_{t_p-\tau} \cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F |^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F | (| F |^2 + | F |) dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \lesssim \| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_{t_p-\tau} \cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F |^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} (| \mathbf{D}^{(A)} F |^2 + | F |^4 + | F |^2) dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \lesssim \| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_{t_p-\tau} \cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} (| F |^4 + | F |^2) dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F |^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \lesssim 1 + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | F |^4 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
& \quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | F |^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} | \mathbf{D}^{(A)} F |^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t}
\end{aligned}$$

(where we absorbed the quantity $\| \mathbf{D}^{(A)} F \|_{L^2(\Sigma_{t_p-\tau} \cap J^-(p))}^2$ into the constant in the symbol \lesssim , since the quantity is assumed to be finite by assumption)

$$\begin{aligned}
&\lesssim 1 + \int_{t_p-\tau}^t (|F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 \int_{\Sigma_{\bar{t}} \cap J^-(p)} |F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)}) d\bar{t} \\
&\quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} |F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
&\lesssim 1 + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 E_{t=0}^{\hat{t}}(\Sigma \cap J^-(p)) d\bar{t} + \int_{t_p-\tau}^t E_{t=0}^{\hat{t}}(\Sigma \cap J^-(p)) d\bar{t} \\
&\quad + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \tag{3.63}
\end{aligned}$$

From (3.63), we get

$$\begin{aligned}
&\int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_t \cap J^-(p)} \\
&\lesssim 1 + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} + \int_{t_p-\tau}^t \left(\int_{\Sigma_{\bar{t}} \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} \right) d\bar{t}
\end{aligned}$$

Using Grönwall lemma, we obtain

$$\int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_t \cap J^-(p)} \lesssim C(t) + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} \tag{3.64}$$

where $C(t)$ is a finite constant that depends on t .

Injecting (3.64) in (3.63), we have

$$\begin{aligned}
&\int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_t \cap J^-(p)} + \|\mathbf{D}_a^{(A)} F\|_{L^2(N_{t_p-\tau, t}^-(p))}^2 \\
&\lesssim 1 + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{\Sigma_{\bar{t}} \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dV ol_{\Sigma_{\bar{t}} \cap J^-(p)} d\bar{t} \\
&\lesssim 1 + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} + \int_{t_p-\tau}^t (C(t) + \int_{t_p-\tau}^{t^*} |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t}) dt^* \\
&\lesssim c(t) + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} dt^*
\end{aligned}$$

Finally,

$$\begin{aligned}
& \int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)} F|^2 dVol_{\Sigma_t \cap J^-(p)} + \|\mathbf{D}_a^{(A)} F\|_{L^2(N_{t_p-\tau, t}^-(p))}^2 \\
\lesssim & c(t) + \int_{t_p-\tau}^t |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} |F|_{L^\infty_{\Sigma_{\bar{t}} \cap J^-(p)}}^2 d\bar{t} dt^* \tag{3.65}
\end{aligned}$$

3.2.17 The proof

Let $p \in \Sigma_{t_p}$.

Let $q \in \Sigma_t$ where $t_p - \tau \leq t \leq t_p$, with τ small enough depending on p .

Let $\Omega_q = J^-(q) \cap J^+(\Sigma_{t_p-\tau})$.

Let $\Sigma_t^p = \Sigma_t \cap J^-(p)$.

Using an adaptation of the Klainerman-Rodnianski parametrix in [KR1] to the Yang-Mills setting, see Appendix (6.103), we have

$$\begin{aligned}
& 4\pi \langle \mathbf{J}_{\alpha\beta}, F^{\alpha\beta} \rangle (q) \\
= & - \int_{\Omega_q} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle + \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta} \\
& + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] - \frac{1}{2} R_{\alpha}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle + C_{t_p-\tau}
\end{aligned}$$

where $\hat{\Delta}^{(A)} \lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$, of $\lambda_{\alpha\beta}$, as in (6.49), and where $C_{t_p-\tau}$ depends on the value of F on $\Sigma_{t_p-\tau}$. Let $\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_1, e_2\}$.

Hence,

$$\begin{aligned}
& |F_{\hat{\mu}\hat{\nu}}(q)|^2 \\
\lesssim & \int_{\Omega_q} | \langle \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle | + \int_{\Omega_q} | \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \\
& + \int_{\Omega_q} | \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | + \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \\
& + \int_{\Omega_q} | \delta(u) \langle [F_{L\underline{L}}, \lambda_{\alpha\beta}], F^{\alpha\beta} \rangle | + \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} R_{\alpha}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta}, F^{\alpha\beta} \rangle | \\
& + \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} R_{\beta}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle | + C_{t_p-\tau}
\end{aligned}$$

We have,

$$\begin{aligned}
& \int_{\Omega_q} | \langle \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle | = \int_{\Omega_q} | \langle \lambda^{\mu\nu}\delta(u), \square_{\mathbf{g}}^{(A)} F_{\mu\nu} \rangle | \\
= & \int_{\Omega_q} | \langle \lambda^{\mu\nu}\delta(u), -2R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} - R_{\mu\gamma} F_{\nu}{}^{\gamma} - R_{\nu\gamma} F^{\gamma}{}_{\mu} - 2[F^{\alpha}{}_{\mu}, F_{\nu\alpha}] \rangle | \\
& \text{(using (3.2))} \\
\lesssim & \int_{\Omega_q} | \langle \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle | + \int_{\Omega_q} | \langle \lambda\delta(u), R_{\mu\gamma} F_{\nu}{}^{\gamma} \rangle | \\
& + \int_{\Omega_q} | \langle \lambda\delta(u), R_{\nu\gamma} F^{\gamma}{}_{\mu} \rangle | + \int_{\Omega_q} | \langle \lambda^{\mu\nu}\delta(u), [F^{\alpha}{}_{\mu}, F_{\nu\alpha}] \rangle |
\end{aligned}$$

Finally,

$$\begin{aligned}
& \sup_{q \in \Sigma_t \cap J^-(p)} |F_{\hat{\mu}\hat{\nu}}(q)|^2 \\
\lesssim & \sup_{q \in \Sigma_t \cap J^-(p)} \left[\int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle| + \int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\mu\gamma} F_\nu^\gamma \rangle| \right. \\
& + \int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\nu\gamma} F^\gamma_\mu \rangle| + \int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), [F^\alpha_\mu, F_\nu^\alpha] \rangle| \\
& + \left| \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| + \left| \int_{\Omega_q} \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| \\
& + \left| \int_{\Omega_q} \delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| + \left| \int_{\Omega_q} \delta(u) \langle [F_{LL}, \lambda_{\alpha\beta}], F^{\alpha\beta} \rangle \right| \\
& + \left| \int_{\Omega_q} \delta(u) \langle \frac{1}{2} R_\alpha^\gamma \underline{LL} \lambda_{\gamma\beta}, F^{\alpha\beta} \rangle \right| + \left| \int_{\Omega_q} \delta(u) \langle \frac{1}{2} R_\beta^\gamma \underline{LL} \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle \right| \\
& + C_{t_p - \tau} \tag{3.66}
\end{aligned}$$

In what follows, we note $\Sigma_t \cap J^-(p)$ as Σ_t^p .

Lemma 3.2.18. *We have,*

$$\begin{aligned}
& \sup_{q \in \Sigma_t^p} \left[\int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle| + \int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\mu\gamma} F_\nu^\gamma \rangle| \right. \\
& \quad + \int_{\Omega_q} |\langle \lambda^{\mu\nu} \delta(u), R_{\nu\gamma} F^\gamma_\mu \rangle| + \left| \int_{\Omega_q} \delta(u) \langle \frac{1}{2} R_\alpha^\gamma \underline{LL} \lambda_{\gamma\beta}, F^{\alpha\beta} \rangle \right| \\
& \quad + \left| \int_{\Omega_q} \delta(u) \langle \frac{1}{2} R_\beta^\gamma \underline{LL} \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle \right| \Big] \\
\lesssim & \tau^{\frac{3}{2}} + \left[\int_{t_p - \tau}^t \|F\|_{L^\infty(\Sigma_t^p)}^2 d\bar{t} \right] \tau^{\frac{3}{2}}
\end{aligned}$$

Proof.

We have,

$$\begin{aligned}
& \sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle | \\
& \lesssim \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_{\bar{t}}} | \langle s \lambda^{\mu\nu}, R_{\gamma\mu\nu\alpha} s^{-1} F^{\alpha\gamma} \rangle | \phi da_{\bar{t}} d\bar{t} \\
& \lesssim \sup_{q \in \Sigma_t^p} \|\phi\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \sup_{q \in \Sigma_t^p} \|s\lambda\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \sup_{q \in \Sigma_t^p} \|R\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \\
& \cdot \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}})} \int_{S_{\bar{t}}} s^{-1} da_{\bar{t}}) d\bar{t}.
\end{aligned}$$

We have,

$$\sup_{q \in \Sigma_t^p} \|R\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

(since the metric is smooth)

$$\|s\lambda\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim |J|$$

thus

$$\sup_{q \in \Sigma_t} \|s\lambda\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

$$\sup_{q \in \Sigma_t^p} \|\phi\|_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

(since ϕ is smooth and bounded)

$$\begin{aligned}
& \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)} \int_{S_{\bar{t}}} s^{-1} da_{\bar{t}}) d\bar{t} \\
& \lesssim \sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t})^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t \left(\int_{S_{\bar{t}}} s^{-1} da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}} \right] \quad (3.67)
\end{aligned}$$

We get,

$$\int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle | \lesssim \left[\int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2) d\bar{t} \right]^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t \left(\int_{S_{\bar{t}}} s^{-1} da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}}$$

$$\left[\int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}})}^2) d\bar{t} \right]^{\frac{1}{2}} \lesssim 1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}})}^2 d\bar{t}$$

Recall that

$$A_t(p) = O((t_p - t)^2)$$

(see (6.59)), and

$$t_p - t = s + o(s)$$

(see (6.64)).

Thus,

$$A_t(p) = O(s^2)$$

We get,

$$\sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t \left(\int_{S_{\bar{t}}} s^{-1}(\bar{t}) da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}} \lesssim \left[\int_{t_p-\tau}^{t_p} s_p(t)^2 dt \right]^{\frac{1}{2}} \lesssim \tau^{\frac{3}{2}}$$

Thus,

$$\int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} \rangle | \lesssim \tau^{\frac{3}{2}} + \left[\int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{3}{2}} \quad (3.68)$$

In the same manner this controls the terms

$$\begin{aligned} & \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), R_{\mu\gamma} F_\nu^\gamma \rangle |, \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), R_{\nu\gamma} F^\gamma_\mu \rangle |, \\ & \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} R_\alpha^\gamma \underline{L} L \lambda_{\gamma\beta}, F^{\alpha\beta} \rangle |, \text{ and } \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} R_\beta^\gamma \underline{L} L \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle | \end{aligned}$$

□

Lemma 3.2.19. *We have,*

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \lesssim \tau^{\frac{1}{2}} + [\int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t}] \tau^{\frac{1}{2}}$$

Proof.

The term

$$\begin{aligned} & \sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \lesssim \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_{\bar{t}}} | \langle \frac{1}{2} \hat{\mu} s \lambda_{\alpha\beta}, s^{-1} F^{\alpha\beta} \rangle | \phi da_{\bar{t}} d\bar{t} \\ & \lesssim \sup_{q \in \Sigma_t^p} \|s\lambda\|_{L^\infty(N_{t_p-\tau, t}^-(q))} \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)} \int_{S_{\bar{t}}} |\hat{\mu}| s^{-1} \phi da_{\bar{t}}) d\bar{t} \end{aligned}$$

We get,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \lesssim [\int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2) d\bar{t}]^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_{\bar{t}}} |\hat{\mu}| s^{-1} \phi da_{\bar{t}})^2 d\bar{t}]^{\frac{1}{2}}$$

And we have

$$\sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (\int_{S_{\bar{t}}} |\hat{\mu}| s^{-1} \phi da_{\bar{t}})^2 d\bar{t} \lesssim \int_{t_p-\tau}^t \sup_{q \in \Sigma_t^p} (\int_{S_{\bar{t}}} |\hat{\mu}|^2 \phi da_{\bar{t}}) (\int_{S_{\bar{t}}} s^{-2} \phi da_{\bar{t}}) d\bar{t}$$

We have $\hat{\mu} = o(s^{-1})$ (see proposition 3.1 in [Wang])

Thus,

$$(\sup_{q \in \Sigma_t^p} \int_{S_{\bar{t}}} |\hat{\mu}|^2 \phi da_{\bar{t}}) \lesssim \sup_{q \in \Sigma_t^p} \|\phi\|_{L^\infty(N_{t_p-\tau, t}^-(q))} (\sup_{q \in \Sigma_t^p} \int_{S_{\bar{t}}} s^{-2} da_{\bar{t}}) \lesssim 1$$

$$\left(\sup_{q \in \Sigma_t^p} \int_{S_{\bar{t}}} s^{-2} \phi da_{\bar{t}}\right) \lesssim 1$$

Thus,

$$\sup_{q \in \Sigma_t^p} \left[\int_{t_p - \tau}^t \left(\int_{S_{\bar{t}}} |\hat{\mu}| s^{-1} \phi da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}} \lesssim \tau^{\frac{1}{2}}$$

We obtain,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} |\delta(u) \langle \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle| \lesssim \tau^{\frac{1}{2}} + \left[\int_{t_p - \tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}} \quad (3.69)$$

□

Next, we want to control the term

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] \rangle | = \sup_{q \in \Sigma_t^p} \int_{t_p - \tau}^t \int_{S_{\bar{t}}} | \langle \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] \rangle \phi da_{\bar{t}} | d\bar{t}$$

Lemma 3.2.20. *We have,*

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] \rangle | \lesssim \tau^{\frac{1}{2}} + \left[\int_{t_p - \tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}}$$

Proof.

Following the remark of Eardley and Moncrief in [EM2], we have

$$|[F^\alpha{}_\mu, F_{\nu\alpha}]| \lesssim \|F\|_{L^\infty(S_t)} (|F_{L\underline{L}}| + |F_{La}| + |F_{Lb}| + |F_{ab}|)$$

on $N_{t_p - \tau, t}^-(q) \cap \Sigma_t = S_t$, and therefore,

$$\begin{aligned}
& \sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] \rangle | \\
\lesssim & \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_{\bar{t}}} | \langle s \lambda^{\mu\nu} \delta(u), [s^{-1} F^\alpha{}_\mu, F_{\nu\alpha}] \rangle \phi da_{\bar{t}} d\bar{t} \\
\lesssim & \sup_{q \in \Sigma_t^p} \| \phi \|_{L^\infty(N_{t_p-\tau, t}^-(q))} \sup_{q \in \Sigma_t^p} \| s \lambda \|_{L^\infty(N_{t_p-\tau, t}^-(q))} \\
& \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \| F \|_{L^\infty(\Sigma_{\bar{t}})} \int_{S_{\bar{t}}} (s^{-1} |F_{LL}| + s^{-1} |F_{La}| + s^{-1} |F_{Lb}| + s^{-1} |F_{ab}|) d\bar{t} da_{\bar{t}} \\
\lesssim & \left[\int_{t_p-\tau}^t (\| F \|_{L^\infty(\Sigma_{\bar{t}})}^2) d\bar{t} \right]^{\frac{1}{2}} \\
& \cdot \sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t \left(\int_{S_{\bar{t}}} (s^{-1} |F_{LL}| + s^{-1} |F_{La}| + s^{-1} |F_{Lb}| + s^{-1} |F_{ab}|) da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}}
\end{aligned}$$

(since ϕ is smooth and bounded near p).

And we have,

$$\begin{aligned}
& \left[\int_{S_{\bar{t}}} (s^{-1} |F_{LL}| + s^{-1} |F_{La}| + s^{-1} |F_{Lb}| + s^{-1} |F_{ab}|) da_{\bar{t}} \right]^2 \\
\lesssim & \left(\int_{S_{\bar{t}}} (s^{-2}) da_{\bar{t}} \right) \left(\int_{S_{\bar{t}}} (|F_{LL}|^2 + |F_{La}|^2 + |F_{Lb}|^2 + |F_{ab}|^2) da_{\bar{t}} \right)
\end{aligned}$$

(by Cauchy-Schwarz inequality)

and

$$F^{\hat{t}}(N_{\tau}^-(q)) = \int_{N_{\tau}^-(q)} \frac{1}{8} |F_{LL}|^2 + \frac{1}{2} |F_{Lea}|^2 + \frac{1}{2} |F_{Leb}|^2 + \frac{1}{2} |F_{ab}|^2$$

Thus,

$$\begin{aligned}
& \int_{t_p-\tau}^t \int_{S_{\bar{t}}} (s^{-1} |F_{LL}| + s^{-1} |F_{La}| + s^{-1} |F_{Lb}| + s^{-1} |F_{ab}|) d\bar{t} da_{\bar{t}} \\
\lesssim & \left(\int_{t_p-\tau}^t 1 dt \right)^{\frac{1}{2}} (F^{\hat{t}}(N_{t_p-\tau, t}^-(q)))^{\frac{1}{2}} \\
\lesssim & \tau^{\frac{1}{2}} (E_{t=0}^{\hat{t}})^{\frac{1}{2}}
\end{aligned} \tag{3.70}$$

Thus,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | \langle \lambda^{\mu\nu} \delta(u), [F^\alpha_\mu, F_{\nu\alpha}] \rangle | \lesssim \tau^{\frac{1}{2}} + \left[\int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}} \quad (3.71)$$

□

Lemma 3.2.21. *We have,*

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} | \delta(u) \langle [F_{LL}, \lambda_{\alpha\beta}], F^{\alpha\beta} \rangle | \lesssim \tau^{\frac{1}{2}} + \left[\int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}} \quad (3.72)$$

Proof.

By same as previously, the term

$$\begin{aligned} & \sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} | \delta(u) \langle [F_{LL}, \lambda_{\alpha\beta}], F^{\alpha\beta} \rangle | \\ & \lesssim \left[\int_{t_p-\tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2) d\bar{t} \right]^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} \left[\int_{t_p-\tau}^t \left(\int_{S_{\bar{t}}} (s^{-1} |F_{LL}|) da_{\bar{t}} \right)^2 d\bar{t} \right]^{\frac{1}{2}} \\ & \lesssim \left[1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} (F^{\hat{t}}(N_{t_p-\tau, t}^-(q)))^{\frac{1}{2}} \\ & \lesssim \left[1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^{\frac{1}{2}} (E_{t=0}^{\hat{t}})^{\frac{1}{2}} \end{aligned}$$

□

Lemma 3.2.22. *We have,*

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} | \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle | \lesssim \tau^4 + \left[\int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^4$$

Proof.

The term

$$\begin{aligned}
& \sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle| \\
&= \sup_{q \in \Sigma_t \cap J^-(p)} \int_{t_p - \tau}^t (\|F\|_{L^\infty(S_{\bar{t}})}) \int_{S_{\bar{t}}} |\zeta_a \mathbf{D}_a^{(A)} \lambda| \phi da_{\bar{t}} d\bar{t} \\
&\lesssim \left[\int_{t_p - \tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t})^{\frac{1}{2}} \sup_{q \in \Sigma_t \cap J^-(p)} \left[\int_{t_p - \tau}^t \left(\int_{S_{\bar{t}}} |\zeta_a|^2 \phi da_{\bar{t}} \right) \left(\int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} \lambda|^2 \phi da_{\bar{t}} \right) d\bar{t} \right]^{\frac{1}{2}}
\end{aligned}$$

We have

$$\zeta_a = O(s)$$

(see proposition 3.1 in [Wang]).

Thus,

$$\int_{S_{\bar{t}}} |\zeta_a|^2 \phi da_{\bar{t}} \lesssim \int_{S_t} s^2 da_t \lesssim s^4 \lesssim \tau^4$$

Therefore,

$$\begin{aligned}
& \sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle| \\
&\lesssim \left[\int_{t_p - \tau}^t (\|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t})^{\frac{1}{2}} \tau^4 \sup_{q \in \Sigma_t^p \cap J^-(p)} \left[\int_{t_p - \tau}^t \int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} \lambda|^2 \phi da_{\bar{t}} d\bar{t} \right]^{\frac{1}{2}}
\end{aligned}$$

We showed previously that $\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_\tau^-(q))} \lesssim 1$, thus

$$\sup_{q \in \Sigma_t^p \cap J^-(p)} \|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_\tau^-(q))} \lesssim 1$$

Finally,

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle| \lesssim \tau^4 + \left[\int_{t_p - \tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} \right] \tau^4 \quad (3.73)$$

□

We are left with the term $\sup_{q \in \Sigma_t^p} |\int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle|$. We recall that $\hat{\Delta}^{(A)} \lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$,

Lemma 3.2.23. *We have,*

$$\begin{aligned} & \sup_{q \in \Sigma_t^p} \left| \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| \\ & \lesssim 1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} dt^* \end{aligned}$$

Proof.

$$\begin{aligned} \sup_{q \in \Sigma_t^p} \left| \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| &= \sup_{q \in \Sigma_t^p} \left| \int_{t_p-\tau}^t \int_{S_{\bar{t}}} \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \phi da_{\bar{t}} d\bar{t} \right| \\ &\lesssim \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \left| \int_{S_{\bar{t}}} \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \phi da_{\bar{t}} \right| d\bar{t} \end{aligned}$$

Definition 3.2.24. We define a restriction of the covariant derivative of $\nabla_b e_a$ to the span of $\{e_a\}$, $a \in \{1, 2\}$ at $q \in N^-(p) \setminus \{p\}$ as being $\nabla_b e_a$.

Definition 3.2.25. We define,

$$\mathbf{D}_b^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} = \mathbf{D}_b^{(A)} (\mathbf{D}_a^{(A)} \lambda)_{\alpha\beta} - \mathbf{D}_{\nabla_b e_a}^{(A)} \lambda_{\alpha\beta} \quad (3.74)$$

whereas,

$$\mathbf{D}_b^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} = \mathbf{D}_b^{(A)} (\mathbf{D}_a^{(A)} \lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_b e_a}^{(A)} \lambda_{\alpha\beta}$$

We have

$$\begin{aligned} & \hat{\Delta}^{(A)} \lambda_{\alpha\beta} \\ &= (\mathbf{D}^{(A)a} \mathbf{D}_a^{(A)} \lambda)(e_\alpha, e_\beta) \\ &= \partial^a [(\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] \\ & \quad - (\mathbf{D}_a^{(A)} \Psi)(\nabla^a e_\alpha, e_\beta) - (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, \nabla^a e_\beta) - (\mathbf{D}_{\nabla^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta) \end{aligned}$$

Hence,

$$\begin{aligned}
& \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\
&= \langle \partial^a [(\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)], F^{\alpha\beta} \rangle \\
&\quad - \langle (\mathbf{D}_a^{(A)} \Psi)(\nabla^a e_\alpha, e_\beta), F^{\alpha\beta} \rangle - \langle (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, \nabla^a e_\beta), F^{\alpha\beta} \rangle \\
&\quad - \langle (\mathbf{D}_{\nabla^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta), F^{\alpha\beta} \rangle
\end{aligned}$$

To compute $\langle (\mathbf{D}_{\nabla^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta), F^{\alpha\beta} \rangle$, since $\nabla^a e_a$ is a full contraction on the 2-spheres $S_{\bar{t}}$, we can choose a normal frame with respect to the induced metric on $S_{\bar{t}}$, i.e. a frame where the restricted covariant derivative of elements of the frame $\nabla^a e_a$ vanish at that point. Hence, this term vanishes.

Whereas to the terms

$$\langle -(\mathbf{D}_a^{(A)} \Psi)(\nabla^a e_\alpha, e_\beta), F^{\alpha\beta} \rangle$$

and

$$\langle (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, \nabla^a e_\beta), F^{\alpha\beta} \rangle$$

since they are full contractions with respect to the space-time metric \mathbf{g} , we can compute those with respect to a normal frame where $\nabla_\alpha e_\beta = 0$ at that point. We can then express ∇^a as a combination of covariant derivatives at that frame ∇^α , and hence we get that $\nabla^a e_\alpha$ vanish.

Consequently,

$$\langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = \langle \partial^a [(\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)], F^{\alpha\beta} \rangle$$

Similarly,

$$\langle \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, \mathbf{D}^{(A)a} F^{\alpha\beta} \rangle = \langle \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, \partial^a F^{\alpha\beta} + [A^a, F^{\alpha\beta}] \rangle$$

Using the fact that the scalar product $\langle \cdot, \cdot \rangle$ is Ad-invariant, we get

$$\langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = \nabla^a \langle \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle - \langle \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, \mathbf{D}^{(A)a} F^{\alpha\beta} \rangle$$

Integrating on $S_{\bar{t}}$, then applying the divergence theorem, and using the fact that we have no boundary terms since it is an integral on $S_{\bar{t}}$, we get

$$\begin{aligned} & \left| \int_{S_{\bar{t}}} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \phi da_{\bar{t}} \right| \\ &= \left| \int_{S_{\bar{t}}} - \langle \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, \mathbf{D}^{(A)a} F^{\alpha\beta} \rangle \phi da_{\bar{t}} \right| \\ &\lesssim \left(\int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} \lambda|^2 \phi da_{\bar{t}} \right)^{\frac{1}{2}} \left(\int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} F|^2 \phi da_{\bar{t}} \right)^{\frac{1}{2}} \end{aligned} \quad (3.75)$$

Thus,

$$\begin{aligned} & \left| \int_{t_p-\tau}^t \int_{S_{\bar{t}}} \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F_{\hat{\mu}\hat{\nu}} \rangle \phi da_{\bar{t}} d\bar{t} \right| \\ &\lesssim \|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_{\tau}^-(q))} \left(\int_{t_p-\tau}^t \int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} F|^2 \phi da_{\bar{t}} d\bar{t} \right)^{\frac{1}{2}} \end{aligned}$$

We proved that $\|\mathbf{D}_a^{(A)} \lambda\|_{L^2(N_{\tau}^-(p))} \lesssim 1$. We also have

$$\begin{aligned} \left(\int_{t_p-\tau}^t \int_{S_{\bar{t}}} |\mathbf{D}_a^{(A)} F|^2 \phi da_{\bar{t}} d\bar{t} \right)^{\frac{1}{2}} &= \|\mathbf{D}_a^{(A)} F\|_{L^2(N_{t_p-\tau, t}^-(q))} \\ &\lesssim 1 + \|\mathbf{D}_a^{(A)} F\|_{L^2(N_{t_p-\tau, t}^-(q))}^2 \end{aligned}$$

We get,

$$\sup_{q \in \Sigma_t^p} \left| \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| \lesssim 1 + \sup_{q \in \Sigma_t^p} \|\mathbf{D}_a^{(A)} F\|_{L^2(N_{t_p-\tau, t}^-(q))}^2 \quad (3.76)$$

We proved that,

$$\|\mathbf{D}_a^{(A)}F\|_{L^2(N_{t_p-\tau,t}^-(q))}^2 \lesssim c(t) + \int_{t_p-\tau}^t |F|_{L^\infty(\Sigma_{\bar{t}}^q)}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} |F|_{L^\infty(\Sigma_{\bar{t}}^q)}^2 d\bar{t} dt^*$$

where $c(t)$ is a finite constant for all t .

$$\begin{aligned} & \sup_{q \in \Sigma_t^p} \|\mathbf{D}_a^{(A)}F\|_{L^2(N_{t_p-\tau,t}^-(q))}^2 \\ & \lesssim 1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} dt^* \end{aligned}$$

Thus,

$$\begin{aligned} & \sup_{q \in \Sigma_t^p} \left| \int_{\Omega_q} \delta(u) \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| \\ & \lesssim 1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} dt^* \end{aligned} \quad (3.77)$$

□

Finally, summing over all the indices we obtain,

$$\|F\|_{L^\infty(\Sigma_t^p)}^2 \lesssim 1 + \int_{t_p-\tau}^t \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} \|F\|_{L^\infty(\Sigma_{\bar{t}}^p)}^2 d\bar{t} dt^* \quad (3.78)$$

Using the result of Pachpatte in [Pach], we get,

$$\|F\|_{L^\infty(\Sigma_t^p)} \lesssim 1, \text{ for all } t \in [t_p - \tau, t_p]$$

The pointwise estimate above prove that

$$|F|(p) \lesssim C(p) \quad (3.79)$$

where $C(p)$ depends on the geometry of the space-time at the point p and values depending

on F on $\Sigma_{t_p - \tau}^p$.

Considering a local solution defined on $[t_0, T)$, at any point q in the space-time at time $t_q = T$, taking the limit in the above, we obtain

$$\lim_{p \rightarrow q, t_p \in [t_0, T)} |F|(p) < \infty \quad (3.80)$$

and this can be done to any point q in the space-time at time T , under the assumption of global hyperbolicity.

Injecting (3.80) in (3.65), we get

$$\lim_{t \rightarrow T, t < T} E_{\mathbf{D}^{(A)}F}^{\hat{t}}(t) < \infty ,$$

We have already showed that the energy will stay finite and this was used to show that the previous limits are finite; hence, we have already proved that

$$\lim_{t \rightarrow T, t < T} E_F^{\hat{t}}(t) < \infty .$$

Chapter 4

On Uniform Decay of Solutions to the Maxwell Equations on Black Hole Space-Times

4.1 Introduction

4.1.1 Background and motivation

In this chapter, we study the Maxwell equations on the Schwarzschild black hole. In a recent paper, [DR1]-[DR2], Dafermos and Rodnianski proved decay for solutions of the free scalar wave equation $\square_{\mathbf{g}}\phi = 0$ in the exterior of the Schwarzschild black hole, up to points on the event horizon. We do not know how to make these methods work for $\square_{\mathbf{g}}\phi = c\cdot\phi$ or for $\square_{\mathbf{g}}\phi = \phi^2$ using the energy-momentum tensor of the wave equation. Thus, this rules out the possibility of using these methods for the Maxwell equations in a hyperbolic formulation where the source terms would be $(\square_{\mathbf{g}}F)_{\mu\nu}$, where $F_{\mu\nu}$ is the Maxwell field. In a recent paper, [Bl], Blue proved decay for the Maxwell fields in the exterior of the Schwarzschild background. The proof of Blue required a study of a wave equation on the Schwarzschild

space-time for the so-called middle components, which can be separated from the other components in the abelian case of the Maxwell equations. This was later extended by Andersson and Blue to Kerr metrics, [AB]. However, in the non-linear case of the Yang-Mills equations, one cannot decouple the middle components from the others. Yet, it seems difficult to generalize the results of Dafermos and Rodnianski for the free scalar wave equation to the Maxwell equations using the Maxwell energy-momentum tensor directly, without referring to the scalar wave equation, combined with suitable Sobolev inequalities. A key step to achieve this would be to bound the conformal energy without separating the middle components of the Maxwell fields. This would provide a new independent proof and improves the result of Blue, and would be in particular useful for the non abelian case of the Yang-Mills equations where such separation cannot occur. I tried to do this in my thesis in the goal of proving uniform boundedness for Yang-Mills fields on the exterior of the Schwarzschild black hole and the Kerr metric. However, as Klainerman pointed out to me later, many people tried to get a more conceptual proof of decay for the Maxwell fields, without passing through the scalar wave equation, and achieving this would be very significant. While I was not able to solve this yet, in this chapter I write a proof of decay for the Maxwell fields on the exterior of the Schwarzschild space-time, directly without separating the middle components or referring to the wave equation, assuming that we have a Morawetz estimate at the 0-derivative level on a compact region in space. To explain, let us recall (see Appendix) that in the exterior, the Schwarzschild metric can be written as,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

If we define,

$$\begin{aligned} r^* &= r + 2m \log(r - 2m) \\ v &= t + r^* \\ w &= t - r^* \end{aligned}$$

then, we have,

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2m}{r}\right)dvdw + r^2 d\sigma^2 \\
&= -\frac{\left(1 - \frac{2m}{r}\right)}{2}dv \otimes dw - \frac{\left(1 - \frac{2m}{r}\right)}{2}dw \otimes dv + r^2 d\sigma^2
\end{aligned}$$

Let,

$$\frac{\hat{\partial}}{\partial w} = \frac{1}{\left(1 - \frac{2m}{r}\right)} \frac{\partial}{\partial w} \quad (4.1)$$

$$\frac{\hat{\partial}}{\partial v} = \frac{\partial}{\partial v} \quad (4.2)$$

$$\frac{\hat{\partial}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \quad (4.3)$$

$$\frac{\hat{\partial}}{\partial \phi} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (4.4)$$

and at a point in the Schwarzschild space-time, let e_1, e_2 be a normalized basis of S^2 , which verifies for all $A, B \in \{1, 2\}$,

$$\mathbf{g}\left(e_A, \frac{\hat{\partial}}{\partial w}\right) = \mathbf{g}\left(e_A, \frac{\partial}{\partial v}\right) = 0 \quad (4.5)$$

$$\mathbf{g}(e_A, e_B) = \delta_{AB} \quad (4.6)$$

In this context, by "assuming a Morawetz estimate", we mean exactly that for r_0, R_0 as in the proof of (4.55), and for all $t_i = (1.1)^i t_0$, where t_0 is any real positive number and i is any natural number, if we define:

$$J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |F_{\hat{\phi}\hat{\theta}}|^2] . dr^* d\sigma^2 dt \quad (4.7)$$

then, we assume that the non-stationary solutions verify

$$\begin{aligned}
& J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \\
& \lesssim |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1})| + \sum_{j=1}^k (|\hat{E}_{\mathcal{L}_{K_j} F}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\mathcal{L}_{K_j} F}^{(\frac{\partial}{\partial t})}(t_{i+1})|)
\end{aligned} \tag{4.8}$$

where $K_j, j \in \{1, \dots, k\}$, are Killing vector fields, and where

$$\begin{aligned}
\hat{E}_F^{(\frac{\partial}{\partial t})}(t) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2(1-\frac{2m}{r})} |F_{t\theta}|^2 + \frac{1}{r^2(1-\frac{2m}{r}) \sin^2 \theta} |F_{t\phi}|^2 \right. \\
&\quad \left. + \frac{1}{r^2(1-\frac{2m}{r})} |F_{r^*\theta}|^2 + \frac{1}{r^2(1-\frac{2m}{r}) \sin^2 \theta} |F_{r^*\phi}|^2 \right] r^2 \left(1 - \frac{2m}{r}\right) d\sigma^2 dr^*
\end{aligned} \tag{4.9}$$

that is the energy without the middle components $F_{tr^*}, F_{\theta\phi}$. The only stationary solutions of the Maxwell equations on the exterior of the Schwarzschild black hole, are the so-called Coulomb solutions. Hence, the assumption above is assumed for the non-Coulomb solutions. For simplicity, we write our proof with $k = 3$ and $K_j = \Omega_j$, where $\Omega_j, j \in \{1, 2, 3\}$, is a basis of angular momentum operators, although the proof would still work with any product of Killing Lie derivatives on the right hand side of (4.8), with adjusting accordingly the quantities in the theorem that depend on the initial data.

We will prove that if the middle components of the non-stationary solutions verify a Morzwetz type estimate at the zero-derivative level, (4.8), then we can prove uniform decay properties of solutions to the Maxwell equations in the domain of outer-communication of the Schwarzschild black hole space-time, including the event horizon, by making use of suitable Sobolev inequalities combined with energy estimates using the energy momentum tensor of the Maxwell fields. We do not make any use of decomposition into spherical harmonics. We start with a Cauchy hypersurface prescribed by $t = t_0$ where the initial data has to verify certain regularity conditions (there is no vanishing condition on the bifurcate sphere for F). Away from the horizon (in the region $r \geq R > 2m$, for a fixed R),

we will prove that

$$\begin{aligned} |F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\leq \frac{C}{(1 + |v|)} \\ |F_{\hat{\nu}\hat{\mu}}|(w, v, \omega) &\leq \frac{C}{(1 + |w|)} \end{aligned}$$

for all $\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$, where $F_{\mu\nu}$ is the Maxwell field. Near the horizon, and in the entire exterior region $r \geq 2m$, up to points on the event horizon, we will prove that

$$\begin{aligned} |F_{\hat{v}\hat{w}}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}}, & |F_{e_1 e_2}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}} \\ |F_{\hat{v}e_a}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}}, & |\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_a}(v, w, \omega)| &\leq \frac{C}{\max\{1, v\}} \end{aligned}$$

To explain really thoroughly:

If one tries to generalize the proof of Dafermos and Rodnianski for the free scalar wave equation to the Maxwell equations, on the Schwarzschild space-time, using the Maxwell energy-momentum tensor instead, some of the difficulties that appear are:

1. In order to bound the conformal energy of the Maxwell fields and their derivatives in the direction of Killing vector fields, one needs to control a space-time integral near the trapped surface $r = 3m$, that involves the so-called middle components of the Maxwell field (see (4.44)). In fact, the terms in the space-time integral $J^{(K)}$, obtained by applying the divergence theorem on the Morawetz vector field $K = -w^2 \frac{\partial}{\partial w} - v^2 \frac{\partial}{\partial v}$ contracted with the energy momentum tensor of the Maxwell fields, are non-negative in a region $r_0 \leq r \leq R_0$ that contains $r = 3m$. It seems that this cannot be controlled using these methods due to the presence of the other components with the "wrong" sign in the space-time integral generated from a space type vector field of control (see (4.49)). Indeed, using the divergence theorem with the vector field $G = f(r^*) \frac{\partial}{\partial r^*}$, where f is a bounded function, we will obtain a space-time integral and boundary terms. However, the space-time integral obtained from G has independent terms

that do not appear in $J^{(K)}$ that enter with the wrong sign, and hence it cannot be made positive. We will prove that if we get past this, see assumption (4.8), we can then write a gauge independent proof of uniform decay of the Maxwell fields in the exterior of the Schwarzschild black hole up to points on the horizon, using the Maxwell equations directly.

2. One needs to construct a new field which verifies the Maxwell equations and the Bianchi identities, that coincides with the original field in some region and vanishes identically outside another specific region (see the proof of (4.58)). In the case of the wave equation $\square_{\mathbf{g}}\phi = 0$, one can multiply the initial data in the Cauchy problem by a cut-off function, and consider the evolution of such data to obtain a solution that verifies the wave equation and the properties stated previously. In the case of the Maxwell and the Yang-Mills equations, if one multiplies the initial data by a cut-off function then the constraint equations would not be satisfied anymore. It seems at first sight that one cannot get a new field that verifies the needed properties. While this is true if one wanted to do this for all components, nevertheless, one can do this for all the components except to the F_{rt} and $F_{\theta\phi}$ components, where the multiplication should be at the level of the space derivative of the components. Hence, one can construct a new field that can be made to coincide with F in a certain region, and vanish outside another except to these last components. The somewhat good news is that the calculations show that these "bad" terms do not appear in the boundary terms generated from the divergence theorem applied to a space-like vector field contracted with the energy-momentum tensor (see (4.52)). Hence, in our assumption (4.8), we suppose that a certain space-time integral near the the trapped surface $r = 3m$ of the middle components, can be bounded by the energy without the middle components, which would be the case if this estimate was obtained by controlling the space-time integral by boundary terms generated from space type vector fields multiplied by a bounded function, as shown in estimate (4.53). This is

crucial to establish (4.58) that is the main estimate to bound the conformal energy in (4.74).

3. In order to prove decay for a generalized energy that would control the L^2 norm of the Maxwell fields near the horizon, one is confronted to a situation where it seems crucial to control a space-time integral supported on a bounded region in space near the event horizon, that contains all the components. We overcome this by our assumption (4.8); it can also be used to bound the space-time integral containing the other components as in (4.104).

Also, in addition to the above:

4. As opposed to the case of the wave equation, the flux of a generalized energy that controls the L^2 norm of the Maxwell fields near the horizon do not contain all of their components. On $v = \text{constant}$ hypersurfaces it contains only $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}\hat{\theta}}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}\hat{\phi}}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{v}\hat{w}}$ and $\sqrt{1 - \frac{2m}{r}} F_{\hat{\theta}\hat{\phi}}$, and on $w = \text{constant}$ hypersurfaces, it contains only the components $F_{\hat{v}\hat{w}}$, $F_{\hat{\theta}\hat{\phi}}$, $F_{\hat{v}\hat{\theta}}$ and $F_{\hat{v}\hat{\phi}}$. In addition, while using Sobolev inequalities near the horizon, since $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$ are not Killing vector fields this would add an additional difficulty, while in the case of the wave equation, the squares of these derivatives appear in the fluxes which are easily controlled. We get around these problems by using suitable Sobolev type inequalities for each component, combined with the Bianchi identities and the field equations, in a way that the derivatives in the direction of $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$ can be controlled by Killing derivatives, ∂_t and ∂_{Ω_j} , of the components that appear in the flux to which we would have proved decay.

In the case of the Yang-Mills equations there is the additional impediment that is the equations are non-linear. If one gets an energy identity for the Yang-Mills fields, one cannot write directly the same energy identity for the derivatives of the field in the direction of Killing vector fields, as opposed to the Maxwell fields, due to the non-linearity of the equations.

More precisely, we will prove the following theorem,

4.1.2 The statement

Theorem 4.1.3. *Let $(\overline{M}, \overline{\mathbf{g}})$ be a maximally extended Schwarzschild space-time. We know by then that the exterior of the black hole space-time, $(\overline{M}, \overline{\mathbf{g}})$, is isometric to (M, \mathbf{g}) where,*

$$\begin{aligned} \mathbf{g} &= -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \\ &= -\frac{\left(1 - \frac{2m}{r}\right)}{2}dv \otimes dw - \frac{\left(1 - \frac{2m}{r}\right)}{2}dw \otimes dv + r^2d\sigma^2 \end{aligned}$$

Let, $\Sigma_{t=t_0}$ be a Cauchy hypersurface prescribed by $t = t_0$. In a system of coordinates, let $F_{\mu\nu}(w, v, \omega)$ be the components of the Maxwell field defined as the solution of the Cauchy problem of the Maxwell equations:

$$\nabla^\alpha F_{\alpha\beta} = 0 \tag{4.10}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{4.11}$$

where the initial data prescribed on the Cauchy hypersurface $\Sigma_{t=t_0}$ verifies the Maxwell constraint equations:

$$\nabla^\nu F_{t\nu}(t_0, r, \omega) = \nabla^\nu F_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu(t_0, r, \omega) = 0$$

$$[\nabla_r F_{\theta\phi} + \nabla_\theta F_{\phi r} + \nabla_\phi F_{r\theta}](t_0, r, \omega) = 0$$

We assume that for r_0 and R_0 as in the proof of (4.55), ($r_0 \leq 3m \leq R_0$), and for all $t_i = (1.1)^i t_0$, where t_0 is any real positive number and i is any natural number, the

non-stationary solutions verify

$$\begin{aligned} & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|F_{\hat{\phi}\hat{\theta}}|^2].dr^*d\sigma^2dt \\ & \lesssim |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1})| + \sum_{j=1}^3 (|\hat{E}_{\mathcal{L}\Omega_j F}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\mathcal{L}\Omega_j F}^{(\frac{\partial}{\partial t})}(t_{i+1})|) \end{aligned} \quad (4.12)$$

where Ω_j , $j \in \{1, 2, 3\}$, is a basis of angular momentum operators, and where

$$\hat{E}_F^{(\frac{\partial}{\partial t})}(t) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 2[(1 - \frac{2m}{r})^2(|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2].r^2 \sin(\theta)d\theta d\phi dr^*$$

that is the energy without the middle components $F_{\hat{v}\hat{w}}$ and $F_{\hat{\theta}\hat{\phi}}$.

Remark 4.1.4. Our proof would work with any arbitrary product of Killing Lie derivatives of F on the right hand side of assumption (4.12), with an adjustment on the initial data accordingly.

Then, we have,

$$\begin{aligned} |F_{\hat{v}\hat{w}}(v, w, \omega)| & \leq \frac{C}{\max\{1, v\}}, & |F_{e_1 e_2}(v, w, \omega)| & \leq \frac{C}{\max\{1, v\}} \\ |F_{\hat{v}e_a}(v, w, \omega)| & \leq \frac{C}{\max\{1, v\}}, & |\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_a}(v, w, \omega)| & \leq \frac{C}{\max\{1, v\}} \end{aligned}$$

in the entire exterior region, up to points on the horizon, under certain regularity conditions on the initial data prescribed in what follows, to which we also add the assumption that in an orthonormal basis the limit of the initial data, and of its derivatives which appear in the following, goes to zero at spatial infinity on the initial slice $\Sigma_{t=t_0}$.

More precisely, away from the horizon (in the region $r \geq R > 2m$), we have,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |w|)}$$

for all normalized components $\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$, and where E_F is defined by,

$$\begin{aligned} E_F &= \left[\sum_{i=0}^1 \sum_{j=0}^5 E_{r^j(\mathcal{L}_t)^i F}^{(\frac{\partial}{\partial t})}(t = t_0) + E_{r^6(\mathcal{L})^6 F}^{(\frac{\partial}{\partial t})}(t = t_0) \right. \\ &\quad \left. + \sum_{i=0}^1 \sum_{j=0}^4 E_{r^j(\mathcal{L}_t)^i F}^{(K)}(t = t_0) + E_{r^5(\mathcal{L})^5 F}^{(K)}(t = t_0) \right]^{\frac{1}{2}} \end{aligned}$$

where \mathcal{L} is the Lie derivative restricted on the 2-spheres, and where,

$$\begin{aligned} E_F^{(K)}(t_i) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (w^2(1 - \frac{2m}{r})^2 [|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2] + v^2 [|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2] \\ &\quad + (\omega^2 + v^2)(1 - \frac{2m}{r}) [|F_{\hat{v}\hat{w}}|^2 + |F_{\hat{\phi}\hat{\theta}}|^2]) r^2 \sin(\theta) d\theta d\phi dr^* \end{aligned}$$

and where,

$$\begin{aligned} E_F^{(\frac{\partial}{\partial t})}(t = t_0) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 2[(1 - \frac{2m}{r})^2 (|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + |F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2 \\ &\quad + (1 - \frac{2m}{r})(|F_{\hat{v}\hat{w}}|^2 + |F_{\hat{\phi}\hat{\theta}}|^2)] \cdot r^2 \sin(\theta) d\theta d\phi dr^* \end{aligned}$$

We expect the angular momentum derivatives, or any other Killing derivatives, in the assumption (4.8) to come out due to the presence of the trapped surface $r = 3m$. Hence, we also assume - although we can make the proof without the following assumption by using only (4.8) with the price of losing more derivatives on the initial data - that the solutions

we are looking at verify

$$\int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |F_{\hat{\phi}\hat{\theta}}|^2] \cdot |r^* - (3m)^*| \cdot dr^* d\sigma^2 dt \lesssim |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1})| \quad (4.13)$$

Then, near the horizon (in the region $2m \leq r \leq R$), we have,

$$\begin{aligned} |F_{\hat{v}\hat{w}}(v, w, \omega)| &\lesssim \frac{E_1}{\max\{1, v\}}, & |F_{e_1 e_2}(v, w, \omega)| &\lesssim \frac{E_1}{\max\{1, v\}} \\ |F_{\hat{v}e_a}(v, w, \omega)| &\lesssim \frac{E_2}{\max\{1, v\}}, & |\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_a}(v, w, \omega)| &\lesssim \frac{E_2}{\max\{1, v\}} \end{aligned}$$

for $a \in \{1, 2\}$, and where,

$$E_1 = \left[\sum_{j=0}^6 E_{r^j(\mathcal{L})^j F}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^5 E_{r^j(\mathcal{L})^j F}^{(K)}(t = t_0) + \sum_{j=1}^3 E_{r^j(\mathcal{L})^j F}^{\#(\frac{\partial}{\partial t})}(t = t_0) \right]^{\frac{1}{2}}$$

where,

$$\begin{aligned} E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\left(1 - \frac{2m}{r}\right) (|F_{\hat{w}\hat{\theta}}|^2 + |F_{\hat{w}\hat{\phi}}|^2) + (|F_{\hat{v}\hat{\theta}}|^2 + |F_{\hat{v}\hat{\phi}}|^2) \right. \\ &\quad \left. + (|F_{\hat{v}\hat{w}}|^2 + |F_{\hat{\phi}\hat{\theta}}|^2) \right] \cdot r^2 d\sigma^2 dr^*(t = t_0) \end{aligned}$$

and where,

$$\begin{aligned}
E_2 &= [E_F^2 + \sum_{i=0}^1 \sum_{j=1}^2 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3 F}^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}} \\
&= [\sum_{i=0}^1 \sum_{j=0}^5 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{(\frac{\partial}{\partial t})}(t=t_0) + E_{r^6(\mathcal{L})^6 F}^{(\frac{\partial}{\partial t})}(t=t_0) \\
&\quad + \sum_{i=0}^1 \sum_{j=0}^4 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{(K)}(t=t_0) + E_{r^5(\mathcal{L})^5 F}^{(K)}(t=t_0) \\
&\quad + \sum_{i=0}^1 \sum_{j=1}^2 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3 F}^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}}
\end{aligned}$$

4.1.5 Strategy of the proof

We decompose our proof of decay for the Maxwell fields, on the Schwarzschild black hole, into two parts. The first proves decay away from the horizon (in the region $r \geq R$ where $R > 2m$, arbitrarily fixed), and the second part deals with the region near the horizon ($2m \leq r \leq R$).

In the first part

All integrations will be done on spacelike hypersurfaces prescribed by $t = \text{constant}$, and space-time integrals will be understood as integrals on a region bounded by those hypersurfaces. The starting point of our proof is a suitable use of Sobolev inequalities. Sobolev inequalities permit one to bound the L^∞ norm of the square of the Maxwell field (the square is taken with respect to the scalar product on the Lie algebra \langle, \rangle), by the L^2 norm of the Maxwell field and its derivatives up to some order. And yet, the L^2 norm of the Maxwell fields and their derivatives in the direction of Killing vector fields can be controlled away from the horizon by the energy $E^{(\frac{\partial}{\partial t})}$ of those (energy obtained from the vector field $\frac{\partial}{\partial t}$). We lose control on the L^2 norm of the components $F_{\hat{v}\hat{w}}, F_{\hat{\theta}\hat{\phi}}, F_{\hat{w}\hat{\theta}}, F_{\hat{\theta}\hat{\phi}}$ of those fields near the horizon due to the presence of the $(1 - \frac{2m}{r})$ term (that vanishes at the

horizon) that appears in the expression of the energy. And so, since the covariant derivatives of the Maxwell fields in the direction of non-Killing vector fields can be transformed into covariant derivatives in the direction of Killing vector fields by using the field equations and the Bianchi identities, one can bound the L^2 norm of those fields by their energy which is conserved, because we have conservation of energy for the Maxwell equations and therefore for their Killing derivatives because the Maxwell equations are linear. This way, we can bound the Maxwell fields away from horizon.

However, if we prove decay of the local L^2 norms of the Maxwell fields and their derivatives in the direction of Killing vector fields, we can prove decay of the Maxwell fields. The key point is that the L^2 norms here can be taken to be space integrals on only a bounded region. This way, away from the horizon, they can be controlled by a piece of the energy integral, that is the energy as a space integral without integrating on the whole space, but only on the bounded region (this is because the terms that appear in the space integral of the energy are exactly the squares of the Maxwell fields, multiplied by the term $(1 - \frac{2m}{r})$ for some components). These energies taken on a bounded region of space, can be bounded by the conformal energy (obtained by taking the Morawetz vector field $K = -\omega^2 \frac{\partial}{\partial \omega} - v^2 \frac{\partial}{\partial v}$) divided by the minimum on that region of v^2 and w^2 (see (4.72)). Consequently, if we bound the conformal energy, we have shown so far how one can possibly obtain decay from this of solutions to the Maxwell equations away from the horizon.

To bound the conformal energy of the Maxwell fields and their derivatives in the direction of Killing vector fields, we proceed as in the following enumerated two steps:

1. We will use the space-time integral $J_F^{(G)}$ in our assumption (4.8), of which the terms are positive, to control $J_F^{(K)}$ (the space-time integral obtained by using the divergence theorem on the Morawetz vector field K contracted with the energy momentum tensor of the Maxwell fields) in the following sense, see estimate (4.55):

$$J_F^{(K)}(t_i \leq t \leq t_{i+1}) \lesssim t_{i+1} J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \quad (4.14)$$

and from our assumption (4.8), we have

$$\begin{aligned} & J_F^{(G)}(t_i \leq t \leq t_{i+1}) \\ & \lesssim |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1})| + \sum_{j=1}^3 (|\hat{E}_{\mathcal{L}\Omega_j F}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\mathcal{L}\Omega_j F}^{(\frac{\partial}{\partial t})}(t_{i+1})|) \end{aligned}$$

where Ω_j , $j \in \{1, 2, 3\}$, is a basis of angular momentum derivatives, and $\hat{E}_F^{(\frac{\partial}{\partial t})}(t)$ is the energy without the middle components, see (4.9).

2. We construct a new field, \hat{F} , such that it verifies the Maxwell equations and the Bianchi identities, and it coincides with the original field in some region and vanishes identically outside another specific region for the components which appear in the boundary terms $\hat{E}_F^{(\frac{\partial}{\partial t})}(t)$, so that we could write:

$$|\hat{E}_{\hat{F}}^{(\frac{\partial}{\partial t})}(t)| \lesssim \frac{E_F^{(K)}(t)}{t^2} \quad (4.15)$$

and consequently (see (4.58)):

$$\begin{aligned} & J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \\ & \lesssim \frac{1}{t_i^2} E_F^{(K)}(t = t_i) + \frac{1}{t_i^2} \sum_j E_{\Omega_j F}^{(K)}(t = t_i) \\ & \quad + \frac{1}{t_{i+1}^2} E_F^{(K)}(t = t_{i+1}) + \frac{1}{t_{i+1}^2} \sum_j E_{\Omega_j F}^{(K)}(t = t_{i+1}) \end{aligned} \quad (4.16)$$

For t_i such that $t_{i+1} = (1.1)t_i$ (where i is an integer), and $|r^*(r_0)| + |r^*(R)| \leq 0.4t_i$, we

can make use of the divergence theorem to properly commutate these inequalities, and use the fact that the series $\sum \frac{i}{t_i}$ converges, to establish a uniform bound on the conformal energy that depends on the initial data and its Killing Lie derivatives derivatives. We obtain (4.74):

$$\begin{aligned}
& E_F^{(K)}(t) \\
\lesssim & E_{F,r}^{(\frac{\partial}{\partial t})} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_{F}(t=t_0) + E_{F,r}^{(K)} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F}(t=t_0) = E_F^M
\end{aligned} \tag{4.17}$$

In the second part

To obtain decay near the horizon, we are going to integrate in rectangles in the Penrose diagram representing the exterior of the Schwarzschild black hole, of which one side is included in the horizon. We will apply the divergence theorem with the vector field $H = -\frac{h(r^*)}{(1-\frac{2m}{r})} \frac{\partial}{\partial w} - h(r^*) \frac{\partial}{\partial v}$ contracted with the energy momentum tensor of the Maxwell fields, where $h \geq 0$ is supported in the region $2m \leq r \leq (1.2)r_1$ for r_1 chosen such that, $2m < r_0 \leq r_1 < (1.2)r_1 < 3m$, and where h is such that $\lim_{r^* \rightarrow -\infty} h(r^*) = 1$, and for $r \leq r_1$, we have $h > 0$, $h' \geq 0$, $h' \leq \frac{2m}{r^2}h$, $(1 - \frac{2m}{r})\frac{3}{r}h \leq h'$, and $(1 + \frac{6m}{r^2})h \leq \frac{2mh'}{(r-2m)}$. By applying the divergence theorem in the rectangles described previously with H defined as such, we get that the flux through the hypersurfaces prescribed by $v = constant$ is roughly speaking the L^2 norms of $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}\hat{\theta}}$ and $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}\hat{\phi}}$, and the L^2 norms of $\sqrt{1 - \frac{2m}{r}}F_{\hat{v}\hat{w}}$ and $\sqrt{1 - \frac{2m}{r}}F_{\hat{\theta}\hat{\phi}}$. In addition, we get a space-time integral supported near the event horizon $-I_F^{(H)}$, of which the terms are roughly speaking the squares of $F_{\hat{w}\hat{\theta}}$, and $F_{\hat{w}\hat{\phi}}$, and roughly a factor that goes to zero when r goes to $2m$ multiplied by the squares of $F_{\hat{v}\hat{w}}$, $F_{\hat{\theta}\hat{\phi}}$, $F_{\hat{v}\hat{\theta}}$, and $F_{\hat{v}\hat{\phi}}$.

3. In order to prove decay for the flux of H , that is a generalized energy that would

control the L^2 norm of some components of the fields near the horizon, one is confronted to a situation where it seems crucial to control a space-time integral supported on a bounded region in space near the event horizon, that contains the non-middle components. This could be overcome by the assumed Morawetz estimate, (4.8), that could be used to bound the space-time integral of the non-middle components as well. Just to simplify the calculations we assume that we have (4.13), since we expect the derivatives in assumption (4.8) to come out due to the trapped surface $r = 3m$, i.e.

$$\begin{aligned} & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|F_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |F_{\hat{\phi}\hat{\theta}}|^2] \cdot |r^* - (3m)^*| \cdot dr^* d\sigma^2 dt \\ & \lesssim | \hat{E}_F^{(\frac{\partial}{\partial t})}(t_i) | + | \hat{E}_F^{(\frac{\partial}{\partial t})}(t_{i+1}) | \end{aligned} \quad (4.18)$$

Hence, away from the horizon, in $r \geq r_1$, the space-time integral, $|I_F^{(H)}|$, can be bounded by the standard energy supported on a bounded region in space, to which one can prove decay due to the boundedness of the conformal energy that we would have already established in the first part.

Near the horizon, in $r \leq r_1$, the choices $h' \geq 0$, $h' \leq \frac{2m}{r^2}h$, $(1 - \frac{2m}{r})\frac{3}{r}h \leq h'$ were constructed on purpose to obtain $0 \leq -I_F^{(H)}(r \leq r_1)$. This last fact will lead to an inequality on $-I_F^{(H)}$, that involve the flux of the standard energy that one can bound, and that from the vector field H on $v = \text{constant}$ hypersurface, (4.107), for $v_i = t_i + r_1^*$, and $w_i = t_i - r_1^*$, where t_i is defined as in the first part, $t_i = (1.1)^i t_0$:

$$\begin{aligned} & -I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\ & -F_F^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) - F_F^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) \\ & \lesssim F_F^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_F^{(H)}(v = v_i)(w_i \leq w \leq \infty) \\ & + |E_F^{(\frac{\partial}{\partial t})}(-0.85t_i \leq r^* \leq 0.85t_i)(t = t_i)| \end{aligned} \quad (4.19)$$

Using the Cauchy stability one can bound the flux from H by the initial data prescribed on the initial Cauchy hypersurface, and hence bound the space-time integral $-I_F^{(H)}(r \leq r_1)$. In addition, we can prove inequality (4.111):

$$\begin{aligned}
& \inf_{v_i \leq v \leq v_{i+1}} -F_F^{(H)}(v)(w_i \leq w \leq \infty) \\
\lesssim & \frac{-I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)}{(v_{i+1} - v_i)} \\
& + \sup_{v_i \leq v \leq v_{i+1}} F_F^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \tag{4.20}
\end{aligned}$$

To prove decay to the flux from H on $v = \text{constant}$ hypersurfaces near the horizon and on $w = \text{constant}$ hypersurfaces on a segment of fixed length in v , we apply the last two inequalities above in the rectangle prescribed by $[v_i, v_{i+1}].[w_i, \infty]$, and we commute them properly. This will lead to decay in $v_+ = \max\{1, v\}$ of the flux from H on $v = \text{constant}$ near the horizon, and on $w = \text{constant}$ hypersurfaces with a fixed length in v , as shown in estimates (4.119) and (4.120)):

$$-F_F^{(H)}(v)(2m \leq r \leq R) \lesssim \frac{[|E_F^{(\frac{\partial}{\partial t})}| + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^M]}{v_+^2} \tag{4.21}$$

and,

$$-F_F^{(H)}(w)(v - 1 \leq \bar{v} \leq v) \lesssim \frac{[|E_F^{(\frac{\partial}{\partial t})}| + E_F^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_F^M]}{v_+^2} \tag{4.22}$$

Finally:

4. To prove decay for the normalized components $F_{\hat{v}\hat{w}}$ and $F_{e_1 e_2}$ we make use of a Sobolev inequality restricted on $w = \text{constant}$ hypersurfaces with a fixed length in v . Using the field equations, the L^2 norms of derivatives in the direction of $\frac{\partial}{\partial v}$ can be controlled by the L^2 norms of $F_{\hat{v}e_1}$, $F_{\hat{v}e_2}$ and of their angular derivatives, and of

$F_{\hat{v}\hat{w}}$ and $F_{e_1e_2}$. This leads to a bound by the flux obtained from H on $w = \text{constant}$ hypersurfaces, of F and its angular momentum derivatives. Since, those are Killing derivatives, and since the Maxwell equations are linear, and we proved decay of for the flux, this leads to the desired result.

We do the same to prove decay for the components $F_{\hat{v}e_1}$ and $F_{\hat{v}e_2}$ except that this time, the L^2 norms of the derivatives in the direction of $\frac{\partial}{\partial v}$ can be bounded by the L^2 norms of the time derivatives of $F_{\hat{v}e_1}$, $F_{\hat{v}e_2}$, and of the angular derivatives of $F_{\hat{v}\hat{w}}$, $F_{e_1e_2}$, and of $F_{\hat{v}e_1}$, $F_{\hat{v}e_2}$, using the field equations and the Bianchi identities.

The components $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_1}$ and $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_2}$ can be controlled by using a Sobolev inequality where we integrate on the hypersurfaces $v = \text{constant}$. As a result of direct computation using the field equations and the Bianchi identities, the L^2 norms of derivatives in the direction of $\frac{\partial}{\partial w}$ can be controlled by the L^2 norms of time derivatives of $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_1}$, $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_2}$, of angular derivatives of $\sqrt{1 - \frac{2m}{r}}F_{\hat{v}\hat{w}}$, $\sqrt{1 - \frac{2m}{r}}F_{e_1e_2}$, and of $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_1}$, $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_2}$. Making use of the decay of the flux from the vector field H on $v = \text{constant}$, we obtain decay of these local L^2 norms, and hence we prove pointwise decay for $\sqrt{1 - \frac{2m}{r}}F_{\hat{w}e_a}$, $a \in \{1, 2\}$.

In order to cover the whole exterior region in theorem (4.1.3), we chose in the first part $R = r_1$, and we can choose $r_1 = r_0$.

4.2 Conservation Laws

Let $\Psi_{\mu\nu}$ be a an anti-symmetric two field valued in the Lie algebra. Consider the energy-momentum tensor

$$T_{\mu\nu}(\Psi) = \langle \Psi_{\mu\beta}, \Psi_{\nu}^{\beta} \rangle - \frac{1}{4} \mathbf{g}_{\mu\nu} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \quad (4.23)$$

Considering the Schwarzschild time t' (see (6.112) in Appendix), and considering two space-like hypersurfaces $\Sigma_{t'_1}, \Sigma_{t'_2}, t'_2 > t'_1$. We consider the region $B = J^+(\Sigma_{t'_1}) \cap J^-(\Sigma_{t'_2}) \cap D$, where D is the closure of the exterior region of the black hole, known as the domain of outer-communication of the black hole.

Considering a vector field V^ν we let

$$J_\mu(V) = V^\nu T_{\mu\nu} = T_{\mu V}$$

We have,

$$\nabla^\mu J_\mu(V) = \partial^\mu T_{\mu V} - T(\nabla^\mu e_\mu, V) = \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V)$$

Hence,

$$\begin{aligned} \nabla^\mu J_\mu(V) &= \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V) \\ &= V^\nu (\nabla^\mu T_{\mu\nu}) + (\nabla^\mu V^\nu) T_{\mu\nu} \\ &= V^\nu (\nabla^\mu T_{\mu\nu}) + \pi^{\mu\nu} T_{\mu\nu} \end{aligned}$$

(by symmetry of T)

Applying the divergence theorem on $J_\mu(V)$ in the region B bounded to the past by $\Sigma_{t'_1}$ and to the future by $\Sigma_{t'_2}$, and by a null hypersurface N , we obtain:

$$\begin{aligned} \int_B V^\nu (\nabla^\mu T_{\mu\nu}) dV_B + \int_B \pi^{\mu\nu} T_{\mu\nu} dV_B &= \int_{\Sigma_{t'_1}} J_\mu(V) n^\mu dV_{\Sigma_{t'_1}} - \int_{\Sigma_{t'_2}} J_\mu(V) n^\mu dV_{\Sigma_{t'_2}} \\ &\quad - \int_N J_\mu(V) n_N^\mu dV_N \end{aligned} \quad (4.24)$$

where n^μ are the unit normal to the hypersurfaces Σ , n_N^μ is any null generator of N , dV_Σ are

the induced volume forms and dV_N is defined such that the divergence theorem applies.

Considering the Maxwell field F , as it verifies (4.10) and (4.11), we have

$$\nabla^\nu T_{\mu\nu}(F) = 0 \tag{4.25}$$

Also, considering the spherical symmetry of the Schwarzschild black hole, the angular momentum operators are Killing and therefore $\mathcal{L}_{\Omega_j}F$ verifies the Maxwell equations(4.10) and (4.11), and thus

$$\nabla^\nu T_{\mu\nu}(\mathcal{L}_{\Omega_j}F) = 0$$

where Ω_j is a basis of angular momentum operators, $j \in \{1, 2, 3\}$.

And given that $\frac{\partial}{\partial t}$ is Killing, we have

$$\nabla^\nu T_{\mu\nu}(\mathcal{L}_t F) = 0$$

Taking Ψ any product of \mathcal{L}_t , \mathcal{L}_{Ω_j} , and F , where Ω_j , $j \in \{1, 2, 3\}$, is a basis of angular momentum operators, we get that Ψ satisfies the Maxwell equations (4.10) and (4.11) since $\frac{\partial}{\partial t}$ and Ω_j , $j \in \{1, 2, 3\}$ are Killing vector fields. Hence,

$$\nabla^\nu T_{\mu\nu}(\Psi) = 0 \tag{4.26}$$

Let,

$$\mu = \frac{2m}{r} \tag{4.27}$$

Now, let's compute,

$$\begin{aligned}
& \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{w\alpha}, \Psi^{w\alpha} \rangle + \langle \Psi_{v\alpha}, \Psi^{v\alpha} \rangle + \langle \Psi_{\theta\alpha}, \Psi^{\theta\alpha} \rangle + \langle \Psi_{\phi\alpha}, \Psi^{\phi\alpha} \rangle \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{w\alpha}, \Psi_v^\alpha \rangle + \frac{-2}{(1-\mu)} \langle \Psi_{v\alpha}, \Psi_w^\alpha \rangle + \frac{1}{r^2} \langle \Psi_{\theta\alpha}, \Psi_\theta^\alpha \rangle \\
&\quad + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{\phi\alpha}, \Psi_\phi^\alpha \rangle \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{w\theta}, \Psi_v^\theta \rangle - \frac{2}{(1-\mu)} \langle \Psi_{w\phi}, \Psi_v^\phi \rangle - \frac{2}{(1-\mu)} \langle \Psi_{wv}, \Psi_v^v \rangle \\
&\quad - \frac{2}{(1-\mu)} \langle \Psi_{v\theta}, \Psi_w^\theta \rangle - \frac{2}{(1-\mu)} \langle \Psi_{v\phi}, \Psi_w^\phi \rangle - \frac{2}{(1-\mu)} \langle \Psi_{vw}, \Psi_w^w \rangle \\
&\quad + \frac{1}{r^2} \langle \Psi_{\theta v}, \Psi_\theta^v \rangle + \frac{1}{r^2} \langle \Psi_{\theta w}, \Psi_\theta^w \rangle + \frac{1}{r^2} \langle \Psi_{\theta\phi}, \Psi_\theta^\phi \rangle \\
&\quad + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{\phi v}, \Psi_\phi^v \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{\phi w}, \Psi_\phi^w \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{\phi\theta}, \Psi_\phi^\theta \rangle \\
&= \frac{-2}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{2}{r^2 \sin^2 \theta(1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle + \frac{4}{(1-\mu)^2} \langle \Psi_{wv}, \Psi_{vw} \rangle \\
&\quad - \frac{2}{r^2(1-\mu)} \langle \Psi_{v\theta}, \Psi_{w\theta} \rangle - \frac{2}{r^2 \sin^2 \theta(1-\mu)} \langle \Psi_{v\phi}, \Psi_{w\phi} \rangle + \frac{4}{(1-\mu)^2} \langle \Psi_{vw}, \Psi_{wv} \rangle \\
&\quad - \frac{2}{r^2(1-\mu)} \langle \Psi_{\theta v}, \Psi_{\theta v} \rangle - \frac{2}{r^2(1-\mu)} \langle \Psi_{\theta w}, \Psi_{\theta w} \rangle + \frac{1}{r^4 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \\
&\quad - \frac{2}{r^2 \sin^2 \theta(1-\mu)} \langle \Psi_{\phi v}, \Psi_{\phi w} \rangle - \frac{2}{r^2 \sin^2 \theta(1-\mu)} \langle \Psi_{\phi w}, \Psi_{\phi v} \rangle + \frac{1}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \\
&= \frac{-8}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{8}{r^2 \sin^2 \theta(1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle - \frac{8}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2
\end{aligned}$$

Computing,

$$\begin{aligned}
T_{ww} &= \langle \Psi_{w\beta}, \Psi_w^\beta \rangle - \frac{1}{2} \mathbf{g}_{ww} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{wv}, \Psi_w^v \rangle + \langle \Psi_{w\theta}, \Psi_w^\theta \rangle + \langle \Psi_{w\phi}, \Psi_w^\phi \rangle \\
&= \frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2
\end{aligned}$$

$$\begin{aligned}
T_{vv} &= \langle \Psi_{v\alpha}, \Psi_v^\alpha \rangle \\
&= \langle \Psi_{vw}, \Psi_v^w \rangle + \langle \Psi_{v\theta}, \Psi_v^\theta \rangle + \langle \Psi_{v\phi}, \Psi_v^\phi \rangle \\
&= \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2
\end{aligned}$$

$$\begin{aligned}
&T_{vw} \\
&= \langle \Psi_{v\alpha}, \Psi_w^\alpha \rangle - \frac{1}{4} \mathbf{g}_{vw} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{vw}, \Psi_w^w \rangle + \langle \Psi_{v\theta}, \Psi_w^\theta \rangle + \langle \Psi_{v\phi}, \Psi_w^\phi \rangle - \frac{1}{4} \mathbf{g}_{vw} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{vw}, \Psi_{vw} \rangle + \frac{1}{r^2} \langle \Psi_{v\theta}, \Psi_{w\theta} \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{v\phi}, \Psi_{w\phi} \rangle \\
&\quad + \frac{(1-\mu)}{8} \left[\frac{-8}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{8}{r^2 \sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \right. \\
&\quad \left. - \frac{8}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
&= \frac{2}{(1-\mu)} |\Psi_{vw}|^2 + \frac{1}{r^2} \langle \Psi_{v\theta}, \Psi_{w\theta} \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{v\phi}, \Psi_{w\phi} \rangle \\
&\quad - \frac{1}{r^2} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \\
&\quad - \frac{1}{(1-\mu)} |\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \\
&= \frac{1}{(1-\mu)} |\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2
\end{aligned}$$

$$\begin{aligned}
& T_{\theta\theta} \\
&= \langle \Psi_{\theta\alpha}, \Psi_{\theta}^{\alpha} \rangle - \frac{1}{4} \mathbf{g}_{\theta\theta} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{\theta v}, \Psi_{\theta}^v \rangle + \langle \Psi_{\theta w}, \Psi_{\theta}^w \rangle + \langle \Psi_{\theta\phi}, \Psi_{\theta}^{\phi} \rangle \\
&\quad - \frac{r^2}{4} \left[\frac{-8}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{8}{r^2 \sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \right. \\
&\quad \left. - \frac{8}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{\theta v}, \Psi_{\theta w} \rangle - \frac{2}{(1-\mu)} \langle \Psi_{\theta w}, \Psi_{\theta v} \rangle + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \\
&\quad + \frac{2}{(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle + \frac{2}{\sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \\
&\quad + \frac{2r^2}{(1-\mu)^2} |\Psi_{vw}|^2 - \frac{1}{2r^2 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{\theta v}, \Psi_{\theta w} \rangle + \frac{2}{\sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \\
&\quad + \frac{2r^2}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{2r^2 \sin^2 \theta} |\Psi_{\phi\theta}|^2
\end{aligned}$$

$$\begin{aligned}
& T_{\phi\phi} \\
&= \langle \Psi_{\phi\alpha}, \Psi_{\phi}^{\alpha} \rangle - \frac{1}{4} \mathbf{g}_{\phi\phi} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{\phi v}, \Psi_{\phi}^v \rangle + \langle \Psi_{\phi w}, \Psi_{\phi}^w \rangle + \langle \Psi_{\phi\theta}, \Psi_{\phi}^{\theta} \rangle \\
&\quad - \frac{r^2 \sin^2 \theta}{4} \left[\frac{-8}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle - \frac{8}{r^2 \sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \right. \\
&\quad \left. - \frac{8}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{\phi v}, \Psi_{\phi w} \rangle - \frac{2}{(1-\mu)} \langle \Psi_{\phi w}, \Psi_{\phi v} \rangle + \frac{1}{r^2} |\Psi_{\theta\phi}|^2 \\
&\quad + \frac{2 \sin^2 \theta}{(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle + \frac{2}{(1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \\
&\quad + \frac{2r^2 \sin^2 \theta}{(1-\mu)^2} |\Psi_{vw}|^2 - \frac{1}{2r^2} |\Psi_{\phi\theta}|^2 \\
&= \frac{-2}{(1-\mu)} \langle \Psi_{\phi v}, \Psi_{\phi w} \rangle + \frac{2 \sin^2 \theta}{(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle \\
&\quad + \frac{2r^2 \sin^2 \theta}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{2r^2} |\Psi_{\phi\theta}|^2
\end{aligned}$$

This gives,

$$\begin{aligned}
& \pi^{\alpha\beta}T_{\alpha\beta} \\
&= T_{\omega\omega}\pi^{\omega\omega} + T_{vv}\pi^{vv} + 2T_{v\omega}\pi^{v\omega} + T_{\theta\theta}\pi^{\theta\theta} + T_{\phi\phi}\pi^{\phi\phi} \\
&= \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_v V^w\right) \\
&\quad + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_w V^v\right) \\
&\quad + 2\left[\frac{1}{(1-\mu)}|\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2\right]\left(\frac{-1}{(1-\mu)}[\partial_v V^v + \partial_w V^w + \frac{m}{r^2}(V^v - V^w)]\right) \\
&\quad + \left[\frac{-2}{(1-\mu)}\langle \Psi_{\theta v}, \Psi_{\theta w} \rangle + \frac{2}{\sin^2\theta(1-\mu)}\langle \Psi_{w\phi}, \Psi_{v\phi} \rangle\right] \\
&\quad + \frac{2r^2}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{2r^2\sin^2\theta}|\Psi_{\phi\theta}|^2\left(\frac{(1-\mu)}{2r^3}(V^v - V^w)\right) \\
&\quad + \left[\frac{-2}{(1-\mu)}\langle \Psi_{\phi v}, \Psi_{\phi w} \rangle + \frac{2\sin^2\theta}{(1-\mu)}\langle \Psi_{w\theta}, \Psi_{v\theta} \rangle\right] \\
&\quad + \frac{2r^2\sin^2\theta}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{2r^2}|\Psi_{\phi\theta}|^2\left(\frac{(1-\mu)}{2r^3\sin^2\theta}(V^v - V^w)\right) \\
&= \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_v V^w\right) \\
&\quad + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_w V^v\right) \\
&\quad + \left[\frac{1}{(1-\mu)}|\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2\right]\left(\frac{-2}{(1-\mu)}[\partial_v V^v + \partial_w V^w + \frac{m}{r^2}(V^v - V^w)]\right) \\
&\quad + \left[\frac{4r^2}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{\phi\theta}|^2\right]\left(\frac{(1-\mu)}{2r^3}(V^v - V^w)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
& \pi^{\alpha\beta}(V)T_{\alpha\beta}(\Psi) \\
&= \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_v V^w\right) \tag{4.28} \\
&\quad + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2\right]\left(\frac{-2}{(1-\mu)}\partial_w V^v\right) \\
&\quad + \left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2\right]\left(-2[\partial_v V^v + \partial_w V^w + \frac{(3\mu-2)}{2r}(V^v - V^w)]\right)
\end{aligned}$$

4.2.1 The vector field $\frac{\partial}{\partial t}$

Let,

$$t^\gamma = \left(\frac{\partial}{\partial t}\right)^\gamma$$

We have,

$$\begin{aligned} \frac{\partial}{\partial v} &= \frac{\partial t}{\partial v} \frac{\partial}{\partial t} + \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} \\ &= \frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*} \end{aligned} \quad (4.29)$$

$$\begin{aligned} \frac{\partial}{\partial w} &= \frac{\partial t}{\partial w} \frac{\partial}{\partial t} + \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*} \\ &= \frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*} \end{aligned} \quad (4.30)$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial v} + \frac{\partial}{\partial w} \\ &= t^v \frac{\partial}{\partial v} + t^w \frac{\partial}{\partial w} \end{aligned}$$

where $t^v = 1$ and $t^w = 1$. Thus,

$$\begin{aligned} &\pi^{\alpha\beta} \left(\frac{\partial}{\partial t}\right) T_{\alpha\beta}(\Psi) \\ &= \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_v t^w \right) \\ &\quad + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_w t^v \right) \\ &\quad + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left(-2[\partial_v t^v + \partial_w t^w + \frac{(3\mu-2)}{2r}(t^v - t^w)] \right) \\ &= 0 \end{aligned}$$

In fact, since $\frac{\partial}{\partial t}$ is Killing, its deformation tensor vanishes, i.e. $\pi^{\alpha\beta}(\frac{\partial}{\partial t}) = 0$, therefore

$$\pi^{\alpha\beta}(\frac{\partial}{\partial t})T_{\alpha\beta}(\Psi) = 0 \quad (4.31)$$

Let,

$$\frac{\hat{\partial}}{\partial t} = \frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial t} \quad (4.32)$$

$$\frac{\hat{\partial}}{\partial r^*} = \frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial r^*} \quad (4.33)$$

We also assume that,

$$\lim_{r \rightarrow \infty} \Psi_{\hat{\mu}\hat{\nu}}(t = t_0) = 0 \quad (4.34)$$

From a local existence result that ensures that a certain regularity will be conserved, one can prove that the condition above will be satisfied for all time, i.e.

$$\lim_{r \rightarrow \infty} \Psi_{\hat{\mu}\hat{\nu}}(t) = 0 \quad (4.35)$$

Thus, we will have no integrals on spatial infinity.

Applying the divergence theorem to the vector $t^\mu T_{\mu\nu}$ in the region B bounded by two hypersurfaces Σ_{t_1} and Σ_{t_2} defined by $t = \text{constant}$, where $t_2 \geq t_1$, we get,

$$\begin{aligned} & \int_B \pi^{\alpha\beta}(\frac{\partial}{\partial t})T_{\alpha\beta}(\Psi)dV_B + \int_B t^\nu(\nabla^\mu T_{\mu\nu}(\Psi))dV_B \\ &= \int_B t^\nu(\nabla^\mu T_{\mu\nu}(\Psi))dV_B \end{aligned}$$

$$\begin{aligned}
&= \int_{\Sigma_{t_1}} J_\mu \left(\frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial \hat{t}} \right)^\mu dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} J_\mu \left(\frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial \hat{t}} \right)^\mu dV_{\Sigma_{t_2}} \\
&= \int_{\Sigma_{t_1}} J_{\hat{t}} \left(\frac{\partial}{\partial t} \right) dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} J_{\hat{t}} \left(\frac{\partial}{\partial t} \right) dV_{\Sigma_{t_2}} \\
&= \int_{\Sigma_{t_1}} T_{\hat{t}\hat{t}}(\Psi) dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} T_{\hat{t}\hat{t}}(\Psi) dV_{\Sigma_{t_2}} \\
&= \int_{\Sigma_{t_1}} T_{\hat{t}\hat{t}}(\Psi) (\sqrt{1-\mu}) dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} T_{\hat{t}\hat{t}}(\Psi) (\sqrt{1-\mu}) dV_{\Sigma_{t_2}}
\end{aligned}$$

We have,

$$dV_{\Sigma_t} = (\sqrt{(1-\mu)r^4 \sin^2(\theta)}) dr^* d\theta d\phi = (\sqrt{1-\mu}) r^2 dr^* \sin(\theta) d\theta d\phi$$

(we have $\sin(\theta) \geq 0$ because $0 \leq \theta \leq \pi$). Thus,

$$dV_{\Sigma_t} = (\sqrt{1-\mu}) r^2 dr^* d\sigma^2$$

On the other hand,

$$\begin{aligned}
&T_{\hat{t}\hat{t}}(\Psi) \\
&= \langle \Psi_{\hat{t}\beta}, \Psi_{\hat{t}}^\beta \rangle - \frac{1}{4} \mathbf{g}_{\hat{t}\hat{t}} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle \\
&= \langle \Psi_{\hat{t}\beta}, \Psi_{\hat{t}}^\beta \rangle + \frac{1}{4} \left[\frac{-8}{r^2(1-\mu)} \langle \Psi_{w\theta}, \Psi_{v\theta} \rangle + \frac{-8}{r^2 \sin^2 \theta (1-\mu)} \langle \Psi_{w\phi}, \Psi_{v\phi} \rangle \right. \\
&\quad \left. + \frac{-8}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
&= \mathbf{g}^{\hat{\beta}\hat{\gamma}} \langle \Psi_{\hat{t}\beta}, \Psi_{\hat{t}\hat{\gamma}} \rangle + [-2 \langle \Psi_{\hat{w}\hat{\theta}}, \Psi_{\hat{v}\hat{\theta}} \rangle - 2 \langle \Psi_{\hat{w}\hat{\phi}}, \Psi_{\hat{v}\hat{\phi}} \rangle - 2 |\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{2} |\Psi_{\hat{\phi}\hat{\theta}}|^2]
\end{aligned}$$

$$\begin{aligned}
\langle \Psi_{\hat{t}\beta}, \Psi_{\hat{t}}^\beta \rangle &= - \langle \Psi_{\hat{t}\hat{t}}, \Psi_{\hat{t}\hat{t}} \rangle + \langle \Psi_{\hat{t}\hat{r}^*}, \Psi_{\hat{t}\hat{r}^*} \rangle + \langle \Psi_{\hat{t}\hat{\theta}}, \Psi_{\hat{t}\hat{\theta}} \rangle + \langle \Psi_{\hat{t}\hat{\phi}}, \Psi_{\hat{t}\hat{\phi}} \rangle \\
&= |\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2
\end{aligned}$$

From (4.29) and (4.30), we get,

$$\begin{aligned}
\Psi_{v\theta} &= \Psi_{\mu\nu} \left(\frac{\partial}{\partial v}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu = \Psi_{\mu\nu} \left(\frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu \\
&= \frac{1}{2} \Psi_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu + \frac{1}{2} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu \\
&= \frac{1}{2} \Psi_{t\theta} + \frac{1}{2} \Psi_{r^*\theta}
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\Psi_{w\theta} &= \Psi_{\mu\nu} \left(\frac{\partial}{\partial w}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu = \Psi_{\mu\nu} \left(\frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu \\
&= \frac{1}{2} \Psi_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu - \frac{1}{2} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial \theta}\right)^\nu \\
&= \frac{1}{2} \Psi_{t\theta} - \frac{1}{2} \Psi_{r^*\theta}
\end{aligned} \tag{4.37}$$

and similarly,

$$\Psi_{v\phi} = \frac{1}{2} \Psi_{t\phi} + \frac{1}{2} \Psi_{r^*\phi} \tag{4.38}$$

$$\Psi_{w\phi} = \frac{1}{2} \Psi_{t\phi} - \frac{1}{2} \Psi_{r^*\phi} \tag{4.39}$$

Thus,

$$\begin{aligned}
\langle \Psi_{\hat{w}\hat{\theta}}, \Psi_{\hat{v}\hat{\theta}} \rangle &= \left\langle \frac{\Psi_{\hat{t}\hat{\theta}} - \Psi_{\hat{r}^*\hat{\theta}}}{2}, \frac{\Psi_{\hat{t}\hat{\theta}} + \Psi_{\hat{r}^*\hat{\theta}}}{2} \right\rangle \\
&= \frac{1}{4} [|\Psi_{\hat{t}\hat{\theta}}|^2 - |\Psi_{\hat{r}^*\hat{\theta}}|^2]
\end{aligned}$$

and,

$$\langle \Psi_{\hat{w}\hat{\phi}}, \Psi_{\hat{v}\hat{\phi}} \rangle = \frac{1}{4} [|\Psi_{\hat{t}\hat{\phi}}|^2 - |\Psi_{\hat{r}^*\hat{\phi}}|^2]$$

We have,

$$\begin{aligned} & \Psi_{vw} \\ = & \Psi_{\mu\nu} \left(\frac{\partial}{\partial v}\right)^\mu \left(\frac{\partial}{\partial w}\right)^\nu = \Psi_{\mu\nu} \left(\frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*}\right)^\mu \left(\frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*}\right)^\nu \\ = & \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r^*}\right)^\nu + \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r^*}\right)^\nu \\ = & \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu \left(\frac{\partial}{\partial t}\right)^\nu - \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial t}\right)^\mu \left(\frac{\partial}{\partial r^*}\right)^\nu + \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial t}\right)^\nu - \frac{1}{4} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial r^*}\right)^\nu \end{aligned}$$

Since $\Psi_{\mu\nu}$ is anti-symmetric two tensor, we get,

$$\begin{aligned} \Psi_{vw} &= \frac{1}{2} \Psi_{\mu\nu} \left(\frac{\partial}{\partial r^*}\right)^\mu \left(\frac{\partial}{\partial t}\right)^\nu \\ &= \frac{1}{2} \Psi_{r^*t} \end{aligned} \tag{4.40}$$

Therefore,

$$\begin{aligned} & T_{\hat{t}\hat{t}}(\Psi) \\ = & |\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{2} [|\Psi_{\hat{r}^*\hat{\theta}}|^2 - |\Psi_{\hat{t}\hat{\theta}}|^2] + \frac{1}{2} [|\Psi_{\hat{r}^*\hat{\phi}} - |\Psi_{\hat{t}\hat{\phi}}|^2|^2] - \frac{1}{2} |\Psi_{\hat{r}^*\hat{t}}|^2 + \frac{1}{2} |\Psi_{\hat{\phi}\hat{\theta}}|^2 \\ = & \frac{1}{2} [|\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2] \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_B T^\nu (\nabla^\mu T_{\mu\nu}(\Psi)) dV_B \\
&= \int_{\Sigma_{t_1}} T_{\hat{t}\hat{t}}(T)(\sqrt{1-\mu}) dV_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} J_{\hat{t}\hat{t}}(T)(\sqrt{1-\mu}) dV_{\Sigma_{t_2}} \\
&= \int_{\Sigma_{t_1}} \frac{1}{2} [|\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2] (1-\mu) r^2 dr^* d\sigma^2 \\
&\quad - \int_{\Sigma_{t_2}} \frac{1}{2} [|\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2] (1-\mu) r^2 dr^* d\sigma^2 \\
&= E_\Psi^{(\frac{\partial}{\partial t})}(t=t_2) - E_\Psi^{(\frac{\partial}{\partial t})}(t=t_1) \tag{4.41}
\end{aligned}$$

where,

$$\begin{aligned}
& E_\Psi^{(\frac{\partial}{\partial t})}(t=t_i) \tag{4.42} \\
&= \int_{\Sigma_{t_i}} -\frac{1}{2} [|\Psi_{\hat{t}\hat{r}^*}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2] (1-\mu) r^2 dr^* d\sigma^2
\end{aligned}$$

Taking Ψ any product of \mathcal{L}_t , \mathcal{L}_{Ω_j} , and F , where Ω_j , $j \in \{1, 2, 3\}$, is a basis of angular momentum operators, since $\int_B T^\nu (\nabla^\mu T_{\mu\nu}(\Psi)) dV_B = 0$, we have conservation of the energy generated from the vector field $\frac{\partial}{\partial t}$.

4.2.2 The vector field K

Let

$$\begin{aligned}
K &= -\omega^2 \frac{\partial}{\partial \omega} - v^2 \frac{\partial}{\partial v} \tag{4.43} \\
&= K^\omega \frac{\partial}{\partial \omega} + K^v \frac{\partial}{\partial v}
\end{aligned}$$

We have,

$$\begin{aligned}
\partial_v K^\omega &= -\partial_v \omega^2 \\
&= 0 \\
\partial_\omega K^v &= -\partial_\omega v^2 \\
&= 0
\end{aligned}$$

Computing,

$$\begin{aligned}
&\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) \\
&= \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_v K^w \right) \\
&\quad + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_w K^v \right) \\
&\quad + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left(-2[\partial_v K^v + \partial_w K^w + \frac{(3\mu-2)}{2r}(K^v - K^w)] \right) \\
&= \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left(-2[-2v - 2w + \frac{(3\mu-2)}{2r}(-v^2 + w^2)] \right) \\
&= \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left[4(v+w) + \frac{(3\mu-2)}{r}(w^2 - v^2) \right]
\end{aligned}$$

Thus,

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) = (v+w) \left[4 + \frac{(3\mu-2)}{r}(v-w) \right] \cdot \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right]$$

Recall that v and w are defined as in (6.108) and (6.109), thus,

$$\begin{aligned}
v + \omega &= 2t \\
v - \omega &= 2r^*
\end{aligned}$$

Therefore, we also have,

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) = 4t[2 + \frac{(3\mu - 2)r^*}{r}] \cdot [\frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2] \quad (4.44)$$

We define,

$$J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) dVol \quad (4.45)$$

Computing,

$$E^{(K)}(t_i) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} J_{\alpha}(K)n^{\alpha} dVol_{t=t_i}(t = t_i) \quad (4.46)$$

where

$$n^{\alpha} = -\frac{\frac{\partial}{\partial t}}{\sqrt{(1 - \mu)}}$$

and

$$dVol_{t=t_i} = r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^*$$

$$\begin{aligned} & E_{\Psi}^{(K)}(t_i) \\ &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -\frac{1}{\sqrt{(1 - \mu)}} [(\frac{\partial}{\partial v})^{\alpha} + (\frac{\partial}{\partial \omega})^{\alpha}] J_{\alpha}(K) r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^* \\ &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -\frac{1}{\sqrt{(1 - \mu)}} [-v^2 T_{vv} - \omega^2 T_{v\omega} - v^2 T_{\omega v} - \omega^2 T_{\omega\omega}] r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^* \\ &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} (w^2 [\frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2] \\ &\quad + v^2 [\frac{1}{r^2(1 - \mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{v\phi}|^2] \\ &\quad + (\omega^2 + v^2) [\frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2]) r^2 (1 - \mu) d\sigma^2 dr^* \end{aligned} \quad (4.47)$$

4.2.3 The vector field G

Let

$$\begin{aligned} G &= -f(r^*) \frac{\partial}{\partial \omega} + f(r^*) \frac{\partial}{\partial v} \\ &= G^\omega \frac{\partial}{\partial \omega} + G^v \frac{\partial}{\partial v} \end{aligned} \tag{4.48}$$

where $f(r^*)$ depends only on r^* .

Computing,

$$\begin{aligned} \frac{\partial}{\partial \omega} G^\omega &= \frac{\partial r^*}{\partial \omega} \frac{\partial}{\partial r^*} G^\omega + \frac{\partial t}{\partial \omega} \frac{\partial}{\partial t} G^\omega \\ &= -\frac{1}{2} \frac{\partial}{\partial r^*} G^\omega + 0 \\ &= \frac{1}{2} f' \end{aligned}$$

where $f' = \frac{\partial}{\partial r^*} f$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial v} G^v &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} G^v + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} G^v \\ &= \frac{1}{2} \frac{\partial}{\partial r^*} G^v + 0 \\ &= \frac{1}{2} f' \\ \frac{\partial}{\partial v} G^\omega &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} G^\omega + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} G^\omega \\ &= \frac{1}{2} \frac{\partial}{\partial r^*} G^\omega + 0 \\ &= -\frac{1}{2} f' \\ \frac{\partial}{\partial \omega} G^v &= \frac{\partial r^*}{\partial \omega} \frac{\partial}{\partial r^*} G^v + \frac{\partial t}{\partial \omega} \frac{\partial}{\partial t} G^v \\ &= -\frac{1}{2} \frac{\partial}{\partial r^*} G^v + 0 \\ &= -\frac{1}{2} f' \end{aligned}$$

$$\begin{aligned}
& \pi^{\alpha\beta}(G)T_{\alpha\beta}(\Psi) \\
= & \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2 \right] \left(\frac{-2}{(1-\mu)}\partial_v G^w \right) \\
& + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2 \right] \left(\frac{-2}{(1-\mu)}\partial_w G^v \right) \\
& + \left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2 \right] \left(-2[\partial_v G^v + \partial_w G^w + \frac{(3\mu-2)}{2r}(G^v - G^w)] \right) \\
= & \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2 \right] \left(\frac{-2}{(1-\mu)}\left(-\frac{1}{2}f'\right) \right) \\
& + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2 \right] \left(\frac{-2}{(1-\mu)}\left(-\frac{1}{2}f'\right) \right) \\
& + \left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2 \right] \left(-2\left[\frac{1}{2}f' + \frac{1}{2}f' + \frac{(3\mu-2)}{2r}(f+f)\right] \right) \\
= & \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2 \right] \left(\frac{f'}{(1-\mu)} \right) \\
& + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2 \right] \left(\frac{f'}{(1-\mu)} \right) \\
& + \left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2 \right] \cdot \left(-2\left[f' + \frac{(3\mu-2)}{r}f\right] \right)
\end{aligned}$$

Finally, we obtain,

$$\begin{aligned}
& T^{\alpha\beta}(\Psi_{\mu\nu})\pi_{\alpha\beta}(G) \\
= & \left[\frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{w\phi}|^2 + \frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2 \right] \frac{f'}{(1-\mu)} \\
& - 2\left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2 \right] \left(f' + \frac{f}{r}(3\mu-2) \right) \tag{4.49}
\end{aligned}$$

Computing,

$$E_{\Psi}^{(G)}(t_i) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} J_{\alpha}(G)(\Psi_{\mu\nu})n^{\alpha}dVol_{t=t_i}(t=t_i) \tag{4.50}$$

where $n^\alpha = -\frac{\partial}{\partial t} / \sqrt{(1-\mu)}$ and $dVol_{t=t_i} = r^2 \sqrt{(1-\mu)} d\sigma^2 dr^*$. Thus,

$$E_\Psi^{(G)}(t_i) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -\frac{1}{\sqrt{(1-\mu)}} \left(\frac{\partial}{\partial t}\right)^\alpha J_\alpha(G)(\Psi_{\mu\nu}) r^2 \sqrt{(1-\mu)} d\sigma^2 dr^*$$

Recall (4.29) and (4.30), thus,

$$G = f(r^*) \frac{\partial}{\partial r^*} \tag{4.51}$$

Therefore,

$$E_\Psi^{(G)}(t_i) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -f T_{tr^*}(\Psi_{\mu\nu}) r^2 d\sigma^2 dr^*$$

$$\begin{aligned} T_{tr^*} &= \langle \Psi_{t\alpha}, \Psi_{r^*\alpha} \rangle \\ &= \langle \Psi_{tr^*}, \Psi_{r^*r^*} \rangle + \langle \Psi_{t\theta}, \Psi_{r^*\theta} \rangle + \langle \Psi_{t\phi}, \Psi_{r^*\phi} \rangle \\ &= \frac{1}{r^2} \langle \Psi_{t\theta}, \Psi_{r^*\theta} \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{t\phi}, \Psi_{r^*\phi} \rangle \end{aligned}$$

Thus,

$$\begin{aligned} &E_\Psi^{(G)}(t_i) \tag{4.52} \\ &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -f \left[\frac{1}{r^2} \langle \Psi_{t\theta}, \Psi_{r^*\theta} \rangle + \frac{1}{r^2 \sin^2 \theta} \langle \Psi_{t\phi}, \Psi_{r^*\phi} \rangle \right] r^2 d\sigma^2 dr^* \end{aligned}$$

4.3 Bounding the Conformal Energy on $t = \text{constant}$ Hypersurfaces

Let Ψ be any product of \mathcal{L}_t , \mathcal{L}_{Ω_j} , and F , where Ω_j , $j \in \{1, 2, 3\}$, is a basis of angular momentum operators. We know by then that we have (4.26).

4.3.1 Estimate for $E_{\Psi}^{(G)}$

Let f in (4.51) be a bounded function of r^* . Then, we have

$$|E_{\Psi}^{(G)}(t = t_i)| \lesssim |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t = t_i)| \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(t = t_i)| \quad (4.53)$$

where,

$$\begin{aligned} \hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2(1-\mu)} |\Psi_{t\theta}|^2 + \frac{1}{r^2(1-\mu)\sin^2\theta} |\Psi_{t\phi}|^2 + \frac{1}{r^2(1-\mu)} |\Psi_{r^*\theta}|^2 \right. \\ &\quad \left. + \frac{1}{r^2(1-\mu)\sin^2\theta} |\Psi_{r^*\phi}|^2 \right] r^2(1-\mu) d\sigma^2 dr^* \end{aligned} \quad (4.54)$$

Proof.

We have,

$$\begin{aligned} &|E_{\Psi}^{(G)}(t_i)| \\ &= \left| \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -f \left[\frac{1}{r^2} \langle \Psi_{t\theta}, \Psi_{r^*\theta} \rangle + \frac{1}{r^2 \sin^2\theta} \langle \Psi_{t\phi}, \Psi_{r^*\phi} \rangle \right] r^2 d\sigma^2 dr^* \right| \\ &\lesssim \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} |f| \left[\frac{1}{r^2} |\Psi_{t\theta}|^2 + \frac{1}{r^2} |\Psi_{r^*\theta}|^2 + \frac{1}{r^2 \sin^2\theta} |\Psi_{t\phi}|^2 + \frac{1}{r^2 \sin^2\theta} |\Psi_{r^*\phi}|^2 \right] r^2 d\sigma^2 dr^* \\ &\quad (\text{by using } a.b \lesssim a^2 + b^2) \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} |f| \left[\frac{1}{r^2(1-\mu)} |\Psi_{t\theta}|^2 + \frac{1}{r^2(1-\mu)} |\Psi_{r^*\theta}|^2 + \frac{1}{r^2(1-\mu)\sin^2\theta} |\Psi_{t\phi}|^2 \right. \\
&\quad \left. + \frac{1}{r^2(1-\mu)\sin^2\theta} |\Psi_{r^*\phi}|^2 \right] r^2(1-\mu) d\sigma^2 dr^* \\
&\lesssim |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t=t_i)|
\end{aligned}$$

(because f is bounded).

Then, we have,

$$\begin{aligned}
&E_{\Psi}^{(\frac{\partial}{\partial t})}(t=t_i) \\
&= - \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left(\frac{1}{2} |\Psi_{\hat{t}r^*}|^2 + \frac{1}{2} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{2} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{2} |\Psi_{\hat{r}^*\hat{\theta}}|^2 + \frac{1}{2} |\Psi_{\hat{r}^*\hat{\phi}}|^2 + \frac{1}{2} |\Psi_{\hat{\theta}\hat{\phi}}|^2 \right) \cdot r^2(1-\mu) d\sigma^2 dr^*
\end{aligned}$$

and therefore,

$$\begin{aligned}
|\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t=t_i)| &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} [|\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2] r^2(1-\mu) d\sigma^2 dr^* \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(t=t_i)|
\end{aligned}$$

□

4.3.2 Controlling $J_{\Psi}^{(K)}$ in terms of $J_{\Psi}^{(G)}$

$$J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \lesssim t_{i+1} J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \quad (4.55)$$

where $r_0 \leq 3m \leq R_0$. And we have,

$$E_{\Psi}^{(K)}(t=t_{i+1}) \leq J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) + E_{\Psi}^{(K)}(t=t_i) \quad (4.56)$$

Proof.

We have,

$$\begin{aligned} & J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \\ = & \int_{t=t_i}^{t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} 4t \left[2 + \frac{(3\mu-2)r^*}{r} \right] \cdot \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] r^2 (1-\mu) d\sigma^2 dr^* dt \end{aligned}$$

We remark that $[2 + \frac{(3\mu-2)r^*}{r}]$ is positive only in a bounded interval $[r_0, R_0]$ where $r_0 > 2m$ and $R_0 > 3m$. To see this, notice that,

$$\lim_{r^* \rightarrow -\infty} \left[2 + \frac{(3\mu-2)r^*}{r} \right] = -\infty$$

and,

$$\lim_{r^* \rightarrow \infty} \left[2 + \frac{(3\mu-2)r^*}{r} \right] = 2 - 2 \cdot (1) = 0$$

and

$$\lim_{r \rightarrow 3m} \left[2 + \frac{(3\mu-2)r^*}{r} \right] = 2 + 0 = 2 > 0$$

More precisely, let's look for the region where $J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1})$ is negative:

$$2 + \frac{r^*}{r}(-2 + 3\mu) \leq 0$$

when

$$r^*(3\mu - 2) \leq -2r$$

Choosing r_0 small enough such that $(3\mu_0 - 2) \geq 0$, then we need r_0 such that for $r \leq r_0$,

$$r^* \leq -\frac{2r}{(3\mu - 2)}$$

so choose r_0 such that

$$r_0^* \leq -\frac{2r_0}{(3\mu_0 - 2)} < 0$$

or choose \hat{R}_0 large such that $(3\mu(\hat{R}_0) - 2) \leq 0$, and such that for $r \geq \hat{R}_0$, we get,

$$r^* \geq -\frac{2r}{(3\mu - 2)} > 0$$

then choose \hat{R} such that

$$\hat{R}_0^* \geq -\frac{2\hat{R}_0}{(3\mu(\hat{R}_0) - 2)} > 0$$

In conclusion choose r_0 such that

$$r_0 < -\frac{2r_0}{(3\mu_0 - 2)} < 0$$

and choose R_0 as the infimum of all \hat{R}_0 such that

$$\hat{R}_0^* \geq -\frac{2\hat{R}_0}{(3\mu(\hat{R}_0) - 2)} > 0 \tag{4.57}$$

Then in the region $r \leq r_0$ or $r \geq R_0$ we know that the integrand in $J_F^{(K)}(t_i \leq t \leq t_{i+1})$ is negative.

Thus,

$$\begin{aligned}
& J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \\
& \leq \int_{t=t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} 4t \left[2 + \frac{(3\mu-2)r^*}{r} \right] \cdot \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] r^2 (1-\mu) d\sigma^2 dr^* dt \\
& \leq t_{i+1} \int_{t=t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} 4 \left[2 + \frac{(3\mu-2)r^*}{r} \right] \cdot \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] r^2 (1-\mu) d\sigma^2 dr^* dt \\
& \lesssim t_{i+1} \int_{t=t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] d\sigma^2 dr^* dt \\
& \lesssim t_{i+1} \int_{t=t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} \left(\int_{r^*}^{\infty} \chi_{[r_0^*, (R_0+1)^*]} d\bar{r}^* \right) \left[|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |\Psi_{\hat{\phi}\hat{\theta}}|^2 \right] \cdot 6\mu r (1-\mu)^2 dr^* d\sigma^2 dt \\
& \lesssim t_{i+1} J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0^* \leq r^* \leq R_0^*)
\end{aligned}$$

□

4.3.3 Estimate for $J_{\Psi}^{(K)}$ in terms of $E_{\Psi}^{(K)}$

For,

$$t_{i+1} \leq t_i + (0.1)t_i$$

and,

$$|r^*(r_0)| + |r^*(R)| \leq 0.4t_i$$

We have:

$$\begin{aligned}
J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) & \lesssim t_{i+1} J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \\
& \lesssim t_{i+1} \left[\frac{1}{t_i^2} E_{\Psi}^{(K)}(t = t_i) + \frac{1}{t_i^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j}^{(K)}} \Psi(t = t_i) \right. \\
& \quad \left. + \frac{1}{t_{i+1}^2} E_{\Psi}^{(K)}(t = t_{i+1}) + \frac{1}{t_{i+1}^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j}^{(K)}} \Psi(t = t_{i+1}) \right]
\end{aligned} \tag{4.58}$$

Proof.

We have, from assumption (4.8),

$$J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1}) \lesssim |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t_{i+1})| + |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}(t_{i+1})| + |\hat{E}_{\mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}(t_i)|$$

Let $\hat{\Psi}$ be an anti-symmetric 2-tensor, defined by the following,

$$\hat{\Psi}_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\Psi_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) \quad (4.59)$$

$$\hat{\Psi}_{\hat{t}\hat{\phi}}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\Psi_{\hat{t}\hat{\phi}}(t = t_i, r^*, \theta, \phi) \quad (4.60)$$

$$\hat{\Psi}_{\hat{r}^*\hat{\theta}}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\Psi_{\hat{r}^*\hat{\theta}}(t = t_i, r^*, \theta, \phi) \quad (4.61)$$

$$\hat{\Psi}_{\hat{r}^*\hat{\phi}}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\Psi_{\hat{r}^*\hat{\phi}}(t = t_i, r^*, \theta, \phi) \quad (4.62)$$

$$\nabla_{r^*}\hat{\Psi}_{\hat{t}\hat{r}^*}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\nabla_{r^*}\Psi_{\hat{t}\hat{r}^*}(t = t_i, r^*, \theta, \phi) \quad (4.63)$$

$$\nabla_{r^*}\hat{\Psi}_{\hat{\theta}\hat{\phi}}(t = t_i, r^*, \theta, \phi) = \hat{\chi}\left(\frac{2r^*}{t_i}\right)\nabla_{r^*}\Psi_{\hat{\theta}\hat{\phi}}(t = t_i, r^*, \theta, \phi) \quad (4.64)$$

where $\hat{\chi}$ is a smooth cut-off function equal to one on $[-1, 1]$ and zero outside $[-\frac{3}{2}, \frac{3}{2}]$.

And for,

$$-\frac{t_i}{2} \leq r^* \leq \frac{t_i}{2} \quad (4.65)$$

$$\hat{\Psi}_{\hat{r}^*\hat{t}}(t = t_i, r^*, \theta, \phi) = \Psi_{\hat{r}^*\hat{t}}(t = t_i, r^*, \theta, \phi) \quad (4.66)$$

$$\hat{\Psi}_{\hat{\theta}\hat{\phi}}(t = t_i, r^*, \theta, \phi) = \Psi_{\hat{\theta}\hat{\phi}}(t = t_i, r^*, \theta, \phi) \quad (4.67)$$

And for,

$$t_i \leq t \leq t_{i+1} \quad (4.68)$$

$$\nabla^\mu \hat{\Psi}_{\mu\nu} = 0 \quad (4.69)$$

$$\nabla_\alpha \hat{\Psi}_{\mu\nu} + \nabla_\mu \hat{\Psi}_{\nu\alpha} + \nabla_\nu \hat{\Psi}_{\alpha\mu} = 0 \quad (4.70)$$

Then, we have that for,

$$-\frac{t_i}{2} \leq r^* \leq \frac{t_i}{2}$$

$$\hat{\Psi}_{\mu\nu}(t = t_i, r^*, \theta, \phi) = \Psi_{\mu\nu}(t = t_i, r^*, \theta, \phi)$$

And for

$$r^* \leq -\frac{3t_i}{4} \text{ and } , r^* \geq \frac{3t_i}{4}$$

for,

$$(k, l) \in \{(r^*, \theta), (r^*, \phi), (t, \theta), (t, \phi)\}$$

we have,

$$\hat{\Psi}_{kl}(t = t_i, r^*, \theta, \phi) = 0$$

Now, considering the region

$$t_i \leq t \leq t_{i+1}, \text{ and } r_0 \leq r \leq R_0$$

where

$$t_{i+1} \leq t_i + (0.1)t_i, \text{ and } |r^*(r_0)| + |r^*(R_0)| \leq 0.4t_i$$

clearly, on $t = t_i$, we have $\hat{\Psi}_{\mu\nu}(t = t_i) = \Psi_{\mu\nu}(t = t_i)$, in the specified region. However, since the information from the initial data propagates no faster than the speed of light, i.e. along the null cones $t = r^*$, and $t = -r^*$, then in the region $t_i \leq t \leq t_{i+1}$ we have $\hat{\Psi}_{\mu\nu} = \Psi_{\mu\nu}$ if $r_0^* \geq -\frac{t_i}{2} + (0.1)t_i = -(0.4)t_i$ and $R_0^* \leq \frac{t_i}{2} - (0.1)t_i = (0.4)t_i$ which is satisfied in the specified region because of the condition that $|r^*(r_0)| + |r^*(R_0)| \leq 0.4t_i$. Thus,

$$\Psi_{\mu\nu} = \hat{\Psi}_{\mu\nu}$$

in the specified region. Therefore, we have,

$$J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \lesssim J_{\hat{\Psi}}^{(G)}(t_i \leq t \leq t_{i+1}) \quad (4.71)$$

And for the same reason, on $t = t_{i+1}$, we have $\hat{\Psi}_{\mu\nu}(t = t_{i+1}) = 0$ if $r^* \geq -\frac{3t_i}{4} - (0.1)t_i = -(0.85)t_i$ and $r^* \leq \frac{3t_i}{4} + (0.1)t_i = (0.85)t_i$.

$$\begin{aligned} & \hat{E}_{\hat{\Psi}}^{(\frac{\partial}{\partial t})}(t = t_{i+1}) \\ &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} (|\hat{\Psi}_{\hat{t}\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{t}\hat{\phi}}|^2 + |\hat{\Psi}_{\hat{r}^*\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{r}^*\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^* \\ &\lesssim \int_{r^*=-0.85t_i}^{r^*=0.85t_i} \int_{\mathbf{S}^2} (|\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}^*\hat{\theta}}|^2 + |\Psi_{\hat{r}^*\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^* \\ &\quad (\text{where we used the boundedness of } \chi) \\ &\lesssim \int_{r^*=-0.85t_i}^{r^*=0.85t_i} \int_{\mathbf{S}^2} (|\Psi_{\hat{v}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\theta}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^* \end{aligned}$$

We also have,

$$\begin{aligned}
& \hat{E}_{\hat{\Psi}}^{(\frac{\partial}{\partial t})}(t = t_{i+1}) \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} (|\hat{\Psi}_{i\hat{\theta}}|^2 + |\hat{\Psi}_{i\hat{\phi}}|^2 + |\hat{\Psi}_{r^*\hat{\theta}}|^2 + |\hat{\Psi}_{r^*\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^* \\
&\lesssim \int_{r^*=-\frac{3t_i}{4}}^{r^*=\frac{3t_i}{4}} \int_{\mathbf{S}^2} (|\Psi_{i\hat{\theta}}|^2 + |\Psi_{i\hat{\phi}}|^2 + |\Psi_{r^*\hat{\theta}}|^2 + |\Psi_{r^*\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^* \\
&\quad (\text{using the boundedness of } \chi) \\
&\lesssim \int_{r^*=-0.85t_i}^{r^*=0.85t_i} \int_{\mathbf{S}^2} (|\Psi_{i\hat{\theta}}|^2 + |\Psi_{i\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\theta}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2) \cdot r^2(1-\mu) d\sigma^2 dr^*
\end{aligned}$$

Lemma 4.3.4.

$$\begin{aligned}
& \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} \left(\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 \right. \\
& \left. + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \cdot (1-\mu) r^2 d\sigma^2 dr^*(t) \\
&\lesssim \frac{E_{\hat{\Psi}}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\hat{\Psi}}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2} \tag{4.72}
\end{aligned}$$

Proof.

We proved that,

$$\begin{aligned}
& E_{\hat{\Psi}}^{(K)}(t_i) \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left(w^2 \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right] + v^2 \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \right. \\
& \left. + (1-\mu) (\omega^2 + v^2) \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \right) r^2 d\sigma^2 dr^*
\end{aligned}$$

Because of this we have,

$$\begin{aligned}
& \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} ((1-\mu)w^2 [\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2] \\
& + (1-\mu)w^2 [\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2]) . r^2 d\sigma^2 dr^*(t_i) \\
& \lesssim E_{\Psi}^{(K)}(t_i)
\end{aligned}$$

and,

$$\begin{aligned}
& \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} ((1-\mu)v^2 [\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2] \\
& + (1-\mu)v^2 [\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2]) . r^2 d\sigma^2 dr^*(t_i) \\
& \lesssim E_{\Psi}^{(K)}(t_i)
\end{aligned}$$

and thus,

$$\begin{aligned}
& \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} ((1-\mu) [\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2] \\
& + (1-\mu) [\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2]) . r^2 d\sigma^2 dr^*(t) \\
& \lesssim \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2}
\end{aligned}$$

and,

$$\begin{aligned}
& \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} ((1-\mu) [\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2] \\
& + (1-\mu) [\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2]) . r^2 d\sigma^2 dr^*(t) \\
& \lesssim \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}
\end{aligned}$$

Summing, we obtain,

$$\begin{aligned}
& \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_{w\phi}|^2 \right] + \left[\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 \right. \right. \\
& \left. \left. + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_{v\phi}|^2 \right] + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \right) \cdot (1-\mu)r^2 d\sigma^2 dr^*(t) \\
\lesssim & \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}
\end{aligned}$$

□

Inequality (4.72) gives,

$$\begin{aligned}
& \int_{r^*=-0.85t_i}^{r^*=(0.85)t_i} \int_{\mathbf{S}^2} \left(\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 \right. \\
& \left. + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_{v\phi}|^2 + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \cdot (1-\mu)r^2 d\sigma^2 dr^*(t) \\
\lesssim & \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{-(0.85)t_i \leq r^* \leq (0.85)t_i\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{-(0.85)t_i \leq r^* \leq (0.85)t_i\}} v^2} \\
\lesssim & \frac{E_{\Psi}^{(K)}(t)}{t_i^2} \tag{4.73}
\end{aligned}$$

Examining now the term,

$$\begin{aligned}
|\hat{E}_{\mathcal{L}_{\Omega_i} \hat{\Psi}}^{(\frac{\partial}{\partial t})}(t = t_i)| & \lesssim \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left(|\mathcal{L}_{\Omega_i} \hat{\Psi}_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_{\Omega_i} \hat{\Psi}_{\hat{w}\hat{\phi}}|^2 + |\mathcal{L}_{\Omega_i} \hat{\Psi}_{\hat{v}\hat{\theta}}|^2 \right. \\
& \left. + |\mathcal{L}_{\Omega_i} \hat{\Psi}_{\hat{v}\hat{\phi}}|^2 \right) \cdot (1-\mu)r^2 d\sigma^2 dr^*(t = t_i)
\end{aligned}$$

For,

$$(k, l) \in \{(r^*, \theta), (r^*, \phi), (t, \theta), (t, \phi)\}$$

$$\begin{aligned}
\mathcal{L}_{\Omega_i} \hat{\Psi}_{kl}(t = t_i, r^*, \theta, \phi) &= \mathcal{L}_{\Omega_i} \chi\left(\frac{2r^*}{t_i}\right) \Psi_{kl}(t = t_i, r^*, \theta, \phi) \\
&= \chi\left(\frac{2r^*}{t_i}\right) \mathcal{L}_{\Omega_i} \Psi_{kl}(t = t_i, r^*, \theta, \phi)
\end{aligned}$$

Thus,

$$|\hat{E}_{\mathcal{L}_{\Omega_i} \hat{\Psi}}^{(\frac{\partial}{\partial t})}(t = t_i)| \lesssim E_{\mathcal{L}_{\Omega_i} \Psi}^{(\frac{\partial}{\partial t})}\left(-\frac{3t_i}{4} \leq r^* \leq \frac{3t_i}{4}\right)(t = t_i)$$

□

4.3.5 Estimate for $E_{\Psi}^{(K)}$

$$\begin{aligned}
E_{\Psi}^{(K)}(t) &\lesssim \sum_{k=1}^3 \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_k} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
&\lesssim E_F^M
\end{aligned} \tag{4.74}$$

where,

$$\begin{aligned}
E_{\Psi}^M &= \sum_{k=1}^3 \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_k} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
&= \sum_{i=0}^3 E_{r^j(\mathcal{L})^j \Psi}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{i=0}^2 E_{r^j(\mathcal{L})^j \Psi}^{(K)}(t = t_0)
\end{aligned} \tag{4.75}$$

Proof.

Let,

$$t_{i+1} = t_i + (0.1)t_i = (1.1)t_i \quad (4.76)$$

For t_0 big enough, we will have

$$|r^*(r_0)| + |r^*(R_0)| \leq 0.4t_0$$

and therefore will be able to apply (4.58).

In view of (4.24) and (4.26) applied to the vector field K in the region $t \in [t_i, t_{i+1}]$, we get

$$\begin{aligned} E_{\Psi}^{(K)}(t = t_{i+1}) &\leq J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1}) + E_{\Psi}^{(K)}(t = t_0) \\ &\leq t_{i+1} J_{\Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi}^{(K)}(t = t_0) \end{aligned}$$

(from (4.55))

and,

$$\begin{aligned} E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_{i+1}) \\ \leq t_{i+1} J_{\mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \end{aligned}$$

(from (4.55))

Since $\hat{\Psi}$ verifies the Maxwell equations, we have

$$\nabla^\alpha T_{\alpha\beta}(\hat{\Psi}) = 0 \quad (4.77)$$

and thus, we can estimate,

$$\begin{aligned} & J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\ \lesssim & \frac{1}{t_i^2} E_{\Psi}^{(K)}(t = t_i) + \frac{1}{t_i^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_i) + \frac{1}{t_{i+1}^2} E_{\Psi}^{(K)}(t = t_{i+1}) + \frac{1}{t_{i+1}^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_{i+1}) \end{aligned}$$

(from (4.58))

$$\begin{aligned} & J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\ \lesssim & \frac{t_i}{t_i^2} J_{\Psi}^{(G)}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} E_{\Psi}^{(K)}(t = t_0) \\ & + \frac{t_i}{t_i^2} \sum_{j=1}^3 J_{\mathcal{L}_{\Omega_j} \Psi}^{(G)}(X)(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\ & + \frac{t_{i+1}}{t_{i+1}^2} J_{\Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} E_{\Psi}^{(K)}(t = t_0) \\ & + \frac{t_{i+1}}{t_{i+1}^2} \sum_{j=1}^3 J_{\mathcal{L}_{\Omega_j} \Psi}^{(G)}(X)(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \end{aligned}$$

We will use the notation $J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)} = J_{\Psi}^{(G)} + J_{\mathcal{L}_{\Omega_j} \Psi}^{(G)}$, for all letters such as J , and for different summations, so as to lighten the notation and be able to write:

$$\begin{aligned}
& J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
\lesssim & \frac{1}{t_i} \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
& + \frac{1}{t_{i+1}} \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

From this we can deduce the following,

$$\begin{aligned}
& J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \\
\lesssim & t_{i+1} J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
& \text{(from (4.55))} \\
\lesssim & \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}} \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

(from the estimate above, and using the positivity of $J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}$)

Since,

$$t_{i+1} = (1.1)t_i$$

we have,

$$t_{i+1} = (1.1)^{i+1}t_0$$

and thus,

$$\sum_i \frac{1}{t_{i+1}} = \sum_i \frac{1}{(1.1)^{i+1}t_0} \lesssim 1$$

Therefore,

$$\begin{aligned}
& J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1}) \\
= & \sum_{i=0}^{i+1} J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \\
\lesssim & (i+1) \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
& \text{(from the above)}
\end{aligned}$$

and thus,

$$\begin{aligned}
& E_{\Psi}^{(K)}(t = t_{i+1}) \\
\lesssim & J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1}) + E_{\Psi}^{(K)}(t = t_0) \\
\lesssim & (i+1) \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

In the same manner, this leads to,

$$\begin{aligned}
& E_{\mathcal{L}_{\Omega_l} \Psi}^{(K)}(t = t_{i+1}) \\
\lesssim & J_{\mathcal{L}_{\Omega_l} \Psi}^{(K)}(t_0 \leq t \leq t_{i+1}) + E_{\mathcal{L}_{\Omega_l} \Psi}^{(K)}(t = t_0) \\
\lesssim & (i+1) \sum_{j=1}^3 J_{\mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

(since Ω_l , $l \in \{1, 2, 3\}$ are Killing, and therefore $\mathcal{L}_{\Omega_l} \Psi$ verifies the Maxwell equations).

Repeating the procedure again, we get,

$$\begin{aligned}
& J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
\lesssim & \frac{1}{t_i^2} E_{\Psi}^{(K)}(t = t_i) + \frac{1}{t_i^2} \sum_{l=1}^3 E_{\mathcal{L}_{\Omega_l} \Psi}^{(K)}(t = t_i) + \frac{1}{t_{i+1}^2} E_{\Psi}^{(K)}(t = t_{i+1}) + \frac{1}{t_{i+1}^2} \sum_{l=1}^3 E_{\mathcal{L}_{\Omega_l} \Psi}^{(K)}(t = t_{i+1}) \\
& \text{(from (4.58))} \\
\lesssim & \frac{i}{t_i^2} \sum_{l=1}^3 \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
& + \frac{(i+1)}{t_{i+1}^2} \sum_{l=1}^3 \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
& + \frac{1}{t_{i+1}^2} \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
\lesssim & \frac{(i+1)}{t_{i+1}^2} \sum_{l=1}^3 \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
& + \frac{1}{t_{i+1}^2} \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

Thus,

$$\begin{aligned}
& J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) \\
\lesssim & t_{i+1} J_{\Psi}^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
& \text{(from (4.55))} \\
\lesssim & \frac{(i+1)}{t_{i+1}} \sum_{l=1}^3 \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
& + \frac{1}{t_{i+1}} \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)
\end{aligned}$$

(using the above).

We have,

$$\begin{aligned}
\frac{(i+1)}{t_{i+1}} &= \frac{(i+1)}{(t_{i+1})^{\frac{1}{2}}} \cdot \frac{1}{(t_{i+1})^{\frac{1}{2}}} \leq C \cdot \frac{1}{(t_{i+1})^{\frac{1}{2}}} \\
&\quad (\text{where we used the fact that } \frac{(i+1)}{(t_{i+1})^{\frac{1}{2}}} \leq C) \\
&\lesssim \frac{1}{(1.1)^{\frac{i+1}{2}}} \\
&\lesssim \left(\sqrt{\frac{1}{1.1}}\right)^i
\end{aligned}$$

$\sum_i (\sqrt{\frac{1}{1.1}})^i$ is a geometric series with $(\sqrt{\frac{1}{1.1}}) < 1$, and therefore,

$$\sum_{i=0}^{\infty} \left(\sqrt{\frac{1}{1.1}}\right)^i \lesssim 1$$

Finally, we have,

$$\begin{aligned}
&J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) \\
&= \sum_{i=0}^{i+1} J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) \\
&\lesssim \sum_{i=0}^{i+1} \left(\sqrt{\frac{1}{1.1}}\right)^i \sum_{l=1}^3 \sum_{j=1}^3 J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \\
&\quad + \sum_{i=0}^{i+1} \frac{1}{t_{i+1}} \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
&\lesssim \sum_{l=1}^3 \sum_{j=1}^3 [J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)]
\end{aligned}$$

(from the above)

which gives,

$$\begin{aligned}
& E_{\Psi}^{(K)}(t = t_{i+1}) \\
& \lesssim J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) + E_{\Psi}^{(K)}(t = t_0) \\
& \lesssim \sum_{l=1}^3 \sum_{j=1}^3 [J_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0)]
\end{aligned}$$

From assumption (4.8),

$$J_{\Psi}^{(G)}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R) \lesssim \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}$$

Thus,

$$\begin{aligned}
& E_{\Psi}^{(K)}(t = t_{i+1}) \\
& \lesssim \sum_{k=1}^3 \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_k} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi, \mathcal{L}_{\Omega_k} \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{l=1}^3 \sum_{j=1}^3 E_{\Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_j} \Psi}^{(K)}(t = t_0) \\
& \lesssim E_{\Psi}^M
\end{aligned}$$

because,

$$\sum_{j=1}^3 |\mathcal{L}_{\Omega_j} \Psi|^2 = r^2 |\mathcal{L} \Psi|^2 = |r \mathcal{L} \Psi|^2 \quad (4.78)$$

We have $t_{i+1} = (1.1)^{i+1} t_0$, however, our proofs work with any a such that $1 < a < 2$, and $t_{i+1} = a^i t_0$. Since for all $t > t_0$ there exist i and a such that $t = a^i t_0$, $t_0 > 1$, we get

that,

$$E_{\Psi}^{(K)}(t) \lesssim E_{\Psi}^M$$

for all $t > t_0$, and similarly for all $t < -t_0$. And since the region $-t_0 \leq t \leq t_0$ is a bounded region, the above inequality holds in this region. Thus, for all t , we have,

$$E_{\Psi}^{(K)}(t) \lesssim E_{\Psi}^M$$

□

4.4 The Proof of Decay Away from the Horizon

We will prove that for all $\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$, all the components of the Maxwell fields satisfy

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |w|)}$$

in $r \geq R > 2m$, for an arbitrarily fixed R , and where E_F is defined by

$$\begin{aligned}
E_F &= \left[\sum_{i=0}^1 \sum_{j=0}^5 E_{r^j(\mathcal{L}_t)^j(\mathcal{L}_t)^i F}^{(\frac{\partial}{\partial t})}(t=t_0) + \sum_{i=0}^1 \sum_{j=0}^4 E_{r^j(\mathcal{L}_t)^j(\mathcal{L}_t)^i F}^{(K)}(t=t_0) \right. \\
&\quad \left. + E_{r^6(\mathcal{L}_t)^6 F}^{(\frac{\partial}{\partial t})}(t=t_0) + E_{r^5(\mathcal{L}_t)^5 F}^{(K)}(t=t_0) \right]^{\frac{1}{2}}
\end{aligned}$$

Proof.

We will prove that,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E_{F, \mathcal{L}_t F, r \mathcal{L}_t F, r^2(\mathcal{L}_t)^2 F, \mathcal{L}_t r^2(\mathcal{L}_t)^2 F, r^3(\mathcal{L}_t)^3 F}^{(K)}(t)]^{\frac{1}{2}}}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E_{F, \mathcal{L}_t F, r \mathcal{L}_t F, r^2(\mathcal{L}_t)^2 F, \mathcal{L}_t r^2(\mathcal{L}_t)^2 F, r^3(\mathcal{L}_t)^3 F}^{(K)}]^{\frac{1}{2}}}{(1 + |w|)}$$

Recall that,

$$E_{\Psi}^{(K)} \lesssim \sum_{j=0}^3 E_{r^j(\mathcal{L}_t)^j \Psi}^{(\frac{\partial}{\partial t})}(t=t_0) + \sum_{j=0}^2 E_{r^j(\mathcal{L}_t)^j \Psi}^{(K)}(t=t_0)$$

thus,

$$\begin{aligned}
& E_{F, \mathcal{L}_t F, r \mathcal{L}_t F, r^2(\mathcal{L}_t)^2 F, \mathcal{L}_t r^2(\mathcal{L}_t)^2 F, r^3(\mathcal{L}_t)^3 F}^{(K)} \\
\lesssim & \sum_{i=0}^1 \sum_{j=0}^5 E_{r^j(\mathcal{L}_t)^j(\mathcal{L}_t)^i F}^{(\frac{\partial}{\partial t})}(t=t_0) + \sum_{i=0}^1 \sum_{j=0}^4 E_{r^j(\mathcal{L}_t)^j(\mathcal{L}_t)^i F}^{(K)}(t=t_0) \\
& + E_{r^6(\mathcal{L}_t)^6 F}^{(\frac{\partial}{\partial t})}(t=t_0) + E_{r^5(\mathcal{L}_t)^5 F}^{(K)}(t=t_0) \\
\lesssim & E_F^2
\end{aligned}$$

Definition 4.4.1. We define positive definite Riemannian metric in the following manner:

$$h(e_\alpha, e_\beta) = \mathbf{g}(e_\alpha, e_\beta) + 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) \quad (4.79)$$

where

$$\frac{\hat{\partial}}{\partial t} = (-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}))^{-\frac{1}{2}} \frac{\partial}{\partial t} \quad (4.80)$$

Definition 4.4.2. For any \mathcal{G} -valued 2-tensor K , we let

$$|K|^2 = h_{\alpha\mu} h_{\beta\nu} |K^{\mu\nu}| \cdot |K^{\alpha\beta}| \quad (4.81)$$

Lemma 4.4.3. *We have,*

$$\nabla_\sigma h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) + 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \nabla_\sigma \frac{\hat{\partial}}{\partial t}) \quad (4.82)$$

Proof.

$$\begin{aligned} \nabla_\sigma h(e_\alpha, e_\beta) &= \partial_\sigma h(e_\alpha, e_\beta) - h(\nabla_\sigma e_\alpha, e_\beta) - h(e_\alpha, \nabla_\sigma e_\beta) \\ &= \nabla_\sigma \mathbf{g}(e_\alpha, e_\beta) + 2\nabla_\sigma [\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t})] \\ &= 2\partial_\sigma [\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t})] - 2\mathbf{g}(\nabla_\sigma e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) \\ &\quad - 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(\nabla_\sigma e_\beta, \frac{\hat{\partial}}{\partial t}) \end{aligned}$$

(since the metric \mathbf{g} is Killing)

$$\begin{aligned}
&= 2\partial_\sigma \mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) + 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \partial_\sigma \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) \\
&\quad - 2\mathbf{g}(\nabla_\sigma e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) - 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(\nabla_\sigma e_\beta, \frac{\hat{\partial}}{\partial t}) \\
&= 2[\partial_\sigma \mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) - \mathbf{g}(\nabla_\sigma e_\alpha, \frac{\hat{\partial}}{\partial t})] \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) \\
&\quad + 2[\partial_\sigma \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) - \mathbf{g}(\nabla_\sigma e_\beta, \frac{\hat{\partial}}{\partial t})] \cdot \mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t})
\end{aligned}$$

Using the fact that $\nabla g = 0$, we get,

$$\nabla_\sigma h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) + 2\mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \nabla_\sigma \frac{\hat{\partial}}{\partial t})$$

□

Let,

$$\hat{t}_\alpha = (\frac{\hat{\partial}}{\partial t})_\alpha = \mathbf{g}_{\mu\alpha} (\frac{\hat{\partial}}{\partial t})^\mu$$

Hence, we can write (6.52) as,

$$h_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + 2(\frac{\hat{\partial}}{\partial t})_\alpha (\frac{\hat{\partial}}{\partial t})_\beta \tag{4.83}$$

and (6.67) as,

$$\nabla_\sigma h_{\alpha\beta} = 2[\nabla_\sigma \hat{t}_\alpha \cdot \hat{t}_\beta + \hat{t}_\alpha \cdot \nabla_\sigma \hat{t}_\beta] \tag{4.84}$$

4.4.4 The region $\omega \geq 1, r \geq R$

We consider the region $w \geq 1, r \geq R$, where R is fixed.

Let Ψ be a tensor, and $|\Psi_{\mu\nu}| = \langle \Psi_{\mu\nu}, \Psi_{\mu\nu} \rangle^{\frac{1}{2}}$. We can compute

$$\begin{aligned} |\nabla|\Psi_{\mu\nu}|| &= \left| \frac{2 \langle \mathcal{L}\Psi_{\mu\nu}, \Psi_{\mu\nu} \rangle}{2 \langle \Psi_{\mu\nu}, \Psi_{\mu\nu} \rangle^{\frac{1}{2}}} \right| \leq \frac{|\mathcal{L}\Psi_{\mu\nu}| |\Psi_{\mu\nu}|}{|\Psi_{\mu\nu}|} \\ &\leq |\mathcal{L}\Psi_{\mu\nu}| \end{aligned}$$

We have the Sobolev inequality,

$$r^2|F|^2 \lesssim \int_{\mathbf{S}^2} r^2|F|^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2|\nabla|F||^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2|\nabla\nabla|F||^2 d\sigma^2$$

where ∇ is the covariant derivative restricted on the 2-spheres. We have

$$\begin{aligned} |\mathcal{L}h_{\alpha\beta}|^2 &= \frac{1}{r^2} \sum_{j=1}^3 |\Omega_j h_{\alpha\beta}|^2 = \frac{1}{r^2} \sum_{i=1}^3 \left| \Omega_j \mathbf{g}(e_\alpha, e_\beta) + 2\Omega_j \mathbf{g}(e_\alpha, \frac{\hat{\partial}}{\partial t}) \cdot \mathbf{g}(e_\beta, \frac{\hat{\partial}}{\partial t}) \right|^2 \\ &= 0 \end{aligned}$$

Since $\mathcal{L}h_{\mu\nu} = 0$, we have

$$r^2|F|^2 \lesssim \int_{\mathbf{S}^2} r^2|F|^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2|\mathcal{L}F|^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2|\mathcal{L}\mathcal{L}F|^2 d\sigma^2$$

Let, r_F be a value of r such that $R \leq r_F \leq R+1$, and to be determined later. we have,

$$\begin{aligned}
\int_{\mathbf{S}^2} r^2 |F|^2(t, r, \omega) d\sigma^2 &\lesssim \int_{\mathbf{S}^2} r^2 |F|^2(t, r_F, \omega) d\sigma^2 + \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} \nabla_{r^*} [r^2 |F|^2](t, r, \omega) d\bar{r}^* d\sigma^2 \\
&\lesssim \int_{\mathbf{S}^2} r_F^2 |F|^2(t, r_F, \omega) d\sigma^2 + \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} 2r |F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
&\quad + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

By the same,

$$\begin{aligned}
&\int_{\mathbf{S}^2} r^2 |\not\partial F|^2(t, r, \omega) d\sigma^2 \\
&\lesssim \int_{\mathbf{S}^2} r^2 \not\partial_F |\not\partial F|^2(t, r, \not\partial_F, \omega) d\sigma^2 + \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} 2r |\not\partial F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
&\quad + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |\not\partial F|^2(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

and,

$$\begin{aligned}
&\int_{\mathbf{S}^2} r^2 |\not\partial \not\partial F|^2(t, r, \omega) d\sigma^2 \\
&\lesssim \int_{\mathbf{S}^2} r^2 \not\partial \not\partial_F |\not\partial \not\partial F|^2(t, r, \not\partial \not\partial_F, \omega) d\sigma^2 + \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} 2r |\not\partial \not\partial F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
&\quad + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |\not\partial \not\partial F|^2(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

We showed the following estimate,

$$\begin{aligned} & \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} (|\Psi_{\hat{w}\hat{\theta}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2 + |\Psi_{\hat{v}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\phi}}|^2 + |\Psi_{\hat{v}\hat{w}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2) \cdot (1-\mu)r^2 d\sigma^2 dr^*(t) \\ \lesssim & \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2} \end{aligned}$$

Thus,

$$\int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} |\Psi|^2(t, \bar{r}, \omega) (1-\mu)r^2 d\sigma^2 d\bar{r}^* \lesssim \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}$$

Therefore,

$$\int_{\bar{r}^*=R^*}^{\bar{r}^*=(R+1)^*} \int_{\mathbf{S}^2} |\Psi|^2(t, \bar{r}, \omega) (1-\mu)r^2 d\sigma^2 d\bar{r}^* \lesssim \frac{E_{\Psi}^{(K)}(t)}{t^2}$$

or,

$$\int_{\bar{r}=R}^{\bar{r}=R+1} \int_{\mathbf{S}^2} |\Psi|^2(t, \bar{r}, \omega) r^2 d\sigma^2 d\bar{r} \lesssim \frac{E_{\Psi}^{(K)}(t)}{t^2}$$

There exists r_{Ψ} , such that $R \leq r_{\Psi} \leq R+1$ and,

$$\int_{\mathbf{S}^2} r_F^2 |F|^2(t, r_F, \omega) d\sigma^2 \lesssim \frac{E_F^{(K)}(t)}{t^2(R+1-R)}$$

which gives,

$$\int_{\mathbf{S}^2} r_F^2 |F|^2(t, r_F, \omega) d\sigma^2 \lesssim \frac{E_F^{(K)}(t)}{t^2}$$

By same,

$$\begin{aligned}
\int_{\mathbf{S}^2} r^2 \mathcal{Q}_F |\mathcal{Q}_F F|^2(t, r \mathcal{Q}_F, \omega) d\sigma^2 &= \sum_{j=1}^3 \frac{1}{r^2 \mathcal{Q}_F} \int_{\mathbf{S}^2} r^2 \mathcal{Q}_F |\mathcal{L}_{\Omega_j} F|^2(t, r \mathcal{Q}_F, \omega) d\sigma^2 \\
&\lesssim \sum_{j=1}^3 \frac{1}{R^2} \int_{\mathbf{S}^2} r^2 \mathcal{Q}_F |\mathcal{L}_{\Omega_j} F|^2(t, r \mathcal{Q}_F, \omega) d\sigma^2 \\
&\lesssim \sum_{j=1}^3 \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2}
\end{aligned}$$

and,

$$\int_{\mathbf{S}^2} r^2 \mathcal{Q}_F |\mathcal{Q}_F F|^2(t, r_0, \omega) d\sigma^2 \lesssim \sum_{i=1}^3 \sum_{j=1}^3 \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2}$$

On the other hand,

$$\begin{aligned}
\int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r} |F|^2(1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* &\lesssim \int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} \frac{\bar{r}}{R} \int_{\mathbf{S}^2} \bar{r} |F|^2(1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}
\end{aligned}$$

Thus,

$$\int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r} |F|^2(1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

and,

$$\begin{aligned}
& \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathcal{L}_F} \int_{\mathbf{S}^2} \bar{r} |\not\partial F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&= \sum_{j=1}^3 \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathcal{L}_F} \int_{\mathbf{S}^2} \bar{r} \frac{1}{\bar{r}^2} |\mathcal{L}_{\Omega_j} F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\leq \sum_{j=1}^3 \frac{1}{R^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathcal{L}_F} \int_{\mathbf{S}^2} \bar{r} |\mathcal{L}_{\Omega_j} F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \sum_{j=1}^3 \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2}
\end{aligned}$$

By same,

$$\int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r} |\not\partial F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \sum_{i=1}^3 \sum_{j=1}^3 \left[\frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \right]$$

Now, we want to estimate the term,

$$\begin{aligned}
& \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\bar{r}^* d\sigma^2 \\
&= \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} r^2 \nabla_{\hat{r}^*} |F|^2(t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2 \\
&= \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_F^*}^{\bar{r}^*=r^*} r^2 [\nabla_{\hat{r}^*} (h^{\mu\alpha} h^{\nu\beta}) |F_{\mu\nu}| |F_{\alpha\beta}| + h^{\mu\alpha} h^{\nu\beta} \nabla_{\hat{r}^*} (|F_{\mu\nu}| |F_{\alpha\beta}|)](t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2
\end{aligned}$$

We have,

$$\nabla_{\hat{r}^*} h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \frac{\partial}{\partial \hat{t}}) \cdot \mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}) \cdot \mathbf{g}(e_\beta, \nabla_\sigma \frac{\partial}{\partial \hat{t}})$$

and

$$\nabla_{r^*} \frac{\hat{\partial}}{\partial t} = 0$$

Hence,

$$\begin{aligned} \nabla_{r^*} h^{\alpha\beta} &= \nabla_{r^*} h(e^\alpha, e^\beta) = 2\mathbf{g}(e^\alpha, \nabla_{r^*} \frac{\partial}{\partial t}) \cdot \mathbf{g}(e^\beta, \frac{\partial}{\partial t}) + 2\mathbf{g}(e^\alpha, \frac{\partial}{\partial t}) \cdot \mathbf{g}(e^\beta, \nabla_{r^*} \frac{\partial}{\partial t}) \\ &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\bar{r}^* d\sigma^2 \\ &= \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} r^2 h^{\mu\alpha} h^{\nu\beta} \nabla_{r^*} (|F_{\mu\nu}| |F_{\alpha\beta}|)(t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2 \\ &\lesssim \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} r^2 [|h^{\mu\alpha} h^{\nu\beta}| |\nabla_{r^*} |F_{\mu\nu}| \cdot |F_{\alpha\beta}| + |h^{\mu\alpha} h^{\nu\beta}| |F_{\mu\nu}| \cdot |\nabla_{r^*} |F_{\alpha\beta}|](t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2 \\ &\lesssim \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} r^2 [|h^{\mu\alpha} h^{\nu\beta}| |\mathcal{L}_{r^*} F_{\mu\nu}| \cdot |F_{\alpha\beta}| + |h^{\mu\alpha} h^{\nu\beta}| |F_{\mu\nu}| \cdot |\mathcal{L}_{r^*} F_{\alpha\beta}|](t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2 \\ &\lesssim \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} r^2 [h^{\hat{\mu}\hat{\alpha}} h^{\hat{\nu}\hat{\beta}} |\mathcal{L}_{r^*} F_{\hat{\mu}\hat{\nu}}| \cdot |F_{\hat{\alpha}\hat{\beta}}| + h^{\hat{\mu}\hat{\alpha}} h^{\hat{\nu}\hat{\beta}} |F_{\hat{\mu}\hat{\nu}}| \cdot |\mathcal{L}_{r^*} F_{\hat{\alpha}\hat{\beta}}|](t, r, \omega) \sqrt{(1-\mu)} d\bar{r}^* d\sigma^2 \end{aligned}$$

where $\hat{\mu}, \hat{\nu}, \hat{\alpha}, \hat{\beta} \in \{\frac{\hat{\partial}}{\partial t}, \frac{\hat{\partial}}{\partial r^*}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$.

As we have (see (6.2.3) in the Appendix),

$$\begin{aligned} \nabla_{r^*} \frac{\hat{\partial}}{\partial t} &= 0 \\ \nabla_{r^*} \frac{\hat{\partial}}{\partial r^*} &= 0 \\ \nabla_{r^*} \frac{\hat{\partial}}{\partial \theta} &= 0 \\ \nabla_{r^*} \frac{\hat{\partial}}{\partial \phi} &= 0 \end{aligned}$$

we get,

$$\mathcal{L}_{\hat{r}^*} F_{\hat{\mu}\hat{\nu}} = \nabla_{\hat{r}^*} F_{\hat{\mu}\hat{\nu}}$$

Consequently, using Cauchy-Schwarz, we obtain

$$\begin{aligned} & \int_{\mathbf{S}^2} \int_{\bar{r}^*=r_0}^{\bar{r}^*=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\bar{r}^* d\sigma^2 \\ & \lesssim \left[\int_{\bar{r}^*=r_0}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |\nabla_{\hat{r}^*} F|^2 (1 - \mu) d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \cdot \left[\int_{\bar{r}^*=r_0}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \end{aligned}$$

We have,

$$\int_{\bar{r}^*=r_0}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 (1 - \mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

Thus,

$$\left[\int_{\bar{r}^*=r_0}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \lesssim \frac{\sqrt{E_F^{(K)}(t)}}{t} + \frac{\sqrt{E_F^{(K)}(t)}}{w}$$

Now, considering the case where $\hat{\mu} = \hat{r}^*$, (or similarly if $\hat{\nu} = \hat{r}^*$), we can compute,

$$\nabla^{\hat{r}^*} F_{\hat{r}^*\hat{\nu}} = -\nabla^{\hat{t}} F_{\hat{t}\hat{\nu}} - \nabla^{\hat{\theta}} F_{\hat{\theta}\hat{\nu}} - \nabla^{\hat{\phi}} F_{\hat{\phi}\hat{\nu}}$$

(since the Maxwell fields are divergence free)

Therefore,

$$\nabla_{\hat{r}^*} F_{\hat{r}^*\hat{\nu}} = \nabla_{\hat{t}} F_{\hat{t}\hat{\nu}} - \nabla_{\hat{\theta}} F_{\hat{\theta}\hat{\nu}} - \nabla_{\hat{\phi}} F_{\hat{\phi}\hat{\nu}}$$

And if both $\mu \neq \hat{r}$, and $\nu \neq \hat{r}$, we can compute,

$$\nabla^{\hat{r}^*} F_{\hat{\mu}\hat{\nu}} = -\nabla_{\hat{\mu}} F_{\hat{\nu}\hat{r}^*} - \nabla_{\hat{\nu}} F_{\hat{r}^*\hat{\mu}}$$

(by using the Bianchi identities)

and therefore,

$$\nabla_{\hat{r}^*} F_{\hat{\mu}\hat{\nu}} = -\nabla_{\hat{\mu}} F_{\hat{\nu}\hat{r}^*} - \nabla_{\hat{\nu}} F_{\hat{r}^*\hat{\mu}}$$

where $\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial t}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$.

In all cases, we get,

$$|\nabla_{\hat{r}^*} F_{\hat{\mu}\hat{\nu}}|^2(1 - \mu) \lesssim [|\nabla_{\hat{t}} F|^2 + |\nabla F|^2](1 - \mu)$$

(using the triangular inequality and $a.b \lesssim a^2 + b^2$)

Hence,

$$\begin{aligned} |\nabla_{\hat{r}^*} F|^2(1 - \mu) &\lesssim [|\nabla_{\hat{t}} F|^2 + \frac{1}{r^2} \sum_{j=1}^3 |\nabla_{\Omega_j} F|^2](1 - \mu) \\ &\lesssim |\nabla_{\hat{t}} F|^2 + \sum_{j=1}^3 |\nabla_{\Omega_j} F|^2 \\ &\lesssim h^{\mu\alpha} h^{\nu\beta} [|\nabla_{\hat{t}} F_{\mu\nu}| \cdot |\nabla_{\hat{t}} F_{\alpha\beta}| + \sum_{j=1}^3 h^{\mu\alpha} h^{\nu\beta} [|\nabla_{\Omega_j} F_{\mu\nu}| \cdot |\nabla_{\Omega_j} F_{\alpha\beta}|] \end{aligned}$$

We have,

$$\begin{aligned}
& |\nabla_t F|^2 \\
&= h^{\mu\alpha} h^{\nu\beta} [|\nabla_t F_{\mu\nu}| \cdot |\nabla_t F_{\alpha\beta}|] \\
&= h^{\mu\alpha} h^{\nu\beta} [|\mathcal{L}_t F_{\mu\nu} - F(\nabla_t \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) - F(\frac{\partial}{\partial\mu}, \nabla_t \frac{\partial}{\partial\nu})| \cdot |\mathcal{L}_t F_{\alpha\beta} - F(\nabla_t \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}) - F(\frac{\partial}{\partial\alpha}, \nabla_t \frac{\partial}{\partial\beta})|] \\
&\lesssim |\mathcal{L}_t F|^2 + |h^{\mu\alpha} h^{\nu\beta}| |F(\nabla_t \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu})| \cdot |F(\nabla_t \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta})| + |h^{\mu\alpha} h^{\nu\beta}| |F(\frac{\partial}{\partial\mu}, \nabla_t \frac{\partial}{\partial\nu})| \cdot |F(\frac{\partial}{\partial\alpha}, \nabla_t \frac{\partial}{\partial\beta})|
\end{aligned}$$

(using Cauchy-Schwarz inequality and $a \cdot b \lesssim a^2 + b^2$)

Since, $\frac{\partial}{\partial t}$ is a smooth vector field away from the horizon, choosing a system of coordinates to compute the contractions above, we get

$$|\nabla_t F|^2 (1 - \mu) \lesssim |\mathcal{L}_t F|^2 + |F|^2$$

Similarly, we have,

$$\begin{aligned}
& |\nabla_{\Omega_j} F|^2 \\
&\lesssim |\mathcal{L}_{\Omega_j} F|^2 + |h^{\mu\alpha} h^{\nu\beta}| |F(\nabla_{\Omega_j} \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu})| \cdot |F(\nabla_{\Omega_j} \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta})| \\
&\quad + |h^{\mu\alpha} h^{\nu\beta}| |F(\frac{\partial}{\partial\mu}, \nabla_{\Omega_j} \frac{\partial}{\partial\nu})| \cdot |F(\frac{\partial}{\partial\alpha}, \nabla_{\Omega_j} \frac{\partial}{\partial\beta})|
\end{aligned}$$

Since Ω_j , $j \in \{1, 2, 3\}$ are smooth vector fields, computing the contractions above in a system of coordinates, we obtain

$$\sum_{j=1}^3 |\nabla_{\Omega_j} F_{\hat{\mu}\hat{\nu}}|^2 \lesssim \sum_{j=1}^3 |\mathcal{L}_{\Omega_j} F|^2 + |F|^2$$

Finally, we have

$$|\nabla_{r^*} F|^2(1 - \mu) \lesssim |F|^2 + |\mathcal{L}_t F|^2 + \sum_{j=1}^3 |\mathcal{L}_{\Omega_j} F|^2 \quad (4.85)$$

Thus,

$$\begin{aligned} & \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |\mathcal{L}_{r^*} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 [|\mathcal{L}_t F|^2 + |\not\mathcal{L} F|^2 + |F|^2] d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 [|F|^2 + |\mathcal{L}_t F|^2] d\sigma^2 d\bar{r}^* + \sum_{j=1}^3 \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} |\mathcal{L}_{\Omega_j} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 [|F|^2 + |\mathcal{L}_t F|^2] d\sigma^2 d\bar{r}^* + \sum_{j=1}^3 \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \frac{\bar{r}^2}{R^2} |\mathcal{L}_{\Omega_j} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \end{aligned}$$

and therefore,

$$\begin{aligned} & \left[\int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 |\mathcal{L}_{r^*} F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \\ \lesssim & \frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|t|} + \frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|w|} \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{\bar{r}^*=r_0^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 \langle \mathcal{L}_{r^*} F, F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\ \lesssim & \left(\frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|t|} + \frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|w|} \right) \\ & \cdot \left(\frac{\sqrt{E_F^{(K)}(t)}}{t} + \frac{\sqrt{E_F^{(K)}(t)}}{w} \right) \\ \lesssim & \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \end{aligned}$$

By same,

$$\begin{aligned} & \int_{\bar{r}^*=r^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 \langle \mathcal{L}_{r^*} \not{L} F, \not{L} F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\ \lesssim & \sum_{i=1}^3 \sum_{j=1}^3 \left[\frac{E_{\mathcal{L}_{\Omega_j} F, \mathcal{L}_t \mathcal{L}_{\Omega_j} F, \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_j} F, \mathcal{L}_t \mathcal{L}_{\Omega_j} F, \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \right] \end{aligned}$$

and,

$$\begin{aligned} & \int_{\bar{r}^*=r^*}^{\bar{r}^*=r^*} \int_{\mathbf{S}^2} \bar{r}^2 \langle \mathcal{L}_{r^*} \not{L} \not{L} F, \not{L} \not{L} F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\ \lesssim & \sum_{l=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \left[\frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F, \mathcal{L}_t \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F, \mathcal{L}_t \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F, \mathcal{L}_{\Omega_l} \mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \right] \end{aligned}$$

Using the fact that,

$$\sum_{j=1}^3 |\mathcal{L}_{\Omega_j} \Psi|^2 = r^2 |\not{L} \Psi|^2 = |r \not{L} \Psi|^2$$

We have,

$$\begin{aligned} |r \not{L} r \not{L} \Psi|^2 &= r^2 [r^2 |\not{L} \not{L} \Psi|^2] \\ &= r^2 \left[\sum_{j=1}^3 |\mathcal{L}_{\Omega_j} \not{L} \Psi|^2 \right] \\ &= \sum_{j=1}^3 r^2 |\not{L} \mathcal{L}_{\Omega_j} \Psi|^2 \\ &= \sum_{i=1}^3 \sum_{j=1}^3 |\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} \Psi|^2 \end{aligned}$$

Finally, we obtain,

$$\begin{aligned}
& r^2 |F|^2(t, r, \omega) \\
\lesssim & \frac{E_{F, \mathcal{L}_t F, r}^{(K)} \not\!{D}_F(t)}{t^2} + \frac{E_{F, \mathcal{L}_t F, r}^{(K)} \not\!{D}_F(t)}{w^2} \\
& + \frac{E_{r \not\!{D}_F, \mathcal{L}_t r \not\!{D}_F, r^2(\not\!{D})^2 F}^{(K)}(t)}{t^2} + \frac{E_{r \not\!{D}_F, \mathcal{L}_t r \not\!{D}_F, r^2(\not\!{D})^2 F}^{(K)}(t)}{w^2} \\
& + \frac{E_{r^2(\not\!{D})^2 F, \mathcal{L}_t r^2(\not\!{D})^2 F, r^3(\not\!{D})^3 F}^{(K)}(t)}{t^2} + \frac{E_{r^2(\not\!{D})^2 F, \mathcal{L}_t r^2(\not\!{D})^2 F, r^3(\not\!{D})^3 F}^{(K)}(t)}{w^2}
\end{aligned}$$

Since what is on the left hand side of the previous inequality is a contraction, it can be computed in the basis $\{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}$. Thus,

$$\begin{aligned}
& \sum_{\hat{\mu}, \hat{\nu} \in \{\frac{\hat{\partial}}{\partial w}, \frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial \theta}, \frac{\hat{\partial}}{\partial \phi}\}} |F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \\
\lesssim & \frac{[E_{F, \mathcal{L}_t F, r}^{(K)} \not\!{D}_F, \mathcal{L}_t r \not\!{D}_F, r^2(\not\!{D})^2 F, \mathcal{L}_t r^2(\not\!{D})^2 F, r^3(\not\!{D})^3 F}(t)]^{\frac{1}{2}}}{rt} \\
& + \frac{[E_{F, \mathcal{L}_t F, r}^{(K)} \not\!{D}_F, \mathcal{L}_t r \not\!{D}_F, r^2(\not\!{D})^2 F, \mathcal{L}_t r^2(\not\!{D})^2 F, r^3(\not\!{D})^3 F}(t)]^{\frac{1}{2}}}{rw}
\end{aligned}$$

For R fixed, consider first the region where $t \geq 1$, and thus we have $r + t \lesssim rt$. Consequently, $v = r^* + t \lesssim 2r + t \lesssim 2r + 2t \lesssim rt$, and $v \lesssim t + r \lesssim r + t - r^* \lesssim r + w \lesssim rw$ (since $w \geq 1$).

For $t \leq 1$, the region, $\omega \geq 1$, $r \geq R$, $t \leq 1$ is a bounded compact region, and therefore,

in this region

$$\begin{aligned}
|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\lesssim E_F^{(\frac{\partial}{\partial t})}(t = t_0) + E_{\nabla F}^{(\frac{\partial}{\partial t})}(t = t_0) \\
&\quad (\text{see [G]}) \\
&\lesssim E_{F, \mathcal{L}_t F, r \not\partial F}^{(\frac{\partial}{\partial t})}(t = t_0)
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r \not\partial F, \mathcal{L}_t r \not\partial F, r^2(\not\partial)^2 F, \mathcal{L}_t r^2(\not\partial)^2 F, r^3(\not\partial)^3 F}(t)]^{\frac{1}{2}}}{(1 + |v|)} \\
&\lesssim \frac{E_F}{(1 + |v|)}
\end{aligned}$$

4.4.5 The region $w \leq -1$, $r \geq R$, $|t| \geq 1$

Again, we have the Sobolev inequality

$$r^2 |F_{\hat{\mu}\hat{\nu}}|^2 \lesssim \int_{\mathbf{S}^2} r^2 |F|^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2 |\not\partial F|^2 d\sigma^2 + \int_{\mathbf{S}^2} r^2 |\not\partial \not\partial F|^2 d\sigma^2$$

We have,

$$\begin{aligned}
\int_{\mathbf{S}^2} r^2 |F|^2(t, r, \omega) d\sigma^2 &\lesssim \int_{\mathbf{S}^2} r^2 |F|^2(t, \infty, \omega) d\sigma^2 + \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \nabla_{r^*} [r^2 |F|^2](t, r, \omega) d\bar{r}^* d\sigma^2 \\
&\lesssim \int_{\mathbf{S}^2} \int_{\bar{r}^*}^{\bar{r}^*=\infty} 2r |F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
&\quad + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} r^2 \langle \nabla_{r^*} F, F \rangle_h(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

By the same,

$$\begin{aligned}
& \int_{\mathbf{S}^2} r^2 |\not\partial F|^2(t, r, \omega) d\sigma^2 \\
\lesssim & \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} 2r |\not\partial F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
& + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} r^2 \langle \nabla_{r^*} \not\partial F, \not\partial F \rangle_h(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

and,

$$\begin{aligned}
& \int_{\mathbf{S}^2} r^2 |\not\partial \not\partial F|^2(t, r, \omega) d\sigma^2 \\
\lesssim & \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} 2r |\not\partial \not\partial F|^2(t, r, \omega) (1 - \mu) d\bar{r}^* d\sigma^2 \\
& + 2 \int_{\mathbf{S}^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} r^2 \langle \nabla_{r^*} \not\partial \not\partial F, \not\partial \not\partial F \rangle_h(t, r, \omega) d\bar{r}^* d\sigma^2
\end{aligned}$$

We have shown,

$$\begin{aligned}
& \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} (|\Psi_{\hat{w}\hat{\theta}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2 + |\Psi_{\hat{v}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\phi}}|^2 + |\Psi_{\hat{v}\hat{w}}|^2 + |\Psi_{\hat{\phi}\hat{\theta}}|^2) \cdot (1 - \mu) r^2 d\sigma^2 dr^*(t) \\
\lesssim & \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}
\end{aligned}$$

Thus,

$$\int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} |\Psi|^2(t, \bar{r}, \omega) (1 - \mu) r^2 d\sigma^2 d\bar{r}^* \lesssim \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}$$

We have,

$$\begin{aligned}
\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} |F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* &\lesssim \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \frac{\bar{r}}{R} \int_{\mathbf{S}^2} \bar{r} |F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}
\end{aligned}$$

Thus,

$$\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} |F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

and,

$$\begin{aligned}
\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} |\not\partial F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* &= \sum_{j=1}^3 \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} \frac{1}{\bar{r}^2} |\mathcal{L}_{\Omega_j} F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \sum_{j=1}^3 \frac{1}{R^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} |\mathcal{L}_{\Omega_j} F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \sum_{j=1}^3 \left[\frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \right]
\end{aligned}$$

By same,

$$\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r} |\not\partial \not\partial F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \sum_{i=1}^3 \sum_{j=1}^3 \left[\frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \right]$$

Now, we can estimate the term,

$$\begin{aligned}
&\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 \langle \nabla_{r^*} F, F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\
&\lesssim \left[\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |\nabla_{r^*} F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \cdot \left[\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}}
\end{aligned}$$

We have,

$$\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 (1-\mu)(t, \bar{r}, \omega) d\sigma^2 d\bar{r}^* \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2}$$

Thus,

$$\left[\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}} \lesssim \frac{\sqrt{E_F^{(K)}(t)}}{t} + \frac{\sqrt{E_F^{(K)}(t)}}{w}$$

We also have the estimate:

$$\begin{aligned} & \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |\nabla_{r^*} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 (|F|^2 + |\mathcal{L}_t F|^2 + |\not\mathcal{L} F|^2) d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 [|F|^2 + |\mathcal{L}_t F|^2] d\sigma^2 d\bar{r}^* + \sum_{j=1}^3 \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 \frac{1}{\bar{r}^2} |\mathcal{L}_{\Omega_j} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 [|F|^2 + |\mathcal{L}_t F|^2] d\sigma^2 d\bar{r}^* + \sum_{j=1}^3 \frac{1}{R^2} \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |\mathcal{L}_{\Omega_j} F|^2 d\sigma^2 d\bar{r}^* \\ \lesssim & \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \end{aligned}$$

and therefore,

$$\lesssim \frac{\left[\int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 |\nabla_{r^*} F|^2 d\sigma^2 d\bar{r}^* \right]^{\frac{1}{2}}}{\sqrt{\frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{|t|}}} + \frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{\sqrt{|w|}}$$

Thus,

$$\begin{aligned}
& \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 \langle \nabla_{r^*} F, F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\
\lesssim & \left(\frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|t|} + \frac{\sqrt{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}}{|w|} \right) \\
& \cdot \left(\frac{\sqrt{E_F^{(K)}(t)}}{t} + \frac{\sqrt{E_F^{(K)}(t)}}{w} \right) \\
\lesssim & \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + \sum_{j=1}^3 E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{w^2} \\
\lesssim & \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + E_{r \not\ll F}^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t) + E_{\mathcal{L}_t F}^{(K)}(t) + E_{r \not\ll F}^{(K)}(t)}{w^2}
\end{aligned}$$

By same,

$$\begin{aligned}
& \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 \langle \nabla_{r^*} \not\ll F, \not\ll F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\
\lesssim & \frac{E_{r \not\ll F, \mathcal{L}_t r \not\ll F, r^2(\not\ll)^2 F}^{(K)}(t)}{t^2} + \frac{E_{r \not\ll F, \mathcal{L}_t r \not\ll F, r^2(\not\ll)^2 F}^{(K)}(t)}{w^2}
\end{aligned}$$

and,

$$\begin{aligned}
& \int_{\bar{r}^*=r^*}^{\bar{r}^*=\infty} \int_{\mathbf{S}^2} \bar{r}^2 \langle \nabla_{r^*} \not\ll \not\ll F, \not\ll \not\ll F \rangle_h(t, r, \omega) d\sigma^2 d\bar{r}^* \\
\lesssim & \frac{E_{r^2(\not\ll)^2 F, \mathcal{L}_t r^2(\not\ll)^2 F, r^3(\not\ll)^3 F}^{(K)}(t)}{t^2} + \frac{E_{r^2(\not\ll)^2 F, \mathcal{L}_t r^2(\not\ll)^2 F, r^3(\not\ll)^3 F}^{(K)}(t)}{w^2}
\end{aligned}$$

Finally, we obtain,

$$\begin{aligned}
& r^2 |F|^2(t, r, \omega) \\
\lesssim & \frac{E_{F, \mathcal{L}_t F, r}^{(K)}(t)}{t^2} + \frac{E_{F, \mathcal{L}_t F, r}^{(K)}(t)}{w^2} \\
& + \frac{E_{r \not\mathcal{L} F, \mathcal{L}_t r \not\mathcal{L} F, r^2(\not\mathcal{L})^2 F}^{(K)}(t)}{t^2} + \frac{E_{r \not\mathcal{L} F, \mathcal{L}_t r \not\mathcal{L} F, r^2(\not\mathcal{L})^2 F}^{(K)}(t)}{w^2} \\
& + \frac{E_{r^2(\not\mathcal{L})^2 F, \mathcal{L}_t r^2(\not\mathcal{L})^2 F, r^3(\not\mathcal{L})^3 F}^{(K)}(t)}{t^2} + \frac{E_{r^2(\not\mathcal{L})^2 F, \mathcal{L}_t r^2(\not\mathcal{L})^2 F, r^3(\not\mathcal{L})^3 F}^{(K)}(t)}{w^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
|F|(w, v, \omega) & \lesssim \frac{[E_{F, \mathcal{L}_t F, r}^{(K)}(t)]^{\frac{1}{2}}}{r|t|} \\
& + \frac{[E_{r \not\mathcal{L} F, \mathcal{L}_t r \not\mathcal{L} F, r^2(\not\mathcal{L})^2 F, \mathcal{L}_t r^2(\not\mathcal{L})^2 F, r^3(\not\mathcal{L})^3 F}^{(K)}(t)]^{\frac{1}{2}}}{r|w|}
\end{aligned}$$

We have,

$$r^* = \frac{v - w}{2}$$

Thus, for $w \leq -1$, we have, $r^* \geq v$, and hence $r \geq v$. Therefore,

$$\frac{1}{r} \lesssim \frac{1}{|v|}$$

Since in this region we have $|w| \geq 1$, and $|t| \geq 1$, we get,

$$\begin{aligned}
|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^{(K)} \mathcal{L}_{F, \mathcal{L}_t r} \mathcal{L}_{F, r^2}(\mathcal{L})^2 F, \mathcal{L}_t r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F(t)]^{\frac{1}{2}}}{(1 + |v|)} \\
&\lesssim \frac{E_F}{(1 + |v|)}
\end{aligned}$$

We also have for fixed R , and $|t| \geq 1$, $|w| = |t - r^*| \lesssim |r^*| + |t| \lesssim 2r + t \lesssim 2r + 2|t| \lesssim r|t|$. Thus, in this region, we also have,

$$\begin{aligned}
|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^{(K)} \mathcal{L}_{F, \mathcal{L}_t r} \mathcal{L}_{F, r^2}(\mathcal{L})^2 F, \mathcal{L}_t r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F(t)]^{\frac{1}{2}}}{(1 + |w|)} \\
&\lesssim \frac{E_F}{(1 + |w|)}
\end{aligned}$$

4.4.6 The region $w \leq -1$, $r \geq R$, $-1 \leq t \leq 1$

Let,

$$t^\# = t - 2$$

When

$$-1 \leq t \leq 1$$

we have

$$-3 \leq t^\# \leq -1$$

Let,

$$w^\# = t^\# - r^* = t - r^* - 2$$

When

$$w \leq -1$$

we have

$$w^\# \leq -3$$

Thus, the region $w \leq -1$, $r \geq R$, $-1 \leq t \leq 1$, is in the new system of coordinates included in the region $w^\# \leq -1$, $r \geq R$, $t^\# \leq -1$.

$\frac{\partial}{\partial t}$ is a Killing vector field, therefore, the time translation will keep the metric invariant, i.e. in the new system of coordinates $\{t^\#, r, \theta, \phi\}$ the metric is written exactly as in the former system of coordinates $\{t, r, \theta, \phi\}$. Consequently, we will have the same results proven previously, i.e., in the region $w^\# \leq -1$, $r \geq R$, $t^\# \leq -1$, we have:

$$\begin{aligned} |F_{\hat{\mu}\hat{\nu}}|(w^\#, v^\#, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \not{F}_{F, \mathcal{L}_t r} \not{F}_{F, r^2}(\not{F})^2 F, \mathcal{L}_t r^2(\not{F})^2 F, r^3(\not{F})^3 F]^{\frac{1}{2}}}{(1 + |v^\#|)} \\ &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \not{F}_{F, \mathcal{L}_t r} \not{F}_{F, r^2}(\not{F})^2 F, \mathcal{L}_t r^2(\not{F})^2 F, r^3(\not{F})^3 F]^{\frac{1}{2}}}{(1 + |v|)} \end{aligned}$$

and,

$$\begin{aligned} |F_{\hat{\mu}\hat{\nu}}|(w^\#, v^\#, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \not{F}_{F, \mathcal{L}_t r} \not{F}_{F, r^2}(\not{F})^2 F, \mathcal{L}_t r^2(\not{F})^2 F, r^3(\not{F})^3 F]^{\frac{1}{2}}}{(1 + |w^\#|)} \\ &\lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \not{F}_{F, \mathcal{L}_t r} \not{F}_{F, r^2}(\not{F})^2 F, \mathcal{L}_t r^2(\not{F})^2 F, r^3(\not{F})^3 F]^{\frac{1}{2}}}{(1 + |w|)} \end{aligned}$$

which gives,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \not{F}_{F, \mathcal{L}_t r} \not{F}_{F, r^2}(\not{F})^2 F, \mathcal{L}_t r^2(\not{F})^2 F, r^3(\not{F})^3 F]^{\frac{1}{2}}}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E_{F, \mathcal{L}_t F, r}^M \mathcal{L}_t F, \mathcal{L}_t r \mathcal{L}_t F, r^2(\mathcal{L}_t)^2 F, \mathcal{L}_t r^2(\mathcal{L}_t)^2 F, r^3(\mathcal{L}_t)^3 F]^{\frac{1}{2}}}{(1 + |w|)}$$

in the region $w \leq -1$, $r \geq R$, $-1 \leq t \leq 1$.

4.4.7 The region $-1 \leq w \leq 1$, $r \geq R$

Let,

$$r^{*\triangleright} = r^* + 2$$

Then, when

$$-1 \leq w \leq 1$$

we have,

$$-3 \leq w^\# \leq -1$$

and, when

$$r^* \geq R^*$$

then,

$$r^{*\triangleright} \geq R^* + 2 \geq R^*$$

Thus, the region $-1 \leq w \leq 1$, $r \geq R$ is included, in the new system of coordinates, in the region $w^\triangleright \leq -1$, $r^{*\triangleright} \geq R^*$.

Notice that r^* is defined up to a constant. With the new definition of r^* everything we have proven with r^* works with $r^{*\triangleright}$ by replacing in (4.8), $J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0)$

by $J_F^{(G)\triangleright}(t_i \leq t \leq t_{i+1})(r_0^\triangleright < r < R_0^\triangleright)$ defined by

$$\begin{aligned}
& J_\Psi^{(G)\triangleright}(r_0^\triangleright < r < R_0^\triangleright)(t_i \leq t \leq t_{i+1}) \\
&= \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^{*\triangleright}}^{r^*=R_0^{*\triangleright}} \int_{\mathbf{S}^2} [|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|\Psi_{\hat{\phi}\hat{\theta}}|^2]. dr^{*\triangleright} d\sigma^2 dt \\
&= \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^{*+2}}^{r^*=R_0^{*+2}} \int_{\mathbf{S}^2} [|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|\Psi_{\hat{\phi}\hat{\theta}}|^2]. dr^* d\sigma^2 dt
\end{aligned}$$

However, since the length of the interval $w \in [-1, 1]$ was arbitrary; we only wanted in the previous subsections to avoid $w = 0$, what is actually only needed is

$$\begin{aligned}
& J_\Psi^{(G)}(r_0 < r < R_0 + \epsilon)(t_i \leq t \leq t_{i+1}) \\
&= \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*+\epsilon} \int_{\mathbf{S}^2} [|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|\Psi_{\hat{\phi}\hat{\theta}}|^2]. dr^* d\sigma^2 dt \tag{4.86}
\end{aligned}$$

in assumption (4.8) with ϵ arbitrary small. Since we assume (4.8), we would obtain the above, or take R_0 as being the infimum of all \hat{R}_0 in (4.57) plus ϵ fixed. Therefore, in the region $w^\triangleright \leq -1$, $r^{*\triangleright} \geq R^*$. we have,

$$\begin{aligned}
|F_{\hat{\mu}\hat{\nu}}|(w^\triangleright, v^\#, \omega) &\lesssim \frac{[E_{F, \mathcal{L}_t F, r \not\llcorner F, \mathcal{L}_t r \not\llcorner F, r^2(\not\llcorner)^2 F, \mathcal{L}_t r^2(\not\llcorner)^2 F, r^3(\not\llcorner)^3 F}]^M}{(1 + |v^\triangleright|)} \\
&\lesssim \frac{[E_{F, \mathcal{L}_t F, r \not\llcorner F, \mathcal{L}_t r \not\llcorner F, r^2(\not\llcorner)^2 F, \mathcal{L}_t r^2(\not\llcorner)^2 F, r^3(\not\llcorner)^3 F}]^M}{(1 + |v|)}
\end{aligned}$$

which gives,

$$|F|(w, v, \omega) \lesssim \frac{E_F}{(1 + |v|)}$$

in the region $-1 \leq w \leq 1$, $r \geq R$.

□

4.5 Decay of the Energy to Observers Traveling to the Black Hole on $v = \text{constant}$ Hypersurfaces Near the Horizon

4.5.1 The vector field H

Let

$$\begin{aligned} H &= -\frac{h(r^*)}{(1-\mu)} \frac{\partial}{\partial w} - h(r^*) \frac{\partial}{\partial v} \\ &= H^w \frac{\partial}{\partial w} + H^v \frac{\partial}{\partial v} \end{aligned} \tag{4.87}$$

Computing,

$$\begin{aligned} \frac{\partial}{\partial w} h^w &= \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*} H^w + \frac{\partial t}{\partial w} \frac{\partial}{\partial t} H^w \\ &= -\frac{1}{2} \frac{\partial}{\partial r^*} H^w + 0 \\ &= \frac{1}{2(1-\mu)} \left[h' - \frac{\mu}{r} h \right] \end{aligned}$$

where $h' = \frac{\partial}{\partial r^*} h$

And,

$$\begin{aligned} \frac{\partial}{\partial v} H^w &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} H^w + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} H^w \\ &= \frac{1}{2} \frac{\partial}{\partial r^*} H^w + 0 \\ &= \frac{-1}{2(1-\mu)} \left[h' - \frac{\mu}{r} h \right] \end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial v} H^v &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} H^v + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} H^v \\
&= \frac{1}{2} \frac{\partial}{\partial r^*} H^v + 0 \\
&= -\frac{1}{2} h' \\
\frac{\partial}{\partial w} H^v &= \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*} H^v + \frac{\partial t}{\partial w} \frac{\partial}{\partial t} H^v \\
&= -\frac{1}{2} \frac{\partial}{\partial r^*} H^v + 0 \\
&= \frac{1}{2} h'
\end{aligned}$$

$$\begin{aligned}
&\pi^{\alpha\beta}(H)T_{\alpha\beta}(\Psi) \\
&= \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_v H^w \right) \\
&\quad + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{-2}{(1-\mu)} \partial_w H^v \right) \\
&\quad + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left(-2[\partial_v H^v + \partial_w H^w + \frac{(3\mu-2)}{2r}(H^v - H^w)] \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\pi^{\alpha\beta}(H)T_{\alpha\beta}(\Psi) \\
&= \left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] \left(\frac{1}{(1-\mu)} [h' - \frac{\mu}{r} h] \right) \\
&\quad + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{-1}{(1-\mu)} h' \right) \\
&\quad + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
&\quad \cdot \left[-2 \left(\frac{1}{2(1-\mu)} [(h' - (1-\mu)h') - \frac{\mu}{r} h] \right) + \frac{(2-3\mu)}{(1-\mu)r} (h - (1-\mu)h) \right]
\end{aligned} \tag{4.88}$$

We have,

$$F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) = \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} J_{\alpha}(H)n^{\alpha}dVol_{w=w_i}(w = w_i) \quad (4.89)$$

where

$$\begin{aligned} n^{\alpha} &= \mathbf{g}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial}{\partial v}\right)^{\alpha} \\ &= \frac{-2}{(1-\mu)}\left(\frac{\partial}{\partial v}\right)^{\alpha} \end{aligned}$$

and

$$\begin{aligned} dVol_{w=w_i} &= \mathbf{g}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)r^2d\sigma^2dv \\ &= -(1-\mu)r^2d\sigma^2dv \end{aligned}$$

We get

$$\begin{aligned} &F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} -2\left[\frac{h(r^*)}{(1-\mu)}T_{wv} + h(r^*)T_{vv}\right]r^2d\sigma^2dv \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} -2\left[\frac{h(r^*)}{(1-\mu)}\left(\frac{1}{(1-\mu)}|\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2\right) \right. \\ &\quad \left. + h(r^*)\left(\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2\right)\right]r^2d\sigma^2dv \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} -2\left[h(r^*)\left(\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4\sin^2\theta}|\Psi_{\phi\theta}|^2\right) \right. \\ &\quad \left. + h(r^*)\left(\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2\sin^2\theta}|\Psi_{v\phi}|^2\right)\right]r^2d\sigma^2dv \end{aligned} \quad (4.90)$$

and,

$$F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1}) = \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} J_{\alpha}(H)n^{\alpha}dVol_{v=v_i}(v = v_i) \quad (4.91)$$

where

$$\begin{aligned} n^\alpha &= \mathbf{g}\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial t}\right)^{-1} \left(\frac{\partial}{\partial w}\right)^\alpha \\ &= \frac{-2}{(1-\mu)} \left(\frac{\partial}{\partial w}\right)^\alpha \end{aligned}$$

and

$$\begin{aligned} dVol_{v=v_i} &= \mathbf{g}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) r^2 d\sigma^2 dw \\ &= -(1-\mu) r^2 d\sigma^2 dw \end{aligned}$$

Thus,

$$\begin{aligned} &F_\Psi^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1}) \\ &= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} -2 \left[\frac{h(r^*)}{(1-\mu)} T_{ww} + h(r^*) T_{vw} \right] r^2 d\sigma^2 dw \\ &= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} -2 \left[\frac{h(r^*)}{(1-\mu)} \left(\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right) \right. \\ &\quad \left. + h(r^*) \left(\frac{1}{(1-\mu)} |\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \right] r^2 d\sigma^2 dw \end{aligned}$$

We get

$$\begin{aligned} &F_\Psi^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1}) \\ &= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} -2 \left[(1-\mu) h(r^*) \left(\frac{1}{r^2 (1-\mu)^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)^2} |\Psi_{w\phi}|^2 \right) \right. \\ &\quad \left. + (1-\mu) h(r^*) \left(\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \right] r^2 d\sigma^2 dw \end{aligned} \quad (4.92)$$

Applying the divergence theorem for $\Psi_{\mu\nu}$ in a rectangle in the Penrose diagram representing the exterior of the Schwarzschild space-time of which one side contains the horizon, say in the region $[w_i, \infty] \cdot [v_i, v_{i+1}]$:

$$\begin{aligned}
& \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] \left(\frac{1}{(1-\mu)} \left[h' - \frac{\mu}{r} h \right] \right) \right. \\
& + \left. \left[\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right] (-h') \right. \\
& + \left. \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \right. \\
& \quad \cdot \left. \left[\frac{-1}{(1-\mu)} \left[(h' - (1-\mu)h') - \frac{\mu}{r} h \right] + \frac{(2-3\mu)}{(1-\mu)r} (h - (1-\mu)h) \right] \right) r^2 d\sigma^2 (1-\mu) dw dv \\
= & -F_{\Psi}^{(H)}(w=\infty)(v_i \leq v \leq v_{i+1}) + F_{\Psi}^{(H)}(w=w_i)(v_i \leq v \leq v_{i+1}) \\
& -F_{\Psi}^{(H)}(v=v_{i+1})(w_i \leq w \leq \infty) + F_{\Psi}^{(H)}(v=v_i)(w_i \leq w \leq \infty)
\end{aligned}$$

We are going to choose h such that

$$h(r^* = -\infty) = 1$$

and for all $r > 2m$:

$$h \geq 0$$

Furthermore, we let h be supported in the region $2m \leq r \leq (1.2)r_1$ for r_1 chosen such that, $2m < r_0 \leq r_1 < (1.2)r_1 < 3m$. We choose h such that, for all $r \leq r_1$, we have

$$\frac{\mu}{r} h - h' \geq 0 \tag{4.93}$$

$$h > 0 \tag{4.94}$$

$$h' \geq 0 \tag{4.95}$$

$$\frac{-1}{(1-\mu)} h' + \frac{3}{r} h \leq 0 \tag{4.96}$$

$$\mu \left[\frac{-1}{(1-\mu)} h' + \frac{3}{r} h \right] \leq -h \tag{4.97}$$

Computing

$$\begin{aligned}
& \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
& \cdot \left[\frac{-1}{(1-\mu)} \left[(h' - (1-\mu)h') - \frac{\mu}{r}h \right] + \frac{(2-3\mu)}{(1-\mu)r} (h - (1-\mu)h) \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
= & \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
& \cdot \left[\frac{-1}{(1-\mu)} \left[\mu h' - \frac{\mu}{r}h \right] + \frac{(2-3\mu)}{(1-\mu)r} (\mu h) \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
= & \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \\
& \cdot \frac{\mu}{(1-\mu)} \left[\left(\frac{h}{r} - h' \right) + \frac{(2-3\mu)}{r} h \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
= & \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \\
& \mu \left[\frac{-1}{(1-\mu)} h' + \frac{3}{r} h \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv \tag{4.98}
\end{aligned}$$

Let,

$$\begin{aligned}
& I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
= & \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)^2} |\Psi_{w\phi}|^2 \right] \left(h' - \frac{\mu}{r}h \right) \right. \\
& \left. + \left(\left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{-h'}{(1-\mu)} \right) \right) \right. \\
& \left. + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{-1}{(1-\mu)} h' + \frac{3}{r} h \right] \right) \cdot r^2 d\sigma^2 (1-\mu) dw dv \tag{4.99}
\end{aligned}$$

Then, we have,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty) \\
= & -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
& -F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty)
\end{aligned} \tag{4.100}$$

We choose r_1 small enough such that $(1.2)r_1 < 3m$.

4.5.2 Estimate 1

For (w_i, v_i) such that $r(w_i, v_i) = r_1$, where r_1 is as determined in the construction of the vector field H , and for $v_{i+1} \geq v_i$, we have

$$-F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \lesssim F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) \tag{4.101}$$

Proof.

We have,

$$F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) = \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} J_{\alpha}(\frac{\partial}{\partial t})n^{\alpha} dV ol_{w=w_i}(w = w_i)$$

where

$$\begin{aligned}
n^{\alpha} &= \mathbf{g}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial t}\right)^{-1} \left(\frac{\partial}{\partial v}\right)^{\alpha} \\
&= \frac{-2}{(1-\mu)} \left(\frac{\partial}{\partial v}\right)^{\alpha}
\end{aligned}$$

and

$$\begin{aligned} dVol_{w=w_i} &= \mathbf{g}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)r^2 d\sigma^2 dv \\ &= -(1-\mu)r^2 d\sigma^2 dv \end{aligned}$$

We get

$$\begin{aligned} F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} 2[T_{vv} + T_{vw}]r^2 d\sigma^2 dv \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} 2\left[\frac{1}{(1-\mu)}|\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right. \\ &\quad \left. + \frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta}|\Psi_{v\phi}|^2\right]r^2 d\sigma^2 dv \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} 2\left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right. \\ &\quad \left. + \frac{1}{r^2(1-\mu)}|\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{v\phi}|^2\right]r^2(1-\mu)d\sigma^2 dv \end{aligned}$$

We showed that,

$$\begin{aligned} &F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \\ &= \int_{v=v_i}^{v=v_{i+1}} \int_{\mathbf{S}^2} -2[h(r^*)\left(\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right) \\ &\quad + h(r^*)\left(\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta}|\Psi_{v\phi}|^2\right)]r^2 d\sigma^2 dv \end{aligned}$$

The region $w = w_i$ and $v_i \leq v \leq v_{i+1}$ is in the region $r \geq r_1$ as $r(w_i, v_i) = r_1$, and $v_{i+1} \geq v_i$. Thus, in this region

$$\frac{h(r^*)}{(1-\mu)} \lesssim 1$$

which gives immediately (4.101). □

4.5.3 Estimate 2

Let

$$t_{i+1} = (1.1)t_i$$

Define,

$$J_{\Psi}^{(C)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{\mathbf{S}^2} [|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4}|\Psi_{\hat{\phi}\hat{\theta}}|^2] \cdot |r^* - (3m)^*| \cdot dr^* d\sigma^2 dt \quad (4.102)$$

From (4.13), we have

$$J_{\Psi}^{(C)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \lesssim |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t_i)| + |\hat{E}_{\Psi}^{(\frac{\partial}{\partial t})}(t_{i+1})| \quad (4.103)$$

Recall that

$$2m < r_0 \leq r_1 < (1.2)r_1 < 3m$$

We have,

$$\begin{aligned} & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} [|\Psi_{r^*\hat{\theta}}|^2 + |\Psi_{r^*\hat{\phi}}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2] \cdot \chi_{[r_1^*, (1.2)r_1^*]} r^2 (1 - \mu) d\sigma^2 dr^* dt \\ & \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \end{aligned} \quad (4.104)$$

Proof.

Let,

$$f(r^*) = \int_{r^*=-\infty}^{r^*} \chi_{[r_1^*, (1.2)r_1^*]}(r^*) dr^* \quad (4.105)$$

where χ is the sharp cut-off function, such that,

$$f(r^*) = 1, \quad \text{for} \quad r_1^* < r^* < (1.2)r_1^*$$

and

$$f(r) = 0, \quad \text{for} \quad r^* \in]-\infty, r_1^*] \cup [(1.2)r_1^*, \infty[$$

We get from (4.49) applied to f ,

$$\begin{aligned} & T^{\alpha\beta}(\Psi_{\mu\nu})\pi_{\alpha\beta}(G) \\ = & \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \frac{\chi_{[r_1^*, (1.2)r_1^*]}}{(1-\mu)} \\ & - 2 \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] (\chi_{[r_1^*, (1.2)r_1^*]}) + \frac{(3\mu-2)}{r} \int_{r^*=-\infty}^{r^*} \chi_{[r_1^*, (1.2)r_1^*]} dr^* \end{aligned}$$

Applying the divergence theorem between the two hypersurfaces $\{t = t_i\}$ and $\{t = t_{i+1}\}$,

we obtain

$$\begin{aligned} & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \frac{\chi_{[r_1^*, (1.2)r_1^*]}}{(1-\mu)} \\ & \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\ = & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} 2 \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] (\chi_{[r_1^*, (1.2)r_1^*]}) \\ & + \frac{(3\mu-2)}{r} \int_{r^*=-\infty}^{r^*} \chi_{[r_1^*, (1.2)r_1^*]} dr^* \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\ & + E_{\Psi}^{(G)}(t_{i+1}) - E_{\Psi}^{(G)}(t_i) \\ \leq & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_1^*}^{r^*=(1.2)r_1^*} \int_{\mathbf{S}^2} 2 \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\ & + \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=r_0^*}^{r^*=(3m)^*} \int_{\mathbf{S}^2} 2 \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left(\frac{1}{r} [(3m)^* - r_0^*] \right) \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\ & + E_{\Psi}^{(G)}(t_{i+1}) - E_{\Psi}^{(G)}(t_i) \\ \lesssim & J_{\Psi}^{(C)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi}^{(G)}(t_{i+1}) - E_{\Psi}^{(G)}(t_i) \end{aligned}$$

Instead of Ψ , take $\hat{\Psi}$ as in the proof of (4.58), we get,

$$\begin{aligned}
& \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \frac{\chi_{[r_1^*, (1.2)r_1^*]}}{(1-\mu)} \\
& \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\
\leq & \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2} |\hat{\Psi}_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{w\phi}|^2 + \frac{1}{r^2} |\hat{\Psi}_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{v\phi}|^2 \right] \frac{\chi_{[r_1^*, (1.2)r_1^*]}}{(1-\mu)} \\
& \cdot r^2 (1-\mu) dr^* d\sigma^2 dt \\
\lesssim & J_{\hat{\Psi}}^{(C)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\hat{\Psi}}^{(G)}(t_{i+1}) - E_{\hat{\Psi}}^{(G)}(t_i)
\end{aligned}$$

Recall that we have,

$$|E_{\hat{\Psi}}^{(G)}(-\infty \leq r^* \leq \infty)(t_i)| \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)|$$

Thus,

$$\begin{aligned}
& \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \frac{1}{2} [|\hat{\Psi}_{r^* \hat{t}}|^2 + |\hat{\Psi}_{\hat{\theta} \hat{\phi}}|^2] \chi_{[r_1^*, 1.2r_1^*]} r^2 (1-\mu) dr^* d\sigma^2 dt \\
\lesssim & J_{\hat{\Psi}}^{(C)}(t_i \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) \\
\lesssim & |E_{\hat{\Psi}}^{(G)}(-\infty \leq r^* \leq \infty)| \\
\lesssim & |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)|
\end{aligned}$$

□

4.5.4 Estimate 3

For

$$w_i = t_i - r_1^*$$

$$v_i = t_i + r_1^*$$

where r_1 is as determined in the construction of the vector field H , we have

$$\begin{aligned}
& -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
& -F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) \\
\leq & CF_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty) \tag{4.106}
\end{aligned}$$

(where C is a constant)

And,

$$\begin{aligned}
& -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\
& -F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) - F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) \\
\lesssim & F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty) \\
& + |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \tag{4.107}
\end{aligned}$$

Proof.

We showed that,

$$\begin{aligned}
& -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
& -F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) \\
= & -F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty)
\end{aligned}$$

From (4.101), we have,

$$-F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \lesssim F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1})$$

This proves (4.106). On the other hand, for all $r \leq r_1$, we have,

$$\frac{-1}{(1-\mu)}h' + \frac{3}{r}h \leq 0 \quad (4.108)$$

Thus,

$$\begin{aligned} & - \int_{v=v_i, r \leq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{-1}{(1-\mu)}h' + \frac{3}{r}h \right] \\ & \cdot r^2 d\sigma^2 (1-\mu) dw dv \\ & \geq 0 \end{aligned} \quad (4.109)$$

whereas,

$$\begin{aligned} & \left| \int_{v=v_i, r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{-1}{(1-\mu)}h' + \frac{3}{r}h \right] \right. \\ & \left. \cdot r^2 (1-\mu) d\sigma^2 dw dv \right| \\ & \lesssim J_{\Psi}^{(C)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r_1 \leq r \leq (1.2)r_1) \\ & \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \end{aligned} \quad (4.110)$$

$$\begin{aligned} & J_{\Psi}^{(C)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r_1 \leq r \leq 1.2r_1) \\ & \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \end{aligned}$$

and,

$$\begin{aligned}
& \left| \int_{v=v_i, r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left(\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right) \left(\frac{1}{(1-\mu)} |h' - \frac{\mu}{r} h| \right) \right. \\
& \quad \left. + \left(\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right) \left(\frac{-h'}{(1-\mu)} \right) \right| \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
\lesssim & \int_{v=v_i, r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right. \\
& \quad \left. + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \cdot \chi_{[r_1^*, 1.2r_1^*]} \cdot r^2 d\sigma^2 dw dv
\end{aligned}$$

where $\chi_{[r_1, 1.2r_1]}$ is a smooth positive cut-off function supported on $[r_1^*, 1.2r_1^*]$.

From (4.104),

$$\begin{aligned}
& \int_{v=v_i, r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right. \\
& \quad \left. + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \cdot \chi_{[r_1^*, 1.2r_1^*]} \cdot r^2 d\sigma^2 dw dv \\
\lesssim & |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)|
\end{aligned}$$

We get,

$$|I_{\Psi}^{(H)}|(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \geq r_1) \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)|$$

This proves (4.107). □

4.5.5 Estimate 4

Let $v_{i+1} \geq v_i$. We have

$$\begin{aligned}
& \inf_{v_i \leq v \leq v_{i+1}} -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) \tag{4.111} \\
\lesssim & \frac{-I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)}{(v_{i+1} - v_i)} + \sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial i})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

Proof.

We have,

$$\begin{aligned}
-F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) &= \int_{w=w_1}^{w=\infty} \int_{\mathbf{S}^2} 2[h(r^*) \left(\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right) \\
&\quad + (1-\mu)h(r^*) \left(\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta (1-\mu)} |\Psi_{\theta\phi}|^2 \right)] r^2 d\sigma^2 dw
\end{aligned}$$

and we have

$$\begin{aligned}
& -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
= & \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] \left(\frac{1}{(1-\mu)} \left[\frac{\mu}{r} h - h' \right] \right) \right. \\
& \quad \left. + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{h'}{(1-\mu)} \right) \right. \\
& \quad \left. + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{1}{(1-\mu)} h' - \frac{3}{r} h \right] \right) r^2 d\sigma^2 (1-\mu) dw dv
\end{aligned}$$

Given the expression of h, h' , in the region $r \leq r_1$, we get for $v \geq v_i$,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty)(r \leq r_1) \\
\lesssim & \int_{w=w_i, r \leq r_1}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] \left(\frac{1}{(1-\mu)} \left[\frac{\mu}{r} h - h' \right] \right) \right. \\
& \quad \left. + \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{h'}{(1-\mu)} \right) \right. \\
& \quad \left. + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{1}{(1-\mu)} h' - \frac{3}{r} h \right] \right) r^2 d\sigma^2 (1-\mu) dw
\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}
& F_{\Psi}^{(\frac{\partial}{\partial t})}(v = v_i)(w_i \leq w \leq w_{i+1}) \\
&= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} 2[T_{ww} + T_{wv}]r^2 d\sigma^2 dw \\
&= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} 2\left[\frac{1}{(1-\mu)}|\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right. \\
&\quad \left. + \frac{1}{r^2}|\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta}|\Psi_{w\phi}|^2\right]r^2 d\sigma^2 dw \\
&= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbf{S}^2} 2\left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right. \\
&\quad \left. + \frac{1}{r^2(1-\mu)}|\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{w\phi}|^2\right]r^2(1-\mu) d\sigma^2 dw
\end{aligned}$$

Thus, from the boundedness of h, h' , we have in $r \geq r_1$,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty)(r \geq r_1) \\
&\lesssim \int_{w=w_i, r \geq r_1}^{w=\infty} \int_{\mathbf{S}^2} \left(\frac{1}{r^2(1-\mu)}|\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{w\phi}|^2\right. \\
&\quad \left. + \frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right).r^2 d\sigma^2(1-\mu) dw \\
&\lesssim F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

Thus,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) \\
&\lesssim \int_{w=w_i, r \leq r_1}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)}|\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{w\phi}|^2\right]\left(\frac{1}{(1-\mu)}\left[\frac{\mu}{r}h - h'\right]\right)\right. \\
&\quad \left. + \left[\frac{1}{r^2}|\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta}|\Psi_{v\phi}|^2\right]\left(\frac{h'}{(1-\mu)}\right)\right. \\
&\quad \left. + \left[\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2\right].\mu\left[\frac{1}{(1-\mu)}h' - \frac{3}{r}h\right]\right).r^2 d\sigma^2(1-\mu) dw \\
&\quad + F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

We have,

$$\begin{aligned}
& (v_{i+1} - v_i) \inf_{v_i \leq v \leq v_{i+1}} -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) \\
\leq & \int_{v=v_i}^{v=v_{i+1}} -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) dv \\
\lesssim & \int_{v=v_i, r \leq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] \left(\frac{1}{(1-\mu)} \left[\frac{\mu}{r} h - h' \right] \right) \right. \\
& + \left(\left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \left(\frac{h'}{(1-\mu)} \right) \right. \\
& + \left. \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \cdot \mu \left[\frac{1}{(1-\mu)} h' - \frac{3}{r} h \right] \right) \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
& + \int_{v=v_i}^{v=v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) dv \\
\lesssim & -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\
& + (v_{i+1} - v_i) \sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
\end{aligned}$$

□

4.5.6 Estimate 5

For

$$\begin{aligned}
w_i &= t_i + r_1 \\
v_i &= t_i + r_1
\end{aligned}$$

where r_1 is as determined in the construction of the vector field H , we have

$$0 \leq -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) \tag{4.112}$$

where,

$$\begin{aligned}
& E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right. \\
&\quad \left. + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] r^2 d\sigma^2 dr^*(t = t_0)
\end{aligned} \tag{4.113}$$

Proof.

Computing,

$$E_{\Psi}^{(H)}(t) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} J_{\alpha}(H)(\Psi_{\mu\nu}) n^{\alpha} dV ol_t \tag{4.114}$$

where $n^{\alpha} = -\frac{(\frac{\partial}{\partial t})^{\alpha}}{\sqrt{(1-\mu)}}$ and $dV ol_{t=t_i} = r^2 \sqrt{(1-\mu)} d\sigma^2 dr^*$, we get,

$$E_{\Psi}^{(H)}(t) = \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -\frac{1}{\sqrt{(1-\mu)}} \left(\frac{\partial}{\partial t}\right)^{\alpha} J_{\alpha}(H)(\Psi_{\mu\nu}) r^2 \sqrt{(1-\mu)} d\sigma^2 dr^*$$

Therefore,

$$\begin{aligned}
E_{\Psi}^{(H)}(t) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} -(H)^{\alpha} T_{t\alpha}(\Psi_{\mu\nu}) r^2 d\sigma^2 dr^* \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{h}{(1-\mu)} T_{tw} + h T_{tv} \right] r^2 d\sigma^2 dr^* \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{h}{(1-\mu)} T_{ww} + \frac{h}{(1-\mu)} T_{vw} + h T_{wv} + h T_{vv} \right] r^2 d\sigma^2 dr^* \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} \left[\frac{h}{(1-\mu)} \left(\frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right) \right. \\
&\quad \left. + \frac{h}{(1-\mu)} \left(\frac{1}{(1-\mu)} |\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \right. \\
&\quad \left. + h \left(\frac{1}{(1-\mu)} |\Psi_{vw}|^2 + \frac{(1-\mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \right. \\
&\quad \left. + h \left(\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right) \right] r^2 d\sigma^2 dr^*
\end{aligned}$$

Hence,

$$\begin{aligned}
E_{\Psi}^{(H)}(t) &= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} [h(1-\mu)(|\Psi_{\hat{w}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\hat{w}\hat{\phi}}|^2) \\
&\quad + h(|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |\Psi_{\hat{\phi}\hat{\theta}}|^2) + h(1-\mu)(|\Psi_{\hat{v}\hat{w}}|^2 + \frac{1}{4} |\Psi_{\hat{\phi}\hat{\theta}}|^2) \\
&\quad + h(|\Psi_{\hat{v}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\phi}}|^2)] r^2 d\sigma^2 dr^*
\end{aligned}$$

By using the divergence theorem in the region $(v \leq v_0)(t_0 \leq t \leq \infty)(r \leq r_1)$, we

$$\begin{aligned}
&-I_{\Psi}^{(H)}(v \leq v_0)(t_0 \leq t \leq \infty)(r \leq r_1) \\
&-F_{\Psi}^{(H)}(v = v_0)(w_0 \leq w \leq \infty) - F_{\Psi}^{(H)}(w = \infty)(-\infty \leq v \leq v_0) \\
&= E_{\Psi}^{(H)}(t_0)
\end{aligned} \tag{4.115}$$

Due to the positivity of the terms on the left hand side, we get

$$\begin{aligned}
-F_{\Psi}^{(H)}(v = v_0)(w_0 \leq w \leq \infty) &\lesssim E_{\Psi}^{(H)}(t_0) \\
&\lesssim E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)
\end{aligned} \tag{4.116}$$

where,

$$\begin{aligned}
&E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) \\
&= \int_{r^*=-\infty}^{r^*=\infty} \int_{\mathbf{S}^2} [\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \\
&\quad + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2] r^2 d\sigma^2 dr^*(t = t_0)
\end{aligned}$$

From the divergence theorem and the fact that $\frac{\partial}{\partial t}$ is Killing, it is easy to see that by integrating in a suitable region and using the positivity of the energy we get,

$$\begin{aligned}
F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) &= F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1})(r \geq r_1) \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)|
\end{aligned} \tag{4.117}$$

From (4.107), we get,

$$\begin{aligned}
&-I_{\Psi}^{(H)}(v_0 \leq v \leq v_1)(w_0 \leq w \leq \infty)(r \leq r_1) \\
&-F_{\Psi}^{(H)}(v = v_1)(w_0 \leq w \leq \infty) - F_{\Psi}^{(H)}(w = \infty)(v_0 \leq v \leq v_1) \\
\lesssim &F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_0)(v_0 \leq v \leq v_1) - F_{\Psi}^{(H)}(v = v_0)(w_0 \leq w \leq \infty) \\
&+ |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_0 \leq r^* \leq (0.85)t_0(t = t_0)| \\
\lesssim &|E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_0 \leq r^* \leq (0.85)t_0(t = t_0)| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)
\end{aligned}$$

(from (4.116) and (4.117)).

By recurrence from inequality (4.107), and using (4.117), we obtain for all i integer

$$\begin{aligned}
&-I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\
&-F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) - F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) \\
\lesssim &F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty) \\
&+ |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \\
\lesssim &|E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)
\end{aligned}$$

Due to sign of h , and the definition of h , we have that the terms in each of the integrands on the left hand side are positive, hence, we obtain (4.112).

□

4.5.7 Estimate 6

For all v , let

$$w_0(v) = v - 2r_1^*$$

Let

$$v_+ = \max\{1, v\} \tag{4.118}$$

We have,

$$-F_{\Psi}^{(H)}(v)(w_0(v) \leq w \leq \infty) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{v_+^2} \tag{4.119}$$

and,

$$-F_{\Psi}^{(H)}(w)(v - 1 \leq \bar{v} \leq v) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{v_+^2} \tag{4.120}$$

Proof.

Let,

$$v_i = t_i + r_1^* \tag{4.121}$$

$$w_i = t_i - r_1^* \tag{4.122}$$

where t_i is defined as in (4.76):

$$t_i = (1.1)^i t_0$$

We have shown that,

$$\begin{aligned} & \inf_{v_i \leq v \leq v_{i+1}} -F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) \\ \lesssim & \frac{-I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)}{(v_{i+1} - v_i)} + \sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \end{aligned}$$

Lemma 4.5.8. *We have,*

$$\sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2} \quad (4.123)$$

Proof. By integrating in a well chosen region and using the divergence theorem we get that,

$$\begin{aligned} & \sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \\ \lesssim & \int_{r^*=cr_1^*}^{r^*=Cr_1^*} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] + \left[\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 \right. \right. \\ & \left. \left. + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right] + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \right) \cdot (1-\mu) r^2 d\sigma^2 dr^*(t) \end{aligned} \quad (4.124)$$

We showed that,

$$\begin{aligned} & \int_{r^*=r_1^*}^{r^*=r_2^*} \int_{\mathbf{S}^2} \left(\left[\frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right] + \left[\frac{1}{r^2(1-\mu)} |\Psi_{v\theta}|^2 \right. \right. \\ & \left. \left. + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right] + \left[\frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \right) \cdot (1-\mu) r^2 d\sigma^2 dr^*(t) \\ \lesssim & \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{v \in \{t\} \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2} \end{aligned}$$

Thus, (4.124) gives,

$$\sup_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(\frac{\partial}{\partial t})}(v)(w_i \leq w \leq \infty)(r \geq r_1) \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2} \quad (4.125)$$

□

And we showed that,

$$-I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)$$

Thus, we obtain,

$$\inf_{v_i \leq v \leq v_{i+1}} F_{\Psi}^{(H)}(v)(w_i \leq w \leq \infty) \lesssim \frac{1}{(v_{i+1} - v_i)} [|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)] + \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2} \quad (4.126)$$

and thus, there exists a $v_i^{\#} \in [v_i, v_{i+1}]$ where above inequality holds.

We have,

$$\begin{aligned} v_{i+1} - v_i &= t_{i+1} + r_1^* - (t_i + r_1^*) = t_{i+1} - t_i = (1.1)^{i+1}t_0 - (1.1)^i t_0 = (1.1)^i t_0 (1.1 - 1) \\ &= (0.1)(1.1)^i t_0 \\ &= 0.1 t_i \end{aligned} \quad (4.127)$$

Let,

$$w_i^\# = v_i^\# - 2r_1^* \quad (4.128)$$

Therefore,

$$\begin{aligned} -F_\Psi^{(H)}(v_i^\#)(w_i^\# \leq w \leq \infty) &\lesssim -F_\Psi^{(H)}(v_i^\#)(w_i \leq w \leq \infty) \\ &\quad (\text{due to the positivity of } -F_\Psi^{(H)}(v_i^\#)(w_i \leq w \leq \infty)) \\ &\lesssim \frac{1}{t_i} [|E_\Psi^{(\frac{\partial}{\partial t})}| + E_\Psi^\#(\frac{\partial}{\partial t})(t = t_0)] + \frac{E_\Psi^{(K)}(t_i)}{t_i^2} \\ &\quad (\text{by (4.126)}). \end{aligned}$$

From (4.107), applied in the region $[w_i^\#, \infty].[v_i^\#, v_{i+1}]$, we get due to the positivity of $-I_\Psi^{(H)}(v_i^\# \leq v \leq v_{i+1})(w_i^\# \leq w \leq \infty)(r \leq r_1)$, and $-F_\Psi^{(H)}(w = \infty)(v_i^\# \leq v \leq v_{i+1})$, that,

$$\begin{aligned} &-F_\Psi^{(H)}(v = v_{i+1})(w_i^\# \leq w \leq \infty) \\ &\lesssim F_\Psi^{(\frac{\partial}{\partial t})}(w = w_i^\#)(v_i^\# \leq v \leq v_{i+1}) - F_\Psi^{(H)}(v = v_i^\#)(w_i^\# \leq w \leq \infty) \\ &\quad + |E_\Psi^{(\frac{\partial}{\partial t})}(-0.85)t_i \leq r^* \leq (0.85)t_i(t = t_i)| \end{aligned}$$

Lemma 4.5.9.

$$F_\Psi^{(\frac{\partial}{\partial t})}(w = w_i^\#)(v_i^\# \leq v \leq v_{i+1}) \lesssim \frac{E_\Psi^{(K)}(t_i)}{t_i^2}$$

Proof. By applying the divergence theorem in a well chosen region, we get,

$$\begin{aligned}
F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i^{\#})(v_i^{\#} \leq v \leq v_{i+1}) &\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq \frac{v_{i+1} - (2t_i - v_{i+1})}{2})(t = t_i)| \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq v_{i+1} - t_i)(t = t_i)| \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq t_{i+1} + r_1^* - t_i)(t = t_i)| \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (1.1)t_i + r_1^* - t_i)(t = t_i)| \\
&\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i)|
\end{aligned} \tag{4.129}$$

We proved that,

$$\begin{aligned}
&|E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i)| \\
= &\int_{r^*=r_1^*}^{r^*=(0.1)t_i+r_1^*} \int_{\mathbf{S}^2} ([\frac{1}{r^2(1-\mu)}|\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{w\phi}|^2] + [\frac{1}{r^2(1-\mu)}|\Psi_{v\theta}|^2 \\
&+ \frac{1}{r^2 \sin^2 \theta(1-\mu)}|\Psi_{v\phi}|^2] + [\frac{1}{(1-\mu)^2}|\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta}|\Psi_{\phi\theta}|^2]).(1-\mu)r^2 d\sigma^2 dr^*(t) \\
\lesssim &\frac{E_{\Psi}^{(K)}(t_i)}{\min_{w \in \{t_i\} \cap \{-r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} w^2} + \frac{E_{\Psi}^{(K)}(t_i)}{\min_{v \in \{t_i\} \cap \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} v^2} \\
\lesssim &\frac{E_{\Psi}^{(K)}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2} + \frac{E_{\Psi}^{(K)}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i + r^*|^2}
\end{aligned} \tag{4.130}$$

For $r^* \in [r_1^*, (0.1)t_i + r_1^*]$,

$$t_i - r_1^* \geq t_i - r^* \geq t_i - [(0.1)t_i + r_1^*] = (0.9)t_i - r_1^*$$

We have,

$$(0.9)t_i - r_1^* \geq 0$$

(for t_i large enough)

Thus,

$$|(0.9)t_i - r_1^*|^2 \leq |t_i - r^*|^2 \leq |t_i - r_1^*|^2$$

$$\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2 \geq |(0.9)t_i - r_1^*|^2$$

Therefore,

$$\frac{E_{\Psi}^{(K)}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2} \gtrsim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2}$$

and thus,

$$|E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i)| \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2}$$

Therefore,

$$F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_i^{\#})(v_i^{\#} \leq v \leq v_{i+1}) \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2}$$

□

We also have,

$$|E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85t_i \leq r^* \leq 0.85t_i)(t = t_i)| \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2}$$

Thus,

$$\begin{aligned} -F_{\Psi}^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty) &\lesssim \frac{1}{t_i}[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0)] + \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2} \\ &\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^{(K)}(t_i)]}{t_i} \end{aligned}$$

(from the above)

and thus,

$$\begin{aligned} -F_{\Psi}^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) &\lesssim -F_{\Psi}^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty) \\ &\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^{(K)}(t_i)]}{t_i} \end{aligned} \tag{4.131}$$

(due to the positivity of $-F_{\Psi}^{(H)}(v = v_{i+1})(w_i^{\#} \leq w \leq \infty)$).

Repeating the same procedure,

Lemma 4.5.10.

$$-I_{\Psi}^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_i}$$

Proof. We get from (4.107),

$$\begin{aligned} & -I_{\Psi}^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1) \\ & \lesssim F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) - F_{\Psi}^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) \\ & \quad + |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})| \end{aligned}$$

(due to the positivity of $-F_{\Psi}^{(H)}(w = \infty)(v_{i+1} \leq v \leq v_{i+2})$, $-F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+1} \leq w \leq \infty)$ and $-I_{\Psi}^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1)$)

We have,

$$|E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})| \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2}$$

and,

$$-F_{\Psi}^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^{(K)}(t_i)]}{t_i}$$

(from above)

And,

$$\begin{aligned} F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) &\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq \frac{v_{i+2} - (2t_{i+1} - v_{i+2})}{2})(t = t_{i+1})| \\ &\lesssim |E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_{i+1} + r_1^*)(t = t_{i+1})| \end{aligned}$$

We proved,

$$|E_{\Psi}^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_{i+1} + r_1^*)(t = t_{i+1})| \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2}$$

hence,

$$F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2}$$

Therefore,

$$-I_{\Psi}^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_i}$$

□

Using (4.111), we get,

$$\begin{aligned} \inf_{v_{i+1} \leq v \leq v_{i+2}} F_{\Psi}^{(H)}(v)(w_{i+1} \leq w \leq \infty) &\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_{i+1}(v_{i+2} - v_{i+1})} + \frac{E_{\Psi}^M}{t_{i+1}^2} \\ &\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_{i+1}^2} \end{aligned}$$

and thus, there exists a $v_{i+1}^{\#} \in [v_{i+1}, v_{i+2}]$ where above inequality holds. We get,

$$\begin{aligned} F_{\Psi}^{(H)}(v_{i+1}^{\#})(w_{i+1}^{\#} \leq w \leq \infty) &\lesssim F_{\Psi}^{(H)}(v_{i+1}^{\#})(w_{i+1} \leq w \leq \infty) \\ &\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_{i+1}^2} \end{aligned}$$

As before, we let,

$$w_{i+1}^{\#} = v_{i+1}^{\#} - 2r_1^*$$

From (4.107), applied in the region $[w_{i+1}^{\#}, \infty]. [v_{i+1}^{\#}, v_{i+2}]$, we get due to the positivity of $-I_{\Psi}^{(H)}(v_{i+1}^{\#} \leq v \leq v_{i+2})(w_{i+1}^{\#} \leq w \leq \infty)$, and $-F_{\Psi}^{(H)}(w = \infty)(v_{i+1}^{\#} \leq v \leq v_{i+2})$, that,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+1}^{\#} \leq w \leq \infty) \\
\lesssim & F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_{i+1}^{\#})(v_{i+1}^{\#} \leq v \leq v_{i+2}) - F_{\Psi}^{(H)}(v = v_{i+1}^{\#})(w_{i+1}^{\#} \leq w \leq \infty) \\
& + |E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})|
\end{aligned}$$

We proved that,

$$F_{\Psi}^{(\frac{\partial}{\partial t})}(w = w_{i+1}^{\#})(v_{i+1}^{\#} \leq v \leq v_{i+2}) \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2}$$

We also have,

$$|E_{\Psi}^{(\frac{\partial}{\partial t})}(-0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})| \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2}$$

Therefore,

$$-F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+1}^{\#} \leq w \leq \infty) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_{i+1}^2}$$

and finally,

$$\begin{aligned}
-F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+2} \leq w \leq \infty) &\lesssim -F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+1}^{\#} \leq w \leq \infty) \\
&\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{t_{i+1}^2} \\
&\lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{v_{+(i+1)}^2}
\end{aligned}$$

(due to the positivity of $-F_{\Psi}^{(H)}(v = v_{i+2})(w_{i+1}^{\#} \leq w \leq \infty)$).

□

4.6 The Proof of Decay Near the Horizon

Let v_+ be as defined in (4.118), we will prove that for all $r \leq r_1$,

$$\begin{aligned}
|F_{\hat{v}\hat{w}}(v, w, \omega)| &\lesssim \frac{E_1}{v_+} \\
|F_{e_1 e_2}(v, w, \omega)| &\lesssim \frac{E_1}{v_+}
\end{aligned}$$

where,

$$E_1 = \left[\sum_{j=0}^6 E_{r^j(\mathcal{L})^j F}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^5 E_{r^j(\mathcal{L})^j F}^{(K)}(t = t_0) + \sum_{j=1}^3 E_{r^j(\mathcal{L})^j F}^{\#(\frac{\partial}{\partial t})}(t = t_0) \right]^{\frac{1}{2}}$$

and,

$$\begin{aligned}
|F_{\hat{v}e_1}(v, w, \omega)| &\lesssim \frac{E_2}{v_+} \\
|F_{\hat{v}e_2}(v, w, \omega)| &\lesssim \frac{E_2}{v_+} \\
|\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_1}(v, w, \omega)| &\lesssim \frac{E_2}{v_+} \\
|\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_2}(v, w, \omega)| &\lesssim \frac{E_2}{v_+}
\end{aligned}$$

where,

$$\begin{aligned}
E_2 &= [E_F^2 + \sum_{i=0}^1 \sum_{j=1}^2 E_{r^j(\mathcal{L}_t)^j}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3}^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}} \\
&= [\sum_{i=0}^1 \sum_{j=0}^5 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^6(\mathcal{L})^6}^{\#(\frac{\partial}{\partial t})}(t=t_0) \\
&\quad + \sum_{i=0}^1 \sum_{j=0}^4 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i}^{(K)}(t=t_0) + E_{r^5(\mathcal{L})^5}^{(K)}(t=t_0) \\
&\quad + \sum_{i=0}^1 \sum_{j=1}^2 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3}^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}}
\end{aligned}$$

Proof.

4.6.1 Decay for $F_{\hat{v}\hat{w}}$ and $F_{e_1e_2}$

We have the Sobolev inequality,

$$\begin{aligned}
|F_{\hat{v}\hat{w}}(v, w, \omega)|^2 &\lesssim \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{w}}|^2 + |\mathcal{L} F_{\hat{v}\hat{w}}|^2 \\
&\quad + |\mathcal{L} \mathcal{L}_v F_{\hat{v}\hat{w}}|^2 + |(\mathcal{L})^2 F_{\hat{v}\hat{w}}|^2 + |(\mathcal{L})^2 \mathcal{L}_v F_{\hat{v}\hat{w}}|^2) d\sigma^2 d\bar{v}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\hat{v}}\Psi_{\hat{v}\hat{w}} &= \nabla_{\hat{v}}\Psi_{\hat{v}\hat{w}} - \Psi\left(\nabla_{\hat{v}}\frac{\hat{\partial}}{\partial v}, \frac{\hat{\partial}}{\partial w}\right) - \Psi\left(\frac{\hat{\partial}}{\partial v}, \nabla_{\hat{v}}\frac{\hat{\partial}}{\partial w}\right) \\
&= -\frac{1}{2}\nabla^{\hat{w}}\Psi_{\hat{v}\hat{w}} - \frac{\mu}{2r}\Psi_{\hat{v}\hat{w}} + \frac{\mu}{2r}\Psi_{\hat{v}\hat{w}} \\
&= \frac{1}{2}[\nabla^{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a} + \nabla^{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b}] \\
&= \frac{1}{2}[\nabla_{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a} + \nabla_{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b}] \\
&= \frac{1}{2}[\mathcal{L}_{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a} - \Psi(\nabla_{\hat{e}_a}\hat{v}, \hat{e}_a) - \Psi(\hat{v}, \nabla_{\hat{e}_a}\hat{e}_a) + \mathcal{L}_{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b} - \Psi(\nabla_{\hat{e}_b}\hat{v}, \hat{e}_b) - \Psi(\hat{v}, \nabla_{\hat{e}_b}\hat{e}_b)] \\
&= \frac{1}{2}[\mathcal{L}_{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a} - \Psi(\hat{v}, \nabla_{\hat{e}_a}\hat{e}_a) + \mathcal{L}_{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b} - \Psi(\hat{v}, \nabla_{\hat{e}_b}\hat{e}_b)]
\end{aligned}$$

(by using the field equations)

Thus,

$$\begin{aligned}
&\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} |\mathcal{L}_v\Psi_{\hat{v}\hat{w}}(v, w, \omega)|^2 d\sigma^2 d\bar{v} \\
\lesssim &\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} |-\nabla_{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a} - \nabla_{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b}|^2 d\sigma^2 d\bar{v} \\
\lesssim &\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|\nabla_{\hat{e}_a}\Psi_{\hat{v}\hat{e}_a}|^2 + |\nabla_{\hat{e}_b}\Psi_{\hat{v}\hat{e}_b}|^2) d\sigma^2 d\bar{v} \\
\lesssim &\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|\not\mathcal{L}\Psi_{\hat{v}\hat{e}_a}|^2 + |\not\mathcal{L}\Psi_{\hat{v}\hat{e}_b}|^2 + |\Psi_{\hat{v}\hat{w}}|^2 + |\Psi_{\hat{v}\hat{e}_a}|^2 + |\Psi_{\hat{v}\hat{e}_b}|^2) d\sigma^2 d\bar{v} \\
\lesssim &\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|r\not\mathcal{L}\Psi_{\hat{v}\hat{e}_a}|^2 + |r\not\mathcal{L}\Psi_{\hat{v}\hat{e}_b}|^2 + |\Psi_{\hat{v}\hat{w}}|^2 + |\Psi_{\hat{v}\hat{e}_a}|^2 + |\Psi_{\hat{v}\hat{e}_b}|^2) r^2 d\sigma^2 d\bar{v}
\end{aligned}$$

(for $r \geq 2m > 0$)

Recall that,

$$\begin{aligned}
& -F_{\Psi}^{(H)}(w)(v-1 \leq \bar{v} \leq v) \\
= & -F_{\Psi}^{(H)}(w)(v-1 \leq \bar{v} \leq v) - F_{\Psi}^{(H_2)}(w)(v-1 \leq \bar{v} \leq v) \\
= & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} 2h(r^*) \left[\frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right. \\
& \left. + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \right] r^2 d\sigma^2 d\bar{v} \\
= & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} 2h(r^*) \left[|\Psi_{ve_1}|^2 + |\Psi_{ve_2}|^2 + \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4} |\Psi_{e_1 e_2}|^2 \right] r^2 d\sigma^2 d\bar{v}
\end{aligned}$$

(by computing (4.90) using the orthonormal basis e_a , $a \in \{1, 2\}$ instead of $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ which are singular at $\theta = 0, \pi$).

Consequently,

$$\int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} |\mathcal{L}_v \Psi_{\hat{v}\hat{w}}(v, w, \omega)|^2 d\sigma^2 d\bar{v} \lesssim -F_{\Psi, r \mathcal{L} \Psi}^{(H)}(w)(v-1 \leq \bar{v} \leq v)$$

Therefore, we have,

$$\begin{aligned}
|F_{\hat{v}\hat{w}}(v, w, \omega)| & \lesssim \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{w}}|^2 + |\mathcal{L} F_{\hat{v}\hat{w}}|^2 \\
& + |\mathcal{L} \mathcal{L}_v F_{\hat{v}\hat{w}}|^2 + |(\mathcal{L})^2 F_{\hat{v}\hat{w}}|^2 + |(\mathcal{L})^2 \mathcal{L}_v F_{\hat{v}\hat{w}}|^2) d\sigma^2 d\bar{v} \\
& \lesssim -F_{F, r \mathcal{L} F, r^2 (\mathcal{L})^2 F, r^3 (\mathcal{L})^3 F}^{(H)}(w)(v-1 \leq \bar{v} \leq v)
\end{aligned}$$

From (4.120), we proved that,

$$-F_{\Psi}^{(H)}(w)(v-1 \leq \bar{v} \leq v) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{\Psi}^M]}{v_+^2}$$

Thus,

$$\begin{aligned}
& -F_{F,r}^{(H)} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(w)(v-1 \leq \bar{v} \leq v) \\
\lesssim & \frac{E_{F,r}^{(\frac{\partial}{\partial t})} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0) + E_{F,r}^{\#(\frac{\partial}{\partial t})} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0)}{v_+^2} \\
& + \frac{E_{F,r}^M \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0)}{v_+^2}
\end{aligned}$$

$$\begin{aligned}
& E_{F,r}^{(\frac{\partial}{\partial t})} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0) + E_{F,r}^{\#(\frac{\partial}{\partial t})} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0) \\
& + E_{F,r}^M \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_F(t=t_0) \\
= & \sum_{j=0}^3 E_{r^j}^{(\frac{\partial}{\partial t})}(\mathcal{L})^j_F(t=t_0) + \sum_{j=0}^3 E_{r^j}^{\#(\frac{\partial}{\partial t})}(\mathcal{L})^j_F(t=t_0) + \sum_{j=0}^3 E_{r^j}^M(\mathcal{L})^j_F
\end{aligned}$$

Recall that,

$$E_F^M = \sum_{j=0}^3 E_{r^j}^{(\frac{\partial}{\partial t})}(\mathcal{L})^j_F(t=t_0) + \sum_{j=0}^2 E_{r^j}^{(K)}(\mathcal{L})^j_F(t=t_0)$$

Thus,

$$\begin{aligned}
\sum_{j=0}^3 E_{r^j}^{(M)}(\mathcal{L})^j_F & = E_{F,r}^{(\frac{\partial}{\partial t}+K)} \mathcal{L}_{F,r^2}(\mathcal{L})^2_{F,r^3}(\mathcal{L})^3_{F,r^4}(\mathcal{L})^4_{F,r^5}(\mathcal{L})^5_F(t=t_0) + E_{r^6}^{(\frac{\partial}{\partial t})}(\mathcal{L})^6_F(t=t_0) \\
& \lesssim \sum_{j=1}^6 E_{r^j}^{(\frac{\partial}{\partial t})}(\mathcal{L})^j_F(t=t_0) + \sum_{j=1}^5 E_{r^j}^{(K)}(\mathcal{L})^j_F(t=t_0)
\end{aligned}$$

We get,

$$\begin{aligned} & |F_{\hat{v}\hat{w}}(v, w, \omega)|^2 \\ \lesssim & \frac{\sum_{j=0}^3 E_{r^j(\mathcal{L})^{jF}}^{\#(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^6 E_{r^j(\mathcal{L})^{jF}}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^5 E_{r^j(\mathcal{L})^{jF}}^{(K)}(t = t_0)}{v_+} \end{aligned}$$

Finally we get,

$$|F_{\hat{v}\hat{w}}(v, w, \omega)| \lesssim \frac{E_1}{v_+}$$

where E_1 is defined as follows,

$$E_1 = \left[\sum_{j=0}^6 E_{r^j(\mathcal{L})^{jF}}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^5 E_{r^j(\mathcal{L})^{jF}}^{(K)}(t = t_0) + \sum_{j=1}^3 E_{r^j(\mathcal{L})^{jF}}^{\#(\frac{\partial}{\partial t})}(t = t_0) \right]^{\frac{1}{2}}$$

Concerning the component $F_{e_1 e_2}$, similarly, we have the Sobolev inequality,

$$\begin{aligned} |F_{e_1 e_2}(v, w, \omega)|^2 & \lesssim \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{e_1 e_2}|^2 + |\mathcal{L}_v F_{e_1 e_2}|^2 + |\mathcal{L} F_{e_1 e_2}|^2 \\ & \quad + |\mathcal{L} \mathcal{L}_v F_{e_1 e_2}|^2 + |(\mathcal{L})^2 F_{e_1 e_2}|^2 + |(\mathcal{L})^2 \mathcal{L}_v F_{e_1 e_2}|^2) d\sigma^2 d\bar{v} \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\hat{v}} \Psi_{e_1 e_2} & = \nabla_{\hat{v}} \Psi_{e_1 e_2} - \Psi(\nabla_{\hat{v}} e_1, e_2) - \Psi(e_1, \nabla_{\hat{v}} e_2) \\ & = -\nabla_{e_1} \Psi_{e_2 \hat{v}} - \nabla_{e_2} \Psi_{\hat{v} e_1} \\ & = \nabla_{e_1} \Psi_{\hat{v} e_2} - \nabla_{e_2} \Psi_{\hat{v} e_1} \\ & = \mathcal{L}_{e_1} \Psi_{\hat{v} e_2} - \Psi(\nabla_{e_1} \hat{v}, e_2) - \Psi(\hat{v}, \nabla_{e_1} e_2) - \mathcal{L}_{e_2} \Psi_{\hat{v} e_1} + \Psi(\nabla_{e_2} \hat{v}, e_1) + \Psi(\hat{v}, \nabla_{e_2} e_1) \end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} |\mathcal{L}_v \Psi_{e_1 e_2}(v, w, \omega)|^2 d\sigma^2 d\bar{v} \\
\lesssim & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|\mathcal{L}_{e_1} \Psi_{\hat{v} e_2}|^2 + |\mathcal{L}_{e_2} \Psi_{\hat{v} e_1}|^2 + |\Psi_{e_1 e_2}|^2 + |\Psi_{\hat{v} e_2}|^2) d\sigma^2 d\bar{v} \\
\lesssim & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|\mathcal{L} \Psi_{\hat{v} e_1}|^2 + |\mathcal{L} \Psi_{\hat{v} e_2}|^2 + |\Psi_{e_1 e_2}|^2 + |\Psi_{\hat{v} e_2}|^2) d\sigma^2 d\bar{v} \\
\lesssim & -F_{\Psi, r}^{(H)} \mathcal{L} \Psi(w) (v-1 \leq \bar{v} \leq v)
\end{aligned}$$

(using what we already proved, and the fact that r is bounded in the region $0 < 2m \leq r \leq R$).

Consequently, we have,

$$|F_{e_1 e_2}(v, w, \omega)|^2 \lesssim -F_{F, r}^{(H)} \mathcal{L} F, r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F(w) (v-1 \leq \bar{v} \leq v)$$

We proved that,

$$\begin{aligned}
& -F_{F, r}^{(H)} \mathcal{L} F, r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F(w) (v-1 \leq \bar{v} \leq v) \\
\lesssim & \frac{E_{F, r}^{\#(\frac{\partial}{\partial t})} F, r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F(t=t_0) + E_{F, r}^M \mathcal{L} F, r^2(\mathcal{L})^2 F, r^3(\mathcal{L})^3 F}{v_+^2}
\end{aligned}$$

Thus,

$$|F_{e_1 e_2}(v, w_0, \omega)| \lesssim \frac{E_1}{v_+}$$

4.6.2 Decay for $F_{\hat{v} e_1}$ and $F_{\hat{v} e_2}$

We have the Sobolev inequality,

$$|F_{\hat{v}\hat{\theta}}(v, w, \omega)|^2 \lesssim \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L} F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L} \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |(\mathcal{L})^2 F_{\hat{v}\hat{\theta}}|^2 + |(\mathcal{L})^2 \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2) d\sigma^2 d\bar{v}$$

On one hand, we can compute,

$$\begin{aligned} \mathcal{L}_{\hat{v}} \Psi_{\hat{v}\hat{\theta}} &= \nabla_{\hat{v}} \Psi_{\hat{v}\hat{\theta}} + \Psi(\nabla_{\hat{v}} \hat{v}, \hat{\theta}) + \Psi(\hat{v}, \nabla_{\hat{v}} \hat{\theta}) \\ &= \nabla_{\hat{v}} \Psi_{\hat{v}\hat{\theta}} + \frac{\mu}{2r} \Psi_{\hat{v}\hat{\theta}} \end{aligned}$$

$$\begin{aligned} \nabla_{\hat{v}} \Psi_{\hat{v}\hat{\theta}} &= \frac{1}{2} \nabla_{\hat{v}} (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) = \frac{1}{2} \nabla_{(\frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*})} (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) \\ &= \frac{1}{4} \nabla_t (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) + \frac{1}{4} \nabla_{r^*} (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) \\ &= \frac{1}{4} \nabla_t (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) + \frac{1}{4} (-\nabla_t \Psi_{\hat{\theta}r^*} - \nabla_{\hat{\theta}} \Psi_{r^*t} + \nabla_t \Psi_{t\hat{\theta}} - (1-\mu) \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \\ &= \frac{1}{2} \nabla_t (\Psi_{t\hat{\theta}} + \Psi_{r^*\hat{\theta}}) - \frac{1}{4} (\nabla_{\hat{\theta}} \Psi_{r^*t} + (1-\mu) \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \\ &= \nabla_t \Psi_{\hat{v}\hat{\theta}} - \frac{1}{4} (1-\mu) (\nabla_{\hat{\theta}} \Psi_{r^*t} + \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \end{aligned}$$

(where we used the field equations and the Bianchi identities).

Thus,

$$\begin{aligned} \nabla_{\hat{v}} \Psi_{\hat{v}\hat{\theta}} &= \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \Psi(\nabla_t \hat{v}, \hat{\theta}) - \Psi(\hat{v}, \nabla_t \hat{\theta}) \\ &\quad + \frac{(1-\mu)}{4} [-\mathcal{L}_{\hat{\theta}} \Psi_{r^*t} + \Psi(\nabla_{\hat{\theta}} r^*, \hat{t}) + \Psi(r^*, \nabla_{\hat{\theta}} \hat{t}) \\ &\quad - \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}} + \Psi(\nabla_{\hat{\phi}} \hat{\phi}, \hat{\theta}) + \Psi(\hat{\phi}, \nabla_{\hat{\phi}} \hat{\theta})] \\ &= \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \frac{\mu}{2r} \Psi_{\hat{v}\hat{\theta}} \\ &\quad + \frac{1}{4} (1-\mu) [-\mathcal{L}_{\hat{\theta}} \Psi_{r^*t} + \frac{\sqrt{1-\mu}}{r} \Psi_{\hat{\theta}t} - \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}} - \frac{\sqrt{1-\mu}}{r} \Psi_{r^*\hat{\theta}}] \end{aligned}$$

$$\begin{aligned}
&= \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \frac{1}{4}(1-\mu)\mathcal{L}_{\hat{\theta}}\Psi_{r^*\hat{t}} - \frac{1}{4}(1-\mu)\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}} \\
&\quad - \frac{1}{2}\left(\frac{\mu}{r} + \frac{(1-\mu)}{r}\right)\Psi_{\hat{v}\hat{\theta}} \\
&= \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \frac{(1-\mu)}{2}\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4}\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}} \\
&\quad - \frac{1}{2}\left(\frac{\mu}{r} + \frac{(1-\mu)}{r}\right)\Psi_{\hat{v}\hat{\theta}}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}_{\hat{v}}\Psi_{\hat{v}\hat{\theta}} &= \nabla_{\hat{v}}\Psi_{\hat{v}\hat{\theta}} + \frac{\mu}{2r}\Psi_{\hat{v}\hat{\theta}} \\
&= \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \frac{(1-\mu)}{2}\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4}\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}} - \frac{(1-\mu)}{2r}\Psi_{\hat{v}\hat{\theta}}
\end{aligned}$$

and similarly, we obtain,

$$\mathcal{L}_{\hat{v}}\Psi_{\hat{v}\hat{\phi}} = \mathcal{L}_t \Psi_{\hat{v}\hat{\phi}} - \frac{(1-\mu)}{2}\mathcal{L}_{\hat{\phi}}\Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4}\mathcal{L}_{\hat{\theta}}\Psi_{\hat{\theta}\hat{\phi}} - \frac{(1-\mu)}{2r}\Psi_{\hat{v}\hat{\phi}}$$

Thus, for $r \geq 2m > 0$,

$$\begin{aligned}
|\mathcal{L}_v \Psi_{\hat{v}\hat{\theta}}|^2 &= \left| \mathcal{L}_t \Psi_{\hat{v}\hat{\theta}} - \frac{(1-\mu)}{2}\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4}\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}} - \frac{(1-\mu)}{2r}\Psi_{\hat{v}\hat{\theta}} \right|^2 \\
&\lesssim |\mathcal{L}_t \Psi_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\theta}}|^2
\end{aligned}$$

(by using $a.b \lesssim a^2 + b^2$)

We get,

$$\begin{aligned}
|\mathcal{L}_v \Psi_{\hat{v}\hat{\theta}}|^2 &\lesssim |\mathcal{L}_t \Psi_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\theta}}|^2 \\
&\lesssim |\mathcal{L}_t \Psi_{\hat{v}\hat{\theta}}|^2 + |r\mathcal{L}_{\hat{\theta}}\Psi_{\hat{v}\hat{w}}|^2 + |r\mathcal{L}_{\hat{\phi}}\Psi_{\hat{\phi}\hat{\theta}}|^2 + |\Psi_{\hat{v}\hat{\theta}}|^2
\end{aligned}$$

(in the region $0 < 2m \leq r \leq R$).

Therefore,

$$\begin{aligned}
& |F_{\hat{v}\hat{\theta}}(v, w, \omega)|^2 \\
\lesssim & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L} F_{\hat{v}\hat{\theta}}|^2 \\
& + |\mathcal{L} \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |(\mathcal{L})^2 F_{\hat{v}\hat{\theta}}|^2 + |(\mathcal{L})^2 \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2) d\sigma^2 d\bar{v} \\
\lesssim & \int_{\bar{v}=v-1}^{\bar{v}=v} \int_{\mathbf{S}^2} (|F_{\hat{v}\hat{\theta}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |r \mathcal{L} F_{\hat{v}\hat{\theta}}|^2 \\
& + |r \mathcal{L} \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2 + |r^2 (\mathcal{L})^2 F_{\hat{v}\hat{\theta}}|^2 + |r^2 (\mathcal{L})^2 \mathcal{L}_v F_{\hat{v}\hat{\theta}}|^2) d\sigma^2 d\bar{v} \\
\lesssim & -F_{F, r \mathcal{L} F, r^2 (\mathcal{L})^2 F, \mathcal{L}_t F, r \mathcal{L} \mathcal{L}_t F, r^2 (\mathcal{L})^2 \mathcal{L}_t F, r^3 (\mathcal{L})^3 F}^{(H)}(w)(v-1 \leq \bar{v} \leq v) \\
\lesssim & \sum_{j=0}^2 \sum_{i=0}^1 -F_{r^j (\mathcal{L})^j (\mathcal{L}_t)^i F}^{(H)}(w)(v-1 \leq \bar{v} \leq v) - F_{r^3 (\mathcal{L})^3 F}^{(H)}(w)(v-1 \leq \bar{v} \leq v)
\end{aligned}$$

From estimate (4.120),

$$-F_{\Psi}^{(H)}(w)(v-1 \leq \bar{v} \leq v) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{\Psi}^M]}{v_+^2}$$

Recall that,

$$E_{\Psi}^M = \sum_{j=0}^3 E_{r^j (\mathcal{L})^j \Psi}^{(\frac{\partial}{\partial t})}(t=t_0) + \sum_{j=0}^2 E_{r^j (\mathcal{L})^j \Psi}^{(K)}(t=t_0)$$

Therefore,

$$|F_{\hat{\theta}}(v, w, \omega)| \lesssim \frac{E_2}{v_+}$$

where,

$$\begin{aligned} E_2 &= \left[\sum_{j=0}^5 \sum_{i=0}^1 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{(\frac{\partial}{\partial t})}(t=t_0) + E_{r^6(\mathcal{L})^6 F}^{(\frac{\partial}{\partial t})}(t=t_0) \right. \\ &\quad + \sum_{j=0}^4 \sum_{i=0}^1 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{(K)}(t=t_0) + E_{r^5(\mathcal{L})^5 F}^{(K)}(t=t_0) \\ &\quad \left. + \sum_{j=1}^2 \sum_{i=0}^1 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3 F}^{\#(\frac{\partial}{\partial t})}(t=t_0) \right]^{\frac{1}{2}} \\ &= [E_F^2 + \sum_{j=1}^2 \sum_{i=0}^1 E_{r^j(\mathcal{L})^j(\mathcal{L}_t)^i F}^{\#(\frac{\partial}{\partial t})}(t=t_0) + E_{r^3(\mathcal{L})^3 F}^{\#(\frac{\partial}{\partial t})}(t=t_0)]^{\frac{1}{2}} \end{aligned}$$

and similarly,

$$|F_{\hat{\phi}}(v, w, \omega)| \lesssim \frac{E_2}{v_+}$$

4.6.3 Decay for $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_1}$ and $\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_2}$

We have the Sobolev inequality,

$$\begin{aligned}
& |\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}}(v, w, \omega)|^2 \\
\lesssim & \int_{\bar{w}=w_0}^{\bar{w}=\infty} \int_{\mathbf{S}^2} (|\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_w(\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}})|^2 + |\mathcal{L}(\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}})|^2 \\
& + |\mathcal{L}\mathcal{L}_w(\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}})|^2 + |(\mathcal{L})^2(\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}})|^2 \\
& + |(\mathcal{L})^2\mathcal{L}_w(\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}})|^2) d\sigma^2 d\bar{w}
\end{aligned}$$

We have,

$$\begin{aligned}
\mathcal{L}\hat{w}\Psi_{\hat{w}\hat{\theta}} &= \nabla_{\hat{w}}\Psi_{\hat{w}\hat{\theta}} + \Psi(\nabla_{\hat{w}}\hat{w}, \hat{\theta}) + \Psi(\hat{w}, \nabla_{\hat{w}}\hat{\theta}) \\
&= \nabla_{\hat{w}}\Psi_{\hat{w}\hat{\theta}} \\
&= \frac{1}{(1-\mu)^2} \nabla_w \Psi_{w\hat{\theta}}
\end{aligned}$$

Computing,

$$\begin{aligned}
\nabla_w \Psi_{w\hat{\theta}} &= \frac{1}{2} \nabla_w (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) = \frac{1}{2} \nabla_{(\frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial r^*})} (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) - \frac{1}{4} \nabla_{r^*} (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) - \frac{1}{4} (-\nabla_t \Psi_{\hat{\theta}r^*} - \nabla_{\hat{\theta}} \Psi_{r^*t} - \nabla_t \Psi_{t\hat{\theta}} + (1-\mu) \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) + \frac{1}{4} (-\nabla_t \Psi_{r^*\hat{\theta}} + \nabla_t \Psi_{t\hat{\theta}} + \nabla_{\hat{\theta}} \Psi_{r^*t} - (1-\mu) \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \\
&= \frac{1}{2} \nabla_t (\Psi_{t\hat{\theta}} - \Psi_{r^*\hat{\theta}}) + \frac{(1-\mu)}{4} (\nabla_{\hat{\theta}} \Psi_{r^*\hat{t}} - \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}})
\end{aligned}$$

(using the field equations and the Bianchi identities).

Thus,

$$\begin{aligned}
\nabla_w \Psi_{w\hat{\theta}} &= \frac{1}{2}(\nabla_t \Psi_{t\hat{\theta}} - \nabla_t \Psi_{r^*\hat{\theta}}) + \frac{(1-\mu)}{4}(\nabla_{\hat{\theta}} \Psi_{r^*\hat{t}} - \nabla_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) \\
&= \frac{1}{2}(\mathcal{L}_t \Psi_{t\hat{\theta}} - \frac{\mu}{2r} \Psi_{r^*\hat{\theta}} - \mathcal{L}_t \Psi_{r^*\hat{\theta}} + \frac{\mu}{2r} \Psi_{t\hat{\theta}}) \\
&\quad + \frac{(1-\mu)}{4}(\mathcal{L}_{\hat{\theta}} \Psi_{r^*\hat{t}} - \frac{\sqrt{1-\mu}}{r} \Psi_{\hat{\theta}\hat{t}} - \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}} - \frac{\sqrt{1-\mu}}{r} \Psi_{r^*\hat{\theta}}) \\
&= \mathcal{L}_t \Psi_{w\hat{\theta}} + \frac{(1-\mu)}{4}(\mathcal{L}_{\hat{\theta}} \Psi_{r^*\hat{t}} - \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}) + \frac{\mu}{2r} \Psi_{\hat{w}\hat{\theta}} + \frac{(1-\mu)}{2r} \Psi_{w\hat{\theta}} \\
&= \mathcal{L}_t \Psi_{w\hat{\theta}} + \frac{(1-\mu)}{2} \mathcal{L}_{\hat{\theta}} \Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4} \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}} + \Psi_{w\hat{\theta}}
\end{aligned}$$

and similarly, we obtain,

$$\nabla_w \Psi_{w\hat{\phi}} = \mathcal{L}_t \Psi_{w\hat{\phi}} + \frac{(1-\mu)}{2} \mathcal{L}_{\hat{\phi}} \Psi_{\hat{v}\hat{w}} - \frac{(1-\mu)}{4} \mathcal{L}_{\hat{\theta}} \Psi_{\hat{\theta}\hat{\phi}} + \Psi_{w\hat{\phi}}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_w \Psi_{\hat{w}\hat{\theta}} &= (1-\mu) \mathcal{L}_{\hat{w}} \Psi_{\hat{w}\hat{\theta}} = \frac{1}{(1-\mu)} \nabla_w \Psi_{w\hat{\theta}} \\
&= \mathcal{L}_t \Psi_{\hat{w}\hat{\theta}} + \frac{1}{2} \mathcal{L}_{\hat{\theta}} \Psi_{\hat{v}\hat{w}} - \frac{1}{4} \mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}} + \Psi_{\hat{w}\hat{\theta}}
\end{aligned}$$

Hence,

$$|\mathcal{L}_w \Psi_{\hat{w}\hat{\theta}}|^2 \lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_{\hat{\theta}} \Psi_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_{\hat{\phi}} \Psi_{\hat{\phi}\hat{\theta}}|^2 + |\Psi_{\hat{w}\hat{\theta}}|^2$$

(by using $a.b \lesssim a^2 + b^2$)

We get,

$$\begin{aligned} |\mathcal{L}_w \Psi_{\hat{w}\hat{\theta}}|^2 &\lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L} \Psi_{\hat{v}\hat{w}}|^2 + |\mathcal{L} \Psi_{\hat{\theta}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\theta}}|^2 \\ &\lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\theta}}|^2 + |r \mathcal{L} \Psi_{\hat{v}\hat{w}}|^2 + |r \mathcal{L} \Psi_{\hat{\theta}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\theta}}|^2 \end{aligned}$$

(in the region $0 < 2m \leq r \leq R$).

We also have,

$$\begin{aligned} \frac{\partial}{\partial w} \sqrt{(1-\mu)} &= \left(\frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*} \right) \sqrt{(1-\mu)} = -\frac{(1-\mu)}{2} \frac{\partial}{\partial r} \sqrt{(1-\mu)} = -\frac{(1-\mu)}{2} \frac{\mu}{2r \sqrt{(1-\mu)}} \\ &= -\frac{\mu}{4r} \sqrt{(1-\mu)} \end{aligned}$$

Thus,

$$\begin{aligned} &|\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}}(v, w, \omega)|^2 \\ &\lesssim \int_{\bar{w}=w_0}^{\bar{w}=\infty} \int_{\mathbf{S}^2} (|\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_w(\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}})|^2 + |\mathcal{L}(\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}})|^2 \\ &\quad + |\mathcal{L} \mathcal{L}_w(\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}})|^2 + |(\mathcal{L})^2(\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}})|^2 \\ &\quad + |(\mathcal{L})^2 \mathcal{L}_w(\sqrt{(1-\mu)} F_{\hat{w}\hat{\theta}})|^2) d\sigma^2 d\bar{w} \\ &\lesssim \int_{\bar{w}=w_0}^{\bar{w}=\infty} \int_{\mathbf{S}^2} (1-\mu)(|F_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_w F_{\hat{w}\hat{\theta}}|^2 + |r \mathcal{L} F_{\hat{w}\hat{\theta}}|^2 \\ &\quad + |r \mathcal{L} \mathcal{L}_w F_{\hat{w}\hat{\theta}}|^2 + |r^2 (\mathcal{L})^2 F_{\hat{w}\hat{\theta}}|^2 + |r^2 (\mathcal{L})^2 \mathcal{L}_w F_{\hat{w}\hat{\theta}}|^2) d\sigma^2 d\bar{w} \\ &\lesssim -F_{F,r \mathcal{L} F, r^2 (\mathcal{L})^2 F, \mathcal{L}_t F, r \mathcal{L} \mathcal{L}_t F, r^2 (\mathcal{L})^2 \mathcal{L}_t F, r^3 (\mathcal{L})^3 F}^{(H)}(v)(w_0 \leq w \leq \infty) \\ &\lesssim \sum_{j=0}^2 \sum_{i=0}^1 -F_{r^j (\mathcal{L})^j (\mathcal{L}_t)^i F}^{(H)}(v)(w_0 \leq w \leq \infty) - F_{r^3 (\mathcal{L})^3 F}^{(H)}(v)(w_0 \leq w \leq \infty) \end{aligned}$$

From estimate 6,

$$-F_{\Psi}^{(H)}(v)(w_0 \leq w \leq \infty) \lesssim \frac{[|E_{\Psi}^{(\frac{\partial}{\partial t})}| + E_{\Psi}^{\#(\frac{\partial}{\partial t})}(t = t_0) + E_{\Psi}^M]}{v_+^2}$$

As

$$E_{\Psi}^M = \sum_{j=0}^3 E_{r^j(\mathcal{L})^j\Psi}^{(\frac{\partial}{\partial t})}(t = t_0) + \sum_{j=0}^2 E_{r^j(\mathcal{L})^j\Psi}^{(K)}(t = t_0)$$

we have,

$$|\sqrt{(1-\mu)}F_{\hat{w}\hat{\theta}}(v, w, \omega)| \lesssim \frac{E_2}{v_+}$$

and similarly,

$$|\sqrt{(1-\mu)}F_{\hat{w}\hat{\phi}}(v, w, \omega)| \lesssim \frac{E_2}{v_+}$$

□

Chapter 5

Opening to Different Problems in PDE

5.1 Introduction

The problem of breakdown of solutions of the critical quasi-geostrophic equation with arbitrary smooth initial data was suggested by S. Klainerman in [K12] as one of the most challenging problems in partial differential equations of the twenty-first century. In an elegant paper, [KNV], A. Kiselev, F. Nazarov, and A. Voldberg, proved global well-posedness of the critical 2-dimensional dissipative quasi-geostrophic equation with smooth periodic initial data. This chapter is a remark that by using an adaptation of the technique introduced by Kiselev, Nazarov, and Voldberg, in [KNV], with a modified scaling argument, we can immediately prove global regularity of the critical 2-dimensional dissipative quasi-geostrophic equation with smooth periodic force, under the assumption that the initial data is smooth and periodic, and the force is bounded in space and time, and α -Hölder continuous in space, $\alpha > 0$.

5.2 Global Regularity for the Critical 2-D Dissipative Quasi-Geostrophic Equation with Force on the Torus

5.2.1 The statement

We consider the critical surface quasi-geostrophic equation with force, which we will write as the following:

$$\partial_t \theta(x, t) = u \cdot \nabla \theta(x, t) - (-\Delta)^{\frac{1}{2}} \theta(x, t) + f(x, t) \quad (5.1)$$

where $x \in \mathbb{R}^2$, $u(x, t) = (-R_2 \theta, R_1 \theta)$, where R_1 and R_2 are the usual Riesz transforms in \mathbb{R}^2 , $\theta(x, t) : \mathbb{R}^2 \cdot \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function, and $f(x, t) : \mathbb{R}^2 \cdot \mathbb{R} \rightarrow \mathbb{R}$ is the force function.

We assume f smooth and periodic on \mathbb{R}^2 (in space), and bounded in space and time, i.e.

$$\|f(x, t)\|_{L^\infty} < \infty \quad (5.2)$$

We also assume f to be α -Hölder continuous with $\alpha > 0$, i.e. there exist constants $C_1 \geq 0$ and $\alpha > 0$ which do not depend on t , such that for all x, y in \mathbb{R}^2 ,

$$|f(x, t) - f(y, t)| \leq C_1 |x - y|^\alpha \quad (5.3)$$

The goal of section (5.3) is to prove the following theorem,

Theorem 5.2.2. *The gradient of local solutions of the critical surface dissipative quasi-geostrophic equation with smooth periodic force, (5.1), with smooth periodic initial data, will not blow up in time under assumptions (5.2) and (5.3) on the force.*

5.2.3 Strategy of the proof

We will prove theorem (5.2.2) by proving that for θ a solution of (5.1) with smooth periodic initial data θ_0 , $\|\nabla\theta\|_{L^\infty}$ is bounded by a constant depending on $\|f\|_{L^\infty}$, on C_1 and α as defined in (5.3), on $\|\nabla\theta_0\|_{L^\infty}$, and on the period of θ_0 and f . Once this is achieved, one can show that local solutions can be extended to global solutions in time by adapting the argument shown by A. Kiselev in [K]. To prove such an estimate on $\|\nabla\theta\|_{L^\infty}$ we will use the method of modulus of continuity of A. Kiselev, F. Nazarov, and A. Volberg in [KNV], with a modified scaling argument.

Definition 5.2.4. We say that a function ω is a modulus of continuity if $\omega : R_+ \rightarrow R_+$ is increasing, continuous, concave, and $\omega(0) = 0$.

Definition 5.2.5. We say that θ has modulus of continuity ω , or ω is preserved by θ , at time t , if for all $x, y \in R^2$,

$$|\theta(x, t) - \theta(y, t)| \leq \omega(|x - y|) \tag{5.4}$$

Observe now that if at time t , θ has ω as modulus of continuity, then

$$\frac{|\theta(x + h, t) - \theta(x, t)|}{|h|} \leq \frac{\omega(|h|)}{|h|}$$

By taking the limit when $|h| \rightarrow 0$ in the above inequality, we obtain for all $x \in R^2$

$$|\nabla\theta(x, t)| \leq \omega'(0)$$

Therefore, by taking the supremum in space in the above inequality, we get that

$$\|\nabla\theta(x, t)\|_{L^\infty} \leq \omega'(0) \tag{5.5}$$

Consequently, if we manage to find one special function ω , modulus of continuity, such that given A large enough depending on $\|\nabla\theta_0\|_{L^\infty}$, where θ_0 is the initial data, on $\|f\|_{L^\infty}$, and on the period of θ_0 and of f , such that

$$\omega_A(\zeta) = \omega(A\zeta) \tag{5.6}$$

is a modulus of continuity for θ_0 , and ω_A remains preserved for all time t by θ , a smooth solution of (5.1) with θ_0 as initial data, in the sense of (5.4), then

$$\|\nabla\theta(x, t)\|_{L^\infty} \leq A\omega'(0) \tag{5.7}$$

Let's look for such ω :

If

$$\omega'(0) = 1 \tag{5.8}$$

and

$$\lim_{\zeta \rightarrow \infty} \omega(\zeta) = \infty \tag{5.9}$$

then we notice that since any smooth periodic function θ_0 is bounded, we can choose $A > 0$ large enough such that θ_0 has $\omega_A(\zeta) = \omega(A\zeta)$ as modulus of continuity, with A depending on $\|\nabla\theta_0\|_{L^\infty}$ and on the period of θ_0 .

If we also impose on ω to have

$$\lim_{\zeta \rightarrow 0^+} \omega''(\zeta) = -\infty$$

then, since θ is smooth because θ_0 and f are smooth, the only way for ω to stop being a modulus of continuity for θ after some time is that there exists a time T , and $x, y \in R^2$,

$x \neq y$, such that

$$\theta(x, T) - \theta(y, T) = \omega_A(|x - y|) \quad (5.10)$$

and

$$\partial_t(\theta(x, T) - \theta(y, T)) \geq 0 \quad (5.11)$$

Hence, we are going to look for ω verifying (5.8) and (5.9) such that

$$\omega''(0) = -\infty \quad (5.12)$$

and such that at $x, y \in R^2$ where (5.10) is verified, we have

$$\partial_t(\theta(x, T) - \theta(y, T)) < 0 \quad (5.13)$$

Because of (5.11), inequality (5.13) will prove that ω_A is preserved by θ for all time t , and consequently we will have our estimate.

5.3 Estimate for $\|\nabla\theta(x, t)\|_{L^\infty}$

Let ω a modulus of continuity, in the sense of (5.2.4), such that,

$$\lim_{\zeta \rightarrow \infty} \omega(\zeta) = \infty \quad (5.14)$$

$$\omega'(0) = 1 \quad (5.15)$$

$$\omega''(0) = -\infty \quad (5.16)$$

Given an arbitrary smooth periodic initial data θ_0 , since it is a C^1 function on a compact, we can choose A large enough depending on $\|\nabla\theta_0\|_{L^\infty}$ and the period of θ_0 , such

that θ_0 has ω_A as modulus of continuity, i.e. for all $x, y \in R^2$, we have

$$|\theta_0(x) - \theta_0(y)| \leq \omega_A(|x - y|) \quad (5.17)$$

This gives for all $x, y \in R^2$

$$|\theta_0(\frac{x}{A}) - \theta_0(\frac{y}{A})| \leq \omega(|x - y|) \quad (5.18)$$

Definition 5.3.1. Let A such that we have (5.17), we define

$$\hat{\theta}(x, t) = \theta(\frac{x}{A}, \frac{t}{A}) \quad (5.19)$$

If $\theta(x, t)$ solves (5.1), then $\hat{\theta}(x, t)$ satisfies

$$\partial_t \hat{\theta}(x, t) = \hat{u} \cdot \nabla \hat{\theta}(x, t) - (-\Delta)^{\frac{1}{2}} \hat{\theta}(x, t) + \frac{1}{A} f(\frac{x}{A}, \frac{t}{A}) \quad (5.20)$$

We would want to find ω preserved by $\hat{\theta}$ for all time t . For this, we will proceed as explained in (5.2.3):

Let $x, y \in R^2$, $x \neq y$, be such that $\hat{\theta}$ has ω as modulus of continuity for all time $t \leq T$, and

$$\hat{\theta}(x, T) - \hat{\theta}(y, T) = \omega(|x - y|) \quad (5.21)$$

Let

$$\zeta = |x - y| \quad (5.22)$$

As explained in (5.2.3), we want to find ω such that for $x, y \in R^2$ as in (5.21), we have

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0 \quad (5.23)$$

This will give that ω is preserved by $\hat{\theta}$ for all time t , and consequently ω_A is preserved by θ for all time, and therefore we will have our desired estimate (5.7).

Computing,

$$\begin{aligned} & \partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \\ = & \hat{u} \cdot \nabla \hat{\theta}(x, T) - \hat{u} \cdot \nabla \hat{\theta}(y, T) - [(-\Delta)^{\frac{1}{2}} \hat{\theta}(x, T) - (-\Delta)^{\frac{1}{2}} \hat{\theta}(y, T)] \\ & + \frac{1}{A} f\left(\frac{x}{A}, \frac{T}{A}\right) - \frac{1}{A} f\left(\frac{y}{A}, \frac{T}{A}\right) \end{aligned} \quad (5.24)$$

Lemma 5.3.2. *If the function $\hat{\theta}$ has modulus of continuity ω , then $\hat{u} = (-R_2 \hat{\theta}, R_1 \hat{\theta})$ has modulus of continuity $\Omega(\zeta)$, where*

$$\Omega(\zeta) = B \left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \right) \quad (5.25)$$

with some universal constant $B > 0$.

Proof. A sketch of the proof of (5.3.2) is in the Appendix of [KNV].

□

Lemma 5.3.3. *For x, y and T as in (5.21), and ζ defined as in (5.22), we have*

$$\hat{u} \cdot \nabla \hat{\theta}(x, T) - \hat{u} \cdot \nabla \hat{\theta}(y, T) \leq \Omega(\zeta) \omega'(\zeta) \geq 0 \quad (5.26)$$

$$\begin{aligned} -[(-\Delta)^{\frac{1}{2}} \hat{\theta}(x, T) - (-\Delta)^{\frac{1}{2}} \hat{\theta}(y, T)] & \leq \frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \\ & + \frac{1}{\pi} \int_{\frac{\zeta}{2}}^\infty \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \end{aligned} \quad (5.27)$$

$$\frac{1}{A} \left[f\left(\frac{x}{A}, \frac{T}{A}\right) - f\left(\frac{y}{A}, \frac{T}{A}\right) \right] \leq \frac{C_1}{A^{1+\alpha}} \zeta^\alpha \quad (5.28)$$

for some $\alpha > 0$ and $C_1 \geq 0$ as in (5.3).

Proof.

To prove (5.26), we compute

$$\hat{u} \cdot \nabla \hat{\theta}(x, T) - \hat{u} \cdot \nabla \hat{\theta}(y, T) = \frac{d}{dh} \Big|_{h=0} [\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T)] \quad (5.29)$$

We have

$$\begin{aligned} \hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) &\leq \omega(x + hu(x) - y - hu(y)) \\ &\quad (\text{because } \omega \text{ is preserved by } \hat{\theta} \text{ at time } T) \\ &\leq \omega(\zeta + h|\hat{u}(x) - \hat{u}(y)|) \end{aligned} \quad (5.30)$$

and

$$|\hat{u}(x) - \hat{u}(y)| \leq \Omega(\zeta) \quad (5.31)$$

(by (5.3.2))

Since ω is increasing, (5.30) and (5.31) give

$$\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) \leq \omega(\zeta + h\Omega(\zeta)) \quad (5.32)$$

(5.32) and (5.21) lead to

$$\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) - [\hat{\theta}(x, T) - \hat{\theta}(y, T)] \leq \omega(\zeta + h\Omega(\zeta)) - \omega(\zeta)$$

Hence,

$$\lim_{h \rightarrow 0} \frac{\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) - [\hat{\theta}(x, T) - \hat{\theta}(y, T)]}{h} \leq \lim_{h \rightarrow 0} \frac{\omega(\zeta + h\Omega(\zeta)) - \omega(\zeta)}{h}$$

Consequently,

$$\frac{d}{dh}\Big|_{h=0} [\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T)] \leq \Omega(\zeta)\omega'(\zeta) \quad (5.33)$$

(5.26) comes out after injecting (5.33) in (5.29).

(5.27) is proved in [KNV].

(5.28) comes out from assumption (5.3), that f is α -Hölder continuous, with $\alpha > 0$:

$$\frac{1}{A} [f(\frac{x}{A}, \frac{T}{A}) - f(\frac{y}{A}, \frac{T}{A})] \leq \frac{C_1}{A} \frac{|x - y|^\alpha}{A^\alpha} = \frac{C_1}{A^{1+\alpha}} \zeta^\alpha$$

□

5.3.4 Construction of ω

Let $\delta > 0$ small enough to be chosen later, $\beta = \min\{\frac{1}{2}, \alpha\}$, where $\alpha > 0$ is defined as in (5.3), and $0 < \gamma \leq \frac{\delta}{2}$.

For $0 \leq \zeta \leq \delta$, let

$$\omega(\zeta) = \zeta - \zeta^{1+\beta} \quad (5.34)$$

For $\zeta > \delta$, let

$$\omega'(\zeta) = \frac{\gamma}{\zeta(4 + \log(\frac{\zeta}{\delta}))} \quad (5.35)$$

Remark 5.3.5. For δ small enough, and $0 < \gamma \leq \frac{\delta}{2}$, ω is a modulus of continuity verifying (5.14), (5.15), and (5.16).

Lemma 5.3.6. *Let $x, y \in R^2$ be as in (5.21) with ω as defined in (5.34) and (5.35), and let $\zeta = |x - y| > 0$. If we choose δ and γ small enough, with $0 < \gamma \leq \frac{\delta}{2}$, then for all*

$0 < \zeta \leq A.D$, where D is the period of θ , we have (5.23), i.e.

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0$$

Proof.

Checking inequality (5.23) for $0 < \zeta \leq \delta$

Injecting (5.25) in (5.26), we get

$$\hat{u}.\nabla\hat{\theta}(x, T) - \hat{u}.\nabla\hat{\theta}(y, T) \leq B\left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta\right)\omega'(\zeta) \quad (5.36)$$

From (5.34), we have

$$\omega(\eta) \leq \eta$$

Thus,

$$\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta \leq \zeta \quad (5.37)$$

On the other hand,

$$\begin{aligned} \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta &= \int_\zeta^\delta \frac{\omega(\eta)}{\eta^2} d\eta + \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \\ &= \int_\zeta^\delta \left(\frac{1}{\eta} - \eta^{-1+\beta}\right) d\eta + \left[\frac{-\omega(\eta)}{\eta}\right]_\delta^\infty - \int_\delta^\infty \frac{-\omega'(\eta)}{\eta} d\eta \\ &\quad \text{(by integrating by parts in the second integral)} \\ &= [\ln \eta]_\zeta^\delta - \left[\frac{\eta^\beta}{\beta}\right]_\zeta^\delta + \frac{\omega(\delta)}{\delta} + \gamma \int_\delta^\infty \frac{1}{\eta^2(4 + \ln(\frac{\eta}{\delta}))} d\eta \\ &\leq \ln\left(\frac{\delta}{\zeta}\right) + 1 + \gamma \int_\delta^\infty \frac{1}{\eta^2} d\eta \\ &\leq \ln\left(\frac{\delta}{\zeta}\right) + 1 + \frac{\gamma}{\delta} \end{aligned}$$

If we choose $\gamma \leq \delta$, we obtain

$$\int_{\zeta}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta \leq 2 + \ln\left(\frac{\delta}{\zeta}\right) \quad (5.38)$$

We also have from (5.34),

$$\omega'(\zeta) \leq 1 \quad (5.39)$$

Injecting (5.37), (5.38), and (5.39) in (5.36), we get

$$\begin{aligned} \hat{u} \cdot \nabla \hat{\theta}(x, T) - \hat{u} \cdot \nabla \hat{\theta}(y, T) &\leq B[\zeta + \zeta(2 + \ln(\frac{\delta}{\zeta}))].1 \\ &\leq B[3\zeta + \zeta \ln(\frac{\delta}{\zeta})] \end{aligned} \quad (5.40)$$

On the other hand, (5.27) has two terms and they are both negative due to the concavity of ω . Indeed, the first term in (5.27) is

$$\frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta$$

If we choose δ small enough, then ω is concave. In addition, $\omega'''(\zeta) > 0$ due to the choice of β . Hence, using the Taylor series, we can estimate

$$\begin{aligned} \omega(\zeta + 2\eta) &\leq \omega(\zeta) + 2\omega'(\zeta)\eta \\ \omega(\zeta - 2\eta) &\leq \omega(\zeta) - 2\omega'(\zeta)\eta + 2\omega''(\zeta)\eta^2 \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta &\leq \frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{2\omega(\zeta) + 2\omega''(\zeta)\eta^2 - 2\omega(\zeta)}{\eta^2} d\eta \\
&\leq \frac{1}{\pi} \int_0^{\frac{\zeta}{2}} 2\omega''(\zeta) d\eta \\
&\leq \frac{\zeta}{\pi} \omega''(\zeta) \\
&\leq -\frac{\beta(1+\beta)}{\pi} \zeta^\beta
\end{aligned} \tag{5.41}$$

Whereas to the second term in (5.27),

$$\frac{1}{\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta$$

since ω is concave, we have

$$\begin{aligned}
\omega(2\eta + \zeta) &= \omega(2\eta - \zeta + \zeta + \zeta) \\
&\leq \omega(2\eta - \zeta) + \omega(\zeta + \zeta) \\
&\leq \omega(2\eta - \zeta) + 2\omega(\zeta)
\end{aligned}$$

Hence,

$$\frac{1}{\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq 0 \tag{5.42}$$

Injecting (5.41) and (5.42) in (5.27), we obtain

$$-[(-\Delta)^{\frac{1}{2}} \hat{\theta}(x, T) - (-\Delta)^{\frac{1}{2}} \hat{\theta}(y, T)] \leq -\frac{\beta(1+\beta)}{\pi} \zeta^\beta \tag{5.43}$$

Finally, injecting (5.28), (5.40), and (5.43) in (5.24), we obtain for $0 < \zeta \leq \delta$

$$\begin{aligned} \partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) &\leq B[\zeta(3 + \ln(\frac{\delta}{\zeta}))] - \frac{\beta(1 + \beta)}{\pi} \zeta^\beta + \frac{C_1}{A^{1+\alpha}} \zeta^\alpha \\ &\leq 3B\zeta + B\zeta \ln(\delta) - B\zeta \ln(\zeta) - \frac{\beta(1 + \beta)}{\pi} \zeta^\beta + \frac{C_1}{A^{1+\alpha}} \zeta^\alpha \end{aligned}$$

Choosing $\delta \leq 1$ and $A \geq 1$, we have

$$\begin{aligned} \zeta^\alpha &\leq \zeta^\beta \\ A^{1+\beta} &\leq A^{1+\alpha} \end{aligned}$$

Consequently,

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq B(3\zeta + \zeta \ln(\delta)) - B\zeta \ln(\zeta) - \zeta^\beta \left(\frac{\beta(1 + \beta)}{\pi} - \frac{C_1}{A^{1+\beta}} \right) \quad (5.44)$$

Choosing δ small enough, and A large enough depending on C_1 and on β , and therefore on f , then (5.44) would lead to

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0$$

Checking inequality (5.23) for $\delta \leq \zeta \leq A.D$, where D is the period of θ

From (5.24), (5.25), (5.26) and (5.27), we have

$$\begin{aligned} \partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) &\leq \omega'(\zeta)B \left[\left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \right) \right] \\ &\quad + \frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \\ &\quad + \frac{1}{\pi} \int_{\frac{\zeta}{2}}^\infty \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \\ &\quad + \frac{1}{A} \left(f\left(\frac{x}{A}, \frac{T}{A}\right) - f\left(\frac{y}{A}, \frac{T}{A}\right) \right) \end{aligned} \quad (5.45)$$

We have

$$\frac{1}{A}(f(\frac{x}{A}, \frac{T}{A}) - f(\frac{y}{A}, \frac{T}{A})) \leq \frac{2}{A}\|f\|_{L^\infty} \quad (5.46)$$

(from assumption (5.2) on the force).

Whereas to the term

$$\frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta$$

since ω is concave, using the Taylor series, we can estimate

$$\omega(\zeta - 2\eta) \leq \omega(\zeta) - 2\eta\omega'(\zeta)$$

$$\omega(\zeta + 2\eta) \leq \omega(\zeta) + 2\eta\omega'(\zeta)$$

Therefore,

$$\frac{1}{\pi} \int_0^{\frac{\zeta}{2}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \leq 0 \quad (5.47)$$

Now, we want to evaluate the term

$$\frac{1}{\pi} \int_{\frac{\zeta}{2}}^\infty \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta$$

We have

$$\begin{aligned} \omega(2\eta + \zeta) &= \omega(2\eta - \zeta + 2\zeta) \\ &\leq \omega(2\eta - \zeta) + \omega(2\zeta) \end{aligned}$$

(by concavity).

Hence,

$$\frac{1}{\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq \frac{1}{\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(2\zeta) - 2\omega(\zeta)}{\eta^2} d\eta \quad (5.48)$$

Since ω is concave, we also have

$$\begin{aligned} \omega(2\zeta) &\leq \omega(\zeta) + \zeta\omega'(\zeta) \\ &\leq \omega(\zeta) + \frac{\gamma}{4 + \ln(\frac{\zeta}{\delta})} \\ &\leq \omega(\zeta) + \frac{\gamma}{4} \end{aligned} \quad (5.49)$$

If we choose $\gamma < \frac{\delta}{2}$, (5.49) will lead to

$$\omega(2\zeta) \leq \omega(\zeta) + \frac{\delta}{8} \quad (5.50)$$

If we choose δ small enough, we will have

$$\delta^{1+\beta} \leq \frac{\delta}{2} \quad (5.51)$$

then, from (5.34) and (5.35) we will get

$$\frac{\delta}{2} \leq \omega(\delta) \leq \omega(\zeta) \quad (5.52)$$

Injecting (5.52) in (5.50), we obtain

$$\omega(2\zeta) \leq \frac{3}{2}\omega(\zeta)$$

Consequently,

$$\frac{1}{\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq -\frac{1}{2\pi} \int_{\frac{\zeta}{2}}^{\infty} \frac{\omega(\zeta)}{\eta^2} d\eta = -\frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} \quad (5.53)$$

(from (5.48)).

Now, we would want to evaluate the term

$$\left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \right)$$

We have,

$$\begin{aligned} \int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta &\leq \int_0^\delta \frac{\omega(\eta)}{\eta} d\eta + \int_\delta^\zeta \frac{\omega(\eta)}{\eta} d\eta \\ &\leq \delta + \omega(\zeta) \ln\left(\frac{\zeta}{\delta}\right) \end{aligned}$$

If we choose δ small enough as before in (5.51), so that $\omega(\zeta) \geq \omega(\delta) \geq \frac{\delta}{2}$, we obtain

$$\begin{aligned} \int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta &\leq 2\omega(\zeta) + \omega(\zeta) \ln\left(\frac{\zeta}{\delta}\right) \\ &\leq \omega(\zeta)(2 + \ln\left(\frac{\zeta}{\delta}\right)) \end{aligned} \tag{5.54}$$

On the other hand, integrating by parts and using (5.35), we can evaluate

$$\begin{aligned} \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta &= \frac{\omega(\zeta)}{\zeta} + \gamma \int_\zeta^\infty \frac{1}{\eta^2(4 + \ln(\frac{\eta}{\delta}))} d\eta \\ &\leq \frac{\omega(\zeta)}{\zeta} + \frac{\gamma}{\zeta} \end{aligned}$$

Consequently, if we choose $\gamma \leq \frac{\delta}{2}$, with δ small enough as in (5.51), then from (5.52) we get

$$\int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \leq 2\frac{\omega(\zeta)}{\zeta} \tag{5.55}$$

Hence, from (5.54) and (5.55), we get

$$\begin{aligned} \left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \right) &\leq \omega(\zeta) \left(2 + \ln\left(\frac{\zeta}{\delta}\right) \right) + 2\omega(\zeta) \\ &\leq \omega(\zeta) \left(4 + \ln\left(\frac{\zeta}{\delta}\right) \right) \end{aligned} \quad (5.56)$$

Finally, injecting (5.46), (5.47), (5.53), and (5.56) in (5.45), we obtain

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq B\omega(\zeta) \left(4 + \ln\left(\frac{\zeta}{\delta}\right) \right) \omega'(\zeta) - \frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} + 2 \frac{\|f\|_{L^\infty}}{A}$$

Therefore, from (5.35) we have

$$\begin{aligned} \partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) &\leq B\gamma \frac{\omega(\zeta)}{\zeta} - \frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} + 2 \frac{\|f\|_{L^\infty}}{A} \\ &\leq \frac{\omega(\zeta)}{\zeta} \left(B\gamma - \frac{1}{\pi} \right) + 2 \frac{\|f\|_{L^\infty}}{A} \end{aligned}$$

If we choose γ small enough, we get

$$B\gamma - \frac{1}{\pi} < 0$$

then, we get for all $\delta \leq \zeta \leq A.D$, where D is the period of θ ,

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq \frac{\omega(A.D)}{A.D} \left(B\gamma - \frac{1}{\pi} \right) + 2 \frac{\|f\|_{L^\infty}}{A}$$

Since ω is increasing, we can choose A large enough depending on D and $\|f\|_{L^\infty}$, such that

$$\frac{\omega(A.D)}{A.D} \left(B\gamma - \frac{1}{\pi} \right) + 2 \frac{\|f\|_{L^\infty}}{A} < 0$$

□

Remark (5.3.5) and lemma (5.3.6) show that for δ and γ chosen small enough, with $0 < \gamma \leq \frac{\delta}{2}$, ω is preserved by $\hat{\theta}$ for $0 \leq \zeta \leq A.D$, where D is the period of θ , for all time t .

Since θ is periodic of period D depending on the period of θ_0 and of f , then $\hat{\theta}$ is periodic of period $A.D$, and since ω is increasing, we have by then that for all $\zeta \geq A.D$ and for all time t ,

$$\hat{\theta}(x, t) - \hat{\theta}(y, t) \leq \omega(\zeta)$$

Therefore, ω_A is preserved by θ for all time. Consequently, from (5.7) we have

$$\|\nabla\theta\|_{L^\infty} \leq A\omega'(0) \leq A$$

where A depends only on $\|f\|_{L^\infty}$, on C_1 and $\beta = \min\{\frac{1}{2}, \alpha\}$, on $\|\nabla\theta_0\|_{L^\infty}$, on the period of θ_0 , and on the period D of θ (which is given by the period of θ_0 and the period of f). If A is finite, this gives that local solutions of (5.1) can be extended globally in time.

Chapter 6

Appendix

6.1 Kirchoff-Sobolev Parametrix for $\square_{\mathbf{g}}^{(A)} F_{\mu\nu}$

We assume (M, \mathbf{g}) to be globally hyperbolic, i.e. it admits a Cauchy surface Σ , which means a space-like hypersurface $\Sigma \subset M$, that is intersected precisely once by every inextendible causal curve. We also assume that the null cones are regular past the space-like hypersurface Σ .

We sketch an adaptation to the Yang-Mills setting of the original construction by S. Klainerman and I. Rodnianski in [KR1] of the Kirchoff-Sobolev parametrix. Recall that the original construction presented in [KR1] was done in the context of a one tensor with values in the tangent bundle verifying the tensorial wave equation, and cannot be applied directly as it is to the the Yang-Mills equations. However, as noted in [KR1] this construction could be systematically generalized to \mathcal{G} -valued tensors of arbitrary order verifying the gauge covariant tensorial wave equations with a compatible Ad-invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{G} , leading to a representation formula suitable to present a gauge invariant proof of the global existence of Yang-Mills fields on the 4-dimensional Minkowski background. We are sketching the adaptation in this appendix so as to use it to give a proof of the global existence of Yang-Mills fields on curved backgrounds.

Definition 6.1.1. Let p be a point to the future of Σ . The affine parameter s on $N^-(p)$

is defined by fixing a future unit time-like vector \mathbf{T}_p at p and considering for every ω in \mathbb{S}^2 , the null vector l_ω in $T_p(M)$, such that

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1 \quad (6.1)$$

and associate to it the null geodesic $\gamma_\omega(s)$ such that $\gamma_\omega(0) = p$, $\frac{\partial}{\partial s}\gamma_\omega(0) = l_\omega$, and $L = \frac{\partial}{\partial s}\gamma_\omega(s)$ where s is chosen so that $\nabla_L L = 0$. Thus, $L(s) = 1, s(p) = 0$.

Definition 6.1.2. Let $N^-(p)$ be the boundary of the causal past of p . Let χ denote the null second fundamental form of $N^-(p)$, that is, for all $q \in N^-(p) \setminus \{p\}$,

$$\chi(X, Y)(q) = \mathbf{g}(\nabla_X L, Y)(q) \quad (6.2)$$

for all X, Y in $T_q N^-(p)$.

Lemma 6.1.3. χ is symmetric and thus χ is diagonalisable, moreover $\chi(L, X) = \chi(L, L) = 0$, for all $X \in T_q N^-(p)$, consequently, we can define $\text{tr}\chi = \chi(L, L) + \chi(e_1, e_1) + \chi(e_2, e_2) = \chi_{11} + \chi_{22}$.

Proof.

Given a point $q \in N^-(p) \setminus \{p\}$, we can define a null frame $\{L, \underline{L}, e_1, e_2\}$ - where e_1 and e_2 are tangent to $N^-(p) \cap \{s = \text{constant}\}$ 2-surfaces - that forms a basis of $T_q M$, such that at $q \in N^-(p) \setminus \{p\}$,

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0 \quad (6.3)$$

$$\mathbf{g}(L, \underline{L}) = -2 \quad (6.4)$$

$$\mathbf{g}(e_a, e_b) = \delta_{ab}, \quad a, b \in \{1, 2\} \quad (6.5)$$

$$\mathbf{g}(L, e_a) = \mathbf{g}(\underline{L}, e_a) = 0, \quad a, b \in \{1, 2\} \quad (6.6)$$

This null frame can be extended locally in a neighbourhood of $q \in N^-(p) \setminus \{p\}$ such that $L, \underline{L}, e_1, e_2$ are vector fields in the neighbourhood and $\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0$ in the neigh-

bourhood.

Let, $X, Y \in T_q N^-(p)$. Since the metric is Killing, we have,

$$\begin{aligned}
0 &= \nabla_X \mathbf{g}(L, Y) = X \mathbf{g}(L, Y) - \mathbf{g}(\nabla_X L, Y) - \mathbf{g}(L, \nabla_X Y) \\
&= -\mathbf{g}(\nabla_X L, Y) - \mathbf{g}(L, \nabla_X Y) \\
&\quad (\text{since } \mathbf{g}(L, Y) = 0 \text{ for } Y \in T_q N^-(p))
\end{aligned} \tag{6.7}$$

Thus, $\mathbf{g}(\nabla_X L, Y) = -\mathbf{g}(L, \nabla_X Y)$, and since we have $[X, Y] \in T_q N^-(p)$, we get,

$$0 = \mathbf{g}(L, [X, Y]) = \mathbf{g}(L, \nabla_X Y - \nabla_Y X)$$

which gives, $\mathbf{g}(L, \nabla_X Y) = \mathbf{g}(L, -\nabla_Y X)$, we finally get,

$$\mathbf{g}(\nabla_X L, Y) = \mathbf{g}(L, \nabla_Y X)$$

Again,

$$\mathbf{g}(\nabla_Y L, X) = -\mathbf{g}(L, \nabla_Y X)$$

(by inverting the roles of X and Y in before)

This gives,

$$\mathbf{g}(\nabla_X L, Y) = \mathbf{g}(\nabla_Y L, X)$$

Consequently χ is symmetric, and hence $\chi(L, X) = \chi(X, L) = \mathbf{g}(\nabla_L L, X) = 0$ because $\nabla_L L = 0$ by construction. \square

Definition 6.1.4. Let \mathbf{J}_p be a fixed \mathcal{G} -valued anti-symmetric 2-tensor at p , and let $\lambda_{\alpha\beta}$ be

the unique 2-tensor field along $N^-(p)$, that verifies the linear transport equation:

$$\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\text{tr}\chi\lambda_{\alpha\beta} = 0 \quad (6.8)$$

$$(s\lambda_{\alpha\beta})(p) = \mathbf{J}_{\alpha\beta}(p) \quad (6.9)$$

$\lambda_{\alpha\beta}$ can be extended smoothly to be defined in a similar way in a neighborhood away from $N^-(p)\setminus\{p\}$.

Definition 6.1.5. For small $\epsilon > 0$, let $T_\epsilon : (1 - \epsilon, 1 + \epsilon) \mapsto M$ be the timelike geodesic from p such that $T_\epsilon(1) = p$ and $T'_\epsilon(1) = \mathbf{T}_p$. We define u , optical function, as $u|_{N^-(q)} = t - 1$ for each $q = T_\epsilon(t)$, where $N^-(q)$ is the boundary of the past set of q , assumed to be regular.

Definition 6.1.6. The following integral in Σ^+ , future of Σ , for any \mathcal{G} -valued 2-tensors $\lambda_{\alpha\beta}$ and $\Lambda_{\alpha\beta}$ supported in Σ^+ , is defined as,

$$\int_{\Sigma^+} \langle \lambda_{\alpha\beta}\delta(u), \Lambda^{\alpha\beta} \rangle = \langle \delta(u), \langle \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} \rangle \rangle$$

in the sense of the distribution, where u is defined in a neighborhood D_ϵ of $N^-(p) \cap \Sigma^+$ as in above, and

$$\langle \delta(u), \langle \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} \rangle \rangle = \int_0^\infty \int_{\mathbb{S}^2} \langle \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} \rangle (t = 1, s, \omega) ds dA_{S^2} \quad (6.10)$$

where dA_{S^2} is the induced volume form on the 2-surfaces defined by $s = \text{constant}$, and $t = 1$. This integral depends only on $\langle \lambda_{\alpha\beta}, \Lambda_{\mu\nu} \rangle$ on $N^-(p)$ and the normalisation condition

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1 \quad (\mathbf{g}(l_\omega, l_\omega) = 0).$$

Therefore, for any continuous function f supported in Σ^+ , we can define $\int_{N^-(p)} f$ as $\langle \delta(u), f \rangle$ in the sense of the distribution.

6.1.7 Computing $\int_{J^-(p) \cap \Sigma^+} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle$

Now, our goal is to compute $\int_{J^-(p) \cap \Sigma^+} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle$ for F supported in Σ^+ .

Definition 6.1.8. We define a time foliation near p , by extending locally the parameter t near p by starting with a fixed spacelike hypersurface Σ_1 passing through p and orthogonal to the future unit timelike vectorfield \mathbf{T}_p and considering the timelike geodesics orthogonal to Σ_1 .

Definition 6.1.9. We define $\Omega_\epsilon = (J^-(p) \cap \Sigma^+) \setminus \cup_{t \in [1-\epsilon, 1]} \Sigma_t$.

We have

$$\int_{J^-(p) \cap \Sigma^+} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle = \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle \quad (6.11)$$

On the other hand,

$$\int_{\Omega_\epsilon} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle = \int_{\Omega_\epsilon} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}^{(A)\gamma} \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle .$$

Lemma 6.1.10. *Given any two \mathcal{G} -valued tensors K and G , since $\langle \cdot, \cdot \rangle$ is Ad-invariant, we have,*

$$\nabla_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle = \langle \mathbf{D}_\gamma^{(A)} K_{\alpha\beta}, G^{\alpha\beta} \rangle + \langle K_{\alpha\beta}, \mathbf{D}_\gamma^{(A)} G^{\alpha\beta} \rangle \quad (6.12)$$

Proof.

Now, given any two \mathcal{G} -valued tensors $K_{\alpha\beta}$ and $G_{\alpha\beta}$, we have

$$\partial_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle \partial_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle K_{\alpha\beta}, \partial_\gamma G^{\alpha\beta} \rangle = 0 \quad (6.13)$$

Since $\langle K_{\alpha\beta}, G^{\alpha\beta} \rangle$ does not depend on the choice of the basis, one can choose a normal frame as in (2.11) to compute (6.13). In such a frame $\partial_\gamma K_{\alpha\beta} = \nabla_\gamma K_{\alpha\beta}$ and

$\partial_\gamma G_{\alpha\beta} = \nabla_\gamma K_{\alpha\beta}$, and by abuse of notation, we will write $\partial_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle$ as $\nabla_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle$. Hence, we have

$$\nabla_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} \rangle = 0$$

So we have

$$\begin{aligned} \nabla_\gamma \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle &= \langle \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle + \langle K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} \rangle \\ &= \langle \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] \rangle \\ &\quad + \langle K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] \rangle + \langle K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} \rangle \\ &= \langle \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle - \langle [K_{\alpha\beta}, A_\gamma], G^{\alpha\beta} \rangle \\ &\quad + \langle K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] \rangle + \langle K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} \rangle \end{aligned}$$

(since $\langle \cdot, \cdot \rangle$ is Ad-invariant)

$$\begin{aligned} &= \langle \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} \rangle + \langle [A_\gamma, K_{\alpha\beta}], G^{\alpha\beta} \rangle + \langle K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] + \nabla_\gamma G^{\alpha\beta} \rangle \\ &= \langle \mathbf{D}_\gamma^{(A)} K_{\alpha\beta}, G^{\alpha\beta} \rangle + \langle K_{\alpha\beta}, \mathbf{D}_\gamma^{(A)} G^{\alpha\beta} \rangle \end{aligned}$$

□

Let $D_\epsilon = (\{-\epsilon' \leq u(t) \leq \epsilon'\} \setminus \Sigma_{1-\epsilon}^+) \cap \Sigma_0^+$, for ϵ' chosen small enough so that $u(t)$ would be defined on $[-\epsilon', \epsilon']$. Also, recall that λ is smooth in a neighborhood away from p , and thus, by choosing ϵ' small enough, λ is smooth in D_ϵ .

Given this, we have,

$$\begin{aligned} &\int_{D_\epsilon} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}^{\gamma(A)} \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle \\ &= \int_{D_\epsilon} \nabla^\gamma \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \int_{D_\epsilon} \nabla^\gamma \langle \mathbf{D}_\gamma^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle \\ &\quad + \int_{D_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle \end{aligned}$$

so,

$$\begin{aligned}
& \int_{D_\epsilon} \langle \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle \\
&= \int_{D_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle \\
& \quad + \int_{D_\epsilon} \nabla^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] \quad (6.14)
\end{aligned}$$

By divergence theorem,

$$\begin{aligned}
& \int_{D_\epsilon} \nabla^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] \\
&= - \int_{\Sigma_t} T^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle]_{t=0}^{t=1-\epsilon} \\
& \quad + \int_{(N^-(T(-\epsilon')) \setminus \Sigma_{1-\epsilon}^+ \cap \Sigma_0^+)} L^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] \\
& \quad + \int_{(N^-(T(\epsilon')) \setminus \Sigma_{1-\epsilon}^+ \cap \Sigma_0^+)} L^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle]
\end{aligned}$$

Since the distributions δ and δ' are supported on $N^-(T(0)) = N^-(p)$, we get,

$$\begin{aligned}
& \int_{(N^-(T(-\epsilon')) \setminus \Sigma_{1-\epsilon}^+ \cap \Sigma_0^+)} L^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] = 0 \\
& \int_{(N^-(T(\epsilon')) \setminus \Sigma_{1-\epsilon}^+ \cap \Sigma_0^+)} L^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] = 0 \\
& \int_{D_\epsilon} \nabla^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] \\
&= \int_{\Omega_\epsilon} \nabla^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle]
\end{aligned}$$

This yields to,

$$\begin{aligned}
& \int_{\Omega_\epsilon} \nabla^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle] \\
&= - \int_{J^-(p) \cap \Sigma_t} T^\gamma [\langle \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} \rangle - \langle \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle]_{t=0}^{t=1-\epsilon}
\end{aligned}$$

where $\Sigma_0 = \Sigma$, and where T is defined on Σ as being the unit normal timelike vectorfield on Σ .

We get,

$$\begin{aligned}
\int_{\Omega_\epsilon} \langle \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} \rangle &= \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle \\
&\quad - \left[\int_{J^-(p) \cap \Sigma_t} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \right]_{t=0}^{t=1-\epsilon} \\
&\quad + \left[\int_{J^-(p) \cap \Sigma_t} \langle \mathbf{D}_T^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle \right]_{t=0}^{t=1-\epsilon}
\end{aligned} \tag{6.15}$$

6.1.11 Computing $\int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle$

Now, we would like to compute $\int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle = \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda \delta(u)), F \rangle$ in (6.15).

We start by computing $\square_{\mathbf{g}}^{(A)} (\lambda \delta(u))$. We have:

$$\begin{aligned}
\square_{\mathbf{g}}^{(A)} (\lambda \delta(u)) &= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} \mathbf{D}_\nu^{(A)} (\lambda \delta(u)) \\
&= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} (\nabla_\nu (\lambda \delta(u)) + [A_\nu, (\lambda \delta(u))]) \\
&= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} (\nabla_\nu (\lambda) \delta(u) + [A_\nu, (\lambda \delta(u))] + \lambda \nabla_\nu (\delta(u))) \\
&= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} [(\nabla_\nu (\lambda) + [A_\nu, \lambda]) \delta(u) + \lambda \nabla_\nu (\delta(u))] \\
&= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} [\mathbf{D}_\nu^{(A)} (\lambda) \delta(u) + \lambda \delta'(u) \nabla_\nu (u)] \\
&= \mathbf{g}^{\mu\nu} \mathbf{D}_\mu^{(A)} \mathbf{D}_\nu^{(A)} (\lambda) \delta(u) + \delta'(u) \nabla_\mu (u) \mathbf{g}^{\mu\nu} \mathbf{D}_\nu^{(A)} (\lambda) \\
&\quad + \delta'(u) \mathbf{g}^{\mu\nu} \nabla_\mu (\lambda \nabla_\nu (u)) + \mathbf{g}^{\mu\nu} \delta''(u) \nabla_\mu (u) \nabla_\nu (u) \lambda
\end{aligned}$$

(by the symmetry of the metric tensor).

Thus,

$$\begin{aligned}\square_{\mathbf{g}}^{(A)}(\lambda\delta(u)) &= \square_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + \delta'(u)(\square_{\mathbf{g}}(u)\lambda + 2g^{\mu\nu}\nabla_{\nu}u\mathbf{D}_{\mu}^{(A)}\lambda) \\ &\quad + \delta''(u)(\mathbf{g}^{\mu\nu}\nabla_{\mu}u\nabla_{\nu}u)\lambda\end{aligned}\tag{6.16}$$

Now, we want to compute $\square_{\mathbf{g}}(u) = \nabla^{\alpha}\nabla_{\alpha}u$, at $q \in N^{-}(p)\setminus\{p\}$.

Lemma 6.1.12. *We have, $\square_{\mathbf{g}}u = \text{tr}\chi$, at $q \in N^{-}(p)\setminus\{p\}$.*

Proof.

Now, let $\nabla u = \nabla^{\nu}u\partial_{\nu}$, defined in a neighbourhood D_{ϵ} of $N^{-} \cap \Sigma^{+}$. Since

$$L = \frac{d}{ds}\gamma_{\omega}(s)$$

where $\gamma_{\omega}(s)$ is the null geodesic initiating at p , we have $L \in T_q(N^{-}(p))$ for $q \in N^{-}(p)\setminus\{p\}$ and since u is constant on $N^{-}(m)$, for $m \in T_{\epsilon}(t)$, $t \in [1 - \epsilon, 1 + \epsilon]$, we have

$$L(u) = 0 = du(L) = \mathbf{g}(\nabla u, L)\tag{6.17}$$

And since $e_a \in T_q(N^{-}(p))$, $a \in \{1, 2\}$. We also have,

$$e_a(u) = du(e_a) = \mathbf{g}(\nabla u, e_a) = 0\tag{6.18}$$

(6.17) and (6.18) give that,

$$\nabla u(q) = f(q)L$$

Hence,

$$\nabla^{\nu}u\nabla_{\nu}u = (\nabla u)u = fL(u) = 0\tag{6.19}$$

At a point q of the space-time, one can choose a normal frame, which means a frame such that $\mathbf{g}(e_\alpha, e_\beta)(q) = \text{diag}(-1, 1, \dots, 1)$, and $\frac{\partial}{\partial \sigma} \mathbf{g}(e_\alpha, e_\beta)(q) = 0$. Hence, in such a frame $\Gamma_{kl}^i = \frac{1}{2} \mathbf{g}^{im} (\frac{\partial \mathbf{g}_{mk}}{\partial x^l} + \frac{\partial \mathbf{g}_{ml}}{\partial x^k} - \frac{\partial \mathbf{g}_{kl}}{\partial x^m}) = 0$. Computing in such a frame,

$$\begin{aligned}
(\nabla_{\nabla u} \nabla u)^\gamma(q) &= (\nabla_{\nabla^\nu u \partial_\nu} (\nabla^\mu u \partial_\mu))^\gamma = (\nabla^\nu u \nabla_\nu (\nabla^\mu u \partial_\mu))^\gamma \\
&= (\nabla^\nu u (\nabla_\nu \nabla^\mu u) \partial_\mu)^\gamma + (\nabla^\nu u \nabla^\mu u \nabla_\nu \partial_\mu)^\gamma \\
&= \nabla^\nu u (\nabla_\nu \nabla^\mu u) \delta_\mu^\gamma + \nabla^\nu u \nabla^\mu u \Gamma_{\nu\mu}^\gamma \\
&= \nabla^\nu u (\nabla^\mu \nabla_\nu u) \delta_\mu^\gamma + \nabla^\nu u \nabla^\mu u \Gamma_{\nu\mu}^\gamma \\
&\quad \text{(using that the metric is compatible. i.e. } \nabla \mathbf{g} = 0) \\
&= \frac{1}{2} \nabla^\mu (\nabla_\nu u \nabla^\nu u) \delta_\mu^\gamma \\
&= 0
\end{aligned}$$

(in view of (6.19)).

Therefore, ∇u is parallel, this gives,

$$\nabla u = cL$$

where c is a constant. We have,

$$T(u)(p) = 1 = \mathbf{g}(\nabla u, \mathbf{T}_p) = \mathbf{g}(cL, \mathbf{T}_p)$$

In view of (6.1), we have,

$$\mathbf{g}(cL, \mathbf{T}_p) = c$$

Thus,

$$c = 1$$

which gives,

$$\nabla u = L = \nabla^\nu u \partial_\nu \quad (6.20)$$

Computing,

$$\square_{\mathbf{g}} u = \nabla^\alpha \nabla_\alpha u = \nabla^L \nabla_L u + \nabla^{\underline{L}} \nabla_{\underline{L}} u + \nabla^a \nabla_a u$$

$a = \{1, 2\}$, at $q \in N^-(p) \setminus \{p\}$. Therefore, at $q \in N^-(p) \setminus \{p\}$,

$$\begin{aligned} \square_{\mathbf{g}} u &= \mathbf{g}^{L\alpha} \nabla_\alpha \nabla_L u + \mathbf{g}^{\underline{L}\alpha} \nabla_\alpha \nabla_{\underline{L}} u + \mathbf{g}^{a\alpha} \nabla_\alpha \nabla_a u \\ &= -\frac{1}{2} \nabla_{\underline{L}} \nabla_L u - \frac{1}{2} \nabla_L \nabla_{\underline{L}} u + \nabla_a \nabla_a u \\ &= -\frac{1}{2} \nabla_{\underline{L}} (\nabla_L u) + \frac{1}{2} \nabla_{\nabla_{\underline{L}} L} u - \frac{1}{2} \nabla_L (\nabla_{\underline{L}} u) + \frac{1}{2} \nabla_{\nabla_L \underline{L}} u + \nabla_a (\nabla_a u) - \nabla_{\nabla_a e_a} u \end{aligned}$$

and at $q \in N^-(p) \setminus \{p\}$,

$$\nabla_L u = L(u) = du(L) = \mathbf{g}(L, L),$$

$$\nabla_{\underline{L}} u = \underline{L}(u) = du(\underline{L}) = \mathbf{g}(L, \underline{L}),$$

$$\nabla_a u = e_a(u) = du(e_a) = \mathbf{g}(L, e_a).$$

We get

$$\nabla_{\underline{L}} (\nabla_L u) = \underline{L} \mathbf{g}(L, L) = \mathbf{g}(\nabla_{\underline{L}} L, L) + \mathbf{g}(L, \nabla_{\underline{L}} L)$$

thus

$$\nabla_{\underline{L}} (\nabla_L u) = 2g(\nabla_{\underline{L}} L, L)$$

$$\nabla_L (\nabla_{\underline{L}} u) = Lg(L, \underline{L}) = 0$$

$$\nabla_a (\nabla_a u) = e_a \mathbf{g}(L, e_a) = 0$$

Therefore,

$$\square_{\mathbf{g}}u = -\mathbf{g}(\nabla_{\underline{L}}L, L) + \frac{1}{2}\nabla_{\nabla_{\underline{L}}L}u + \frac{1}{2}\nabla_{\nabla_L\underline{L}}u - \nabla_{\nabla_a e_a}u$$

We recall that in the frame $\{L, \underline{L}, e_1, e_2\}$ a vector field X can be written as:

$$X = -\frac{1}{2}\mathbf{g}(X, \underline{L})L - \frac{1}{2}\mathbf{g}(X, L)\underline{L} + \mathbf{g}(X, e_a)e_a \quad (6.21)$$

Thus, taking $X = \nabla_L\underline{L}$, we get

$$\nabla_L\underline{L} = -\frac{1}{2}\mathbf{g}(\nabla_L\underline{L}, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_L\underline{L}, L)\underline{L} + \mathbf{g}(\nabla_L\underline{L}, e_a)e_a$$

Therefore,

$$\frac{1}{2}\nabla_{\nabla_L\underline{L}}u = \frac{1}{2}\mathbf{g}(L, \nabla_L\underline{L}) = -\frac{1}{4}\mathbf{g}(\mathbf{g}(\nabla_L\underline{L}, L)\underline{L}, L) = \frac{1}{2}\mathbf{g}(\nabla_L\underline{L}, L) = 0$$

$$\frac{1}{2}\nabla_{\nabla_L L}u = \frac{1}{2}\mathbf{g}(L, \nabla_L L) = -\frac{1}{4}\mathbf{g}(\mathbf{g}(\nabla_L L, L)\underline{L}, L) = \frac{1}{2}\mathbf{g}(\nabla_L L, L) = 0$$

We are left with $-\nabla_{\nabla_a e_a}u$

Taking $X = \nabla_a e_a$, we obtain,

$$\begin{aligned} \nabla_a e_a &= -\frac{1}{2}\mathbf{g}(\nabla_a e_a, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_a e_a, L)\underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b \\ &= \frac{1}{2}\mathbf{g}(e_a, \nabla_a \underline{L})L + \frac{1}{2}\mathbf{g}(e_a, \nabla_a L)\underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b \\ &= \frac{1}{2}tr\chi L + \frac{1}{2}tr\chi \underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b \end{aligned} \quad (6.22)$$

Finally,

$$-\nabla_{\nabla_a e_a}u = -\mathbf{g}(L, \nabla_a e_a) = -\mathbf{g}(L, \frac{1}{2}tr\chi \underline{L}) = tr\chi$$

This yields to

$$\square_{\mathbf{g}}u = tr\chi \quad (6.23)$$

□

Going back to (6.16), we have now shown that

$$\begin{aligned} & \square_{\mathbf{g}}^{(A)}(\lambda\delta(u)) \\ &= \square_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)\left(\frac{TrX}{2}\lambda + \mathbf{g}^{\mu\nu}\nabla_{\nu}u\mathbf{D}_{\mu}^{(A)}\lambda\right) + \delta''(u)(\mathbf{g}^{\mu\nu}\nabla_{\mu}u\nabla_{\nu}u)\lambda \end{aligned} \quad (6.24)$$

at $q \in N^-(p) \setminus \{p\}$.

We recall that u is constant on $N^-(p)$ and $L \in T_q N^-(p)$ for $q \in N^-(p) \setminus \{p\}$.

(6.17) and (6.20) yield to

$$L(u) = 0 = \mathbf{g}^{\mu\nu}\nabla_{\mu}u\nabla_{\nu}u$$

at q . Also,

$$\mathbf{g}^{\mu\nu}\nabla_{\nu}u\mathbf{D}_{\mu}^{(A)}\lambda = L^{\mu}\mathbf{D}_{\nu}^{(A)}\lambda$$

(since $L = \mathbf{g}^{\mu\nu}\nabla_{\mu}u\partial_{\nu}$ gives $L^{\nu} = \mathbf{g}^{\mu\nu}\nabla_{\mu}u$).

Thus (6.24) becomes,

$$\square_{\mathbf{g}}^{(A)}(\lambda\delta(u)) = \square_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)(\mathbf{D}_L^{(A)}\lambda + \frac{trX}{2}\lambda) \quad (6.25)$$

Hence,

$$\int_{\Omega_{\epsilon}} \langle \square_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F \rangle = \int_{\Omega_{\epsilon}} \langle \square_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)(\mathbf{D}_L^{(A)}\lambda + \frac{trX}{2}\lambda), F \rangle \quad (6.26)$$

6.1.13 Evaluating $\int_{\Omega_{\epsilon}} \delta'(u) \langle \mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{trX}{2}\lambda_{\alpha\beta}, F^{\alpha\beta} \rangle$

We have $\nabla_{\underline{L}}\delta(u) = \delta'(u)\nabla_{\underline{L}}(u)$.

$$\nabla_{\underline{L}}(u) = du(\underline{L}) = \mathbf{g}(\nabla u, \underline{L}) = \mathbf{g}(\underline{L}, \underline{L}) = -2$$

Thus,

$$\nabla_{\underline{L}}\delta(u) = -2\delta'(u)$$

or

$$\delta'(u) = -\frac{1}{2}\nabla_{\underline{L}}\delta(u)$$

This yields to,

$$\begin{aligned}
& 2 \int_{\Omega_\epsilon} \delta'(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle \\
= & 2\left(\frac{-1}{2}\right) \int_{\Omega_\epsilon} \nabla_{\underline{L}}\delta(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle \\
= & - \int_{J^-(p) \cap \Sigma_t} \mathbf{g}(\underline{L}, T)\delta(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle \\
& + \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda), F \rangle \\
& + \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, \mathbf{D}_{\underline{L}}^{(A)}F \rangle \\
& + \int_{\Omega_\epsilon} \delta(u) \nabla_{\alpha}\underline{L}^\alpha \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle
\end{aligned} \tag{6.27}$$

(by integration by parts)

The integrals

$$\begin{aligned}
& - \int_{J^-(p) \cap \Sigma_t} \mathbf{g}(\underline{L}, T)\delta(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle \\
& \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, \mathbf{D}_{\underline{L}}^{(A)}F \rangle
\end{aligned}$$

and

$$\int_{\Omega_\epsilon} \delta(u) \nabla_{\alpha}\underline{L}^\alpha \langle \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F \rangle$$

depend only on the values of the integrated function on $N^-(p)$ and on the normalisation condition $\mathbf{g}(L, T)(p) = 1$. As $\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda = 0$ on $N^-(p)$, and due to the presence of $\delta(u)$,

these three terms vanish. Thus, (6.27) can be written as,

$$2 \int_{\Omega_\epsilon} \delta'(u) \langle \mathbf{D}_L^{(A)} \lambda + \frac{tr\chi}{2} \lambda, F \rangle = \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_L^{(A)} (\mathbf{D}^{(A)}_L \lambda) + \mathbf{D}_L^{(A)} \left(\frac{tr\chi}{2} \lambda \right), F \rangle \quad (6.28)$$

and thus,

$$\begin{aligned} & \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda \delta(u)), F \rangle \\ &= \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda) \delta(u), F \rangle \\ & \quad + \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_L^{(A)} (\mathbf{D}^{(A)}_L \lambda) + \mathbf{D}_L^{(A)} \left(\frac{tr\chi}{2} \lambda \right), F \rangle \end{aligned} \quad (6.29)$$

6.1.14 Evaluating $\int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)} (\lambda_{\alpha\beta}) \delta(u), F^{\alpha\beta} \rangle$

Now, in its turn, we would like to compute the tensorial

$$\square_{\mathbf{g}}^{(A)} (\lambda) = \mathbf{g}^{\alpha\beta} \mathbf{D}_{\alpha\beta}^{(A)2} \lambda$$

where

$$\mathbf{g}^{\alpha\beta} \mathbf{D}_{\alpha\beta}^{(A)2} \lambda = \mathbf{g}^{\alpha\beta} \mathbf{D}_\alpha^{(A)} \mathbf{D}_\beta^{(A)} \lambda$$

which we will distinguish from $\mathbf{g}^{\alpha\beta} \mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \lambda)$ as $\mathbf{D}_{\alpha\beta}^{(A)2}$ is the tensorial second order derivative defined by

$$\mathbf{D}_{\alpha\beta}^{(A)2} \lambda_{\mu\nu} = (\mathbf{D}_\alpha^{(A)} (\mathbf{D}_\beta^{(A)} \lambda))_{\mu\nu} - (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)} \lambda)_{\mu\nu} \quad (6.30)$$

where the tensorial derivative $\mathbf{D}_\alpha^{(A)} \lambda$ is defined by,

$$(\mathbf{D}_\alpha^{(A)} \lambda)(X, Y) = \mathbf{D}_\alpha^{(A)} (\lambda(X, Y)) - \lambda(\nabla_\alpha X, Y) - \lambda(X, \nabla_\alpha Y) \quad (6.31)$$

for any $X, Y \in TM$. We have,

$$\square_{\mathbf{g}}^{(A)} \lambda_{\alpha\beta} = -\frac{1}{2} \mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\alpha\beta} - \frac{1}{2} \mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\alpha\beta} + \delta^{ab} \mathbf{D}_{ab}^{(A)2} \lambda_{\alpha\beta} \quad (6.32)$$

$$\begin{aligned} \mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\alpha\beta} - \mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\alpha\beta} &= \nabla_{\underline{L}} \nabla_{\underline{L}} \lambda_{\alpha\beta} - \nabla_{\underline{L}} \nabla_{\underline{L}} \lambda_{\alpha\beta} + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] \\ &= R_{\alpha}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} + R_{\beta}{}^{\gamma}{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma} + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] \end{aligned}$$

$$\mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\mu\nu} = \mathbf{D}_{\underline{L}}^{(A)} (\mathbf{D}_{\underline{L}}^{(A)} \lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_{\underline{L}} \underline{L}}^{(A)} \lambda_{\alpha\beta} \quad (6.33)$$

Lemma 6.1.15. *We have,*

$$\nabla_{\underline{L}} L = 2\zeta_a e_a - 2\omega L \quad (6.34)$$

where,

$$\zeta_a = \frac{1}{2} \mathbf{g}(\nabla_a L, \underline{L}) \quad (6.35)$$

$$\omega = -\frac{1}{4} \mathbf{g}(\nabla_{\underline{L}} \underline{L}, L) \quad (6.36)$$

Proof.

Using (6.21),

$$\begin{aligned} \nabla_{\underline{L}} L &= -\frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} L, \underline{L}) L - \frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} L, L) \underline{L} + \mathbf{g}(\nabla_{\underline{L}} L, e_a) e_a, \quad a \in 1, 2 \\ &= \frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} \underline{L}, L) L + 0 - \mathbf{g}(L, \nabla_{\underline{L}} e_a) e_a \end{aligned}$$

To show that $\mathbf{g}(L, [\underline{L}, e_a]) = 0$ (see [C]) let q_m be the 1-parameter group generated by \underline{L} , θ_r the 1-parameter group generated by e_a . Let $\Omega(r, m) = \theta_{-r} \circ q_{-m} \circ \theta_r \circ q_m$. We have,

$$[\underline{L}, e_a](p) = \frac{\partial^2}{\partial r \partial m} \Omega(0, 0)(p)$$

θ_r maps $J^-(p) \cap \Sigma_t$ into itself for all t , and q_m maps $J^-(p) \cap \Sigma_t$ into say $Q_m(t)$, where the vector field e_a can still be constructed to be tangent to $Q_m(t)$ for all t in a neighborhood of p , and therefore θ_r maps $Q_m(t)$ into itself. We get that $\theta_{-r} \circ q_{-m} \circ \theta_r \circ q_m$ maps $J^-(p) \cap \Sigma_t$ into itself for each t and consequently, $\Omega(r, m)$ maps $J^-(p) \cap \Sigma_t$ into itself. Therefore $[\underline{L}, e_a]$ is tangential to $J^-(p) \cap \Sigma_t$. Hence,

$$\mathbf{g}(L, [\underline{L}, e_a]) = 0$$

Thus,

$$\begin{aligned} \nabla_{\underline{L}} L &= \frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} \underline{L}, L) L - \mathbf{g}(L, \nabla_{e_a} \underline{L}) e_a \\ &= \frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} \underline{L}, L) L + \mathbf{g}(\nabla_{e_a} L, \underline{L}) e_a \end{aligned}$$

With the notation (6.35) and (6.36), we get (6.34).

□

Therefore (6.33) can be written as,

$$\begin{aligned} &\mathbf{D}_{\underline{L}\underline{L}}^{(A)2} \lambda_{\alpha\beta} \\ = &\mathbf{D}_{\underline{L}}^{(A)} (\mathbf{D}_L^{(A)} \lambda_{\alpha\beta}) - 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + 2\omega \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} \\ &+ R_\alpha^\gamma \underline{L} L \lambda_{\gamma\beta} + R_\beta^\gamma \underline{L} L \lambda_{\alpha\gamma} + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] \end{aligned} \tag{6.37}$$

We define,

$$\Delta^{(A)} \lambda_{\alpha\beta} = \delta^{ab} \mathbf{D}_{ab}^{(A)2} \lambda_{\alpha\beta} \tag{6.38}$$

Injecting (6.37) in (6.32), we obtain,

$$\begin{aligned}\square_{\mathbf{g}}^{(A)}\lambda_{\alpha\beta} &= -\frac{1}{2}\mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) - \frac{1}{2}\mathbf{D}_{\underline{LL}}^{(A)2}\lambda_{\alpha\beta} + \zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} \\ &\quad - \frac{1}{2}R_{\alpha}^{\gamma}\underline{L}L\lambda_{\gamma\beta} - \frac{1}{2}R_{\beta}^{\gamma}\underline{L}L\lambda_{\alpha\gamma} + [F_{\underline{LL}}, \lambda_{\alpha\beta}] + \Delta^{(A)}\lambda_{\alpha\beta}\end{aligned}\quad (6.39)$$

Recall (6.34), then (6.33) can be written as,

$$\mathbf{D}_{\underline{LL}}^{(A)2}\lambda_{\alpha\beta} = \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) - 2\zeta_a\mathbf{D}_{e_a}^{(A)}\lambda_{\alpha\beta} + 2\omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}\quad (6.40)$$

Injecting (6.40) in (6.39), we get

$$\begin{aligned}\square_{\mathbf{g}}^{(A)}\lambda_{\alpha\beta} &= -\mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) + \zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}R_{\alpha}^{\gamma}\underline{L}L\lambda_{\gamma\beta} \\ &\quad - \frac{1}{2}R_{\beta}^{\gamma}\underline{L}L\lambda_{\alpha\gamma} + [F_{\underline{LL}}, \lambda_{\alpha\beta}] + \Delta^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mathbf{D}_{\nabla\underline{L}}^{(A)}\lambda_{\alpha\beta}\end{aligned}\quad (6.41)$$

6.1.16 Revisiting $\int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle$

We showed (6.29) that is,

$$\begin{aligned}\int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F \rangle &= \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda)\delta(u), F \rangle \\ &\quad + \int_{\Omega_\epsilon} \delta(u) \langle \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda) + \mathbf{D}_{\underline{L}}^{(A)}\left(\frac{trX}{2}\lambda\right), F \rangle \\ \mathbf{D}_{\underline{L}}^{(A)}\left(\frac{trX}{2}\lambda_{\alpha\beta}\right) &= \nabla_{\underline{L}}\left(\frac{trX}{2}\lambda_{\alpha\beta}\right) + \frac{trX}{2}\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta}\end{aligned}$$

Recall (6.33), we also have $\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{trX}{2}\lambda_{\alpha\beta} = 0$ at $q \in N^-(p) \setminus \{p\}$, using (6.42) we

obtain,

$$\begin{aligned}
& \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle \\
&= \int_{\Omega_\epsilon} \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} - \omega \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} R_{\alpha}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma} \\
&\quad + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] + \Delta^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \text{tr} \chi \mathbf{D}_{\underline{L}}^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mathbf{D}_{\nabla_{\underline{L}} L}^{(A)} \lambda_{\alpha\beta} + \nabla_{\underline{L}} \left(\frac{\text{tr} \chi}{2} \right) \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\
&= \int_{\Omega_\epsilon} \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} - 2\omega \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} + \omega \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} \text{tr} \underline{\chi} \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} \\
&\quad + \frac{1}{2} \text{tr} \underline{\chi} \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} R_{\alpha}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma} + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] + \Delta^{(A)} \lambda_{\alpha\beta} \\
&\quad + \frac{1}{2} \text{tr} \chi \mathbf{D}_{\underline{L}}^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mathbf{D}_{\nabla_{\underline{L}} L}^{(A)} \lambda_{\alpha\beta} + \nabla_{\underline{L}} \left(\frac{\text{tr} \chi}{2} \right) \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle
\end{aligned}$$

Let μ be the mass aspect function defined by

$$\mu = \nabla_{\underline{L}} \text{tr} \chi + \frac{1}{2} \text{tr} \chi \text{tr} \underline{\chi} + 2\omega \text{tr} \chi \quad (6.42)$$

We have,

$$\begin{aligned}
& \int_{\Omega_\epsilon} \langle \square_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F \rangle \\
&= \int_{\Omega_\epsilon} \delta(u) \langle \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \omega \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \text{tr} \underline{\chi} \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} R_{\alpha}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma} \\
&\quad + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] + \Delta^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \text{tr} \chi \mathbf{D}_{\underline{L}}^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mathbf{D}_{\nabla_{\underline{L}} L}^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mu \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\
&= \int_{\Omega_\epsilon} \delta(u) \langle 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \text{tr} \underline{\chi} \mathbf{D}_L^{(A)} \lambda_{\alpha\beta} - \frac{1}{2} R_{\alpha}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^\gamma{}_{\underline{L}\underline{L}} \lambda_{\alpha\gamma} \\
&\quad + [F_{\underline{L}\underline{L}}, \lambda_{\alpha\beta}] + \Delta^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \text{tr} \chi \mathbf{D}_{\underline{L}}^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mu \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle
\end{aligned} \quad (6.43)$$

(where we used (6.34))

6.1.17 Estimating $\lim_{\epsilon \rightarrow 0} \left| - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \right|$

Computing $\delta^{ab}\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta}$ we have,

$$\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta} = \mathbf{D}_a^{(A)}(\mathbf{D}_b^{(A)}\lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_a e_b}^{(A)}\lambda_{\alpha\beta} \quad (6.44)$$

Definition 6.1.18. We define a restriction of the covariant derivative to the span of $\{e_a\}$, $a \in \{1, 2\}$ at $q \in N^-(p) \setminus \{p\}$ as being,

$$\nabla_a e_b = \nabla_a e_b - \frac{1}{2}\chi_{ab}\underline{L} - \frac{1}{2}\underline{\chi}_{ab}L \quad (6.45)$$

where

$$\underline{\chi}_{ab} = \mathbf{g}(\nabla_a \underline{L}, e_b) \quad (6.46)$$

We have

$$\mathbf{g}(\nabla_a e_b, L) = \mathbf{g}(\nabla_a e_b, L) - \frac{1}{2}\chi_{ab}\mathbf{g}(\underline{L}, L) - \frac{1}{2}\underline{\chi}_{ab}\mathbf{g}(L, L) = \mathbf{g}(\nabla_a e_b, L) + \chi_{ab}$$

We have $\mathbf{g}(e_b, L) = 0$ along $N^-(p)$ and since e_a is tangent to $N^-(p)$ at q , we get

$$e_a \mathbf{g}(e_b, L) = 0 = \mathbf{g}(\nabla_a e_b, L) + \mathbf{g}(e_b, \nabla_a L)$$

so

$$\mathbf{g}(\nabla_a e_b, L) = -\mathbf{g}(e_b, \nabla_a L) = -\chi_{ab}$$

and hence

$$\mathbf{g}(\nabla_a e_b, L) = 0$$

Therefore, $\nabla_a e_b$ is tangent to $N^-(p)$.

Computing,

$$\mathbf{g}(\nabla_a e_b, \underline{L}) = \mathbf{g}(\nabla_a e_b, \underline{L}) - \frac{1}{2}\chi_{ab}\mathbf{g}(\underline{L}, \underline{L}) - \frac{1}{2}\underline{\chi}_{ab}\mathbf{g}(L, \underline{L}) = \mathbf{g}(\nabla_a e_b, \underline{L}) + \underline{\chi}_{ab}$$

Similarly, we get

$$\mathbf{g}(\nabla_a e_b, \underline{L}) = -\underline{\chi}_{ab}$$

and therefore

$$\mathbf{g}(\nabla_a e_b, \underline{L}) = 0$$

Finally, we get that $\nabla_a e_b$ is in the span of $\{e_a\}$, $a \in \{1, 2\}$.

Going back to $\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta}$, we have

$$\mathbf{D}^{(A)}_{\nabla_a e_b} \lambda_{\alpha\beta} = \mathbf{D}^{(A)}_{\nabla_a e_b + \frac{1}{2}\chi_{ab}\underline{L} + \frac{1}{2}\underline{\chi}_{ab}L} \lambda_{\alpha\beta} = \mathbf{D}^{(A)}_{\nabla_a e_b} \lambda_{\alpha\beta} + \frac{1}{2}\chi_{ab}\mathbf{D}^{(A)}_{\underline{L}} \lambda_{\alpha\beta} + \frac{1}{2}\underline{\chi}_{ab}\mathbf{D}^{(A)}_{L} \lambda_{\alpha\beta}$$

Hence, we have at $q \in N^-(p) \setminus \{p\}$,

$$\delta^{\alpha\beta}\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta} = \delta^{\alpha\beta}\mathbf{D}^{(A)}_a(\mathbf{D}^{(A)}_b \lambda_{\alpha\beta}) - \delta^{ab}\mathbf{D}^{(A)}_{\nabla_a e_b} \lambda_{\alpha\beta} - \frac{1}{2}\text{tr}\chi\mathbf{D}^{(A)}_{\underline{L}} \lambda_{\alpha\beta} - \frac{1}{2}\text{tr}\underline{\chi}\mathbf{D}^{(A)}_{L} \lambda_{\alpha\beta}$$

$\nabla_a e_b$ is in the span of $e_a \in T_q N^-(p)$ for $a \in \{1, 2\}$, we therefore define

$$\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_a(\mathbf{D}^{(A)}_b \lambda_{\alpha\beta}) - \mathbf{D}^{(A)}_{\nabla_a e_b} \lambda_{\alpha\beta} \quad (6.47)$$

where $\hat{\mathbf{D}}$ is the restriction of the gauge covariant derivative $\mathbf{D}^{(A)}$ along to the span of $e_a \in T_q N^-(p)$, $a \in \{1, 2\}$. We get

$$\begin{aligned} \delta^{ab}\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta} &= \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} - \frac{1}{2}\delta^{ab}\chi_{ab}\mathbf{D}^{(A)}_{\underline{L}} \lambda_{\alpha\beta} - \frac{1}{2}\delta^{ab}\underline{\chi}_{ab}\mathbf{D}^{(A)}_{L} \lambda_{\alpha\beta} \\ &= \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} - \frac{1}{2}\text{tr}\chi\mathbf{D}^{(A)}_{\underline{L}} \lambda_{\alpha\beta} - \frac{1}{2}\text{tr}\underline{\chi}\mathbf{D}^{(A)}_{L} \lambda_{\alpha\beta} \end{aligned} \quad (6.48)$$

Definition 6.1.19. Let $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ be the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$ defined by,

$$\hat{\Delta}^{(A)}\lambda_{\alpha\beta} = \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} = \Delta^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\text{tr}\underline{\chi}\mathbf{D}^{(A)}_{L} \lambda_{\alpha\beta} + \frac{1}{2}\text{tr}\chi\mathbf{D}^{(A)}_{\underline{L}} \lambda_{\alpha\beta} \quad (6.49)$$

We obtain after injecting (6.49) and (6.43) in (6.15),

$$\begin{aligned}
& \int_{\Omega_\epsilon} \langle \lambda \delta(u), \square_{\mathbf{g}}^{(A)} F \rangle_{\mathbf{g}} \\
&= \int_{\Omega_\epsilon} \langle \hat{\Delta}^{(A)} \lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mu \lambda_{\alpha\beta} + [F_{LL}, \lambda_{\alpha\beta}] \\
&\quad - \frac{1}{2} R_{\alpha}{}^\gamma{}_{LL} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^\gamma{}_{LL} \lambda_{\alpha\gamma}, F^{\alpha\beta} \rangle \\
&\quad - \left[\int_{J^-(p) \cap \Sigma_t} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \right]_{t=0}^{t=1-\epsilon} \\
&\quad + \left[\int_{J^-(p) \cap \Sigma_t} \langle \mathbf{D}_T^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} \rangle \right]_{t=0}^{t=1-\epsilon}
\end{aligned}$$

We have by definition,

$$\int_{N^-(p)} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle = \int_{\mathbb{S}^2} \int_0^\infty \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle (u=0, s, \omega) ds dA_s$$

We have

$$\frac{dt}{ds} = L(t) = dt(L) = \mathbf{g}(-T, L)$$

We define,

$$\phi = \mathbf{g}(T, L)^{-1} \tag{6.50}$$

the null lapse function. We have,

$$ds = -\phi dt \tag{6.51}$$

We denote by $dA_{S_t(p)}$ the area element of the 2-surface $S_t(p) = N^-(p) \cap \Sigma_t$

Thus,

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{S}^2} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle (u=0, s, \omega) dA_s ds \\
&= \int_{t=1}^{t=-\infty} \int_{S_t} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle (-\phi) dA_t dt \\
&= \int_{t=-\infty}^{t=1} \int_{S_t} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \phi dA_t dt
\end{aligned}$$

We get,

$$-\int_{\mathbf{J}^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle = -\int_{N^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \phi dA_{S_{t=1-\epsilon}(p)}$$

Definition 6.1.20. We define positive definite Riemannian metric in the following manner:

$$h(e_\alpha, e_\beta) = \mathbf{g}(e_\alpha, e_\beta) + 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) \tag{6.52}$$

where

$$\mathbf{g}(\hat{t}, \hat{t}) = -1 \tag{6.53}$$

Definition 6.1.21. For any \mathcal{G} -valued 2-tensor K , we let

$$|K|^2 = h_{\alpha\mu} h_{\beta\nu} |K^{\mu\nu}| \cdot |K^{\alpha\beta}| \tag{6.54}$$

Lemma 6.1.22. For any two \mathcal{G} -valued tensors K and G , we have

$$|\langle K_{\alpha\beta}, G^{\alpha\beta} \rangle| \lesssim (|K|^2)^{\frac{1}{2}} \cdot (|G|^2)^{\frac{1}{2}} \tag{6.55}$$

Proof.

$$\begin{aligned}
| \langle K_{\alpha\beta}, G^{\alpha\beta} \rangle | &= | \mathbf{g}_{\alpha\mu} \mathbf{g}_{\beta\nu} \langle K^{\mu\nu}, G^{\alpha\beta} \rangle | \\
&\leq | \mathbf{g}_{\alpha\mu} | \cdot | \mathbf{g}_{\beta\nu} | \cdot | \langle K^{\mu\nu}, G^{\alpha\beta} \rangle | \\
&\leq h_{\alpha\mu} h_{\beta\nu} | \langle K^{\mu\nu}, G^{\alpha\beta} \rangle | \\
&\leq h_{\alpha\mu} h_{\beta\nu} | K^{\mu\nu} | \cdot | G^{\alpha\beta} | \\
&\leq (h_{\alpha\mu} h_{\beta\nu} | K^{\mu\nu} | \cdot | K^{\alpha\beta} |)^{\frac{1}{2}} \cdot (h_{\alpha\mu} h_{\beta\nu} | G^{\mu\nu} | \cdot | G^{\alpha\beta} |)^{\frac{1}{2}}
\end{aligned}$$

(by applying Cauchy-Schwarz)

□

Hence,

$$\begin{aligned}
&| \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \phi dA_{S_{1-\epsilon}(p)} | \\
&\lesssim \| \phi \|_{L^\infty} \left(\int_{N^-(p) \cap \Sigma_{1-\epsilon}} |\lambda|^2 dA_{S_{1-\epsilon}(p)} \right)^{\frac{1}{2}} \left(\int_{N^-(p) \cap \Sigma_{1-\epsilon}} |\mathbf{D}_T^{(A)} F|^2 dA_{S_{1-\epsilon}(p)} \right)^{\frac{1}{2}}
\end{aligned}$$

(by Cauchy-Schwarz)

$$\lesssim \| \phi \|_{L^\infty} \| \mathbf{D}_T^{(A)} F \|_{L^\infty} \| \lambda \|_{L^2(N^-(p) \cap \Sigma_{1-\epsilon})} |A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}} \quad (6.56)$$

where $A_{S_{1-\epsilon}(p)}$ denote the area of $S_{1-\epsilon}(p) = N^-(p) \cap \Sigma_{t=1-\epsilon}$ and where,

$$\| \lambda \|_{L^2(\Sigma_{1-\epsilon}(p))} = \left(\int_{S_{1-\epsilon}(p)} h_{\alpha\mu} h_{\beta\nu} |\lambda^{\mu\nu}| \cdot |\lambda^{\alpha\beta}| dA_{S_{1-\epsilon}(p)} \right)^{\frac{1}{2}} \quad (6.57)$$

and

$$\| \mathbf{D}_T^{(A)} F \|_{L^\infty} = \| (h_{\alpha\mu} h_{\beta\nu} | \mathbf{D}_T^{(A)} F^{\mu\nu} | \cdot | \mathbf{D}_T^{(A)} F^{\alpha\beta} |)^{\frac{1}{2}} \|_{L^\infty} \quad (6.58)$$

We want to study now the behavior of $|A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}}$ near $\epsilon = 0$.

Lemma 6.1.23. *We have,*

$$\begin{aligned} |A_{S_t}| &= 4\pi s^2 + o(s^2) \\ &= 4\pi(1-t)^2 + o((1-t)^2) \end{aligned} \tag{6.59}$$

Proof.

We have,

$$\frac{d}{dt}|A_{S_t(p)}| = \int_{S_t(p)} \phi \operatorname{tr} \chi dA \tag{6.60}$$

We also have,

$$\frac{d}{ds}(\operatorname{tr} \chi) + \frac{1}{2}(\operatorname{tr} \chi)^2 = -|\hat{\chi}|^2 - \operatorname{Ric}(L, L) \tag{6.61}$$

where $\hat{\chi}$ is the traceless part of χ and $(\operatorname{str} \chi)(p) = 2, \hat{\chi}(p) = 0$. This yields to

$$\lim_{q \rightarrow p} |\operatorname{tr} \chi(q) - \frac{2}{s}| = 0 \tag{6.62}$$

(see [Wang]).

$$\phi = \mathbf{g}(T, L)^{-1} = (\nabla_T u)^{-1}.$$

Since u is smooth and

$$\phi(p) = 1 = (\nabla_T u)^{-1}(p)$$

we also have that ϕ smooth and bounded near p . Thus,

$$\lim_{q \rightarrow p} |\phi(q) - 1| = 0 \tag{6.63}$$

Since, $ds = -\phi dt$, we get,

$$\int_0^s 1.ds = \int_1^t -\phi dt$$

Hence, using (6.63), we get

$$s = 1 - t + o(1 - t) \tag{6.64}$$

(6.60), (6.62), (6.63) and (6.64) yield to (6.59).

□

Injecting (6.59) in (6.56), we get,

$$\left| - \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \phi dA_{S_{1-\epsilon}(p)} \right| \lesssim \epsilon \|\phi\|_{L^\infty} \|\mathbf{D}_T^{(A)} F\|_{L^\infty} \|\lambda\|_{L^2(S_{1-\epsilon})} \tag{6.65}$$

and

$$\|\lambda\|_{L^2(S_{1-\epsilon})} = \left(\int_{S_{1-\epsilon}} |\lambda|^2 dA \right)^{\frac{1}{2}} \lesssim \|\lambda\|_{L^\infty(S_{1-\epsilon})} |S_{1-\epsilon}|^{\frac{1}{2}} \lesssim \epsilon \|\lambda\|_{L^\infty(S_{1-\epsilon})}$$

Now, we would want to study the behavior of $\|\lambda\|_{L^\infty(S_{1-\epsilon})}$ when $\epsilon \mapsto 0$.

Lemma 6.1.24. *Let $\Psi_{\alpha\beta}$ be a \mathcal{G} -valued tensor, and $|\Psi_{\alpha\beta}| = \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}$. Then,*

$$|\nabla_\sigma |\Psi_{\alpha\beta}| | \leq |\mathbf{D}^{(A)} \Psi_{\alpha\beta}| + |\Psi(\nabla_\sigma e_\alpha, e_\beta)| + |\Psi(e_\alpha, \nabla_\sigma e_\beta)| \tag{6.66}$$

Proof.

We can compute

$$\begin{aligned}
\nabla_\sigma |\Psi_{\alpha\beta}| &= \frac{2 \langle \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} \rangle}{2 \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}} \\
&= \frac{2 \langle \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} \rangle - 2 \langle \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] \rangle + 2 \langle \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] \rangle}{2 \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}} \\
&= \frac{2 \langle \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} \rangle - 2 \langle [\Psi_{\alpha\beta}, A_\sigma], \Psi_{\alpha\beta} \rangle + 2 \langle \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] \rangle}{2 \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}}
\end{aligned}$$

(since $\langle \cdot, \cdot \rangle$ is Ad-invariant)

$$\begin{aligned}
&= \frac{2 \langle \nabla_\sigma \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle + 2 \langle [A_\sigma, \Psi_{\alpha\beta}], \Psi_{\alpha\beta} \rangle + 2 \langle \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] \rangle}{2 \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}} \\
&\quad + \frac{2 \langle \Psi(\nabla_\sigma e_\alpha, e_\beta), \Psi_{\alpha\beta} \rangle + 2 \langle \Psi(e_\alpha, \nabla_\sigma e_\beta), \Psi_{\alpha\beta} \rangle}{2 \langle \Psi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle^{\frac{1}{2}}}
\end{aligned}$$

Hence,

$$\begin{aligned}
|\nabla_\sigma |\Psi_{\alpha\beta}|| &\leq \frac{|\mathbf{D}_\sigma^{(A)} \Psi_{\alpha\beta}| |\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} + \frac{|\Psi(\nabla_\sigma e_\alpha, e_\beta)| |\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} + \frac{|\Psi(e_\alpha, \nabla_\sigma e_\beta)| |\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} \\
&\leq |\mathbf{D}^{(A)} \Psi_{\alpha\beta}| + |\Psi(\nabla_\sigma e_\alpha, e_\beta)| + |\Psi(e_\alpha, \nabla_\sigma e_\beta)|
\end{aligned}$$

□

Lemma 6.1.25. *We have,*

$$\nabla_\sigma h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) + 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \nabla_\sigma \hat{t}) \quad (6.67)$$

Proof.

$$\begin{aligned}
\nabla_\sigma h(e_\alpha, e_\beta) &= \partial_\sigma h(e_\alpha, e_\beta) - h(\nabla_\sigma e_\alpha, e_\beta) - h(e_\alpha, \nabla_\sigma e_\beta) \\
&= \nabla_\sigma \mathbf{g}(e_\alpha, e_\beta) + 2\nabla_\sigma [\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t})] \\
&= 2\partial_\sigma [\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t})] - 2\mathbf{g}(\nabla_\sigma e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) \\
&\quad - 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(\nabla_\sigma e_\beta, \hat{t})
\end{aligned}$$

(since the metric \mathbf{g} is Killing)

$$\begin{aligned}
&= 2\partial_\sigma \mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) + 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \partial_\sigma \mathbf{g}(e_\beta, \hat{t}) \\
&\quad - 2\mathbf{g}(\nabla_\sigma e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) - 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(\nabla_\sigma e_\beta, \hat{t}) \\
&= 2[\partial_\sigma \mathbf{g}(e_\alpha, \hat{t}) - \mathbf{g}(\nabla_\sigma e_\alpha, \hat{t})] \cdot \mathbf{g}(e_\beta, \hat{t}) \\
&\quad + 2[\partial_\sigma \mathbf{g}(e_\beta, \hat{t}) - \mathbf{g}(\nabla_\sigma e_\beta, \hat{t})] \cdot \mathbf{g}(e_\alpha, \hat{t})
\end{aligned}$$

Using the fact that $\nabla g = 0$, we get,

$$\nabla_\sigma h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \hat{t}) \cdot \mathbf{g}(e_\beta, \hat{t}) + 2\mathbf{g}(e_\alpha, \hat{t}) \cdot \mathbf{g}(e_\beta, \nabla_\sigma \hat{t})$$

□

Let,

$$\hat{t}_\alpha = (\hat{t})_\alpha = \mathbf{g}_{\mu\alpha}(\hat{t})^\mu$$

Hence, we can write (6.52) as,

$$h_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + 2(\hat{t})_\alpha (\hat{t})_\beta \tag{6.68}$$

and (6.67) as,

$$\nabla_\sigma h_{\alpha\beta} = 2[\nabla_\sigma \hat{t}_\alpha \cdot \hat{t}_\beta + \hat{t}_\alpha \cdot \nabla_\sigma \hat{t}_\beta] \tag{6.69}$$

Lemma 6.1.26. *Let Ψ be a \mathcal{G} -valued tensor, we have,*

$$|\nabla_\sigma |\Psi|^2|(p) \leq C(p)[|\mathbf{D}_\sigma^{(A)} \Psi|^2 + |\Psi|^2] \tag{6.70}$$

where $C(p)$ depends on the space-time geometry on the point p .

Proof.

$$\begin{aligned}\nabla_\sigma |\Psi|^2 &= \nabla_\sigma (h_{\alpha\mu} h_{\beta\nu} |\Psi^{\mu\nu}| |\Psi^{\alpha\beta}|) = \nabla_\sigma (h_{\alpha\mu} h_{\beta\nu}) \cdot |\Psi^{\mu\nu}| |\Psi^{\alpha\beta}| \\ &\quad + h_{\alpha\mu} h_{\beta\nu} \cdot \nabla_\sigma (|\Psi^{\mu\nu}| |\Psi^{\alpha\beta}|)\end{aligned}$$

Therefore,

$$\begin{aligned}|\nabla_\sigma |\Psi|^2| &\leq |(\nabla_\sigma h_{\alpha\mu}) h_{\beta\nu}| \cdot |\Psi^{\mu\nu}| |\Psi^{\alpha\beta}| + |h_{\alpha\mu} (\nabla_\sigma h_{\beta\nu})| \cdot |\Psi^{\mu\nu}| |\Psi^{\alpha\beta}| \\ &\quad + |h_{\alpha\mu} h_{\beta\nu}| \cdot (|\mathbf{D}_\sigma^{(A)} \Psi^{\mu\nu}| + |\Psi(\nabla_\sigma e^\mu, e^\nu)| + |\Psi(e^\mu, \nabla_\sigma e^\nu)|) \cdot |\Psi^{\alpha\beta}| \\ &\quad + |h_{\alpha\mu} h_{\beta\nu}| \cdot |\Psi^{\mu\nu}| \cdot (|\mathbf{D}_\sigma^{(A)} \Psi^{\alpha\beta}| + |\Psi(\nabla_\sigma e^\alpha, e^\beta)| + |\Psi(e^\alpha, \nabla_\sigma e^\beta)|)\end{aligned}$$

(due to (6.66)).

Using (6.67), applying Cauchy-Schwarz, using the fact that the metric is smooth, and the inequality $a \cdot b \lesssim a^2 + b^2$, we get,

$$\begin{aligned}|\nabla_\sigma |\Psi|^2|(p) &\leq C(p) [h_{\alpha\mu} h_{\beta\nu} |\mathbf{D}_\sigma^{(A)} \Psi^{\mu\nu}| \cdot |\mathbf{D}_\sigma^{(A)} \Psi^{\alpha\beta}| + h_{\alpha\mu} h_{\beta\nu} |\Psi^{\mu\nu}| \cdot |\Psi^{\alpha\beta}|] \\ &\lesssim |\mathbf{D}_\sigma^{(A)} \Psi|^2 + |\Psi|^2\end{aligned}$$

□

Finally, we get

Lemma 6.1.27. *We have,*

$$\sup_{0 \leq \bar{s} \leq s} |\bar{s} \lambda|^2 \leq C(p, s) |J|^2 \quad (6.71)$$

Proof.

We also have at $q \in N^-(p) \setminus \{p\}$,

$$\mathbf{D}_L^{(A)}(s\lambda_{\alpha\beta}) = L(s)\lambda_{\alpha\beta} + s\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} = \lambda_{\alpha\beta} + s(-\frac{1}{2}tr\chi\lambda_{\alpha\beta}) = -\frac{s}{2}\lambda_{\alpha\beta}(tr\chi - \frac{2}{s})$$

As $|tr\chi - \frac{2}{s}| \mapsto 0$, we get

$$\mathbf{D}_L^{(A)}(s\lambda_{\alpha\beta}) = O(1)s\lambda_{\alpha\beta} \tag{6.72}$$

Hence,

$$\begin{aligned} |\nabla_L |s\lambda|^2| &\lesssim |\mathbf{D}_L^{(A)}(s\lambda)|^2 + |s\lambda|^2 \\ &\lesssim |s\lambda|^2 \end{aligned}$$

(due to (6.72)).

For all $(u = 0, \bar{s}, \omega) \in N_\tau^-(p)$,

$$\int_0^s \nabla_L |s\lambda|^2 d\bar{s} = |s\lambda|^2(s) - |s\lambda|^2(p) \leq O(s)C(p) \sup_{0 \leq \bar{s} \leq s} |s\lambda|^2$$

As

$$|s\lambda|^2(p) = |J|^2$$

choosing s small depending on p we have (6.1.27). □

Therefore $s\lambda_{\alpha\beta}$ remains bounded near p , and it is also smooth away from p , so $\epsilon \|\lambda\|_{L^\infty(S_{1-\epsilon})}$ remains bounded and therefore $\|\lambda\|_{L^2(S_{1-\epsilon})}$ remains bounded. Therefore (6.65) gives,

$$\lim_{\epsilon \rightarrow 0} \left| - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \langle \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} \rangle \right| = 0 \tag{6.73}$$

6.1.28 Estimating $\lim_{\epsilon \rightarrow 0} \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \langle \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle$

Examining now,

$$\begin{aligned}
& \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \langle \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} \rangle \\
&= \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \langle \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta})\delta(u) + \nabla_T u \delta'(u) \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\
&= \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \langle \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}), F^{\alpha\beta} \rangle + \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta'(u) \phi^{-1} \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\
&= I_\epsilon^1 + I_\epsilon^2 \tag{6.74}
\end{aligned}$$

At $q \in N^-(p) \setminus \{p\}$ define,

$$N = \phi L + T \tag{6.75}$$

We have,

$$\begin{aligned}
\mathbf{g}(N, N) &= \mathbf{g}(\phi L + T, \phi L + T) = \phi \mathbf{g}(T, L) + \phi \mathbf{g}(L, T) + \mathbf{g}(T, T) \\
&= 2\phi\phi^{-1} - 1 = 2 - 1 = 1
\end{aligned}$$

means that N is unit. For all $X \in T_q S_{1-\epsilon}(p)$ tangent to $S_{1-\epsilon}(p)$, i.e. $X \in T_q N^-(p) \cap T_q \Sigma_{1-\epsilon}$, we have

$$\mathbf{g}(N, X) = \mathbf{g}(\phi L + T, X) = \phi \mathbf{g}(L, X) + \mathbf{g}(T, X) \tag{6.76}$$

$$\mathbf{g}(L, X) = 0 \tag{6.77}$$

(since $X \in T_q N^-(p)$), and

$$\mathbf{g}(T, X) = 0 \tag{6.78}$$

(since $X \in T_q \Sigma_{1-\epsilon}$).

(6.77) and (6.78) show that N is the unit normal to $S_{1-\epsilon} = N^-(p) \cap \Sigma_{1-\epsilon}$, it can be extended locally to define a vectorfield.

Thus, we have

$$\begin{aligned} \nabla_N \delta(u) &= \nabla_N(u) \cdot \delta'(u) = \nabla_{\phi L + T}(u) \cdot \delta'(u) \\ &= (\phi \nabla_L(u) + \nabla_T(u)) \cdot \delta'(u) = \nabla_T(u) \cdot \delta'(u) \\ &= \phi^{-1} \delta'(u) \end{aligned}$$

Thus

$$\begin{aligned} I_\epsilon^2 &= \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \nabla_N \delta(u) \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\ &= - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \nabla_N \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \\ &\quad - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \operatorname{div}(N) \cdot \delta(u) \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \end{aligned}$$

(by integration by parts)

We choose e_1, e_2 tangent to $S_{1-\epsilon}$. Since N, e_1, e_2 are unit we have

$$\operatorname{div} N = \mathbf{g}(\nabla_N N, N) + \mathbf{g}(\nabla_a N, e_a), \quad a \in \{1, 2\}$$

$\mathbf{g}(N, N) = 1$ gives $\mathbf{g}(\nabla_N N, N) = 0$. We get $\operatorname{div} N = \mathbf{g}(\nabla_a N, e_a)$ and N is unit normal to $S_{1-\epsilon}$, and $N \in T_q \Sigma_{1-\epsilon}$ since,

$$\mathbf{g}(N, T) = \mathbf{g}(\phi L + T, T) = \phi \mathbf{g}(L, T) + \mathbf{g}(T, T) = \phi \phi^{-1} - 1 = 1 - 1 = 0$$

so we get $\operatorname{div} N = \operatorname{tr} \theta$, where θ is the second fundamental form of the surface $S_{1-\epsilon}$ embedded in $\Sigma_{1-\epsilon}$, defined as,

$$\theta(X, Y) = \mathbf{g}(\nabla_X N, Y) \quad (6.79)$$

for all $X, Y \in T_q S_{1-\epsilon}$. Thus (6.79) becomes,

$$\begin{aligned} I_\epsilon^2 &= - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) (\nabla_N \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle + \operatorname{tr} \theta \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle) \\ &= - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) (\langle \mathbf{D}_N^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle + \operatorname{tr} \theta \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle) \\ &\quad - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \langle \lambda_{\alpha\beta}, \mathbf{D}_N^{(A)} F^{\alpha\beta} \rangle \end{aligned} \quad (6.80)$$

We showed (6.73), in the same manner, we have,

$$\lim_{\epsilon \rightarrow 0} \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \langle \lambda_{\alpha\beta}, \mathbf{D}_N^{(A)} F^{\alpha\beta} \rangle = 0 \quad (6.81)$$

Thus, injecting (6.81) and (6.80) in (6.74), we get,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} I_\epsilon^1 + I_\epsilon^2 &= \lim_{\epsilon \rightarrow 0} \left(\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) [\langle \mathbf{D}_T^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right. \\ &\quad \left. - \langle \mathbf{D}_N^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle + \operatorname{tr} \theta \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle] \right) \\ &= \lim_{\epsilon \rightarrow 0} - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) (\langle \mathbf{D}_{N-T}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle + \operatorname{tr} \theta \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle) \end{aligned} \quad (6.82)$$

We recall that $N = \phi L + T$, thus $\phi L = N - T$, therefore

$$\mathbf{D}_{N-T}^{(A)} \lambda_{\alpha\beta} = \phi \mathbf{D}_L^{(A)} \lambda_{\alpha\beta}$$

and we recall that by construction of λ , we have $\mathbf{D}_L^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \operatorname{tr} \chi \lambda_{\alpha\beta} = 0$ at $q \in N^-(p) \setminus \{p\}$.

We obtain,

$$\mathbf{D}_{N-T}^{(A)}\lambda_{\alpha\beta} = -\frac{1}{2}\text{tr}\chi\phi\lambda_{\alpha\beta} \quad (6.83)$$

at $q \in N^-(p) \setminus \{p\}$.

Hence, from (6.83) we can write (6.82) as,

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^1 + I_\epsilon^2 = -\lim_{\epsilon \rightarrow 0} \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \left(-\frac{1}{2}\phi \text{tr}\chi + \text{tr}\theta\right) \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \quad (6.84)$$

$$\begin{aligned} \theta_{aa} &= \mathbf{g}(\nabla_a N, e_a) = \mathbf{g}(\nabla_a(\phi L + T), e_a) \\ &= \mathbf{g}(e_a(\phi)L + \phi \nabla_a L + \nabla_a T, e_a) \\ &= e_a(\phi)\mathbf{g}(L, e_a) + \phi \mathbf{g}(\nabla_a L, e_a) + \mathbf{g}(\nabla_a T, e_a). \end{aligned}$$

We have $\mathbf{g}(L, e_a) = 0$, therefore,

$$\theta_{aa} = \phi \chi_{aa} + k_{aa} \quad (6.85)$$

where,

$$k_{aa} = \mathbf{g}(\nabla_a T, e_a) \quad (6.86)$$

Injecting (6.85) and (6.86) in (6.84) we get,

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^1 + I_\epsilon^2 = -\lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \left(\frac{1}{2}\phi \text{tr}\chi + k_{aa}\right) \cdot \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \quad (6.87)$$

where the repeated index k_{aa} means summation $\sum_{a=1,2} k_{aa} = \delta^{ab} k_{ab}$.

We get,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} I_\epsilon^1 + I_\epsilon^2 \tag{6.88} \\ = & -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \phi \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle - \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \delta^{ab} k_{ab} \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \end{aligned}$$

Lemma 6.1.29. *We have,*

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \delta^{ab} k_{ab} \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = 0$$

Proof. :

$$\begin{aligned} & \left| \int_{\Sigma_{1-\epsilon}} \delta^{ab} k_{ab} \delta(u) \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \right| = \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \delta^{ab} k_{ab} \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle \phi dA_{S_{t=1-\epsilon}} \\ & \lesssim \|k\|_{L^\infty} \|F\|_{L^\infty} \|\phi\|_{L^\infty} \|\lambda\|_{L^2(S_{1-\epsilon}(p))} |A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}} \\ & \lesssim \epsilon \|k\|_{L^\infty} \|F\|_{L^\infty} \|\phi\|_{L^\infty} \|\lambda\|_{L^2(S_{1-\epsilon}(p))} \end{aligned}$$

And as we showed previously $\|\lambda\|_{L^2(S_{1-\epsilon}(p))}$ remains bounded as $\epsilon \rightarrow 0$. Thus,

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \delta^{ab} k_{ab} \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = 0 \tag{6.89}$$

□

We are left to estimate $-\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \phi \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle$ in (6.88).

Lemma 6.1.30. *We have,*

$$-\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \phi \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta}(p) \rangle dA$$

Proof. :

$$-\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \phi \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{t=1-\epsilon}} \phi^2 \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} \rangle_{\mathbf{g}} dA$$

We have

$$\begin{aligned} & \left| \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) \rangle dA \right| \\ &= \left| \int_{S_{1-\epsilon}(p)} \phi^2 \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) \rangle dA \right| \\ &\lesssim \|\phi\|_{L^\infty}^2 \|A\|_{L^2(S_{1-\epsilon}(p))} \|F - F(p)\|_{L^\infty(S_{1-\epsilon}(p))} |S_{1-\epsilon}(p)|^{\frac{1}{2}} \|\operatorname{tr} \chi\|_{L^\infty(S_{1-\epsilon}(p))} \end{aligned}$$

As,

$$|S_{1-\epsilon}(p)| \sim 4\pi\epsilon^2 \quad \text{as } \epsilon \mapsto 0$$

and

$$\|\operatorname{tr} \chi\| \sim \frac{2}{\epsilon} \quad \text{as } \epsilon \mapsto 0$$

we get,

$$|S_{1-\epsilon}(p)|^{\frac{1}{2}} \|\operatorname{tr} \chi\|_{L^\infty(S_{1-\epsilon}(p))} \sim 8\pi \quad \text{as } \epsilon \mapsto 0$$

This yields to

$$\begin{aligned} & \left| \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} \chi \langle \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) \rangle dA \right| \\ &\lesssim \|\phi\|_{L^\infty}^2 \|A\|_{L^2(S_{1-\epsilon}(p))} \|F - F(p)\|_{L^\infty(S_{1-\epsilon}(p))} \end{aligned}$$

Since,

$$\lim_{\epsilon \rightarrow 0} \|F - F(p)\|_{L^\infty(S_{1-\epsilon}(p))} = 0$$

and as we showed $\|\lambda_{\alpha\beta}\|_{L^2(S_{1-\epsilon}(p))}$ remains bounded as $\epsilon \rightarrow 0$, we get

$$\lim_{\epsilon \rightarrow 0} \left| \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} \chi < \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) >_{\mathbf{g}} dA \right| = 0$$

Therefore,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} X < \lambda_{\alpha\beta}, F^{\alpha\beta} > dA \\ &= \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} X < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > dA \end{aligned} \quad (6.90)$$

□

Lemma 6.1.31. *We have,*

$$\lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} X < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > dA = 8\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) >$$

Proof.

We have,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \operatorname{tr} \chi < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > \\ &= \lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 (s^{-1} \operatorname{tr} \chi) < (s\lambda_{\alpha\beta}), F^{\alpha\beta}(p) > \end{aligned} \quad (6.91)$$

As,

$$\left| \operatorname{tr} \chi - \frac{2}{s} \right| = O(s^2) \quad (6.92)$$

where O depends on the geometry of the space-time (see for example proposition 3.2 in the thesis of Q. Wang [Wang]), we get,

$$\lim_{s \rightarrow 0} \sup_{S_{1-\epsilon}(p)} \left| s^{-1} \operatorname{tr} \chi - \frac{2}{s^2} \right| = 0 \quad (6.93)$$

and we know that,

$$\lim_{s \rightarrow 0} \sup_{S_{1-\epsilon}(p)} |\phi - 1| = 0 \quad (6.94)$$

and,

$$\lim_{s \rightarrow 0} (s\lambda_{\alpha\beta}) = \mathbf{J}_p \quad (6.95)$$

This yields to

$$|s^{-1}tr\chi| \sim_{\epsilon \rightarrow 0} \frac{2}{\epsilon^2} \quad (6.96)$$

$$|\phi| \sim_{\epsilon \rightarrow 0} 1 \quad (6.97)$$

$$(s\lambda_{\alpha\beta}) \sim_{\epsilon \rightarrow 0} \mathbf{J}_p \quad (6.98)$$

and therefore,

$$\begin{aligned} & \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 (s^{-1}tr\chi) \langle (s\lambda_{\alpha\beta}), F^{\alpha\beta}(p) \rangle dA \\ & \sim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \frac{2}{\epsilon^2} \langle (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) \rangle dA \end{aligned} \quad (6.99)$$

and since,

$$|N^-(p) \cap \Sigma_{1-\epsilon}| = |S_{1-\epsilon}(p)| \sim_{\epsilon \rightarrow 0} 4\pi\epsilon^2 \quad (6.100)$$

we get,

$$\int_{N^-(p) \cap \Sigma_{1-\epsilon}} \frac{2}{\epsilon^2} \langle (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) \rangle dA = \frac{2}{\epsilon^2} (4\pi)\epsilon^2 \langle (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) \rangle + O(\epsilon) \quad (6.101)$$

where $O(\epsilon) \mapsto 0$ as $\epsilon \mapsto 0$

Given (6.91), this yields to

$$\lim_{\epsilon \rightarrow 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 \text{tr} \chi < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > = 8\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > \quad (6.102)$$

□

From (6.1.30) we get,

$$-\frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\Sigma_{1-\epsilon}} \delta(u) \phi \text{tr} \chi < \lambda_{\alpha\beta}, F^{\alpha\beta} > = -4\pi < (\mathbf{J}_p)_{\alpha\beta}, F(p)^{\alpha\beta} >$$

6.1.32 The parametrix

Finally, combining (6.11), (6.15), (6.43), (6.73), (6.74), (6.88), (6.1.29), (6.1.30) and (6.1.31), we get,

$$\begin{aligned} & \int_{\Omega} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > \\ = & \lim_{\epsilon \rightarrow 0} \left[\int_{\Omega_{\epsilon}} \delta(u) < \hat{\Delta}^{(A)} \lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mu \lambda_{\alpha\beta} \right. \\ & \left. + [F_{LL}, \lambda_{\alpha\beta}] - \frac{1}{2} R_{\alpha}^{\gamma}{}_{LL} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}^{\gamma}{}_{LL} \lambda_{\alpha\gamma}, F^{\alpha\beta} > \right] + 0 - 4\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > \\ & + \int_{J^-(p) \cap \Sigma_t} < \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} > |_{t=0} - \int_{J^-(p) \cap \Sigma_t} < \mathbf{D}_T^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} > |_{t=0} \end{aligned}$$

Therefore,

$$\begin{aligned} 4\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > & = - \int_{\Omega} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > \\ & + \int_{\Omega} \delta(u) < \hat{\Delta}^{(A)} \lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} \mu \lambda_{\alpha\beta} \\ & + [F_{LL}, \lambda_{\alpha\beta}] - \frac{1}{2} R_{\alpha}^{\gamma}{}_{LL} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}^{\gamma}{}_{LL} \lambda_{\alpha\gamma}, F^{\alpha\beta} > \\ & + \int_{J^-(p) \cap \Sigma} < \lambda_{\alpha\beta} \delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} > \\ & - \int_{J^-(p) \cap \Sigma} < \mathbf{D}_T^{(A)} (\lambda_{\alpha\beta} \delta(u)), F^{\alpha\beta} > \quad (6.103) \end{aligned}$$

where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$, of $\lambda_{\alpha\beta}$, defined by (6.49), and where the last two terms are the contribution of the initial data, the first term is the contribution of the source term in the tensorial wave equation, and the middle term is related to the geometry of the problem.

6.2 The Schwarzschild Space-Time and Black Holes

General Relativity postulates that space-time is a 4-dimensional Lorentzian manifold (M, \mathbf{g}) that satisfies the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}R = 8\pi T_{\mu\nu}$$

where $T_{\mu\nu}$ is a symmetric 2-tensor on M that is the stress-energy-momentum tensor of matter. In vacuum $T_{\mu\nu} = 0$, thus the Einstein vacuum equations are $R_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}R = 0$. In vacuum, this yields to $R_{\mu\nu} = \frac{1}{2}\mathbf{g}_{\mu\nu}R$ and since $R = R^i_i$ by definition, we get $R = \mathbf{g}^{ij}R_{ij} = \frac{1}{2}\mathbf{g}^{ij}\mathbf{g}_{ij}R = \frac{4}{2}R = 2R$. This means that in vacuum $R = 0$ and therefore the Einstein vacuum equations can be written as the following:

$$R_{\mu\nu} = 0 \tag{6.104}$$

The simplest solution of the Einstein vacuum equations is the 4-dimensional Minkowski metric, it represents a flat space-time. The first black hole solution of the Einstein vacuum equations was discovered by Karl Schwarzschild about a month after the publication of Einstein's theory of General Relativity. However, it took nearly 50 years from then for it to be fully understood as a black hole space-time. When it was first discovered, it was to represent the gravitational field outside a spherical, uncharged, non-rotating star with

mass m and was written under the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (6.105)$$

It turned out that this solution could be extended, as solutions to the Einstein vacuum equations (6.104), to describe the gravitational field inside the star, created by the mass of the star, and thought of as being there in vacuum, i.e. without the matter inside the star. The extended Schwarzschild solution is what became to be known as a black hole space-time. It is also good to note that according to Birkhoff's theorem, any spherically symmetric solution of the Einstein vacuum equations is locally isometric to the Schwarzschild solution. In this sense the Schwarzschild solution is unique although it can have a different form in a different system of coordinates.

6.2.1 The extended Schwarzschild solution

To obtain the extended Schwarzschild solution, as explained in [HE], first let,

$$r^* = \int \frac{dr}{\left(1 - \frac{2m}{r}\right)} = r + 2m \log(r - 2m) \quad (6.106)$$

We have,

$$\begin{aligned} dr^* &= dr + \frac{2m}{r - 2m}dr = \left(1 + \frac{1}{\frac{r}{2m} - 1}\right)dr = \left(\frac{\frac{r}{2m}}{\frac{r}{2m} - 1}\right)dr \\ &= \left(\frac{1}{1 - \frac{2m}{r}}\right)dr \end{aligned}$$

Thus, the Schwarzschild space-time in the exterior (6.105) can be also written as:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)dr^{*2} + r^2d\sigma^2 \quad (6.107)$$

where $d\sigma^2$ is the usual metric on the unit sphere.

Let,

$$v = t + r^* \quad (6.108)$$

$$w = t - r^* \quad (6.109)$$

We have

$$\begin{aligned} t &= \frac{v + w}{2} \\ dt &= \frac{dv + dw}{2} \\ (dt)^2 &= \frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2} = \frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dv \otimes dw}{4} + \frac{dw \otimes dv}{4} \end{aligned}$$

Injecting in (6.107), we get

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\left[\frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2}\right] + \left(1 - \frac{2m}{r}\right)(dr^*)^2 + r^2 d\sigma^2$$

We also have

$$\begin{aligned} r^* &= \frac{v - w}{2} \\ (dr^*)^2 &= \left(\frac{dv - dw}{2}\right)^2 = \frac{dv^2}{4} + \frac{dw^2}{4} - \frac{dvdw}{2} \end{aligned}$$

Therefore,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\left[\frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2}\right] + \left(1 - \frac{2m}{r}\right)^{-1}\left(1 - \frac{2m}{r}\right)^2\left[\frac{dv^2}{4} + \frac{dw^2}{4} - \frac{dvdw}{2}\right] + r^2 d\sigma^2$$

Thus,

$$\begin{aligned}
 ds^2 &= -\left(1 - \frac{2m}{r}\right)dvdw + r^2d\sigma^2 \\
 &= -\frac{\left(1 - \frac{2m}{r}\right)}{2}dv \otimes dw - \frac{\left(1 - \frac{2m}{r}\right)}{2}dw \otimes dv + r^2d\sigma^2
 \end{aligned} \tag{6.110}$$

Kruskal coordinates

Let $v' = v'(v)$, $w' = w'(w)$ where v', w' are arbitrary C^1 functions.

$$dv = \frac{dv}{dv'}dv'$$

$$dw = \frac{dw}{dw'}dw'$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\frac{dv}{dv'}\frac{dw}{dw'}dv'dw' + r^2d\sigma^2 \tag{6.111}$$

Define,

$$t' = \frac{v' + w'}{2} \tag{6.112}$$

$$x' = \frac{v' - w'}{2} \tag{6.113}$$

We get

$$v' = t' + x'$$

$$w' = t' - x'$$

and

$$dv' = dt' + dx'$$

$$dw' = dt' - dx'$$

Thus,

$$\begin{aligned} dv' d\omega' &= (dt' + dx')(dt' - dx') = (dt')^2 - dt' dx' + dx' dt' - (dx')^2 = (dt')^2 - (dx')^2 \\ &= -(-(dt')^2 + (dx')^2) \end{aligned}$$

Let,

$$F^2 = -\left(-\left(1 - \frac{2m}{r}\right) \frac{dv}{dv'} \frac{dw}{dw'}\right) = \left(1 - \frac{2m}{r}\right) \frac{dv}{dv'} \frac{dw}{dw'}$$

From (6.111), we have

$$(ds')^2 = F^2(t', x')(-dt')^2 + (dx')^2 + r^2 d\sigma^2$$

Kruskal's choice is:

$$v' = \exp\left(\frac{v}{4m}\right) \quad (6.114)$$

$$w' = -\exp\left(-\frac{w}{4m}\right) \quad (6.115)$$

$$\begin{aligned} t' &= \frac{1}{2}(v' + w') \\ (t')^2 &= \frac{1}{4}[(v')^2 + (w')^2 + 2v'w'] = \frac{1}{4}\left(\exp\left(\frac{v}{2m}\right) + \exp\left(-\frac{w}{2m}\right) + -2\exp\left(\frac{v-w}{4m}\right)\right) \\ x' &= \frac{1}{2}(v' - w') \\ (x')^2 &= \frac{1}{4}\left(\exp\left(\frac{v}{2m}\right) + \exp\left(-\frac{w}{2m}\right) + 2\exp\left(\frac{v-w}{4m}\right)\right) \end{aligned}$$

Computing,

$$(t')^2 - (x')^2 = -\frac{2}{4}\left(\exp\left(\frac{v-w}{2m}\right) + \exp\left(\frac{v-w}{2m}\right)\right) = -\frac{2}{2}\exp\left(\frac{v-w}{4m}\right) = -\exp\left(\frac{v-w}{4m}\right)$$

$$v - w = 2r^* = 2r + 4m \log(r - 2m)$$

Thus,

$$(t')^2 - (x')^2 = -\exp\left(\frac{r}{2m}\right) \cdot (r - 2m) \quad (6.116)$$

Computing,

$$\begin{aligned} \frac{dv'}{dv} &= \frac{1}{4m} \exp\left(\frac{v}{4m}\right), \quad \frac{dw'}{dw} = \frac{1}{4m} \exp\left(-\frac{w}{4m}\right) \\ \frac{dv}{dv'} &= 4m \cdot \exp\left(-\frac{v}{4m}\right), \quad \frac{dw}{dw'} = 4m \cdot \exp\left(\frac{w}{4m}\right) \\ \frac{dv}{dv'} \frac{dw}{dw'} &= 16 \cdot m^2 \exp\left(\frac{w-v}{4m}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} F^2 &= \left(1 - \frac{2m}{r}\right) 16 \cdot m^2 \exp\left(-\frac{2r^*}{4m}\right) \\ &= \frac{(r-2m)}{r} 16 \cdot m^2 \exp\left(-\frac{2}{4m}(r + 2m \log(r-2m))\right) \\ &= \frac{(r-2m)}{r} 16 \cdot m^2 \exp\left(-\frac{2r}{4m}\right) \exp(-\log(r-2m)) \\ &= \frac{1}{r} \cdot 16 \cdot m^2 \exp\left(-\frac{r}{2m}\right) \\ &= \frac{16 \cdot m^2}{r} \cdot \exp\left(-\frac{r}{2m}\right) \end{aligned}$$

Finally, we obtain,

$$ds^2 = \frac{16m^2}{r} \exp\left(-\frac{r}{2m}\right) \left(-dt'\right)^2 + (dx')^2 + r^2(t', x') d\sigma^2 \quad (6.117)$$

6.2.2 The Penrose diagram

The Penrose diagram, see [HE], is constructed by taking,

$$v'' = \arctan\left(\frac{v'}{2m}\right) \quad (6.118)$$

$$w'' = \arctan\left(\frac{w'}{2m}\right) \quad (6.119)$$

$$-\pi < v'' + w'' < \pi \quad (6.120)$$

$$-\frac{\pi}{2} < v'' < \frac{\pi}{2} \quad (6.121)$$

$$-\frac{\pi}{2} < w'' < \frac{\pi}{2} \quad (6.122)$$

6.2.3 The compatible symmetric connection

Computing the Christoffel symbols for the Schwarzschild metric, we have

$$\Gamma_{kl}^i = \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{mk}}{\partial x^l} + \frac{\partial \mathbf{g}_{ml}}{\partial x^k} - \frac{\partial \mathbf{g}_{kl}}{\partial x^m} \right)$$

Hence,

$$\Gamma_{\theta\theta}^i = \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\theta}}{\partial x^\theta} + \frac{\partial \mathbf{g}_{m\theta}}{\partial x^\theta} - \frac{\partial \mathbf{g}_{\theta\theta}}{\partial x^m} \right)$$

We get

$$\Gamma_{\theta\theta}^\theta = \frac{1}{2} \mathbf{g}^{\theta\theta} \left(2 \frac{\partial \mathbf{g}_{\theta\theta}}{\partial x^\theta} - \frac{\partial \mathbf{g}_{\theta\theta}}{\partial x^\theta} \right)$$

$$= 0$$

$$\Gamma_{\theta\theta}^r = \frac{1}{2} \mathbf{g}^{rr} \left(- \frac{\partial \mathbf{g}_{\theta\theta}}{\partial x^r} \right)$$

$$= \frac{1}{2} (1 - \mu) \left(- \frac{\partial r^2}{\partial r} \right)$$

$$= \frac{1}{2} (1 - \mu) (-2r)$$

$$= -(1 - \mu)r$$

$$\Gamma_{\theta\theta}^t = \Gamma_{\theta\theta}^\phi = 0$$

Consequently,

$$\begin{aligned}\nabla_{\theta} \frac{\partial}{\partial \theta} &= -(1 - \mu)r \frac{\partial}{\partial r} \\ &= -r \frac{\partial}{\partial r^*}\end{aligned}\tag{6.123}$$

We have,

$$\Gamma_{\phi\phi}^i = \frac{1}{2} \mathbf{g}^{im} \left(2 \frac{\partial \mathbf{g}_{m\phi}}{\partial x^{\phi}} - \frac{\partial \mathbf{g}_{\phi\phi}}{\partial x^m} \right)$$

therefore

$$\begin{aligned}\Gamma_{\phi\phi}^r &= \frac{1}{2} \mathbf{g}^{rr} \left(2 \frac{\partial \mathbf{g}_{m\phi}}{\partial x^{\phi}} - \frac{\partial \mathbf{g}_{\phi\phi}}{\partial x^r} \right) \\ &= \frac{1}{2} \mathbf{g}^{rr} \left(- \frac{\partial \mathbf{g}_{\phi\phi}}{\partial x^r} \right) \\ &= \frac{1}{2} (1 - \mu) (-2r \sin^2(\theta)) \\ &= -r(1 - \mu) \sin^2(\theta) \\ \Gamma_{\phi\phi}^{\theta} &= \frac{1}{2} \mathbf{g}^{\theta\theta} \left(- \frac{\partial \mathbf{g}_{\phi\phi}}{\partial x^{\theta}} \right) \\ &= \frac{1}{2r^2} \mathbf{g}^{im} \left(-r^2 \frac{\partial \sin^2(\theta)}{\partial \theta} \right) \\ &= \frac{1}{2} (-2 \sin(\theta) \cos(\theta)) \\ &= -\sin(\theta) \cos(\theta)\end{aligned}$$

We obtain

$$\begin{aligned}\nabla_{\phi} \frac{\partial}{\partial \phi} &= -(1 - \mu)r \sin^2(\theta) \frac{\partial}{\partial r} - \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \\ &= -r \sin^2(\theta) \frac{\partial}{\partial r^*} - \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta}\end{aligned}\tag{6.124}$$

We also have

$$\begin{aligned}\Gamma_{\theta\phi}^i &= \frac{1}{2}\mathbf{g}^{im}\left(\frac{\partial\mathbf{g}_{m\theta}}{\partial x^\phi} + \frac{\partial\mathbf{g}_{m\phi}}{\partial x^\theta} - \frac{\partial\mathbf{g}_{\theta\phi}}{\partial x^m}\right) \\ &= \frac{1}{2}\mathbf{g}^{im}\left(\frac{\partial\mathbf{g}_{m\phi}}{\partial x^\theta}\right)\end{aligned}$$

hence

$$\begin{aligned}\Gamma_{\theta\phi}^\phi &= \frac{1}{2}\mathbf{g}^{\phi\phi}\left(\frac{\partial\mathbf{g}_{\phi\phi}}{\partial x^\theta}\right) \\ &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial r^2\sin^2(\theta)}{\partial\theta}\right) \\ &= \frac{1}{2\sin^2(\theta)}(2\sin(\theta)\cos(\theta)) \\ &= \frac{\cos(\theta)}{\sin(\theta)}\end{aligned}$$

We get

$$\nabla_\theta\frac{\partial}{\partial\phi} = \frac{\cos(\theta)}{\sin(\theta)}\frac{\partial}{\partial\phi} \tag{6.125}$$

We also compute

$$\begin{aligned}\Gamma_{\theta t}^i &= \frac{1}{2}\mathbf{g}^{im}\left(\frac{\partial\mathbf{g}_{m\theta}}{\partial x^t} + \frac{\partial\mathbf{g}_{mt}}{\partial x^\theta} - \frac{\partial\mathbf{g}_{\theta t}}{\partial x^m}\right) \\ &= 0\end{aligned}$$

from which we derive

$$\nabla_\theta\frac{\partial}{\partial t} = 0 \tag{6.126}$$

Similarly,

$$\Gamma_{\phi t}^i = 0$$

gives

$$\nabla_{\phi} \frac{\partial}{\partial t} = 0 \tag{6.127}$$

Computing

$$\begin{aligned} \Gamma_{tt}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{mt}}{\partial x^t} + \frac{\partial \mathbf{g}_{mt}}{\partial x^t} - \frac{\partial \mathbf{g}_{tt}}{\partial x^m} \right) \\ &= \frac{1}{2} \mathbf{g}^{im} \left(-\frac{\partial \mathbf{g}_{tt}}{\partial x^m} \right) \end{aligned}$$

$$\begin{aligned} \Gamma_{tt}^r &= \frac{1}{2} \mathbf{g}^{rr} \left(-\frac{\partial \mathbf{g}_{tt}}{\partial x^r} \right) \\ &= \frac{1}{2} (1 - \mu) \left(-\frac{\partial (1 - \mu)}{\partial r} \right) \\ &= \frac{\mu(1 - \mu)}{2r} \end{aligned}$$

we derive

$$\begin{aligned} \nabla_t \frac{\partial}{\partial t} &= \frac{\mu(1 - \mu)}{2r} \frac{\partial}{\partial r} \\ &= \frac{\mu}{2r} \frac{\partial}{\partial r^*} \end{aligned} \tag{6.128}$$

Computing

$$\begin{aligned}
\Gamma_{r^*r^*}^i &= \frac{1}{2}\mathbf{g}^{im}\left(\frac{\partial\mathbf{g}_{mr^*}}{\partial x^{r^*}} + \frac{\partial\mathbf{g}_{mr^*}}{\partial x^{r^*}} - \frac{\partial\mathbf{g}_{r^*r^*}}{\partial x^m}\right) \\
\Gamma_{r^*r^*}^{r^*} &= \frac{1}{2}\mathbf{g}^{r^*r^*}\left(2\frac{\partial\mathbf{g}_{r^*r^*}}{\partial x^{r^*}} - \frac{\partial\mathbf{g}_{r^*r^*}}{\partial x^{r^*}}\right) \\
&= \frac{1}{2(1-\mu)}\left(2\frac{\partial(1-\mu)}{\partial r^*} - \frac{\partial(1-\mu)}{\partial r^*}\right) \\
&= \frac{1}{2(1-\mu)}\left(\frac{\partial(1-\mu)}{\partial r^*}\right) \\
&= \frac{1}{2}\left(\frac{\partial(1-\mu)}{\partial r}\right) \\
&= \frac{\mu}{2r}
\end{aligned}$$

we obtain

$$\nabla_{r^*}\frac{\partial}{\partial r^*} = \frac{\mu}{2r}\frac{\partial}{\partial r^*} \quad (6.129)$$

Computing

$$\begin{aligned}
\Gamma_{tr^*}^i &= \frac{1}{2}\mathbf{g}^{im}\left(\frac{\partial\mathbf{g}_{mt}}{\partial x^{r^*}} + \frac{\partial\mathbf{g}_{mr^*}}{\partial x^t} - \frac{\partial\mathbf{g}_{tr^*}}{\partial x^m}\right) \\
&= \frac{1}{2}\mathbf{g}^{it}\left(\frac{\partial\mathbf{g}_{tt}}{\partial x^{r^*}}\right) \\
\Gamma_{tr^*}^t &= \frac{1}{2}\mathbf{g}^{tt}\left(\frac{\partial\mathbf{g}_{tt}}{\partial x^{r^*}}\right) \\
&= \frac{1}{2(1-\mu)}\left(\frac{\partial(1-\mu)}{\partial r^*}\right) \\
&= \frac{1}{2}\left(\frac{\partial(1-\mu)}{\partial r}\right) \\
&= \frac{\mu}{2r}
\end{aligned}$$

we get

$$\nabla_t\frac{\partial}{\partial r^*} = \frac{\mu}{2r}\frac{\partial}{\partial t} \quad (6.130)$$

We have

$$\begin{aligned}
\Gamma_{\theta r^*}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\theta}}{\partial x^{r^*}} + \frac{\partial \mathbf{g}_{mr^*}}{\partial x^\theta} - \frac{\partial \mathbf{g}_{\theta r^*}}{\partial x^m} \right) \\
&= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\theta}}{\partial r^*} \right) \\
\Gamma_{\theta r^*}^\theta &= \frac{1}{2} \mathbf{g}^{\theta\theta} \left(\frac{\partial \mathbf{g}_{\theta\theta}}{\partial r^*} \right) \\
&= \frac{1}{2r^2} \left(\frac{\partial r^2}{\partial r^*} \right) \\
&= \frac{(1-\mu)}{2r^2} \left(\frac{\partial r^2}{\partial r} \right) \\
&= \frac{(1-\mu)}{2r^2} (2r) \\
&= \frac{(1-\mu)}{r}
\end{aligned}$$

therefore,

$$\nabla_\theta \frac{\partial}{\partial r^*} = \frac{(1-\mu)}{r} \frac{\partial}{\partial \theta} \tag{6.131}$$

Computing

$$\begin{aligned}
\Gamma_{\phi r^*}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\phi}}{\partial x^{r^*}} + \frac{\partial \mathbf{g}_{mr^*}}{\partial x^\phi} - \frac{\partial \mathbf{g}_{\phi r^*}}{\partial x^m} \right) \\
&= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\phi}}{\partial r^*} \right) \\
\Gamma_{\phi r^*}^\phi &= \frac{1}{2} \mathbf{g}^{\phi\phi} \left(\frac{\partial \mathbf{g}_{\phi\phi}}{\partial r^*} \right) \\
&= \frac{1}{2r^2 \sin^2(\theta)} \left(\frac{\partial r^2 \sin^2(\theta)}{\partial r^*} \right) \\
&= \frac{(1-\mu)}{2r^2} \left(\frac{\partial r^2}{\partial r} \right) \\
&= \frac{(1-\mu)}{2r^2} (2r) \\
&= \frac{(1-\mu)}{r}
\end{aligned}$$

we obtain

$$\nabla_{\phi} \frac{\partial}{\partial r^*} = \frac{(1-\mu)}{r} \frac{\partial}{\partial \phi} \quad (6.132)$$

By letting,

$$\begin{aligned} \frac{\hat{\partial}}{\partial t} &= \frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial t} \\ \frac{\hat{\partial}}{\partial r^*} &= \frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial r^*} \end{aligned} \quad (6.133)$$

$$\begin{aligned} \frac{\hat{\partial}}{\partial \theta} &= \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\hat{\partial}}{\partial \phi} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{aligned} \quad (6.134)$$

we can compute,

$$\begin{aligned} \nabla_{\hat{\theta}} \frac{\hat{\partial}}{\partial \theta} &= \frac{1}{r} \nabla_{\theta} \frac{\hat{\partial}}{\partial \theta} = \frac{1}{r^2} \nabla_{\theta} \frac{\partial}{\partial \theta} = -\frac{1}{r} \frac{\partial}{\partial r^*} \\ &= -\frac{\sqrt{(1-\mu)}}{r} \frac{\hat{\partial}}{\partial r^*} \end{aligned} \quad (6.135)$$

$$\begin{aligned} \nabla_{\hat{\phi}} \frac{\hat{\partial}}{\partial \phi} &= \frac{1}{r^2 \sin^2(\theta)} \nabla_{\phi} \frac{\partial}{\partial \phi} \\ &= -\frac{\sqrt{(1-\mu)}}{r} \frac{\hat{\partial}}{\partial r^*} - \frac{\cos(\theta)}{r \sin(\theta)} \frac{\hat{\partial}}{\partial \theta} \end{aligned} \quad (6.136)$$

$$\begin{aligned}
\nabla_{\hat{\theta}} \frac{\hat{\partial}}{\partial \phi} &= \frac{1}{r} \nabla_{\theta} \frac{\hat{\partial}}{\partial \phi} = \frac{1}{r^2} \nabla_{\theta} \left(\frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \right) \\
&= -\frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \phi} + \frac{1}{r^2 \sin(\theta)} \nabla_{\theta} \frac{\partial}{\partial \phi} \\
&= -\frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \phi} + \frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \phi} \\
&= 0
\end{aligned} \tag{6.137}$$

$$\begin{aligned}
\nabla_{\hat{\phi}} \frac{\hat{\partial}}{\partial \theta} &= \frac{1}{r \sin(\theta)} \nabla_{\phi} \frac{\hat{\partial}}{\partial \theta} = \frac{1}{r^2 \sin(\theta)} \nabla_{\phi} \frac{\partial}{\partial \theta} \\
&= \frac{1}{r^2 \sin(\theta)} \frac{\cos(\theta)}{\sin(\theta)} \frac{\partial}{\partial \phi} \\
&= \frac{1 \cos(\theta)}{r \sin(\theta)} \frac{\hat{\partial}}{\partial \phi}
\end{aligned} \tag{6.138}$$

$$\begin{aligned}
\nabla_{\hat{\theta}} \frac{\hat{\partial}}{\partial t} &= \frac{1}{r \sqrt{(1-\mu)}} \nabla_{\theta} \frac{\partial}{\partial t} \\
&= 0
\end{aligned} \tag{6.139}$$

$$\begin{aligned}
\nabla_{\hat{\phi}} \frac{\hat{\partial}}{\partial t} &= \frac{1}{r \sin(\theta) \sqrt{(1-\mu)}} \nabla_{\phi} \frac{\partial}{\partial t} \\
&= 0
\end{aligned} \tag{6.140}$$

$$\begin{aligned}
\nabla_{\hat{t}} \frac{\hat{\partial}}{\partial t} &= \frac{1}{(1-\mu)} \nabla_t \frac{\partial}{\partial t} \\
&= \frac{\mu}{2r \sqrt{(1-\mu)}} \frac{\hat{\partial}}{\partial r^*}
\end{aligned} \tag{6.141}$$

$$\begin{aligned}
\nabla_{\hat{r}^*} \frac{\hat{\partial}}{\partial r^*} &= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \frac{\hat{\partial}}{\partial r^*} = \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial r^*} \right) \\
&= \sqrt{(1-\mu)} \nabla_r \left(\frac{1}{\sqrt{(1-\mu)}} \right) \frac{\partial}{\partial r^*} + \frac{1}{(1-\mu)} \nabla_{r^*} \frac{\partial}{\partial r^*} \\
&= \sqrt{(1-\mu)} \frac{(-\mu)}{2r(1-\mu)^{\frac{3}{2}}} \frac{\partial}{\partial r^*} + \frac{1}{(1-\mu)} \nabla_{r^*} \frac{\partial}{\partial r^*} \\
&= \frac{-\mu}{2r(1-\mu)} \frac{\partial}{\partial r^*} + \frac{1}{(1-\mu)} \frac{\mu}{2r} \frac{\partial}{\partial r^*} \\
&= 0
\end{aligned} \tag{6.142}$$

$$\begin{aligned}
\nabla_{\hat{t}} \frac{\hat{\partial}}{\partial r^*} &= \frac{1}{(1-\mu)} \nabla_t \frac{\partial}{\partial r^*} \\
&= \frac{\mu}{2r\sqrt{(1-\mu)}} \frac{\hat{\partial}}{\partial t}
\end{aligned} \tag{6.143}$$

$$\begin{aligned}
\nabla_{\hat{r}^*} \frac{\hat{\partial}}{\partial t} &= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{\sqrt{(1-\mu)}} \frac{\partial}{\partial t} \right) \\
&= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{\sqrt{(1-\mu)}} \right) \frac{\partial}{\partial t} + \frac{1}{(1-\mu)} \nabla_t \frac{\partial}{\partial r^*} \\
&= \sqrt{(1-\mu)} \frac{(-\mu)}{2r(1-\mu)^{\frac{3}{2}}} \frac{\partial}{\partial t} + \frac{1}{(1-\mu)} \frac{\mu}{2r} \frac{\partial}{\partial t} \\
&= \frac{-\mu}{2r(1-\mu)} \frac{\partial}{\partial t} + \frac{1}{(1-\mu)} \frac{\mu}{2r} \frac{\partial}{\partial t} \\
&= 0
\end{aligned} \tag{6.144}$$

$$\begin{aligned}
\nabla_{\hat{\theta}} \frac{\hat{\partial}}{\partial r^*} &= \frac{1}{r\sqrt{(1-\mu)}} \nabla_{\theta} \frac{\partial}{\partial r^*} \\
&= \frac{\sqrt{(1-\mu)}}{r} \frac{\hat{\partial}}{\partial \theta}
\end{aligned} \tag{6.145}$$

$$\begin{aligned}
\nabla_{\hat{r}^*} \frac{\hat{\partial}}{\partial \theta} &= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \\
&= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{r} \right) \frac{\partial}{\partial \theta} + \frac{1}{r \sqrt{(1-\mu)}} \nabla_{\theta} \frac{\partial}{\partial r^*} \\
&= \sqrt{(1-\mu)} \nabla_r \left(\frac{1}{r} \right) \frac{\partial}{\partial \theta} + \frac{\sqrt{(1-\mu)}}{r} \frac{\hat{\partial}}{\partial \theta} \\
&= \frac{-\sqrt{(1-\mu)}}{r^2} \frac{\partial}{\partial \theta} + \frac{\sqrt{(1-\mu)}}{r^2} \frac{\partial}{\partial \theta} \\
&= 0
\end{aligned} \tag{6.146}$$

$$\begin{aligned}
\nabla_{\hat{\phi}} \frac{\hat{\partial}}{\partial r^*} &= \frac{1}{r \sin(\theta) \sqrt{(1-\mu)}} \nabla_{\phi} \frac{\partial}{\partial r^*} \\
&= \frac{\sqrt{(1-\mu)}}{r} \frac{\hat{\partial}}{\partial \phi}
\end{aligned} \tag{6.147}$$

$$\begin{aligned}
\nabla_{\hat{r}^*} \frac{\hat{\partial}}{\partial \phi} &= \frac{1}{\sqrt{(1-\mu)}} \nabla_{r^*} \left(\frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right) \\
&= \sqrt{(1-\mu)} \nabla_r \left(\frac{1}{r \sin(\theta)} \right) \frac{\partial}{\partial \phi} + \frac{1}{r \sin(\theta) \sqrt{(1-\mu)}} \nabla_{\phi} \frac{\partial}{\partial r^*} \\
&= \frac{-\sqrt{(1-\mu)}}{r^2 \sin(\theta)} \frac{\partial}{\partial \phi} + \frac{1}{r \sin(\theta) \sqrt{(1-\mu)}} \nabla_{\phi} \frac{\partial}{\partial r^*} \\
&= \frac{-\sqrt{(1-\mu)}}{r^2 \sin(\theta)} \frac{\partial}{\partial \phi} + \frac{\sqrt{(1-\mu)}}{r^2 \sin(\theta)} \frac{\partial}{\partial \phi} \\
&= 0
\end{aligned} \tag{6.148}$$

6.2.4 The deformation tensor

We start by evaluating

$$\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial w} = \Gamma_{vw}^i e_i$$

We have,

$$\Gamma_{kl}^i = \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{mk}}{\partial x^l} + \frac{\partial \mathbf{g}_{ml}}{\partial x^k} - \frac{\partial \mathbf{g}_{kl}}{\partial x^m} \right)$$

Computing,

$$\Gamma_{vw}^i = \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{mv}}{\partial w} + \frac{\partial \mathbf{g}_{mw}}{\partial v} - \frac{\partial \mathbf{g}_{vw}}{\partial x^m} \right)$$

we have,

$$\begin{aligned} \Gamma_{vw}^v &= \frac{1}{2} \mathbf{g}^{vm} \left(\frac{\partial \mathbf{g}_{vw}}{\partial w} + 0 - \frac{\partial \mathbf{g}_{vw}}{\partial w} \right) \\ &= 0 \\ \Gamma_{vw}^w &= \frac{1}{2} \mathbf{g}^{wv} \left(\frac{\partial \mathbf{g}_{vw}}{\partial w} + \frac{\partial \mathbf{g}_{vw}}{\partial v} - \frac{\partial \mathbf{g}_{vw}}{\partial v} \right) \\ &= \frac{1}{2} \mathbf{g}^{wv} \left(0 + \frac{\partial \mathbf{g}_{vw}}{\partial v} - \frac{\partial \mathbf{g}_{vw}}{\partial v} \right) \\ &= 0 \\ \Gamma_{vw}^\theta &= 0 \\ \Gamma_{vw}^\phi &= 0 \end{aligned}$$

Thus,

$$\nabla_v \frac{\partial}{\partial w} = 0$$

also means,

$$\nabla^w \frac{\partial}{\partial w} = 0$$

Now, we want to compute

$$\begin{aligned} \Gamma_{ww}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{mw}}{\partial w} + \frac{\partial \mathbf{g}_{mw}}{\partial w} - \frac{\partial \mathbf{g}_{ww}}{\partial x^m} \right) \\ &= \mathbf{g}^{im} \frac{\partial \mathbf{g}_{mw}}{\partial w} \\ \Gamma_{vv}^i &= \mathbf{g}^{im} \frac{\partial \mathbf{g}_{mv}}{\partial v} \end{aligned}$$

We have,

$$\begin{aligned}\Gamma_{ww}^v &= \mathbf{g}^{vw} \frac{\partial \mathbf{g}_{ww}}{\partial w} \\ &= 0\end{aligned}$$

and

$$\Gamma_{ww}^w = \mathbf{g}^{vw} \frac{\partial \mathbf{g}_{vw}}{\partial w}$$

We have,

$$\mathbf{g}_{vw} = -\frac{(1 - \frac{2m}{r})}{2}$$

and

$$r^* = \frac{v - w}{2}$$

Computing,

$$\frac{\partial r}{\partial w} = \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial w} = \frac{-1}{2}(1 - \mu)$$

where we define $\mu = \frac{2m}{r}$. Computing,

$$\frac{\partial r}{\partial v} = \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial v} = \frac{1}{2}(1 - \mu)$$

$$\begin{aligned}\frac{\partial \mathbf{g}_{vw}}{\partial r} &= -\frac{1}{2} \frac{\partial(1 - \frac{2m}{r})}{\partial r} \\ &= -\frac{m}{r^2}\end{aligned}$$

Thus,

$$\begin{aligned}
\Gamma_{ww}^w &= \mathbf{g}^{vw} \frac{\partial \mathbf{g}_{vw}}{\partial w} = -\frac{2}{1-\mu} \frac{\partial \mathbf{g}_{vw}}{\partial r} \frac{\partial r}{\partial w} \\
&= -\frac{1}{1-\mu} \frac{-2m}{r^2} \left(-\frac{1}{2}(1-\mu)\right) = -\frac{m(1-\mu)}{r^2(1-\mu)} \\
&= -\frac{m}{r^2}
\end{aligned}$$

We also have

$$\Gamma_{ww}^\theta = \Gamma_{ww}^\phi = \Gamma_{ww}^v = 0$$

hence,

$$\nabla_w \frac{\partial}{\partial w} = -\frac{m}{r^2} \frac{\partial}{\partial w}$$

On the other hand,

$$\begin{aligned}
\Gamma_{vv}^v &= \mathbf{g}^{wv} \left(\frac{\partial \mathbf{g}_{wv}}{\partial v}\right) = -\frac{2}{1-\mu} \frac{\partial \mathbf{g}_{wv}}{\partial r} \frac{\partial r}{\partial v} \\
&= \frac{2m}{r^2(1-\mu)} \frac{\partial r}{\partial v} = \frac{m(1-\mu)}{r^2(1-\mu)} \\
&= \frac{m}{r^2}
\end{aligned}$$

We also have,

$$\Gamma_{vv}^w = \mathbf{g}^{wv} \left(\frac{\partial \mathbf{g}_{vv}}{\partial v}\right) = 0$$

and,

$$\Gamma_{vv}^\theta = \Gamma_{vv}^\phi = 0$$

from which

$$\nabla_v \frac{\partial}{\partial v} = \frac{m}{r^2} \frac{\partial}{\partial v}$$

Computing,

$$\begin{aligned} \Gamma_{\theta v}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\theta}}{\partial x^v} + \frac{\partial \mathbf{g}_{mv}}{\partial \theta} - \frac{\partial \mathbf{g}_{\theta v}}{\partial x^m} \right) \\ &= \frac{1}{2} \mathbf{g}^{im} \frac{\partial \mathbf{g}_{m\theta}}{\partial x^v} \\ \Gamma_{\theta v}^\theta &= \frac{1}{2} \mathbf{g}^{\theta\theta} \frac{\partial \mathbf{g}_{\theta\theta}}{\partial v} = \frac{1}{2r^2} \frac{\partial(r^2)}{\partial r} \frac{\partial r}{\partial v} = \frac{r}{2r^2} (1 - \mu) \\ &= \frac{(1 - \mu)}{2r} \end{aligned}$$

thus,

$$\nabla_\theta \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta}$$

And,

$$\begin{aligned} \Gamma_{\phi v}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\phi}}{\partial x^v} + \frac{\partial \mathbf{g}_{mv}}{\partial \phi} - \frac{\partial \mathbf{g}_{\phi v}}{\partial x^m} \right) \\ &= \frac{1}{2} \mathbf{g}^{im} \frac{\partial \mathbf{g}_{m\phi}}{\partial v} \\ \Gamma_{\phi v}^\phi &= \frac{1}{2} \mathbf{g}^{\phi\phi} \frac{\partial \mathbf{g}_{\phi\phi}}{\partial v} = \frac{1}{2r^2 \sin^2 \theta} \frac{\partial(r^2 \sin^2 \theta)}{\partial r} \frac{\partial r}{\partial v} = \frac{2r}{2r^2} \frac{1}{2} (1 - \mu) \\ &= \frac{(1 - \mu)}{2r} \end{aligned}$$

thus,

$$\nabla_\phi \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi}$$

Also,

$$\begin{aligned}
\Gamma_{\phi w}^i &= \frac{1}{2} \mathbf{g}^{im} \left(\frac{\partial \mathbf{g}_{m\phi}}{\partial x^w} + \frac{\partial \mathbf{g}_{mw}}{\partial \phi} - \frac{\partial \mathbf{g}_{\phi w}}{\partial x^m} \right) \\
&= \frac{1}{2} \mathbf{g}^{im} \frac{\partial \mathbf{g}_{m\phi}}{\partial w} \\
\Gamma_{\phi w}^\phi &= \frac{1}{2} \mathbf{g}^{\phi\phi} \frac{\partial \mathbf{g}_{\phi\phi}}{\partial w} = \frac{1}{2r^2 \sin^2 \theta} \frac{\partial(r^2 \sin^2 \theta)}{\partial r} \frac{\partial r}{\partial w} = -\frac{r}{2r^2} (1 - \mu) \\
&= -\frac{(1 - \mu)}{2r}
\end{aligned}$$

thus,

$$\nabla_\phi \frac{\partial}{\partial w} = -\frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi}$$

We also have,

$$\nabla_\theta \frac{\partial}{\partial w} = -\frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta}$$

Therefore, in conclusion, we have:

$$\nabla_w \frac{\partial}{\partial w} = -\frac{m}{r^2} \frac{\partial}{\partial w} \quad (6.149)$$

$$\nabla_v \frac{\partial}{\partial v} = \frac{m}{r^2} \frac{\partial}{\partial v} \quad (6.150)$$

$$\nabla_v \frac{\partial}{\partial w} = \nabla_w \frac{\partial}{\partial v} = 0 \quad (6.151)$$

$$\nabla_\theta \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta} \quad (6.152)$$

$$\nabla_\theta \frac{\partial}{\partial w} = \frac{-(1 - \mu)}{2r} \frac{\partial}{\partial \theta} \quad (6.153)$$

$$\nabla_\phi \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi} \quad (6.154)$$

$$\nabla_\phi \frac{\partial}{\partial w} = \frac{-(1 - \mu)}{2r} \frac{\partial}{\partial \phi} \quad (6.155)$$

Now, let

$$V = V^w(v, w) \frac{\partial}{\partial w} + V^v(v, w) \frac{\partial}{\partial v} \quad (6.156)$$

We use the notation $V^w(v, w) = V^w$, and $V^v(v, w) = V^v$.

Computing,

$$\begin{aligned}
\nabla_v V &= (\partial_v V^w) \frac{\partial}{\partial w} + V^w (\nabla_v \frac{\partial}{\partial w}) + (\partial_v V^v) \frac{\partial}{\partial v} + V^v (\nabla_v \frac{\partial}{\partial v}) \\
&= (\partial_v V^w) \frac{\partial}{\partial w} + (\partial_v V^v) \frac{\partial}{\partial v} + V^v \frac{m}{r^2} \frac{\partial}{\partial v} \\
\nabla_w V &= (\partial_w V^w) \frac{\partial}{\partial w} + V^w (\nabla_w \frac{\partial}{\partial w}) + (\partial_w V^v) \frac{\partial}{\partial v} + V^v (\nabla_w \frac{\partial}{\partial v}) \\
&= (\partial_w V^w) \frac{\partial}{\partial w} - V^w \frac{m}{r^2} \frac{\partial}{\partial w} + (\partial_w V^v) \frac{\partial}{\partial v} \\
\nabla_\theta V &= (\partial_\theta V^w) \frac{\partial}{\partial w} + V^w (\nabla_\theta \frac{\partial}{\partial w}) + (\partial_\theta V^v) \frac{\partial}{\partial v} + V^v (\nabla_\theta \frac{\partial}{\partial v}) \\
&= V^v \frac{(1-\mu)}{2r} \frac{\partial}{\partial \theta} - V^w \frac{(1-\mu)}{2r} \frac{\partial}{\partial \theta} \\
&= \frac{(1-\mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \theta} \\
\nabla_\phi V &= V^w (\nabla_\phi \frac{\partial}{\partial w}) + V^v (\nabla_\phi \frac{\partial}{\partial v}) \\
&= V^v \frac{(1-\mu)}{2r} \frac{\partial}{\partial \phi} - V^w \frac{(1-\mu)}{2r} \frac{\partial}{\partial \phi} \\
&= \frac{(1-\mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \phi}
\end{aligned}$$

Therefore,

$$\nabla_v V = (\partial_v V^w) \frac{\partial}{\partial w} + (\partial_v V^v + \frac{m}{r^2} V^v) \frac{\partial}{\partial v} \quad (6.157)$$

$$\nabla_w V = (\partial_w V^v) \frac{\partial}{\partial v} + (\partial_w V^w - \frac{m}{r^2} V^w) \frac{\partial}{\partial w} \quad (6.158)$$

$$\nabla_\theta V = \frac{(1-\mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \theta} \quad (6.159)$$

$$\nabla_\phi V = \frac{(1-\mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \phi} \quad (6.160)$$

The deformation tensor is,

$$\pi^{\alpha\beta}(V) = \frac{1}{2} (\nabla^\alpha V^\beta + \nabla^\beta V^\alpha)$$

Computing,

$$\begin{aligned}\pi^{ww}(V) &= \nabla^w V^w = \mathbf{g}^{w\omega} \nabla_\omega V^w \\ &= \frac{-2}{(1-\mu)} \partial_v V^w\end{aligned}\tag{6.161}$$

$$\begin{aligned}\pi^{vv}(V) &= \nabla^v V^v = \mathbf{g}^{v\omega} \nabla_\omega V^v \\ &= \frac{-2}{(1-\mu)} \partial_w V^v\end{aligned}\tag{6.162}$$

$$\begin{aligned}\pi^{vw}(V) &= \frac{1}{2}(\nabla^v V^w + \nabla^w V^v) \\ &= \frac{1}{2}(\mathbf{g}^{v\omega} \nabla_\omega V^w + \mathbf{g}^{w\omega} \nabla_\omega V^v) \\ &= \frac{-1}{(1-\mu)} [\partial_v V^v + \partial_w V^w + \frac{m}{r^2}(V^v - V^w)]\end{aligned}\tag{6.163}$$

$$= \pi^{wv}(V)\tag{6.164}$$

$$\begin{aligned}\pi^{\theta\theta}(V) &= \nabla^\theta V^\theta = \mathbf{g}^{\theta\theta} \nabla_\theta V^\theta \\ &= \frac{(1-\mu)}{2r^3} (V^v - V^w)\end{aligned}\tag{6.165}$$

$$\begin{aligned}\pi^{\phi\phi}(V) &= \nabla^\phi V^\phi = \mathbf{g}^{\phi\phi} \nabla_\phi V^\phi \\ &= \frac{(1-\mu)}{2r^3 \sin^2 \theta} (V^v - V^w)\end{aligned}\tag{6.166}$$

$$\begin{aligned}\pi^{\theta\phi}(V) &= \frac{1}{2}(\nabla^\theta V^\phi + \nabla^\phi V^\theta) \\ &= 0 = \pi^{\phi\theta}\end{aligned}\tag{6.167}$$

$$\begin{aligned}\pi^{\theta v}(V) &= \frac{1}{2}(\nabla^\theta V^v + \nabla^v V^\theta) \\ &= 0 = \pi^{v\theta}(V)\end{aligned}\tag{6.168}$$

$$\pi^{\phi v}(V) = 0 = \pi^{v\phi}(V)\tag{6.169}$$

$$\pi^{\theta w}(V) = 0 = \pi^{w\theta}(V)\tag{6.170}$$

$$\pi^{\phi w}(V) = 0 = \pi^{w\phi}(V)\tag{6.171}$$

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