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Arnaud Rousselle

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**THÈSE DE DOCTORAT DE
L'UNIVERSITÉ DE ROUEN**

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Présentée par

Arnaud ROUSSELLE

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**Marches au hasard sur des graphes géométriques
aléatoires engendrés par des processus ponctuels**

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après les rapports de :

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Résumé

Les marches aléatoires sur des graphes aléatoires plongés dans \mathbb{R}^d apparaissent naturellement dans de nombreux problèmes issus de la mécanique statistique tels que la description de flux, de diffusions de molécules ou de chaleur dans des milieux aléatoires et irréguliers. L'idée générale est d'étendre des résultats connus sur la grille \mathbb{Z}^d ou des perturbations aléatoires de celle-ci à des graphes engendrés par des processus ponctuels dans \mathbb{R}^d .

Dans cette thèse, on considère des marches au plus proche voisin sur des graphes dépendant de la géométrie d'un ensemble aléatoire et infini de points. Plus précisément, étant donnée une réalisation d'un processus ponctuel simple et stationnaire dans \mathbb{R}^d , un graphe G , connexe, infini et localement fini, est construit. Ce graphe est ensuite muni éventuellement d'une fonction de conductance C , c'est-à-dire une fonction strictement positive définie sur son ensemble d'arêtes. Les exemples de graphes géométriques étudiés dans ce manuscrit sont la triangulation de Delaunay, le graphe de Gabriel, les *creek-crossing graphs* et le squelette de la mosaïque de Voronoï engendrés par le processus ponctuel. On étudie les propriétés la marche simple et la marche associée à la conductance C sur de tels graphes.

Les principaux résultats portent sur la caractérisation de la récurrence ou de la transience presque sûre des marches aléatoires et sur la description de leurs limites diffusives. On montre que, sous des hypothèses convenables sur le processus ponctuel sous-jacent et la fonction de conductance, les marches aléatoires sur la triangulation de Delaunay, le graphe de Gabriel et le squelette de la mosaïque de Voronoï engendrés par presque toute réalisation de ce processus ponctuel sont récurrentes si $d = 2$ et transitoires si $d \geq 3$. On établit aussi un principe d'invariance *annealed* (ou en moyenne) pour les marches simples partant de l'origine sur la triangulation de Delaunay et le graphe de Gabriel engendrés par les mesures de Palm de certains processus ponctuels ainsi qu'un principe d'invariance *quenched* (ou presque sûr) pour les marches simples sur des triangulations de Delaunay engendrées par des processus ponctuels.

Cette thèse exploite à la fois des outils de géométrie aléatoire (processus ponctuels, mesures de Palm, mosaïques et graphes aléatoires...) et de la théorie des marches aléatoires (liens avec les réseaux électriques, l'environnement vu par la particule).

Mots-clés : marches aléatoires en environnements aléatoires, géométrie stochastique, mosaïque de Voronoï, triangulation de Delaunay, graphe de Gabriel, récurrence, transience, principes d'invariance.

Classification AMS : 60K37, 60D05 ; 60G55, 05C81, 60G17, 60F17.

Abstract

Random walks on random graphs embedded in \mathbb{R}^d appear naturally in problems arising from statistical mechanics such that the description of flows, molecules or heat diffusions in random and irregular environments. The general idea is to extend known results for random walks on \mathbb{Z}^d or on random perturbations of the grid to results for random walks on graphs generated by point processes in \mathbb{R}^d .

In this thesis, we consider nearest neighbor random walks on graphs depending on the geometry of a random infinite locally finite set of points. More precisely, given a realisation of a simple stationary point process in \mathbb{R}^d , a connected infinite and locally finite graph G is constructed. This graph is then possibly equipped with a conductance function C , that is a positive function defined on its edge set. Examples of graphs studied in this manuscript are the Delaunay triangulation, the Gabriel graph, the creek-crossing graphs and the skeleton of the Voronoi tiling generated by the point process. We study properties of the simple random walk or of a random walk associated with the conductance C on such graphs.

The main results concern the characterisation of the recurrence or transience of the random walks and the description of their diffusive scaling limits. Under suitable assumptions on the underlying point process and the conductance function, we show that the random walks on the Delaunay triangulation, the Gabriel graph and the skeleton of the Voronoi tiling generated by almost every realisation of the point process are recurrent if $d = 2$ and transient if $d \geq 3$. We state an annealed invariance principle for simple random walks starting from the origin on the Delaunay triangulation, the Gabriel graph and the creek-crossing graphs generated by Palm measures of suitable point processes. Finally, we show a quenched invariance principle for simple random walks on random Delaunay triangulations.

This thesis uses tools from both stochastic geometry (point processes, Palm measures, random graphs ...) and the theory of random walks (links with electrical networks theory, the environment seen from the particle,...).

Key words: random walks in random environments, stochastic geometry, Voronoi tiling, Delaunay triangulation, Gabriel graph, recurrence, transience, invariance principles.

AMS classification: 60K37, 60D05; 60G55, 05C81, 60G17, 60F17.

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Introduction

Dans le cadre classique des marches au hasard, on dispose d'un espace d'états déterministe, la grille \mathbb{Z}^d par exemple, et les déplacements du marcheur sont régis par des probabilités de transition fixées à l'avance. Une des premières questions que l'on peut se poser est celle du caractère *récurrent* (le marcheur retourne infiniment souvent à son point de départ) ou, au contraire, *transitoire* (le marcheur ne revient qu'un nombre fini de fois à son point de départ) de la marche. En 1921, Pólya [Pól21] a montré que les marches simples au plus proche voisin sur \mathbb{Z}^d sont récurrentes en dimensions 1 et 2 et transitoires en dimension plus grande. Un deuxième objet d'étude est la détermination d'un processus stochastique continu vers lequel la marche, convenablement renormalisée en temps et en espace, va converger. En 1952, Donsker [Don52] a justifié que les marches simples sur \mathbb{Z}^d convergent, dans une limite d'échelle diffusive, vers un mouvement brownien. Un tel résultat est appelé *principe d'invariance* ou *TCL fonctionnel*.

Le cadre des marches au hasard sur un ensemble d'états déterministe admet certaines limites. En effet, les marches aléatoires servent à modéliser des flux ou des diffusions de toutes sortes (thermiques, électriques, ...) qui peuvent avoir lieu dans des environnements qu'il est impossible de décrire précisément. Imaginons, par exemple, que l'on souhaite étudier les déplacements d'une particule à l'échelle microscopique dans un matériau macroscopique globalement homogène. On ne peut *a priori* pas décrire la structure microscopique exacte du matériau en question et il faut trouver un moyen pour décrire des marches aléatoires dans des milieux localement irréguliers et possédant une certaine homogénéité globale de grande échelle. Une solution est de voir cette structure comme une réalisation particulière d'un phénomène aléatoire dont la loi a pu être choisie grâce à des informations émanant d'observations macroscopiques. Pour de telles raisons, l'étude des marches au hasard dans des environnements aléatoires a été très largement développée au cours des trente dernières années. La plupart des modèles apparaissant dans la littérature sont construits à partir d'un ensemble dénombrable de points formant un réseau, généralement \mathbb{Z}^d muni de sa structure de graphe naturelle (c'est-à-dire avec des liens entre les paires de points à distance 1 l'un de l'autre) que l'on perturbe aléatoirement. La perturbation de l'environnement provient alors soit de poids aléatoires affectés soit aux arêtes soit aux sites de \mathbb{Z}^d eux-mêmes. On peut notamment citer les marches au hasard en conductances aléatoires, sur des amas de percolation ou en pièges aléatoires et les marches aléatoires en milieu aléatoire (MAMA). Nous allons maintenant décrire plus précisément les marches au hasard en conductances aléatoires puisque ce modèle est certainement le plus proche de ceux que nous étudions

dans cette thèse parmi les différents modèles que nous venons d'évoquer.

Marches aléatoires en conductances aléatoires et sur l'amas infini de percolation. On commence par se donner une mesure de probabilité \mathbb{P} sur l'ensemble Ω des familles de réels positifs doublement indicés par les points de \mathbb{Z}^d , stationnaire et ergodique par rapport au groupe des translations de \mathbb{Z}^d agissant sur Ω . On note $\omega = \{c_{x,y} : x, y \in \mathbb{Z}^d\}$ un élément générique de Ω et on appelle \mathbb{P} la loi de l'environnement. On suppose que \mathbb{P} est telle que les conditions suivantes sont satisfaites pour presque tout ω :

- pour tous $x, y \in \mathbb{Z}^d$, $c_{x,y} = c_{y,x}$;
- pour tout $x \in \mathbb{Z}^d$, $w_\omega(x) := \sum_{y \in \mathbb{Z}^d} c_{x,y} > 0$;
- $c_{x,y} = 0$ si x et y ne sont pas voisins dans la grille \mathbb{Z}^d .

De cette façon, pour presque toute réalisation de l'environnement, la fonction $C(x, y) := c_{x,y}$ définit une *conductance* sur \mathbb{Z}^d dans la terminologie introduite dans le paragraphe 1.2 du Chapitre 1. On impose parfois une condition additionnelle d'*ellipticité uniforme* c'est-à-dire l'existence d'un nombre $\alpha \in]0, 1[$ tel que pour tous x et y voisins, on ait :

$$\alpha < c_{x,y} < \frac{1}{\alpha}.$$

Il est possible de relâcher quelque peu la condition de positivité de $w_\omega(\cdot)$ en tout site de \mathbb{Z}^d à condition qu'il existe presque sûrement une unique composante connexe infinie $\mathcal{C}_\infty(\omega)$ constituée d'arêtes de conductances strictement positives et de restreindre notre attention à celle-ci. On peut, en particulier, considérer le cas où les conductances sont choisies indépendamment et uniformément sur chaque arête selon une loi de Bernoulli de paramètre p suffisamment proche de 1. Rappelons que, si $d \geq 2$, il existe une valeur critique $p_c = p_c(d) < 1$, telle que le schéma de Bernoulli de paramètre p , décrit ci-dessus, admette une unique composante connexe infinie, appelée *amas de percolation* (voir [Gri89, Théorème 1.10]). Les marches aléatoires sur l'amas infini de percolation en régime sur-critique constituent un exemple fondamental de marches aléatoires dans un milieu aléatoire.

Les marches aléatoires dans ce milieu aléatoire sont alors les marches associées à la conductance C : la marche à temps discret $(X_n^\omega)_{n \in \mathbb{N}}$ est la chaîne de Markov homogène en temps dont les probabilités de transition sont données par :

$$P_x^\omega[X_1^\omega = y] = P^\omega[X_1^\omega = y | X_0^\omega = x] := \frac{c_{x,y}}{w_\omega(x)},$$

et son pendant, à temps continu et vitesse variable, est le processus de Markov $(X_t^\omega)_{t \geq 0}$ de taux de saut $c_{x,y}$. Une propriété remarquable de ces marches est la réversibilité qui est une conséquence directe de leur définition. En effet, la condition de réversibilité, aussi connue sous le nom de *detailed balance condition* :

$$w_\omega(x)c_{x,y} = w_\omega(y)c_{y,x},$$

est satisfaite.

Reprenons l'exemple des marches aléatoires sur l'amas infini de percolation. Si la question de la récurrence en dimension 2 de la marche sur l'amas de percolation ne se pose pas vraiment puisque celle-ci découle immédiatement de la récurrence de la marche simple sur \mathbb{Z}^2 tout entier, celle de la transience en dimension plus grande est bien moins évidente. Grimmett, Kesten et Zhang ont démontré, dans [GKZ93], que pour presque toute réalisation de l'amas de percolation en régime sur-critique en dimension plus grande que 3, la marche aléatoire simple sur celui-ci est transitoire.

Une deuxième question que l'on peut se poser est celle de la convergence faible du processus mis à l'échelle diffusive :

$$X_t^{\omega,\varepsilon} := \varepsilon \left(X_{\lfloor \varepsilon^{-2}t \rfloor}^\omega + \left(\varepsilon^{-2}t - \lfloor \varepsilon^{-2}t \rfloor \right) \left(X_{\lfloor \varepsilon^{-2}t \rfloor + 1}^\omega - X_{\lfloor \varepsilon^{-2}t \rfloor}^\omega \right) \right)$$

vers un mouvement brownien. On peut donner deux sens différents à la convergence précédente en regardant la distribution des trajectoires au travers de deux mesures de probabilité différentes. Premièrement, on peut considérer que les trajectoires sont distribuées, pour presque tout ω et pour $x \in \mathcal{C}_\infty(\omega)$, selon la mesure P_x^ω . On parle de *principe d'invariance presque sûr, fort* ou *quenched* lorsque pour \mathbb{P} -presque tout ω , pour tout $x \in \mathcal{C}_\infty(\omega)$, sous P_x^ω , le processus mis à l'échelle $(X_t^{\omega,\varepsilon})_{t \geq 0}$ converge en loi, quand ε tend vers 0, vers un mouvement brownien non dégénéré. Une alternative est de voir les trajectoires sous une mesure moyennée sur les environnements. Pour définir celle-ci, notons \mathbb{P}_0 la probabilité obtenue en conditionnant \mathbb{P} par l'événement $\ll 0 \in \mathcal{C}_\infty(\omega) \gg$ et \mathbb{Q}_0 la probabilité admettant pour densité $\frac{d\mathbb{Q}_0}{d\mathbb{P}_0}(\omega) = \frac{w_\omega(0)}{\int w_\omega(0)\mathbb{P}_0(d\omega)}$ par rapport à \mathbb{P}_0 . La mesure moyennée sur les environnements ou *annealed* est alors donnée par $P[\cdot] = \int P_0^\omega[\cdot]\mathbb{Q}_0(d\omega)$, On parle de *principe d'invariance en moyenne* ou *annealed* lorsque, sous P , le processus mis à l'échelle $(X_t^{\omega,\varepsilon})_{t \geq 0}$ converge en loi, quand ε tend vers 0, vers un mouvement brownien non dégénéré. Des principes d'invariance *annealed* ont été établis dans [KV86] et [DMFGW89] tandis que des principes d'invariance *quenched* ont été obtenus plus récemment dans divers contextes. Citons notamment le cas où \mathbb{P} est ergodique et les conductances sont uniformément elliptiques [SS04] et le cas de conductances i.i.d. surcritiques et intégrables [SS04, BB07, BP07, MP07, Mat08, BD10, ABDH13].

Il n'est pas possible de dresser une liste exhaustive des résultats obtenus sur les marches aléatoires en conductances aléatoires en quelques lignes. Le lecteur souhaitant une introduction plus étoffée à ce sujet pourra consulter l'article de survol très complet [Bis11].

Présentation des modèles. Il n'y a *a priori* pas de raison pour que les sites du milieu microscopique que l'on souhaite modéliser soient répartis le long d'une grille fixée à l'avance telle que \mathbb{Z}^d . Une solution, pour éviter d'avoir recours à une telle grille, est de considérer des perturbations aléatoires de celle-ci ou plus généralement que les sites sont répartis selon une réalisation d'un processus ponctuel stationnaire dans l'espace. Dans cette thèse, on considère des marches au hasard sur des graphes plongés dans \mathbb{R}^d construits à partir de réalisations de processus ponctuels selon des règles faisant intervenir la géométrie de l'ensemble de points tiré. Nous décrivons ici les modèles étudiés dans le reste du manuscrit.

Commençons par définir la mosaïque de Voronoï d'un ensemble ξ localement fini de points de \mathbb{R}^d . Étant donné ξ , la *cellule de Voronoï* de germe $\mathbf{x} \in \xi$ est l'ensemble :

$$\text{Vor}_\xi(\mathbf{x}) := \left\{ x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \xi \right\}.$$

La collection des cellules de Voronoï $\{\text{Vor}_\xi(\mathbf{x}), \mathbf{x} \in \xi\}$ est appelée *mosaïque de Voronoï*. Son histoire remonte à Descartes qui étudie la répartition de la matière dans le système solaire en 1644. En 1850, Dirichlet [Dir50] réutilise ce concept, en dimension 2 et dans un cadre déterministe, pour résoudre des problèmes de minimisation de formes quadratiques prises sur des vecteurs à coordonnées entières. En 1908, Voronoï étend la notion en dimension supérieure dans [Vor08] et lui laisse son nom. Il faut attendre 1953 et Meijering [Mei53] pour que les premières cellules de Voronoï aléatoires apparaissent dans la littérature. L'auteur souhaite alors modéliser la formation des cristaux et considère que les germes sont répartis selon la réalisation d'un processus ponctuel de Poisson homogène. Meijering introduit ainsi un modèle maintenant connu sous le nom de *mosaïque de Poisson-Voronoï*. La cellule de Voronoï de germe \mathbf{x} représentant le territoire de celui-ci lorsque les germes étendent leurs territoires respectifs à la même vitesse ou encore l'ensemble des points de l'espace préférant, par proximité, se rattacher à \mathbf{x} plutôt qu'à un autre site, le modèle a trouvé des applications dans des contextes variés : en géographie [Boo73, Boo75], en biologie [GTL95], en astronomie [RBFN01] ou encore dans le domaine des télécommunications [FZ96, BB09, GVS11].

Le modèle de Poisson-Voronoï apparaissant naturellement dans des situations concrètes, il est souhaitable d'étudier des marches aléatoires se déplaçant à chaque étape d'une cellule de Voronoï vers l'une de ses voisines ou encore d'un germe de la mosaïque vers un autre germe tel que les deux cellules ont une frontière commune. Il s'agit, en d'autres termes, d'étudier les marches au hasard sur le graphe dual de la mosaïque de Voronoï qui est connu sous le nom de *triangulation de Delaunay*. Dans la suite, on considèrera des triangulations de Delaunay engendrées par des processus ponctuels de Poisson homogènes ou d'autres processus ponctuels possédant des propriétés d'attractivité ou de répulsivité bien différentes. On étudiera aussi les marches au hasard sur certains sous-graphes de la triangulation de Delaunay tels que le graphe de Gabriel (trouvant des applications dans les domaines de la géographie, des stratégies de routage ou de la biologie [BBD02, GS69, MS80a]) ou les *creek-crossing graphs* (voir le paragraphe §1.1.4 du Chapitre 1).

Par la définition même des modèles considérés, pour réaliser notre étude, nous avons besoin de notions issues de la géométrie stochastique. Une notion importante est celle de *mesure de Palm* d'un processus ponctuel, permettant d'observer les graphes considérés depuis « un point typique ». Il faut, en particulier, contrôler des moments polynomiaux ou exponentiels de certaines caractéristiques géométriques des graphes aléatoires sous la mesure de Palm associée au processus ponctuel. On s'intéresse notamment au degré d'un tel point dans la triangulation de Delaunay ou à la distance maximale d'un tel point à l'un de ses voisins dans le graphe (donnant de l'information sur la longueur euclidienne parcourue par la marche lors d'un saut typique).

Présentation des résultats et contexte. Dans ce paragraphe, nous décrivons les principaux résultats de cette thèse et nous les mettons en parallèle avec les résultats existants dans la littérature sur un modèle de marches au hasard à longue portée sur des processus ponctuels (c'est-à-dire de marches aléatoires sur les graphes complets engendrés par les réalisations de processus ponctuels avec des probabilités de transition décroissant rapidement avec la distance euclidienne et éventuellement modifiées par la présence de marques d'énergie).

Récurrence et transience. En 2005, Addario-Berry et Sarkar ont annoncé, dans un manuscrit non publié [ABS05], des résultats de récurrence et transience pour les marches aléatoires sur des triangulations de Poisson-Delaunay. Leurs preuves s'appuyaient sur un résultat, plus fort que ce qui est nécessaire, portant sur le *stabbing number* de la triangulation de Poisson-Delaunay, c'est-à-dire le nombre maximal de cellules de Delaunay intersectées par un segment contenu dans le cube $[-n, n]^d$. La source alors citée est malheureusement invérifiable et semble avoir disparu de la littérature accessible. Il faut toutefois noter que le résultat portant sur le *stabbing number* a depuis été démontré dans [PR12]. Ces résultats de récurrence et transience sont couverts par ceux obtenus dans le Chapitre 2 que nous présentons ci-dessous.

En 2009, Caputo, Faggionato et Gaudillière ont obtenu des résultats de récurrence et transience pour le modèle des marches au hasard à longue portée sur des processus ponctuels.

Dans le Chapitre 2, on montre que, sous des hypothèses sur les variables de comptage du processus ponctuel sous-jacent et de rang de dépendance fini, les marches aléatoires sur la triangulation de Delaunay, le graphe de Gabriel et le squelette de la mosaïque de Voronoï engendrés par presque toute réalisation de ce processus ponctuel sont récurrentes si $d = 2$ et transitoires si $d \geq 3$. Ces résultats sont établis en appliquant deux critères, bien adaptés au cadre étudié, pour la récurrence ou la transience presque sûre de marches aléatoires. Ces critères peuvent être utilisés pour obtenir des résultats pour d'autres graphes plongés dans \mathbb{R}^d (amas de percolation continue, *creek-crossing graphs*, ...).

Principe d'invariance *annealed*. Dans [FSBS06], Faggionato, Schulz-Baldes et Spohner ont démontré un principe d'invariance *annealed* (ou en moyenne) pour les marches au hasard à longue portée sur des processus ponctuels. Ils ont également minoré la matrice de diffusion du mouvement brownien limite par la loi de Mott. Une majoration de cette matrice a été obtenue par Faggionato et Mathieu dans [FM08].

Dans le Chapitre 3, on étudie des marches aléatoires sur des graphes construits à partir d'un ensemble aléatoire de points dans \mathbb{R}^d choisi selon la probabilité de Palm \mathcal{P}_0 associée à un processus ponctuel stationnaire. Pour \mathcal{P}_0 -presque toute réalisation ξ^0 , on considère la marche simple à temps continu et à vitesse variable (VSRW) partant de 0 sur un graphe $G(\xi^0) = (\xi^0, E_{G(\xi^0)})$ pouvant être la triangulation de Delaunay ou le graphe de Gabriel engendrés par ξ^0 . On suppose que le degré de l'origine et la distance maximale entre l'origine et l'un de ses voisins dans $G(\xi^0)$ admettent des moments polynomiaux de tout

ordre sous la mesure de Palm du processus ponctuel et que celui-ci est ergodique et satisfait une certaine hypothèse de domination stochastique. On démontre un principe d'invariance *annealed* ou *en moyenne*, c'est-à-dire la convergence faible en \mathcal{P}_0 -probabilité du processus mis à l'échelle diffusive $(\varepsilon X_{\varepsilon^{-2}t}^{\xi_0})_{t \geq 0}$ vers un mouvement brownien non dégénéré.

Principe d'invariance *quenched*. En 2012, dans l'article [FGG12], lié à la thèse du deuxième auteur [Gri09], Ferrari, Grisi et Groisman ont montré l'existence de déformations harmoniques de triangulations de Delaunay engendrées par des processus ponctuels convexes dans \mathbb{R}^d et ont donné un moyen explicite de les construire. Ils en ont déduit un principe d'invariance presque sûr pour les marches aléatoires sur ces triangulations de Delaunay en dimension 2 et ont proposé d'établir des bornes sur le noyau de la chaleur de celles-ci afin d'étendre ce principe d'invariance en dimension plus grande que 3. Nous obtenons, par une approche différente, ce principe d'invariance en toute dimension.

En 2013, Caputo, Faggionato et Prescott ont démontré un principe d'invariance presque sûr dans [CFP13] pour le modèle des marches aléatoires à longue portée sur des processus ponctuels.

Dans le Chapitre 4, on démontre sous des hypothèses sur le processus ponctuel similaires à celles du Chapitre 3, que pour presque toute réalisation ξ , pour tout point de départ $x \in \xi$, la marche simple sur la triangulation de Delaunay converge en loi, à l'échelle diffusive et sous la mesure *quenched* P_x^ξ , vers un mouvement brownien dont la matrice de covariance est un multiple de l'identité et ne dépend pas de ξ . Autrement dit, on démontre dans ce chapitre un principe d'invariance presque sûr pour les marches au hasard sur des triangulations de Delaunay aléatoires.

Organisation du manuscrit. Ce manuscrit comporte un premier chapitre introductif se décomposant en deux parties. La première est consacrée à des rappels de géométrie stochastique et la seconde à des éléments de la théorie des marches aléatoires et à ses liens avec la théorie des réseaux électriques. Le lecteur, familier avec une de ces thématiques, pourra sans difficulté ignorer une des parties composant ce chapitre ou son tout ! Les trois chapitres suivants constituent le cœur de cette thèse. Ils sont rédigés sous forme d'articles, en anglais, et sont précédés d'introductions partielles. Ceux-ci correspondent aux trois travaux évoqués ci-dessus et sont présentés dans l'ordre dans lequel ils ont été réalisés (récurrence/transience, principe d'invariance *annealed* et principe d'invariance *quenched*). Leur présentation sous la forme d'articles rend ces trois chapitres essentiellement auto-suffisants au prix d'un certain nombre de redites. Une dernière partie fournit une liste de problèmes ouverts et de pistes de travail.

Chapitre 1

Éléments de géométrie stochastique et de la théorie des marches au hasard

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Dans ce chapitre, on rappelle quelques notions de géométrie aléatoire et de la théorie des marches au hasard qui seront utiles pour les chapitres suivants.

1.1 Notions de géométrie stochastique

1.1.1 Processus ponctuels

Les modèles que nous allons considérer dans la suite sont construits à partir de processus ponctuels simples dans \mathbb{R}^d . C'est pourquoi, pour présenter les processus ponctuels, nous nous plaçons ici dans le cadre particulier $E = \mathbb{R}^d$ plutôt que dans celui, plus général, d'un

espace localement compact E . Les preuves seront omises et le lecteur désireux de plus de détails ou d'une présentation dans un cadre général pourra consulter par exemple [DVJ03], [DVJ08], [SW08] ou [CSKM13].

L'espace des mesures de comptage et la définition des processus ponctuels. On désigne par \mathcal{N} l'ensemble des mesures de comptage dans \mathbb{R}^d , c'est-à-dire des mesures ξ vérifiant :

1. ξ est *localement finie* : $\xi(C) < +\infty$, pour tout compact C ;
2. $\xi(B) \in \mathbb{N} \cup \{+\infty\}$ pour tout borélien B .

Un élément générique ξ de \mathcal{N} peut être identifié à un sous-ensemble de \mathbb{R}^d , $\{x_i; i \in I\}$, I fini ou dénombrable, possédant éventuellement des points multiples de sorte que $\xi = \sum_{i \in I} \delta_{x_i}$ où δ_{x_i} désigne la masse de Dirac en x_i . Lorsque la mesure ξ est *simple*, c'est-à-dire telle que $\xi(\{x\}) = 0$ ou 1 pour tout $x \in \mathbb{R}^d$, l'ensemble $\{x_i; i \in I\}$ n'est autre que le support de ξ . On munit \mathcal{N} de la plus petite tribu $\mathcal{F}(\mathcal{N})$ rendant mesurables les applications :

$$\xi \longrightarrow \xi(B), \quad B \in \mathcal{B}_b(\mathbb{R}^d),$$

où $\mathcal{B}_b(\mathbb{R}^d)$ désigne l'ensemble des boréliens bornés de \mathbb{R}^d . On peut maintenant définir formellement la notion de processus ponctuel correspondant à un choix aléatoire d'un élément de \mathcal{N} :

Définition 1.1. *Un processus ponctuel est une application mesurable Ξ d'un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ dans $(\mathcal{N}, \mathcal{F}(\mathcal{N}))$. La loi de Ξ est donnée par la mesure image $\mathcal{P} = \mathcal{P}_\Xi := \Xi_*\mathbb{P}$.*

Des exemples de processus ponctuels sont donnés dans la Section 1.1.3. On confondra parfois dans les chapitres suivants un processus ponctuel avec ses réalisations et on notera abusivement ξ au lieu de Ξ . Dans toute la suite, on considèrera des processus ponctuels *simples*, c'est-à-dire chargeant uniquement le sous-espace de \mathcal{N} constitué de mesures simples que l'on continuera de noter \mathcal{N} et que l'on identifiera à l'ensemble des sous-ensembles localement finis de \mathbb{R}^d . De tels processus ponctuels sont caractérisés par leurs probabilités de vide (*void probabilities*) : si $\mathbb{P}[\Xi(C) = 0] = \mathbb{P}[\Xi'(C) = 0]$ pour tout compact C , alors Ξ et Ξ' sont égaux en loi (voir [SW08, Theorem 3.1.1.]).

Stationnarité, isotropie et ergodicité. Un processus ponctuel Ξ (ou sa loi \mathcal{P}) est dit *stationnaire* s'il est invariant par translations : pour toute translation τ_x de vecteur x , Ξ et $\tau_x \Xi$ ont la même loi. De la même façon, un processus ponctuel Ξ est dit *isotrope* s'il est invariant par toute rotation autour de l'origine. Un processus ponctuel stationnaire Ξ est dit *ergodique* si pour tout événement $A \in \mathcal{F}(\mathcal{N})$ invariant par l'action des translations $\{\tau_x : x \in \mathbb{R}^d\}$, $\mathcal{P}[A] = 0$ ou 1 .

Mesure d'intensité et théorème de Campbell. La mesure d'intensité Λ de Ξ est la mesure sur \mathbb{R}^d définie par :

$$\Lambda(B) := \mathbb{E} [\Xi(B)] = \int_{\mathcal{N}} \xi(B) \mathcal{P}(d\xi),$$

pour tout borélien $B \in \mathcal{B}(\mathbb{R}^d)$. La mesure d'intensité d'un processus ponctuel stationnaire est nécessairement invariante par toute translation. Si celle-ci est localement finie, il s'agit, à un facteur multiplicatif $0 < \lambda < +\infty$ près, de la mesure de Lebesgue dans \mathbb{R}^d ; λ est appelé l'*intensité* du processus ponctuel et s'interprète comme le nombre moyen de points par unité de volume.

En suivant la méthode standard on obtient le théorème de Campbell (voir par exemple [SW08, Theorem 3.1.2]).

Théorème 1.2 (Campbell). *Soit Ξ un processus ponctuel de mesure d'intensité Λ et $f : \mathbb{R}^d \rightarrow \mathbb{R}$ une fonction mesurable positive. Alors, $\sum_{x \in \Xi} f(x)$ est mesurable et :*

$$\mathbb{E} \left[\sum_{x \in \Xi} f(x) \right] = \int_{\mathcal{N}} \sum_{x \in \xi} f(x) \mathcal{P}(d\xi) = \int_{\mathbb{R}^d} f(x) \Lambda(dx).$$

Fonctions de corrélation. On appelle, sous réserve d'existence, *fonctions de corrélation* d'un processus ponctuel Ξ (ici par rapport à la mesure de Lebesgue dans \mathbb{R}^d) les fonctions ρ_m , $m \in \mathbb{N}^*$, telles que pour toute famille B_1, \dots, B_m de boréliens disjoints :

$$\mathbb{E} \left[\prod_{i=1}^m \Xi(B_i) \right] = \int_{\prod_{i=1}^m B_i} \rho_m(x_1, \dots, x_m) dx_1 \dots dx_m,$$

autrement dit, ρ_m est la densité du moment d'ordre m du processus ponctuel. Ces fonctions de corrélation s'interprètent comme la densité de probabilité pour qu'il y ait un point du processus ponctuel en chacun des sites x_1, \dots, x_m . Plus formellement, on vérifie que :

$$\rho_m(x_1, \dots, x_m) = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{P} [\#(\xi \cap B(x_i, \epsilon)) \neq 0, i = 1, \dots, m]}{\text{Vol}_{\mathbb{R}^d}(B(0, \epsilon))^m},$$

où $B(x, r)$ désigne la boule euclidienne de centre x et de rayon r .

Position générale et apériodicité. Deux conditions géométriques vont être utilisées à de nombreuses reprises dans ce manuscrit. La première est que presque toute réalisation ξ du processus ponctuel soit en position générale (en fait en position générale renforcée au sens de [Zes08]), c'est-à-dire que l'on ne trouve pas $d+1$ points (resp. $d+2$ points) sur un hyperplan (resp. sur une sphère). Cette condition permet d'assurer la bonne définition de la triangulation de Delaunay de ξ . La seconde est une condition d'apériodicité du processus ponctuel : pour presque tout ξ , pour tout $x \in \mathbb{R}^d$ on a $\xi \neq \tau_x \xi$. Ceci permet la définition et surtout l'utilisation du processus de *l'environnement vu par la particule* ou *par le marcheur*. En fait, sous la condition d'apériodicité, on peut reconstruire la trajectoire d'un marcheur

dans l'environnement ξ à partir des différents translatés de ξ que celui-ci aura vus depuis ses positions successives (voir §1.2.3). Remarquons que si ξ est en position générale, alors ξ est apériodique. En effet, supposons que ξ soit périodique et choisissons $x \in \mathbb{R}^d$ tel que $\tau_x \xi = \xi$. On obtient, par une récurrence immédiate, $\tau_{kx} \xi = \xi$, pour tout $k \in \mathbb{Z}$. Ainsi, si $y \in \xi$, alors ξ contient tous les $y + kx$, $k \in \mathbb{Z}$ et ξ ne peut pas être en position générale.

1.1.2 Mesure de Palm

Comme nous venons de le mentionner, nous aurons plus tard besoin de regarder l'environnement depuis un point particulier ou typique du processus ponctuel. Autrement dit, on voudrait observer Ξ depuis un point choisi au hasard et uniformément dans Ξ , ou encore, en utilisant la stationnarité de Ξ , observer Ξ depuis 0 sachant que 0 appartient à Ξ . L'événement « $0 \in \Xi$ » étant de probabilité nulle, il faut prendre quelques précautions pour donner à ceci un sens précis et rigoureux. En 1943, Palm a introduit dans [Pal43] des mesures adéquates, qui portent son nom, en dimension 1, dont l'étude a été approfondie et généralisée au début de la seconde moitié du XXème siècle. Nous définissons ici les mesures de Palm associées à des processus ponctuels stationnaires dans \mathbb{R}^d et donnons quelques résultats essentiels concernant celles-ci.

Mesure de Campbell et mesure de Palm. Soit Ξ un processus ponctuel stationnaire dans \mathbb{R}^d de loi \mathcal{P} et d'intensité $0 < \lambda < +\infty$. On définit sur $\mathbb{R}^d \times \mathcal{N}$, muni de la tribu produit $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{N})$, une mesure \mathcal{C} , appelée *mesure de Campbell* associée à Ξ , par la formule :

$$\int_{\mathcal{N}} \sum_{x \in \xi} f(x, \xi) \mathcal{P}(dx) = \int_{\mathbb{R}^d \times \mathcal{N}} f(x, \xi) \mathcal{C}(dx, d\xi),$$

pour toute fonction f mesurable et positive sur $\mathbb{R}^d \times \mathcal{N}$. Pour tout ensemble produit $B \times Y \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{N})$, on a :

$$\mathcal{C}(B \times Y) = \int_{\mathcal{N}} \xi(B) \mathbf{1}_Y(\xi) \mathcal{C}(dx, d\xi) = \mathbb{E}[\Xi(B) \mathbf{1}_Y(\Xi)].$$

Dans ce cadre, pour tout $Y \in \mathcal{F}(\mathcal{N})$, la mesure sur \mathbb{R}^d qui associe à tout borélien B la quantité $\mathcal{C}(B, Y)$ est absolument continue par rapport à la mesure de Lebesgue et admet, d'après le théorème de Radon-Nikodym, une densité $\lambda \mathcal{P}_x[Y]$ par rapport à celle-ci :

$$\mathcal{C}(B \times Y) = \lambda \int_B \mathcal{P}_x[Y] dx.$$

Pour x fixé, \mathcal{P}_x définit une distribution sur $(\mathcal{N}, \mathcal{F}(\mathcal{N}))$, appelée distribution de Palm de \mathcal{P} par rapport à x . Puisque \mathcal{P} est stationnaire, il nous suffit de connaître \mathcal{P}_0 qui peut être définie de façon équivalente par :

$$\mathcal{P}_0[Y] = \frac{1}{\lambda \text{Vol}_{\mathbb{R}^d}(B)} \mathbb{E} \left[\sum_{x \in \Xi \cap B} \mathbf{1}_{\tau_x \Xi \in Y} \right], \quad \forall B \in \mathcal{B}_b(\mathbb{R}^d).$$

Notons que \mathcal{P}_0 ne charge que le sous-ensemble \mathcal{N}_0 de \mathcal{N} constitué des sous-ensembles (simples et localement finis) de points de \mathbb{R}^d contenant 0 (ou des mesures de comptage correspondantes).

Formule de Campbell et formule d'échange de Neveu. Nous allons maintenant énoncer un raffinement du théorème de Campbell également connu sous le nom de *formule de Campbell*.

Théorème 1.3 (voir [SW08, Théorème 3.3.3. et Théorème 3.4.3]). *Soit Ξ un processus ponctuel stationnaire dans \mathbb{R}^d d'intensité $0 < \lambda < +\infty$ et soit $f : \mathbb{R}^d \times \mathcal{N} \mapsto \mathbb{R}_+$ une fonction mesurable. Alors,*

$$\mathbb{E} \left[\sum_{x \in \Xi} f(x, \Xi) \right] = \lambda \int_{\mathbb{R}^d} \int_{\mathcal{N}_0} f(x, \tau_{-x}\xi^0) \mathcal{P}_0(d\xi^0) dx.$$

La *formule d'échange de Neveu* relie des intégrales par rapport aux mesures de Palm associées à deux processus ponctuels stationnaires dans \mathbb{R}^d (ou plus généralement à deux mesures aléatoires, voir [SW08, Théorème 3.4.5.]). Cette formule donne, dans le cas où ces processus ponctuels ont la même loi, un résultat non trivial que l'on peut qualifier de *principe de transport de masse* et qui nous sera utile dans les Chapitres 3 et 4.

Théorème 1.4 (Formule d'échange de Neveu dans le cas de deux processus égaux en loi). *Soit \mathcal{P}_0 la mesure de Palm associée à un processus ponctuel stationnaire dans \mathbb{R}^d d'intensité finie. Alors, pour toute fonction mesurable $f : \mathbb{R}^d \times \mathcal{N}_0 \mapsto \mathbb{R}_+$, on a :*

$$\int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(x, \tau_x \xi^0) \mathcal{P}_0(d\xi^0) = \int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(-x, \xi^0) \mathcal{P}_0(d\xi^0).$$

Lien entre événements \mathcal{P} -presque sûrs et \mathcal{P}_0 -presque sûrs. Nous citons, pour terminer cette section, un résultat qui relie des événements \mathcal{P} -presque sûrs à des événements \mathcal{P}_0 -presque sûrs et qui nous sera d'une grande utilité dans le Chapitre 4. Ce résultat n'est généralement pas présenté dans les références standard mais apparaît par exemple dans l'annexe de [CFP13]. Il se déduit (presque) immédiatement de la définition de la mesure de Palm.

Proposition 1.5 (voir [CFP13, Lemme B.2]). *Soit $Y_0 \subset \mathcal{N}_0$ un ensemble mesurable et soit*

$$Y := \{\xi \in \mathcal{N} : \forall x \in \xi, \tau_x \xi \in Y_0\}.$$

Alors, $\mathcal{P}_0[Y_0] = 1$ si, et seulement si, $\mathcal{P}[Y] = 1$.

1.1.3 Exemples de processus ponctuels

On fournit ici quelques exemples de processus ponctuels dans \mathbb{R}^d auxquels on peut appliquer les résultats des chapitres suivants. On donne également des propriétés classiques de ceux-ci ainsi qu'une description des mesures de Palm associées.

Processus ponctuel de Poisson. Étant donnée une mesure σ -finie Λ sur \mathbb{R}^d , le *processus ponctuel de Poisson* (PPP) Ξ d'intensité Λ dans \mathbb{R}^d est défini par les deux propriétés suivantes :

1. pour tout borélien borné $B \in \mathcal{B}_b(\mathbb{R}^d)$, le nombre de points de Ξ tombant dans B suit une loi de Poisson de paramètre $\Lambda(B)$,
2. pour toute famille de boréliens bornés deux à deux disjoints B_1, \dots, B_k , les variables aléatoires $\Xi(B_1), \dots, \Xi(B_k)$ sont indépendantes.

Ce processus ponctuel constitue un exemple fondamental et facile à manipuler grâce à sa définition même. Celui-ci est simple si, et seulement si, Λ est sans atome et est stationnaire lorsque Λ est un multiple de la mesure de Lebesgue. Dans ce cas, on parle de processus ponctuel de Poisson *homogène* (d'intensité $\lambda \in \mathbb{R}_+$). Tout PPP homogène est isotrope et presque sûrement en position générale. Une simulation se trouve dans la figure 1.1.

Un théorème dû à Mecke (voir [SW08, Théorème 3.2.5.]) affirme que, si Λ est sans atome, Ξ est un processus ponctuel de Poisson d'intensité Λ si, et seulement si,

$$\mathbb{E} \left[\sum_{x \in \Xi} f(\Xi, x) \right] = \int_{\mathbb{R}^d} \mathbb{E} [f(\Xi + \delta_x, x)] \Lambda(dx), \quad (1.1.1)$$

pour toute fonction mesurable positive f sur $\mathcal{N} \times \mathbb{R}^d$.

En itérant la formule (1.1.1), on obtient un autre outil pratique de calcul : la *formule de Slivnyak-Mecke*.

Théorème 1.6 (Formule de Slivnyak-Mecke, voir [SW08, Corollaire 3.2.3.]). *Soit Ξ un processus ponctuel de Poisson d'intensité Λ , soit $m \in \mathbb{N}^*$ et soit f une fonction mesurable positive sur $\mathcal{N} \times (\mathbb{R}^d)^m$. Alors,*

$$\begin{aligned} \mathbb{E} \left[\sum_{(x_1, \dots, x_m) \in \Xi_{\neq}^m} f(\Xi, x_1, \dots, x_m) \right] \\ = \iint_{(\mathbb{R}^d)^m} \mathbb{E} \left[f \left(\Xi + \sum_{i=1}^m \delta_{x_i}, x_1, \dots, x_m \right) \right] \Lambda(dx_1) \dots \Lambda(dx_m), \end{aligned}$$

où la notation $(x_1, \dots, x_m) \in \Xi_{\neq}^m$ indique que les points x_1, \dots, x_m sont choisis deux à deux distincts dans Ξ .

Une autre conséquence importante du théorème de Mecke est que la version de Palm d'un PPP homogène Ξ a la même loi que $\Xi + \delta_0$ (ou que $\Xi \cup \{0\}$ si l'on voit Ξ comme un ensemble aléatoire plutôt qu'une mesure aléatoire, voir [SW08, Théorème 3.3.5.]).

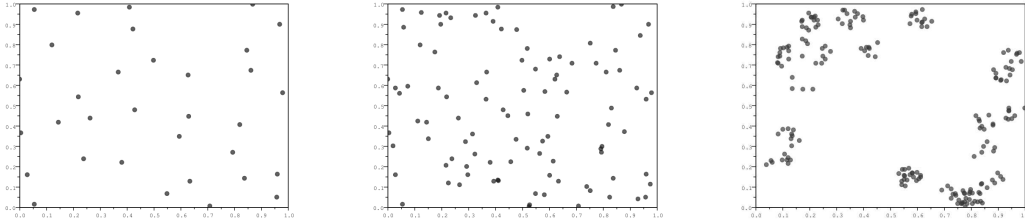


FIGURE 1.1 – Processus ponctuel de Poisson (centre), de Matérn « Hardcore » (gauche) et de Matérn « Cluster » (droite)

Processus de Matérn « Cluster ». Les *processus de Matérn « Cluster »* (MCP) sont des processus de Neyman-Scott particuliers (voir par exemple [CSKM13, p. 171]) présentant une certaine attractivité entre les points. De tels processus ont été utilisés pour modéliser la position des galaxies dans l'espace [KPBS⁺99] ou des épacentres de (petits) séismes [VJ70]. Présentons-en la construction. On commence par tirer une réalisation ξ_{par} d'un PPP homogène Ξ_{par} d'intensité λ , appelé le processus « *parents* ». Pour tout $x \in \xi_{\text{par}}$, on tire ensuite une réalisation d'un processus centré « *enfants* » ξ_x de telle sorte que les ξ_x , $x \in \xi_{\text{par}}$ sont des PPP homogènes d'intensité μ dans $B(0, R)$ indépendants les uns des autres. Alors, $\xi := \bigcup_{x \in \xi_{\text{par}}} (x + \xi_x)$ est distribué selon un processus de Matérn « Cluster » de paramètres λ, μ et R (voir la figure 1.1). Ces processus peuvent être vus comme des processus de Cox (voir [CSKM13, p. 166] pour une définition). Leurs mesures aléatoires d'intensité $\Lambda_{\Xi_{\text{par}}}$ sont diffusives et admettent une densité par rapport à la mesure de Lebesgue donnée par $\sum_{y \in \Xi_{\text{par}}} \mathbf{1}_{B(y, R)}(x)$. En particulier, les hyperplans et les sphères sont négligeables par rapport à $\Lambda_{\Xi_{\text{par}}}$ et, grâce aux résultats de [Zes08], les MCP sont presque sûrement en position générale. La construction explicite des MCP rend nombre de calculs possibles. On est par exemple capable de décrire précisément les mesures de Palm associées à ces processus. Notons \mathcal{C} la loi commune des « amas d'enfants » ξ_x et désignons par \mathcal{C}_0 la mesure de Palm associée :

$$\mathcal{C}_0(Y) = \frac{\mathbb{E}_{\mathcal{C}} \left[\sum_{x \in \Xi_0} \mathbf{1}_Y(\Xi_0 - x) \right]}{\mathbb{E}_{\mathcal{C}} \left[\#(\Xi_0) \right]}.$$

Il découle de [SKM87, §5.3, p.142] que $\mathcal{P}_0 = \mathcal{P} * \mathcal{C}_0$.

Processus de Matérn « Hardcore ». En 1960, Matérn a introduit plusieurs modèles de processus ponctuels qui sont construits en supprimant de manière dépendante des points de réalisations de PPP. Les points de ces processus ont tendance à se répartir plus régulièrement dans l'espace que ceux des processus ponctuels de Poisson. Ils sont utilisés pour modéliser des situations dans lesquelles les points sont en compétition pour utiliser des ressources comme par exemple la répartition des villes ou des arbres (voir [Mat86] et les

références citées dans cet article). On présente maintenant la construction des *processus de Matérn « Hardcore » de type I* (MHP I). Étant donné une réalisation ξ d'un PPP initial d'intensité λ et un réel $R > 0$, notons ξ_I l'ensemble défini par :

$$\xi_I := \{x \in \xi : \|x - y\| > R, \forall y \in \xi \setminus \{x\}\}.$$

Alors, ξ_I est distribué selon un processus de Matérn « Hardcore » de type I. Celui-ci est clairement presque sûrement en position générale puisque le PPP sous-jacent l'est. Il est également stationnaire et isotrope.

Puisqu'une réalisation ξ_I d'un MHP I est obtenue par l'application d'une fonction déterministe de suppression de points th_I à une réalisation ξ d'un PPP, on peut expliciter la mesure de Palm \mathcal{P}_0 . Désignons par Ξ_I^0 (resp. Ξ^0) la version de Palm d'un MHP I (resp. du PPP initial). Nous allons voir que :

$$\mathcal{L}oi(\Xi_I^0) = \mathcal{L}oi(\text{th}_I(\Xi^0) | 0 \in \text{th}_I(\Xi^0)).$$

Notons $\lambda_I := \lambda \mathcal{P}_0^{\text{ini}}[0 \in \text{th}_I(\xi^0)]$ l'intensité de Ξ_I où $\mathcal{P}_0^{\text{ini}}$ est la mesure de Palm du PPP initial. Rappelons que les *void probabilities* caractérisent le processus ponctuel. Il nous suffit donc d'écrire en utilisant la définition de la mesure de Palm et la formule de Slivnyak-Mecke que :

$$\begin{aligned} \mathcal{P}_0[\#(A \cap \xi_I^0) = 0] &= \frac{1}{\lambda_I \text{Vol}_{\mathbb{R}^d}(B)} \mathbb{E} \left[\sum_{x \in B \cap \Xi_I} \mathbf{1}_{\#((x+A) \cap \Xi_I) = 0} \right] \\ &= \frac{1}{\lambda_I \text{Vol}_{\mathbb{R}^d}(B)} \mathbb{E} \left[\sum_{x \in B \cap \Xi} \mathbf{1}_{\#((x+A) \cap \text{th}_I(\Xi)) = 0} \mathbf{1}_{x \in \text{th}_I(\Xi)} \right] \\ &= \frac{\lambda}{\lambda_I \text{Vol}_{\mathbb{R}^d}(B)} \int_B \mathcal{P}_x^{\text{ini}} [\#((x+A) \cap \text{th}_I(\xi^x)) = 0, x \in \text{th}_I(\xi^x)] dx \\ &= \frac{\mathcal{P}_0^{\text{ini}} [\#(A \cap \text{th}_I(\xi^0)) = 0, 0 \in \text{th}_I(\xi^0)]}{\mathcal{P}_0^{\text{ini}} [0 \in \text{th}_I(\xi^0)]} \\ &= \mathcal{P}_0^{\text{ini}} [\#(A \cap \text{th}_I(\xi^0)) = 0 | 0 \in \text{th}_I(\xi^0)]. \end{aligned}$$

Les *processus de Matérn « Hardcore » de type II* (MHP II) permettent, par l'introduction de marques de temps, d'éliminer moins de points du PPP initial que dans le cas des MHP I. Si l'on imagine que les points du processus ponctuel initial représentent des voitures souhaitant se garer, le MHP II donne les places de celles qui ont effectivement pu le faire puisqu'il y avait la place nécessaire à leur arrivée. Formellement, étant donné une réalisation ξ d'un PPP d'intensité λ , des marques $\{T_x\}_{x \in \xi}$ distribuées uniformément et indépendamment dans $[0, 1]$ et un réel $R > 0$, notons ξ_{II} l'ensemble défini par :

$$\xi_{II} := \{x \in \xi : T_x < T_y, \forall y \in \xi \cap B(x, R)\}.$$

Alors, ξ_{II} est distribué selon un MHP II. On obtient comme ci-dessus que :

$$\mathcal{L}oi(\Xi_{II}^0) = \mathcal{L}oi(\text{th}_{II}(\Xi^0) | 0 \in \text{th}_{II}(\Xi^0)),$$

où Ξ_{II}^0 (resp. Ξ^0) est la version de Palm d'un MHP II (resp. du PPP initial) et th_{II} est l'opérateur permettant d'obtenir ξ_{II} à partir de ξ .

Processus déterminantaux. Les *processus ponctuels déterminantaux* (DPP) ont été introduits en 1975 par Macchi [Mac75] qui souhaitait modéliser des fermions pour des applications en mécanique quantique. Ces processus ont une définition plus abstraite que ceux présentés ci-dessus mais apparaissent néanmoins naturellement dans des contextes variés comme l'étude des valeurs propres de matrices aléatoires, des arbres couvrants aléatoires ou des processus de retenues lors de l'addition de grands nombres aléatoires (voir [BHKPV06, Dia03, ST03a, ST03b, BP93, Gin65]).

On considère un opérateur K à noyau agissant sur $L^2(\mathbb{R}^d)$. Pour $B \in \mathcal{B}_b(\mathbb{R}^d)$, on note $K|_B$ la restriction de K à $L^2(B)$. On suppose que K est positif auto-adjoint et *localement de trace finie*, c'est-à-dire que pour tout $B \in \mathcal{B}_b(\mathbb{R}^d)$ les valeurs propres de $K|_B$ notées λ_i^B vérifient $\sum_i \lambda_i^B < +\infty$. On suppose également que le noyau intégral k de K a été choisi, ce qui est toujours possible, de sorte qu'il satisfasse la *formule des traces locales* :

$$\mathrm{tr} K|_B = \int_B k(x, x) dx, \quad \forall B \in \mathcal{B}_b(\mathbb{R}^d) \quad (1.1.2)$$

Le DPP associé à K est le processus ponctuel simple Ξ dont les fonctions de corrélation ρ_m satisfont :

$$\rho_m(x_1, \dots, x_m) = \det \left(k(x_i, x_j) \right)_{1 \leq i, j \leq m}, \quad \forall m \geq 1, \forall x_1, \dots, x_m \in \mathbb{R}^d.$$

Lorsque Ξ est stationnaire, il existe une constante K_0 telle que $K_0 := k(0, 0) = k(x, x) > 0$ pour presque tout x et on peut réécrire la formule 1.1.2 sous la forme :

$$\mathrm{tr} K|_B = K_0 \mathrm{Vol}_{\mathbb{R}^d}(B). \quad (1.1.3)$$

Le lecteur intéressé par plus de détails pourra consulter [BHKPV06], [Sos00] ou les annexes de [GY05].

Le Théorème 7 de [BHKPV06] se révèle très pratique pour effectuer des calculs avec les DPP. Celui-ci établit que le nombre de points d'un DPP Ξ tombant dans un ensemble borné B a la loi de la somme de v.a.i.i.d. de Bernoulli de paramètres les valeurs propres λ_j de la restriction $K|_B$ de K à B . Ceci permet, en particulier, de majorer les *void probabilities* et d'obtenir des moments exponentiels pour le nombre de points d'un DPP tombant dans un ensemble borné B . Plus précisément, en utilisant (1.1.3) et que $1 \pm s \leq e^{\pm s}$, on obtient :

$$\mathcal{P}[\#(\xi \cap B) = 0] = \prod_j (1 - \lambda_j) \leq \prod_j e^{-\lambda_j} = e^{-\mathrm{tr}(K|_B)} = e^{-K_0 \mathrm{Vol}_{\mathbb{R}^d}(B)},$$

et

$$\mathcal{E} \left[\binom{3}{2}^{\#(\xi \cap B)} \right] = \prod_j \left(1 + \frac{\lambda_j}{2} \right) \leq \prod_j e^{\frac{\lambda_j}{2}} = e^{\frac{1}{2} \mathrm{tr}(K|_B)} = e^{\frac{1}{2} K_0 \mathrm{Vol}_{\mathbb{R}^d}(B)}.$$

Les processus ponctuels présentés dans les paragraphes précédents possèdent clairement un rang de dépendance fini R , c'est-à-dire que, dans chaque cas, si deux boréliens

CHAPITRE 1. GÉOMÉTRIE STOCHASTIQUE ET MARCHES AU HASARD

sont distants de plus de R , le processus ponctuel se comporte indépendamment dans chacun d'eux. Ce n'est généralement pas le cas pour les processus déterminantaux. On peut toutefois formuler des hypothèses quelque peu artificielles sous lesquelles cette propriété est satisfaite. Pour B fixé, l'opérateur d'interaction locale $J_{[B]}$ du DPP est défini par :

$$J_{[B]} := K_{|B}(I - K_{|B})^{-1},$$

et l'opérateur d'interaction globale est défini par :

$$J := K(I - K)^{-1}.$$

On dit que J a un rang fini $R < \infty$ si son noyau j satisfait :

$$j(x, y) = 0 \text{ quand } \|x - y\| > R.$$

Dans ce cas, $j_{[B]}(x, y) = 0$ dès $\|x - y\| > R$ puisque $J_{[B]} \leq J$ au sens des opérateurs.

Grâce à [Sos00, Equation (1.36)], on sait que les densités de Janossy locales σ_B (voir [DVJ03, Définition 5.4.IV]) du DPP dans B sont données par l'identité :

$$\sigma_B(x_1, \dots, x_m) = \det(I - K_{|B}) \det \left(j_{[B]}(x_i, x_j) \right)_{1 \leq i, j \leq m}, \quad (1.1.4)$$

où la constante de normalisation $\det(I - K_{|B})$ doit être comprise comme un déterminant de Fredholm.

Rappelons (cf. [DVJ03, Exercice 5.3.8]) que, pour toute fonction f positive ou bornée définie sur l'ensemble des sous-ensembles finis de points de B , on a :

$$\begin{aligned} \mathbb{E}[f(\Xi)] &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{B^m} f(x_1, \dots, x_m) \sigma_B(x_1, \dots, x_m) dx_1 \dots dx_m \\ &= \det(I - K_{|B}) \sum_{m=0}^{\infty} \frac{1}{m!} \int_{B^m} f(x_1, \dots, x_m) \det \left(j_{[B]}(x_i, x_j) \right)_{1 \leq i, j \leq m} dx_1 \dots dx_m. \end{aligned} \quad (1.1.5)$$

Supposons que J a un rang fini R et donnons nous deux compacts $A, B \subset \mathbb{R}^d$ distants de plus de R . Nous allons voir que $\Xi_{|A}$ et $\Xi_{|B}$ sont indépendants. Pour $x \in A$ et $y \in B$, on

a $j_{[A]}(x, y) = 0$. Grâce à (1.1.5) et au théorème de Fubini, on a :

$$\begin{aligned}
 & \frac{\mathbb{E}\left[f(\Xi_{|A})g(\Xi_{|B})\right]}{\det(I - K_{|A \cup B})} \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j=0}^m \left\{ \frac{m!}{j!(m-j)!} \int_{A^j} f(x_1, \dots, x_j) \det\left(j_{[A]}(x_{i_1}, x_{i_2})\right)_{1 \leq i_1, i_2 \leq j} dx_1 \dots dx_j \right. \\
 & \quad \left. \times \int_{B^{m-j}} g(x_1, \dots, x_{m-j}) \det\left(j_{[B]}(x_{i_1}, x_{i_2})\right)_{1 \leq i_1, i_2 \leq m-j} dx_1 \dots dx_{m-j} \right\} \\
 &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{A^m} f(x_1, \dots, x_m) \det\left(j_{[A]}(x_{i_1}, x_{i_2})\right)_{1 \leq i_1, i_2 \leq m} dx_1 \dots dx_m \\
 & \quad \times \sum_{j=0}^{\infty} \frac{1}{j!} \int_{B^j} g(x_1, \dots, x_j) \det\left(j_{[B]}(x_{i_1}, x_{i_2})\right)_{1 \leq i_1, i_2 \leq j} dx_1 \dots dx_j \\
 &= \frac{\mathbb{E}\left[f(\Xi_{|A})\right] \mathbb{E}\left[g(\Xi_{|B})\right]}{\det(I - K_{|A}) \det(I - K_{|B})}. \tag{1.1.6}
 \end{aligned}$$

Notre assertion sera démontrée dès que l'on aura vérifié que $\det(I - K_{|A \cup B}) = \det(I - K_{|A}) \det(I - K_{|B})$. Ceci découle de (1.1.6) en prenant $f = g \equiv 1$.

Le Corollaire 3.4 de [GY05] assure que si $\text{spec}(K) \subset [0, 1[$, alors le DPP associé à K est dominé stochastiquement par un PPP d'intensité $\lambda := \limsup_{\|x\| \rightarrow \infty} j(x, x) < \infty$. Il s'ensuit qu'un tel DPP est presque sûrement en position générale.

Nous ne discutons pas des mesures de Palm associées aux processus déterminantaux puisque nous ne les utiliserons pas dans la suite. Le lecteur souhaitant se documenter sur cette question pourra consulter [ST03a] et [Gol10].

1.1.4 Exemples de mosaïques et de graphes géométriques

Nous donnons les définitions et quelques propriétés des graphes que nous allons considérer dans les chapitres à venir. Ces graphes sont construits à partir d'ensembles localement finis de points dans \mathbb{R}^d selon des règles (déterministes) basées sur la géométrie des ensembles de points considérés.

Mosaïque de Voronoï. Étant donné un sous-ensemble ξ localement fini de points de \mathbb{R}^d , la *cellule de Voronoï* de germe $\mathbf{x} \in \xi$ est l'ensemble :

$$\text{Vor}_{\xi}(\mathbf{x}) := \left\{ x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \xi \right\}.$$

Lorsque l'enveloppe convexe de ξ est \mathbb{R}^d tout entier, ce que l'on supposera dans toute la suite, la collection des cellules de Voronoï $\{\text{Vor}_{\xi}(\mathbf{x}), \mathbf{x} \in \xi\}$ forme une partition de l'espace en des polyèdres convexes appelée *mosaïque de Voronoï* ou *tessellation de Voronoï* de ξ . Il s'agit d'une mosaïque face-à-face, c'est-à-dire que, lorsqu'elle est non vide, l'intersection

de deux cellules est une face de chacune de celles-ci. Si ξ est en position générale, cette mosaïque est de plus normale c'est-à-dire que chaque face d'ordre s est dans l'intersection de $d - s + 1$ cellules. On notera $VS(\xi)$ le *squelette* de la mosaïque de Voronoï de ξ , c'est-à-dire le graphe dont l'ensemble de sommets (resp. d'arêtes) est l'ensemble des sommets (resp. des arêtes) se trouvant sur les bords des cellules de Voronoï.

Dans la suite nous allons considérer des mosaïques de Voronoï (et d'autres graphes) construits à partir de réalisations de processus ponctuels dans \mathbb{R}^d . Une littérature assez riche existe dans le cas où le processus sous-jacent est un processus ponctuel de Poisson. On parle alors de mosaïque de Poisson-Voronoï. Citons quelques résultats marquants concernant ce modèle. En 1962, Gilbert donne dans [Gil62] un encadrement de la loi du volume de la cellule typique qui est égale en loi à la cellule de l'origine de la mosaïque de Voronoï du processus ponctuel auquel on a ajouté un point en 0. Zuyev explicite dans [Zuy92] la loi conditionnelle, sachant le nombre de faces, du volume de la *fleur de Voronoï* de la cellule typique, c'est-à-dire de l'ensemble :

$$\mathcal{F}\left(\text{Vor}_{\Xi \cup \{0\}}(0)\right) := \bigcup_{x \in \text{Vor}_{\Xi \cup \{0\}}(0)} B(x, \|x\|).$$

Nous adapterons, dans le Chapitre 4, certains des arguments de cet article dans le cas d'autres processus ponctuels pour obtenir des moments exponentiels, sous la mesure de Palm, pour le degré de l'origine dans la triangulation de Delaunay et le maximum des distances entre l'origine et un de ses voisins dans ce même graphe. Il est assez naturel de vouloir contrôler ces quantités dans le contexte de l'étude des marches au hasard sur des triangulations de Delaunay aléatoires. En effet, elles apportent de l'information sur le nombre typique de choix de déplacements possibles pour la marche lors d'un saut et sur la longueur d'un saut typique. En 2002, dans sa thèse [Cal02a], Calka étudie certaines caractéristiques géométriques de la cellule typique de la mosaïque de Poisson-Voronoï. Il obtient, en particulier, la loi du nombre de faces de la cellule typique en dimension 2 dans [Cal03] et celle du rayon circonscrit, en toute dimension, dans [Cal02b]. Notons que la loi du rayon inscrit est également connue et bien plus facile à déterminer. Très récemment, dans sa thèse [Che13], Chenavier a obtenu des théorèmes limite pour les valeurs extrêmes de diverses caractéristiques géométriques de mosaïques aléatoires, incluant les mosaïques de Poisson-Voronoï.

On pourra également consulter [Møl94, Cal10] pour des compléments sur ces tessellations.

Triangulation de Delaunay. Soit ξ un sous-ensemble localement fini de points de \mathbb{R}^d en position générale (renforcée) dont l'enveloppe convexe est \mathbb{R}^d tout entier. La mosaïque duale de la mosaïque de Voronoï de ξ est constituée des *cellules de Delaunay* :

$$\text{Del}_\xi(\mathbf{v}) := \text{conv} \{ \mathbf{x} \in \xi : \mathbf{v} \in \text{Vor}_\xi(\mathbf{x}) \},$$

où \mathbf{v} est un sommet de $VS(\xi)$. La collection des cellules de Delaunay forme la *mosaïque de Delaunay*. Il s'agit d'une mosaïque face-à-face dont toutes les cellules sont des simplexes

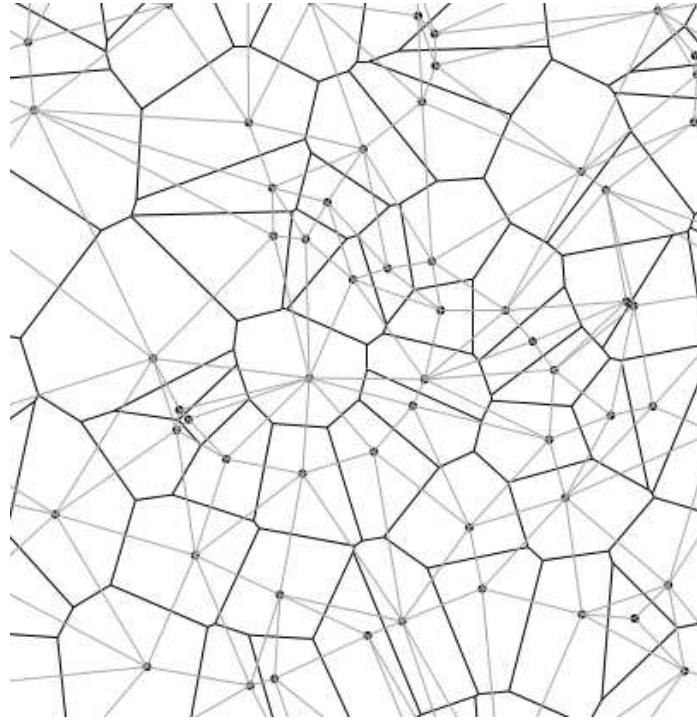


FIGURE 1.2 – Mosaïque de Voronoï (noir) et triangulation de Delaunay (gris)

puisque la mosaïque de Voronoï est normale. Le graphe ayant pour ensemble de sommets ξ et dont l'ensemble d'arêtes est constitué des arêtes des cellules de Delaunay est la *triangulation de Delaunay*. Il est clair qu'il y a alors une arête entre deux points de ξ , \mathbf{x} et \mathbf{y} , si et seulement si leurs cellules de Voronoï $\text{Vor}_\xi(\mathbf{x})$ et $\text{Vor}_\xi(\mathbf{y})$ partagent une face $(d - 1)$ -dimensionnelle. Les cellules de Delaunay sont également caractérisées par la propriété suivante : $d + 1$ points de ξ forment un simplexe de Delaunay si et seulement si leur sphère circonscrite ne contient pas de point de ξ en son intérieur. Cette caractérisation est une généralisation en toute dimension de la condition originale donnée par Delaunay en 1934. Elle coïncide avec la définition que nous venons de donner puisque ξ est supposé en position générale.

Quelques sous-graphes de la triangulation de Delaunay. Le *graphe de Gabriel* de ξ , noté $\text{Gab}(\xi)$ est obtenu à partir de la triangulation de Delaunay en ne conservant que les arêtes $\{\mathbf{x}, \mathbf{y}\}$ telles que la boule de diamètre $[\mathbf{x}, \mathbf{y}]$ ne contient pas d'autre point de ξ .

Les *creek-crossing graphs* engendrés par ξ , notés $G_n(\xi)$, $n \geq 2$ sont tels qu'il existe une arête entre $\mathbf{x}, \mathbf{y} \in \xi$ dans $G_n(\xi)$ si et seulement s'il n'existe pas un entier $k \leq n$ et des sommets $u_0 = \mathbf{x}, \dots, u_k = \mathbf{y} \in \xi$ tels que $\|u_i - u_{i+1}\| < \|\mathbf{x} - \mathbf{y}\|$ pour tout $i \in \{0, \dots, k - 1\}$. Le *creek-crossing graph* $G_2(\xi)$ est également connu sous le nom de *graphe des voisins relatifs*. Ces graphes ont été introduits dans [HNS13] pour approcher la *forêt couvrante minimale* (MSF). Les auteurs de cet article donnent des conditions sur les processus ponctuels pour

que les $G_n(\xi)$ soient connexes (voir aussi [HNGS15]). Notons que la question de la connexité de la MSF en dimension plus grande que 2 reste ouverte.

1.2 Autour des marches aléatoires

Dans cette section on rappelle des notions usuelles concernant les marches au hasard, ainsi que les liens entre ces marches et les réseaux électriques (voir [DS84, LP14]). On introduit également un outil fondamental dans l'étude des marches aléatoires dans des milieux aléatoires : *l'environnement vu par la particule*.

1.2.1 Marches aléatoires et réseaux électriques

Étant donné un graphe non orienté et connexe $G = (S_G, A_G)$ dans lequel tous les sommets sont de degré fini, une fonction de *conductance* est une application $C : S_G \times S_G \rightarrow \mathbb{R}_+$ vérifiant :

1. si $\{x, y\} \notin A_G$, alors $C(x, y) = 0$,
2. pour tous $x, y \in S_G$, $C(x, y) = C(y, x)$,
3. pour $x \in S_G$, $w_G(x) \neq 0$ où $w_G = \sum_{y \in S_G} C(x, y)$.

On supposera dans la suite que $C(x, y) > 0$ dès que $\{x, y\} \in A_G$. Alternativement, une conductance peut être définie comme une fonction strictement positive sur A_G et la résistance correspondante R est définie sur A_G comme l'inverse de la conductance. On utilisera dans la suite ces deux définitions. Le couple (G, C) forme un réseau électrique et permet de définir plusieurs marches aléatoires. Tout d'abord, la marche aléatoire à temps discret $(X_n)_{n \in \mathbb{N}}$ associée à (G, C) est la chaîne de Markov homogène en temps sur S_G dont les probabilités de transition sont données par :

$$\pi(x, y) := \frac{C(x, y)}{w_G(x)}.$$

Lorsque $C(x, y) = \mathbf{1}_{\{x, y\} \in A_G}$, $(X_n)_{n \in \mathbb{N}}$ n'est autre que la marche simple au plus proche voisin sur G . Deux marches à temps continu et à vitesses respectivement variable et constante (VSRW et CSRW) sont aussi définies à partir de (G, C) . Il s'agit des processus de Markov $(X_t^{\text{var}})_{t \geq 0}$ et $(X_t^{\text{const}})_{t \geq 0}$ dont les générateurs infinitésimaux sont donnés par :

$$\mathcal{L}^{\text{var}} f(x) = \sum_{y \in S_G} C(x, y) (f(y) - f(x)), \quad \forall x \in S_G,$$

et

$$\mathcal{L}^{\text{const}} f(x) = \sum_{y \in S_G} \pi(x, y) (f(y) - f(x)), \quad \forall x \in S_G.$$

CHAPITRE 1. GÉOMÉTRIE STOCHASTIQUE ET MARCHES AU HASARD

Ces marches peuvent être définies alternativement par changement de temps à partir de $(X_n)_{n \in \mathbb{N}}$. La CSRW est une version à temps continu de $(X_n)_{n \in \mathbb{N}}$ obtenue par une poissonisation du temps : le marcheur, partant de x , attend un temps exponentiel de paramètre 1, puis se déplace vers un nouveau site y choisi selon les probabilités $\pi(x, y)$. Le procédé est ensuite répété indéfiniment et indépendamment. La VSRW est le processus dont les taux de saut instantanés sont donnés directement par la conductance et des explosions en temps fini pourraient avoir lieu (voir §1.2.2). Son comportement est le suivant : le marcheur, partant de x , attend un temps exponentiel de paramètre $w_G(x)$, puis se déplace vers un nouveau site y choisi selon les probabilités $\pi(x, y)$. Le procédé est ensuite répété indéfiniment et indépendamment.

D'autre part, la *detailed balance condition* :

$$w_G(x)\pi(x, y) = w_G(y)\pi(y, x), \quad \forall x, y \in S_G,$$

est satisfaite. Ainsi, w_G est, à une constante multiplicative près, la mesure stationnaire et réversible pour $(X_n)_{n \in \mathbb{N}}$ et $(X_t^{\text{const}})_{t \geq 0}$ et la mesure de comptage sur S_G est stationnaire et réversible pour $(X_t^{\text{var}})_{t \geq 0}$.

Fonctions harmoniques. On dit qu'une fonction f est *harmonique en x* si elle vérifie :

$$f(x) = \sum_{y \sim x} \pi(x, y) f(y),$$

et *harmonique sur $B \subset S_G$* si elle l'est en tout point de B . On dit simplement que f est *harmonique* si elle l'est sur S_G . Les fonctions harmoniques satisfont un *principe du maximum*, un *principe d'existence* et un *principe d'unicité*. Nous donnons l'énoncé de ce dernier qui permettra, dans le paragraphe suivant, d'interpréter un *potentiel électrique* comme une certaine probabilité.

Théorème 1.7 (Principe d'unicité). *Soit B un sous-ensemble propre et fini de S_G et $f, g : S_G \rightarrow \mathbb{R}$ deux fonctions harmoniques sur B .*

Si f et g coïncident en dehors de B , alors elles coïncident sur S_G .

Les fonctions harmoniques vont jouer un rôle fondamental dans le Chapitre 4. En fait, si f est harmonique, $(f(X_n))_{n \in \mathbb{N}}$ est une martingale. Ainsi, pour établir un principe d'invariance, on va exhiber une fonction harmonique convenable φ , décomposer X_n sous la forme $X_n = \varphi(X_n) + \chi(X_n)$ où $\chi(X_n)$ est une petite correction que l'on montrera être négligeable dans la limite diffusive. Le principe d'invariance pour $(X_n)_{n \in \mathbb{N}}$ découlera alors du principe d'invariance pour la martingale $(\varphi(X_n))_{n \in \mathbb{N}}$.

Potentiel et intensité. Étant donnés deux sous-ensembles disjoints de S_G , on appelle *potentiel électrique* entre A et Z toute fonction V définie sur S_G et harmonique en dehors de $A \cup Z$. L'*intensité* associée au potentiel V est la fonction i définie sur $S_G \times S_G$ par :

$$i(x, y) := C(x, y) (V(x) - V(y)).$$

Il apparaît clairement que $i(x, y) = -i(y, x)$ et que pour tout $x \notin A \cup Z$, $\sum_{y \sim x} i(x, y) = 0$. Autrement dit, l'intensité est un *flux*. Ces définitions sont en accord avec les notions physiques correspondantes et permettent de retrouver les lois usuelles d'électricité (*loi d'Ohm*, *lois des nœuds* et *des mailles de Kirchhoff*, voir [LP14, §2.1]) ainsi que les règles de réduction de réseaux habituelles (*loi des séries*, *loi des parallèles* et *transformation triangle-étoile*, voir [LP14, §2.3]).

Donnons maintenant une interprétation probabiliste du potentiel électrique. Pour cela, fixons deux sous-ensembles disjoints de sommets A et Z , notons P_x la loi de la marche $(X_n)_{n \in \mathbb{N}}$ partant de x , $T_A := \inf\{n \geq 0 : X_n \in A\}$ le temps d'entrée dans A et T_Z le temps d'entrée dans Z . Alors, $P_x[T_A < T_Z]$ est harmonique en dehors de $A \cup Z$ et vaut 1 sur A et 0 sur Z . Grâce au principe d'unicité, il s'agit du potentiel électrique obtenu en fixant la différence de potentiel entre A et Z à 1 (c'est-à-dire en imposant $V|_A \equiv 1$ et $V|_Z \equiv 0$).

Conductance et résistance efficaces. Lorsque le potentiel V est fixé à v_A sur A et à 0 sur Z , la quantité totale de courant sortant de A est donnée par $I_A := \sum_{x \in A} \sum_{y \notin A} i(x, y)$ et dépend de toutes les résistances du réseau. La *résistance efficace* entre A et Z est alors définie par $R_{\text{eff}}(A \leftrightarrow Z) := v_A / I_A$ et la *conductance efficace* $C_{\text{eff}}(A \leftrightarrow Z)$ entre A et Z est son inverse. La résistance efficace et la conductance efficace satisfont des formules variationnelles appelées respectivement *principe de Thomson* et *principe de Dirichlet*. Le principe de Dirichlet que nous énonçons ci-dessous permettra, dans le Chapitre 3, de comparer des coefficients de diffusion avec les résistances efficaces de réseaux adéquats.

Théorème 1.8 (Principe de Dirichlet). *On a :*

$$C_{\text{eff}}(A \leftrightarrow Z) = \inf \left\{ \frac{1}{2} \sum_{x, y \in S_G} C(x, y) (f(x) - f(y))^2 : f : S_G \rightarrow \mathbb{R}, f|_A \equiv 1, f|_Z \equiv 0 \right\}$$

et la borne inférieure est réalisée pour $h = V$ le potentiel valant 1 sur A et 0 sur Z .

Donnons-nous maintenant une racine (ou origine) $a \in S_G$ du réseau et $Z \subset S_G$ tels que $a \notin Z$ et fixons le potentiel à v_a en a et à 0 sur Z . On va interpréter la conductance efficace $C_{\text{eff}}(a \leftrightarrow Z)$ entre a et Z en termes d'une probabilité d'échappement. Notons $\mathbf{P}[a \rightarrow Z] := P_a[T_Z < T_a^1]$ la probabilité d'entrer dans Z avant de retourner en a , où $T_a^1 := \inf\{n \geq 1 : X_n = a\}$ est le temps de retour en a . Par le principe d'unicité, on a que $P_x[T_a < T_Z] = \frac{V(x)}{V(a)}$ et donc :

$$\begin{aligned}
 \mathbf{P}[a \rightarrow Z] &= \sum_{x \in S_G \setminus \{a\}} \pi(a, x) (1 - P_x[T_a < T_Z]) \\
 &= \sum_{x \in S_G \setminus \{a\}} \frac{C(a, x)}{w_G(a)} \left(1 - \frac{V(x)}{V(a)}\right) \\
 &= \frac{1}{V(a)w_G(a)} \sum_{x \in S_G \setminus \{a\}} i(a, x) \\
 &= \frac{1}{w_G(a)} C_{\text{eff}}(a \leftrightarrow Z),
 \end{aligned}$$

soit

$$C_{\text{eff}}(a \leftrightarrow Z) = w_G(a) \mathbf{P}[a \rightarrow Z].$$

Il s'ensuit que le nombre de retours en a de la marche $(X_n)_{n \in \mathbb{N}}$ avant de toucher Z suit une loi géométrique de moyenne $\mathbf{P}[a \rightarrow Z]^{-1} = w_G(a) R_{\text{eff}}(a \leftrightarrow Z)$.

Récurrence et transience. Considérons maintenant un réseau électrique infini (G, C) . On souhaite caractériser la transience de la marche associée. Pour cela, on considère une *suite exhaustive* de sous-graphes finis de G , c'est-à-dire une suite $(G_n)_{n \in \mathbb{N}}$ de sous-graphes finis de G telle que :

1. pour tout $n \in \mathbb{N}$, $G_n \subset G_{n+1}$,
2. $\bigcup_{n \in \mathbb{N}} G_n = G$.

En munissant les arêtes de chaque G_n de leur conductance dans G , on obtient une suite exhaustive de sous-réseaux de G . Notons $Z_n := G \setminus G_n$. Pour tout $a \in G$, $(\{a \rightarrow Z_n\})_{n \in \mathbb{N}}$ est une suite décroissante d'événements. Ainsi, par convergence monotone, $\mathbf{P}[a \rightarrow Z_n] \downarrow \mathbf{P}[a \rightarrow \infty]$ la probabilité d'échappement à l'infini depuis a . La probabilité $\mathbf{P}[a \rightarrow \infty]$ ne dépend pas de la suite exhaustive de sous-graphes considérée. Il est clair que celle-ci est strictement positive si, et seulement si, la marche aléatoire $(X_n)_{n \in \mathbb{N}}$ est transitoire. On définit alors la *conductance efficace entre a et l'infini* par :

$$C_{\text{eff}}(a \leftrightarrow \infty) := \lim_{n \rightarrow \infty} C_{\text{eff}}(a \leftrightarrow Z_n).$$

Puisque G est connexe et les conductances sont strictement positives sur les arêtes de G , la marche $(X_n)_{n \in \mathbb{N}}$ est irréductible, et nous venons de montrer que :

Théorème 1.9. *La marche aléatoire sur un réseau électrique infini (G, C) est transitoire si, et seulement si, il existe $a \in S_G$ tel que $C_{\text{eff}}(a \leftrightarrow \infty) > 0$.*

Le *principe de monotonie de Rayleigh* permet de comparer les conductances efficaces associées à deux fonctions de conductance sur le même graphe et donc de transporter des résultats de récurrence ou de transience d'un réseau électrique à un autre.

Théorème 1.10 (Principe de monotonie de Rayleigh). *Soit G un graphe connexe muni de deux fonctions de conductance C et C' telles que $C \leq C'$.*

1. *Si G est fini et $A, Z \subset S_G$ sont deux ensembles de sommets disjoints, alors*

$$C_{\text{eff}}(A \leftrightarrow Z; C) \leq C_{\text{eff}}(A \leftrightarrow Z; C').$$

2. *Si G est infini et $a \in S_G$, alors $C_{\text{eff}}(a \leftrightarrow \infty; C) \leq C_{\text{eff}}(a \leftrightarrow \infty; C')$; en particulier, si (G, C) est transitoire, (G, C') l'est aussi.*

On déduit aisément de ce principe un lien intuitif entre récurrence ou transience d'un graphe infini et connexe G et d'un sous-graphe (infini et connexe) H de celui-ci. En effet, la marche sur H peut être vue comme la marche sur G avec la conductance C définie par $C = C'$ sur H et $C = 0$ en dehors de H , où C' est la conductance sur G . Le principe de monotonie assure alors que si (H, C) est transitoire, il en est de même pour (G, C') . De même, on peut voir que si (G, C') est récurrent, il en est de même pour (H, C) .

Un autre résultat, généralement dénommé *rough embedding theorem* et dû à Kanai [Kan86] dans sa forme originale, permet de transporter la transience d'un réseau électrique vers un autre. La preuve du critère de transience établi dans le Chapitre 2 repose en grande partie sur celui-ci.

Théorème 1.11 (*rough embedding theorem*, voir [LP14, Théorème 2.17]). *Soient (G, C) et (G', C') deux réseaux électriques et soient R, R' les résistances associées. On suppose que (G, C) est transitoire.*

S'il existe une application $\phi : S_G \rightarrow S_{G'}$, des constantes $\alpha, \beta < +\infty$ et une application Φ de A_G dans l'ensemble des chemins non orientés dans G' telle que :

1. *pour toute arête $\{x, y\} \in A_G$, $\Phi(\{x, y\})$ est un chemin simple (non vide) dans G' entre $\phi(x)$ et $\phi(y)$ vérifiant :*

$$\sum_{e' \in \Phi(\{x, y\})} R'(e') \leq \alpha R(\{x, y\});$$

2. *toute arête $e' \in A_{G'}$ est dans l'image par Φ d'au plus β arêtes de G ,*

alors (G', C') est transitoire.

1.2.2 Conditions pour la bonne définition des marches à temps continu

La marche à vitesse variable associée au réseau électrique (G, C) peut être mal définie dans des cas dégénérés. On veut ici assurer, en particulier, qu'il n'y a pas d'explosion en temps fini, c'est-à-dire que la marche ne saute qu'un nombre fini de fois dans un intervalle de temps fini. On rappelle des conditions permettant de montrer que $(X_t^{\text{var}})_{t \geq 0}$ existe et est unique en tant que processus de saut pur.

Pour cela, on introduit quelques notations permettant de voir $(X_t^{\text{var}})_{t \geq 0}$ comme obtenue à partir de $(X_n)_{n \in \mathbb{N}}$ par changement de temps. On se donne une famille de variables aléatoires indépendantes $\{T_n(x) : x \in S_G, n \in \mathbb{N}\}$ telle que les $\{T_n(x) : n \in \mathbb{N}\}$ sont de loi exponentielle de paramètre $w_G(x)$. On définit alors les temps de n ième saut par :

$$R_0 := 0, \quad R_n := R_{n-1} + T_{n-1}(X_{n-1}),$$

et la variable aléatoire comptant le nombre de sauts de $(X_t^{\text{var}})_{t \geq 0}$ jusqu'au temps t par :

$$n_*(t) = n, \quad \text{si } R_n \leq t < R_{n+1}.$$

On a ainsi $X_t^{\text{var}} = X_{n_*(t)}$.

Le Théorème 15.37 de [Bre68] assure que $(X_t^{\text{var}})_{t \geq 0}$ est un processus de saut pur dès que $n_*(t)$ est bien défini tandis que la Proposition 15.38 de ce même ouvrage en établit l'unicité.

Cette condition est impliquée par un critère plus simple à mettre en œuvre dans le cadre présent pour vérifier que $(X_t^{\text{var}})_{t \geq 0}$ est un processus de saut pur, se trouve dans [Bre68, Proposition 15.43]. Celui-ci affirme que $(X_t^{\text{var}})_{t \geq 0}$ est un processus de saut pur si et seulement si pour tout $x \in S_G$:

$$P_x \left[\sum_{n=0}^{\infty} \frac{1}{w_G(X_n)} < +\infty \right] = 0.$$

Dans les Chapitres 3 et 4, on considèrera des marches à vitesses variables sur des graphes engendrés par des processus ponctuels. Sous une hypothèse convenable d'ergodicité et une hypothèse de finitude du moment d'ordre 1 du degré de l'origine sous la mesure de Palm \mathcal{P}_0 associée au processus ponctuel, valides dans ces chapitres, le critère précédant peut être vérifié pour \mathcal{P}_0 -presque tout ξ^0 en utilisant les arguments donnés dans la preuve de [FSBS06, Proposition 10]. Ceci assure la bonne définition presque sûre des marches y apparaissant.

1.2.3 L'environnement vu par la particule

Depuis les travaux de Kozlov [Koz85], Kipnis et Varadhan [KV86] et De Masi, Ferrari, Goldstein et Wick [DMFGW89], l'*environnement vu par la particule* est devenu un outil fondamental dans l'étude des marches aléatoires en milieux aléatoires. Celui-ci a été abondamment utilisé dans le contexte des marches sur \mathbb{Z}^d associées à des conductances aléatoires et peut également être employé pour étudier des marches sur des graphes dont l'ensemble de sommets est donné par une réalisation d'un processus ponctuel dans \mathbb{R}^d . Les buts de cette section sont de motiver l'introduction de cette notion et de la définir formellement dans le dernier cadre évoqué. Nous nous restreignons ici au cas du temps discret, la discussion s'adaptant sans difficulté au cas du temps continu.

Étant donné ξ^0 tiré selon la mesure de Palm \mathcal{P}_0 d'un processus ponctuel stationnaire, considérons la marche simple au plus proche voisin $(X_n^{\xi^0})_{n \in \mathbb{N}}$ sur un graphe $G(\xi^0) =$

$(\xi^0, A(\xi^0))$. Notons $P_x^{\xi^0}$ la loi de la marche partant de x . Le graphe $G(\xi^0)$ est en général très irrégulier et il peut être avantageux de regarder les trajectoires de la marche sous la mesure dite *annealed* $P[\cdot] := \int P_x^{\xi^0}[\cdot] \mathcal{P}_0(d\xi^0)$ plutôt que sous la loi *quenched* $P_x^{\xi^0}$. En effet, la moyennisation, due à l'intégration contre \mathcal{P}_0 , homogénéise le milieu et gomme en un certain sens ses irrégularités. Cependant, au cours du temps, la marche gagne la connaissance des positions des points de ξ^0 dans \mathbb{R}^d et des relations de voisinage entre ceux-ci ce qui entraîne que $(X_n^{\xi^0})_{n \in \mathbb{N}}$ n'est plus markovien sous P . On souhaite donc introduire un processus auxiliaire qui soit markovien sous la mesure *annealed* et qui permette de retrouver toute la trajectoire de $(X_n^{\xi^0})_{n \in \mathbb{N}}$ partant de $0 \in \xi^0$. Il s'agit du processus de l'*environnement vu par la particule* ou le *marcheur* $(\xi_n^0)_{n \in \mathbb{N}}$ défini par :

$$\xi_n^0 := \tau_{X_n} \xi^0,$$

où les τ_x , $x \in \mathbb{R}^d$, forment le groupe des translations dans \mathbb{R}^d agissant sur \mathcal{N}_0 . On note \mathbf{P}_{ξ^0} la loi de ce processus ayant pour état initial ξ^0 . Une manière moins formelle de décrire ce processus est d'imaginer, qu'au lieu de se déplacer dans ξ^0 , le marcheur reste statique en l'origine et enregistre les vues successives qu'il a sur l'environnement qui se déplace (se translate en fait) autour de lui.

Expliquons maintenant comment la marche originale $(X_n^{\xi^0})_{n \in \mathbb{N}}$ peut être reconstruite à partir du processus de l'environnement. Pour cela, supposons que ξ^0 est apériodique (voir §1.1.1) et définissons Δ sur $\mathcal{N}_0 \times \mathcal{N}_0$ par :

$$\Delta(\xi^0, \tilde{\xi}^0) := \begin{cases} x & \text{si } \tilde{\xi}^0 = \tau_x \xi^0 \\ 0 & \text{sinon} \end{cases}.$$

Remarquons que Δ est bien défini grâce à l'apériodicité de ξ^0 ; on donne une valeur arbitraire à Δ dans le cas périodique. Définissons aussi la fonctionnelle de la trajectoire de l'environnement $(\xi_n^0)_{n \in \mathbb{N}}$:

$$Y_0 = 0 \quad \text{et} \quad Y_n(\underline{\xi}^0) := \sum_{i=0}^{n-1} \Delta(\xi_i^0, \xi_{i+1}^0), \quad n \geq 1.$$

Supposons que \mathcal{P}_0 -presque toute réalisation de ξ^0 soit apériodique. Alors, pour presque tout ξ^0 , Y_n est bien définie pour tout n et la loi de $(Y_n(\underline{\xi}^0))_{n \in \mathbb{N}}$ sous \mathbf{P}_{ξ^0} coïncide avec $P_0^{\xi^0}$, la loi de la marche originale partant de 0 dans ξ^0 .

Les arguments permettant d'obtenir la réversibilité et l'ergodicité du processus de l'environnement vu par la particule par rapport à la mesure *annealed* $\mathbf{P}[\cdot] = \int \mathbf{P}_{\xi^0}[\cdot] \mathcal{P}_0(d\xi^0)$ sont relativement classiques et rappelés dans le paragraphe 3.2.3 du Chapitre 3.

Chapitre 2

Récurrance et transience

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Dans ce chapitre, on étudie des marches simples ou associées à des conductances sur des graphes aléatoires plongés dans \mathbb{R}^d . Le but est de déterminer, pour presque toute réalisation du graphe (connexe), si ces marches sont récurrentes ou transitoires. Les graphes aléatoires que nous considérons dépendent de la géométrie d'un ensemble aléatoire, infini et localement fini de points. Plus précisément, étant donnée une réalisation ξ d'un processus ponctuel simple dans \mathbb{R}^d , un graphe $G(\xi) = (V_{G(\xi)}, E_{G(\xi)})$, connexe, infini, localement fini, est construit à partir de la géométrie de cette réalisation et de règles déterministes (squelette de la mosaïque de Voronoï, triangulation de Delaunay, graphe de Gabriel, ...). Ce graphe est ensuite muni d'une fonction de conductance C , c'est-à-dire une fonction strictement positive sur son ensemble d'arêtes $E_{G(\xi)}$. Rappelons que la marche aléatoire sur $G(\xi)$ associée à C est la chaîne de Markov homogène en temps $(X_n)_{n \in \mathbb{N}}$ dont les probabilités de transition sont données par :

$$\mathcal{P}[X_{n+1} = v | X_n = u] = \frac{C(u, v)}{w(u)},$$

où $w(u) := \sum_{v \sim u} C(u, v)$.

Dans le Théorème 2.2, on obtient que, sous des hypothèses sur les variables de comptage du processus ponctuel sous-jacent et des hypothèses sur la fonction de conductance, les marches aléatoires sur la triangulation de Delaunay, le graphe de Gabriel et le squelette de la mosaïque de Voronoï engendrés par presque toute réalisation de ce processus ponctuel sont récurrentes si $d = 2$ et transitoires si $d \geq 3$. On donne dans le Théorème 2.1 des exemples de processus ponctuels satisfaisant les conditions données dans le Théorème 2.2; ceux-ci illustrent le fait que les résultats sont valables pour des processus ponctuels ayant des interactions entre points très différentes (attractivité ou répulsivité). Afin de prouver ces résultats, on établit (et applique) deux critères assez simples et esthétiques pour la récurrence ou la transience presque sûre de marches aléatoires. Les preuves de ceux-ci s'appuient sur une analogie bien connue entre les marches aléatoires et les réseaux électriques. Le critère de récurrence (Critère 2.4) est basé sur un découpage de l'espace en anneaux successifs et une minoration de la résistance efficace entre un sommet fixé du graphe et l'infini. Notons que ce critère s'applique à des graphes déterministes plongés dans \mathbb{R}^d , $d \geq 2$. L'idée principale de la preuve du critère de transience (Critère 2.8) est de découper l'espace en boîtes et d'injecter grossièrement, pour presque toute réalisation du graphe aléatoire, une réalisation d'un amas infini de percolation en régime sur-critique dans \mathbb{Z}^d et d'exploiter le fait qu'en dimension 3 et plus, la marche simple sur un tel amas de percolation est transitoire. Mentionnons pour terminer que ce dernier critère peut être utilisé pour obtenir la transience de marches sur les *creek-crossing graphs* (voir Chapitre 3 §3.4 pour des arguments similaires).

Le travail présenté dans ce chapitre fait l'objet d'un article accepté sous réserve de modifications dans *Stochastic Processes and their Applications*.

Recurrence or transience of random walks on random graphs generated by point processes in \mathbb{R}^d

2.1 Introduction and main results

We deal with the question of recurrence or transience for random walks on Delaunay triangulations, Gabriel graphs and skeletons of Voronoi tilings which are generated by point processes in \mathbb{R}^d . We obtain recurrence or transience results for random walks on these graphs for a.a. realization of the underlying point process. Random walks on random geometric networks are natural for describing flows, molecular diffusions, heat conduction or other problems from statistical mechanics in random and irregular media. In the last decades, most authors considered models in which the underlying graphs were the lattice \mathbb{Z}^d or a subgraph of \mathbb{Z}^d . Particular examples include random walks on percolation clusters (see [GKZ93] for the question of recurrence or transience and [BB07, Mat08, MP07] for invariance principles) and the so-called random conductance model (see [Bis11] and references therein). In recent years, techniques were developed to analyze random walks on complete graphs generated by point processes with jump probability which is a decreasing function of the distance between points. The main results on this model include an annealed invariance principle [FSBS06, FM08], isoperimetric inequalities [CF09], recurrence or transience results [CFG09], a quenched invariance principle and heat kernel estimates [CFP13]. If the underlying graph is not the complete graph but, for example, a Delaunay triangulation or a Gabriel graph, additional difficulties appear due to the lack of information on the structure of the graphs (distribution and correlations of the degrees or of the edge lengths, volume growth, ...). Very recently, Ferrari, Grisi and Groisman [FGG12] constructed harmonic deformations of Delaunay triangulations generated by point processes which could be used to prove a quenched invariance principle via the corrector method, at least in the 2-dimensional case. Unfortunately, the sublinearity of the corrector does not directly follow from their proofs in higher dimensions. Precise heat kernel estimates, similar to those derived by Barlow [Bar04] in the percolation setting, are still to be obtained.

Under suitable assumptions on the point process and the transition probabilities, we obtain in this paper almost sure recurrence or transience results, namely Theorem 2.1 and Theorem 2.2, for random walks on the Delaunay triangulation, the Gabriel graph or the skeleton of the Voronoi tiling of this point process. We hope that the different methods developed in the papers [FSBS06, FM08, CF09, CFP13] can be adapted to our setting to obtain annealed and quenched invariance principles. This will be the subject of future work.

2.1.1 Conditions on the point process

In what follows the point process ξ is supposed to be simple, stationary and almost surely in general position (see [Zes08]): a.s. there are no $d + 1$ points (resp. $d + 2$ points) in any $(d - 1)$ -dimensional affine subspace (resp. in a sphere).

In this paper, c_1, c_2, \dots denote positive and finite constants. We will need the following assumptions on the void probabilities **(V)** and on the deviation probabilities **(D₂)** and **(D₃₊)**:

Assumptions.

(V) *There exists a constant c_1 such that for L large enough:*

$$\mathcal{P}\left[\#\left([0, L]^d \cap \xi\right) = 0\right] \leq e^{-c_1 L^d}.$$

(D₂) *If $d = 2$, there are constants c_2, c_3 such that for L, l large enough:*

$$\mathcal{P}\left[\#\left(\left([0, L] \times [0, l]\right) \cap \xi\right) \geq c_2 L l\right] \leq e^{-c_3 L l}.$$

(D₃₊) *If $d \geq 3$, there exists c_4 such that for L large enough and all $m > 0$:*

$$\mathcal{P}\left[\#\left([0, L]^d \cap \xi\right) \geq m\right] \leq e^{c_4 L^d - m}.$$

For transience results, the following additional assumptions are needed:

(FR_k) *ξ has a finite range of dependence k , i.e., for any disjoint Borel sets $A, B \subset \mathbb{R}^d$ with $d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} \geq k$, $\xi \cap A$ and $\xi \cap B$ are independent.*

(ND) *Almost surely, ξ does not have any descending chain, where a descending chain is a sequence $(u_i)_{i \in \mathbb{N}} \subset \xi$ such that:*

$$\forall i \in \mathbb{N}, \|u_{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\|.$$

As discussed in Section 2.6, these assumptions are in particular satisfied if ξ is:

- a homogeneous Poisson point process (PPP),
- a Matérn cluster process (MCP),
- a Matérn hardcore process I or II (MHP I/II).

Moreover, assumptions **(V)** and **(D₂)** hold if ξ is a stationary determinantal point process (DPP).

A brief overview on each of these point processes is given in Section 2.6. Note that these processes have different interaction properties: for PPPs there is no interaction between points, MCPs exhibit clustering effects whereas points in MHPs and DPPs repel each other.

2.1.2 The graph structures

Write $\text{Vor}_\xi(\mathbf{x}) := \{x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \xi\}$ for the Voronoi cell of $\mathbf{x} \in \xi$; \mathbf{x} is called the nucleus or the seed of the cell. The Voronoi diagram of ξ is the collection of the Voronoi cells. It tessellates \mathbb{R}^d into convex polyhedra. See [Møl94, Cal10] for an overview of these tessellations.

The graphs considered in the sequel are:

$\text{VS}(\xi)$ the skeleton of the Voronoi tiling of ξ . Its vertex (resp. edge) set consists of the collection of the vertices (resp. edges) on the boundaries of the Voronoi cells. Note that this is the only graph with bounded degree considered in the sequel. Actually, if ξ is in general position in \mathbb{R}^d , any vertex of $\text{VS}(\xi)$ has degree $d + 1$.

$\text{DT}(\xi)$ the Delaunay triangulation of ξ . It is the dual graph of its Voronoi tiling. It has ξ as vertex set and there is an edge between \mathbf{x} and \mathbf{y} in $\text{DT}(\xi)$ if $\text{Vor}_\xi(\mathbf{x})$ and $\text{Vor}_\xi(\mathbf{y})$ share a $(d - 1)$ -dimensional face. Another useful characterization of $\text{DT}(\xi)$ is the following: a simplex Δ is a cell of $\text{DT}(\xi)$ iff its circumscribed sphere has no point of ξ in its interior. Note that this triangulation is well defined since ξ is assumed to be in general position.

These two graphs are widely used in many fields such as astrophysics [RBFN01], cellular biology [Pou04], ecology [Roq97] and telecommunications [BB09].

$\text{Gab}(\xi)$ the Gabriel graph of ξ . Its vertex set is ξ and there is an edge between $u, v \in \xi$ if the ball of diameter $[u, v]$ contains no point of ξ in its interior. Note that $\text{Gab}(\xi)$ is a subgraph of $\text{DT}(\xi)$ (see [MS80b]). It contains the Euclidean minimum spanning forest of ξ (in which there is an edge between two points of ξ \mathbf{x} and \mathbf{y} iff there do not exist an integer m and vertices $u_0 = \mathbf{x}, \dots, u_m = \mathbf{y} \in \xi$ such that $\|u_i - u_{i+1}\| < \|\mathbf{x} - \mathbf{y}\|$ for all $i \in \{0, \dots, m - 1\}$, see [AS92]) and the relative neighborhood graph (in which there is an edge between two points x and y of ξ whenever there does not exist a third point that is closer to both x and y than they are to each other). It has for example applications in geography, routing strategies, biology and tumor growth analysis (see [BBD02, GS69, MS80a]).

2.1.3 Conductance function

Given a realization ξ of a point process and an unoriented graph $G(\xi) = (V_{G(\xi)}, E_{G(\xi)})$ obtained from ξ by one of the constructions given above, a *conductance* is a positive function on $E_{G(\xi)}$. For any vertex u set $w(u) := \sum_{v \sim u} C(u, v)$ and $R = 1/C$ the associated *resistance*. Then $(G(\xi), C)$ is an infinite *electrical network* and the random walk on $G(\xi)$ associated with C is the (time homogeneous) Markov chain $(X_n)_n$ with transition probabilities given by:

$$\mathcal{P}[X_{n+1} = v | X_n = u] = \frac{C(u, v)}{w(u)}.$$

We refer to [DS84, LP14] for introductions to electrical networks and random walks associated with conductances. In our context, the cases where C is either constant on the edge set (simple random walk on $G(\xi)$) or given by a decreasing positive function of edge length are of particular interest. Our models fit into the broader context of random walks on random graphs with conductances, but due to the geometry of the problem we need to adapt the existing techniques.

2.1.4 Main results

The obtained random graph is now equipped with the conductance function C possibly depending on the graph structure. Our main result is:

Theorem 2.1. *Let ξ be a homogeneous Poisson point process, a Matérn cluster process or a Matérn hardcore process of type I or II.*

1. *Let $d = 2$. Assume that, for almost any realization of ξ , C is bounded from above by a finite constant whose value possibly depends on ξ . Then for almost any realization of ξ the random walks on $\text{DT}(\xi)$, $\text{Gab}(\xi)$ and $\text{VS}(\xi)$ associated with C are recurrent.*
2. *Let $d \geq 3$. If C is uniformly bounded from below or a decreasing positive function of the edge length, then for almost any realization of ξ the random walks on $\text{DT}(\xi)$, $\text{Gab}(\xi)$ and $\text{VS}(\xi)$ associated with C are transient.*

Moreover, (1) holds if ξ is a stationary determinantal point process in \mathbb{R}^2 .

Note that, in (1), the bounded conductances case follows immediately from the unweighted case by Rayleigh monotonicity principle and we can restrict our attention to this last case in the proofs.

To the best of our knowledge, recurrence or transience of random walks on this kind of graphs has been sparsely considered in the literature. Only in the unpublished manuscript [ABS05] Addario-Berry and Sarkar announced similar results in the setting of simple random walks on the Delaunay triangulation generated by a PPP and they noticed that their method can be applied to more general point processes. Their proofs relied on a deviation result for the so-called *stabbing number* of $\text{DT}(\xi)$ contained in a second unpublished manuscript (*The slicing number of a Delaunay triangulation* by Addario-Berry, Broutin, and Devroye). This last work is unfortunately unavailable. Note that the deviation result for the *stabbing number* has been proved since then in [PR12]. We develop a new method, which avoids the use of such a strong estimate and is thus more tractable. This allows us to obtain recurrence and transience results for a large class of point processes and geometric graphs.

Besides, several works show that random walks on distributional limits of finite rooted planar random graph are almost surely recurrent (see Benjamini and Schramm [BS01] and Gurel-Gurevich and Nachmias [GGN13]). Let us briefly explain how the results of [GGN13] could be used to obtain the recurrence of simple random walks on Delaunay triangulations generated by Palm measures associated with point processes in the plane. Given ξ and

n , consider the sub-graph $G_n(\xi)$ of $\text{DT}(\xi)$ with vertex set given by $V_n(\xi) := \{\mathbf{x} \in \xi : \text{Vor}_\xi(\mathbf{x}) \cap [-n, n]^2 \neq \emptyset\}$ and with edge set induced by the edges of $\text{DT}(\xi)$. If the point process is stationary, one could obtain Delaunay triangulations generated by the Palm version of point processes rooted at 0 as distributional limits of (G_n, ρ_n) where, for each n , ρ_n is chosen uniformly at random in $V_n(\xi)$. Thanks to the results of Zuyev [Zuy92], we know that the degree of the origin in the Delaunay triangulation generated by the Palm measure of a PPP has an exponential tail. The proof can be adapted in the non-Poissonian case under assumptions similar to **(V)** and **(D₂)** for the Palm version of the point process and an assumption of finite range of dependence. Using a similar construction, one could expect to apply the results of [BS01] to the skeleton of the Voronoi tiling $\text{VS}(\xi)$ in the plane in which each vertex has degree 3. In this case, one must consider the Palm measure of the point process of the vertices of $\text{VS}(\xi)$ on which there is no information. The main problem of this approach is that it provides a recurrence result for random walks on the Palm version of the point process of VS -vertices. It has no clear connection with the initial point process of the nuclei of the Voronoi cells. We do think that the recurrence criterion stated below is well adapted to the particular geometric graphs that we consider in this paper.

In the sequel, we will actually prove Theorem 2.2 which implies Theorem 2.1 and deals with general point processes as described in Subsection 2.1.1:

Theorem 2.2. *Let ξ be a stationary simple point process in \mathbb{R}^d almost surely in general position.*

1. *Let $d = 2$. Assume that ξ satisfies **(V)** and **(D₂)**. If C is uniformly bounded from above, then for almost any realization of ξ the random walks on $\text{DT}(\xi)$, $\text{Gab}(\xi)$ and $\text{VS}(\xi)$ associated with C are recurrent.*
2. *Let $d \geq 3$. Assume that ξ satisfies **(V)**, **(D₃₊)** and **(FR_k)**. If C is uniformly bounded from below or a decreasing positive function of the edge length, then for almost any realization of ξ the random walks on $\text{DT}(\xi)$ and $\text{VS}(\xi)$ associated with C are transient.*

*If in addition ξ satisfies **(ND)**, the same conclusion holds on $\text{Gab}(\xi)$.*

Remark 2.3. *The stationarity assumption is not required. Indeed, one can derive similar results when the underlying point process is not stationary and **(V)**, **(D₂)** and **(D₃₊)** are replaced by:*

(V') *There exists a constant c'_1 such that for L large enough:*

$$\mathcal{P}\left[\#\left(a + [0, L]^d \cap \xi\right) = 0\right] \leq e^{-c'_1 L^d}, \quad \forall a \in \mathbb{R}^d.$$

(D'₂) *If $d = 2$, there are constants c'_2, c'_3 such that for L, l large enough:*

$$\mathcal{P}\left[\#\left(a + \left([0, L] \times [0, l]\right) \cap \xi\right) \geq c'_2 L l\right] \leq e^{-c'_3 L l}, \quad \forall a \in \mathbb{R}^d.$$

(D'3+) If $d \geq 3$, there exists c'_4 such that for L large enough and all $m > 0$:

$$\mathcal{P}\left[\#\left(a + [0, L]^d \cap \xi\right) \geq m\right] \leq e^{c'_4 L^{d-m}}, \quad \forall a \in \mathbb{R}^d.$$

Let us also point out that conditions **(V)**, **(D3+)** and **(FR_k)** can be replaced by the domination assumption (3) of Criterion 2.8 for the processes of good boxes defined in Subsections 2.5.1 and 2.5.3. In particular, this allows to relax the finite range of dependence assumption and to obtain results for point processes with good mixing properties (see also Remark 2.9).

2.1.5 Outline of the paper

The use of the theory of electrical networks allows us to derive recurrence and transience criteria, namely Criteria 2.4 and 2.8, which are well suited to the study of random walks on random geometric graphs embedded in the Euclidean space. These general criteria are proved in a concise way in Sections 2.2 and 2.3. Sections 2.4 and 2.5 are devoted to the proof of Theorem 2.2. They constitute the heart of the paper and rely on arguments from stochastic geometry. In Section 2.6, assumptions of Theorem 2.2 are proved to hold for the point processes considered in Theorem 2.1.

2.2 A recurrence criterion

In this section, we give a recurrence criterion for random walks on a graph $G = (V_G, E_G)$ embedded in \mathbb{R}^d and equipped with a conductance C . In the sequel G is assumed to be connected, infinite and locally finite. The criterion is established on deterministic graphs, but it gives a way to obtain almost sure results in the setting of random graphs (see Section 2.4). Note that it is a slight generalisation of the proof of [ABS05, Theorem 4] and is close in spirit to the Nash-Williams criterion (see [LP14, §2.5]).

Let us define:

$$\begin{aligned} A_i &:= \{x \in \mathbb{R}^d : i - 1 \leq \|x\|_\infty < i\}, \\ B_i &:= [-i, i]^d, \\ Ed_G(i) &:= \{e = (u, v) \in E_G : u \in B_i, v \notin B_i\}. \end{aligned}$$

Criterion 2.4. Assume that the conductance C is bounded from above and there exist functions L and N defined on the set \mathbb{N}^* of positive integers and with values in the set \mathbb{R}_+^* of positive real numbers such that:

1. $\sum_i \frac{1}{L(i)N(i)} = +\infty$,
2. the length of any edge in $Ed_G(i)$ is less than $L(i)$,
3. the number of edges in $Ed_G(i)$ is less than $N(i)$.

Then the random walk on G with conductance C is recurrent.

Remark 2.5. 1. One can check that $w(u) = \sum_{v \sim u} C(u, v)$, $u \in V_G$, is a reversible measure of infinite mass, hence the random walk is null recurrent.

2. The estimates appearing in the proof of this criterion lead to lower bounds for the effective resistance between the closest point x_0 to the origin and $\xi \cap B_n^c$. This quantity is of interest because it is related to the expected number of visits of the random walk at x_0 before leaving B_n . For each example given in this paper, we could obtain lower bounds to be of order $\log(\log(n))$; this is not expected to be the correct order. It seems credible that the correct order is, as for simple walks on \mathbb{Z}^2 , $\log(n)$.

Proof. The basic idea is inherited from electrical network theory. One can define the effective resistance to infinity of the network which is known to be infinite if and only if the associated random walk is recurrent (see [LP14, §2.2]). We will reduce the network so that the resistance of any edge does not increase. Thanks to the Rayleigh monotonicity principle, the effective resistance to infinity does not increase either. It then suffices to show that the reduced network has infinite resistance to infinity.

First each edge $e = (u, v)$ with $u \in A_{i_1}, v \in A_{i_2}, i_1 < i_2$ is cut into $j = i_2 - i_1$ resistors connected in series with endpoints in consecutive annuli A_{i_1}, \dots, A_{i_2} each one having resistance $(jC(e))^{-1}$. Secondly, points of a annulus A_i are merged together into a single point a_i . We hence obtain a new network with vertices $(a_i)_{i \geq i_0}$ where i_0 is the lowest index such that A_{i_0} contains a vertex of the original graph G . The resistance to infinity of the new network is lower than the original one and is equal to $\sum_{i=i_0}^{+\infty} r_i$, where r_i stands for the effective resistance between a_i and a_{i+1} .

It remains to show that:

$$\sum_{i=i_0}^{+\infty} r_i = +\infty.$$

Let us subdivide $Ed_G(i)$ into the following subsets:

$$Ed_G(i, j) := \left\{ e = (u, v) \in E_G : u \in A_{i_1}, v \in A_{i_2}, i_1 \leq i < i_2, i_2 - i_1 = j \right\}.$$

Note that an edge $e \in Ed_G(i, j)$ provides a conductance of $jC(e)$ between a_i and a_{i+1} in the new network. Thanks to the usual reduction rules and conditions (2) and (3), we get:

$$\begin{aligned} \frac{1}{r_i} &= \sum_{j=1}^{+\infty} \sum_{e \in Ed_G(i, j)} jC(e) \leq (\sup C) \sum_{j=1}^{+\infty} j \# Ed_G(i, j) \\ &\leq (\sup C) \sum_{j=1}^{L(i)} j \# Ed_G(i, j) \leq (\sup C) L(i) \sum_{j=1}^{L(i)} \# Ed_G(i, j) \\ &\leq (\sup C) L(i) \sum_{j=1}^{+\infty} \# Ed_G(i, j) \leq (\sup C) L(i) N(i). \end{aligned}$$

This concludes the proof since:

$$\sum_{i=i_0}^{+\infty} r_i \geq \sum_{i=i_0}^{+\infty} \frac{1}{(\sup C)L(i)N(i)}$$

and the r.h.s. is infinite by condition (1). □

2.3 A transience criterion

In this section, the graph $G = (V_G, E_G)$ is obtained from a point process in \mathbb{R}^d and is equipped with a conductance C (and therefore a resistance R) on E_G . The key idea is to combine *discretization techniques* with a *rough embedding method*.

Let us define *rough embeddings* for unoriented networks in a similar way to [LP14, §2.6]:

Definition 2.6. *Let H and H' be two networks with resistances r and r' .*

We say that a map $\phi : V_H \rightarrow V_{H'}$ is a rough embedding from (H, r) to (H', r') if there exist $\alpha, \beta < +\infty$ and a map Φ from (unoriented) edges of H to (unoriented) paths in H' such that:

1. *for every edge $(u, v) \in E_H$, $\Phi(u, v)$ is a non-empty simple (i.e. with no repeating vertices) path of edges of H' between $\phi(u)$ and $\phi(v)$ with*

$$\sum_{e' \in \Phi(u, v)} r'(e') \leq \alpha r(u, v);$$

2. *any edge $e' \in E_{H'}$ is in the image under Φ of at most β edges of H .*

A version of [LP14, Theorem 2.17] without orientation is needed to establish the criterion. Lyons and Peres attribute this result to Kanai [Kan86].

Theorem 2.7 (see [LP14]). *If there is a rough embedding from (H, r) to (H', r') and (H, r) is transient, then (H', r') is transient.*

To deduce Theorem 2.7 from [LP14, Theorem 2.17], one can do the following. We construct oriented graphs \vec{H} and \vec{H}' from H and H' such that if there is an unoriented edge between x and y in H (resp. in H') with resistance $r(\{x, y\})$, there are oriented edges from x to y and from y to x in \vec{H} (resp. in \vec{H}') with resistance $r(\{x, y\})$. It is then easy to see that if there is a rough embedding from (H, r) to (H', r') in the sense of Definition 2.6, there is also a rough embedding from (\vec{H}, r) to (\vec{H}', r') according to the definition given in [LP14].

Let us divide \mathbb{R}^d into boxes of side $M \geq 1$:

$$B_{\mathbf{z}} = B_{\mathbf{z}}^M := M\mathbf{z} + \left[-\frac{M}{2}, \frac{M}{2} \right)^d, \mathbf{z} \in \mathbb{Z}^d.$$

We will prove the following criterion:

Criterion 2.8. *If $d \geq 3$ and if one can find a subset of boxes, called good boxes, such that:*

1. *in each good box $B_{\mathbf{z}}$, one can choose a reference vertex $v_{\mathbf{z}} \in B_{\mathbf{z}} \cap V_G$,*
2. *there exist K, L such that to each pair of neighboring good boxes $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$, one can associate a path $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$ in G between the respective reference vertices $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ of these boxes satisfying:*
 - (a) $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2}) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$,
 - (b) *any edge of $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$ has resistance at most K ,*
 - (c) *the length of $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$ in the graph distance is bounded by L ,*
3. *the process $X = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\} := \{\mathbb{1}_{B_{\mathbf{z}}} \text{ is good}, \mathbf{z} \in \mathbb{Z}^d\}$ stochastically dominates a supercritical independent Bernoulli site percolation process on \mathbb{Z}^d ,*

then the random walk on G with resistance R is almost surely transient.

Remark 2.9. *Thanks to [LSS97, Theorem 0.0] (see also [Gri89, Theorem (7.65)]), in order to show (3), it suffices to check that $X = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ is a k -dependent process so that $\mathcal{P}[X_{\mathbf{z}} = 1] \geq p^* \in]0, 1[$, where p^* depends on k and d but not on M . The results of Liggett, Schonmann and Stacey are still valid for processes which are not k -dependent but have correlations decaying rapidly. This could be used to relax the finite range of dependence assumption for the point process in Theorem 2.2. More precisely, in order to ensure (3), it is enough to verify that there exists $k > 0$ such that:*

$$\mathcal{P}\left[X_{\mathbf{z}} = 1 \mid \{X_{\mathbf{z}'}\}_{\|\mathbf{z}' - \mathbf{z}\| \geq k}\right] \geq p$$

with p as close to 1 as desired if the parameters of good boxes are well chosen. Unfortunately, we do not find any example of point process without a finite range of dependance which satisfies the assumptions of Criterion 2.8.

Proof. Thanks to condition (3), one can define a random field $(\sigma_1, \sigma_2) \in \{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d}$, where σ_1 has the distribution of the supercritical (independent) Bernoulli site percolation process, σ_2 has the law of X , and the pair satisfies $\sigma_1 \leq \sigma_2$ almost surely. Let π_∞ stands for the (a.s. unique) infinite percolation cluster in σ_1 . It is known that the simple random walk on π_∞ is a.s. transient when $d \geq 3$ (see [GKZ93]). By Theorem 2.7, it is enough to exhibit an a.s. rough embedding from π_∞ (with resistance 1 on each edge) to (G, R) .

By stochastic domination, for any open site $\mathbf{z} \in \pi_\infty$ the corresponding box $B_{\mathbf{z}}$ is good. We set $\phi(\mathbf{z}) := v_{\mathbf{z}}$ the reference vertex of $B_{\mathbf{z}}$ given by (1). Fix $\mathbf{z}_1 \sim \mathbf{z}_2 \in \pi_\infty$. Then $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$ are two neighboring good boxes and one can find a path $\Phi(\mathbf{z}_1, \mathbf{z}_2)$ between $v_{\mathbf{z}_1} = \phi(\mathbf{z}_1)$ and $v_{\mathbf{z}_2} = \phi(\mathbf{z}_2)$ fully included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Note that $\Phi(\mathbf{z}_1, \mathbf{z}_2)$ can be assumed to be simple and that by (2)(b),(c), one has:

$$\sum_{e \in \Phi(\mathbf{z}_1, \mathbf{z}_2)} R(e) \leq \alpha := KL.$$

Moreover, an edge of E_G is in the image of at most $\beta := 2d$ edges of π_∞ since a good box has at most $2d$ neighboring good boxes and $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2}) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. \square

2.4 Recurrence in dimension 2

Let us assume that ξ is a stationary simple point process, almost surely in general position and satisfying **(V)** and **(D₂)**. The aim of this section is to prove the case $d = 2$ of Theorem 2.2, i.e. almost sure recurrence of walks on $\text{VS}(\xi)$, $\text{DT}(\xi)$ and $\text{Gab}(\xi)$.

Since $\text{Gab}(\xi)$ is a subgraph of $\text{DT}(\xi)$, recurrence on $\text{DT}(\xi)$ implies recurrence on $\text{Gab}(\xi)$ by Rayleigh monotonicity principle. Consequently, we only need to find functions $L_{\text{DT}(\xi)}$, $L_{\text{VS}(\xi)}$ and $N_{\text{DT}(\xi)}$, $N_{\text{VS}(\xi)}$ so that assumptions of Criterion 2.4 are satisfied for $\text{DT}(\xi)$ and $\text{VS}(\xi)$ respectively. In fact, for almost any realization of ξ , $L(i)$ can be chosen to be of order $\sqrt{\log(i)}$ and $N(i)$ to be of order $i\sqrt{\log(i)}$.

2.4.1 Delaunay triangulation case

We first get an upper bound on the length of the edges in $\text{Ed}_{\text{DT}(\xi)}(i)$ (i.e. the set of edges with only one endpoint in B_i) via an extended version of [ABS05, Lemma 1].

We write $\mathcal{A}_{\text{DT}}(i)$ for the event ‘there exists an edge of $\text{Ed}_{\text{DT}(\xi)}(i)$ with length greater than $8c_1^{-\frac{1}{2}}\sqrt{\log i}$ ’ where c_1 is the constant appearing in **(V)**, and $\mathcal{B}_{\text{DT}}(i, j)$ for the event ‘there exists an edge of $\text{Ed}_{\text{DT}(\xi)}(i)$ with length between $8c_1^{-\frac{1}{2}}\sqrt{j \log i}$ and $8c_1^{-\frac{1}{2}}\sqrt{(j+1) \log i}$ ’.

Lemma 2.10. *Let ξ be a stationary simple point process, almost surely in general position and such that **(V)** holds. Then there exists a constant $c_5 > 0$ such that for i large enough:*

$$\mathcal{P}[\mathcal{A}_{\text{DT}}(i)] \leq \frac{c_5}{i^2}.$$

Proof. Assume that $\mathcal{B}_{\text{DT}}(i, j)$ occurs and let e be an edge of length between $8c_1^{-\frac{1}{2}}\sqrt{j \log i}$ and $8c_1^{-\frac{1}{2}}\sqrt{(j+1) \log i}$ having an endpoint in B_i . This edge is fully contained in $B_{i'}$, where $i' := \lceil i + 8c_1^{-\frac{1}{2}}\sqrt{(j+1) \log i} \rceil$. Let $m \leq i'$ be as large as possible such that $s := \lfloor 2c_1^{-\frac{1}{2}}\sqrt{j \log i} \rfloor$ divides m and set $l := m/s$.

Now, divide B_m into l^2 squares Q_1, \dots, Q_{l^2} of side s . Let Δ be one of the two Delaunay triangles having e as an edge. The circumscribed sphere of Δ contains one of the two half disks \mathfrak{D} with diameter $\|e\|$ located on one side or the other of e . Since the circumscribed sphere of the Delaunay triangle Δ contains no point of ξ in its interior, \mathfrak{D} contains no point of ξ in its interior. Note that \mathfrak{D} is included in $B_{i'}$ because e has an endpoint in B_i

and has length at most $8c_1^{-\frac{1}{2}}\sqrt{(j+1)\log i}$. Thanks to the choice of s , \mathfrak{D} contains one of the squares Q_1, \dots, Q_{l^2} , say Q_k . So $Q_k \cap \xi$ is empty. By stationarity of ξ , we obtain:

$$\mathcal{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq l^2 \mathcal{P}[Q_1 \cap \xi = \emptyset] \leq c_6 i^2 \mathcal{P}[Q_1 \cap \xi = \emptyset], \quad (2.4.1)$$

for some constant $c_6 > 0$. But, thanks to **(V)**, for i large enough, one has:

$$\mathcal{P}[Q_1 \cap \xi = \emptyset] \leq \mathcal{P}\left[\left[0, 2c_1^{-\frac{1}{2}}\sqrt{j\log i}\right]^2 \cap \xi = \emptyset\right] \leq e^{-4j\log i} = \left(\frac{1}{i^4}\right)^j.$$

Hence, with (2.4.1):

$$\mathcal{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq c_6 i^2 \left(\frac{1}{i^4}\right)^j.$$

Finally, for i large enough, one obtains:

$$\mathcal{P}[\mathcal{A}_{\text{DT}}(i)] \leq \sum_{j=1}^{+\infty} \mathcal{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq c_6 i^2 \sum_{j=1}^{+\infty} \left(\frac{1}{i^4}\right)^j \leq \frac{c_6 i^2}{i^4 - 1} \leq \frac{c_5}{i^2}.$$

□

Thanks to the Borel-Cantelli lemma, for almost any realization of ξ , $\mathcal{A}_{\text{DT}}(i)$ holds for only finitely many i . Thus, for almost any realization of ξ , one can choose $L_{\text{DT}}(i)$ to be of order $\sqrt{\log i}$ in Criterion 2.4.

We will see that we can choose $N_{\text{DT}}(i)$ to be of order $i\sqrt{\log i}$. To do so, we show the following lemma and conclude with the Borel-Cantelli lemma as before.

Lemma 2.11. *Let ξ be a stationary simple point process, almost surely in general position and such that **(V)** and **(D₂)** hold.*

Then, there exists a constant $c_7 > 0$ such that for i large enough:

$$\mathcal{P}[\mathcal{C}_{\text{DT}}(i)] \leq \frac{c_7}{i^2},$$

where $\mathcal{C}_{\text{DT}}(i)$ is the event ‘ $\#(Ed_{\text{DT}(\xi)}(i)) \geq 384c_1^{-\frac{1}{2}}c_2i\sqrt{\log i}$ ’.

Proof. Note that:

$$\mathcal{P}[\mathcal{C}_{\text{DT}}(i)] = \mathcal{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)] + \mathcal{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] \leq \mathcal{P}[\mathcal{A}_{\text{DT}}(i)] + \mathcal{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c],$$

with $\mathcal{A}_{\text{DT}}(i)$ as in Lemma 2.10. So, it remains to show:

$$\mathcal{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] \leq \frac{c_8}{i^2},$$

for some constant c_8 .

On the event $\mathcal{A}_{\text{DT}}(i)^c$, edges in $Ed_{\text{DT}(\xi)}(i)$ have lengths at most $8c_1^{-\frac{1}{2}}\sqrt{\log i}$, thus these edges are fully included in the annulus $R(i) := [-i - 8c_1^{-\frac{1}{2}}\sqrt{\log i}, i + 8c_1^{-\frac{1}{2}}\sqrt{\log i}]^2 \setminus [-i + 8c_1^{-\frac{1}{2}}\sqrt{\log i}, i - 8c_1^{-\frac{1}{2}}\sqrt{\log i}]^2$. The restriction of $\text{DT}(\xi)$ to $R(i)$ is a planar graph with $\#(\xi \cap R(i))$ vertices. Thanks to a corollary of Euler's formula (see [Bol98, Theorem 16, p.22]), it has at most $3\#(\xi \cap R(i)) - 6$ edges. Thus $\#(Ed_{\text{DT}(\xi)}(i))$ is bounded from above by $3\#(\xi \cap R(i))$.

So, with **(D₂)** and for i large enough:

$$\begin{aligned} \mathcal{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] &\leq \mathcal{P}\left[\#(\xi \cap R(i)) \geq 128c_1^{-\frac{1}{2}}c_2i\sqrt{\log i}\right] \\ &\leq 4\mathcal{P}\left[\#(\xi \cap ([0, 2i] \times [0, 16c_1^{-\frac{1}{2}}\sqrt{\log i}])) \geq 32c_1^{-\frac{1}{2}}c_2i\sqrt{\log i}\right] \\ &\leq 4e^{-32c_1^{-\frac{1}{2}}c_3i\sqrt{\log i}} \leq \frac{c_8}{i^2}. \end{aligned}$$

□

2.4.2 Skeleton of the Voronoi tiling case

In order to estimate the lengths of edges in $Ed_{\text{VS}(\xi)}(i)$, the following analogue of Lemma 2.10 is needed:

Lemma 2.12. *Let ξ be a stationary simple point process, almost surely in general position and such that **(V)** holds.*

Then, there exists a positive constant c_9 such that for i large enough:

$$\mathcal{P}[\mathcal{A}_{\text{VS}}(i)] \leq \frac{c_9}{i^2}$$

where $\mathcal{A}_{\text{VS}}(i)$ is the event ‘there exists an edge of $Ed_{\text{VS}(\xi)}(i)$ with length greater than $4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ ’.

Proof. Fix $\varepsilon_0 > 0$ and set $\overline{B}_i := B_i + B(0, 4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon_0)$ where $B(x, r)$ stands for the Euclidean ball centered at x and of radius r . We say that $\mathcal{B}_{\text{VS}}(i)$ holds if, when \overline{B}_i is covered with $O(i^2/\log i)$ disjoint squares of side $2c_1^{-\frac{1}{2}}\sqrt{\log i}$ and such that one has a corner at $(-i, -i)$, each of these squares contains a point of ξ .

We will show that, on $\mathcal{B}_{\text{VS}}(i)$, every Voronoi cell intersecting ∂B_i is contained in a ball centered at its nucleus and of radius $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ (see Figure 2.1). This will imply that $\mathcal{B}_{\text{VS}}(i) \subset \mathcal{A}_{\text{VS}}(i)^c$.

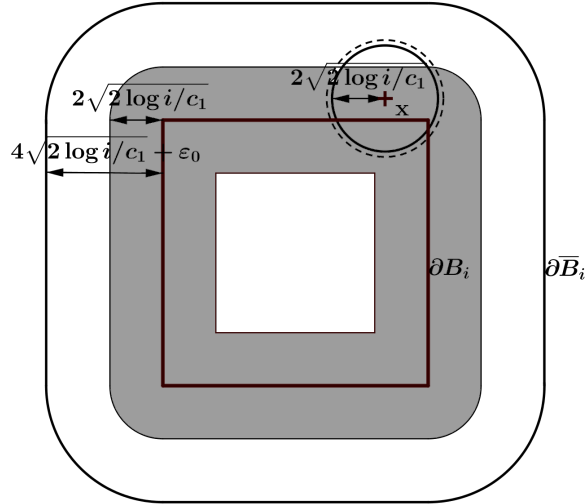


Figure 2.1: If $\mathcal{B}_{\text{VS}}(i)$ holds, nuclei of Voronoi cells intersecting lie in the grey region. Since the dashed circle is included in \bar{B}_i , points of this circle are separated by at most $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ from the respective nuclei of their Voronoi cells. Hence, the Voronoi cell with nucleus $\mathbf{x} \in \xi$ is contained in $B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i})$.

Let us assume that $\mathcal{B}_{\text{VS}}(i)$ holds. Since there is at least one point of ξ in each of the squares of side $2c_1^{-\frac{1}{2}}\sqrt{\log i}$ covering \bar{B}_i , any point in \bar{B}_i is separated by at most $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ from the nucleus of its Voronoi cell (of each cell in which it is if it belongs to the borders of two or three Voronoi cells). In particular, nuclei of Voronoi cells intersecting ∂B_i and $\partial \bar{B}_i$ itself are separated by at most $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$. Fix \mathbf{x} nucleus of a Voronoi cell intersecting ∂B_i and note that, for $0 < \varepsilon < \varepsilon_0$, $B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon) \subset \bar{B}_i$. Hence, points in $\partial B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon)$ are within a distance of at most $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ from nuclei of their respective Voronoi cells which cannot be \mathbf{x} . Thus $\text{Vor}_\xi(\mathbf{x}) \subset B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i})$. So, if $\mathcal{A}_{\text{VS}}(i)$ holds, $\mathcal{B}_{\text{VS}}(i)$ fails.

Finally, with **(V)**, for i large enough:

$$\begin{aligned} \mathcal{P}[\mathcal{A}_{\text{VS}}(i)] &\leq \mathcal{P}[\mathcal{B}_{\text{VS}}(i)^c] \leq c_9 \frac{i^2}{\log i} \mathcal{P}[[0, 2c_1^{-\frac{1}{2}}\sqrt{\log i}]^2 \cap \xi = \emptyset] \\ &\leq c_9 i^2 e^{-4 \log i} = \frac{c_9}{i^2}. \end{aligned}$$

□

By the Borel-Cantelli lemma, for almost any realization of ξ , $\mathcal{A}_{\text{VS}}(i)$ holds only finitely many times. Thus one can choose $L_{\text{VS}(\xi)}$ to be of order $\sqrt{\log i}$.

It remains to show that one can choose $N_{\text{VS}}(i)$ to be of order $i\sqrt{\log i}$. To do this, we state the following lemma and conclude as usual with the Borel-Cantelli lemma.

Lemma 2.13. *Let ξ be a stationary simple point process, almost surely in general position and such that **(V)** and **(D₂)** hold.*

Then there exists a positive constant c_{10} such that for i large enough:

$$\mathcal{P}[\mathcal{C}_{\text{VS}}(i)] \leq \frac{c_{10}}{i^2},$$

where $\mathcal{C}_{\text{VS}}(i)$ is the event ‘ $\#(Ed_{\text{VS}(\xi)}(i)) \geq 64\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}$ ’.

Proof. One has:

$$\mathcal{P}[\mathcal{C}_{\text{VS}}(i)] = \mathcal{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)^c] + \mathcal{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] \leq \mathcal{P}[\mathcal{B}_{\text{VS}}(i)^c] + \mathcal{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)],$$

with $\mathcal{B}_{\text{VS}}(i)$ as in the proof of Lemma 2.12. It remains to show that:

$$\mathcal{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] \leq \frac{c_{11}}{i^2},$$

for some constant $c_{11} > 0$

One can see that edges in $Ed_{\text{VS}(\xi)}(i)$ intersect ∂B_i and belong to boundaries of Voronoi cells intersecting ∂B_i . Note that, since Voronoi cells are convex, each of the four line segments constituting ∂B_i intersects at most two sides of a given Voronoi cell and the number of edges in $Ed_{\text{VS}(\xi)}(i)$ intersecting one of these line segments is bounded by the number of Voronoi cells intersecting this line segment. As noticed during the proof of Lemma 2.12, on the event $\mathcal{B}_{\text{VS}}(i)$, points of ∂B_i are within a distance of at most $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ from nuclei of their respective Voronoi cells. Thus, if i is large enough, a given Voronoi cell intersects at most two of the four line segments constituting ∂B_i and nuclei of cells intersecting ∂B_i are in the annulus:

$$R(i) := \left[-i - 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}, i + 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} \right]^2 \setminus \left[-i + 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}, i - 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} \right]^2.$$

So, on $\mathcal{B}_{\text{VS}}(i)$, for i large enough, the number of edges in $Ed_{\text{VS}(\xi)}(i)$ is bounded by twice the number of Voronoi cells intersecting ∂B_i which is less than $2\#(\xi \cap R(i))$.

Thus, with **(D₂)**, for i large enough:

$$\begin{aligned} \mathcal{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] &\leq \mathcal{P}[\#(\xi \cap R(i)) \geq 32\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}] \\ &\leq 4\mathcal{P}[\#(\xi \cap ([0, 2i] \times [0, 4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}]]) \geq 8\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}] \\ &\leq 4e^{-8\sqrt{2}c_3c_1^{-\frac{1}{2}}i\sqrt{\log i}} \leq \frac{c_{11}}{i^2}. \end{aligned}$$

□

2.5 Transience in higher dimensions

This section is devoted to the proof of the second part of Theorem 2.2, i.e. almost sure transience of walks on $\text{VS}(\xi)$, $\text{DT}(\xi)$ and $\text{Gab}(\xi)$ under the assumptions on ξ .

The assumption that ξ has almost surely no descending chain is only used in the proof of transience on $\text{Gab}(\xi)$. Since $\text{Gab}(\xi)$ is a subgraph of $\text{DT}(\xi)$, transience on $\text{Gab}(\xi)$ implies transience on $\text{DT}(\xi)$. Actually, one can directly prove transience on $\text{DT}(\xi)$ without this additional condition. We give a separate proof of transience on $\text{DT}(\xi)$ without this additional condition.

In each case, we define ‘good boxes’ and use Criterion 2.8 for $\text{Gab}(\xi)$ and $\text{VS}(\xi)$: we construct paths needed in Criterion 2.8 and check that $\mathcal{P}[X_{\mathbf{z}} = 1]$ is large enough if parameters are well chosen. Thanks to the finite range of dependence assumptions (say of range k) and to the definitions of ‘good boxes’ it will be clear that $\{X_{\mathbf{z}}\}$ is a k -dependent Bernoulli process on \mathbb{Z}^d .

2.5.1 Skeleton of the Voronoi tiling case

Good boxes

For $M \geq 1$ to be determined later, consider a partition of \mathbb{R}^d into boxes of side M :

$$B_{\mathbf{z}} = B_{\mathbf{z}}^M := M\mathbf{z} + \left[-\frac{M}{2}, \frac{M}{2} \right)^d, \mathbf{z} \in \mathbb{Z}^d.$$

We say that a box $B_{\mathbf{z}}$ is M -good for $\text{VS}(\xi)$ if the following conditions are satisfied:

- i- $\#(B_{\mathbf{z}} \cap \xi) \leq 2c_4 M^d$,
- ii- when $B_{\mathbf{z}}$ is (regularly) cut into $\alpha_d^d := (6\lceil\sqrt{d}\rceil)^d$ sub-boxes $b_i^{\mathbf{z}}$ of side M/α_d , each of these sub-boxes contains at least one point of ξ .

Construction of paths

First of all, for each good box $B_{\mathbf{z}_i}$, one can fix a reference vertex $v_{\mathbf{z}_i}$ on the boundary of the Voronoi cell of $M\mathbf{z}_i$ as required by condition (1) in Criterion 2.8. Note that, thanks to -ii- in the definition of good boxes, the cell containing $M\mathbf{z}_i$ is included in $B_{\mathbf{z}_i}$ so that $v_{\mathbf{z}_i} \in B_{\mathbf{z}_i}$.

Consider two neighboring good boxes $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$. One must show that there exists a path in $\text{VS}(\xi)$ between $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ so that condition (2) of Criterion 2.8 holds. To do so, note that one can find a self-avoiding path between $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ with edges belonging to boundaries of the Voronoi cells crossing the line segment $[M\mathbf{z}_1, M\mathbf{z}_2]$. Clearly, such a path is contained in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ as soon as Voronoi cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ are included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Since sub-boxes $b_1^{\mathbf{z}_1}, \dots, b_{\alpha_d^d}^{\mathbf{z}_1}, b_1^{\mathbf{z}_2}, \dots, b_{\alpha_d^d}^{\mathbf{z}_2}$ are non-empty (of points of ξ), points in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ are distant by at most $\sqrt{d}M/\alpha_d$ from nuclei of their respective Voronoi cell. As

in the proof of Lemma 2.12, one can see that the Voronoi cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ are included in $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 2\sqrt{d}M/\alpha_d) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Actually, since there is at least one point of ξ in each of the sub-boxes of side M/α_d contained in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$, nuclei of Voronoi cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ are within a distance of at most $\sqrt{d}M/\alpha_d$ from this line segment. Fix \mathbf{x} the nucleus of a Voronoi cell intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ and note that, for $\varepsilon > 0$ small enough, $B(\mathbf{x}, \sqrt{d}M/\alpha_d + \varepsilon) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Hence, points in $\partial B(\mathbf{x}, \sqrt{d}M/\alpha_d + \varepsilon)$ are within a distance of at most $\sqrt{d}M/\alpha_d$ from the nuclei of their respective Voronoi cells which cannot be \mathbf{x} . Repeating the arguments above with $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 2\sqrt{d}M/\alpha_d)$ instead of $[M\mathbf{z}_1, M\mathbf{z}_2]$, one obtains that Voronoi cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 2\sqrt{d}M/\alpha_d)$ have their nuclei in $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 3\sqrt{d}M/\alpha_d)$. Since Voronoi cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ are contained in $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 2\sqrt{d}M/\alpha_d)$, it follows that nuclei of Voronoi cells which are neighbors of cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$ are in $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 3\sqrt{d}M/\alpha_d) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Note that the path between $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ has chemical length (i.e. for the graph distance) bounded by the number of vertices on the boundaries of Voronoi cells crossing $[M\mathbf{z}_1, M\mathbf{z}_2]$. This is less than the number of Voronoi cells crossing $[M\mathbf{z}_1, M\mathbf{z}_2]$ times the maximal number of vertices on such a cell. Since these cells are included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ and $B_{\mathbf{z}_1}, B_{\mathbf{z}_2}$ are good, there are no more than $4c_4M^d$ cells intersecting $[M\mathbf{z}_1, M\mathbf{z}_2]$. Each of these cells has at most $4c_4M^d$ neighboring cells whose nuclei are in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. The total number of vertices on the boundary of such a cell is generously bounded by $\binom{4c_4M^d}{d}$. Indeed, any of these vertices is obtained as the intersection of d bisecting hyperplanes separating the cell from one of its neighbors (which are at most $4c_4M^d$). So, one can choose $L := 4c_4M^d \binom{4c_4M^d}{d}$ in Criterion 2.8 (2)(c).

Finally, if C is uniformly bounded from below set $K := \max 1/C$ in Criterion 2.8 (2)(b). If C is given by a decreasing positive function φ of edge length, set $K := 1/\varphi(\sqrt{d} + 3M)$. Indeed, by construction, any edge in the path from $v_{\mathbf{z}_1}$ to $v_{\mathbf{z}_2}$ has (Euclidean) length at most $\sqrt{d} + 3M$ since it is included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$.

$\mathcal{P}[X_{\mathbf{z}} = 1]$ is large enough

It remains to show that if M is appropriately chosen (3) in Criterion 2.8 is satisfied. Since ξ has a finite range of dependence k , assuming that $M \geq 2$, the process $\{X_{\mathbf{z}}\}$ is k -dependent, so, as noticed in Remark 2.9, it suffices to show that for M large enough:

$$\mathcal{P}[X_{\mathbf{z}} = 1] \geq p^*$$

where $p^* = p^*(d, k) < 1$ is large enough to ensure that $\{X_{\mathbf{z}}\}$ dominates site percolation on \mathbb{Z}^d .

Indeed, with **(V)** and **(D₃₊)**, one has:

$$\begin{aligned} \mathcal{P}[X_{\mathbf{z}} = 0] &\leq \mathcal{P}\left[\#(B_{\mathbf{z}} \cap \xi) > 2c_4M^d \text{ or one of the } b_i^{\mathbf{z}}\text{s is empty}\right] \\ &\leq \mathcal{P}\left[\#([0, M]^d \cap \xi) > 2c_4M^d\right] + \alpha_d^d \mathcal{P}\left[\#\left(\left[0, \frac{M}{\alpha_d}\right]^d \cap \xi\right) = 0\right] \\ &\leq \exp\left(-c_4M^d\right) + \left(6\lceil\sqrt{d}\rceil\right)^d \exp\left(-c_1\frac{M^d}{6^d\lceil\sqrt{d}\rceil^d}\right) \end{aligned}$$

which is as small as we wish for large M .

2.5.2 Delaunay triangulation case

We only define the ‘good boxes’ and construct paths appearing in Criterion 2.8. The arguments showing that $\mathcal{P}[X_{\mathbf{z}} = 1]$ is large enough are the same as in the case of the skeleton of the Voronoi tiling.

Definition of the good boxes

For $s \in \mathbb{N}^*$, let us part \mathbb{R}^d into boxes of side $M := (2\lceil\sqrt{d}\rceil + 1)s$:

$$B_{\mathbf{z}} = B_{\mathbf{z}}^M := M\mathbf{z} + \left[-\frac{M}{2}, \frac{M}{2}\right]^d, \quad \mathbf{z} \in \mathbb{Z}^d. \quad (2.5.1)$$

Each box $B_{\mathbf{z}}$ is then separated into smaller sub-boxes $b_i^{\mathbf{z}}$ of side s . In the sequel, we will refer to:

$$\bigcup_{j=1}^d \left[M\left(\mathbf{z} - \frac{1}{2}e_j\right), M\left(\mathbf{z} + \frac{1}{2}e_j\right) \right] \text{ where } (e_1, \dots, e_d) \text{ is the standard basis of } \mathbb{R}^d \quad (2.5.2)$$

as the ‘central cross’ of the box $B_{\mathbf{z}}$.

We are now ready to define a notion of ‘good boxes’ which depends only on the behavior of the point process ξ inside the box. A box $B_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{Z}^d$ is called *good* if:

- i- each sub-box $b_i^{\mathbf{z}}$ of side s which intersects the central cross of $B_{\mathbf{z}}$ contains at least one point of ξ ,
- ii- $B_{\mathbf{z}}$ contains at most $2c_4M^d$ points of ξ .

A box $B_{\mathbf{z}}$ is said *bad* if it is not good. We call *reference vertex* of a good box $B_{\mathbf{z}}$ the nucleus $v_{\mathbf{z}} \in \xi$ of the Voronoi cell of $M\mathbf{z}$.

Construction of the paths

Assume that $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$ are two neighboring good boxes. We claim that one can construct a (self-avoiding) path $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ from the reference vertex $v_{\mathbf{z}_1}$ of $B_{\mathbf{z}_1}$ to the reference vertex $v_{\mathbf{z}_2}$ of $B_{\mathbf{z}_2}$ in the Delaunay triangulation of ξ satisfying:

- (a) $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ is fully included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$,
- (b) each edge of $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ has (Euclidean) length at most $2\sqrt{d}s$,
- (c) $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ has (graph) length at most $4c_4M^d$.

To construct $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$, it suffices to link the nuclei of the consecutive Voronoi cells which intersect the line segment $[M\mathbf{z}_1, M\mathbf{z}_2]$. Due to *-i-* in the definition of good boxes, each small sub-box intersecting this line segment contains a point of ξ . Thus, the nucleus of the Voronoi cell containing a point $x \in [M\mathbf{z}_1, M\mathbf{z}_2]$ is within a distance at most $\sqrt{d}s$ from this point x . Hence, such a path $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ satisfies (a) and (b). Moreover, thank to (a) and *-ii-* in the definition of good boxes, the graph length of $\gamma_{\mathbf{z}_1, \mathbf{z}_2}$ is generously bounded by $\#(\xi \cap (B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2})) \leq 2c_4M^d$.

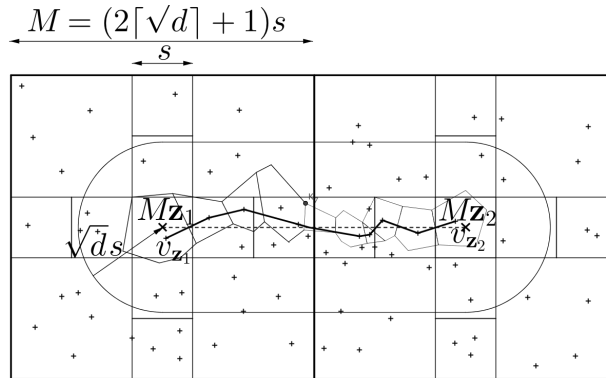


Figure 2.2: Two neighboring good boxes $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$. The path from $v_{\mathbf{z}_1}$ to $v_{\mathbf{z}_2}$ is actually contained in the grey part.

2.5.3 Gabriel graph case

A geometric lemma

We shall state a generalization of [BBD02, Lemma 1] which allows us to control the behavior of paths as needed in Criterion 2.8.

Lemma 2.14. *Let ξ be a locally finite subset of \mathbb{R}^d without descending chain.*

Then, for any $x, y \in \xi$, there exists a path $(x_1 = x, \dots, x_n = y)$ in $\text{Gab}(\xi)$ from x to y such that:

$$\sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2 \leq \|y - x\|^2. \quad (2.5.3)$$

Proof. Assume that ξ is a locally finite subset of \mathbb{R}^d in which there are $x, y \in \xi$ such that there is no path between x and y in $\text{Gab}(\xi)$ satisfying (2.5.3). We fix such x and y and we obtain a contradiction by constructing a descending chain $(u_i)_{i \in \mathbb{N}}$ in ξ . To this end, we prove by induction that for any $i \geq 1$ there exist $u_0, \dots, u_i, z_1^{i+1} \in \xi$ such that:

- i- $\|u_{j+1} - u_j\| < \|u_j - u_{j-1}\|$ for $j = 1, \dots, i - 1$,
- ii- $\|z_1^{i+1} - u_i\| < \|u_i - u_{i-1}\|$,
- iii- there is no path between u_i and z_1^{i+1} satisfying (2.5.3).

This eventually proves that ξ has a descending chain $(u_i)_{i \in \mathbb{N}}$.

Base step. Since there is no path between x and y in $\text{Gab}(\xi)$ satisfying (2.5.3), the open ball $B([x, y])$ with diameter $[x, y]$ contains at least a point of ξ . Note that, otherwise, there would be an edge between x and y which would be a particular path between these points trivially satisfying (2.5.3). Let us denote by z_2^1 the smallest point in the lexicographic order in $B([x, y]) \cap \xi$. Then, either there is no path satisfying (2.5.3) between x and z_2^1 or between z_2^1 and y . Indeed, the section of $B([x, y])$ by the plane generated by x, y, z_2^1 is a circle with diameter $[x, y]$. Thus we have:

$$\|z_2^1 - x\|^2 + \|y - z_2^1\|^2 \leq \|y - x\|^2.$$

If there were a path $(x_1 = x, \dots, x_k = z_2^1)$ between x and z_2^1 and a path $(x_k = z, \dots, x_n = y)$ between z_2^1 and y satisfying (2.5.3), then their concatenation would be a path between x and y satisfying (2.5.3):

$$\begin{aligned} \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2 &= \sum_{i=1}^{k-1} \|x_{i+1} - x_i\|^2 + \sum_{i=k}^{n-1} \|x_{i+1} - x_i\|^2 \\ &\leq \|z_2^1 - x\|^2 + \|y - z_2^1\|^2 \leq \|y - x\|^2. \end{aligned}$$

Note also that $\|z_2^1 - x\|, \|y - z_2^1\| < \|y - x\|$. Without loss of generality, we may assume that there is no path satisfying (2.5.3) between z_2^1 and y and we set $u_0 := x$, $u_1 := y$ and $z_1^2 := z_2^1$.

Inductive step. We assume that $u_0, \dots, u_i, z_1^{i+1} \in \xi$ satisfying properties -i- to -iii- listed above are constructed, and we construct u_{i+1} and z_1^{i+2} such that:

- i- $\|z_1^{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\| < \|u_i - u_{i-1}\|$,
- ii- there is no path between u_{i+1} and z_1^{i+2} satisfying (2.5.3).

Since there is no path between u_i and z_1^{i+1} in $\text{Gab}(\xi)$ satisfying (2.5.3), $B([u_i, z_1^{i+1}]) \cap \xi \neq \emptyset$ (otherwise, there were an edge between u_i and z_1^{i+1} in $\text{Gab}(\xi)$). We write z_2^{i+1} for the smallest point in the lexicographic order in $B([u_i, z_1^{i+1}]) \cap \xi$. There are then two possibilities:

1) there is no path between z_1^{i+1} and z_2^{i+1} satisfying (2.5.3) We then set $u_{i+1} := z_1^{i+1}$ and $z_1^{i+2} := z_2^{i+1}$. Using that z_1^{i+2} is inside the ball $B([u_i, u_{i+1}])$ and the induction hypothesis, we have:

$$\|z_1^{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\| = \|z_1^{i+1} - u_i\| < \|u_i - u_{i-1}\| < \dots < \|u_1 - u_0\|.$$

Moreover, there is no path between u_{i+1} and z_1^{i+2} satisfying (2.5.3).

2) there is a path between z_1^{i+1} and z_2^{i+1} satisfying (2.5.3) Then there is no path between u_i and z_2^{i+1} satisfying (2.5.3), otherwise there would be a path between u_i and z_1^{i+1} satisfying (2.5.3). In particular, $B([u_i, z_2^{i+1}]) \cap \xi \neq \emptyset$ and we denote by z_3^{i+1} the smallest point in the lexicographic order in $B([u_i, z_2^{i+1}]) \cap \xi$. If there is no path between z_3^{i+1} and z_2^{i+1} satisfying (2.5.3), we proceed as in **1)**. Otherwise, we repeat the procedure until we find $z_{n_{i+1}}^{i+1}$ and $z_{n_{i+1}+1}^{i+1}$ such that there is no path between $z_{n_{i+1}}^{i+1}$ and $z_{n_{i+1}+1}^{i+1}$ in $\text{Gab}(\xi)$ satisfying (2.5.3). One can see that:

$$\|z_{n_{i+1}+1}^{i+1} - z_{n_{i+1}}^{i+1}\| < \|z_{n_{i+1}}^{i+1} - u_i\| < \dots < \|z_1^{i+1} - u_i\| < \|u_i - u_{i-1}\|.$$

In particular, such n_{i+1} does exist. Indeed, if not, there would be infinitely many points of ξ in the ball $B(u_i, \|u_i - u_{i-1}\|)$ and ξ would not be locally finite. We set $u_{i+1} := z_{n_{i+1}}^{i+1}$ and $z_1^{i+2} := z_{n_{i+1}+1}^{i+1}$. Then:

$$\|z_1^{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\| < \|u_i - u_{i-1}\| < \dots < \|u_1 - u_0\|,$$

and there is no path between u_{i+1} and z_1^{i+2} satisfying (2.5.3).

Finally, ξ has a descending chain. This contradicts the original assumption. □

Good boxes

For $M \geq 1$, consider as before a partition of \mathbb{R}^d into boxes of side M , $\{B_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$. For $m \in \mathbb{N}^*$, write $\alpha_{d,m}$ for the odd integer such that:

$$\beta_d m^2 + \sqrt{d} + 1 \leq \alpha_{d,m} < \beta_d m^2 + \sqrt{d} + 3,$$

where $\beta_d := 2^{2d+2}(d+3+2(d+3)^{3/2})$. We say that a box $B_{\mathbf{z}}$ is (M, m) -good for the Gabriel graph if when $B_{\mathbf{z}}$ is cut into $\alpha_{d,m}^d$ sub-boxes $b_i^{\mathbf{z}}$ of side $M/\alpha_{d,m}$, each of these sub-boxes contains at least one and at most m points of ξ .

Construction of paths

Let $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$ be two neighboring good boxes of side M and $b_1^{\mathbf{z}_1, \mathbf{z}_2}, \dots, b_{\alpha_{d,m}+1}^{\mathbf{z}_1, \mathbf{z}_2}$ be the sub-boxes of side $M/\alpha_{d,m}$ intersecting the line segment $[M\mathbf{z}_1, M\mathbf{z}_2]$. Write c_i (resp. v_i) for the center of $b_i^{\mathbf{z}_1, \mathbf{z}_2}$ (resp. the point of $b_i^{\mathbf{z}_1, \mathbf{z}_2} \cap \xi$ which is the closest to c_i). Vertices v_1 and $v_{\alpha_{d,m}+1}$ are reference vertices of $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$ respectively. One must prove that there exists a Gabriel path from v_1 to $v_{\alpha_{d,m}+1}$ which is included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. To this end, it suffices to check that, for all $i \in \{1, \dots, \alpha_{d,m}\}$, there exists a Gabriel path from v_i to v_{i+1} which is included in $B(c_i, M/2)$. Proceeding along the same lines as in the proof of [BBD02, Lemma 3], we show that the path $\gamma_i := (x_1 = v_i, \dots, x_n = v_{i+1})$ given by Lemma 2.14 for v_i and v_{i+1} satisfies this property. Observe that this path can be assumed to be simple and satisfies:

$$\sum_{j=1}^{n-1} \|x_{j+1} - x_j\|^2 \leq \|v_{i+1} - v_i\|^2 \leq (d+3) \frac{M^2}{\alpha_{d,m}^2}, \quad (2.5.4)$$

where the last bound is obtained by the Pythagorean theorem using that v_i and v_{i+1} belong to two neighboring boxes of side $M/\alpha_{d,m}$. In particular, γ_i does not contain any edge with length greater than $\sqrt{d+3}M/\alpha_{d,m}$ and contains at most $2^{2d+2}(d+3)m^2$ edges with length between $M/(2^{d+1}\alpha_{d,m}m)$ and $\sqrt{d+3}M/\alpha_{d,m}$, called *long* edges in the following. Indeed, with (2.5.4),

$$\frac{M^2}{2^{2d+2}\alpha_{d,m}^2 m^2} \#\{e : \text{long edge of } \gamma_i\} \leq \sum_{e: \text{long edge}} \|e\|^2 \leq (d+3) \frac{M^2}{\alpha_{d,m}^2}.$$

Hence, γ_i consists of at most $2^{2d+2}(d+3)m^2$ long edges and at most $2^{2d+2}(d+3)m^2 + 1$ groups of consecutive *short* edges with length lower than $M/(2^{d+1}\alpha_{d,m}m)$. Let us consider a simple path starting at v_i and having $N_l \in \{0, \dots, 2^{2d+2}(d+3)m^2\}$ long edges and $N_s \in \{0, \dots, 2^{2d+2}(d+3)m^2 + 1\}$ groups of consecutive short edges. Each group of short edges has total Euclidean length bounded by $M/(2\alpha_{d,m})$. Indeed, one can prove by induction that the vertex at the beginning of each group of consecutive short edges satisfies the assumption of Lemma 2.15, stated at the end of this subsection for sake of readability. Consequently, the distance from c_i to the farthest point of the path is less than:

$$\begin{aligned} \|v_i - c_i\| &+ N_l \frac{\sqrt{d+3}M}{\alpha_{d,m}} + N_s \frac{M}{2\alpha_{d,m}} \\ &\leq \sqrt{d} \frac{M}{2\alpha_{d,m}} + N_l \frac{\sqrt{d+3}M}{\alpha_{d,m}} + N_s \frac{M}{2\alpha_{d,m}} \\ &\leq \sqrt{d} \frac{M}{2\alpha_{d,m}} + 2^{2d+2}(d+3)^{\frac{3}{2}} m^2 \frac{M}{\alpha_{d,m}} + \left(2^{2d+2}(d+3)m^2 + 1\right) \frac{M}{2\alpha_{d,m}} \\ &= \frac{M}{2} \frac{\beta_d m^2 + \sqrt{d} + 1}{\alpha_{d,m}} \leq \frac{M}{2}. \end{aligned}$$

Thus γ_i is included in $B(c_i, M/2)$ and there exists a Gabriel path $\gamma := (v_1, \dots, v_{\alpha_{d,m}+1})$ from v_1 to $v_{\alpha_{d,m}+1}$ contained in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$.

It remains to choose L and K in Criterion 2.8 (2)(b) and (2)(c). Since $\gamma \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ and can be supposed simple, it has chemical length at most $\#((B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}) \cap \xi) - 1 \leq 2\alpha_{d,m}^d m - 1$. Thus, one can set $L := 2\alpha_{d,m}^d m - 1$. If $C(\cdot) = \varphi(\|\cdot\|)$ is a decreasing positive function of edges lengths, we set $K := 1/\varphi(\sqrt{d} + 3M)$; if C is uniformly bounded from below, we set $K := \max 1/C$.

Lemma 2.15. *Let $B_{\mathbf{z}_1}, B_{\mathbf{z}_2}$ be two neighboring good boxes and γ be a simple Gabriel path with edges of length bounded by $M/(2^{d+1}\alpha_{d,m}m)$ with a vertex u such that $B(u, M/(2\alpha_{d,m}))$ is included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$.*

Then, γ consists in at most $2^d m - 1$ edges. In particular, it has total Euclidean length bounded by $M/(2\alpha_{d,m})$.

Proof. Note that $B(u, M/(2\alpha_{d,m}))$ intersects at most 2^d sub-boxes of side $M/\alpha_{d,m}$. Since it is furthermore included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ which are good boxes, it contains at most $2^d m$ points of ξ . Assume that γ contains more than $2^d m$ edges. Hence, γ has a (sub-)path γ' of $2^d m$ edges (and $2^d m + 1$ vertices) such that $u \in \gamma'$. But γ' is included in $B(u, M/(2\alpha_{d,m}))$ which provides a contradiction. \square

$\mathcal{P}[X_{\mathbf{z}} = 1]$ is large enough

Assuming that $M \geq 2$, the process $\{X_{\mathbf{z}}\}$ is k -dependent because of the definition of good boxes and the fact that ξ has a finite range of dependence. It remains to show that if M, m are suitably chosen:

$$\mathcal{P}[X_{\mathbf{z}} = 1] \geq p^*$$

where $p^* = p^*(d, k) < 1$ is large enough to ensure that $\{X_{\mathbf{z}}\}$ dominates supercritical site percolation on \mathbb{Z}^d .

Thanks to the choice of $\alpha_{d,m} \sim \beta_d m^2$, for $m \in \mathbb{N}^*$ large enough, we can choose M so that:

$$\frac{\alpha_{d,m}^d}{c_1} \log \left(\frac{2\alpha_{d,m}^d}{1-p^*} \right) \leq M^d \leq \frac{\alpha_{d,m}^d}{c_4} \left(m - \log \left(\frac{2\alpha_{d,m}^d}{1-p^*} \right) \right).$$

With (V) and (D₃₊):

$$\mathcal{P} \left[\# \left(\left[0, \frac{M}{\alpha_{d,m}} \right]^d \cap \xi \right) = 0 \right] \leq \exp \left(-c_1 \frac{M^d}{\alpha_{d,m}^d} \right) \leq \frac{1-p^*}{2\alpha_{d,m}^d},$$

and

$$\mathcal{P} \left[\# \left(\left[0, \frac{M}{\alpha_{d,m}} \right]^d \cap \xi \right) > m \right] \leq \exp \left(c_4 \frac{M^d}{\alpha_{d,m}^d} - m \right) \leq \frac{1-p^*}{2\alpha_{d,m}^d}.$$

Finally, using stationarity of ξ :

$$\begin{aligned} \mathcal{P}[X_{\mathbf{z}} = 0] &\leq \sum_{i=1}^{\alpha_{d,m}^d} \left\{ \mathcal{P}\left[\#(b_i^{\mathbf{z}} \cap \xi) = 0\right] + \mathcal{P}\left[\#(b_i^{\mathbf{z}} \cap \xi) > m\right] \right\} \\ &= \alpha_{d,m}^d \left\{ \mathcal{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \xi\right) = 0\right] + \mathcal{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \xi\right) > m\right] \right\} \\ &\leq 1 - p^*. \end{aligned}$$

2.6 Examples of point processes

In this section, an overview on point processes appearing in Theorem 2.1 is given. Assumptions of Theorem 2.2 are checked for these processes. Note that the probability estimates given here are rough but good enough to verify **(V)**-**(D₃₊)**.

2.6.1 Poisson point processes

For homogeneous Poisson point processes (PPPs), stationarity and the finite range of dependence condition are clear. Note that stationary Poisson point processes are almost surely in general position. We refer to [HM96] or [DL05] for the almost sure absence of descending chains. Moreover, we check by standard computations that, for any Borel set A , a PPP ξ of intensity λ satisfies:

$$\mathcal{P}\left[\#(A \cap \xi) = 0\right] = e^{-\lambda \text{Vol}_{\mathbb{R}^d}(A)}, \quad (2.6.1)$$

and

$$\mathcal{P}\left[\#(A \cap \xi) > m\right] \leq e^{\lambda(e-1) \text{Vol}_{\mathbb{R}^d}(A) - m}. \quad (2.6.2)$$

This implies **(V)**-**(D₃₊)**.

Results similar to (2.6.1) and (2.6.2) are satisfied when the intensity measure μ of the PPP is comparable to Lebesgue measure on \mathbb{R}^d in the sense that there exists a positive constant c_{12} such that for every measurable subset A of \mathbb{R}^d :

$$\frac{1}{c_{12}} \text{Vol}_{\mathbb{R}^d}(A) \leq \mu(A) \leq c_{12} \text{Vol}_{\mathbb{R}^d}(A).$$

This implies that the conclusions of Theorem 2.1 also hold for such non-stationary PPPs (see Remark 2.3).

2.6.2 Matérn cluster processes

Matérn cluster processes (MCPs) are particular cases of Neyman-Scott Poisson processes (see [CSKM13, p. 171]). Cluster processes are used as models for spatial phenomena, e.g. galaxy locations in space [KPBS⁺99] or epicenters of micro-earthquake locations [VJ70].

MCPs are constructed as follows. Let $\lambda, \mu, R > 0$. One first chooses a PPP Y of intensity λ called the *parent process*. For any $y \in Y$, a centered *daughter process* ξ_y is then chosen such that, given Y , $\{\xi_y\}_{y \in Y}$ are mutually independent PPP with intensity μ in $B(0, R)$. Then, $\xi := \bigcup_{y \in Y} (y + \xi_y)$ is a MCP with parameters λ, μ, R . It is clear that such processes are stationary. Since its parent process has a finite range of dependence and daughter processes have bounded supports, any MCP has a finite range of dependence. Thanks to [HNS13, Proposition 2.3] or [DL05, Theorem 7.2], MCPs have almost surely no descending chains. MCPs can be seen as doubly stochastic processes or Cox processes (see [CSKM13, p. 166] for a definition). Their (diffusive) random intensity measures μ_Y have densities $\sum_{y \in Y} \mathbb{1}_{B(y, R)}(x)$ w.r.t. Lebesgue measure. In particular, $(d-1)$ dimensional hyperplanes and spheres are μ_Y -null sets and MCPs are almost surely in general position.

It remains to check assumptions **(V)**-**(D₃₊)**. Let $L > 2R$, then,

$$\begin{aligned} \mathcal{P}[\#(\xi \cap [0, L]^d) = 0] &= \mathbb{E}[\mathcal{P}[\#(\xi \cap [0, L]^d) = 0 | Y]] \\ &= \mathbb{E}\left[\mathcal{P}\left[\bigcap_{y \in Y} \{\#(\xi_y \cap [0, L]^d) = 0\} \middle| Y\right]\right] \\ &= \mathbb{E}\left[\prod_{y \in Y} \mathcal{P}\left[\{\#(\xi_y \cap [0, L]^d) = 0\} \middle| Y\right]\right] \\ &= \mathbb{E}\left[\prod_{y \in Y} \exp\left(-\mu \text{Vol}_{\mathbb{R}^d}(B(y, R) \cap [0, L]^d)\right)\right] \end{aligned}$$

so, with [SW08, Theorem 3.2.4],

$$\begin{aligned} \mathcal{P}[\#(\xi \cap [0, L]^d) = 0] &= \exp\left(-\lambda \int_{[0, L]^d + B(0, R)} \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}([0, L]^d \cap B(y, R)))\} dy\right) \\ &\leq \exp\left(-\lambda \int_{[0, L]^d - B(0, R)} \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(0, R)))\} dy\right) \end{aligned}$$

where, for $A, B \subset \mathbb{R}^d$, $A - B := \{x \in \mathbb{R}^d : \forall y \in B, x + y \in A\}$. Thus,

$$\mathcal{P}[\#(\xi \cap [0, L]^d) = 0] \leq \exp\left(-\lambda \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(0, R)))\} (L - 2R)^d\right)$$

and **(V)** holds. Alternatively, one can see that the MCP with parameters λ, μ and R satisfies the hypothesis **(V)** because it stochastically dominates a PPP with intensity λ' where $\lambda' = \lambda(1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(0, R))))$, obtained by deleting all but one uniformly chosen point of each non-empty daughter process ξ_y .

In order to show **(D₂)** and **(D₃₊)** for MCPs one can use exponential Markov inequality

and the following estimate. For any bounded Borel set A :

$$\begin{aligned}
 \mathbb{E}\left[e^{\#(\xi \cap A)}\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\sum_{y \in Y} \#(\xi_y \cap A)\right) \middle| Y\right]\right] \\
 &= \mathbb{E}\left[\prod_{y \in Y \cap (A+B(0,R))} \mathbb{E}\left[e^{\#(\xi_y \cap A)} \middle| Y\right]\right] \\
 &\leq \mathbb{E}\left[\prod_{y \in Y \cap (A+B(0,R))} \mathbb{E}\left[e^{\#\xi_y} \middle| Y\right]\right] \\
 &= \mathbb{E}\left[\exp\left(\mu \text{Vol}_{\mathbb{R}^d}(B(0,R))(e-1)\#\{y \in Y \cap (A+B(0,R))\}\right)\right] \\
 &= e^{c \text{Vol}_{\mathbb{R}^d}(A+B(0,R))}
 \end{aligned}$$

where $c = c(\lambda, \mu, R)$.

2.6.3 Matérn hardcore processes

In 1960, Matérn introduced several hardcore models for point processes. These processes are dependent thinnings of PPPs and spread more regularly in space than PPPs. Such models are useful when competition for resources exists (e.g. tree or city locations, see [Mat86] and references therein).

Let ξ be a marked PPP of intensity λ , with independent marks $\{T_x\}_{x \in \xi}$ uniformly distributed in $[0, 1]$. Then, Matérn I/II hardcore processes (MHP I/II) ξ_{I} and ξ_{II} are defined, for a given $R > 0$ by:

$$\begin{aligned}
 \xi_{\text{I}} &:= \{x \in \xi : \|x - y\| > R, \forall y \in \xi \setminus \{x\}\}, \\
 \xi_{\text{II}} &:= \{x \in \xi : T_x < T_y, \forall y \in \xi \cap B(x, R)\}.
 \end{aligned}$$

Clearly, MHPs are stationary and have finite range of dependence. Note that $\xi_{\text{I}} \subset \xi_{\text{II}} \subset \xi$. Hence, the facts that $\xi_{\text{I}}, \xi_{\text{II}}$ are almost surely in general position and that they have almost surely no descending chains are inherited from these properties for PPPs. Moreover, inequalities **(D₂)** and **(D₃₊)** are immediate from those for PPPs, and it suffices to show **(V)** for ξ_{I} . To this end, first assume there is an integer n such that $L = 3nR$ and cut $[0, L]^d$ into $3^d n^d$ disjoint sub-boxes b_j of side R . Note that if there is a sub-box b_j with $\#(b_j \cap \xi) = 1$ having all neighboring sub-boxes empty, then $[0, L]^d \cap \xi_{\text{I}} \neq \emptyset$. Thus,

$$\mathcal{P}\left[\#([0, L]^d \cap \xi_{\text{I}}) = 0\right] \leq \mathcal{P}\left[\bigcap_j A_j\right],$$

where A_j stands for the event: ‘ $\#(b_j \cap \xi) \neq 1$ or a neighbor of b_j contains at least a point of ξ ’. One can choose a collection of n^d sub-boxes b_j so that, if $i \neq j$, events A_i and A_j are independent. So, there exists a constant $c_{13} > 0$ such that:

$$\mathcal{P}\left[\#([0, L]^d \cap \xi_{\text{I}}) = 0\right] \leq \mathcal{P}\left[A_{j_0}\right]^{n^d} = e^{-c_{13}n^d} = e^{-\frac{c_{13}}{3^d}L^d}.$$

For general $L \geq 3R$, it suffices to fix n so that $3nR \leq L \leq 3(n+1)R$ to obtain:

$$\mathcal{P}[\#\left([0, L]^d \cap \xi_{\mathbb{I}}\right) = 0] \leq \mathcal{P}[\#\left([0, 3nR]^d \cap \xi\right) = 0] \leq e^{-\frac{c_{13}}{6^d} L^d}.$$

2.6.4 Determinantal processes

In 1975, Macchi [Mac75] introduced determinantal point processes (DPPs) in order to model fermions in quantum mechanics. These processes also arise in various other settings such as eigenvalues of random matrices, random spanning trees, carries processes when adding a list of random numbers; see for example [BP93, Dia03, Gin65, ST03a, ST03b].

Let K be a self-adjoint non-negative locally trace class operator acting on $L^2(\mathbb{R}^d)$ with integral kernel k . For a bounded Borel set $A \subset \mathbb{R}^d$, let us denote by P_A the projection operator from $L^2(\mathbb{R}^d)$ onto $L^2(A)$ and by $K|_A := P_A K P_A$ the restriction of K onto $L^2(A)$. The integral kernel k can be properly chosen such that it satisfies the so called *local trace formula*:

$$\mathrm{tr} K|_A = \int_A k(x, x) dx \tag{2.6.3}$$

for all bounded Borel set $A \subset \mathbb{R}^d$. A DPP with kernel k is a simple point process ξ whose correlation functions ρ_m satisfy:

$$\rho_m(x_1, \dots, x_m) = \det \left(k(x_i, x_j) \right)_{1 \leq i, j \leq m}$$

for all $m \geq 1$ and all $x_1, \dots, x_m \in \mathbb{R}^d$. If ξ is stationary, one can fix $K_0 := k(0, 0) = k(x, x) > 0$ for almost all x . Thus, formula (2.6.3) reduces in this case to:

$$\mathrm{tr} K|_A = K_0 \mathrm{Vol}_{\mathbb{R}^d}(A), \tag{2.6.4}$$

where $\mathrm{Vol}_{\mathbb{R}^d}(A)$ denotes the volume of A for the Lebesgue measure on \mathbb{R}^d . We refer to [BHKPV06], [Sos00] and the appendix in [GY05] for more details.

Theorem 7 in [BHKPV06] provides the following useful fact for DPPs: the number of points of a DPP ξ with kernel k falling in a bounded set A has the law of the sum of independent Bernoulli random variables with parameters the eigenvalues of the restriction $K|_A$ of K to A . Thus writing λ_j for the eigenvalues of $K|_A$, using that for $s \geq 0$, $1 - s \leq e^{-s}$ and formula (2.6.4), we have:

$$\begin{aligned} \mathcal{P}[\#\left(A \cap \xi\right) = 0] &= \prod_j (1 - \lambda_j) \\ &\leq \prod_j e^{-\lambda_j} = e^{-\mathrm{tr}(K|_A)} \\ &= e^{-K_0 \mathrm{Vol}_{\mathbb{R}^d}(A)} \end{aligned}$$

which implies **(V)**. Similarly, using that $1 + s \leq e^s$ and formula (2.6.4):

$$\begin{aligned} \mathbb{E} \left[\left(\frac{3}{2} \right)^{\#(\xi \cap A)} \right] &= \prod_j \left(1 + \frac{\lambda_j}{2} \right) \\ &\leq \prod_j e^{\frac{\lambda_j}{2}} = e^{\frac{1}{2} \text{tr}(K|_A)} \\ &= e^{\frac{1}{2} K_0 \text{Vol}_{\mathbb{R}^d}(A)}. \end{aligned}$$

This implies **(D₂)** using exponential Markov inequality. Actually, one can obtain **(D₃₊)** in a similar way, but we are unable to check assumption (3) in Criterion 2.8 because of the lack of independence in this case.

Chapitre 3

Principes d'invariance *annealed*

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Dans ce chapitre, comme dans le précédent, on étudie des marches aléatoires sur des graphes construits à partir d'un ensemble aléatoire dans \mathbb{R}^d et de la géométrie de celui-ci tels que sa triangulation de Delaunay, le graphe de Gabriel ou les *creek-crossing graphs* associés. Cette fois, l'ensemble aléatoire des sommets du graphe est choisi selon la probabilité de Palm \mathcal{P}_0 associée à un processus ponctuel stationnaire. Pour \mathcal{P}_0 -presque toute réalisation ξ^0 , on considère la marche simple à temps continu et à vitesse variable (VSRW) partant de 0 sur $G(\xi^0) = (\xi^0, E_{G(\xi^0)})$. Son comportement peut être décrit comme suit : le marcheur, étant arrivé en x , attend un temps exponentiel de paramètre $\deg_{G(\xi^0)}(x)$ puis choisit uniformément un de ses voisins et se déplace vers lui. La procédure est alors répétée indépendamment du passé.

Sous des hypothèses sur le processus ponctuel sous-jacent similaires à celles du chapitre précédent et une hypothèse d'ergodicité, on établit, dans le Théorème 3.1, un principe d'invariance *annealed* ou *en moyenne*, c'est-à-dire la convergence faible en \mathcal{P}_0 -probabilité du processus mis à l'échelle diffusive $(\varepsilon X_{\varepsilon^{-2}t}^{\xi^0})_{t \geq 0}$ vers un mouvement brownien non dégénéré. Pour cela, on adapte, dans le cadre présent, les arguments utilisés dans [FSBS06] dans le cadre des marches aléatoires sur des graphes complets engendrés par des processus ponctuels dans \mathbb{R}^d avec des probabilités de transition décroissant avec la distance entre les points (voir aussi [DMFGW89] dans le cadre de marches associées à des conductances aléatoires dans \mathbb{Z}^d). Dans ce chapitre, on omettra les preuves ou parties de preuves trop proches de ces deux derniers articles et on donnera plus de détails dès qu'une preuve diffèrera significativement ou que cela paraîtra nécessaire à la compréhension. On peut distinguer deux grandes parties dans la méthode utilisée. On commence par montrer la convergence du processus mis à l'échelle vers un mouvement brownien de matrice de covariance explicite dans la Proposition 3.2. La première difficulté dans la preuve de celle-ci réside dans le fait que le processus $(X_t^{\xi^0})_{t \geq 0}$ n'est pas markovien sous la mesure *annealed* puisque la connaissance de la trajectoire jusqu'à un certain temps t apporte de l'information sur l'environnement tiré. On suit la solution, maintenant traditionnelle et déjà présente dans [Koz85] et [KV86], pour pallier cette difficulté : on utilise le processus de *l'environnement vu par le marcheur*. Celui-ci permet de reconstruire le processus d'origine comme fonctionnelle additive d'un processus markovien et l'utilisation de résultats de [DMFGW89]. La seconde grande étape est de montrer que le mouvement brownien obtenu à la limite n'est pas dégénéré. Ceci est établi dans la Proposition 3.4 dont la preuve repose sur la comparaison du coefficient de diffusion avec les résistances efficaces de réseaux électriques bien choisis. Remarquons que la méthode pourrait être adaptée pour étudier, comme dans le Chapitre 2, des marches associées à des conductances dépendant par exemple de la longueur des arêtes sur les graphes que nous considérons.

Annealed invariance principle for random walks on random graphs generated by point processes in \mathbb{R}^d

3.1 Introduction and results

3.1.1 Introduction

We are interested in stating annealed invariance principles for simple random walks on random graphs generated by point processes. More precisely, given a realization ξ of a point process in \mathbb{R}^d , a connected graph $G(\xi) = (\xi, E_{G(\xi)})$, in which each vertex has finite degree, is constructed. While its vertex set is ξ itself, its edge set $E_{G(\xi)}$ is obtained using deterministic rules based on the geometry of ξ . For instance, we will consider *Delaunay triangulations*, *Gabriel graphs* or *creek-crossing graphs* generated by point processes. Their constructions will be described below. We then consider continuous time random walks on $G(\xi)$ and investigate their annealed or averaged scaling limits.

In the sequel, we will write \mathcal{N} (resp. \mathcal{N}_0) for the collection of all infinite locally finite subsets of \mathbb{R}^d (resp. infinite locally finite subsets of \mathbb{R}^d containing the origin). We will assume that the infinite locally finite random set ξ^0 which generates the graph contains the origin. To make it precise, ξ^0 will be chosen according to the Palm version of a stationary point process. It is well known that the Palm probability of a Poisson point process (PPP) is simply obtained by adding a point at the origin (*i.e.* $\xi^0 = \xi \cup \{0\}$ in distribution, where ξ is distributed according to a PPP). We write \mathcal{P} (resp. \mathcal{P}_0) for the law of the point process (resp. its Palm version). We denote by \mathcal{E} (resp. \mathcal{E}_0) the expectation with respect to \mathcal{P} (resp. \mathcal{P}_0). We assume in all what follows that \mathcal{P} is stationary, ergodic and simple. We also assume that \mathcal{P}_0 -almost all ξ^0 is aperiodic, *i.e.*, with \mathcal{P}_0 probability 1:

$$\xi^0 \neq \tau_x \xi^0, \forall x \in \mathbb{R}^d,$$

where τ_x is the translation $\tau_x \xi^0 = \sum_{y \in \xi^0} \delta_{y-x}$.

For almost any realization of ξ^0 , we define a variable speed random walk (VSRW) on $G(\xi^0)$ as follows. Let $D(\mathbb{R}_+; \xi^0)$ be the space of càdlàg ξ^0 -valued functions on \mathbb{R}_+ endowed with the Skorohod topology, and $X_t^{\xi^0} = X^{\xi^0}(t)$, $t \geq 0$ be the coordinate maps from $D(\mathbb{R}_+; \xi^0)$ to ξ^0 . For $x \in \xi^0$, we write $P_x^{\xi^0}$ for the probability measure on $\Omega_{\xi^0} := D(\mathbb{R}_+; \xi^0)$ under which the coordinate process $(X_t^{\xi^0})_{t \geq 0}$ is the continuous time Markov process starting from $X_0^{\xi^0} = x$ with infinitesimal generator given by:

$$\mathcal{L}^{\xi^0} f(y) := \sum_{z \in \xi^0} c_{y,z}^{\xi^0} (f(z) - f(y)), \quad y \in \xi^0, \quad (3.1.1)$$

where $c_{y,z}^{\xi^0}$ is equal to the indicator function of the event ' $(y, z) \in E_{G(\xi^0)}$ '. We denote by $E_x^{\xi^0}$ the expectation with respect to $P_x^{\xi^0}$. Note that $P_x^{\xi^0}$ can be seen as a probability measure on $\Omega := D(\mathbb{R}_+; \mathbb{R}^d)$ concentrated on Ω_{ξ^0} .

The behavior of $X_t^{\xi^0}$ under $P_x^{\xi^0}$ can be described as follows: starting from the point x , the particle waits for an exponential time of parameter $\deg_{G(\xi^0)}(x)$ and then chooses, uniformly at random, one of its neighbors in $G(\xi^0)$ and moves to it. This procedure is then iterated with independent times (of parameter the degree of the current location in $G(\xi^0)$). The same arguments as in [FSBS06, Appendix A] show that these random walks are well-defined (*i.e.* that no explosion occurs) as soon as $\mathcal{E}_0[\deg_{G(\xi^0)}(0)] < \infty$.

Let P be the so-called annealed semi-direct product measure on $\mathcal{N}_0 \times \Omega$:

$$P[F(\xi^0, X^{\xi^0})] = \int_{\mathcal{N}_0} P_0^{\xi^0}[F(\xi^0, X^{\xi^0})] \mathcal{P}_0(d\xi^0).$$

Note that $(X_t^{\xi^0})_{t \geq 0}$ is no longer Markovian under P . Following the line of the proofs in [DMFGW89, Section 4] and [FSBS06], we will use the *point of view of the particle* and [DMFGW89, Theorem 2.2] to obtain annealed invariance principles for random walks on specific examples of random geometric graphs with suitable underlying point processes. Let us describe the precise models.

3.1.2 Conditions on the point process

In all what follows we assume that \mathcal{P} is the distribution of a simple, stationary, ergodic point process ξ satisfying:

$$\mathcal{E}[\#(\xi \cap [0, 1]^d)^8] < \infty \text{ and } \lim_{N \rightarrow \infty} N^8 \mathcal{P}[\#(\xi \cap [-N, N]^d) \geq c_1 N^d] = 0 \text{ for some } c_1 > 0.$$

We also suppose that the second factorial moment measure of the point process ξ :

$$\Lambda^{(2)}(A_1 \times A_2) := \mathcal{E}[\#\{(x_1, x_2) : x_1 \in \xi \cap A_1, x_2 \in \xi \cap A_2 \setminus \{x_1\}\}], \quad A_1, A_2 \in \mathcal{B}(\mathbb{R}^d),$$

is absolutely continuous with respect to $2d$ -Lebesgue measure with bounded density. We assume that the Palm version of \mathcal{P} denoted by \mathcal{P}_0 is aperiodic and that \mathcal{P} and \mathcal{P}_0 are isotropic and almost surely in general position (*i.e.* there are no $d+1$ points (resp. $d+2$ points) in a $(d-1)$ -dimensional affine subspace (resp. in a sphere), see [Zes08]). It is moreover assumed that ξ (resp. ξ^0) has a *finite range of dependence* k , *i.e.*, for any disjoint Borel sets $A, B \subset \mathbb{R}^d$ with $d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} \geq k$, $\xi \cap A$ and $\xi \cap B$ are independent (resp. $\xi^0 \cap A$ and $\xi^0 \cap B$ are independent).

Finally, the following hypothesis on the void probabilities **(V)**, and **(V')**, the deviation probabilities **(D)** and the exponential moments of the number of points in a large cube **(E)** are needed:

(V) There exists a positive constant c_2 such that for L large enough:

$$\mathcal{P}[\#([0, L]^d \cap \xi) = 0] \leq e^{-c_2 L^d}.$$

(V') There exists a positive constant c_3 such that for L large enough:

$$\mathcal{P}_0[\#(\xi^0 \cap C_L) = 0] \leq e^{-c_3 L^d}, \quad \forall C_L \text{ cube of side } L.$$

(D) There exists a positive constant c_4 such that for L large enough and all $m > 0$:

$$\mathcal{P}\left[\#\left([0, L]^d \cap \xi\right) \geq m\right] \leq e^{c_4 L^d - m}.$$

(E) There exists a positive constant c_5 and a positive function $f(\rho)$ which goes to 0 with ρ such that for L large enough:

$$\mathcal{E}_0\left[e^{\rho \#(\xi^0 \cap [-L, L]^d)}\right] \leq c_5 e^{f(\rho)L^d}.$$

These assumptions are in particular satisfied if ξ is:

- a homogeneous Poisson point process (PPP),
- a Matérn cluster process (MCP),
- a Matérn hardcore process I or II (MHP I/II).

We refer to Chapter 1 for an overview on each of these point processes. Note that these processes have different interaction properties: for PPPs there is no interaction between points, MCPs exhibit clustering effects whereas points in MHPs repel each other.

3.1.3 The graph structures

Given a realization ξ of the point process satisfying previous assumptions, we write

$$\text{Vor}_\xi(\mathbf{x}) := \{x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \xi\}$$

for the Voronoi cell of $\mathbf{x} \in \xi$; \mathbf{x} is called the nucleus or the seed of the cell. The Voronoi diagram of ξ is the collection of the Voronoi cells. It tessellates \mathbb{R}^d into convex polyhedra. See [Møl94, Cal10] for an overview on these tessellations.

The graphs considered in the sequel are:

$\text{DT}(\xi)$ the *Delaunay triangulation* of ξ . It is the dual graph of its Voronoi tiling. It has ξ as vertex set and there is an edge between \mathbf{x} and \mathbf{y} in $\text{DT}(\xi)$ if $\text{Vor}_\xi(\mathbf{x})$ and $\text{Vor}_\xi(\mathbf{y})$ share a $(d - 1)$ -dimensional face. Another useful characterization of $\text{DT}(\xi)$ is the following: a simplex Δ is a cell of $\text{DT}(\xi)$ *iff* its circumscribed sphere has no point of ξ in its interior. Note that this triangulation is well defined since ξ is assumed to be in general position.

These graphs are widely used in many fields such as astrophysics [RBFN01], cellular biology [Pou04], ecology [Roq97] and telecommunications [BB09].

$\text{Gab}(\xi)$ the *Gabriel graph* of ξ . Its vertex set is ξ and there is an edge between $\mathbf{x}, \mathbf{y} \in \xi$ if the ball of diameter $[\mathbf{x}, \mathbf{y}]$ contains no point of ξ in its interior. Note that $\text{Gab}(\xi)$ is a subgraph of $\text{DT}(\xi)$ and contains the Euclidean minimum spanning forest of ξ and the nearest neighbor graph. It has for example applications in geography, routing strategies, biology or tumor growth analysis (see [BBD02, GS69, MS80a]).

$G_n(\xi)$, $n \geq 2$ the *creek-crossing graphs* generated by ξ . These are sub-graphs of $DT(\xi)$. Each $G_n(\xi)$ has ξ as vertex set and two vertices $\mathbf{x}, \mathbf{y} \in \xi$ are connected by an edge $G_n(\xi)$ iff there do not exist an integer $k \leq n$ and vertices $u_0 = \mathbf{x}, \dots, u_k = \mathbf{y} \in \xi$ such that $\|u_i - u_{i+1}\| < \|\mathbf{x} - \mathbf{y}\|$ for all $i \in \{0, \dots, k-1\}$. Let us recall that $G_2(\xi)$ is also known as the *relative neighborhood graph* generated by ξ and that the intersection of the $G_n(\xi)$ (in the graph sense) is the minimum spanning forest of ξ (see [HNS13, HNGS15]).

3.1.4 Main results

The aim of this paper is to prove:

Theorem 3.1. *Assume that \mathcal{P} and \mathcal{P}_0 satisfy the conditions described in Subsection 3.1.2.*

Let $(X_t^{\xi^0})_{t \geq 0}$ be the random walk on $DT(\xi^0)$, $\text{Gab}(\xi^0)$ or $G_n(\xi^0)$, $n \geq 2$ with generator (3.1.1).

Then, the rescaled process $\underline{Y}^{\xi^0, \varepsilon} := (\varepsilon X_{\varepsilon^{-2t}}^{\xi^0})_{t \geq 0}$ converges weakly in \mathcal{P}_0 -probability to a nondegenerate Brownian motion $\underline{W}^{\sigma^2 \mathbf{I}_d}$ with:

$$\sigma^2 = \lim_{t \rightarrow \infty} \frac{1}{t} \mathcal{E}_0 \left[E_0^{\xi^0} \left[(X_t^{\xi^0} \cdot e_1)^2 \right] \right], \quad (3.1.2)$$

where e_1 is the first vector of the standard basis of \mathbb{R}^d .

This is the case if the underlying point process is a PPP, a MCP or a MHP I/II.

As mentioned above, we closely follow the method of [FSBS06] and we want to apply the results of De Masi, Ferrari, Goldstein and Wick. A key idea of [DMFGW89], which also appears in the prior works by Kozlov [Koz85] and by Kipnis and Varadhan [KV86], is to introduce an auxiliary Markov process: *the environment seen from the particle*. This allows to deduce properties of the original random walk from properties of the environment process which is Markovian. Actually, thanks to the aperiodicity of the medium, $(X_t^{\xi^0})_{t \geq 0}$ can be reconstructed as a functional of the environment process. This leads to the two following preliminary results stated in the general context of the introduction.

Proposition 3.2. *Let \mathcal{P} be a stationary ergodic simple point process in \mathbb{R}^d . Assume that \mathcal{P}_0 is aperiodic and that $\max_{x \sim 0} \|x\|$ and $\deg_{G(\xi^0)}(0)$ admit polynomial moments of any order under \mathcal{P}_0 .*

Then, the rescaled process $\underline{Y}^{\xi^0, \varepsilon} := (\varepsilon X_{\varepsilon^{-2t}}^{\xi^0})_{t \geq 0}$ converges weakly in \mathcal{P}_0 -probability to a Brownian motion \underline{W}^D with covariance matrix D characterized by equation (3.2.13).

Remark 3.3. 1. *If the underlying point process is isotropic, then $D = \sigma^2 \mathbf{I}_d$ where σ^2 is given by (3.1.2).*

2. *It is not necessary to require the finiteness of all polynomial moment for $\max_{x \sim 0} \|x\|$ and $\deg_{G(\xi^0)}(0)$. Following the proofs line by line, one can determine the maximal orders needed. In the applications presented here $\max_{x \sim 0} \|x\|$ and $\deg_{G(\xi^0)}(0)$ admit exponential moments.*

Note that the diffusion matrix D of \underline{W}^D can be degenerate. We will show that, under additional assumptions, it is not. To this end, we will need to discretize the space \mathbb{R}^d and to consider boxes of side $K \geq 1$:

$$B_{\mathbf{z}} = B_{\mathbf{z}}^K := K\mathbf{z} + \left[-\frac{K}{2}, \frac{K}{2} \right]^d, \mathbf{z} \in \mathbb{Z}^d.$$

We will also define so-called ‘good boxes’ in the same spirit as in [Rou13] which allow us to deduce ‘nice properties’ of $G(\xi)$ from ‘nice properties’ of a suitable percolation process on \mathbb{Z}^d .

Proposition 3.4. *Let \mathcal{P} and \mathcal{P}_0 satisfy the assumptions of Proposition 3.2. Assume in addition that $\mathcal{E} \left[\#(\xi \cap [0, 1]^d)^8 \right] < \infty$ and that there exists $c_1 > 0$ such that:*

$$\lim_{N \rightarrow \infty} N^8 \mathcal{P} \left[\#(\xi \cap [-N, N]^d) \leq c_1 N^d \right] = 0. \quad (3.1.3)$$

If one can associate to a.a. ξ a subset of the boxes $\{B_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$, called (ξ) -good boxes, such that:

1. *in each good box $B_{\mathbf{z}}$, one can choose a reference vertex $v_{\mathbf{z}} \in B_{\mathbf{z}} \cap \xi$,*
2. *there exists L such that to each pair of neighboring good boxes $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$, one can associate a path in $G(\xi)$, $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$, between the respective reference vertices $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ of these boxes satisfying:*
 - (a) $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2}) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$,
 - (b) *the length of $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$ in the graph distance is bounded by L ,*
3. *if ξ is distributed according to \mathcal{P} , the process $\mathbb{X} = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\} := \{\mathbf{1}_{B_{\mathbf{z}}} \text{ is } \xi\text{-good}, \mathbf{z} \in \mathbb{Z}^d\}$ stochastically dominates an independent Bernoulli site percolation process on \mathbb{Z}^d with parameter $p \geq p_c^{\text{site}}(\mathbb{Z}^2)$,*

then the limiting Brownian motion \underline{W}^D in Proposition 3.2 has a nondegenerate covariance matrix.

Sections 3.2 and 3.3 are respectively devoted to the proof of Propositions 3.2 and 3.4. In Section 3.4, we explain how to construct the ‘good boxes’ of Proposition 3.4 in the case of creek-crossing graphs. The non degeneracy of the limiting Brownian motion then follows in the other cases by Rayleigh monotonicity principle. Arguments which allow to deduce exponential moments for $\deg_{\text{DT}(\xi^0)}(0)$ and $\max_{x \sim 0 \text{ in } \text{DT}(\xi^0)} \|x\|$ can be found in Chapter 4.

3.2 Proof of Proposition 3.2

We show how to use [DMFGW89, Theorem 2.2] in our framework. This result was used in the context of random walks in random conductance models (see [DMFGW85, DMFGW89])

and of random walks on complete graphs generated by point processes with jump rates depending on energy marks (see [FSBS06]). It allows us to obtain the convergence of the rescaled process to a Brownian motion in Proposition 3.2 and gives an expression for the diffusion matrix of the limiting Brownian motion. The nondegeneracy of this matrix and the proof of Proposition 3.4 are postponed to Section 3.3. For this section, we assume that the assumptions of Proposition 3.2 are satisfied.

3.2.1 The point of view of the particle

Let us denote by $(\tau_x)_{x \in \mathbb{R}^d}$ the group of translations in \mathbb{R}^d which acts naturally on \mathcal{N}_0 as follows: $\tau_x \xi^0 = \sum_{y \in \xi^0} \delta_{y-x}$. The theory developed in [KV86, DMFGW89] makes extensive use of the process of the *environment seen from the particle* $(\tau_{X_t^{\xi^0}} \xi^0)_{t \geq 0}$. Whereas the original random walk is defined on the probability space $(\Omega_{\xi^0}, P_0^{\xi^0})$, this auxiliary process is defined on $\Xi^0 := D(\mathbb{R}_+; \mathcal{N}_0)$ and we write \mathbf{P}_{ξ^0} for its distribution. A generic element of Ξ^0 is denoted by $\underline{\xi}^0 = (\xi_t^0)_{t \geq 0}$. The environment process can be seen from an ‘annealed’ point of view, that is under the distribution $\mathbf{P} := \int \mathbf{P}_{\xi^0}[\cdot] \mathcal{P}_0(d\xi^0)$. In this setting, this is the continuous time Markov process with initial measure \mathcal{P}_0 and transition probabilities given by:

$$\mathbf{P}[\xi_{s+t}^0 = \xi' | \xi_s^0 = \xi] = \mathbf{P}_{\xi}[\xi_t^0 = \xi'] := \begin{cases} P_0^{\xi}[X_t^{\xi} = x] & \text{if } \xi' = \tau_x \xi, x \in \xi \\ 0 & \text{otherwise} \end{cases}, \forall s, t \geq 0.$$

Let us explain how the original walk can be reconstructed from the process $\underline{\xi}^0$ under the assumption of aperiodicity. For $t \geq 0$, let $X_t : \Xi^0 \rightarrow \mathbb{R}^d$ be the random variable defined by:

$$X_t(\underline{\xi}^0) := \sum_{\substack{s \in [0, t] : \\ \Delta_s(\underline{\xi}^0) \neq 0}} \Delta_s(\underline{\xi}^0), \tag{3.2.1}$$

where $\Delta_s(\underline{\xi}^0) = y$ if there exists y such that $\xi_s^0 = \tau_y \xi_{s-}^0$ and is 0 otherwise. Note that Δ_s is well defined for a.a. realization of the environment process since ξ^0 is almost surely aperiodic. The law of $(X_t(\underline{\xi}^0))_{t \geq 0}$ under \mathbf{P}_{ξ^0} is then equal to $P_0^{\xi^0}$, the distribution of the original random walk starting at 0. In particular, the diffusion matrix can be calculated using the following equality:

$$\mathcal{E}_0[E_0^{\xi^0} [(X_t^{\xi^0} \cdot a)^2]] = \mathbf{E}[(X_t \cdot a)^2].$$

The family $\{X_{[s,t]} := X_t - X_s\}_{t > s \geq 0}$ is an antisymmetric additive covariant family of random variables in the sense of [DMFGW89], and X_t has càdlàg paths. Hence, to apply [DMFGW89, Theorem 2.2], we only need to check that:

1. the environment process $(\xi_t^0)_{t \geq 0}$ is reversible and ergodic w.r.t. \mathbf{P} ;
2. the random variables $X_{[s,t]}$, $t > s \geq 0$ are in $L^1(\Xi^0, \mathbf{P})$;

3. the mean forward velocity:

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{\xi^0}[X_t]$$

exists as a strong limit in $L^1(\mathcal{N}_0, \mathcal{P}_0)$;

4. the martingale $M_t := X_t - \int_0^t \varphi(\xi_s^0) ds$ is square integrable under \mathbf{P} .

The rest of this section is devoted to the verification of these hypothesis.

3.2.2 Underlying discrete-time Markov chains and description of the dynamics

It is convenient to decompose the dynamics in terms of the jump chain and the jump times as described in this subsection. Given $\xi^0 \in \mathcal{N}_0$, let us write $\tilde{\Omega}_{\xi^0} := (\xi^0)^{\mathbb{N}}$ and denote by $(\tilde{X}_n^{\xi^0})_{n \in \mathbb{N}}$ a generic element of $\tilde{\Omega}_{\xi^0}$. For $x \in \xi^0$, let $\tilde{P}_x^{\xi^0}$ be the distribution on $\tilde{\Omega}_{\xi^0}$ of the discrete-time simple nearest neighbor random walk on $G(\xi^0)$. We denote by $\tilde{E}_x^{\xi^0}$ the corresponding expectation. On an other probability space (Θ, \mathbf{Q}) , let $T_{n,x}^{\xi^0}$, $x \in \xi^0$, $n \in \mathbb{N}$ be independent exponential random variables with mean $1/\deg_{G(\xi^0)}(x)$, and define on $(\tilde{\Omega}_{\xi^0} \times \Theta, \tilde{P}_x^{\xi^0} \otimes \mathbf{Q})$ the random variables:

$$R_0^{\xi^0} := 0; R_n^{\xi^0} := \sum_{i=0}^{n-1} T_{i, \tilde{X}_i^{\xi^0}}, n \geq 1;$$

$$n_*^{\xi^0}(t) := n \text{ where } R_n^{\xi^0} \leq t < R_{n+1}^{\xi^0}.$$

These random variables can be thought respectively as the instant of n th jump and the number of jumps up to time t for the continuous-time random walk. The same arguments as in [FSBS06, Appendix A] show that almost surely no explosion occurs:

$$\tilde{P}_x^{\xi^0} \otimes \mathbf{Q} \left[\lim_{n \uparrow \infty} R_n^{\xi^0} = \infty \right] = 1.$$

In particular, for a.a. ξ^0 , $n_*^{\xi^0}(t)$ is well defined for all $t \geq 0$ and $(\tilde{X}_{n_*^{\xi^0}(t)}^{\xi^0})_{t \geq 0}$ is a jump process which satisfies the same dynamics (under $\tilde{P}_x^{\xi^0} \otimes \mathbf{Q}$) as $(X_t^{\xi^0})_{t \geq 0}$ (under $P_x^{\xi^0}$): starting from the point x , the particle waits for an exponential time of parameter $\deg_{G(\xi^0)}(x)$, chooses uniformly one of its neighbors in $G(\xi^0)$ and moves to it.

It is useful to introduce the probability measure \mathcal{Q}_0 on \mathcal{N}_0 given by:

$$\mathcal{Q}_0(d\xi^0) = \frac{\deg_{G(\xi^0)}(0)}{\mathcal{E}_0[\deg_{G(\xi^0)}(0)]} \mathcal{P}_0(d\xi^0),$$

and $\tilde{P} := \int \tilde{P}_{\xi^0} \mathcal{Q}_0(d\xi^0)$. In the same way as in Subsection 3.2.3, one can obtain reversibility and ergodicity of the discrete time process under \tilde{P} . Let \tilde{E} denote the expectation under \tilde{P} .

One can also consider the discrete-time version of the environment process $(\tilde{\xi}_n^0 := \tau_{\tilde{X}_n^{\xi^0}} \xi^0)_{n \geq 0}$ and write $\tilde{\mathbf{P}}_{\xi^0}$ for its distribution on the path space $\tilde{\Xi}^0 := \mathcal{N}_0^{\mathbb{N}}$. When ξ^0 is aperiodic the same identification as in Subsection 3.2.1 is possible. It is easy to see that the processes $(\deg_{G(\xi^0)}(\tilde{X}_n^{\xi^0}))_{n \in \mathbb{N}}$ on $(\tilde{\Omega}_{\xi^0}, \tilde{P}_0^{\xi^0})$ and $(\deg_{G(\tilde{\xi}_n^0)}(0))_{n \in \mathbb{N}}$ on $(\tilde{\Xi}^0, \tilde{\mathbf{P}}_{\xi^0})$ have the same law. One can also verify that the process $(Y_n)_{n \in \mathbb{N}}$ defined by $Y_0 = 0$ and:

$$Y_n := \sum_{i=0}^{n-1} \Delta(\tilde{\xi}_i^0, \tilde{\xi}_{i+1}^0),$$

with $\Delta(\tilde{\xi}_i^0, \tilde{\xi}_{i+1}^0)$ defined in the same way as in (3.2.1), has distribution $P_0^{\xi^0}$ on $\tilde{\Omega}_{\xi^0}$. If there does not exist $x \in \xi$ such that $\xi' = \tau_x \xi$, we set $\Delta(\xi, \xi') := 0$.

Remark 3.5. *By slight modifications of the arguments given in Subsection 3.2.3, one can obtain reversibility and ergodicity of $(\tilde{\xi}_n^0)_{n \in \mathbb{N}}$ under $\tilde{\mathbf{P}} := \int \tilde{\mathbf{P}}_{\xi^0} \mathcal{Q}_0(d\xi^0)$. This can be used to derive an annealed invariance principle for the discrete-time random walk thanks to [DMFGW89, Theorem 2.1].*

3.2.3 Reversibility and ergodicity

Reversibility. We want to show that the environment process $(\xi_t^0)_{t \geq 0}$ is reversible with respect to \mathbf{P} and we follow the arguments of the proof of [FSBS06, Proposition 2]. Using the description of the dynamics given in Subsection 3.2.2, one can check that, for any $n \in \mathbb{N}$ and any $\xi_0 = \xi^0, \xi_1, \dots, \xi_n = \xi^{0'} \in \mathcal{N}_0$:

$$\mathbf{P}_{\xi^0} \left[n_{*}^{\xi^0}(t) = n, \tilde{\xi}_{R_1^{\xi^0}}^0 = \xi_1, \dots, \tilde{\xi}_{R_n^{\xi^0}}^0 = \xi_n \right] = \mathbf{P}_{\xi^{0'}} \left[n_{*}^{\xi^{0'}}(t) = n, \tilde{\xi}_{R_1^{\xi^{0'}}}^0 = \xi_{n-1}, \dots, \tilde{\xi}_{R_n^{\xi^{0'}}}^0 = \xi_0 \right],$$

and deduce that:

$$\mathbf{P}_{\xi^0} \left[\xi_t^0 = \xi^{0'} \right] = \mathbf{P}_{\xi^{0'}} \left[\xi_t^0 = \xi^0 \right], \quad t > 0.$$

Hence, for any positive measurable functions f, g and any $t > 0$, with Neveu's exchange formula (see [SW08, Theorem 3.4.5]), we obtain:

$$\begin{aligned} \mathbf{E} \left[f(\xi_0^0) g(\xi_t^0) \right] &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(\xi^0) g(\tau_x \xi^0) \mathbf{P}_{\xi^0} \left[\xi_t^0 = \tau_x \xi^0 \right] \mathcal{P}_0(d\xi^0) \\ &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(\tau_x \xi^0) g(\xi^0) \mathbf{P}_{\tau_x \xi^0} \left[\xi_t^0 = \xi^0 \right] \mathcal{P}_0(d\xi^0) \\ &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(\tau_x \xi^0) g(\xi^0) \mathbf{P}_{\xi^0} \left[\xi_t^0 = \tau_x \xi^0 \right] \mathcal{P}_0(d\xi^0) \\ &= \mathbf{E} \left[f(\xi_t^0) g(\xi_0^0) \right]. \end{aligned}$$

The reversibility of $(\xi_t^0)_{t \geq 0}$ then follows.

Ergodicity. We follow the proof of [FSBS06, Proposition 2]. Note that it suffices to ascertain that for any measurable $A \subset \mathcal{N}_0$ satisfying $\mathbf{P}_{\xi^0}[\xi_t^0 \in A] = \mathbf{1}_A(\xi^0)$ for a.a. ξ^0 , we have $\mathcal{P}_0(A) = 0$ or 1 . Given such an A , there is a subset $B \subset A$ such that $\mathcal{P}_0[A \setminus B] = 0$ and $\mathbf{P}_{\xi^0}[\xi_t^0 \in A] = 1$ for all $\xi^0 \in B$. For $x \in \xi^0$, $\xi^0 \in B$, one then has $\mathbf{P}_{\xi^0}[\xi_t^0 = \tau_x \xi^0 \in A] = \mathbf{P}_{\xi^0}[\xi_t^0 = \tau_x \xi^0] > 0$ by the positivity of the jump rates. Thanks to [FSBS06, Lemma 1 (iii)], $\mathcal{P}_0(A) = 0$ or 1 which completes the proof.

3.2.4 Integrability of $\mathbf{X}_{[s,t]} = \mathbf{X}_t - \mathbf{X}_s$, $t > s \geq 0$

This is an immediate consequence of:

Lemma 3.6. *Let $\gamma \in \mathbb{N}^*$ and $t_0 > 0$. There exists a constant $0 < c_6 = c_6(\gamma, t_0) < \infty$ such that:*

$$\mathcal{E}_0 \left[E_0^{\xi^0} \left[\|X_t^{\xi^0}\|^\gamma \right] \right] \leq c_6 \sqrt{t}, \quad \text{for all } t \leq t_0. \quad (3.2.2)$$

To obtain this lemma we redo the calculations of the proof of [FSBS06, Proposition 1] with $p = q = 2$ using that $\deg_{G(\xi^0)}(0)$ and $\max_{x \sim 0} \|x\|$ admit polynomial moments and we make the dependence on t apparent.

3.2.5 Existence of the mean forward velocity

We want to verify that the mean forward velocity:

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{\xi^0} [X_t]$$

exists as a strong limit in L^1 . Actually, we prove the following stronger result which is also useful for the square integrability of the martingale M_t in Subsection 3.2.6:

Lemma 3.7. *Let $\varphi : \mathcal{N}_0 \mapsto \mathbb{R}^d$ defined by:*

$$\varphi(\xi^0) := \sum_{x \sim 0} x. \quad (3.2.3)$$

Then,

$$\frac{1}{t} \mathbf{E} [X_t] \xrightarrow[t \rightarrow 0]{L^2(\mathcal{N}_0, \mathcal{P}_0)} \varphi. \quad (3.2.4)$$

Proof. As in Subsection 3.2.2, we denote by $n_*^{\xi^0}(t)$ the number of jumps of the environment process starting at ξ^0 up to time t . First, we use that $X_t \mathbf{1}_{n_*^{\xi^0}(t)=0} = X_0 = 0$ and $(a+b)^2 \leq$

$2(a^2 + b^2)$ to write:

$$\begin{aligned}
 & \frac{1}{t^2} \int_{\mathcal{N}_0} \left\| \mathbf{E}_{\xi^0} [X_t] - t\varphi(\xi^0) \right\|^2 \mathcal{P}_0(d\xi^0) \\
 &= \frac{1}{t^2} \int_{\mathcal{N}_0} \left\| \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t)=1}] - t\varphi(\xi^0) + \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t) \geq 2}] \right\|^2 \mathcal{P}_0(d\xi^0) \\
 &\leq \frac{2}{t^2} \int_{\mathcal{N}_0} \left\| \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t)=1}] - t\varphi(\xi^0) \right\|^2 \mathcal{P}_0(d\xi^0) \\
 &\quad + \frac{2}{t^2} \int_{\mathcal{N}_0} \left\| \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t) \geq 2}] \right\|^2 \mathcal{P}_0(d\xi^0). \tag{3.2.5}
 \end{aligned}$$

We need to show that the two summands in (3.2.5) go to 0 with t . We first deal with the first term in the r.h.s. of (3.2.5).

Note that, with the notations of Subsection 3.2.2, for $x \in \xi^0$:

$$\begin{aligned}
 \mathbf{P}_{\xi^0} [\xi_t^0 = \tau_x \xi^0, n_*^{\xi^0}(t) = 1] &\leq \tilde{P}_0^{\xi^0} \otimes \mathbf{Q} [\tilde{X}_1^{\xi^0} = x, T_{0,0}^{\xi^0} \leq t] \\
 &= \tilde{P}_0^{\xi^0} [\tilde{X}_1^{\xi^0} = x] (1 - e^{-\deg_{G(\xi^0)}(0)t}) \\
 &\leq \frac{\mathbf{1}_{x \sim 0}}{\deg_{G(\xi^0)}(0)} \deg_{G(\xi^0)}(0)t = \mathbf{1}_{x \sim 0} t, \tag{3.2.6}
 \end{aligned}$$

where we used that $1 - e^{-u} \leq u$. Thus, $\mathbf{1}_{x \sim 0} t - \mathbf{P}_{\xi^0} [\xi_t^0 = \tau_x \xi^0, n_*^{\xi^0}(t) = 1]$ is nonnegative. Thanks to the Cauchy-Schwarz inequality and the observation above, we have:

$$\begin{aligned}
 & \frac{1}{t^2} \mathcal{E}_0 \left[\left\| \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t)=1}] - t\varphi(\xi^0) \right\|^2 \right] \\
 &\leq \frac{1}{t^2} \mathcal{E}_0 \left[\left(\sum_{x \in \xi^0} (\mathbf{1}_{x \sim 0} t - \mathbf{P}_{\xi^0} [\xi_t^0 = \tau_x \xi^0, n_*^{\xi^0}(t) = 1]) \right) \right. \\
 &\quad \left. \times \left(\sum_{x \in \xi^0} \|x\|^2 (\mathbf{1}_{x \sim 0} t - \mathbf{P}_{\xi^0} [\xi_t^0 = \tau_x \xi^0, n_*^{\xi^0}(t) = 1]) \right) \right] \\
 &\leq \frac{1}{t^2} \mathcal{E}_0 \left[\left(\sum_{x \underset{G(\xi^0)}{\sim} 0} \int_0^t (1 - e^{-\deg_{G(\xi^0)}(0)s - \deg_{G(\xi^0)}(x)(t-s)}) ds \right) \times \left(t \sum_{x \underset{G(\xi^0)}{\sim} 0} \|x\|^2 \right) \right] \\
 &\leq \frac{1}{t} \mathcal{E}_0 \left[\sum_{x \underset{G(\xi^0)}{\sim} 0} \int_0^t (\deg_{G(\xi^0)}(0)s + \deg_{G(\tau_x \xi^0)}(0)(t-s)) ds \deg_{G(\xi^0)}(0) \left(\max_{x \underset{G(\xi^0)}{\sim} 0} \|x\| \right)^2 \right] \\
 &= \frac{t}{2} \left(\mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^3 \left(\max_{x \underset{G(\xi^0)}{\sim} 0} \|x\| \right)^2 \right] \right. \\
 &\quad \left. + \mathcal{E}_0 \left[\sum_{x \underset{G(\xi^0)}{\sim} 0} \deg_{G(\tau_x \xi^0)}(0) \deg_{G(\xi^0)}(0) \left(\max_{x \underset{G(\xi^0)}{\sim} 0} \|x\| \right)^2 \right] \right).
 \end{aligned}$$

Recall the definitions of $\tilde{\mathbf{P}}_{\xi^0}$ and $\tilde{\mathbf{P}}$ from Subsection 3.2.2 and let us note that, if f and g are positive measurable functions defined on \mathcal{N}_0 , we have:

$$\mathcal{E}_0 \left[f(\xi^0) \sum_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} g(\tau_x \xi^0) \right] = \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0) \tilde{\mathbf{E}}_{\xi^0} \left[f(\xi^0) g(\tilde{\xi}_1^0) \right] \right] = \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0) \tilde{\mathbf{E}} \left[f(\xi^0) g(\tilde{\xi}_1^0) \right] \right].$$

It follows using the Cauchy-Schwarz inequality and the stationarity of the discrete-time environment process $(\tilde{\xi}_n^0)_{n \in \mathbb{N}}$ under $\tilde{\mathbf{P}} = \int \tilde{\mathbf{P}}_{\xi^0} \mathcal{Q}_0(d\xi^0)$ that:

$$\begin{aligned} & \frac{1}{t^2} \mathcal{E}_0 \left[\left\| \mathbf{E}_{\xi^0} \left[X_t \mathbf{1}_{n_*^{\xi^0}(t)=1} \right] - t\varphi(\xi^0) \right\|^2 \right] \\ & \leq \frac{t}{2} \left(\mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^6 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0) \right] \tilde{\mathbf{E}} \left[\deg_{G(\tilde{\xi}_1^0)}(0) \deg_{G(\xi^0)}(0) \left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^2 \right] \right) \\ & \leq \frac{t}{2} \left(\mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^6 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0) \right] \tilde{\mathbf{E}} \left[\left(\deg_{G(\tilde{\xi}_1^0)}(0) \right)^2 \right]^{\frac{1}{2}} \tilde{\mathbf{E}} \left[\left(\deg_{G(\xi^0)}(0) \right)^2 \left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right) \\ & \leq \frac{t}{2} \left(\mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^6 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0) \right]^{\frac{1}{2}} \tilde{\mathbf{E}} \left[\left(\deg_{G(\tilde{\xi}_1^0)}(0) \right)^2 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^3 \left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right) \\ & \leq \frac{t}{2} \left(\mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^6 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^4 \right]^{\frac{1}{2}} \right. \\ & \quad \left. + \mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^3 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\deg_{G(\xi^0)}(0) \right)^6 \right]^{\frac{1}{4}} \mathcal{E}_0 \left[\left(\max_{\substack{x \underset{G(\xi^0)}{\sim} 0 \\ G(\xi^0)}} \|x\| \right)^8 \right]^{\frac{1}{4}} \right). \end{aligned}$$

Let us move to the second term in the r.h.s. of (3.2.5). Applying twice the Hölder inequality, we obtain:

$$\begin{aligned} \mathcal{E}_0 \left[\left\| \mathbf{E}_{\xi^0} \left[X_t \mathbf{1}_{n_*^{\xi^0}(t) \geq 2} \right] \right\|^2 \right] & \leq \mathcal{E}_0 \left[\mathbf{E}_{\xi^0} \left[\|X_t\|^4 \right]^{\frac{1}{2}} \mathbf{P}_{\xi^0} \left[n_*^{\xi^0}(t) \geq 2 \right]^{\frac{3}{2}} \right] \\ & \leq \mathcal{E}_0 \left[\mathbf{E}_{\xi^0} \left[\|X_t\|^4 \right] \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\mathbf{P}_{\xi^0} \left[n_*^{\xi^0}(t) \geq 2 \right]^3 \right]^{\frac{1}{2}}. \end{aligned} \quad (3.2.7)$$

Note that:

$$\begin{aligned}
 \mathbf{P}_{\xi^0} [n_*^{\xi^0}(t) \geq 2] &\leq \mathbf{P}_0^{\xi^0} \otimes \mathbf{Q} [T_{0,0}^{\xi^0} \leq t, T_{1, \tilde{X}_1^{\xi^0}}^{\xi^0} \leq t] \\
 &= (1 - e^{-\deg_{G(\xi^0)}(0)t}) \sum_{x \in \xi^0} \tilde{\mathbf{E}}_{\xi^0} \left[(1 - e^{-\deg_{G(\tilde{\xi}_1^0)}(0)t}) \mathbf{1}_{\tilde{\xi}_1^0 = \tau_x \xi^0} \right] \\
 &\leq t^2 \deg_{G(\xi^0)}(0) \tilde{\mathbf{E}}_{\xi^0} [\deg_{G(\tilde{\xi}_1^0)}(0)], \tag{3.2.8}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbf{P}_{\xi^0} [n_*^{\xi^0}(t) \geq 2] &\leq \mathbf{P}_0^{\xi^0} \otimes \mathbf{Q} [T_{0,0}^{\xi^0} \leq t, T_{1, \tilde{X}_1^{\xi^0}}^{\xi^0} \leq t] \\
 &= (1 - e^{-\deg_{G(\xi^0)}(0)t}) \sum_{x \in \xi^0} \tilde{\mathbf{E}}_{\xi^0} \left[(1 - e^{-\deg_{G(\tilde{\xi}_1^0)}(0)t}) \mathbf{1}_{\tilde{\xi}_1^0 = \tau_x \xi^0} \right] \\
 &\leq t \tilde{\mathbf{E}}_{\xi^0} [\deg_{G(\tilde{\xi}_1^0)}(0)]. \tag{3.2.9}
 \end{aligned}$$

Hence, (3.2.8) multiplied (3.2.9) to the square, the Jensen inequality and the stationarity of the discrete-time environment process under $\tilde{\mathbf{P}} = \int \tilde{\mathbf{P}}_{\xi^0} \mathcal{Q}_0(d\xi^0)$ imply that:

$$\begin{aligned}
 \mathcal{E}_0 [\mathbf{P}_{\xi^0} [n_*^{\xi^0}(t) \geq 2]^3] &\leq t^4 \mathcal{E}_0 [\deg_{G(\xi^0)}(0) \tilde{\mathbf{E}}_{\xi^0} [\deg_{G(\tilde{\xi}_1^0)}(0)]^3] \\
 &\leq t^4 \mathcal{E}_0 [\deg_{G(\xi^0)}(0) \tilde{\mathbf{E}}_{\xi^0} [(\deg_{G(\tilde{\xi}_1^0)}(0))^3]] \\
 &= t^4 \mathcal{E}_0 [\deg_{G(\xi^0)}(0)] \tilde{\mathbf{E}} [(\deg_{G(\tilde{\xi}_1^0)}(0))^3] \\
 &= t^4 \mathcal{E}_0 [(\deg_{G(\xi^0)}(0))^4]. \tag{3.2.10}
 \end{aligned}$$

Finally, with (3.2.2), (3.2.7) and (3.2.10), we obtain that for t small:

$$\frac{1}{t^2} \mathcal{E}_0 \left[\left\| \mathbf{E}_{\xi^0} [X_t \mathbf{1}_{n_*^{\xi^0}(t) \geq 2}] \right\|^2 \right] \leq c_7 t^{\frac{1}{4}},$$

which tends to 0 with t . □

3.2.6 Square integrability of the martingale $M_t = X_t - \int_0^t \varphi(\xi_s^0) ds$

Due to Lemma 3.6, $X_t \in L^2(\Xi^0, \mathbf{P})$, so it is enough to see that:

$$\int_0^t \varphi(\xi_s^0) ds \in L^2(\Xi^0, \mathbf{P}).$$

Using the Cauchy-Schwarz inequality and the stationarity of the environment process under \mathbf{P} , one has:

$$\begin{aligned}
 \mathbf{E} \left[\left\| \int_0^t \varphi(\xi_s^0) ds \right\|^2 \right] &\leq t \mathbf{E} \left[\int_0^t \|\varphi(\xi_s^0)\|^2 ds \right] \\
 &= t \int_0^t \mathbf{E} \left[\|\varphi(\xi_s^0)\|^2 \right] ds \\
 &= t^2 \mathbf{E} \left[\|\varphi(\xi_0^0)\|^2 \right] \\
 &= t^2 \mathcal{E}_0 \left[\|\varphi\|^2 \right] \\
 &\leq t^2 \mathcal{E}_0 \left[\deg_{G(\xi^0)}(0)^4 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\max_{x \sim 0} \|x\| \right)^4 \right]^{\frac{1}{2}} < \infty.
 \end{aligned}$$

3.2.7 Infinitesimal square displacement as a L^2 -limit

By following precisely the same proof as in Lemma 3.7 with higher order moments estimates, one obtains:

Lemma 3.8. *For $\xi^0 \in \mathcal{N}_0$, let ψ be the $d \times d$ symmetric matrix characterized by:*

$$(a \cdot \psi(\xi^0)a) := \sum_{x \sim 0} (a \cdot x)^2, \quad a \in \mathbb{R}^d. \quad (3.2.11)$$

Then, $(a \cdot \psi(\cdot)a) \in L^2(\mathcal{N}_0, \mathcal{P}_0)$ and:

$$\frac{1}{t} \mathbf{E} \left[(a \cdot X_t)^2 \right] \xrightarrow[t \rightarrow 0]{L^2(\mathcal{N}_0, \mathcal{P}_0)} (a \cdot \psi(\cdot)a). \quad (3.2.12)$$

Since its assumptions were verified in this section, Theorem 2.2 of [DMFGW89] implies Proposition 3.2. It gives moreover the following characterization of the diffusion matrix D of the limiting Brownian motion:

$$(a \cdot Da) = \mathcal{E}_0 \left[(a \cdot \psi a) \right] - 2 \int_0^\infty \langle \varphi \cdot a, e^{t\mathcal{L}} \varphi \cdot a \rangle_{\mathcal{P}_0}, \quad a \in \mathbb{R}^d, \quad (3.2.13)$$

where \mathcal{L} is the generator of the environment process and φ, ψ are defined in (3.2.3) and (3.2.11) respectively.

3.3 Proof of Proposition 3.4: the diffusion coefficient is non degenerate

From now on, we assume that the hypotheses of Proposition 3.4 are satisfied.

3.3.1 Variational formula

In order to show that the diffusion matrix of the limiting Brownian motion in Proposition 3.2 is nondegenerate, it is useful to obtain a variational characterization of this matrix. The use of variational formulas to approximate diffusion coefficients is rather classical. It appears in particular in papers based on the works of Kozlov [Koz85] or De Masi, Ferrari, Goldstein and Wick [DMFGW89]. In principle, it allows to derive the nondegeneracy of the diffusion coefficient and to obtain numerical estimates for it. See [FSBS06, Bis11] and references therein.

Lemma 3.9. *The diffusion matrix of the limiting Brownian motion in Proposition 3.2 is given by:*

$$(a \cdot Da) = \inf_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left((a \cdot x) + \nabla_x f(\xi^0) \right)^2 \mathcal{P}_0(d\xi^0), \quad a \in \mathbb{R}^d, \quad (3.3.1)$$

where ∇_x is the discrete gradient $\nabla_x f(\xi^0) := f(\tau_x \xi^0) - f(\xi^0)$.

Proof. Minor changes to the proof of [FSBS06, Proposition 3] show that the generator \mathcal{L} of the environment process is non-positive and self-adjoint with core $L^\infty(\mathcal{N}_0, \mathcal{P}_0)$. Moreover, for any $f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)$, one has:

$$\mathcal{L}f(\xi^0) = \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left(f(\tau_x \xi^0) - f(\xi^0) \right),$$

where $c_{0,x}^{\xi^0}$ equals the indicator function of the event $(0, x) \in Ed(\xi^0)$. We refer for example to [RS75, Chapter X] for basic tools to prove this.

Hence, following the same strategy as in [FSBS06] and invoking for example equations (45) and (46) of this paper, one obtains:

$$\int_0^\infty \langle \varphi \cdot a, e^{t\mathcal{L}} \varphi \cdot a \rangle_{\mathcal{P}_0} dt = \sup_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \left(2 \langle \varphi \cdot a, f \rangle_{\mathcal{P}_0} - \langle f, (-\mathcal{L})f \rangle_{\mathcal{P}_0} \right). \quad (3.3.2)$$

On the other hand, Neveu's exchange formula gives:

$$\begin{aligned} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} f(\xi^0)^2 \mathcal{P}_0(d\xi^0) &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,-x}^{\tau_x \xi^0} f(\tau_x \xi^0)^2 \mathcal{P}_0(d\xi^0) \\ &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} f(\tau_x \xi^0)^2 \mathcal{P}_0(d\xi^0) \end{aligned}$$

thus,

$$\begin{aligned} \langle f, (-\mathcal{L})f \rangle_{\mathcal{P}_0} &= \int_{\mathcal{N}_0} f(\xi^0) \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left(f(\xi^0) - f(\tau_x \xi^0) \right) \mathcal{P}_0(d\xi^0) \\ &= \frac{1}{2} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left(f(\tau_x \xi^0)^2 + f(\xi^0)^2 - 2f(\tau_x \xi^0)f(\xi^0) \right) \mathcal{P}_0(d\xi^0) \\ &= \frac{1}{2} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left(\nabla_x f(\xi^0) \right)^2 \mathcal{P}_0(d\xi^0). \end{aligned} \quad (3.3.3)$$

In the same way, Neveu's exchange formula gives:

$$\int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0}(a \cdot x) f(\xi^0) \mathcal{P}_0(d\xi^0) = - \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0}(a \cdot x) f(\tau_x \xi^0) \mathcal{P}_0(d\xi^0)$$

which implies that:

$$\begin{aligned} \langle \varphi \cdot a, f \rangle_{\mathcal{P}_0} &= \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0}(a \cdot x) f(\xi^0) \mathcal{P}_0(d\xi^0) \\ &= -\frac{1}{2} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0}(a \cdot x) \nabla_x f(\xi^0) \mathcal{P}_0(d\xi^0). \end{aligned} \quad (3.3.4)$$

Finally, with equations (3.2.13), (3.3.2)-(3.3.4), and the definition of ψ , one has:

$$\begin{aligned} (a \cdot Da) &= \mathcal{E}_0[(a \cdot \psi a)] - 2 \sup_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \left(2 \langle \varphi \cdot a, f \rangle_{\mathcal{P}_0} - \langle f, (-\mathcal{L})f \rangle_{\mathcal{P}_0} \right) \\ &= \inf_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \left(\mathcal{E}_0[(a \cdot \psi a)] - 4 \langle \varphi \cdot a, f \rangle_{\mathcal{P}_0} + 2 \langle f, (-\mathcal{L})f \rangle_{\mathcal{P}_0} \right) \\ &= \inf_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} c_{0,x}^{\xi^0} \left\{ (a \cdot x)^2 + 2(a \cdot x) \nabla_x f(\xi^0) + (\nabla_x f(\xi^0))^2 \right\} \mathcal{P}_0(d\xi^0) \end{aligned}$$

which gives the required result. \square

For sake of simplicity, we assume from now on that the underlying point process is isotropic. Nevertheless, the arguments below can be easily adapted to the anisotropic case.

3.3.2 Cut-off on the transition rates and lower bound for the diffusion matrix

In the sequel, we will consider the (largest) subgraph of $G(\xi^0)$ in which each edge has length at most $r_c > 0$. For $r_c > 0$ fixed, we set:

$$\tilde{c}_{x,y}^{\xi^0} := \mathbf{1}_{(x,y) \text{ is an edge of } G(\xi^0) \text{ and } \|y-x\| \leq r_c} \leq c_{x,y}^{\xi^0}. \quad (3.3.5)$$

We denote by $G_{r_c}(\xi^0)$ the sub-graph of $G(\xi^0)$ induced by the conductances $\tilde{c}_{x,y}^{\xi^0}$. Thanks to the variational formula (3.3.1), we immediately obtain that:

$$(a \cdot Da) \geq \inf_{f \in L^\infty(\mathcal{N}_0, \mathcal{P}_0)} \int_{\mathcal{N}_0} \sum_{x \in \xi^0} \tilde{c}_{0,x}^{\xi^0} \left((a \cdot x) + \nabla_x f(\xi^0) \right)^2 \mathcal{P}_0(d\xi^0). \quad (3.3.6)$$

As above, one can show that the operator $\hat{\mathcal{L}}$ defined by:

$$\hat{\mathcal{L}}f(\xi^0) := \sum_{x \in \xi^0} \tilde{c}_{0,x}^{\xi^0} \left(f(\tau_x \xi^0) - f(\xi^0) \right), \quad f \in \mathcal{D}(\hat{\mathcal{L}}),$$

has core $L^\infty(\mathcal{N}_0, \mathcal{P}_0)$, is self-adjoint and is the generator of a Markov process on $D(\mathbb{R}_+; \mathcal{N}_0)$. We write $\widehat{\mathbf{P}}_{\xi^0}$ (resp. $\widehat{\mathbf{P}}$) for the probability measure on $D(\mathbb{R}_+; \mathcal{N}_0)$ associated with this Markov process with initial distribution δ_{ξ^0} (resp. \mathcal{P}_0). The same arguments as in Subsection 3.3.1 show that the r.h.s. of (3.3.6) is actually equal to:

$$(a \cdot \widehat{D}a) := \mathcal{E}_0[(a \cdot \widehat{\psi}a)] - 2 \int_0^\infty \langle \widehat{\varphi} \cdot a, e^{t\widehat{\mathcal{L}}}\widehat{\varphi} \cdot a \rangle_{\mathcal{P}_0} dt, \quad (3.3.7)$$

where

$$\widehat{\varphi}(\xi^0) := \sum_{x \in \xi^0} \widehat{c}_{0,x}^{\xi^0} x \text{ and } (a \cdot \widehat{\psi}(\xi^0)a) := \sum_{x \in \xi^0} \widehat{c}_{0,x}^{\xi^0} (a \cdot x)^2.$$

3.3.3 Periodic approximants and electrical networks

The aim of this subsection is to bound from below \widehat{D} by using diffusion coefficients of random walks in appropriate periodic electrical networks. The method is very similar to the one in [DMFGW89, FSBS06] and relies on the theory of electrical networks (see [SW08, LP14]) and homogenization (see [JKO94, Owh03]). Actually, we will define suitable random resistor networks with diffusion coefficient which can be expressed as an effective conductance and which is lower than \widehat{D} . In the sequel, the medium will be periodized along the first coordinate direction and we will assume that $N > r_c$.

Periodized medium

For $\xi \in \mathcal{N}$ a locally finite subset of \mathbb{R}^d and $N > r_c$, we will construct two finite networks from $G(\xi)$. To this end, let us define the following subsets of \mathbb{R}^d :

$$\begin{aligned} C_{2N} &:= [-N, N]^d, \quad \overset{\circ}{C}_{2N} :=]-N, N[^d, \quad \mathfrak{Q}_N^\xi := \xi \cap \overset{\circ}{C}_{2N}, \\ \Gamma_N^\pm &:= \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_1 = \pm N, \|x_j\| < N, j = 2, \dots, d\}, \\ \overline{\mathfrak{V}}_N^\xi &:= \mathfrak{Q}_N^\xi \cup \Gamma_N^- \cup \Gamma_N^+, \quad B_N^- := \{x \in \overset{\circ}{C}_{2N} : x_1 \in]-N, -N + r_c]\}, \\ B_N^+ &:= \{x \in \overset{\circ}{C}_{2N} : x_1 \in [N - r_c, N[\}, \quad B_N^{\xi, \pm} := \mathfrak{Q}_N^\xi \cap B_N^\pm. \end{aligned}$$

A finite (unoriented) graph $\overline{\mathfrak{G}}_N^\xi = (\overline{\mathfrak{V}}_N^\xi, \overline{\mathfrak{E}}_N^\xi)$ is then constructed; it has an edge between x and y if:

- $x, y \in \mathfrak{Q}_N^\xi$, $\{x, y\}$ is an edge of $G(\xi)$ and $\|x - y\| \leq r_c$,
- $x \in B_N^{\xi, -}$ and $y \in \Gamma_N^-$,
- $x \in B_N^{\xi, +}$ and $y \in \Gamma_N^+$.

Another graph $\mathfrak{G}_N^\xi = (\mathfrak{V}_N^\xi, \mathfrak{E}_N^\xi)$ is obtained from $\overline{\mathfrak{G}}_N^\xi$ by identifying vertices:

$$x_- = (-N, x_2, \dots, x_d) \text{ and } x_+ = (N, x_2, \dots, x_d).$$

We denote by \mathfrak{p} the projection map from $\overline{\mathfrak{V}}_N^\xi$ onto \mathfrak{V}_N^ξ . Note that $\mathfrak{p}|_{\mathfrak{Q}_N^\xi}$ is nothing but the identity. The graph \mathfrak{G}_N^ξ will represent the *periodized medium* in which any vertex in $\Gamma_N := \mathfrak{p}(\Gamma_N^-) (= \mathfrak{p}(\Gamma_N^+))$ is connected to any vertex in $B_N^\xi := B_N^{\xi,-} \cup B_N^{\xi,+}$.

We consider now a continuous-time random walk on \mathfrak{V}_N^ξ with infinitesimal generator given by:

$$\mathcal{L}_N^\xi f(x) := \sum_{y \in \mathfrak{V}_N^\xi} \mathfrak{c}_N^\xi(x, y) (f(y) - f(x)), \quad \forall x \in \mathfrak{V}_N^\xi, \quad (3.3.8)$$

where

$$\mathfrak{c}_N^\xi(x, y) := \begin{cases} 1 & \text{if } \{x, y\} \in \mathfrak{E}_N^\xi \text{ with } x, y \in \mathfrak{Q}_N^\xi \\ \frac{1}{\#(\Gamma_N)} & \text{if } \{x, y\} \in \mathfrak{E}_N^\xi \text{ with } x \text{ or } y \in \Gamma_N \\ 0 & \text{if } \{x, y\} \notin \mathfrak{E}_N^\xi \end{cases} . \quad (3.3.9)$$

One checks that the generator \mathcal{L}_N^ξ is self-adjoint in $L^2(\mathfrak{V}_N^\xi, \mathfrak{m}_N^\xi)$ where \mathfrak{m}_N^ξ is the uniform distribution on \mathfrak{V}_N^ξ . In general \mathfrak{G}_N^ξ is not connected and \mathfrak{m}_N^ξ is not ergodic. In this case, ergodic probability measures are uniform distributions on the connected components of \mathfrak{G}_N^ξ and one can apply the results of De Masi, Ferrari, Goldstein and Wick in each connected component. We denote by \mathbf{P}_N^ξ (resp. $\mathbf{P}_{N,x}^\xi, x \in \mathfrak{V}_N^\xi$) the distribution on the path space $\Omega_N^\xi := D(\mathbb{R}_+; \mathfrak{V}_N^\xi)$ of the random walk with generator \mathcal{L}_N^ξ and initial distribution \mathfrak{m}_N^ξ (resp. δ_x). In order to apply one more time [DMFGW89, Theorem 2.2], let us introduce d_1 the antisymmetric function on \mathfrak{V}_N^ξ such that:

$$d_1(x, y) := \begin{cases} y_1 - x_1 & \text{if } x, y \in \mathfrak{Q}_N^\xi \\ y_1 + N & \text{if } y \in \mathfrak{Q}_N^\xi, y_1 < 0 \text{ and } x \in \Gamma_N \\ y_1 - N & \text{if } y \in \mathfrak{Q}_N^\xi, y_1 > 0 \text{ and } x \in \Gamma_N \end{cases} . \quad (3.3.10)$$

Moreover, for $t \geq 0$, $\underline{\omega} = (\omega_s)_{s \geq 0} \in \Omega_N^\xi$, we define the random variable:

$$X_{N,t}^{(1),\xi}(\underline{\omega}) := \sum_{\substack{s \in [0, t] \\ \omega_{s-} \neq \omega_s}} d_1(\omega_{s-}, \omega_s). \quad (3.3.11)$$

The last quantity is the sum of the increments of the projections along the first coordinate of the positions taken by the particle up to time t . It is then possible to apply [DMFGW89, Theorem 2.2] to the time-covariant antisymmetric family $\{X_{N,[s,t]}^{(1),\xi} := X_{N,t}^{(1),\xi} - X_{N,s}^{(1),\xi}\}_{t > s \geq 0}$ and obtain in particular:

Lemma 3.10. For $N \in \mathbb{N}$, $N > r_c$ and $\xi \in \mathcal{N}$:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbf{E}_N^\xi \left[(X_{N,t}^{(1),\xi})^2 \right] = D_N^\xi := \mathbf{m}_N^\xi(\psi_N^\xi) - 2 \int_0^\infty \langle \varphi_N^\xi, e^{t\mathcal{L}_N^\xi} \varphi_N^\xi \rangle_{\mathbf{m}_N^\xi} dt, \quad (3.3.12)$$

where φ_N^ξ and ψ_N^ξ are the scalar functions defined by:

$$\varphi_N^\xi(x) := \sum_{y \in \mathfrak{V}_N^\xi} \mathbf{c}_N^\xi(x, y) d_1(x, y) \text{ and } \psi_N^\xi(x) := \sum_{y \in \mathfrak{V}_N^\xi} \mathbf{c}_N^\xi(x, y) d_1(x, y)^2. \quad (3.3.13)$$

Relationship between \widehat{D} and the diffusion constants of periodized media

We compare \widehat{D} (and therefore D) with suitable averages of the diffusion coefficients of the periodized media.

Lemma 3.11. It holds that:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi(\psi_N^\xi) \right] = \mathcal{E}_0 \left[\widehat{\psi}_{1,1} \right] \quad (3.3.14)$$

and

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\langle \varphi_N^\xi, e^{t\mathcal{L}_N^\xi} \varphi_N^\xi \rangle_{\mathbf{m}_N^\xi} \right] = \langle \widehat{\varphi}_1, e^{t\widehat{\mathcal{L}}} \widehat{\varphi}_1 \rangle_{\mathcal{P}_0} \quad (3.3.15)$$

where $\widehat{\varphi}_1$ (resp. $\widehat{\psi}_{1,1}$) is the first coordinate element of $\widehat{\varphi}$ (resp. the first diagonal element of $\widehat{\psi}$) defined in Subsection 3.3.2.

In particular,

$$D \geq \widehat{D} \geq \left(\limsup_{N \rightarrow \infty} \mathcal{E} \left[D_N^\xi \right] \right) \mathbf{I}_d. \quad (3.3.16)$$

Proof. We first derive (3.3.16) from (3.3.14) and (3.3.15). By the isotropy of \mathcal{P}_0 , \widehat{D} is a multiple of the identity matrix. Hence, using (3.3.7) and Fatou's lemma:

$$\begin{aligned} D \geq \widehat{D} &= \left(\mathcal{E}_0 \left[\widehat{\psi}_{1,1} \right] - 2 \int_0^\infty \langle \widehat{\varphi}_1, e^{t\widehat{\mathcal{L}}} \widehat{\varphi}_1 \rangle_{\mathcal{P}_0} dt \right) \mathbf{I}_d \\ &= \left(\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi(\psi_N^\xi) \right] - 2 \int_0^\infty \lim_{N \rightarrow \infty} \mathcal{E} \left[\langle \varphi_N^\xi, e^{t\mathcal{L}_N^\xi} \varphi_N^\xi \rangle_{\mathbf{m}_N^\xi} \right] dt \right) \mathbf{I}_d \\ &\geq \left(\limsup_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi(\psi_N^\xi) \right] - 2 \liminf_{N \rightarrow \infty} \int_0^\infty \mathcal{E} \left[\langle \varphi_N^\xi, e^{t\mathcal{L}_N^\xi} \varphi_N^\xi \rangle_{\mathbf{m}_N^\xi} \right] dt \right) \mathbf{I}_d \\ &= \left(\limsup_{N \rightarrow \infty} \mathcal{E} \left[D_N^\xi \right] \right) \mathbf{I}_d. \end{aligned}$$

We now prove (3.3.14). Recall the definition of $\tilde{c}_{x,y}^0$ from (3.3.5) and note that $\tilde{c}_{x,y}^0 = 0$ if $x \in C_{2N-2r_c}$ and $y \notin C_{2N}$. Thus, for $x \in \mathfrak{Q}_{N-r_c}$:

$$\begin{aligned} \widehat{\psi}_{1,1}(\tau_x \xi) &= (e_1 \cdot \widehat{\psi}(\tau_x \xi) e_1) = \sum_{y \in \tau_x \xi} \mathbf{1}_{\{0,y\}} \text{ is an edge of } G(\tau_x \xi), \|y\| \leq r_c y_1^2 \\ &= \sum_{\substack{y \in \mathfrak{V}_N^\xi \\ \{x,y\} \in \mathbf{e}_N^\xi}} (y_1 - x_1)^2 = \psi_N^\xi(x). \end{aligned}$$

This allows us to write:

$$\mathbf{m}_N^\xi(\psi_N^\xi) = \frac{1}{\#(\xi \cap C_{2N}) + \#\Gamma_N} \left(\sum_{x \in \Omega_{N-r_c}^\xi} \widehat{\psi}_{1,1}(\tau_x \xi) + \sum_{x \in \mathfrak{V}_N^\xi \setminus \Omega_{N-r_c}^\xi} \psi_N^\xi(x) \right)$$

and to conclude by taking the \mathcal{P} -expectations in this identity and by using Lemma 3.15 (5) and Lemma 3.16 with $h_N^\xi = \psi_N^\xi$. Before moving to the proof of (3.3.15), let us note that the same method shows that for $1 \leq p \leq 4$:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi(|\varphi_N^\xi|^p) \right] = \mathcal{E}_0[|\widehat{\varphi}_1|^p] \leq r_c \mathcal{E}_0[\#(\xi^0 \cap B_2(0, r_c))^p] < \infty. \quad (3.3.17)$$

In order to prove (3.3.15), let us fix $0 < \alpha < 1$, set $M := N - \lfloor N^\alpha \rfloor$ and define the exit times:

$$\mathcal{T}_N^\xi := \inf\{t \geq 0 : \omega_t \notin C_{2N-2r_c}\}, \quad \omega = (\omega_t)_{t \geq 0} \in \Omega_N^\xi = D(\mathbb{R}_+; \mathfrak{V}_N^\xi). \quad (3.3.18)$$

Since $(e^{t\mathcal{L}_N^\xi} \varphi_N^\xi)(x) = \mathbf{E}_{N,x}^\xi[\varphi_N^\xi(\omega_t)]$, we can write:

$$\mathcal{E} \left[\langle \varphi_N^\xi, e^{t\mathcal{L}_N^\xi} \varphi_N^\xi \rangle_{\mathbf{m}_N^\xi} \right] = \mathcal{E} \left[A_{1,N}^\xi + A_{2,N}^\xi + A_{3,N}^\xi \right], \quad (3.3.19)$$

where

$$\begin{aligned} A_{1,N}^\xi &:= \mathbf{m}_N^\xi \left(\mathbf{1}_{x \notin C_{2M}} \varphi_N^\xi(x) \mathbf{E}_{N,x}^\xi \left[\varphi_N^\xi(\omega_t) \right] \right), \\ A_{2,N}^\xi &:= \mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \varphi_N^\xi(x) \mathbf{E}_{N,x}^\xi \left[\mathbf{1}_{\mathcal{T}_N^\xi \leq t} \varphi_N^\xi(\omega_t) \right] \right), \\ A_{3,N}^\xi &:= \mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \varphi_N^\xi(x) \mathbf{E}_{N,x}^\xi \left[\mathbf{1}_{\mathcal{T}_N^\xi > t} \varphi_N^\xi(\omega_t) \right] \right). \end{aligned}$$

Then (3.3.15) will be established once we show:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[A_{i,N}^\xi \right] = 0, \quad i = 1, 2 \quad \text{and} \quad \lim_{N \rightarrow \infty} \mathcal{E} \left[A_{3,N}^\xi \right] = \langle \widehat{\varphi}_1, e^{t\widehat{\mathcal{L}}} \widehat{\varphi}_1 \rangle_{\mathcal{P}_0}. \quad (3.3.20)$$

For the first limit in (3.3.20), we use the Cauchy-Schwarz inequality and the stationarity of the process ω under \mathbf{m}_N^ξ to write:

$$\begin{aligned} \left| \mathcal{E} \left[A_{1,N}^\xi \right] \right| &\leq \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{1}_{x \notin C_{2M}} |\varphi_N^\xi(x)| \mathbf{E}_{N,x}^\xi \left[|\varphi_N^\xi(\omega_t)| \right] \right) \right] \\ &\leq \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathfrak{V}_N^\xi \setminus C_{2M} \right) \right]^{\frac{1}{2}} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\varphi_N^\xi(x)^2 \mathbf{E}_{N,x}^\xi \left[\varphi_N^\xi(\omega_t)^2 \right] \right) \right]^{\frac{1}{2}} \\ &\leq \mathcal{E} \left[\frac{\#\Gamma_N + \#(\xi \cap C_{2N} \setminus C_{2M})}{\#\Gamma_N + \#(\xi \cap C_{2N})} \right]^{\frac{1}{2}} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\varphi_N^\xi(x)^4 \right) \right]^{\frac{1}{4}} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{E}_{N,x}^\xi \left[\varphi_N^\xi(\omega_t)^4 \right] \right) \right]^{\frac{1}{4}} \\ &= \mathcal{E} \left[\frac{\#\Gamma_N + \#(\xi \cap C_{2N} \setminus C_{2M})}{\#\Gamma_N + \#(\xi \cap C_{2N})} \right]^{\frac{1}{2}} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\varphi_N^\xi(x)^4 \right) \right]^{\frac{1}{2}}. \end{aligned}$$

The first factor of the r.h.s. goes to 0 thanks to the dominated convergence theorem while the second remains bounded thanks to (3.3.17). Let us turn our attention to the second limit in (3.3.20). Applying twice the Cauchy-Schwarz inequality and using stationarity, one has:

$$\begin{aligned}
 |\mathcal{E}[A_{2,N}^\xi]| &\leq \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}|\varphi_N^\xi(x)|\mathbf{E}_{N,x}^\xi\left[\mathbf{1}_{\mathcal{T}_N^\xi \leq t}|\varphi_N^\xi(\omega_t)|\right]\right)\right] \\
 &\leq \mathcal{E}\left[\mathbf{m}_N^\xi\left(\varphi_N^\xi(x)^2\right)\right]^{\frac{1}{2}}\mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\mathbf{E}_{N,x}^\xi\left[\mathbf{1}_{\mathcal{T}_N^\xi \leq t}\varphi_N^\xi(\omega_t)^2\right]\right)\right]^{\frac{1}{2}} \\
 &\leq \mathcal{E}\left[\mathbf{m}_N^\xi\left(\varphi_N^\xi(x)^2\right)\right]^{\frac{1}{2}}\mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{E}_{N,x}^\xi\left[\varphi_N^\xi(x)^4\right]\right)\right]^{\frac{1}{4}}\mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\mathbf{P}_{N,x}^\xi\left[\mathcal{T}_N^\xi \leq t\right]\right)\right]^{\frac{1}{4}} \\
 &= \mathcal{E}\left[\mathbf{m}_N^\xi\left(\varphi_N^\xi(x)^2\right)\right]^{\frac{1}{2}}\mathcal{E}\left[\mathbf{m}_N^\xi\left(\varphi_N^\xi(x)^4\right)\right]^{\frac{1}{4}}\mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\mathbf{P}_{N,x}^\xi\left[\mathcal{T}_N^\xi \leq t\right]\right)\right]^{\frac{1}{4}}.
 \end{aligned}$$

Once again, thanks to (3.3.17), the first two terms are uniformly bounded. The last factor goes to 0 by Lemma 3.17.

To deal with the last limit in (3.3.17), we need to introduce the following exit times:

$$\mathcal{T}_{N,x} := \inf\{t \geq 0 : x + X_t(\underline{\xi}^0) \notin C_{2N-2r_c}\}, \quad x \in \xi^0 \cap C_{2M}, \quad \underline{\xi}^0 = (\xi_t^0)_{t \geq 0} \in \Xi^0, \quad (3.3.21)$$

with $X_t(\underline{\xi}^0)$ as defined in (3.2.1).

Note that, if N is large enough (*i.e.* $\lfloor N^\alpha \rfloor > r_c$), one has for $x \in C_{2M}$:

$$\varphi_N^\xi(x) = \widehat{\varphi}_1(\tau_x \xi) \text{ and } \mathbf{E}_{N,x}^\xi\left[\mathbf{1}_{\mathcal{T}_N^\xi > t}\varphi_N^\xi(\omega_t)\right] = \widehat{\mathbf{E}}_{\tau_x \xi}\left[\mathbf{1}_{\mathcal{T}_{N,x} > t}\widehat{\varphi}_1(\xi_t^0)\right].$$

Hence,

$$\mathcal{E}\left[A_{3,N}^\xi\right] = \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\widehat{\varphi}_1(\tau_x \xi)\widehat{\mathbf{E}}_{\tau_x \xi}\left[\mathbf{1}_{\mathcal{T}_{N,x} > t}\widehat{\varphi}_1(\xi_t^0)\right]\right)\right].$$

In the same spirit as what we did to obtain $\lim_{N \rightarrow \infty} \mathcal{E}[A_{2,N}^\xi] = 0$, we get using (3.4.7):

$$\lim_{N \rightarrow \infty} \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\widehat{\varphi}_1(\tau_x \xi)\widehat{\mathbf{E}}_{\tau_x \xi}\left[\mathbf{1}_{\mathcal{T}_{N,x} \leq t}\widehat{\varphi}_1(\xi_t^0)\right]\right)\right] = 0,$$

so that

$$\lim_{N \rightarrow \infty} \mathcal{E}\left[A_{3,N}^\xi\right] = \lim_{N \rightarrow \infty} \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\widehat{\varphi}_1(\tau_x \xi)\widehat{\mathbf{E}}_{\tau_x \xi}\left[\widehat{\varphi}_1(\xi_t^0)\right]\right)\right].$$

Finally, Lemma 3.15 (6), applied to the $L^2(\mathcal{N}_0, \mathcal{P}_0)$ function

$$h(\xi^0) := \widehat{\varphi}_1(\xi^0)\widehat{\mathbf{E}}_{\xi^0}\left[\widehat{\varphi}_1(\xi_t^0)\right] = \widehat{\varphi}_1(\xi^0)\left(e^{t\widehat{\mathcal{L}}}\widehat{\varphi}_1\right)(\xi^0),$$

gives

$$\lim_{N \rightarrow \infty} \mathcal{E}\left[A_{3,N}^\xi\right] = \lim_{N \rightarrow \infty} \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}}\widehat{\varphi}_1(\tau_x \xi)\widehat{\mathbf{E}}_{\tau_x \xi}\left[\widehat{\varphi}_1(\xi_t^0)\right]\right)\right] = \langle \widehat{\varphi}_1, e^{t\widehat{\mathcal{L}}}\widehat{\varphi}_1 \rangle_{\mathcal{P}_0}.$$

□

Relationship between D_N^ξ and the effective conductance of suitable resistor networks

We want to establish a link between the diffusion constant D_N^ξ and the effective conductance of an appropriate electrical network. To this end, recall definitions of Subsection 3.3.3 and equip edges of $\overline{\mathfrak{G}}_N^\xi$ with conductances:

$$\overline{\mathfrak{c}}_N^\xi(x, y) := \mathfrak{c}_N^\xi(\mathfrak{p}(x), \mathfrak{p}(y)), \quad \{x, y\} \in \overline{\mathfrak{E}}_N^\xi,$$

where \mathfrak{c}_N^ξ is given by (3.3.9) and \mathfrak{p} is the projection map from $\overline{\mathfrak{V}}_N^\xi$ onto \mathfrak{V}_N^ξ . Then, the effective conductivity $\overline{\kappa}_N^\xi$ of this electrical network is defined as the total amount of current flowing from Γ_N^- to Γ_N^+ when a unit potential difference is imposed between these two sets. From Dirichlet's principle (see [LP14, §2.4] for example), we know that:

$$\overline{\kappa}_N^\xi = \frac{1}{2} \inf \left\{ \sum_{x \in \overline{\mathfrak{V}}_N^\xi} \sum_{y: \{x, y\} \in \overline{\mathfrak{E}}_N^\xi} \overline{\mathfrak{c}}_N^\xi(x, y) (\nabla_{(x, y)} g)^2 : g : \overline{\mathfrak{V}}_N^\xi \mapsto \mathbb{R}, g|_{\Gamma_N^-} \equiv 0, g|_{\Gamma_N^+} \equiv 1 \right\} \quad (3.3.22)$$

where $\nabla_{(x, y)} g := g(y) - g(x)$.

The same method as the one used in Subsection 3.3.1 shows that the diffusion coefficient D_N^ξ (defined in Lemma 3.10) satisfies the variational formula:

$$\begin{aligned} D_N^\xi &= \frac{1}{\#\mathfrak{V}_N^\xi} \inf_{f \in L^2(\overline{\mathfrak{V}}_N^\xi)} \left\{ \sum_{x \in \overline{\mathfrak{V}}_N^\xi} \sum_{y: \{x, y\} \in \overline{\mathfrak{E}}_N^\xi} \mathfrak{c}_N^\xi(x, y) (d_1(x, y) + \nabla_{(x, y)} f)^2 \right\} \\ &= \frac{1}{\#\mathfrak{V}_N^\xi} \inf_{\substack{f \in L^2(\overline{\mathfrak{V}}_N^\xi) \\ f|_{\mathfrak{p}^{-1}(\{x\})} = C_x, \forall x \in \Gamma_N}} \left\{ \sum_{x \in \overline{\mathfrak{V}}_N^\xi} \sum_{y: \{x, y\} \in \overline{\mathfrak{E}}_N^\xi} \overline{\mathfrak{c}}_N^\xi(x, y) (\nabla_{(x, y)} (\pi_1 + f))^2 \right\}, \end{aligned} \quad (3.3.23)$$

where π_1 is the projection along the first coordinate.

We claim that:

$$D_N^\xi = \frac{1}{\#\mathfrak{V}_N^\xi} \inf_{\substack{f \in L^2(\overline{\mathfrak{V}}_N^\xi) \\ f|_{\Gamma_N^- \cup \Gamma_N^+} = N}} \left\{ \sum_{x \in \overline{\mathfrak{V}}_N^\xi} \sum_{y: \{x, y\} \in \overline{\mathfrak{E}}_N^\xi} \overline{\mathfrak{c}}_N^\xi(x, y) (\nabla_{(x, y)} (\pi_1 + f))^2 \right\}. \quad (3.3.24)$$

To see this, let us fix the values of such a function f on \mathfrak{Q}_N^ξ , say $f|_{\mathfrak{Q}_N^\xi} = h|_{\mathfrak{Q}_N^\xi}$. We will verify that the function minimizing (3.3.23) subject to this condition is actually constant on $\Gamma_N^- \cup \Gamma_N^+$. Note that, when the values of f on \mathfrak{Q}_N^ξ are fixed, we know the value of:

$$\sum_{x \in \mathfrak{Q}_N^\xi} \sum_{y \in \mathfrak{Q}_N^\xi} \mathbf{1}_{\{x, y\} \in \overline{\mathfrak{E}}_N^\xi} (\nabla_{(x, y)} (\pi_1 + f))^2,$$

and we are led to find C_x , $x \in \Gamma_N$ minimizing:

$$\sum_{x \in \Gamma_N} \left\{ \sum_{y \in B_N^{\xi,-}} (y_1 + N + h(y) - C_x)^2 + \sum_{y \in B_N^{\xi,+}} (y_1 - N + h(y) - C_x)^2 \right\}.$$

The summand depends on x only through C_x and we must choose C_x as the (unique) minimizer of this quantity for each $x \in \Gamma_N$. Thus the infimum in (3.3.23) is achieved for functions which are constant on $\Gamma_N^- \cup \Gamma_N^+$. Since f appears in the variational formula in a gradient form, one can assume that f is equal to N on $\Gamma_N^- \cup \Gamma_N^+$ and (3.3.24) is proved.

Now, let us note that the functions $g : \overline{\mathfrak{V}}_N^\xi \mapsto \mathbb{R}$ satisfying $g|_{\Gamma_N^-} \equiv 0$ and $g|_{\Gamma_N^+} \equiv 1$ can be expressed as:

$$g = \frac{1}{2N} (\pi_1 + f),$$

where f is a function on $\overline{\mathfrak{V}}_N^\xi$ such that $f|_{\Gamma_N^- \cup \Gamma_N^+} \equiv N$.

Hence,

$$D_N^\xi = \frac{4N^2}{\#(\mathfrak{V}_N^\xi)} \inf \left\{ \sum_{x \in \overline{\mathfrak{V}}_N^\xi} \sum_{y: \{x,y\} \in \overline{\mathfrak{E}}_N^\xi} \overline{\mathfrak{c}}_N^\xi(x,y) (\nabla_{(x,y)} g)^2 : g : \overline{\mathfrak{V}}_N^\xi \mapsto \mathbb{R}, g|_{\Gamma_N^-} \equiv 0, g|_{\Gamma_N^+} \equiv 1 \right\}.$$

Thanks to Dirichlet's principle (3.3.22), we have just proved:

Lemma 3.12.

$$D_N^\xi = \frac{8N^2}{\#(\mathfrak{V}_N^\xi)} \overline{\kappa}_N^\xi. \tag{3.3.25}$$

Lower bounds on $\overline{\kappa}_N^\xi$ are obtained in Subsection 3.3.4

3.3.4 Lower bound on $\overline{\kappa}_N^\xi$

In order to obtain lower bounds on $\overline{\kappa}_N^\xi$ for large N , we use the discretization of the space and the ‘good boxes’ introduced in Proposition 3.4. These ‘good boxes’ will allow us to link nice properties of the random graphs we consider with nice properties of independent Bernoulli processes on \mathbb{Z}^d . The rest of the proof then relies on techniques from percolation theory which also appear in [FSBS06, Section 6]. This method is actually quite classical and is partially described in the 2-dimensional case in [Kes82, Chapter 11].

Percolation estimates and LR-crossings

Let us consider an independent Bernoulli site percolation process $\mathbb{Y} = \{Y_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ on \mathbb{Z}^d with parameter $p > p_c^{\text{site}}(\mathbb{Z}^2)$. We will use fine estimates about the number of left-right crossings (LR-crossings) of $R_{2N,2(N-1)} := [-N, N] \times [-N+1, N-1]^{d-1}$. Let us recall that

a LR-crossing with length $k-1$ of $R_{2N,2(N-1)}$ for a realization of $\{Y_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ is a sequence of distinct points $(\mathbf{z}_1, \dots, \mathbf{z}_k)$ in $R_{2N,2(N-1)} \cap \mathbb{Z}^d$ such that $|\mathbf{z}_i - \mathbf{z}_{i+1}| = 1$ for $1 \leq i \leq k-1$, $Y_{\mathbf{z}_i} = 1$ for $1 \leq i \leq k$, $\mathbf{z}_1^{(1)} = -N$, $\mathbf{z}_k^{(1)} = N$, $-N < \mathbf{z}_i^{(1)} < N$ for $1 < i < k$, and $\mathbf{z}_i^{(l)} = \mathbf{z}_j^{(l)}$ for any $l \geq 3$ and for $1 \leq i < j \leq k$. Two crossings are said to be disjoint if all the involved \mathbf{z}_j 's are distinct. Note that, by definition, LR-crossings are contained into 2-dimensional slices. The techniques of [Gri89, Sections 2.6 and 11.3] can be used, in the context of site percolation, to derive:

Proposition 3.13. *If $p > p_c(2)$, there exist positive constants $a = a(p)$, $b = b(p)$, and $c = c(p)$ such that for all $N \in \mathbb{N}^*$:*

$$\mathbb{P}\left[\mathbb{Y} \text{ has less than } bN^{d-1} \text{ disjoint LR-crossings in } R_{2N,2(N-1)}\right] \leq ce^{-aN}. \quad (3.3.26)$$

Thanks to the domination assumption (3) in Proposition 3.4, there exists a coupling of the process of the good boxes \mathbb{X} and the supercritical independent Benoulli proces \mathbb{Y} on \mathbb{Z}^d such that almost surely $Y_{\mathbf{z}} \leq X_{\mathbf{z}}$ for all $\mathbf{z} \in \mathbb{Z}^d$. In other words, any good box corresponds almost surely to an open site in the percolation process \mathbb{Y} .

From now on, fix $r_c := \sqrt{d+3}K$. Consider a LR-crossing $\gamma^{\text{perco}} = (\mathbf{z}_1, \dots, \mathbf{z}_k)$ of $R_{2N,2(N-1)}$. One can associate to γ^{perco} a LR-crossing $\gamma^{\overline{\mathfrak{G}}_{KN}^\xi}$ of $\overline{\mathfrak{G}}_{KN}^\xi$ from a point of Γ_{KN}^- to a point of Γ_{KN}^+ . To construct this LR-crossing, we first link $K\mathbf{z}_1$ to the reference vertex $v_{\mathbf{z}_2}$ of $B_{\mathbf{z}_2}$. Since $v_{\mathbf{z}_2} \in B_{\mathbf{z}_2}$, $\|K\mathbf{z}_1 - v_{\mathbf{z}_2}\| \leq r_c$ and $(K\mathbf{z}_1, v_{\mathbf{z}_2})$ is an edge of $\overline{\mathfrak{G}}_{KN}^\xi$. Next, for $i = 2, \dots, k-2$, we go from $v_{\mathbf{z}_i}$ to $v_{\mathbf{z}_{i+1}}$ using the path given in (2) of the definition of the good boxes in Proposition 3.4. These edges are edges of G_{r_c} (and thus of $\overline{\mathfrak{G}}_{KN}^\xi$) thanks to the choice of r_c . Finally, we link $v_{\mathbf{z}_{k-1}}$ to $K\mathbf{z}_k \in \Gamma_{KN}^+$. We assume that $\gamma^{\overline{\mathfrak{G}}_{KN}^\xi}$ is self-avoiding; if not, we remove loops. Since the boxes $B_{\mathbf{z}_1}, \dots, B_{\mathbf{z}_k}$ are good, it is easy to see that:

- (a) $L(\gamma^{\overline{\mathfrak{G}}_{KN}^\xi}) \leq L \times L(\gamma^{\text{perco}})$ where $L(\cdot)$ denotes the graph length of a path,
- (b) $\gamma^{\overline{\mathfrak{G}}_{KN}^\xi} \subset \bigcup_{\mathbf{z} \in \gamma^{\text{perco}}} (B_{\mathbf{z}} \cap C_{2KN})$,

in particular,

- (c) if γ_1^{perco} and γ_2^{perco} are two disjoint LR-crossings of $R_{2N,2(N-1)}$ for \mathbb{X} , then $\gamma_1^{\overline{\mathfrak{G}}_{KN}^\xi}$ and $\gamma_2^{\overline{\mathfrak{G}}_{KN}^\xi}$ are disjoint LR-crossings of $\overline{\mathfrak{G}}_N^\xi$.

Lower bound for $\overline{\kappa}_{KN}^\xi$ for N -good configurations

We say that a configuration $\xi \in \mathcal{N}$ is N -good if the corresponding realization of the percolation process \mathbb{Y} has at least bN^{d-1} disjoint LR-crossings of $R_{2N,2(N-1)}$. From now on, we assume that ξ is an N -good configuration and we write $\mathcal{C}_N(\xi)$ for a collection of at least bN^{d-1} such disjoint crossings. By the arguments given at the end of Subsection 3.3.4, there are at least bN^{d-1} LR-crossings of C_{2KN} in the graph $\overline{\mathfrak{G}}_N^\xi$. We write as before

$\gamma^{\bar{\mathfrak{G}}_{KN}^\xi}$ for the LR-crossing in $\bar{\mathfrak{G}}_N^\xi$ associated with the LR-crossing γ^{perco} of $R_{2N,2(N-1)}$ for \mathbb{Y} . Since the paths are assumed to be self-avoiding and the γ^{perco} 's in $\mathcal{C}_N(\xi)$ are disjoint, it holds that:

$$\sum_{\gamma^{\text{perco}} \in \mathcal{C}_N(\xi)} L(\gamma^{\bar{\mathfrak{G}}_{KN}^\xi}) \leq L \sum_{\gamma^{\text{perco}} \in \mathcal{C}_N(\xi)} L(\gamma^{\text{perco}}) \leq L(2N+1)^d.$$

With Jensen's inequality, one can deduce:

$$\sum_{\gamma^{\text{perco}} \in \mathcal{C}_N(\xi)} \frac{1}{L(\gamma^{\bar{\mathfrak{G}}_{KN}^\xi})} \geq \frac{\#\left(\mathcal{C}_N(\xi)\right)^2}{\sum_{\gamma^{\text{perco}} \in \mathcal{C}_N(\xi)} L(\gamma^{\bar{\mathfrak{G}}_{KN}^\xi})} \geq \frac{(bN^{d-1})^2}{L(2N+1)^d} \geq c_8(KN)^{d-2}, \quad (3.3.27)$$

with $c_8 = c_8(d, K)$.

We are now ready to bound from below $\bar{\kappa}_{KN}^\xi$. To this end, let us introduce another electrical network $\check{\mathfrak{G}}_{KN}^\xi$ with vertex set $\check{\mathfrak{V}}_{KN}^\xi := \mathfrak{V}_{KN}^\xi \cup \{\check{\Gamma}_{KN}^-, \check{\Gamma}_{KN}^+\}$, where $\check{\Gamma}_{KN}^-, \check{\Gamma}_{KN}^+$ are two extra points. Its edge set $\check{\mathfrak{E}}_{KN}^\xi$ consists in the collection of edges of $\bar{\mathfrak{E}}_{KN}^\xi$ with both endpoints in \mathfrak{V}_{KN}^ξ and extra edges between $\check{\Gamma}_{KN}^\pm$ and each point of $B_{KN}^{\xi, \pm}$. We assign a conductance 1 to each of these edges and we define the effective conductance $\check{\kappa}_{KN}^\xi$ of $\check{\mathfrak{G}}_{KN}^\xi$ as the total amount of current flowing from $\check{\Gamma}_{KN}^-$ to $\check{\Gamma}_{KN}^+$ when a unit difference of potential is imposed. This new network can be seen as obtained from $\bar{\mathfrak{G}}_{KN}^\xi$ by merging together vertices in Γ_{KN}^- and Γ_{KN}^+ respectively. Since the same difference of potential was imposed between Γ_{KN}^- and Γ_{KN}^+ in the definition of $\bar{\kappa}_{KN}^\xi$, it is not difficult to see that $\check{\kappa}_{KN}^\xi = \bar{\kappa}_{KN}^\xi$.

The disjoint LR-crossing $\gamma^{\bar{\mathfrak{G}}_{KN}^\xi}$ of $\bar{\mathfrak{G}}_{KN}^\xi$ exhibited above can be used to bound $\check{\kappa}_{KN}^\xi$ from below. They correspond to disjoint (except at their endpoints) paths from Γ_{KN}^- to Γ_{KN}^+ in the new network. By *Rayleigh monotonicity principle*, one can obtain a lower bound on $\check{\kappa}_{KN}^\xi$ by removing each edge which does not appear in these crossings. Thanks to the *series law* each of these crossings $\gamma^{\bar{\mathfrak{G}}_{KN}^\xi}$ has 'total' resistance $L(\gamma^{\bar{\mathfrak{G}}_{KN}^\xi})$. Finally, due to *parallels law* and (3.3.27), one has:

$$\check{\kappa}_{KN}^\xi = \bar{\kappa}_{KN}^\xi \geq \sum_{\gamma^{\text{perco}} \in \mathcal{C}_N(\xi)} \frac{1}{L(\gamma^{\bar{\mathfrak{G}}_{KN}^\xi})} \geq c_8(KN)^{d-2}. \quad (3.3.28)$$

3.3.5 Conclusion by collecting bounds

Due to (3.3.16), it remains to show that:

$$\limsup_{N \rightarrow \infty} \mathcal{E} \left[D_N^\xi \right] > 0.$$

But, with Lemma 3.12 and (3.3.28), we obtain:

$$\limsup_{N \rightarrow \infty} \mathcal{E} \left[D_N^\xi \right] \geq 8K^2 \limsup_{N \rightarrow \infty} \mathcal{E} \left[\frac{N^2}{\#\left(\mathfrak{V}_{KN}^\xi\right) \bar{\kappa}_{KN}^\xi} \right] \geq c_9 \lim_{N \rightarrow \infty} \mathcal{E} \left[\frac{(KN)^d}{\#\left(\mathfrak{V}_{KN}^\xi\right)} \mathbf{1}_{\xi \text{ is } N\text{-good}} \right],$$

where we used that:

$$\mathcal{E} \left[\frac{(KN)^2}{\#(\mathfrak{B}_{KN}^\xi)} \mathbf{1}_{\xi \text{ is } N\text{-bad}} \right] \geq 0.$$

Observe that thanks to (3.3.26):

$$\begin{aligned} \mathcal{E} \left[\frac{(KN)^d}{\#(\mathfrak{B}_{KN}^\xi)} \mathbf{1}_{\xi \text{ is } N\text{-bad}} \right] &= \mathcal{E} \left[\frac{(KN)^d}{\#(\xi \cap C_{2KN}) + \#(\Gamma_{2KN})} \mathbf{1}_{\xi \text{ is } N\text{-bad}} \right] \\ &\leq c_{10} N^d \mathcal{P}[\xi \text{ is } N\text{-bad}] \leq c_{11} N^d e^{-aN} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Thus, we obtain that:

$$\limsup_{N \rightarrow \infty} \mathcal{E} [D_N^\xi] \geq c_9 \lim_{N \rightarrow \infty} \mathcal{E} \left[\frac{(KN)^d}{\#(\mathfrak{B}_{KN}^\xi)} \right] > 0$$

thanks to Lemma 3.15 (3). Hence, the diffusion coefficient of the limiting Brownian motion in Proposition 3.2 is non degenerate and Proposition 3.4 is proved.

3.4 Good boxes and paths constructions for applications of Proposition 3.4

Due to the sub-graph relation $G_n(\xi) \subset \text{Gab}(\xi) \subset \text{DT}(\xi)$, it is enough to construct good boxes and paths for the creek-crossing graphs G_n , $n \geq 2$ to obtain Theorem 3.1. Note that the constructions can be easily made by hand in the case of Delaunay triangulations as in Chapter 2. These constructions are described in the following subsection.

We will use [HNGS15, Lemma 8] to verify assumptions of Proposition 3.4.

Lemma 3.14 (Lemma 8 in [HNGS15]). *Fix $n \geq 2$. Assume that ξ is distributed according to a simple, stationary, isotropic and m -dependent point process satisfying **(V)** and **(D)**. Assume in addition that its second factorial moment measure is absolutely continuous with respect to $2d$ -Lebesgue measure and its density $\rho(\cdot, \cdot)$ is bounded from above by some positive constant.*

There exist then $0 < c_{12}, c_{13} < \infty$ such that:

$$\mathcal{P}[\mathcal{A}_{G_n, x, L}] \geq 1 - c_{12} \exp(-L^{c_{13}}),$$

where $\mathcal{A}_{G_n, x, L}$ is the event ‘any two vertices of G_n $u, v \in x + [-\frac{L}{2}, \frac{L}{2}]^d$ are connected by a G_n -path contained in $x + [-L, L]^d$ ’.

In the sequel, we assume that the underlying point process satisfies the hypothesis of Lemma 3.14.

Definition of the good boxes

For $K \geq 1$ to be determined later, consider a partition of \mathbb{R}^d into boxes of side K :

$$B_{\mathbf{z}} = B_{\mathbf{z}}^K := K\mathbf{z} + \left[-\frac{K}{2}, \frac{K}{2} \right]^d, \mathbf{z} \in \mathbb{Z}^d.$$

We say that a box $B_{\mathbf{z}_1}$ is *nice* if:

-i- $\#(B_{\mathbf{z}_1} \cap \xi) \leq 2c_4 K^d,$

-ii- when $B_{\mathbf{z}_1}$ is (regularly) cut into $\alpha_d^d := (4\lceil\sqrt{d}\rceil + 5)^d$ sub-boxes $b_i^{\mathbf{z}_1}$ of side $s := K/\alpha_d$, each of these sub-boxes contains at least one point of ξ .

For $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$ two neighboring boxes, we write $b_i^{\mathbf{z}_1, \mathbf{z}_2}$, $i = 1, \dots, \alpha_d + 1$, for the sub-boxes of side s intersecting the line segment $[K\mathbf{z}_1, K\mathbf{z}_2]$. We denote by $c_i^{\mathbf{z}_1, \mathbf{z}_2}$ the center of the small box $b_i^{\mathbf{z}_1, \mathbf{z}_2}$ and by $v_i^{\mathbf{z}_1, \mathbf{z}_2}$ the nucleus of the Voronoi cell of $c_i^{\mathbf{z}_1, \mathbf{z}_2}$.

We finally say that the box $B_{\mathbf{z}_1}$ is *good* for G_n if it is nice and:

-iii- for any $\mathbf{z}_2 \in \mathbb{Z}^d$ with $\|\mathbf{z}_1 - \mathbf{z}_2\|_\infty = 1$ and any $i \in \{1, \dots, 2\lceil\sqrt{d}\rceil + 3\}$, $\mathcal{A}_{G_n, c_i^{\mathbf{z}_1, \mathbf{z}_2}, (1+\sqrt{d})s}$ holds.

Construction of paths

For each good box $B_{\mathbf{z}_i}$, we choose the nucleus of the Voronoi cell of $K\mathbf{z}_i$ as reference vertex and denote it by $v_{\mathbf{z}_i}$. Thanks to -ii- in the definition of good boxes, one can check that $v_{\mathbf{z}_i} \in B_{\mathbf{z}_i}$.

Consider two neighboring good boxes $B_{\mathbf{z}_1}$ and $B_{\mathbf{z}_2}$. We want to verify that there exists a $G_n(\xi)$ -path between $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ so that condition (2) of Proposition 3.4 holds. To this end, let us consider the collection $v_1^{\mathbf{z}_1, \mathbf{z}_2} = v_{\mathbf{z}_1}, \dots, v_{\alpha_d+1}^{\mathbf{z}_1, \mathbf{z}_2} = v_{\mathbf{z}_2}$ of the nuclei of the Voronoi cells of the centers of the sub-boxes $b_i^{\mathbf{z}_1, \mathbf{z}_2}$, $1 \leq i \leq \alpha_d + 1$. We claim that, for $i = 1, \dots, \alpha_d$, there is a $G_n(\xi)$ -path $\gamma_i^{\mathbf{z}_1, \mathbf{z}_2}$ between $v_i^{\mathbf{z}_1, \mathbf{z}_2}$ and $v_{i+1}^{\mathbf{z}_1, \mathbf{z}_2}$ included in $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Condition -ii- in the definition of nice boxes implies that $v_i^{\mathbf{z}_1, \mathbf{z}_2}$ and $v_{i+1}^{\mathbf{z}_1, \mathbf{z}_2}$ belong to $c_i^{\mathbf{z}_1, \mathbf{z}_2} + [-(1+\sqrt{d})s, (1+\sqrt{d})s]^d$. The existence of $\gamma_i^{\mathbf{z}_1, \mathbf{z}_2}$ then follows from -iii- in the definition of good boxes and the fact that $c_i^{\mathbf{z}_1, \mathbf{z}_2} + [-(1+\sqrt{d})s, (1+\sqrt{d})s]^d \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$. Finally, we obtain the path between $v_{\mathbf{z}_1}$ and $v_{\mathbf{z}_2}$ needed in Proposition 3.4 by concatenating the γ_i 's and removing any loop. This path has length at most $4c_4 K^d$ since it is simple and contained in the union of two good boxes.

Domination

Since the underlying point process ξ is supposed m -dependent, the process of good boxes has a finite range of dependence k . Clearly, the fact that a box $B_{\mathbf{z}_1}$ is nice or not depends only on the behavior of ξ in this box. Moreover, due to the definition of the edge set of $G_n(\xi)$, the fact that two vertices $u, v \in \xi$ are connected by an edge in $G_n(\xi)$ only depends

on $B(u, n\|u - v\|) \cap \xi$. Hence, in order to decide if $\mathcal{A}_{G_n, c_i^{z_1, z_2}, (1+\sqrt{d})s}$ holds or not, we only need to know $B(c_i^{z_1, z_2}, (2n+1)\sqrt{d}(1+\sqrt{d})s) \cap \xi$. Thanks to [LSS97, Theorem 0.0], we know that there exists $p^* = p^*(d, k) < 1$ such that if $\mathcal{P}[B_{\mathbf{z}} \text{ is good}] \geq p^*$, then the process of the good boxes $\mathbb{X} = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ dominates an independent site percolation process on \mathbb{Z}^d with parameter greater than $p_c^{\text{site}}(\mathbb{Z}^2)$. We will chose s and thus K such that $\mathcal{P}[B_{\mathbf{z}} \text{ is good}] \geq p^*$.

Using **(V)** and **(D)**, one can show that, for s large enough:

$$\mathcal{P}[B_{\mathbf{z}} \text{ is not nice}] \leq \frac{1-p^*}{2}.$$

Moreover, using stationarity and isotropy of ξ and Lemma 3.14:

$$\begin{aligned} & \mathbb{P}[\exists \mathbf{z}' \sim \mathbf{z}, \exists i \in \{1, \dots, 2\lceil\sqrt{d}\rceil + 3\} \text{ s.t. } \mathcal{A}_{G_n, c_i^{z, z'}, (1+\sqrt{d})s} \text{ does not hold}] \\ & \leq \sum_{\mathbf{z}' \sim \mathbf{z}} \sum_{i=1}^{2\lceil\sqrt{d}\rceil+3} \mathbb{P}[(\mathcal{A}_{G_n, c_i^{z, z'}, (1+\sqrt{d})s})^c] \\ & = 2d(2\lceil\sqrt{d}\rceil + 3)\mathbb{P}[(\mathcal{A}_{G_n, 0, (1+\sqrt{d})s})^c] \\ & = 2d(2\lceil\sqrt{d}\rceil + 3)c_{12} \exp\left(- (1+\sqrt{d})^{c_{13}} s^{c_{13}}\right) \\ & \leq \frac{1-p^*}{2}, \end{aligned}$$

for s large enough. This completes the proof.

Appendix: technical results for the proof of Lemma 3.11

In this appendix we collect preliminary results used in the proof of Lemma 3.11 and the assumptions of Proposition 3.4 are supposed to hold.

Stationary point processes and their Palm versions We first recall useful facts concerning stationary point processes and their Palm versions (see also Chapter 1).

Lemma 3.15. 1. For $p \geq 1$, if A is the disjoint union of k unit cubes:

$$\mathcal{E}[\#(\xi \cap A)^p] \leq k^p \mathcal{E}[\#(\xi \cap C_1)^p].$$

2. Let $1 \leq p \leq 8$. Let $(A_n)_{n \in \mathbb{N}}$ be a convex averaging sequence of bounded Borel sets in \mathbb{R}^d , that is a sequence of convex sets satisfying $A_n \subset A_{n+1}$ and $r(A_n) \rightarrow \infty$ where $r(A) := \sup\{r : A \text{ contains a ball of radius } r\}$. The following ergodic result holds:

$$\frac{\#(\xi \cap A_n)}{\lambda \text{Vol}(A_n)} \xrightarrow{\text{a.s., } L^p} 1.$$

3. Let $1 \leq p \leq 8$. For any $C \in \mathbb{R}$:

$$\frac{\lambda \text{Vol}(C_{2N})}{\#(\xi \cap C_{2N}) + CN^{d-1}} \xrightarrow{L^p} 1.$$

4. For any $h \in L^2(\mathcal{N}_0, \mathcal{P}_0)$:

$$\mathcal{E} \left[\int_A h(\tau_x \xi) \xi(dx) \right] = \lambda \text{Vol}(A) \mathcal{E}_0[h], \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

5. For any $h \in L^2(\mathcal{N}_0, \mathcal{P}_0)$:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi \left(h(\tau_x \xi) \right) \right] = \mathcal{E}_0[h].$$

6. Fix $0 < \alpha < 1$ and define $M = M(N, \alpha) := N - \lfloor N^\alpha \rfloor$. For any $h \in L^2(\mathcal{N}_0, \mathcal{P}_0)$:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\frac{1}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} \int_{C_{2M}} h(\tau_x \xi) \xi(dx) \right] = \mathcal{E}_0[h].$$

Proof. Since the point process is stationary, 1. is immediate. Point 2. is a consequence of [DVJ08, Proposition 12.2.II] and of the ergodicity of \mathcal{P} . Observe that:

$$\begin{aligned} \mathcal{E} \left[\left| \frac{\lambda \text{Vol}(C_{2N})}{\#(\xi \cap C_{2N}) + CN^{d-1}} - 1 \right|^p \right] \\ \leq \left(\frac{\lambda N}{C} \right)^p \mathcal{P} \left[\#(\xi \cap C_{2N}) \leq c_1 N^d \right] \\ + \mathcal{E} \left[\left| \frac{\lambda \text{Vol}(C_{2N})}{\#(\xi \cap C_{2N}) + CN^{d-1}} - 1 \right|^p \mathbf{1}_{\#(\xi \cap C_{2N}) \geq c_1 N^d} \right]. \end{aligned}$$

Thanks to assumption (3.1.3) in Proposition 3.4, the first summand vanishes in the limit. Using point 2., one has:

$$f_N(\xi) := \left| \frac{\lambda \text{Vol}(C_{2N})}{\#(\xi \cap C_{2N}) + CN^{d-1}} - 1 \right|^p \mathbf{1}_{\#(\xi \cap C_{2N}) \geq c_1 N^d} \xrightarrow[N \rightarrow \infty]{a.s.} 0.$$

Since $|f_N|$ is bounded, the second summand goes to 0 by the dominated convergence theorem and point 3. is proved. Point 4. is a special case of Campbell formula (see for example [SW08, Theorem 3.3.3]). Thanks to point 4, in order to derive point 5, it suffices to show that:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\left(\frac{1}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - \frac{1}{\lambda \text{Vol}(C_{2N-2r_c})} \right) \int_{C_{2N-2r_c}} h(\tau_x \xi) \xi(dx) \right] = 0.$$

Using four times the Cauchy-Schwarz inequality, and points 1. and 4., one has:

$$\begin{aligned}
 & \mathcal{E} \left[\left(\frac{1}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - \frac{1}{\lambda \text{Vol}(C_{2N-2r_c})} \right) \int_{C_{2N-2r_c}} h(\tau_x \xi) \xi(dx) \right]^2 \\
 & \leq \mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^2 \frac{\#(\xi \cap C_{2N-2r_c})}{\lambda \text{Vol}(C_{2N-2r_c})} \right] \\
 & \quad \times \frac{1}{\lambda \text{Vol}(C_{2N-2r_c})} \mathcal{E} \left[\frac{1}{\#(\xi \cap C_{2N-2r_c})} \left(\int_{C_{2N-2r_c}} h(\tau_x \xi) \xi(dx) \right)^2 \right] \\
 & \leq \mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^2 \frac{\#(\xi \cap C_{2N-2r_c})}{\lambda \text{Vol}(C_{2N-2r_c})} \right] \\
 & \quad \times \frac{1}{\lambda \text{Vol}(C_{2N-2r_c})} \mathcal{E} \left[\int_{C_{2N-2r_c}} h(\tau_x \xi)^2 \xi(dx) \right] \\
 & = \mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^2 \frac{\#(\xi \cap C_{2N-2r_c})}{\lambda \text{Vol}(C_{2N-2r_c})} \right] \mathcal{E}_0[h^2] \\
 & \leq \frac{1}{\lambda} \mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^4 \right]^{\frac{1}{2}} \mathcal{E} \left[\frac{\#(\xi \cap C_{2N-2r_c})^2}{\text{Vol}(C_{2N-2r_c})^2} \right]^{\frac{1}{2}} \mathcal{E}_0[h^2] \\
 & \leq \frac{1}{\lambda} \mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^4 \right]^{\frac{1}{2}} \mathcal{E}[\#(\xi \cap C_1)^2]^{\frac{1}{2}} \mathcal{E}_0[h^2].
 \end{aligned}$$

This completes the proof since $h \in L^2(\mathcal{N}_0, \mathcal{P}_0)$, $\mathcal{E}[\#(\xi \cap C_1)^2] < \infty$ and by point 3:

$$\mathcal{E} \left[\left(\frac{\lambda \text{Vol}(C_{2N-2r_c})}{\#(\xi \cap C_{2N}) + \#(\Gamma_N)} - 1 \right)^4 \right] \xrightarrow{N \rightarrow \infty} 0.$$

The proof of point 6. is very similar to the previous one. □

The mean boundary contributions are negligible

Lemma 3.16. *Let $h_N^\xi : \mathfrak{X}_N^\xi \mapsto \mathbb{R}$ be a function verifying:*

$$|h_N^\xi(x)| \leq c_{14} \begin{cases} \#(\xi \cap \overline{B_2(x, r_c)}) & \text{if } x \in \Omega_N^\xi \\ \frac{\#(B_N^\xi)}{\#(\Gamma_N)} & \text{if } x \in \Gamma_N \end{cases}, \quad (3.4.1)$$

for some $c_{14} < \infty$ independent of N .

Then the mean boundary contributions vanish in the limit:

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\frac{1}{\#(\xi \cap C_{2N}) + \#\Gamma_N} \sum_{x \in \mathfrak{V}_N^\xi \setminus \mathfrak{Q}_{N-r_c}^\xi} |h_N^\xi(x)|^p \right] = 0, \quad 1 \leq p \leq 4. \quad (3.4.2)$$

In particular, this is the case for $h_N^\xi = \varphi_N^\xi$ or ψ_N^ξ defined in (3.3.13).

Proof. Note that with (3.4.1):

$$\sum_{x \in \mathfrak{V}_N^\xi \setminus \mathfrak{Q}_{N-r_c}^\xi} |h_N^\xi(x)|^p \leq c_{14}^p \left\{ \#\Gamma_N \frac{\#(B_N^\xi)^p}{\#(\Gamma_N)^p} + \sum_{x \in \mathfrak{Q}_N^\xi \setminus \mathfrak{Q}_{N-r_c}^\xi} \#(\xi \cap \overline{B_2(x, r_c)})^p \right\}.$$

But, on the one hand:

$$\begin{aligned} \mathcal{E} \left[\frac{\#\Gamma_N}{\#(\xi \cap C_{2N}) + \#\Gamma_N} \frac{\#(B_N^\xi)^p}{\#(\Gamma_N)^p} \right] &\leq \mathcal{E} \left[\frac{\#\Gamma_N^2}{(\#(\xi \cap C_{2N}) + \#\Gamma_N)^2} \right]^{\frac{1}{2}} \mathcal{E} \left[\frac{\#(B_N^\xi)^{2p}}{\#(\Gamma_N)^{2p}} \right]^{\frac{1}{2}} \\ &\leq c_{15} \mathcal{E} \left[\frac{\#\Gamma_N^2}{(\#(\xi \cap C_{2N}) + \#\Gamma_N)^2} \right]^{\frac{1}{2}}, \end{aligned} \quad (3.4.3)$$

where c_{15} is a constant which does not depend on N thanks to the point (1) of Lemma 3.15. On the other hand, using the Cauchy-Schwarz inequality and Lemma 3.15 (5):

$$\begin{aligned} &\mathcal{E} \left[\frac{1}{\#(\xi \cap C_{2N}) + \#\Gamma_N} \sum_{x \in \mathfrak{Q}_N^\xi \setminus \mathfrak{Q}_{N-r_c}^\xi} \#(\xi \cap \overline{B_2(x, r_c)})^p \right] \\ &\leq \mathcal{E} \left[\frac{\#(\xi \cap C_{2N} \setminus C_{2N-2r_c})}{(\#(\xi \cap C_{2N}) + \#\Gamma_N)^2} \right]^{\frac{1}{2}} \\ &\quad \times \mathcal{E} \left[\frac{1}{\#(\xi \cap C_{2N} \setminus C_{2N-2r_c})} \left(\sum_{x \in \mathfrak{Q}_N^\xi \setminus \mathfrak{Q}_{N-r_c}^\xi} \#(\xi \cap \overline{B_2(x, r_c)})^p \right)^2 \right]^{\frac{1}{2}} \\ &\leq \text{Vol}(C_{2N} \setminus C_{2N-2r_c})^{\frac{1}{2}} \mathcal{E} \left[\frac{\#(\xi \cap C_{2N} \setminus C_{2N-2r_c})}{(\#(\xi \cap C_{2N}) + \#\Gamma_N)^2} \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\#(\xi \cap B_2(0, r_c))^{2p} \right]^{\frac{1}{2}} \end{aligned} \quad (3.4.4)$$

$$\leq \left(\frac{\text{Vol}(C_{2N} \setminus C_{2N-2r_c})}{\lambda \text{Vol}(C_{2N})} \right)^{\frac{1}{2}} \mathcal{E} \left[\frac{\lambda \text{Vol}(C_{2N})}{\#(\xi \cap C_{2N}) + \#\Gamma_N} \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\#(\xi \cap B_2(0, r_c))^{2p} \right]^{\frac{1}{2}}. \quad (3.4.5)$$

Hence, (3.4.2) is proved since the r.h.s. of (3.4.3) and the r.h.s of (3.4.4) go to 0 thanks to Lemma 3.15 (3).

Let us check that φ_N^ξ defined in (3.3.13) satisfies (3.4.1). If $x \in \mathfrak{Q}_N^\xi$, then:

$$\begin{aligned} |\varphi_N^\xi(x)| &\leq \sum_{y \in \mathfrak{Q}_N^\xi} \mathbf{1}_{\{x,y\} \in E_G(\xi), \|x-y\| \leq r_c} |y_1 - x_1| \\ &\quad + \frac{1}{\#(\Gamma_N)} \sum_{y \in \Gamma_N} \left(\mathbf{1}_{x \in B_N^{\xi,-}} |x_1 + N| + \mathbf{1}_{x \in B_N^{\xi,+}} |x_1 - N| \right) \\ &\leq \left(\#(\xi \cap \overline{B_2(x, r_c)}) - 1 \right) r_c + r_c = \#(\xi \cap \overline{B_2(x, r_c)}) r_c. \end{aligned}$$

If $x \in \Gamma_N$, then:

$$|\varphi_N^\xi(x)| \leq \frac{1}{\#(\Gamma_N)} \sum_{y \in B_N^{\xi,-}} |y_1 + N| + \frac{1}{\#(\Gamma_N)} \sum_{y \in B_N^{\xi,+}} |y_1 - N| \leq \frac{r_c \#(B_N^\xi)}{\#(\Gamma_N)}.$$

Hence, φ_N^ξ satisfies (3.4.1) with $c_{14} = r_c$. The same computations show that ψ_N^ξ satisfies (3.4.1) with $c_{14} = r_c^2$. \square

Control of exit times Let us give a sketch of proof of the following result which is a slight rewriting of [FSBS06, Lemma 5].

Lemma 3.17. *Let \mathcal{T}_N^ξ and $\mathcal{T}_{N,x}$ be as in (3.3.18) and (3.3.21), $0 < \alpha < 1$ and $M := N - \lfloor N^\alpha \rfloor$. Then,*

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \mathbf{P}_{N,x}^\xi \left[\mathcal{T}_N^\xi \leq t \right] \right) \right] = 0, \quad (3.4.6)$$

$$\lim_{N \rightarrow \infty} \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \widehat{\mathbf{P}}_{\tau_x \xi} \left[\mathcal{T}_{N,x} \leq t \right] \right) \right] = 0. \quad (3.4.7)$$

Proof. Recall the notations from Subsections 3.3.2 and 3.3.3. We first note that the expectations in (3.4.6) and (3.4.7) coincide. Actually, for each $N \in \mathbb{N}^*$, each locally finite set $\xi \subset \mathbb{R}^d$ and each $x \in \xi$, one can define a probability measure μ on $\Omega_N^\xi \times \Xi^0$ such that:

$$\mu(A \times \Xi^0) = \mathbf{P}_{N,x}^{\xi^0}(A), \quad \forall A \in \mathcal{B}(\Omega_N^\xi),$$

$$\mu(\Omega_N^\xi \times B) = \widehat{\mathbf{P}}_{\tau_x \xi}(B), \quad \forall B \in \mathcal{B}(\Xi^0),$$

and such that, μ -a.s., $\mathcal{T}_N^\xi = \mathcal{T}_{N,x}$ and $\omega_t = x + X_t(\xi^0)$, $0 \leq t < \mathcal{T}_N^\xi$. The existence of such a coupling implies that $\mathbf{P}_{N,x}^\xi[\mathcal{T}_N^\xi \leq t] = \widehat{\mathbf{P}}_{\tau_x \xi}[\mathcal{T}_{N,x} \leq t]$ and we only need to prove (3.4.6).

To this end, let us part $C_{2N-2r_c} \setminus C_{2M}$ into unit cubes $Q_i = Q_{N,i}$, $i \in I = I_N$. For $n \in \mathbb{N}^*$, we set $I_n^* := \{\underline{l} := (l_1, \dots, l_n) \in I^n : l_j \neq l_k, j \neq k\}$. For $\underline{\omega} \in \Omega_N^\xi$ such that $\mathcal{T}_N^\xi(\underline{\omega}) < \infty$, let us define:

$$k = k(\underline{\omega}) := \#\{i \in I : \exists 0 \leq t < \mathcal{T}_N^\xi(\underline{\omega}), \omega_t \in Q_i\},$$

and $\underline{i} := (i_1, \dots, i_k) \in I_k^*$ the list of such indices sorted in the chronological order of first visits. For $j = 1, \dots, k$, we denote by x_j the first site of Q_j visited by $\underline{\omega}$ and by t_j the time spent at x_j during the first visit.

Now let $\{T_i^\xi, i \in I, \xi \in \mathcal{N}\}$ be a family of independent exponential random variables with respective parameters $\#(\xi \cap \tilde{Q}_i)$, where

$$\tilde{Q}_i := \{y \in \mathbb{R}^d : \text{dist}(y, Q_i) \leq r_c\}.$$

Remark that given ξ , k and $\underline{x} = (x_1, \dots, x_k)$, t_j ($1 \leq j \leq k$) are independent exponential variables with respective parameters no larger than $\#(\xi \cap \tilde{Q}_i)$. Thus, n , \underline{y} being fixed:

$$\mathbf{P}_{N,x}^\xi[\mathcal{T}_N^\xi \leq t \mid k = n, \underline{x} = \underline{y}] \leq \mathbb{P}[T_{l_1}^\xi + \dots + T_{l_n}^\xi \leq t]. \quad (3.4.8)$$

Since $k \geq k_{\min} := \lfloor N^\alpha \rfloor - r_c$, one has using (3.4.8):

$$\begin{aligned} & \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}} \mathbf{P}_{N,x}^\xi[\mathcal{T}_N^\xi \leq t]\right)\right] \\ &= \sum_{n=\lfloor N^\alpha \rfloor - r_c}^{\#I} \sum_{\underline{l} \in I_n^*} \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}} \sum_{\underline{y} \in \prod_{j=1}^n Q_{l_j} \cap \mathfrak{W}_N^\xi} \mathbf{P}_{N,x}^\xi[\mathcal{T}_N^\xi \leq t, k = n, \underline{x} = \underline{y}]\right)\right] \\ &\leq \sum_{n=\lfloor N^\alpha \rfloor - r_c}^{\#I} \sum_{\underline{l} \in I_n^*} \mathcal{E}\left[\mathbf{m}_N^\xi\left(\mathbf{1}_{x \in C_{2M}} \mathbf{P}_{N,x}^\xi[k = n, \underline{i} = \underline{l}]\right) \times \mathbb{P}[T_{l_1}^\xi + \dots + T_{l_n}^\xi \leq t]\right]. \end{aligned} \quad (3.4.9)$$

It remains to bound the r.h.s of (3.4.9). To this end, for $\kappa > 0$ and $\underline{l} \in I_n^*$, let us define the event:

$$\mathcal{A} = \mathcal{A}(\kappa, \underline{l}) = \left\{ \xi \in \mathcal{N} : \#\{j : 1 \leq j \leq n, \#(\xi \cap \tilde{Q}_j) > \kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]\} > \frac{n}{2} \right\}.$$

Due to the Markov inequality and the stationarity of \mathcal{P} , one has:

$$\begin{aligned} \mathcal{P}(\mathcal{A}) &\leq \frac{2}{n} \mathcal{E}\left[\#\{j : 1 \leq j \leq n, \#(\xi \cap \tilde{Q}_j) > \kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]\}\right] \\ &\leq 2\mathcal{P}\left[\#(\xi \cap \tilde{Q}_1) > \kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]\right] \xrightarrow{\kappa \rightarrow \infty} 0. \end{aligned}$$

Note that if $\xi \notin \mathcal{A}$, at least $\lceil n/2 \rceil$ of the exponential random variables $T_{l_1}^\xi, \dots, T_{l_n}^\xi$ have parameters smaller than $\kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]$. It follows that on \mathcal{A}^c :

$$\mathbb{P}[T_{l_1}^\xi + \dots + T_{l_n}^\xi \leq t] \leq \rho(n, \kappa) := e^{-\kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]t} \sum_{r=\lceil \frac{n}{2} \rceil}^{\infty} \frac{(\kappa \mathcal{E}[\#(\xi \cap \tilde{Q}_1)]t)^r}{r!}.$$

Finally, we deduce by collecting bounds that:

$$\begin{aligned}
 & \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \mathbf{P}_{N,x}^\xi [\mathcal{T}_N^\xi \leq t] \right) \right] \\
 & \leq \mathcal{E} \left[\mathbf{m}_N^\xi \left(\mathbf{1}_{x \in C_{2M}} \sum_{n=\lfloor N^\alpha \rfloor - r_c}^{\#I} \sum_{l \in I_n^*} \mathbf{P}_{N,x}^\xi [k = n, \underline{i} = \underline{l}] \right) \left(\mathbf{1}_{\mathcal{A}} + \mathbf{1}_{\mathcal{A}^c} \rho(\lfloor N^\alpha \rfloor - r_c, \kappa) \right) \right] \\
 & \leq 2\mathcal{P} \left[\#(\xi \cap \tilde{Q}_1) > \kappa \mathcal{E} \left[\#(\xi \cap \tilde{Q}_1) \right] \right] + \rho(\lfloor N^\alpha \rfloor - r_c, \kappa).
 \end{aligned}$$

This gives the result by letting N and then κ go to the infinity. □

Chapitre 4

Principes d'invariance *quenched*

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Pour presque toute réalisation ξ d'un processus ponctuel convenablement choisi, on considère les marches simples au plus proche voisin, à temps discret et continu, sur la triangulation de Delaunay de ξ et on montre la convergence du processus renormalisé $(\varepsilon X_{\varepsilon^{-2}t}^\xi)_{t \geq 0}$ vers un mouvement brownien en un sens plus fort que dans le chapitre précédent. Plus précisément, on établit, dans le Théorème 4.1, un principe d'invariance *quenched* ou presque sûr : pour presque tout ξ , $(\varepsilon X_{\varepsilon^{-2}t}^\xi)_{t \geq 0}$ converge en loi vers un mouvement brownien dont la matrice de covariance est un multiple de l'identité et ne dépend pas de ξ .

Décrivons maintenant la méthode utilisée qui repose au départ sur une idée fondamentale apparue dans la littérature durant la première moitié des années 1980 avec les travaux de Kozlov [Koz85]. On écrit le processus sous la forme $X_t^\xi = \varphi(\xi, X_t^\xi) + \chi(\xi, X_t^\xi)$ où, pour presque tout ξ , $\varphi(\xi, \cdot)$ est une fonction *harmonique* pour la marche $(X_t^\xi)_{t \geq 0}$ et $\chi(\xi, X_t^\xi)$ est un reste (le *correcteur*) que l'on cherche à contrôler. Le processus $(\varphi(\xi, X_t^\xi))_{t \geq 0}$ est alors une martingale et le TCL fonctionnel pour les martingales permet de traiter le premier terme. Il faut ensuite vérifier que le correcteur est négligeable dans la limite diffusive. On montre pour cela la sous-linéarité presque sûre de celui-ci dans le Théorème 4.3 dont la preuve constitue le cœur du chapitre. Notons que Ferrari, Grisi et Groisman ont obtenu, sous certaines hypothèses sur le processus ponctuel, par une approche constructive (mettant en œuvre un *processus de Harness*) très différente de celle présentée ici, l'existence de la déformation harmonique φ et la sous-linéarité du correcteur en dimension 2. Ces auteurs ont également signalé que l'établissement de bornes sur le noyau de la chaleur, similaires à celles obtenues par Barlow [Bar04] dans le cadre de marche sur l'amas infini de percolation, devraient permettre d'obtenir la sous-linéarité presque sûre du correcteur en dimension plus grande. Nous optons ici pour une approche un peu différente et moins exigeante en terme de bornes sur le noyau de la chaleur. On suit la méthode introduite par Biskup et Prescott [BP07] et on prouve d'abord la sous-linéarité du correcteur dans un sous-graphe infini de la triangulation de Delaunay possédant de bonnes propriétés de régularité (dans lequel, en particulier, le degré est uniformément borné). Cette méthode a également été utilisée par Caputo, Faggionato et Prescott [CFP13] dans le cadre des marches aléatoires sur des graphes complets engendrés par des processus ponctuels dans \mathbb{R}^d avec des probabilités de transition décroissant avec la distance entre les points. Dans le présent chapitre comme dans ce dernier article, le sous-graphe régulier est obtenu, pour presque toute réalisation du processus ponctuel, par un découpage de l'espace en boîtes et un couplage avec un processus de percolation dans \mathbb{Z}^d . Contrairement à ce qui est fait dans [FSBS06], on conserve ici la trace du couplage tout au long de la preuve évitant ainsi quelques complications techniques (en particulier lors de l'établissement d'inégalités isopérimétriques, §4.7.2) et rendant la construction plus transparente. On déduit finalement la sous-linéarité du correcteur $\chi(\xi, \cdot)$ dans le graphe entier de la sous-linéarité de celui-ci dans le sous-graphe régulier grâce à l'harmonicité de $x \mapsto x - \chi(\xi, \cdot)$ et au principe du maximum. Remarquons que, comme dans les deux chapitres précédents, les arguments pourraient être adaptés sans difficulté à d'autres graphes engendrés par des processus ponctuels et construits à partir de la géométrie locale de l'ensemble de points tiré.

Quenched invariance principle for random walks on Delaunay triangulation generated by point processes in \mathbb{R}^d

4.1 Introduction

Given a locally finite subset ξ of \mathbb{R}^d , the Voronoi tessellation associated with ξ is the collection of the Voronoi cells: $\text{Vor}_\xi(\mathbf{x}) := \{x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \xi\}$; the point \mathbf{x} is called the nucleus or the seed of the cell. The Delaunay triangulation $\text{DT}(\xi)$ of ξ is the dual graph of its Voronoi tiling. It has ξ as vertex set and there is an edge between \mathbf{x} and \mathbf{y} in $\text{DT}(\xi)$ if $\text{Vor}_\xi(\mathbf{x})$ and $\text{Vor}_\xi(\mathbf{y})$ share a $(d - 1)$ -dimensional face. Another useful characterization of $\text{DT}(\xi)$ is the following: a simplex Δ is a cell of $\text{DT}(\xi)$ iff its circumscribed sphere has no point of ξ in its interior. Recall that this is a well defined triangulation when ξ is in general position. In the sequel, we denote by \mathcal{N} (resp. \mathcal{N}_0) the set of locally finite subsets of \mathbb{R}^d (resp. the set of locally finite subsets of \mathbb{R}^d containing 0).

Given a realization ξ of a suitable point process with law \mathcal{P} , we consider the variable speed nearest-neighbor random walk $(X_t)_{t \geq 0} = (X_t^\xi)_{t \geq 0}$ on the Delaunay triangulation of ξ , that is the Markov process with generator:

$$\mathcal{L}^\xi f(x) := \sum_{y \in \xi} c_{x,y}^\xi (f(y) - f(x)), \quad x \in \xi, \quad (4.1.1)$$

where $c_{x,y}^\xi$ is the indicator function of ' $y \sim x$ in $\text{DT}(\xi)$ '. We denote by P_x^ξ the law of the walk starting from $x \in \xi$ and by E_x^ξ the corresponding expectation. Recurrence and transience results were obtained for such walks in Chapter 2 and an *annealed* invariance principle was proved in Chapter 3. Here, we study for almost every realization of the point process the behavior of the process at the diffusive scale and we prove the following theorem.

Theorem 4.1. *Assume that ξ is distributed according to a simple, stationary, isotropic point process with law \mathcal{P} a.s. in general position, satisfying assumptions **(V)**, **(SD)**, **(Er)** and **(PM)** (see Subsection 4.1.1) and such that its Palm version is isotropic and a.s. in general position.*

For \mathcal{P} -a.e. ξ , for all $x \in \xi$, under P_x^ξ , the rescaled process $(X_t^\varepsilon)_{t \geq 0} = (\varepsilon X_{\varepsilon^{-2}t})_{t \geq 0}$ converges in law as ε tends to 0 to a Brownian motion with covariance matrix $\sigma^2 I$ where $\sigma^2 = \sigma_{\text{VSRW}}^2$ is positive and does not depend on ξ .

Note that the assumptions of Theorem 4.1 are satisfied by Poisson point processes, Matérn hardcore processes and Matérn cluster processes. The same result holds for the discrete-time nearest-neighbor random walk $(X_n)_{n \in \mathbb{N}} = (X_n^\xi)_{n \in \mathbb{N}}$ with diffusion coefficient σ_{DTRW}^2 related with σ_{VSRW}^2 by the formula:

$$\sigma_{\text{VSRW}}^2 = \mathcal{E}_0 \left[\text{deg}_{\text{DT}(\xi^0)}(0) \right] \sigma_{\text{DTRW}}^2,$$

where \mathcal{E}_0 denotes the expectation w.r.t. the Palm measure \mathcal{P}_0 associated with the (stationary) point process with law \mathcal{P} . Thanks to Proposition 1.5, the theorem above is a direct consequence of the following one.

Theorem 4.2. *Under the assumptions of Theorem 4.1, for \mathcal{P}_0 -a.e. ξ^0 , under $P_0^{\xi^0}$, the rescaled process $(X_t^\varepsilon)_{t \geq 0} = (\varepsilon X_{\varepsilon^{-2}t})_{t \geq 0}$ converges in law as ε tends to 0 to a Brownian motion with covariance matrix $\sigma^2 I$ where σ^2 is positive and does not depend on ξ .*

The main idea for proving such results is to show that the random walk behaves like a martingale up to a correction which is negligible at the diffusive scale. Actually, well-known arguments (see e.g. [CFP13, §3.3.1], [BP07, p. 1340-1341] or [BB07, §6.1 and §6.2]) show that the last claim follows from Theorem 4.3.

Theorem 4.3. *Under the assumptions of Theorem 4.1, there exists a so-called corrector $\chi : \mathcal{N}_0 \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for:*

1. $\varphi(\xi^0, x) := x - \chi(\xi^0, x)$ is harmonic at 0 for \mathcal{P}_0 - a.e. ξ^0 , i.e.:

$$\sum_{x \in \xi^0} c_{0,x}^{\xi^0} \|\varphi(\xi^0, x)\| < \infty \text{ and } \mathcal{L}^{\xi^0} \varphi(\xi^0, 0) = 0 \text{ for } \mathcal{P}_0 \text{ - a.e. } \xi^0;$$

2. χ is a.s. sublinear:

$$\frac{\max_{x \in \xi^0 \cap [-n,n]^d} \|\chi(\xi^0, x)\|}{n} \xrightarrow[n \rightarrow \infty]{\mathcal{P}_0\text{-a.s.}} 0.$$

The arguments to deduce Theorem 4.2 from Theorem 4.3 are rather standard. We have chosen to do not develop the arguments which can be found in the references cited above and we only indicate the main lines of the proof in Section 4.10.2. Theorem 4.3 is proved by following the approach developed in [BP07]. Actually, we first prove the sublinearity of the corrector restricted to a suitable subgraph of the Delaunay triangulation and extend it by harmonicity. It is worth noting that the existence of an harmonic corrector was recently established in [FGG12] by a different (constructive) method. Nevertheless, the authors of this paper obtained the sublinearity of the corrector only in dimension 2. As in previous chapters, the method can be used to obtain quenched invariance principles for random walks on other graphs constructed according to the geometry of a realization of a point process.

4.1.1 Conditions on the point process

In this section, we list the assumptions on the point process needed to obtain the quenched invariance principle.

We assume that ξ is distributed according to a simple and stationary point process with law \mathcal{P} . In the sequel, we will denote by \mathcal{E} the expectation with respect to \mathcal{P} . We suppose that \mathcal{P} and its Palm version \mathcal{P}_0 are isotropic and almost surely in general position (see [Zes08]): there are no $d + 1$ points (resp. $d + 2$ points) in a $(d - 1)$ -dimensional affine subspace (resp. in a sphere). We also assume that the point process satisfies:

(V) there exists a positive constant c_1 such that for L large enough:

$$\mathcal{P}\left[\#\left([0, L]^d \cap \xi\right) = 0\right] \leq e^{-c_1 L^d}.$$

In order to prove the sublinearity of the corrector, we need to work in a subgraph of the Delaunay triangulation of ξ which has good regularity properties. To this end, we will define a notion of ‘good boxes’ that in particular allows us to bound the maximal degree of vertices in an infinite subgraph of the Delaunay triangulation of ξ . Precise definitions and assumptions are given below.

Good boxes, good points and the stochastic domination assumption

For $s \in \mathbb{N}^*$, let us divide \mathbb{R}^d into boxes of side $K := \lceil 3\sqrt{d} \rceil s$:

$$B_{\mathbf{z}} = B_{\mathbf{z}}^K := K\mathbf{z} + \left[-\frac{K}{2}, \frac{K}{2}\right]^d, \quad \mathbf{z} \in \mathbb{Z}^d.$$

Each box $B_{\mathbf{z}}$ is then subdivided into smaller sub-boxes $b_i^{\mathbf{z}}$, $i = 1, \dots, \lceil 3\sqrt{d} \rceil^d$ of side s .

A box $B_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{Z}^d$, is called (α) -nice if each sub-box $b_i^{\mathbf{z}}$ of side s in $B_{\mathbf{z}}$ satisfies $1 \leq \#(\xi \cap b_i^{\mathbf{z}}) \leq \alpha s^d$. A box $B_{\mathbf{z}}$ is then said (α) -good if $B_{\mathbf{z}'}$ is α -nice for each $\mathbf{z}' \in \mathbb{Z}^d$ with $\|\mathbf{z}' - \mathbf{z}\|_{\infty} \leq 1$. Writing $p_c^{\text{site}}(\mathbb{Z}^d)$ for the critical probability for site percolation in \mathbb{Z}^d , the stochastic domination hypothesis is the following.

(SD) For any $p_c^{\text{site}}(\mathbb{Z}^d) < p < 1$, if α and s_0 are well chosen, for any $s \geq s_0$ the process of good boxes $\mathbb{X} := \{X_{\mathbf{z}} := \mathbf{1}_{B_{\mathbf{z}}} \text{ is good}, \mathbf{z} \in \mathbb{Z}^d\}$ dominates an independent site percolation process $\mathbb{Y} := \{Y_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ on \mathbb{Z}^d with parameter p .

If (SD) is satisfied, we can find a coupling $\mathcal{P}^{K,p}$ of the processes \mathbb{X} and \mathbb{Y} such that $B_{\mathbf{z}}$ is good when $Y_{\mathbf{z}} = 1$. With a slight abuse of notation, we will omit the superscript and denote by \mathcal{P} the probability measure $\mathcal{P}^{K,p}$ on $\widehat{\mathcal{N}} := \mathcal{N} \times \{0, 1\}^{\mathbb{Z}^d}$ whose marginal distributions are respectively the law of the point process ξ and the law of the independent site percolation process \mathbb{Y} with parameter p being fixed large enough. The precise value of p is not stated explicitly but we assume that it is large enough to ensure that all the percolation results we need are satisfied. A generic element of $\widehat{\mathcal{N}}$ is denoted by $\widehat{\xi} = (\xi, (y_{\mathbf{z}})_{\mathbf{z} \in \mathbb{Z}^d})$.

Let us denote by $\mathbb{G}_{(L)}$ (resp. \mathbb{G}_{∞}) the largest ($\|\cdot\|_1$ -)connected component of \mathbb{Y} contained in $[-L, L]^d \cap \mathbb{Z}^d$ (resp. the a.s. unique infinite component of \mathbb{Y}). We then define $\mathcal{G}_{(L)} = \mathcal{G}_{(L)}(\widehat{\xi})$ (resp. $\mathcal{G}_{\infty} = \mathcal{G}_{\infty}(\widehat{\xi})$) as the set of points of ξ whose Voronoi cell intersects a K -box with index in $\mathbb{G}_{(L)}$ (resp. \mathbb{G}_{∞}). The points of \mathcal{G}_{∞} are called *good points*. Let us note that $\mathcal{G}_{(L)}$ and \mathcal{G}_{∞} are connected in the Delaunay triangulation of ξ .

We claim that good points have their degrees uniformly bounded. More precisely:

Lemma 4.4. *There exists $D = D(d, K, \alpha)$ such that for every $x \in \xi$ with Voronoi cell intersecting an α -good box:*

$$\text{deg}_{\text{DT}(\xi)}(x) \leq D.$$

Remark 4.5. *As it appears in the following proof, D is also an upper bound for the maximal number of Voronoi cells which intersect a good box. It will be used to bound these two quantities throughout the paper.*

Proof. Let $x \in \xi$ such that its Voronoi cell intersects an α -good box $B_{\mathbf{z}}$. By definition, each box $B'_{\mathbf{z}}$ with $\|\mathbf{z}' - \mathbf{z}\|_{\infty} \leq 1$ is α -nice. Since any sub-box of side s that belongs to a nice box contains at least one point of ξ , the points in nice boxes are within a distance at most $\sqrt{d}s$ from the nucleus of their Voronoi cell. In particular, x is within a distance at most $\sqrt{d}s$ from $B_{\mathbf{z}}$. Similarly, the nuclei of the Voronoi cells which share a face with $\text{Vor}_{\xi}(x)$ are in $B_{\mathbf{z}} + B_2(0, 3\sqrt{d}s) \subset \overline{B_{\mathbf{z}}} := \bigcup_{\mathbf{z}': \|\mathbf{z}' - \mathbf{z}\|_{\infty} \leq 1} B_{\mathbf{z}'}$. Hence, $\deg_{\text{DT}(\xi)}(x)$ is generously bounded by $\#(\xi \cap \overline{B_{\mathbf{z}}}) \leq D := \alpha(3K)^d$. \square

The ergodicity assumption

This assumption is quite similar to assumption (H2) in [CFP13]. We will use it to adapt the method developed in [BB07, §4] for proving the sublinearity of the corrector along coordinate directions.

(Er) For each (K, p) such that **(SD)** holds and each e , $\mathcal{P} = \mathcal{P}^{K,e}$ is ergodic with respect to the transformation:

$$\tau = \tau^{K,e} : (\xi, (y_{\mathbf{z}})_{\mathbf{z} \in \mathbb{Z}^d}) \longrightarrow (\tau_{Ke}\xi, (y_{\mathbf{z}+e})_{\mathbf{z} \in \mathbb{Z}^d}).$$

Note that τ is invertible and \mathcal{P} is invariant with respect to τ due to the stationarity of the point process.

Polynomial moments

We also make the following assumption:

(PM) the number of points in a unit cube admits a polynomial moment of order 2 under \mathcal{P} and that $\deg_{\text{DT}(\xi^0)}(0)$ and $\max_{x \sim 0 \text{ in } \text{DT}(\xi^0)} \|x\|$ admit respectively a moment of order 2 and 4 under the Palm distribution \mathcal{P}_0 associated with the point process.

The case of point processes with a finite range of dependence

We verify that, when the point process has a finite range of dependence, assumption **(Er)** is always satisfied, and assumptions **(SD)** and **(PM)** are implied by assumptions **(V)**, **(D)**, **(V')** and **(EM)** which are described below.

Let us check that **(Er)** is satisfied if the point process has a finite range of dependence. Let \mathcal{A} be a measurable set such that $\tau\mathcal{A} = \mathcal{A}$. Fix $\varepsilon > 0$ and \mathcal{B} with $\mathcal{P}(\mathcal{A} \Delta \mathcal{B}) \leq \varepsilon$ which depends only on ξ restricted to a compact subset of \mathbb{R}^d and on finitely many $y_{\mathbf{z}}$ s. It is thus

possible to find n such that \mathcal{B} and $\tau^n \mathcal{B}$ are independent. Then, $\mathcal{P}[\mathcal{B} \cap \tau^n \mathcal{B}] = \mathcal{P}[\mathcal{B}]^2$ by invariance of \mathcal{P} w.r.t. τ . Hence,

$$\begin{aligned}
 |\mathcal{P}[\mathcal{A}] - \mathcal{P}[\mathcal{A}]^2| &\leq |\mathcal{P}[\mathcal{A}] - \mathcal{P}[\mathcal{B} \cap \tau^n \mathcal{B}]| + |\mathcal{P}[\mathcal{B} \cap \tau^n \mathcal{B}] - \mathcal{P}[\mathcal{A}]^2| \\
 &\leq \mathcal{P}[\mathcal{A} \Delta (\mathcal{B} \cap \tau^n \mathcal{B})] + |\mathcal{P}[\mathcal{B}]^2 - \mathcal{P}[\mathcal{A}]^2| \\
 &\leq \mathcal{P}[(\mathcal{A} \Delta \mathcal{B}) \cup (\mathcal{A} \Delta \tau^n \mathcal{B})] + (\mathcal{P}[\mathcal{B}] + \mathcal{P}[\mathcal{A}]) |\mathcal{P}[\mathcal{B}] - \mathcal{P}[\mathcal{A}]| \\
 &\leq \mathcal{P}[\mathcal{A} \Delta \mathcal{B}] + \mathcal{P}[\mathcal{A} \Delta \tau^n \mathcal{B}] + 2\varepsilon \\
 &\leq \mathcal{P}[\tau^n \mathcal{A} \Delta \tau^n \mathcal{B}] + 3\varepsilon \\
 &= \mathcal{P}[\mathcal{A} \Delta \mathcal{B}] + 3\varepsilon \leq 4\varepsilon
 \end{aligned}$$

Since ε is arbitrary, this implies that $\mathcal{P}[\mathcal{A}]$ is 0 or 1.

Since deciding if a box is good depends only on the behavior of the point process ξ in a neighborhood of the box, the process of good boxes \mathbb{X} is a Bernoulli process on \mathbb{Z}^d with a finite range of dependence when the point process has itself a finite range of dependence. In this case, if $\mathcal{P}[X_{\mathbf{z}} = 1]$ is as close to 1 as we wish for s and α large enough, [LSS97, Theorem 0.0] ensures that **(SD)** holds. One can easily bound $\mathcal{P}[X_{\mathbf{z}} = 1]$ from below when the point process satisfies **(V)** and

(D) there exist positive constants c_2, c_3 such that for L large enough:

$$\mathcal{P}[\#\left([0, L]^d \cap \xi\right) \geq c_2 L^d] \leq e^{-c_3 L^d}.$$

In Lemma 4.31, we prove that $\deg_{\text{DT}(\xi^0)}(0)$ and $\max_{x \sim 0 \text{ in } \text{DT}(\xi^0)} \|x\|$ admit exponential moments when the point process has a finite range of dependence and satisfies:

(V') there exists a positive constant c_4 such that for L large enough:

$$\mathcal{P}_0[\#\left(\xi^0 \cap C_L\right) = 0] \leq e^{-c_4 L^d}, \quad \forall C_L \text{ cube of side } L,$$

and

(EM) there exist a positive constant c_5 and a positive function $f(\rho)$ which goes to 0 with ρ such that for L large enough:

$$\mathcal{E}_0 \left[e^{\rho \#(\xi^0 \cap [-L, L]^d)} \right] \leq c_5 e^{f(\rho) L^d}.$$

Let us finally note that these assumptions are in particular satisfied by homogeneous Poisson point processes, Matérn cluster processes and type I or II Matérn hardcore processes. Indeed, these point processes have finite range of dependence and it is quite classical to check assumptions **(V)**, **(D)**, **(V')** and **(EM)** for these processes (see Chapter 1).

4.1.2 Outline of the paper

As announced at the beginning of the introduction, the crux is to prove Theorem 4.3 and we follow the approach of [BP07]. The main steps of the proof are stated explicitly in Theorem 2.4 of that paper. We prove the existence of the corrector in Section 4.2. Next, we verify that the corrector grows at most polynomially in Section 4.3 and at most linearly in each coordinate direction in Section 4.4. The sublinearity on average is treated in Section 4.5 while diffusive bounds for a related random walk are proved in Sections 4.7 and 4.8. In Section 4.9, we prove the a.s. sublinearity of the corrector in the set of ‘good points’. The proofs of Theorems 4.1, 4.2 and 4.3 are finally completed in Section 4.10.

4.2 Construction of the corrector and harmonic deformation

Let us define the measure μ on $\mathcal{N}_0 \times \mathbb{R}^d$ by:

$$\int u d\mu := \mathcal{E}_0 \left[\sum_{x \in \xi^0} c_{0,x}^{\xi^0} u(\xi^0, x) \right],$$

where $c_{0,x}^{\xi^0} = \mathbf{1}_{0 \sim x \text{ in } \text{DT}(\xi^0)}$. This measure has total mass $\mathcal{E}_0[\text{deg}_{\text{DT}(\xi^0)}(0)]$ which is finite thanks to assumption **(PM)**. We denote by $(\cdot, \cdot)_\mu$ the scalar product in $L^2(\mu)$.

4.2.1 Weyl decomposition of $L^2(\mu)$

As in [MP07], [BB07] or [CFP13], we work with the orthogonal decomposition of $L^2(\mu)$ in the subspaces of square integrable *potential* and *solenoidal* fields. This decomposition is quite standard (see *e.g.* [LP14, Chap. 9]) and generally called *Weyl decomposition*. Let us denote by $(\tau_x)_{x \in \mathbb{R}^d}$ the group of translations in \mathbb{R}^d which acts naturally on \mathcal{N}_0 as follows: $\tau_x \xi^0 = \sum_{y \in \xi^0} \delta_{y-x}$.

Definition 4.6. For $\psi : \mathcal{N}_0 \rightarrow \mathbb{R}$, the gradient field $\nabla\psi : \mathcal{N}_0 \times \mathbb{R}^d \rightarrow \mathbb{R}$ is defined for $x \in \xi^0$ by:

$$\nabla\psi(\xi^0, x) := \psi(\tau_x \xi^0) - \psi(\xi^0)$$

and by 0 if $x \notin \xi^0$.

Note that gradients of measurable bounded functions on \mathcal{N}_0 are elements of $L^2(\mu)$ thanks to assumption **(PM)**.

Definition 4.7. The space $L_{pot}^2(\mu)$ of potential fields is defined as the closure of the subspace of gradients of measurable bounded functions on \mathcal{N}_0 . Its orthogonal complement is the set of solenoidal (or divergence-free) fields and is denoted by $L_{sol}^2(\mu)$.

Let us recall some additional definitions:

Definition 4.8. A function $u : \mathcal{N}_0 \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is called:

1. antisymmetric if

$$u(\xi^0, x) = -u(\tau_x \xi^0, -x), \quad \forall \xi^0 \in \mathcal{N}_0, \forall x \in \xi^0;$$

2. shift-covariant if

$$u(\xi^0, x) = u(\xi^0, y) + u(\tau_y \xi^0, x - y), \quad \forall \xi^0 \in \mathcal{N}_0, \forall x, y \in \xi^0;$$

3. curl-free if it satisfies the following co-cycle relation: for any $\xi^0 \in \mathcal{N}_0$, any $n \in \mathbb{N}^*$, and any collection of points $x_0, \dots, x_n \in \xi^0$ with $x_0 = x_n$, one has

$$\sum_{i=1}^{n-1} u(\tau_{x_i} \xi^0, x_{i+1} - x_i) = 0. \quad (4.2.1)$$

A function of $L^2(\mu)$ is called *antisymmetric* (resp. *shift-covariant*, *curl-free*) if it is antisymmetric (resp. shift-covariant, curl-free) for \mathcal{P}_0 -a.a. ξ^0 . In each case, it admits a representative which satisfies the corresponding property everywhere. By taking $x = y = 0$ in the definition, one can see that any shift-covariant function u must satisfy $u(\xi^0, 0) = 0$ for any $\xi^0 \in \mathcal{N}_0$. Next proposition lists simple but useful links between definitions above:

Proposition 4.9. Let $u \in L^2(\mu)$.

1. If $u \in L^2_{\text{pot}}(\mu)$, then it is curl-free.
2. If u is curl-free, then it is also antisymmetric and shift-covariant.

Proof.

1. Gradients fields are clearly curl-free. The general case is obtained by a standard approximation argument.
2. The case $n = 2$, $x_0 = x_2 = 0$ and $x_1 = x$ in (4.2.1) gives the antisymmetry of u . By (4.2.1) with $n = 3$, $x_0 = x_3 = 0$, $x_1 = y$ and $x_2 = x$, one has:

$$u(\xi^0, y) + u(\tau_y \xi^0, x - y) + u(\tau_x \xi^0, -x) = 0.$$

The shift-covariance of u then follows from its antisymmetry.

□

We can now define the *divergence* of a square integrable field and derive an integration by parts formula.

Definition 4.10. *The divergence of $u \in L^1(\mu)$ is defined by:*

$$\operatorname{div} u(\xi^0) := \sum_{x \in \xi^0} c_{0,x}^{\xi^0} u(\xi^0, x), \quad \xi^0 \in \mathcal{N}_0.$$

Triangle inequality clearly implies that divergences of $L^1(\mu)$ functions are in $L^1(\mathcal{P}_0)$. Moreover, if $u \in L^1(\mu)$ is a positive function, we have the equality:

$$\|u\|_{L^1(\mu)} = \|\operatorname{div} u\|_{L^1(\mathcal{P}_0)}. \quad (4.2.2)$$

We derive the following integration by parts formula:

Lemma 4.11. *Let ψ be a bounded measurable function on \mathcal{N}_0 and let $u \in L^2(\mu)$ be an antisymmetric field. It holds that:*

$$(u, \nabla \psi)_\mu = -2\mathcal{E}_0[\psi \operatorname{div} u]. \quad (4.2.3)$$

Proof. Observe that:

$$c_{0,x}^{\xi^0} = c_{0,-x}^{\tau_x \xi^0}.$$

Due to the antisymmetry of u , one has:

$$\begin{aligned} (u, \nabla \psi)_\mu &= \mathcal{E}_0 \left[\sum_{x \in \xi^0} c_{0,x}^{\xi^0} u(\xi^0, x) \psi(\tau_x \xi^0) \right] - \mathcal{E}_0 \left[\sum_{x \in \xi^0} c_{0,x}^{\xi^0} u(\xi^0, x) \psi(\xi^0) \right] \\ &= -\mathcal{E}_0 \left[\sum_{x \in \xi^0} c_{0,-x}^{\tau_x \xi^0} u(\tau_x \xi^0, -x) \psi(\tau_x \xi^0) \right] - \mathcal{E}_0[\psi \operatorname{div} u]. \end{aligned}$$

By Neveu exchange formula (see [SW08, Theorem 3.4.5] in the special case where $X = Y$), one has for any integrable function f on $\mathcal{N}_0 \times \mathbb{R}^d$:

$$\int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(\tau_x \xi^0, x) \mathcal{P}_0(d\xi^0) = \int_{\mathcal{N}_0} \sum_{x \in \xi^0} f(\xi^0, -x) \mathcal{P}_0(d\xi^0).$$

The conclusion is then obtained by applying the identity above with:

$$f(\xi^0, x) := c_{0,-x}^{\xi^0} u(\xi^0, -x) \psi(\xi^0).$$

□

This lemma implies the immediate following corollary.

Corollary 4.12. *An antisymmetric field $u \in L^2(\mu)$ is solenoidal if and only if $\operatorname{div} u(\xi^0) = 0$ for \mathcal{P}_0 -a.e. ξ^0 .*

4.2.2 Construction of the corrector

In [FGG12], the authors relied on an Harness-type process to obtain harmonic deformations of Delaunay triangulations which corresponds to the existence of the corrector. Here, we recall how the decomposition of $L^2(\mu)$ allows us to derive the existence of the corrector by following the construction of [MP07, CFP13].

For $i = 1, \dots, d$, $\xi^0 \in \mathcal{N}_0$ and $x \in \mathbb{R}^d$, set $u_i(\xi^0, x) := x_i$ the i^{th} coordinate of x . Note that u_i is clearly antisymmetric in the sense of Definition 4.8 and that $u_i \in L^2(\mu)$. Actually, by the Cauchy-Schwarz inequality and assumption **(PM)**, one obtains:

$$\begin{aligned} \int |u_i|^2 d\mu &= \mathcal{E}_0 \left[\sum_{x \in \xi^0} c_{0,x}^{\xi^0} |x_i|^2 \right] \leq \mathcal{E}_0 \left[\left(\max_{x \sim 0 \text{ in DT}(\xi^0)} \|x\| \right)^2 \deg_{\text{DT}(\xi^0)}(0) \right] \\ &\leq \mathcal{E}_0 \left[\left(\max_{x \sim 0 \text{ in DT}(\xi^0)} \|x\| \right)^4 \right]^{\frac{1}{2}} \mathcal{E}_0 \left[\left(\deg_{\text{DT}(\xi^0)}(0) \right)^2 \right]^{\frac{1}{2}} < \infty. \end{aligned}$$

Consider now the orthogonal decomposition of the form $u_i = \chi_i + \varphi_i$ with $\chi_i \in L^2_{\text{pot}}(\mu)$ and $\varphi_i \in L^2_{\text{sol}}(\mu)$. Since $\chi_i \in L^2_{\text{pot}}(\mu)$, it is antisymmetric and φ_i also (as a difference of antisymmetric functions). Hence, it follows from Corollary 4.12 that φ_i is harmonic at 0.

The *corrector field* is the vector-valued function on $\mathcal{N}_0 \times \mathbb{R}^d$ defined by $\chi = (\chi_1, \dots, \chi_d)$. It admits a shift-covariant representative and its norm $\|\chi\|$ is in $L^2(\mu)$. We denote by $\varphi = (\varphi_1, \dots, \varphi_d)$ the harmonic function $(\xi^0, x) \rightarrow x - \chi(\xi^0, x)$. From the harmonicity of $\varphi(\xi^0, \cdot)$ at 0 for \mathcal{P}_0 -a.a. ξ^0 and Proposition 1.5, one deduces that, for \mathcal{P} -a.a. ξ , for any $x \in \xi$:

$$\sum_{y \in \xi} c_{x,y}^{\xi} \|\varphi(\tau_x \xi, y - x)\| < \infty \quad \text{and} \quad \sum_{y \in \xi} c_{x,y}^{\xi} \varphi(\tau_x \xi, y - x) = 0. \quad (4.2.4)$$

Let us define

$$M_n^{\xi} := \varphi(\tau_{X_0} \xi, X_n - X_0) = \sum_{i=1}^n \varphi(\tau_{X_i} \xi, X_{i+1} - X_i).$$

It follows from (4.2.4) that $(M_n^{\xi})_{n \in \mathbb{N}}$ is a martingale under P_x^{ξ} .

4.3 Polynomial growth

Let us define:

$$\mathcal{R}_n = \mathcal{R}_n(\xi) := \max_{\substack{x, y \in \xi \\ \text{Vor}_{\xi}(x) \cap [-n, n]^d \neq \emptyset \\ \text{Vor}_{\xi}(y) \cap [-n, n]^d \neq \emptyset}} \|\chi(\tau_x \xi, y - x)\|. \quad (4.3.1)$$

Proposition 4.13. *For every $\beta > d + 1$, one has:*

$$\frac{\mathcal{R}_n}{n^{\beta}} \xrightarrow[n \rightarrow \infty]{\mathcal{P}\text{-a.s.}} 0.$$

Proof. For n fixed, let us cover $[-2n, 2n]^d$ with disjoint boxes of side $\log n$ and denote by \mathcal{A}_n the event ‘each of these boxes contains at least one point of ξ ’. Note that, thanks to assumption **(V)**, $\mathcal{P}[\mathcal{A}_n^c] = O(n^{-2})$. Let $d + 1 < \beta' < \beta$. Since

$$\mathcal{P}[\mathcal{R}_n \geq n^{\beta'}] \leq \mathcal{P}[\mathcal{R}_n \mathbf{1}_{\mathcal{A}_n} \geq n^{\beta'}] + \mathcal{P}[\mathcal{A}_n^c],$$

we only need to show that $\sum_n \mathcal{P}[\mathcal{R}_n \mathbf{1}_{\mathcal{A}_n} \geq n^{\beta'}] < \infty$. The result then follows using the Borel-Cantelli lemma.

Let $\mathbf{x}, \mathbf{y} \in \xi$ with $\text{Vor}_\xi(\mathbf{x}) \cap [-n, n]^d \neq \emptyset$ and $\text{Vor}_\xi(\mathbf{y}) \cap [-n, n]^d \neq \emptyset$ be such that $\mathcal{R}_n = \|\chi(\tau_{\mathbf{x}}\xi, \mathbf{y} - \mathbf{x})\|$. Consider the simple Delaunay-path (x_0, \dots, x_m) from $x_0 = \mathbf{x}$ to $x_m = \mathbf{y}$ obtained by connecting the nuclei of successive Voronoi cells which intersect the line segment $[\mathbf{x}, \mathbf{y}]$. Observe that, on \mathcal{A}_n , any point of $[-2n, 2n]^d$ is within a distance at most $\sqrt{d} \log n$ from the nucleus of its Voronoi cell. In particular, (x_0, \dots, x_m) is contained in $[-2n, 2n]^d$.

Recall that the corrector χ is shift-covariant. Hence, for $i = 0, \dots, m - 1$, one has:

$$\chi(\tau_{x_i}\xi, x_{i+1} - x_i) = \chi(\tau_{\mathbf{x}}\xi, x_{i+1} - \mathbf{x}) - \chi(\tau_{\mathbf{x}}\xi, x_i - \mathbf{x}),$$

and using that $\chi(\tau_{\mathbf{x}}\xi, 0) = 0$:

$$\chi(\tau_{\mathbf{x}}\xi, \mathbf{y} - \mathbf{x}) = \sum_{i=1}^{m-1} \chi(\tau_{x_i}\xi, x_{i+1} - x_i).$$

We deduce that on \mathcal{A}_n :

$$\begin{aligned} \mathcal{R}_n &\leq \sum_{i=1}^{m-1} \|\chi(\tau_{x_i}\xi, x_{i+1} - x_i)\| \\ &\leq \sum_{x \in \xi \cap [-2n, 2n]^d} \sum_{y \in \xi} c_{x,y}^\xi \|\chi(\tau_x \xi, y - x)\| \\ &= \sum_{x \in \xi \cap [-2n, 2n]^d} \text{div} \|\chi\|(\tau_x \xi). \end{aligned}$$

Together with Markov inequality and Campbell formula, the inequality above leads to:

$$\begin{aligned} \mathcal{P}[\mathcal{R}_n \mathbf{1}_{\mathcal{A}_n} \geq n^{\beta'}] &\leq \frac{\mathcal{E}[\mathcal{R}_n \mathbf{1}_{\mathcal{A}_n}]}{n^{\beta'}} \\ &\leq n^{-\beta'} \int_{\mathcal{N}} \sum_{x \in \xi \cap [-2n, 2n]^d} \text{div} \|\chi\|(\tau_x \xi) \mathcal{P}(d\xi) \\ &\leq cn^{d-\beta'} \left\| \text{div} \|\chi\| \right\|_{L^1(\mathcal{P}_0)} \end{aligned}$$

Since $\beta' > d + 1$ and $\text{div} \|\chi\| \in L^2(\mathcal{P}_0)$, this completes the proof. \square

4.4 Sublinearity along coordinate directions in $\mathcal{G}_\infty(\widehat{\xi})$

In this section, we adapt the arguments of [CFP13, §7.2] which consist in an adaptation of the ‘lattice method’ developed in [BB07, BP07].

Given a unit vector e in the direction of one of the coordinate axes of \mathbb{R}^d and $\widehat{\xi} \in \widehat{\mathcal{N}}$, let us define:

$$n_0(\widehat{\xi}) = n_0^e(\widehat{\xi}) := 0 \quad \text{and} \quad n_{i+1}(\widehat{\xi}) = n_{i+1}^e(\widehat{\xi}) := \min\{j > n_i(\widehat{\xi}) : je \in \mathbb{G}_\infty\}.$$

Recall the definition of τ from assumption **(Er)** and consider the shift τ_* induced on $\mathcal{N}^* := \{\widehat{\xi} \in \widehat{\mathcal{N}} : 0 \in \mathbb{G}_\infty\}$ from τ , that is $\tau_* : \widehat{\xi} \rightarrow \tau^{n_1(\widehat{\xi})}\widehat{\xi}$. Thanks to assumption **(Er)** and the fact that $\mathcal{P}[0 \in \mathbb{G}_\infty] > 0$, standard arguments (see *e.g.* [CFP13, Lemma 7.3] or [BB07, Theorem 3.2]) lead to:

Lemma 4.14. *The probability measure $\mathcal{P}[\cdot | 0 \in \mathbb{G}_\infty]$ is stationary and ergodic w.r.t. τ_* .*

Next, for $\widehat{\xi} \in \mathcal{N}^*$, we write w_i for the point of ξ whose Voronoi cell contains the center of the box $B_{n_i(\widehat{\xi})Ke}$.

Lemma 4.15. *It holds that:*

$$\mathcal{E} \left[\|\chi(\tau_{w_0}\xi, w_1 - w_0)\| \mid 0 \in \mathbb{G}_\infty \right] < \infty \quad \text{and} \quad \mathcal{E} \left[\chi(\tau_{w_0}\xi, w_1 - w_0) \mid 0 \in \mathbb{G}_\infty \right] = 0.$$

Proof. For $\widehat{\xi} \in \mathcal{N}^*$, let $d(\widehat{\xi}) \geq n_1(\widehat{\xi})$ denote the chemical distance between 0 and $n_1(\widehat{\xi})e$ in the infinite cluster \mathbb{G}_∞ . On the event $\{d(\widehat{\xi}) = j\}$, there exists a path $\mathbf{z}_0 = 0, \mathbf{z}_1, \dots, \mathbf{z}_j = n_1(\widehat{\xi})e$ in \mathbb{G}_∞ . For $i = 0, \dots, j-1$, thanks to the definition of the good boxes and assumption **(SD)**, the nuclei of the Voronoi cells intersecting the line segment $[K\mathbf{z}_i, K\mathbf{z}_{i+1}]$ are within a distance at most $\sqrt{d}s$ from this line segment. By connecting the successive nuclei of the Voronoi cells intersecting the broken line $[K\mathbf{z}_0, K\mathbf{z}_1, \dots, K\mathbf{z}_j]$, we obtain a simple path $w_0 = x_0, x_1, \dots, x_m = w_1$ between w_0 and w_1 in $\mathcal{G}_\infty(\widehat{\xi})$ which is contained in $[-K(j + \frac{1}{2}), K(j + \frac{1}{2})]^d$ and has length $m \leq (j+1)D$. Thanks to the shift-covariance of χ , as in the proof of Proposition 4.13, one obtains:

$$\begin{aligned} \|\chi(\tau_{w_0}\xi, w_1 - w_0)\| &\leq \sum_{i=1}^{m-1} \|\chi(\tau_{x_i}\xi, x_{i+1} - x_i)\| \\ &\leq \sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \sum_{y \stackrel{\sim}{\text{DT}(\widehat{\xi})} x} \|\chi(\tau_x\xi, y - x)\|. \end{aligned}$$

Together with the Cauchy-Schwarz inequality, this leads to:

$$\begin{aligned}
 & \mathcal{E}[\|\chi(\tau_{w_0}\xi, w_1 - w_0)\| \mid 0 \in \mathbb{G}_\infty] \\
 &= \sum_{j=1}^{\infty} \mathcal{E} \left[\|\chi(\tau_{w_0}\xi, w_1 - w_0)\| \mathbf{1}_{d(\widehat{\xi})=j} \mid 0 \in \mathbb{G}_\infty \right] \\
 &\leq \sum_{j=1}^{\infty} \mathcal{E} \left[\sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \sum_{y \underset{\text{DT}(\xi)}{\sim} x} \|\chi(\tau_x\xi, y - x)\| \mathbf{1}_{d(\widehat{\xi})=j} \mid 0 \in \mathbb{G}_\infty \right] \\
 &\leq \sum_{j=1}^{\infty} \left\{ \mathcal{E} \left[\sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \sum_{y \underset{\text{DT}(\xi)}{\sim} x} \|\chi(\tau_x\xi, y - x)\|^2 \mid 0 \in \mathbb{G}_\infty \right]^{\frac{1}{2}} \right. \\
 &\quad \left. \times \mathcal{E} \left[\left(\sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \text{deg}_{\text{DT}(\xi)}(x) \right) \mathbf{1}_{d(\widehat{\xi})=j} \mid 0 \in \mathbb{G}_\infty \right]^{\frac{1}{2}} \right\} \\
 &\leq \frac{1}{\mathcal{P}[0 \in \mathbb{G}_\infty]^{\frac{3}{4}}} \sum_{j=1}^{\infty} \left\{ \mathcal{E} \left[\sum_{\substack{x \in \xi : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \text{div} \|\chi\|^2(\tau_x\xi) \right]^{\frac{1}{2}} \right. \\
 &\quad \left. \times \mathcal{E} \left[\left(\sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \text{deg}_{\text{DT}(\xi)}(x) \right)^2 \right]^{\frac{1}{4}} \mathcal{P}[d(\widehat{\xi}) = j \mid 0 \in \mathbb{G}_\infty]^{\frac{1}{4}} \right\}.
 \end{aligned} \tag{4.4.1}$$

It follows from Campbell formula and formula (4.2.2) that

$$\mathcal{E} \left[\sum_{x \in \xi : \|x\|_\infty \leq K(j + \frac{1}{2})} \text{div} \|\chi\|^2(\tau_x\xi) \right] \leq c j^d \|\text{div} \|\chi\|^2\|_{L^1(\mathcal{P}_0)} = c j^d \|\|\chi\|\|_{L^2(\mu)}^2.$$

Since points of $\mathcal{G}_\infty(\widehat{\xi})$ have degrees bounded by D (see Lemma 4.4) and $\#(\xi \cap [0, 1]^d)$ admits a moment of order 2, one has:

$$\mathcal{E} \left[\left(\sum_{\substack{x \in \mathcal{G}_\infty(\widehat{\xi}) : \\ \|x\|_\infty \leq K(j + \frac{1}{2})}} \text{deg}_{\text{DT}(\xi)}(x) \right)^2 \right] \leq D^2 \mathcal{E} \left[\# \left(\xi \cap \left[-K \left(j + \frac{1}{2} \right), K \left(j + \frac{1}{2} \right) \right]^d \right)^2 \right] \leq c j^d.$$

Thanks to [BB07, Lemma 4.4], we know that $\mathcal{P}[d(\widehat{\xi}) = j \mid 0 \in \mathbb{G}_\infty]$ has an exponential decay. Hence, collecting bounds, we obtain that the sum in the r.h.s. of (4.4.1) is finite.

Since $\chi \in L^2_{\text{pot}}(\mu)$, it is the L^2 -limit of gradients of bounded measurable functions $(g_n)_{n \in \mathbb{N}}$ defined on \mathcal{N}_0 . Let us define $\chi_n := \nabla g_n$. By the same arguments as above, one obtains that:

$$\mathcal{E} \left[\|\chi(\tau_{w_0}\xi, w_1 - w_0) - \chi_n(\tau_{w_0}\xi, w_1 - w_0)\| \mid 0 \in \mathbb{G}_\infty \right] \leq c \|\chi - \chi_n\|_{L^2(\mu)}^2 \xrightarrow{n \rightarrow \infty} 0.$$

Note that, for all i , w_i is a deterministic function of $\widehat{\xi}$ and that $w_1(\widehat{\xi}) = w_0(\tau_*\widehat{\xi}) + n_1(\widehat{\xi})Ke$. The conclusion follows since, due to the stationarity of $\mathcal{P}[\cdot|0 \in \mathbb{G}_\infty]$ with respect to τ_* applied to the function $(g_n \circ \tau_{w_0(\cdot)})(\cdot)$, $\mathcal{E}[\chi_n(\tau_{w_0}\xi, w_1 - w_0)|0 \in \mathbb{G}_\infty] = 0$. \square

Combining Lemmas 4.14 and 4.15, one obtains the sublinearity along the direction e in $\mathcal{G}_\infty(\widehat{\xi})$.

Proposition 4.16. *For $\mathcal{P}[\cdot|0 \in \mathbb{G}_\infty] - a.a. \widehat{\xi}$:*

$$\lim_{k \rightarrow \infty} \frac{\chi(\tau_{w_0}\xi, w_k - w_0)}{k} = 0,$$

and

$$\lim_{k \rightarrow \infty} \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \frac{\chi(\tau_{x_0}\xi, w_k - x_0)}{k} = 0.$$

Proof. Thanks to the shift-covariance of the corrector, one has:

$$\begin{aligned} \frac{\chi(\tau_{w_0}\xi, w_k - w_0)}{k} &= \frac{1}{k} \sum_{j=0}^{k-1} (\chi(\tau_{w_0}\xi, w_{j+1} - w_0) - \chi(\tau_{w_0}\xi, w_j - w_0)) \\ &= \frac{1}{k} \sum_{j=0}^{k-1} \chi(\tau_{w_j}\xi, w_{j+1} - w_j). \end{aligned} \quad (4.4.2)$$

Observe that $w_j(\widehat{\xi}) = w_0(\tau_*^j\widehat{\xi}) + n_j(\widehat{\xi})Ke$, $w_{j+1}(\widehat{\xi}) = w_1(\tau_*^j\widehat{\xi}) + n_1(\tau_*^j\widehat{\xi})Ke$ and recall that, by Lemma 4.14, $\mathcal{P}[\cdot|0 \in \mathbb{G}_\infty]$ is ergodic with respect to τ_* . Thanks to Birkhoff's theorem, the last expression in (4.4.2) converges to $\mathcal{E}[\chi(\tau_{w_0}\xi, w_1 - w_0)|0 \in \mathbb{G}_\infty]$ which is 0 by Lemma 4.15.

The second part of the lemma then follows from equality :

$$\chi(\tau_{x_0}\xi, w_k - x_0) = \chi(\tau_{w_0}\xi, w_k - w_0) + \chi(\tau_{x_0}\xi, w_0 - x_0)$$

which is due to the shift-covariance of the corrector. \square

4.5 Sublinearity on average in $\mathcal{G}_\infty(\widehat{\xi})$

We derive the sublinearity on average of the corrector in $\mathcal{G}_\infty(\widehat{\xi})$ from Lemma 4.16. Our approach is close in spirit of [CFP13, §7.3].

Proposition 4.17. *For every $\varepsilon_0 > 0$, for $\mathcal{P}[\cdot|0 \in \mathbb{G}_\infty] - a.a. \widehat{\xi} \in \widehat{\mathcal{N}}$:*

$$\lim_{L \rightarrow \infty} \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \frac{1}{\#\Lambda_L} \sum_{\substack{x \in \xi : \\ \exists z \in \mathbb{G}_\infty \cap \Lambda_L \\ \text{Vor}_\xi(x) \cap B_z^K \neq \emptyset}} \mathbf{1}_{\|\chi(\tau_{x_0}\xi, x_0 - x)\| \geq \varepsilon_0 L} = 0, \quad (4.5.1)$$

where $\Lambda_L := \mathbb{Z}^d \cap [-L, L]^d$.

Let us describe roughly the method which is an alternative to the one of [BB07, §5.2] and relies on multiscale arguments. The first idea is to extend the directional sublinearity result of Proposition 4.16 dimension by dimension. For $\nu \in \{1, \dots, d\}$, we denote by Λ_L^ν the set:

$$\Lambda_L^\nu := \left\{ \mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_d) \in \mathbb{Z}^d : \text{for } 1 \leq i \leq \nu, |\mathbf{z}_i| \leq L, \text{ and for } \nu + 1 \leq i \leq d, \mathbf{z}_i = 0 \right\}.$$

Assume that we have a ‘good’ (sublinear) control of $\chi(\tau_{x_0}\xi, x - x_0)$ for x_0 whose Voronoi cell intersects B_0^K and x whose Voronoi cell intersects $B_{\mathbf{z}}^K$ for some $\mathbf{z} \in \Lambda_L^\nu \cap \mathbb{G}_\infty$. Then, using Proposition 4.16, one obtains a sublinear control on $\chi(\tau_x\xi, x' - x)$ for x whose Voronoi cell intersects $B_{\mathbf{z}}^K$ and x' whose Voronoi cell intersects $B_{\mathbf{z}'}^K$ for any $\mathbf{z}' \in \Lambda_L^{\nu+1} \cap \mathbb{G}_\infty$ which differs from \mathbf{z} only on the $(\nu + 1)$ -th coordinate. By the shift-covariance, this gives a control on $\chi(\tau_{x_0}\xi, x' - x_0)$. As noticed in [BB07], we can not deduce directly Proposition 4.17 from this argument because \mathbb{G}_∞ covers only a fraction of order $p = \mathcal{P}[0 \in \mathbb{G}_\infty]$ of the ν -dimensional section Λ_L^ν . The idea is then to work at a larger scale, say mK , $m \geq 1$. The interest of using the mK scale is that the process of good mK -boxes stochastically dominates a percolation process with parameter as close to one as we wish for m large enough (recall assumption **(SD)**). We follow this strategy at the mK scale and we show in Lemmas 4.19 and 4.20 that it is possible to obtain a good control of the corrector for points in a large fraction of the mK -boxes. Finally, we go back to the K scale by finding a K -box contained in a suitable mK -box from which we can extend the control on the corrector.

In the rest of the section we add the superscripts K and mK to indicate the considered scale.

Let us denote by e_1, \dots, e_d the vectors of the standard basis of \mathbb{R}^d . In order to control the behavior of the corrector at the scale mK , for fixed C, m, ε , let us define the measurable sets:

$$\begin{aligned} \mathcal{A}_{C,m,\varepsilon} := & \left\{ \widehat{\xi} \in \widehat{\mathcal{N}} : \forall i \in \{1, \dots, d\}, \text{ for } e = \pm e_i, \forall N \in \mathbb{N}^*, \right. \\ & \text{if } j \in \{1, \dots, N\} \text{ is s.t. } je \in \mathbb{G}_\infty^{mK}(\widehat{\xi}) \text{ then:} \\ & \exists x \in \xi \text{ with } \text{Vor}_\xi(x) \cap B_{je}^{mK} \neq \emptyset \text{ s.t. } \forall x_0 \in \xi \text{ s.t. } \text{Vor}_\xi(x_0) \cap B_0^{mK} \neq \emptyset \\ & \left. \left\| \chi(\tau_{x_0}\xi, x - x_0) \right\| \leq C + \varepsilon N \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{C,m} := & \left\{ \widehat{\xi} \in \widehat{\mathcal{N}} : \forall x, x' \in \xi \text{ with } \text{Vor}_\xi(x) \cap B_0^{mK} \neq \emptyset, \text{Vor}_\xi(x') \cap B_0^{mK} \neq \emptyset \text{ one has:} \right. \\ & \left. \left\| \chi(\tau_x\xi, x' - x) \right\| \leq C \right\}. \end{aligned}$$

For $\nu \in \{1, \dots, d\}$, $n \in \mathbb{N}^*$ and $\widehat{\xi} \in \widehat{\mathcal{N}}$, let us also define:

$$\Gamma_{n,\nu}^{C,m,\varepsilon} := \left\{ \mathbf{z} \in \Lambda_n^\nu \cap \mathbb{G}_\infty^{mK} : \tau_{mK\mathbf{z}}\widehat{\xi} \in \mathcal{A}_{C,m} \cap \mathcal{A}_{C,m,\varepsilon} \right\},$$

and

$$\Gamma_n^{C,m,\varepsilon} := \bigcap_{\nu=1}^d \left\{ \mathbf{z} \in \Lambda_n^d : \mathbf{z}^{(\nu)} \in \Gamma_{n,\nu}^{C,m,\varepsilon} \right\},$$

where $\mathbf{z}^{(\nu)} = (\mathbf{z}_1, \dots, \mathbf{z}_\nu, 0, \dots, 0)$.

We now prove three intermediary lemmas.

Lemma 4.18. *For each $\delta, \varepsilon > 0$, there are C and m such that for \mathcal{P} – a.a. $\widehat{\xi}$, there exists $n_0 = n_0(\widehat{\xi}, C, m, \varepsilon, \delta) < \infty$ such that:*

$$\frac{\#\Gamma_n^{C,m,\varepsilon}}{\#\Lambda_n^d} \geq 1 - \delta, \quad \forall n \geq n_0. \quad (4.5.2)$$

Proof. Note that, thanks to the union bound, it suffices to show that for any $\nu \in \{1, \dots, d\}$:

$$\frac{\#\Gamma_{n,\nu}^{C,m,\varepsilon}}{\#\Lambda_n^\nu} \geq 1 - \frac{\delta}{d}$$

for $n \geq n_0$. Let $\delta' := \delta/d$.

Thanks to assumption **(SD)**, one has for m large enough:

$$\mathcal{P}\left[0 \in \mathbb{G}_\infty^{mK}\right] \geq 1 - \frac{\delta'}{2}. \quad (4.5.3)$$

Given δ', ε and m such that the inequality above holds, using Proposition 4.16 at the scale mK , we can find C large enough such that:

$$\mathcal{P}\left[\mathcal{A}_{C,m} \cap \mathcal{A}_{C,m,\varepsilon} \mid 0 \in \mathbb{G}_\infty^{mK}\right] \geq 1 - \frac{\delta'}{2}. \quad (4.5.4)$$

Due to the ergodicity assumption **(Er)**, (4.5.3) and (4.5.4), one has:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#\Gamma_{n,\nu}^{C,m,\varepsilon}}{\#\Lambda_n^\nu} &= \lim_{n \rightarrow \infty} \frac{1}{\#\Lambda_n^\nu} \sum_{\mathbf{z} \in \Lambda_n^\nu} \mathbf{1}_{\tau_{mK\mathbf{z}}\widehat{\xi} \in (\mathcal{A}_{C,m} \cap \mathcal{A}_{C,m,\varepsilon} \cap \{0 \in \mathbb{G}_\infty^{mK}\})} \\ &= \mathcal{P}\left[\mathcal{A}_{C,m} \cap \mathcal{A}_{C,m,\varepsilon} \cap \{0 \in \mathbb{G}_\infty^{mK}\}\right] \\ &= \mathcal{P}\left[\mathcal{A}_{C,m} \cap \mathcal{A}_{C,m,\varepsilon} \mid 0 \in \mathbb{G}_\infty^{mK}\right] \mathcal{P}\left[0 \in \mathbb{G}_\infty^{mK}\right] \\ &\geq \left(1 - \frac{\delta'}{2}\right) \left(1 - \frac{\delta'}{2}\right) > 1 - \delta'. \end{aligned}$$

This implies the result. \square

Lemma 4.19. *Given $C, m, \varepsilon > 0$ and $\widehat{\xi} \in \widehat{\mathcal{N}}$, if $x \in \xi$ is such that $\text{Vor}_\xi(x) \cap B_{\mathbf{a}}^{mK} \neq \emptyset$ for some $\mathbf{a} \in \Gamma_n^{C,m,\varepsilon}$, then there exists $x^1 \in \xi$ with $\text{Vor}_\xi(x^1) \cap B_{\mathbf{a}^{(1)}}^{mK} \neq \emptyset$ satisfying:*

$$\|\chi(\tau_x \xi, x^1 - x)\| \leq (d-1)(C + \varepsilon n). \quad (4.5.5)$$

Proof. Since $\mathbf{a} \in \Gamma_n^{C,m,\varepsilon}$, $\mathbf{a} = \mathbf{a}^{(d)} \in \Gamma_{n,d}^{C,m,\varepsilon}$ and $\mathbf{a}^{(d-1)} \in \Gamma_{n,d-1}^{C,m,\varepsilon} \subset \mathbb{G}_\infty^{mK}$. In particular, $\tau_{mK}\widehat{\xi} \in \mathcal{A}_{C,m,\varepsilon}$ and we can write $\mathbf{a}^{(d-1)} = \mathbf{a}^{(d)} + je_d$, $|j| \leq n$. This implies that there exists $x^{d-1} \in \xi$ with $\text{Vor}_\xi(x^{d-1}) \cap B_{\mathbf{a}^{(d-1)}}^{mK} \neq \emptyset$ satisfying:

$$\left\| \chi(\tau_x \xi, x^{d-1} - x) \right\| \leq C + \varepsilon n.$$

Let us write $x^d := x$. One can construct in the same way and by induction x^d, x^{d-1}, \dots, x^1 such that $\text{Vor}_\xi(x^{i-1}) \cap B_{\mathbf{a}^{(i-1)}}^{mK} \neq \emptyset$ and satisfying:

$$\left\| \chi(\tau_{x^i} \xi, x^{i-1} - x^i) \right\| \leq C + \varepsilon n, \quad i = 2, \dots, d.$$

The shift-covariance of the corrector leads to:

$$\chi(\tau_x \xi, x^1 - x) = \sum_{i=2}^d \left\{ \chi(\tau_x \xi, x^{i-1} - x) - \chi(\tau_x \xi, x^i - x) \right\} = \sum_{i=2}^d \chi(\tau_{x^i} \xi, x^{i-1} - x^i).$$

Hence,

$$\left\| \chi(\tau_x \xi, x^1 - x) \right\| \leq \sum_{i=2}^d \left\| \chi(\tau_{x^i} \xi, x^{i-1} - x^i) \right\| \leq (d-1)(C + \varepsilon n).$$

□

Lemma 4.20. *Let $C, m, \varepsilon > 0$, $\widehat{\xi} \in \widehat{\mathcal{N}}$, $x, y \in \xi$ with $\text{Vor}_\xi(x) \cap B_{\mathbf{a}}^{mK} \neq \emptyset$ and $\text{Vor}_\xi(y) \cap B_{\mathbf{b}}^{mK} \neq \emptyset$ for some $\mathbf{a}, \mathbf{b} \in \Gamma_n^{C,m,\varepsilon}$. Then,*

$$\left\| \chi(\tau_x \xi, y - x) \right\| \leq 2d(C + \varepsilon n).$$

Proof. Let $x^1, y^1 \in \xi$ with $\text{Vor}_\xi(x^1) \cap B_{\mathbf{a}^{(1)}}^{mK} \neq \emptyset$ and $\text{Vor}_\xi(y^1) \cap B_{\mathbf{b}^{(1)}}^{mK} \neq \emptyset$ given by Lemma 4.19. Thanks to the shift-covariance and the antisymmetry of the corrector, one has:

$$\begin{aligned} \chi(\tau_x \xi, y - x) &= \chi(\tau_x \xi, x^1 - x) + \chi(\tau_{x^1} \xi, y^1 - x^1) + \chi(\tau_{y^1} \xi, y - y^1) \\ &= \chi(\tau_x \xi, x^1 - x) + \chi(\tau_{x^1} \xi, y^1 - x^1) - \chi(\tau_{y^1} \xi, y^1 - y). \end{aligned}$$

Together with Lemma 4.19, this leads to:

$$\left\| \chi(\tau_x \xi, y - x) \right\| \leq 2(d-1)(C + \varepsilon n) + \left\| \chi(\tau_{x^1} \xi, y^1 - x^1) \right\|. \quad (4.5.6)$$

Since $\mathbf{a}^{(1)} \in \Gamma_{n,1}^{C,m,\varepsilon}$ and $\mathbf{b}^{(1)} \in \mathbb{G}_\infty^{mK}$, there exists a point $\bar{y} \in \xi$ with $\text{Vor}_\xi(\bar{y}) \cap B_{\mathbf{b}^{(1)}}^{mK} \neq \emptyset$ satisfying:

$$\left\| \chi(\tau_{x^1} \xi, \bar{y} - x^1) \right\| \leq C + 2\varepsilon n. \quad (4.5.7)$$

Moreover, $\tau_{mK\mathbf{b}^{(1)}}\widehat{\xi} \in \mathcal{A}_{C,m}$ which implies that:

$$\left\| \chi(\tau_{\bar{y}} \xi, y^1 - \bar{y}) \right\| \leq C. \quad (4.5.8)$$

Bounds (4.5.6)-(4.5.8) and the shift-covariance of the corrector finally give that:

$$\left\| \chi(\tau_x \xi, y - x) \right\| \leq 2(d-1)(C + \varepsilon n) + \left\| \chi(\tau_{x^1} \xi, \bar{y} - x^1) \right\| + \left\| \chi(\tau_{\bar{y}} \xi, y^1 - \bar{y}) \right\| \leq 2d(C + \varepsilon n).$$

□

Proof of Proposition 4.17. We must show that for each $\delta_0, \varepsilon_0 > 0$, for $\mathcal{P}[\cdot | 0 \in \mathbb{G}_\infty] - a.a.$ $\widehat{\xi} \in \widehat{\mathcal{N}}$, for any $x_0 \in \xi$ with $\text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset$:

$$\frac{1}{\#\Lambda_L} \sum_{\substack{x \in \xi: \\ \exists z \in \mathbb{G}_\infty^K \cap \Lambda_L \\ \text{Vor}_\xi(x) \cap B_z^K \neq \emptyset}} \mathbf{1}_{\|\chi(\tau_{x_0} \xi, x - x_0)\| \geq \varepsilon_0 L} \leq \delta_0$$

for every L large enough. Let us define $p := \mathcal{P}[0 \in \mathbb{G}_\infty^K]$ and fix $\delta < \min(p/2, \delta_0/(2D))$ and $\varepsilon < \varepsilon_0/(4d)$. We then choose C and m large enough such that the conclusion of Lemma 4.18 holds. Without loss of generality we restrict our attention to the case $L = mn$, $n \in \mathbb{N}$. We will work at both scales K and mK .

By ergodicity

$$\frac{\#\{j \in \Lambda_L^1 : je_1 \in \mathbb{G}_\infty^K\}}{2L + 1} \xrightarrow{L \rightarrow \infty} p,$$

in particular, for L large enough

$$\#\{j \in \Lambda_L^1 : je_1 \in \mathbb{G}_\infty^K\} \geq \frac{(2L + 1)p}{2}. \quad (4.5.9)$$

On the other hand, denoting by $\pi^{(1)}$ the projection along the first coordinate axis, one has $\pi^{(1)}(\mathbf{\Gamma}_n^{C,m,\varepsilon}) \subset \mathbf{\Gamma}_n^{C,m,\varepsilon}$ and

$$\#\left(\pi^{(1)}(\mathbf{\Gamma}_n^{C,m,\varepsilon})\right) \geq \frac{\#\mathbf{\Gamma}_n^{C,m,\varepsilon}}{(2n + 1)^{d-1}} \geq (1 - \delta)(2n + 1)$$

by Lemma 4.18. It follows that

$$\#\left\{j \in \Lambda_L^1 : \left\lfloor \frac{j}{m} \right\rfloor e_1 \in \mathbf{\Gamma}_n^{C,m,\varepsilon}\right\} \geq (1 - \delta)(2n + 1)m \geq (1 - \delta)(2L + 1). \quad (4.5.10)$$

Due to the choice of δ , $(1 - \delta)(2L + 1) + (2L + 1)p/2 > (2L + 1) = \#\Lambda_L^1$ which implies that

$$\left\{j \in \Lambda_L^1 : je_1 \in \mathbb{G}_\infty^K\right\} \cap \left\{j \in \Lambda_L^1 : \left\lfloor \frac{j}{m} \right\rfloor e_1 \in \mathbf{\Gamma}_n^{C,m,\varepsilon}\right\} \neq \emptyset.$$

Fix j in the intersection above, thanks to the sublinearity in the direction e_1 at scale K (see Proposition 4.16), there exists $x \in \xi$ with $\text{Vor}_\xi(x) \cap B_{je_1}^K \neq \emptyset$ satisfying:

$$\left\| \chi(\tau_{x_0} \xi, x - x_0) \right\| \leq C + \varepsilon L, \quad \forall x_0 \in \xi \text{ with } \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset.$$

Together with Lemma 4.20 applied with $\mathbf{a} = \lfloor \frac{j}{m} \rfloor e_1$ and the shift-covariance of the corrector, this allows us to conclude that for any $y \in \xi$ whose Voronoi cell intersects an mK -box with index \mathbf{b} in $\Gamma_n^{C,m,\varepsilon}$, for any $x_0 \in \xi$ with $\text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset$:

$$\begin{aligned} \|\chi(\tau_{x_0}\xi, y - x_0)\| &\leq \|\chi(\tau_{x_0}\xi, x - x_0)\| + \|\chi(\tau_x\xi, y - x)\| \\ &\leq C + \varepsilon L + 2d(C + \varepsilon n) \leq (2d + 1)(C + \varepsilon L). \end{aligned}$$

Thanks to the choice of ε , the last quantity is smaller than $\varepsilon_0 L$ when L is large enough. For any $x_0 \in \xi$ with $\text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset$, one has for L large enough:

$$\begin{aligned} \frac{1}{\#\Lambda_L} \sum_{\substack{x \in \xi: \\ \exists \mathbf{z} \in \mathbb{G}_\infty^K \cap \Lambda_L \\ \text{Vor}_\xi(x) \cap B_{\mathbf{z}}^K \neq \emptyset}} \mathbf{1}_{\|\chi(\tau_{x_0}\xi, x - x_0)\| \geq \varepsilon_0 L} &\leq \frac{1}{\#\Lambda_L} \sum_{\mathbf{a} \in \Lambda_n \setminus \Gamma_n^{C,m,\varepsilon}} \sum_{\substack{\mathbf{z} \in \mathbb{G}_\infty^K: \\ B_{\mathbf{z}}^K \subset B_{\mathbf{a}}^{mK}}} D \\ &\leq \frac{m^d D \#(\Lambda_n \setminus \Gamma_n^{C,m,\varepsilon})}{\#\Lambda_L} \\ &\leq 2D \left(1 - \frac{\#(\Gamma_n^{C,m,\varepsilon})}{\#\Lambda_n} \right) \\ &\leq 2D\delta \leq \delta_0 \end{aligned}$$

where we used that:

$$\frac{m^d \#\Lambda_n}{\#\Lambda_L} \xrightarrow[n \rightarrow \infty]{} 1$$

and Lemma 4.18. □

4.6 Random walks on $\mathcal{G}_\infty(\hat{\xi})$

In order to derive the strong sublinearity of the corrector in $\mathcal{G}_\infty(\hat{\xi})$ from its sublinearity on average, we need to obtain heat-kernel estimates and bounds on the expected distance between positions of the walker at time t and 0 (see equations (4.7.1) and (4.8.1)). Such estimates cannot be obtained directly for the random walk on the (full) Delaunay triangulation generated by ξ in which the degree is not bounded. Nevertheless, since $\mathcal{G}_\infty(\hat{\xi})$ has good regularity properties, these bounds will be established in Sections 4.7 and 4.8 for restricted random walks described below.

For $\hat{\xi} \in \hat{\mathcal{N}}$, let us consider the Markov chain $(\hat{Y}_n)_{n \in \mathbb{N}} = (\hat{Y}_n^{\hat{\xi}})_{n \in \mathbb{N}}$ on $\mathcal{G}_\infty(\hat{\xi})$ induced from the original (discrete time) random walk $(X_n)_{n \in \mathbb{N}}$ on $\text{DT}(\xi)$. In other words, $(\hat{Y}_n)_{n \in \mathbb{N}}$ is the time-homogeneous Markov chain on $\mathcal{G}_\infty(\hat{\xi})$ with jump probabilities given by:

$$\hat{c}_{x,y}^{\hat{\xi}} := P^{\hat{\xi}}[\hat{Y}_{k+1} = y \mid \hat{Y}_k = x] = P_x^\xi[X_{T_1} = y], \quad x, y \in \mathcal{G}_\infty(\hat{\xi}), \quad (4.6.1)$$

where $T_1 := \inf\{j \geq 1 : X_j \in \mathcal{G}_\infty(\hat{\xi})\}$. Note that the *holes* (i.e. the $(\text{DT}(\xi))$ -connected components of $\xi \setminus \mathcal{G}_\infty(\hat{\xi})$) are a.s. finite. Hence, T_1 is a.s. finite and $(\hat{Y}_n)_{n \in \mathbb{N}}$ is well defined.

Moreover, for any $z \in \mathcal{G}_\infty(\hat{\xi})$, applying the optional stopping theorem to the martingale $(X_n - z - \chi(\tau_z \xi, X_n - z))_{n \in \mathbb{N}}$ starting from z , it appears that, $(\hat{Y}_n - z - \chi(\tau_z \xi, \hat{Y}_n - z))_{n \in \mathbb{N}}$ is a martingale.

We also consider a continuous-time version of the random walk defined above $(\hat{Y}_t)_{t \geq 0} := (\hat{Y}_{N(t)})_{t \geq 0}$ where $N(t)$ is the intensity 1 Poisson process on the half-line \mathbb{R}^+ . It has infinitesimal generator:

$$\hat{\mathcal{L}}^{\hat{\xi}} f(x) := \sum_{y \in \mathcal{G}_\infty(\hat{\xi})} \hat{c}_{x,y}^{\hat{\xi}} (f(y) - f(x)), \quad x, y \in \mathcal{G}_\infty(\hat{\xi}). \quad (4.6.2)$$

It is not difficult to see that $(\hat{Y}_t - z - \chi(\tau_z \xi, \hat{Y}_t - z))_{t \geq 0}$ is also a martingale.

We denote by $P_x^{\hat{\xi}}$ the (quenched) law of this walk starting from x . Note that this walk has speed ‘at most 1’. Observe also that the measure $\text{deg}_{\text{DT}(\xi)}$ is reversible w.r.t. both $(\hat{Y}_n)_{n \in \mathbb{N}}$ and $(\hat{Y}_t)_{t \geq 0}$. Actually, standard computations show that the *detailed balance condition*:

$$\text{deg}_{\text{DT}(\xi)}(x) \hat{c}_{x,y}^{\hat{\xi}} = \text{deg}_{\text{DT}(\xi)}(y) \hat{c}_{y,x}^{\hat{\xi}}, \quad x, y \in \mathcal{G}_\infty(\hat{\xi}),$$

is satisfied.

4.7 Heat-kernel estimates for $(\hat{Y}_t^{\hat{\xi}})_{t > 0}$

The aim of this section is to prove the following heat-kernel bound.

Proposition 4.21. *For a.a. $\hat{\xi} \in \hat{\mathcal{N}}$:*

$$\sup_{n \geq 1} \max_{x \in \mathcal{G}_\infty(\hat{\xi}) \cap [-n, n]^d} \sup_{t \geq n} t^{\frac{d}{2}} P_x^{\hat{\xi}} [\hat{Y}_t = x] < \infty \quad (4.7.1)$$

where $(\hat{Y}_t)_{t \geq 0}$ is the continuous-time random walk on $\mathcal{G}_\infty(\hat{\xi})$ with generator (4.6.2).

The proof of this bound relies on isoperimetric inequalities and the technics developed in [MP05]; it is completed in Subsection 4.7.4. Precise definitions are given in Subsection 4.7.1. Isoperimetric inequalities for random walks confined in large boxes are established in Subsection 4.7.2. Additional technical results are isolated from the proof of Proposition 4.21 and given in Subsection 4.7.3.

4.7.1 Precise definitions

We will state isoperimetric inequalities for random walks confined in large boxes. We need to introduce additional notations and random walks confined in boxes of side L .

Recall the definitions of $\mathbb{G}_{(L)}$ and $\mathcal{G}_{(L)}(\hat{\xi})$ from the introduction and denote by $\overline{\mathbb{G}}_{(L)}$ the complementary in \mathbb{Z}^d of the unique unbounded (l^1 -)connected component of $\mathbb{Z}^d \setminus \mathbb{G}_{(L)}$ and by $\overline{\mathcal{G}}_{(L)}(\hat{\xi})$ the set of points of ξ whose Voronoi cell intersects a K -box with index in $\overline{\mathbb{G}}_{(L)}$.

In other words, $\overline{\mathbb{G}}_{(L)}$ is the union of $\mathbb{G}_{(L)}$ with the (discrete) holes contained in $[-L, L]^d$ and $\overline{\mathcal{G}}_{(L)}(\hat{\xi})$ is the union of $\mathcal{G}_{(L)}(\hat{\xi})$ with the holes (for the DT(ξ) structure) contained in $\left[-(L + \frac{1}{2})K, (L + \frac{1}{2})K\right]^d$. Then, we write $\left(\overline{X}_n^{(L)}\right)_{n \in \mathbb{N}}$ for the random walk in the restriction of DT(ξ) to $\overline{\mathcal{G}}_{(L)}(\hat{\xi})$. Denoting by T_n^* the time of n^{th} visit to $\mathcal{G}_{(L)}(\hat{\xi})$ for this walk, we consider the induced discrete-time random walk $\left(\widehat{Y}_n^{(L)}\right)_{n \in \mathbb{N}} := \left(\overline{X}_{T_n^*}^{(L)}\right)_{n \in \mathbb{N}}$ on $\mathcal{G}_{(L)}(\hat{\xi})$. We also consider its continuous-time counterpart $\left(\widehat{Y}_t^{(L)}\right)_{t \geq 0} := \left(\widehat{Y}_{N(t)}^{(L)}\right)_{t \geq 0}$ where $N(\cdot)$ is an intensity 1 Poisson process on the half-line \mathbb{R}^+ . The random walk $\left(\widehat{Y}_t^{(L)}\right)_{t \geq 0}$ can be thought of as the continuous-time random walk on $\mathcal{G}_{(L)}(\hat{\xi})$ with speed at most 1 and which ‘jumps holes’ of $\mathcal{G}_{(L)}(\hat{\xi})$. Let us denote by $\widehat{\text{deg}}_{L, \hat{\xi}}(\cdot)$ the degree in the restriction of DT(ξ) to $\overline{\mathcal{G}}_{(L)}(\hat{\xi})$ and write $\widehat{\text{deg}}_{L, \hat{\xi}}(A) := \sum_{x \in A} \widehat{\text{deg}}_{L, \hat{\xi}}(x)$, $A \subset \overline{\mathcal{G}}_{(L)}(\hat{\xi})$. Classical computations show that $\widehat{\text{deg}}_{L, \hat{\xi}}(\cdot)$ is reversible for both $\left(\overline{X}_n^{(L)}\right)_{n \in \mathbb{N}}$ and $\left(\widehat{Y}_t^{(L)}\right)_{t \geq 0}$. The *conductance* of the set $A \subset \mathcal{G}_{(L)}(\hat{\xi})$ w.r.t. $\left(\widehat{Y}_t^{(L)}\right)_{t \geq 0}$ is given by:

$$\widehat{I}_A^{(L)} = \widehat{I}_A^{\hat{\xi}, (L)} := \frac{\sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\hat{\xi}) \setminus A} \widehat{\text{deg}}_{L, \hat{\xi}}(x) \overline{P}_x^{\hat{\xi}, (L)} \left[\overline{X}_{T_1^*}^{(L)} = y \right]}{\widehat{\text{deg}}_{L, \hat{\xi}}(A)}, \quad (4.7.2)$$

where $\overline{P}_x^{\hat{\xi}, (L)}$ stands for the law of $\left(\overline{X}_n^{(L)}\right)_{n \in \mathbb{N}}$. The associated *isoperimetric profile* is:

$$\widehat{\varphi}_{(L)}(u) := \inf \left\{ \widehat{I}_A^{(L)} : \widehat{\text{deg}}_{L, \hat{\xi}}(A) \leq u \widehat{\text{deg}}_{L, \hat{\xi}}(\mathcal{G}_{(L)}(\hat{\xi})) \right\}, \quad u \in \left] 0, \frac{1}{2} \right]. \quad (4.7.3)$$

The advantage of $\left(\widehat{Y}_t^{(L)}\right)_{t \geq 0}$ is that this walk coincides with $\left(\widehat{Y}_t\right)_{t \geq 0}$ as long as they do not leave (the interior of) $\overline{\mathcal{G}}_{(L)}(\hat{\xi})$. Nevertheless, we are not able to obtain directly a bound on the isoperimetric profile $\widehat{\varphi}_{(L)}$. In a similar way as in [BBHK08], we compare it with the isoperimetric profile of the constant speed random walk on $\mathcal{G}_{(L)}(\hat{\xi})$, that is the walk $\left(\widetilde{Y}_t^{(L)}\right)_{t \geq 0}$ with generator:

$$\widetilde{\mathcal{L}}_{\hat{\xi}}^{(L)}(x, y) := \begin{cases} \frac{\mathbf{1}_{x \sim y \text{ in DT}(\xi)}}{\widehat{\text{deg}}_{L, \hat{\xi}}(x)} & \text{if } x \neq y \\ -1 & \text{if } x = y \end{cases}, \quad (4.7.4)$$

where $\widehat{\text{deg}}_{L, \hat{\xi}}(\cdot)$ denotes the degree in the restriction of DT(ξ) to $\mathcal{G}_{(L)}(\hat{\xi})$. The measure $\widehat{\text{deg}}_{L, \hat{\xi}}(\cdot)$ is clearly reversible w.r.t. $\left(\widetilde{Y}_t^{(L)}\right)_{t \geq 0}$. The associated *conductance* of the set $A \subset \mathcal{G}_{(L)}(\hat{\xi})$ is given by:

$$\widetilde{I}_A^{(L)} = \widetilde{I}_A^{\hat{\xi}, (L)} := \frac{\sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\hat{\xi}) \setminus A} \mathbf{1}_{x \sim y \text{ in DT}(\xi)}}{\widehat{\text{deg}}_{L, \hat{\xi}}(A)}, \quad (4.7.5)$$

and the corresponding *isoperimetric profile* is:

$$\tilde{\varphi}_{(L)}(u) := \inf \left\{ \tilde{I}_A^{(L)} : \widetilde{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(A) \leq u \widetilde{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(\mathcal{G}_{(L)}(\hat{\xi})) \right\}, \quad u \in \left] 0, \frac{1}{2} \right]. \quad (4.7.6)$$

4.7.2 Isoperimetric inequality

The goal of this section is to obtain a lower bound on the isoperimetric profile $\hat{\varphi}_{(L)}$; it is stated in Corollary 4.25.

Comparison between $\tilde{\varphi}_{(L)}$ and $\hat{\varphi}_{(L)}$

First, note that for $x \in \mathcal{G}_{(L)}(\hat{\xi})$:

$$D \geq \deg_{\text{DT}(\xi)}(x) \geq \widehat{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x) \geq \widetilde{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x) \geq 1 \geq \frac{\widehat{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x)}{D} \geq \frac{\widetilde{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x)}{D}, \quad (4.7.7)$$

and that for $x, y \in \mathcal{G}_{(L)}(\hat{\xi})$

$$\widehat{P}_x^{\hat{\xi},(L)} \left[\overline{X}_{T_1^*}^{(L)} = y \right] \geq \widetilde{P}_x^{\hat{\xi},(L)} \left[\overline{X}_1^{(L)} = y \right] \mathbf{1}_{x \sim y \text{ in DT}(\xi)} = \frac{\mathbf{1}_{x \sim y \text{ in DT}(\xi)}}{\widehat{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x)}. \quad (4.7.8)$$

Hence, for $A \subset \mathcal{G}_{(L)}(\hat{\xi})$, we have:

$$\sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\hat{\xi}) \setminus A} \widehat{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(x) \widehat{P}_x^{\hat{\xi},(L)} \left[\overline{X}_{T_1^*}^{(L)} = y \right] \geq \sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\hat{\xi}) \setminus A} \mathbf{1}_{x \sim y \text{ in DT}(\xi)}.$$

With $\widehat{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(A) \leq D \widetilde{\deg}_{\mathcal{G}_{L,\hat{\xi}}}(A)$, this implies that:

$$\hat{I}_A^{(L)} \geq \frac{\tilde{I}_A^{(L)}}{D}. \quad (4.7.9)$$

Using (4.7.7) and (4.7.9), one deduces:

Lemma 4.22. *For $u \in \left] 0, \frac{1}{2D} \right]$:*

$$\hat{\varphi}_{(L)}(u) \geq \frac{\tilde{\varphi}_{(L)}(Du)}{D}. \quad (4.7.10)$$

Lower bounds for $\tilde{\varphi}_{(L)}$

Our aim is to show the following bound on the isoperimetric profile $\tilde{\varphi}_{(L)}$ associated with $(\tilde{Y}_t^{(L)})_{t \geq 0}$.

Proposition 4.23. *There exists $c = c(d, K, \alpha) > 0$ such that \mathcal{P} – a.s. for L large enough:*

$$\tilde{\varphi}_{(L)}(u) \geq c \min \left\{ \frac{1}{u^{1/d} L}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\}, \quad u \in \left] 0, \frac{1}{2} \right].$$

As in [CF07], we use as much as possible an isoperimetric inequality for the percolation cluster $\mathbb{G}_{(L)}$.

Proposition 4.24 (see [CF07], eq. (2.5)). *There exists $\kappa > 0$ such that almost surely for L large enough, for $\mathbb{A} \subset \mathbb{G}_{(L)}$ with $0 < \#(\mathbb{A}) \leq \frac{1}{2} \#(\mathbb{G}_{(L)})$:*

$$\frac{\#(\partial \mathbb{A})}{\#(\mathbb{A})} \geq \kappa \min \left\{ \frac{1}{\#(\mathbb{A})^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\},$$

where $\partial \mathbb{A} = \{x \in \mathbb{G}_{(L)} \setminus \mathbb{A} : x \sim y \text{ for some } y \in \mathbb{A}\}$ is the (vertex external) boundary of the set \mathbb{A} in $\mathbb{G}_{(L)}$.

This result can be proved by adapting the arguments given in [BBHK08, Appendix] to the context of supercritical site percolation (see also the proof of [BM03, Lemma 2.6] for $p \gg p_c^{\text{site}}(\mathbb{Z}^d)$).

We adapt the proof of [CF07, Theorem 1.1] to the present setting. It is worth noting that the arguments of [CF07] can be used to derive isoperimetric bounds for the Delaunay triangulation confined in cubic boxes at least when the underlying point process is a PPP. This does not lead to sharp enough heat-kernel bounds for the random walk on the full Delaunay triangulation due to the unboundedness of the degree.

Proof of Proposition 4.23. For $A \subset \mathcal{G}_{(L)}(\hat{\xi})$, we define:

$$\mathbb{L}(A) := \left\{ \mathbf{z} \in \mathbb{G}_{(L)} : \exists x \in A \text{ s.t. } \text{Vor}_{\xi}(x) \cap B_{\mathbf{z}} \neq \emptyset \right\}.$$

Let us observe that, thanks to the definition of the good boxes, for any $A \subset \mathcal{G}_{(L)}(\hat{\xi})$:

$$\frac{\#\mathbb{L}(A)}{2^d} \leq \widetilde{\text{deg}}_{L, \hat{\xi}}(A) \leq \#A \max_{x \in A} \text{deg}_{\text{DT}(\xi)}(x) \leq D^2 \#\mathbb{L}(A). \quad (4.7.11)$$

In the first inequality, we used that, for $x \in \mathcal{G}_{(L)}(\hat{\xi})$, $\text{Vor}_{\xi}(x)$ does not intersect more than 2^d good boxes since the diameter of the cell is less than K .

From now on we assume that $\widetilde{\text{deg}}_{L, \hat{\xi}}(A) \leq \frac{1}{2} \widetilde{\text{deg}}_{L, \hat{\xi}}(\mathcal{G}_{(L)}(\hat{\xi}))$. We are going to discuss separately the cases when $\#\mathbb{L}(A)$ is large or small with respect to $\#(\mathbb{G}_{(L)})$. Roughly, if $\#\mathbb{L}(A)$ is large then $\#\mathbb{L}(\mathcal{G}_{(L)}(\hat{\xi}) \setminus A)$ is not too small and $\tilde{I}_A^{(L)}$ is easily bounded from below by some constant. When $\#\mathbb{L}(A)$ is small, a bound is obtained using the isoperimetric inequality for $\mathbb{G}_{(L)}$ given in Proposition 4.24.

The case $\#(\mathbb{L}(A)) > \left(1 - \frac{1}{2^{d+2}D^2}\right)\#(\mathbb{G}_{(L)})$. Using the general bound (4.7.11) and inequality $\widetilde{\deg}_{L,\widehat{\xi}}(A) \leq \frac{1}{2}\widetilde{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi}))$, one obtains:

$$\#\mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A) \geq \frac{\widetilde{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)}{D^2} \geq \frac{\widetilde{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi}))}{2D^2} \geq \frac{\#\mathbb{G}_{(L)}}{2^{d+1}D^2}.$$

It follows that $\mathbb{L}(A)$ and $\mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)$ have a large intersection in this case:

$$\begin{aligned} \#(\mathbb{L}(A) \cap \mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)) &\geq \#\mathbb{L}(A) - \#(\mathbb{G}_{(L)} \setminus \mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)) \\ &= \#\mathbb{L}(A) - \#(\mathbb{G}_{(L)}) + \#(\mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)) \\ &\geq \left(1 - \frac{1}{2^{d+2}D^2}\right)\#(\mathbb{G}_{(L)}) - \#(\mathbb{G}_{(L)}) + \frac{1}{2^{d+1}D^2}\#(\mathbb{G}_{(L)}) \\ &= \frac{1}{2^{d+2}D^2}\#(\mathbb{G}_{(L)}). \end{aligned} \quad (4.7.12)$$

This allows us to bound from below the numerator in $\widetilde{I}_A^{(L)}$ as follows. If $\mathbf{z} \in \mathbb{L}(A) \cap \mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A)$, one can choose $x \in A$ and $y \in \mathcal{G}_{(L)}(\widehat{\xi}) \setminus A$ whose respective Voronoi cells intersect $B_{\mathbf{z}}$. Hence, connecting the nuclei of the Voronoi cells which intersect the line segment $[x, y]$, it is easy to find an edge between a point of A and a point of $\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A$ which is included in $\overline{B}_{\mathbf{z}} = \bigcup_{\mathbf{z}': \|\mathbf{z}' - \mathbf{z}\|_{\infty} \leq 1} B_{\mathbf{z}'}$ thanks to the definition of good boxes. Since a specific edge is associated to at most 3^d boxes by this procedure, it follows using (4.7.12) that:

$$\sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\widehat{\xi}) \setminus A} \mathbf{1}_{x \sim y \text{ in DT}(\xi)} \geq \frac{\#(\mathbb{L}(A) \cap \mathbb{L}(\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A))}{3^d} \geq \frac{\#(\mathbb{G}_{(L)})}{4 \cdot 6^d D^2}.$$

Since $\widetilde{\deg}_{L,\widehat{\xi}}(A) \leq D^2\#(\mathbb{L}(A)) \leq D^2\#(\mathbb{G}_{(L)})$, we obtain that:

$$\widetilde{I}_A^{(L)} \geq \frac{1}{4 \cdot 6^d D^4}.$$

The case $\#(\mathbb{L}(A)) \leq \left(1 - \frac{1}{2^{d+2}D^2}\right)\#(\mathbb{G}_{(L)})$. Let us show that, in this case, the numerator in $\widetilde{I}_A^{(L)}$ can be bounded from below in terms of $\#\partial\mathbb{L}(A)$ or $\#\partial(\mathbb{G}_{(L)} \setminus \mathbb{L}(A))$. If $B_{\mathbf{z}}$ and $B_{\mathbf{z}'}$ are two neighboring good boxes such that $\mathbf{z} \in \mathbb{L}(A)$ and $\mathbf{z}' \in \mathbb{G}_{(L)} \setminus \mathbb{L}(A)$, there exists an edge between a point of A and a point of $\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A$ contained in $\overline{B}_{\mathbf{z}} \cup \overline{B}_{\mathbf{z}'}$. To see this, let us fix a point $x \in A$ whose Voronoi cell intersects $B_{\mathbf{z}}$ and a point $y \in \mathcal{G}_{(L)}(\widehat{\xi}) \setminus A$ whose Voronoi cell intersects $B_{\mathbf{z}'}$. It then suffices to connect the consecutive nuclei of the Voronoi cells which intersect the line segment $[x, y]$ to find an edge between a point of A and a point of $\mathcal{G}_{(L)}(\widehat{\xi}) \setminus A$. This edge is contained in $\overline{B}_{\mathbf{z}} \cup \overline{B}_{\mathbf{z}'}$ thanks to the definition of good boxes. It follows that there exists $\delta = \delta(d)$ such that:

$$\sum_{x \in A} \sum_{y \in \mathcal{G}_{(L)}(\widehat{\xi}) \setminus A} \mathbf{1}_{x \sim y \text{ in DT}(\xi)} \geq \delta \max \left\{ \#\partial\mathbb{L}(A), \#\partial(\mathbb{G}_{(L)} \setminus \mathbb{L}(A)) \right\}.$$

Since $\widetilde{\deg}_{L,\widehat{\xi}}(A) \leq D^2 \#\mathbb{L}(A) \leq D^2(2^{d+2}D^2 - 1)\#(\mathbb{G}_{(L)} \setminus \mathbb{L}(A))$, we deduce that:

$$\widetilde{I}_A^{(L)} \geq \frac{\delta}{(2^{d+2}D^2 - 1)D^2} \frac{\#\partial\mathbb{A}}{\#\mathbb{A}},$$

for $\mathbb{A} = \mathbb{L}(A)$ and $\mathbb{A} = \mathbb{G}_{(L)} \setminus \mathbb{L}(A)$.

Choosing

$$\mathbb{A} := \begin{cases} \mathbb{L}(A) & \text{if } \#\mathbb{L}(A) \leq \frac{1}{2}\#\mathbb{G}_{(L)} \\ \mathbb{G}_{(L)} \setminus \mathbb{L}(A) & \text{otherwise} \end{cases},$$

Proposition 4.24 then implies that almost surely for L large:

$$\begin{aligned} \widetilde{I}_A^{(L)} &\geq \frac{\kappa\delta}{(2^{d+2}D^2 - 1)D^2} \min \left\{ \frac{1}{\#\mathbb{A}^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\} \\ &\geq \frac{\kappa\delta}{(2^{d+2}D^2 - 1)D^2} \min \left\{ \frac{1}{\#\mathbb{L}(A)^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\}. \end{aligned}$$

Using that $\#\mathbb{L}(A) \leq 2^d \widetilde{\deg}_{L,\widehat{\xi}}(A)$, we obtain that:

$$\widetilde{I}_A^{(L)} \geq \frac{\kappa\delta}{2(2^{d+2}D^2 - 1)D^2} \min \left\{ \frac{1}{\widetilde{\deg}_{L,\widehat{\xi}}(A)^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\}.$$

Since $\widetilde{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi})) \leq D^2 \#\mathbb{G}_{(L)} \leq D^2(2L+1)^d$, the conclusion of Proposition 4.23 follows. \square

Combining Lemma 4.22 and Proposition 4.23, we obtain:

Corollary 4.25. *There exists $c = c(d, K, \alpha) > 0$ such that \mathcal{P} – a.s. for L large enough:*

$$\widehat{\varphi}_{(L)}(u) \geq c \min \left\{ \frac{1}{u^{1/d}L}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right\}, \quad u \in \left] 0, \frac{1}{2D} \right].$$

4.7.3 Other technical results

Volume growth for $\widehat{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi}))$

We briefly check that there exist constants c and C such that a.s. for L large enough:

$$cL^d \leq \widehat{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi})) \leq CL^d. \tag{4.7.13}$$

The upper bound is very simple since it suffices to write:

$$\begin{aligned} \widehat{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi})) &\leq D \#(\mathcal{G}_{(L)}(\widehat{\xi})) \leq D \max_{\mathbf{z} \in \mathbb{G}_{(L)}} \#\{x \in \xi : \text{Vor}_\xi(x) \cap B_{\mathbf{z}} \neq \emptyset\} \#(\mathbb{G}_{(L)}) \\ &\leq D^2 \#([-L, L]^d \cap \mathbb{Z}^d) \leq CL^d. \end{aligned}$$

Since any box with index in $\mathbb{G}_{(L)}$ contains at least a point of $\mathcal{G}_{(L)}(\widehat{\xi})$ which has degree at least 1:

$$\widehat{\deg}_{L,\widehat{\xi}}(\mathcal{G}_{(L)}(\widehat{\xi})) \geq \#(\mathcal{G}_{(L)}(\widehat{\xi})) \geq \#(\mathbb{G}_{(L)}).$$

The lower bound follows by using that a.s. for L large enough:

$$\#(\mathbb{G}_{(L)}) \geq cL^d,$$

which is a consequence of the ergodic theorem.

Size of the holes and connectivity of $\mathcal{G}_{\infty}(\widehat{\xi})$ in large boxes

In order to compare $(\widehat{Y}_t)_{t \geq 0}$ with $(\widehat{Y}_t^{(L)})_{t \geq 0}$, we need to control the size of the holes (*i.e.* DT(ξ)-connected components of $\xi \setminus \mathcal{G}_{\infty}(\widehat{\xi})$) and to establish connectivity properties of $\mathcal{G}_{\infty}(\widehat{\xi})$ in large boxes. More precisely, for $C, \gamma > 1$ and $t > 0$, let define the events:

$$\mathcal{A}_t = \mathcal{A}_{t,\gamma,C} := \left\{ \begin{array}{l} \text{any hole contained in } \left[-K \left(\lfloor t^\gamma \rfloor + \frac{1}{2} \right), K \left(\lfloor t^\gamma \rfloor + \frac{1}{2} \right) \right]^d \\ \text{has diameter smaller than } C \log t \end{array} \right\},$$

and

$$\mathcal{B}_t = \mathcal{B}_{t,\gamma,C} := \left\{ \mathcal{G}_{\infty}(\widehat{\xi}) \cap Q_t = \mathcal{G}_{(\lfloor t^\gamma \rfloor)}(\widehat{\xi}) \cap Q_t \right\}$$

where:

$$Q_t := \left[-t \left(\frac{3C}{2} \log t + 1 \right), t \left(\frac{3C}{2} \log t + 1 \right) \right]^d.$$

We prove that:

Lemma 4.26. *Assuming that K is large enough, for each $\gamma > 1$, there exists $C < \infty$ such that almost surely for t large enough \mathcal{A}_t and \mathcal{B}_t are realized.*

Proof. First, observe that any DT(ξ)-connected component A of $\xi \setminus \mathcal{G}_{\infty}(\widehat{\xi})$ is contained in the union of K -boxes with indices in some discrete hole \mathbb{A} (*i.e.* a connected component of $\mathbb{Z}^d \setminus \mathbb{G}_{\infty}$). Hence, in order to show that \mathcal{A}_t holds almost surely, it suffices to verify that a.s. any discrete hole contained in $[-\lfloor t^\gamma \rfloor, \lfloor t^\gamma \rfloor]^d \cap \mathbb{Z}^d$ has diameter at most $C \log t$, for C suitably chosen. Denote by $\mathbb{A}_{\mathbf{z}}$ the (possibly empty) hole at $\mathbf{z} \in \mathbb{Z}^d$ for an independent percolation process of parameter p . Recall that assumption **(SD)** ensures that the process of ‘good boxes’ dominates such a process with p as close to 1 as we wish whenever K is

fixed large enough. Assuming that p is large enough, a standard Peierls argument shows that there exists $c_6 > 0$ such that:

$$\mathcal{P} [\text{diam } \mathbb{A}_0 \geq n] \leq e^{-c_6 n}.$$

Thus,

$$\mathcal{P} [\exists \mathbf{z} \in [-n, n]^d \cap \mathbb{Z}^d \text{ s.t. } \text{diam } \mathbb{A}_{\mathbf{z}} \geq C \log n] \leq (2n + 1)^d e^{-c_6 C \log n}.$$

Our first claim then follows by the Borel-Cantelli lemma if C is well chosen.

As above, in order to show that \mathcal{B}_t is a.s. realized for t large enough, we only need to check the corresponding claim for the percolation process, that is: almost surely for t large enough, $\mathbb{G}_\infty \cap Q_t = \mathbb{G}_{(\lfloor t^\gamma \rfloor)} \cap Q_t$.

Let us justify that, almost surely for t large enough, $\mathbb{G}_{(\lfloor t^\gamma \rfloor)}$ coincides with the largest connected component of $\mathbb{G}_\infty \cap [-\lfloor t^\gamma \rfloor, \lfloor t^\gamma \rfloor]^d$. Thanks to [CF07, Lemma B.1] and the Borel-Cantelli lemma, we know that there exists L_0 a.s. finite such that, for $L \geq L_0$, the maximal open cluster $\mathbb{G}_{(L)}$ in $[-L, L]^d \cap \mathbb{Z}^d$ is the only open cluster in this box with diameter larger than $L/10$ and crosses this box in every coordinate direction (see also [CF07, Remark 7]). In particular, $\mathbb{G}_{(L)}$ has diameter $2L \geq (L + 1)/10$ and is thus included in $\mathbb{G}_{(L+1)}$. So, for $L' \geq L \geq L_0$, $\mathbb{G}_{(L)}$ is contained in $\mathbb{G}_{(L')}$. Hence, it is included in an open cluster with infinite diameter which is \mathbb{G}_∞ .

It remains to verify that any two vertices \mathbf{z} and \mathbf{z}' of $\mathbb{G}_\infty \cap Q_t$ are connected by an open path within $[-\lfloor t^\gamma \rfloor, \lfloor t^\gamma \rfloor]^d$. Call \mathcal{B}'_t this event and write $d_{\mathbb{G}_\infty}(\cdot, \cdot)$ for the graph distance in \mathbb{G}_∞ . Assume that \mathcal{B}'_t fails for some large t and fix $\mathbf{z}, \mathbf{z}' \in \mathbb{G}_\infty \cap Q_t$ which are not connected in $[-\lfloor t^\gamma \rfloor, \lfloor t^\gamma \rfloor]^d$. Considering a shortest path from \mathbf{z} to \mathbf{z}' one can find \mathbf{z}'' in $\mathbb{G}_\infty \cap Q_{3t} \setminus Q_{2t}$ such that $d_{\mathbb{G}_\infty}(\mathbf{z}', \mathbf{z}'') \geq \lfloor t^\gamma \rfloor - t(3C \log(t)/2 + 1)$. Hence, for any κ , for t large enough:

$$\mathcal{P} [(\mathcal{B}'_t)^c] \leq \sum_{\mathbf{z}' \in Q_t} \sum_{\mathbf{z}'' \in Q_{3t} \setminus Q_{2t}} \mathcal{P} [\mathbf{z}', \mathbf{z}'' \in \mathbb{G}_\infty, d_{\mathbb{G}_\infty}(\mathbf{z}', \mathbf{z}'') \geq \kappa \|\mathbf{z}'' - \mathbf{z}'\|].$$

From [AP96], we know that $\mathcal{P} [\mathbf{z}', \mathbf{z}'' \in \mathbb{G}_\infty, d_{\mathbb{G}_\infty}(\mathbf{z}', \mathbf{z}'') \geq \kappa \|\mathbf{z}'' - \mathbf{z}'\|]$ decreases exponentially with $\|\mathbf{z}'' - \mathbf{z}'\|$ which implies that $\mathcal{P} [(\mathcal{B}'_t)^c]$ decreases exponentially with t . One finally concludes thanks to the Borel-Cantelli lemma. \square

4.7.4 Proof of Proposition 4.21

For $t \geq 0$, let us denote by $\widehat{N}(t)$ the number of jumps of $(\widehat{Y}_s)_{s \geq 0}$ up to time t and by \mathcal{C}_t the event: ' $\widehat{N}(t) \leq 3t/2$ '. Since $(\widehat{Y}_s)_{s \geq 0}$ has speed at most 1, $\widehat{N}(\cdot)$ is dominated by a Poisson process of intensity 1 on \mathbb{R}^+ and $P_x^\xi[\mathcal{C}_t^c] \leq c_7 \exp(-c_8 t)$ for some $c_7, c_8 > 0$. This implies that, almost surely for t large enough, \mathcal{C}_t is realized and we only need to obtain the heat-kernel bound on this event.

Recall the definitions of \mathcal{A}_t and \mathcal{B}_t from the previous subsection. On $\mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$, starting from a point of $\mathcal{G}_\infty(\widehat{\xi}) \cap [-t, t]^d$, $(\widehat{Y}_s)_{s \geq 0}$ does at most $3/2t$ jumps of length at most

$C \log t$ up to time t . In particular, it has visited only points of $\mathcal{G}_{(\lfloor t^\gamma \rfloor)}(\widehat{\xi}) \cap Q_t$ and does not depend on $\xi \setminus \widehat{\mathcal{G}}_{(\lfloor t^\gamma \rfloor)}(\widehat{\xi})$ up to time t . Hence, we can find a coupling of $(\widehat{Y}_s)_{s \geq 0}$ and $(\widehat{Y}_s^{(\lfloor t^\gamma \rfloor)})_{s \geq 0}$ such that these two coincide up to time t . Thus, for t large enough, we can write:

$$\begin{aligned} P_x^{\widehat{\xi}} [\widehat{Y}_t = y] &= P_x^{\widehat{\xi}} \left[\left\{ \widehat{Y}_t = y \right\} \cap \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t \right] \\ &= P_x^{\widehat{\xi}, (\lfloor t^\gamma \rfloor)} \left[\left\{ \widehat{Y}_t^{(\lfloor t^\gamma \rfloor)} = y \right\} \cap \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t \right] \\ &= P_x^{\widehat{\xi}, (\lfloor t^\gamma \rfloor)} \left[\widehat{Y}_t^{(\lfloor t^\gamma \rfloor)} = y \right]. \end{aligned}$$

It then remains to bound $P_x^{\widehat{\xi}, (\lfloor t^\gamma \rfloor)} \left[\widehat{Y}_t^{(\lfloor t^\gamma \rfloor)} = y \right]$. To this end, we will rely on the isoperimetric inequality stated in Corollary 4.25 and apply the strategy developed by Morris and Peres in [MP05].

Theorem 4.27 (see [MP05, Theorem 13]). *Let $(X_t)_{t \geq 0}$ be an irreducible continuous-time Markov chain on a finite state space \mathcal{X} with reversible probability measure π and isoperimetric profile φ .*

For all $\varepsilon > 0$ and all $x, y \in \mathcal{X}$, if

$$t \geq \int_{4 \min(\pi(x), \pi(y))}^{4/\varepsilon} \frac{8du}{u\varphi(u)^2} \tag{4.7.14}$$

then

$$P_x [X_t = y] \leq \pi(y) (1 + \varepsilon).$$

Here the reversible probability measure is given by:

$$\widehat{\pi}_{\lfloor t^\gamma \rfloor}(\cdot) := \widehat{\text{deg}}_{\lfloor t^\gamma \rfloor, \widehat{\xi}}(\cdot) / \widehat{\text{deg}}_{\lfloor t^\gamma \rfloor, \widehat{\xi}}(\mathcal{G}_{(\lfloor t^\gamma \rfloor)}(\widehat{\xi})).$$

Choosing ε of the form $\varepsilon = c_9 t^{d(\gamma - \frac{1}{2})}$ (c_9 will be chosen large enough), the conclusion of Theorem 4.27 reads:

$$P_x^{\widehat{\xi}, (\lfloor t^\gamma \rfloor)} \left[\widehat{Y}_t^{(\lfloor t^\gamma \rfloor)} = y \right] \leq \frac{\widehat{\text{deg}}_{\lfloor t^\gamma \rfloor, \widehat{\xi}}(y)}{\widehat{\text{deg}}_{\lfloor t^\gamma \rfloor, \widehat{\xi}}(\mathcal{G}_{(\lfloor t^\gamma \rfloor)}(\widehat{\xi}))} \left(1 + c_9 t^{d(\gamma - \frac{1}{2})} \right).$$

Using Lemma 4.4 and (4.7.13), one deduces that $P_x^{\widehat{\xi}, (\lfloor t^\gamma \rfloor)} \left[\widehat{Y}_t^{(\lfloor t^\gamma \rfloor)} = y \right] \leq c_{10} t^{-\frac{d}{2}}$ for t large enough. It remains to check the validity of (4.7.14) with $\varepsilon = c_9 t^{d(\gamma - \frac{1}{2})}$ for t large enough when c_9 is well chosen.

Assuming that t satisfies $4t^{d(\frac{1}{2} - \gamma)} \leq c_9 / (2D)$, one obtains with Corollary 4.25 and

inequality (4.7.11) that:

$$\begin{aligned}
 & \int_{4 \min(\widehat{\pi}_{\lfloor t^\gamma \rfloor}(x), \widehat{\pi}_{\lfloor t^\gamma \rfloor}(y))}^{4c_9^{-1}t^{d(\frac{1}{2}-\gamma)}} \frac{8du}{u\widehat{\varphi}_{(\lfloor t^\gamma \rfloor)}(u)^2} \\
 & \leq 8 \int_{c_{11}/t^{\gamma d}}^{4c_9^{-1}t^{d(\frac{1}{2}-\gamma)}} \frac{du}{u\widehat{\varphi}_{(\lfloor t^\gamma \rfloor)}(u)^2} \\
 & \leq c_{12} \left((\log t)^{\frac{2d}{d-1}} \int_{c_{11}/t^{\gamma d}}^{(\log \lfloor t^\gamma \rfloor)^{\frac{d^2}{d-1}}/\lfloor t^\gamma \rfloor^d} \frac{du}{u} + t^{2\gamma} \int_{(\log \lfloor t^\gamma \rfloor)^{\frac{d^2}{d-1}}/\lfloor t^\gamma \rfloor^d}^{4c_9^{-1}t^{d(\frac{1}{2}-\gamma)}} u^{\frac{2}{d}-1} du \right) \\
 & \leq c_{12} \left(\frac{d}{2} \left(\frac{4}{c_9} \right)^{\frac{2}{d}} t + c_{13} (\log t)^{\frac{2d}{d-1}} (\log \log t + c_{14}) \right).
 \end{aligned}$$

If c_9 has been fixed large enough, the last expression is smaller than t for every t large enough.

To summarize, we have just proved that for a.a. $\widehat{\xi}$, there exist $c_{15} = c_{15}(\widehat{\xi})$ and $T = T(\widehat{\xi})$ such that for any $t \geq T$, for any $x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-t, t]^d$ and any $y \in \mathcal{G}_\infty(\widehat{\xi})$:

$$P_x^{\widehat{\xi}} [\widehat{Y}_t = y] \leq \frac{c_{15}}{t^{\frac{d}{2}}}.$$

This implies the required result. □

4.8 Expected distance bound for $(\widehat{Y}_t^{\widehat{\xi}})_{t>0}$

It is known that bounds on the expected distance between the position of the walk at time t and its starting point can be derived from the heat-kernel estimate (4.7.1) as soon as the volume grows regularly (see for example [Bas02, Bar04, BP07]). In this section, we use this strategy to prove the following proposition.

Proposition 4.28. *For a.a. $\widehat{\xi} \in \widehat{\mathcal{N}}$:*

$$\sup_{n \geq 1} \max_{x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d} \sup_{t \geq n} \frac{E_x^{\widehat{\xi}} [\|\widehat{Y}_t - x\|]}{\sqrt{t}} < \infty. \quad (4.8.1)$$

Proof. Proposition 4.21 shows that the assumption of [BP07, Proposition 6.2] is satisfied in the present setting. Hence, there exist constants c_{16} and c_{17} such that for a.a. $\widehat{\xi}$, for every $x \in \mathcal{G}_\infty(\widehat{\xi})$, for t large enough:

$$\frac{E_x^{\widehat{\xi}} [\widehat{d}(\widehat{Y}_t, x)]}{\sqrt{t}} \leq c_{16} + c_{17} \sup_{0 < s \leq t^{-\frac{1}{2}}} \left\{ s^d \sum_{y \in \mathcal{G}_\infty(\widehat{\xi})} e^{-s\widehat{d}(x, y)} \right\}, \quad (4.8.2)$$

where $\widehat{d}(x, y) = \widehat{d}_{\mathcal{G}_\infty(\widehat{\xi})}(x, y)$ denotes the ‘natural’ distance between x and y for $(\widehat{Y}_t)_{t>0}$ (i.e. the minimal number of jumps that the random walk needs to do in order to go from x to y).

At this point, we need to compare \widehat{d} with the Euclidean distance and to check that the r.h.s. of (4.8.2) is uniformly bounded. Let us denote by $\widehat{d}' = \widehat{d}'_{\mathbb{G}_\infty}$ the chemical distance in \mathbb{G}_∞ in which we add an edge between every two points on the boundary of a (shared) discrete hole. For $x \in \mathcal{G}_\infty(\widehat{\xi})$, we choose $\mathbf{z}(x) \in \mathbb{G}_\infty$ such that $\text{Vor}_\xi(x)$ intersects $B_{\mathbf{z}(x)}$ to be the minimal one in the lexicographic order. It is not difficult to see that the definitions of \mathbb{G}_∞ and $\mathcal{G}_\infty(\widehat{\xi})$ imply that there are constants c_{18} and c_{19} such that:

$$c_{18}\widehat{d}'(\mathbf{z}(x), \mathbf{z}(y)) \leq \widehat{d}(x, y) \leq c_{19}(\widehat{d}'(\mathbf{z}(x), \mathbf{z}(y)) + 1), \quad \forall x, y \in \mathcal{G}_\infty(\widehat{\xi}). \quad (4.8.3)$$

By the same arguments as in the proof of [BP07, Lemma 3.1], one obtains that:

$$\mathcal{P} \left[\mathbf{z}, \mathbf{z}' \in \mathbb{G}_\infty, \widehat{d}'(\mathbf{z}, \mathbf{z}') \leq c_{20}\|\mathbf{z} - \mathbf{z}'\| \right] \leq e^{-c_{21}\|\mathbf{z} - \mathbf{z}'\|},$$

for suitable constants c_{20} and c_{21} .

Using the estimate above and the Borel-Cantelli lemma, one deduces that there exists a constant C such that, for $n \geq N = N(\widehat{\xi})$, for all $\mathbf{z} \in \mathbb{G}_\infty \cap [-n/K, n/K]^d$, for all $\mathbf{z}' \in \mathbb{G}_\infty$ with $\|\mathbf{z}' - \mathbf{z}\| \geq C \log n$:

$$\widehat{d}'(\mathbf{z}, \mathbf{z}') \geq c_{20}\|\mathbf{z}' - \mathbf{z}\|. \quad (4.8.4)$$

Then, (4.8.2), (4.8.3) and (4.8.4) imply that for every $x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d$, for $t \geq n \geq N$:

$$\frac{E_x^{\widehat{\xi}} \left[\|\widehat{Y}_t - x\| \right]}{\sqrt{t}} \leq c_{22} + c_{23} \sup_{0 < s \leq t^{-\frac{1}{2}}} \left\{ s^d \sum_{y \in \mathcal{G}_\infty(\widehat{\xi})} e^{-s\widehat{d}(x, y)} \right\}. \quad (4.8.5)$$

Finally, observe that once $1/s \geq t^{\frac{1}{2}} \geq n^{\frac{1}{2}} \gg C \log n$ for any $x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d$:

$$\begin{aligned} \sum_{y \in \mathcal{G}_\infty(\widehat{\xi})} e^{-s\widehat{d}(x, y)} &\leq \sum_{\mathbf{z}' \in \mathbb{G}_\infty} \sum_{\substack{y \in \xi \\ \text{Vor}_\xi(y) \cap B_{\mathbf{z}'} \neq \emptyset}} e^{-s\widehat{d}(x, y)} \\ &\leq D \sum_{\mathbf{z}' \in \mathbb{G}_\infty} e^{-c_{18}s\widehat{d}'(\mathbf{z}(x), \mathbf{z}')} \\ &\leq c_{24} \left(s^{-d} + \sum_{\substack{\mathbf{z}' \in \mathbb{G}_\infty : \\ \|\mathbf{z}' - \mathbf{z}(x)\| \geq 1/s}} e^{-c_{25}s\|\mathbf{z}' - \mathbf{z}(x)\|} \right) \\ &\leq c_{26}s^{-d}. \end{aligned}$$

The conclusion then follows thanks to (4.8.5). \square

4.9 Almost sure sublinearity in $\mathcal{G}_\infty(\widehat{\xi})$

The aim of this section is to prove the ‘strong’ sublinearity of the corrector in $\mathcal{G}_\infty(\widehat{\xi})$.

Proposition 4.29. *For $\mathcal{P}[0 \in \mathbb{G}_\infty]$ -a.a. $\widehat{\xi}$:*

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{R}_n}{n} = 0, \quad (4.9.1)$$

where

$$\mathfrak{R}_n = \mathfrak{R}_n(\widehat{\xi}) := \max_{\substack{x_0 \in \xi: \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d} \|\chi(\tau_{x_0}\xi, x - x_0)\|.$$

Following an idea attributed to Yuval Peres in [BP07] and [CFP13], it suffices to show the recursive bound:

Lemma 4.30. *For $\mathcal{P}[0 \in \mathbb{G}_\infty]$ -a.e. $\widehat{\xi}$, for each $\varepsilon, \delta > 0$, there exists $n_0 = n_0(\widehat{\xi}, \varepsilon, \delta) < \infty$ such that:*

$$\mathfrak{R}_n \leq \varepsilon n + \delta \mathfrak{R}_{3n}, \quad \forall n \geq n_0. \quad (4.9.2)$$

For the reader’s convenience, we recall how Proposition 4.29 can be deduced from the previous lemma and Proposition 4.13 (see also [BP07, Proof of Theorem 2.4, p. 1337]). Assume that the conclusion of Proposition 4.29 is false and choose $0 < c_{27} < \limsup_{n \rightarrow \infty} \mathfrak{R}_n/n$, $\varepsilon := c_{27}/2$ and $\delta := 1/3^{\beta+1}$ with β such that:

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{R}_n}{n^\beta} = 0. \quad (4.9.3)$$

For infinitely many n ’s, $\mathfrak{R}_n \geq c_{27}n$ which implies by Lemma 4.30 that:

$$\mathfrak{R}_{3n} \geq \frac{\mathfrak{R}_n - \varepsilon n}{\delta} \geq \frac{(c_{27} - \varepsilon)n}{\delta} \geq 3^\beta c_{27}n$$

when $n \geq n_0$. One then obtains by induction that $\mathfrak{R}_{3^k n} \geq c_{27} 3^{\beta k} n$ which contradicts (4.9.3). We now turn our attention to the proof of Lemma 4.30.

Proof of Lemma 4.30. We adapt the arguments given in [BP07, §5]. Let us define:

$$C_1 = C_1(\widehat{\xi}) := \sup_{n \geq 1} \max_{x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d} \sup_{t \geq n} t^{\frac{d}{2}} P_x^{\widehat{\xi}} [\widehat{Y}_t = x]$$

and

$$C_2 = C_2(\widehat{\xi}) := \sup_{n \geq 1} \max_{x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-n, n]^d} \sup_{t \geq n} \frac{E_x^{\widehat{\xi}} [\|\widehat{Y}_t - x\|]}{\sqrt{t}}.$$

Recall that these two quantities are a.s. finite thanks to Propositions 4.21 and 4.28.

For large n , we choose $y_0 = y_0(n)$ with $\text{Vor}_\xi(y_0) \cap B_0 \neq \emptyset$ and $y = y(n) \in [-n, n]^d$ such that $\mathfrak{R}_n = \|\chi(\tau_{y_0}\xi, y - y_0)\|$ and we define the stopping time:

$$S_n := \inf \left\{ t \geq 0 : \|\widehat{Y}_t - y\| \geq 2n \right\}.$$

For n is large enough, holes have sizes of logarithmic order (see Lemma 4.26) and thus $\|\widehat{Y}_{t \wedge S_n} - y\| \leq 3n$ for all t . Due to the harmonicity of φ , the optional stopping theorem gives:

$$E_y^{\widehat{\xi}} \left[\widehat{Y}_{t \wedge S_n} - y - \chi \left(\tau_y \xi, \widehat{Y}_{t \wedge S_n} - y \right) \right] = E_y^{\widehat{\xi}} \left[\varphi \left(\tau_y \xi, \widehat{Y}_{t \wedge S_n} - y \right) \right] = 0.$$

By the shift-covariance of the corrector, one has:

$$\chi \left(\tau_{y_0} \xi, y - y_0 \right) = \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \chi \left(\tau_y \xi, \widehat{Y}_{t \wedge S_n} - y \right),$$

thus

$$\chi \left(\tau_{y_0} \xi, y - y_0 \right) = E_y^{\widehat{\xi}} \left[\chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right].$$

It follows that:

$$\mathfrak{R}_n = \|\chi \left(\tau_{y_0} \xi, y - y_0 \right)\| \leq E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \right]. \quad (4.9.4)$$

Let us fix $\varepsilon > 0$ and define:

$$\mathcal{O}_n := \left\{ x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-3n, 3n]^d : \exists x_0 \text{ s.t. } \text{Vor}_\xi(x_0) \cap B_0 \neq \emptyset \text{ and } \|\chi \left(\tau_{x_0} \xi, x - x_0 \right)\| \geq \frac{\varepsilon}{2} n \right\}.$$

Note that, by Proposition 4.17, $\#\mathcal{O}_n = o(n^d)$. Restricting our attention to $t = t(n) \geq 4n$ (whose value will be specified at the end), we will decompose the expectation above as:

$$E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \right] = E_1 + E_2,$$

with

$$E_1 = E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \mathbf{1}_{S_n < t} \right],$$

and

$$E_2 = E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \mathbf{1}_{S_n \geq t} \right].$$

We first deal with the term E_1 . Since $t \geq 4n$, Markov inequality shows that:

$$P_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_{2t} - y \right\| \geq \frac{3}{2} n \right] \leq \frac{2E_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_{2t} - y \right\| \right]}{3n} \leq \frac{2\sqrt{2t}C_2}{3n}.$$

Observe that $\left\{\left\|\widehat{Y}_{2t} - y\right\| \leq 3n/2, S_n < t\right\} \subset \left\{\left\|\widehat{Y}_{2t} - \widehat{Y}_{S_n}\right\| \geq n/2, S_n < t\right\}$. On $\{S_n < t\}$, since $s := 2t - S_n \in [t, 2t]$, one has:

$$P_z^{\widehat{\xi}} \left[\left\| \widehat{Y}_s - z \right\| \geq \frac{1}{2}n \right] \leq \frac{2\sqrt{2t}C_2}{n},$$

with $z = \widehat{Y}_{S_n}$, this implies by the strong Markov property that:

$$P_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_{2t} - y \right\| \leq \frac{3}{2}n, S_n < t \right] \leq \frac{2\sqrt{2t}C_2}{n}.$$

Recall that $\left\| \widehat{Y}_{t \wedge S_n} - z \right\| \leq 3n$ for n large enough. It follows that:

$$\begin{aligned} E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \mathbf{1}_{S_n < t} \right] \\ \leq (\mathfrak{R}_{3n} + 3n) P_y^{\widehat{\xi}} [S_n < t] \\ \leq (\mathfrak{R}_{3n} + 3n) \left(P_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_{2t} - y \right\| \geq \frac{3}{2}n \right] + P_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_{2t} - y \right\| \leq \frac{3}{2}n, S_n < t \right] \right) \\ \leq \frac{8\sqrt{2t}C_2}{3n} (\mathfrak{R}_{3n} + 3n). \end{aligned} \quad (4.9.5)$$

Thanks to the definitions of C_2 and \mathcal{O}_n and to the fact that $t \geq n$, one has:

$$\begin{aligned} E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_{t \wedge S_n} - y_0 \right) - \widehat{Y}_{t \wedge S_n} + y \right\| \mathbf{1}_{S_n \geq t} \right] \\ \leq E_y^{\widehat{\xi}} \left[\left\| \widehat{Y}_t - y \right\| \mathbf{1}_{S_n \geq t} \right] + E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_t - y_0 \right) \right\| \mathbf{1}_{S_n \geq t} \right] \\ \leq C_2 \sqrt{t} + E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_t - y_0 \right) \right\| \mathbf{1}_{S_n \geq t, \widehat{Y}_t \notin \mathcal{O}_n} \right] \\ \quad + E_y^{\widehat{\xi}} \left[\left\| \chi \left(\tau_{y_0} \xi, \widehat{Y}_t - y_0 \right) \right\| \mathbf{1}_{S_n \geq t, \widehat{Y}_t \in \mathcal{O}_n} \right] \\ \leq C_2 \sqrt{t} + \frac{\varepsilon}{2}n + \mathfrak{R}_{3n} P_y^{\widehat{\xi}} \left[\widehat{Y}_t \in \mathcal{O}_n \right] = C_2 \sqrt{t} + \frac{\varepsilon}{2}n + \mathfrak{R}_{3n} \sum_{z \in \mathcal{O}_n} P_y^{\widehat{\xi}} \left[\widehat{Y}_t = z \right]. \end{aligned} \quad (4.9.6)$$

But, using that $\text{deg}_{\text{DT}(\xi)}$ is reversible w.r.t $(\widehat{Y}_s)_{s \geq 0}$ and bounded by D on $\mathcal{G}_\infty(\widehat{\xi})$, we

obtain by the Markov property and the Cauchy-Schwarz inequality that:

$$\begin{aligned}
 P_y^{\widehat{\xi}} [\widehat{Y}_t = z]^2 &= \left(\sum_{x \in \mathcal{G}_\infty(\widehat{\xi})} P_y^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = x] P_x^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = z] \right)^2 \\
 &\leq \left(\sum_{x \in \mathcal{G}_\infty(\widehat{\xi})} P_y^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = x]^2 \right) \left(\sum_{x \in \mathcal{G}_\infty(\widehat{\xi})} P_x^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = z]^2 \right) \\
 &\leq \left(\sum_{x \in \mathcal{G}_\infty(\widehat{\xi})} P_y^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = x] \frac{\deg_{\text{DT}(\xi)}(x)}{\deg_{\text{DT}(\xi)}(y)} P_x^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = y] \right) \\
 &\quad \times \left(\sum_{x \in \mathcal{G}_\infty(\widehat{\xi})} \frac{\deg_{\text{DT}(\xi)}(z)}{\deg_{\text{DT}(\xi)}(x)} P_z^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = x] P_x^{\widehat{\xi}} [\widehat{Y}_{\frac{t}{2}} = z] \right) \\
 &\leq D^2 P_y^{\widehat{\xi}} [\widehat{Y}_t = y] P_z^{\widehat{\xi}} [\widehat{Y}_t = z] \leq \frac{D^2 C_1^2}{t^d}.
 \end{aligned} \tag{4.9.7}$$

Combining bounds (4.9.4)-(4.9.7), we get:

$$\begin{aligned}
 \mathfrak{R}_n &\leq \frac{8\sqrt{2t}C_2}{3n} (\mathfrak{R}_{3n} + 3n) + C_2\sqrt{t} + \frac{\varepsilon}{2}n + \frac{DC_1\#\mathcal{O}_n}{t^{\frac{d}{2}}}\mathfrak{R}_{3n} \\
 &= \left(\frac{8\sqrt{2t}C_2}{3n} + \frac{DC_1\#\mathcal{O}_n}{t^{\frac{d}{2}}} \right) \mathfrak{R}_{3n} + \left(\frac{8\sqrt{2t}C_2}{n} + \frac{\varepsilon}{2} \right) n + C_2\sqrt{t}.
 \end{aligned}$$

The conclusion then follows by choosing $t = c_{28}n^2$ for some $c_{28} = c_{28}(\widehat{\xi}, \varepsilon, \delta)$ small enough and using that $\#\mathcal{O}_n = o(n^d)$. \square

4.10 Proof of main results

4.10.1 Proof of Theorem 4.3

We first show that Proposition 4.29 and the control of the diameter of the holes imply that for $\mathcal{P}[\cdot | 0 \in \mathbb{G}_\infty]$ -a.a. $\widehat{\xi}$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \xi \cap [-n, n]^d} \|\chi(\tau_{x_0}\xi, x - x_0)\| = 0. \tag{4.10.1}$$

Recall that, almost surely for n large enough, holes intersecting $[-n, n]^d$ have diameters smaller than $C \log n$ (see Lemma 4.26). Let $\mathcal{H} \subset \xi$ be a hole intersecting $[-n, n]^d$ and denote by $\partial_{\text{ext}}\mathcal{H} \subset \mathcal{G}_\infty(\widehat{\xi})$ its external boundary, that is the set of the points of $\xi \setminus \mathcal{H}$ which are neighbors of a point of \mathcal{H} in $\text{DT}(\xi)$. We can assume that $\partial_{\text{ext}}\mathcal{H}$ is contained in

$[-2n, 2n]^d$. Let us define $S := \inf\{k \geq 0 : X_k \notin \mathcal{H}\}$. Thanks to the harmonicity of φ and the optional stopping theorem, for $x \in \mathcal{H}$, one has:

$$E_x^\xi [X_S - x - \chi(\tau_x \xi, X_S - x)] = 0.$$

But the shift-covariance of the corrector implies that for any $x_0 \in \xi$:

$$\chi(\tau_x \xi, X_S - x) = \chi(\tau_{x_0} \xi, X_S - x_0) - \chi(\tau_{x_0} \xi, x - x_0),$$

and thus

$$\chi(\tau_{x_0} \xi, x - x_0) = E_x^\xi [X_S - x - \chi(\tau_{x_0} \xi, X_S - x_0)].$$

It follows that:

$$\begin{aligned} & \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \xi \cap [-n, n]^d} \|\chi(\tau_{x_0} \xi, x - x_0)\| \\ & \leq \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \mathcal{G}_\infty(\widehat{\xi}) \cap [-2n, 2n]^d} \|\chi(\tau_{x_0} \xi, x - x_0)\| + C \log n. \end{aligned}$$

Together with Proposition 4.29, this implies that (4.10.1) holds for $\mathcal{P}[\cdot | 0 \in \mathbb{G}_\infty]$ -a.a. $\widehat{\xi}$.

We now prove that (4.10.1) actually holds for \mathcal{P} -a.a. ξ . Note that this is the step where we eliminate the coupling. Observe that:

$$\begin{aligned} & \max_{\substack{x_0 \in \tau_{Ke_1} \xi : \\ \text{Vor}_{\tau_{Ke_1} \xi}(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \tau_{Ke_1} \xi \cap [-n, n]^d} \|\chi(\tau_{x_0} \tau_{Ke_1} \xi, x - x_0)\| \\ & = \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_{-e_1}^K \neq \emptyset}} \max_{x \in \xi \cap [-n-K, n-K] \times [-n, n]^{d-1}} \|\chi(\tau_{x_0} \xi, x - x_0)\| \\ & \leq \max_{\substack{x_1 \in \xi : \\ \text{Vor}_\xi(x_1) \cap B_0^K \neq \emptyset}} \max_{x \in \xi \cap [-2n, 2n]^d} \|\chi(\tau_{x_1} \xi, x - x_1)\| \\ & \quad + \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_{-e_1}^K \neq \emptyset}} \max_{\substack{x_1 \in \xi : \\ \text{Vor}_\xi(x_1) \cap B_0^K \neq \emptyset}} \|\chi(\tau_{x_1} \xi, x_0 - x_1)\|. \end{aligned}$$

Hence, the event:

$$\mathcal{A} := \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\substack{x_0 \in \xi : \\ \text{Vor}_\xi(x_0) \cap B_0^K \neq \emptyset}} \max_{x \in \xi \cap [-n, n]^d} \|\chi(\tau_{x_0} \xi, x - x_0)\| = 0 \right\}$$

is shift invariant w.r.t. $\tau = \tau^{K, e_1}$. Thanks to the ergodicity assumption **(Er)**, it is a 0-1 event and we already know that $\mathcal{P}[\mathcal{A}] \geq \mathcal{P}[0 \in \mathbb{G}_\infty] > 0$. Thus, (4.10.1) holds \mathcal{P} -a.s..

In particular, (4.10.1) holds $\mathcal{P}[\cdot | \xi \cap B_0 \neq \emptyset]$ -a.s.. The conclusion then follows by using *e.g.* [CFP13, Lemma 7.14] which state that a $\mathcal{P}[\cdot | \xi \cap B_0 \neq \emptyset]$ -almost sure event is also \mathcal{P}_0 -almost sure.

4.10.2 From Theorem 4.3 to Theorems 4.1 and 4.2

As mentioned in the introduction, the arguments to deduce Theorems 4.1 and 4.2 from Theorem 4.3 are now quite standard and we only sketch the main lines of the proof. The reader is referred to [CFP13, §3.3], [BP07, p. 1340-1341] or [BB07, §6.1 and §6.2] for more details.

Recall that, for \mathcal{P}_0 -a.e. ξ^0 , $\varphi(\xi^0, \cdot)$ is harmonic. Hence, $(M_n)_{n \in \mathbb{N}} := (\varphi(\xi^0, X_n))_{n \in \mathbb{N}}$ is a martingale under $P_0^{\xi^0}$. By the same arguments as in [BB07, pp. 108-109], one can show that $(M_n \cdot e_i)_{n \in \mathbb{N}}$ satisfies the assumptions of the Lindeberg-Feller theorem for martingales (see [Dur96, Theorem (7.3), p. 414]). It follows with the Cramér-Wold device (see [Dur96, Theorem (9.5), p. 170]) that $t \rightarrow \varepsilon M_{\lfloor \varepsilon^{-2} t \rfloor}$ converges weakly to a Brownian motion with explicit covariance matrix proportional to the identity due to the isotropy of the point process. The diffusion coefficient does not depend on the particular realization ξ^0 of the point process and is positive. If it were zero, it would hold that $x = \chi(\xi^0, x)$ for \mathcal{P}_0 -a.e. ξ^0 , for all $x \in \xi^0$, which contradicts the sublinearity of the corrector. The sublinearity of the corrector also implies that $\max_{k \leq n} \|X_k - M_k\| = \max_{k \leq n} \|\chi(\xi^0, X_k)\| = o(\sqrt{n})$ in $P_0^{\xi^0}$ -probability. The ‘discrete time version’ of Theorem 4.2 then follows. One concludes in the continuous-time case by arguing as in [CFP13, p. 666] and by showing that:

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \mathcal{E}_0 \left[\text{deg}_{\text{DT}(\xi^0)}(0) \right],$$

where $N(t)$ denotes the number of jumps of $(X_s)_{s \geq 0}$ up to time t . This also prove the relation between σ_{VSRW}^2 and σ_{DTRW}^2 . One finally deduces Theorem 4.1 using Proposition 1.5.

4.11 Bounds for moments of $\text{deg}_{\text{DT}(\xi^0)}(0)$ and $\max_{x \sim 0} \|x\|$

The method developed in [Zuy92, §2] can be used to derive exponential moments for $\text{deg}_{\text{DT}(\xi^0)}(0)$ and $\max_{x \sim 0 \text{ in } \text{DT}(\xi^0)} \|x\|$ when the point process has a finite range of dependence. More precisely, we show the following lemma.

Lemma 4.31. *Assume that \mathcal{P}_0 is isotropic and satisfies **(V’)**, then there exists $\rho_1 > 0$ such that:*

$$\mathcal{E}_0 \left[e^{\rho \max_{x \sim 0 \text{ in } \text{DT}(\xi^0)} \|x\|} \right] < \infty, \quad \forall \rho < \rho_1, \quad (4.11.1)$$

*Assume moreover that \mathcal{P}_0 has a finite range of dependence m and satisfies **(EM)**, then there exists $\rho_2 > 0$ such that:*

$$\mathcal{E}_0 \left[e^{\rho \text{deg}_{\text{DT}(\xi^0)}(0)} \right] < \infty, \quad \forall \rho < \rho_2. \quad (4.11.2)$$

Proof. We use the method of [Zuy92, §2]. Let us recall some definitions, notations, and facts from this article. The fundamental region of a point $x \in \xi^0$ is the union of the balls centered at the vertices of its Voronoi polygon and having the nucleus x on their boundaries. Let Γ^0 be the union of $2d$ open balls of radii 1 centered in points $\pm e_i$. Let $\Phi_1^0, \dots, \Phi_{2d}^0$ be the intersection of exactly d such balls.

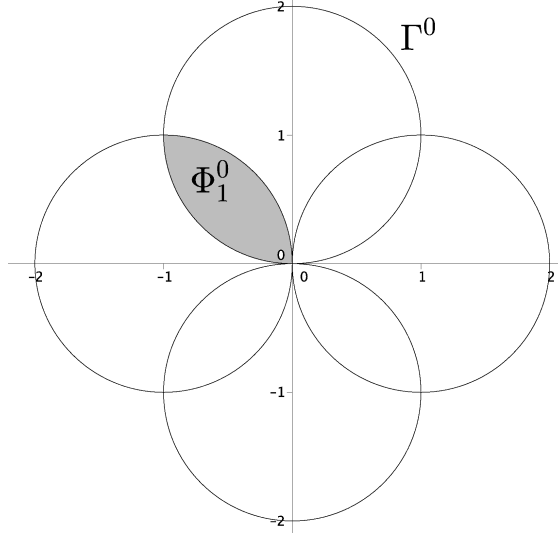


Figure 4.1: Γ^0 and the lens Φ_i^0 .

Let us fix $\beta > 1$ and consider the sequence of sets $\Gamma^n, \Phi_1^n, \dots, \Phi_{2d}^n$ obtained by the homothetic transformation of center 0 and coefficient β^n from $\Gamma^0, \Phi_1^0, \dots, \Phi_{2d}^0$. The important point is that simple geometric arguments show that:

Fact 4.32. *If each d -faced lens $\Phi_i^n, i = 1, \dots, 2^d$, contains a point of ξ^0 , then the fundamental region of the particle at 0 is fully included in Γ^n . In particular, any neighbor of 0 in $\text{DT}(\xi^0)$ is in Γ^n .*

Let \mathcal{A}_0 be the event $\bigcap_{i=1}^{2^d} \{\#(\xi^0 \cap \Phi_i^0) \neq 0\}$ and

$$\mathcal{A}_n := \bigcap_{i=1}^{2^d} \{\#(\xi^0 \cap \Phi_i^n) \neq 0\} \setminus \bigcap_{i=1}^{2^d} \{\#(\xi^0 \cap \Phi_i^{n-1}) \neq 0\}.$$

Note that the \mathcal{A}_n are disjoint and $\mathcal{P}_0\left[\bigcup_{n=0}^{\infty} \mathcal{A}_n\right] = 1$. Thanks to Fact 4.32, one has:

$$\begin{aligned}
 \mathcal{E}_0\left[e^{\rho \max_{x \sim 0} \deg_{\text{DT}} \|x\|}\right] &= \sum_{n=0}^{\infty} \mathcal{E}_0\left[e^{\rho \max_{x \sim 0} \deg_{\text{DT}} \|x\|} \mathbf{1}_{\mathcal{A}_n}\right] \\
 &\leq \sum_{n=0}^{\infty} e^{2\rho\beta^n} \mathcal{P}_0[\mathcal{A}_n] \\
 &\leq e^{2\rho} + \sum_{n=1}^{\infty} e^{2\rho\beta^n} \mathcal{P}_0\left[\bigcup_{i=1}^{2^d} \{\#\left(\xi^0 \cap \Phi_i^{n-1}\right) = 0\}\right] \\
 &\leq e^{2\rho} + 2^d \sum_{n=1}^{\infty} e^{2\rho\beta^n} \mathcal{P}_0\left[\#\left(\xi^0 \cap \Phi_1^{n-1}\right) = 0\right], \tag{4.11.3}
 \end{aligned}$$

where we used the isotropy of the point process in the last inequality.

Now, there exists a constant $c_{29} > 0$ such that Φ_1^{n-1} contains a cube C_{n-1} of side $c_{29}\beta^{n-1}$. Hence, with **(V')** and (4.11.3), we obtain:

$$\begin{aligned}
 \mathcal{E}_0\left[e^{\rho \max_{x \sim 0} \deg_{\text{DT}} \|x\|}\right] &\leq c_{30} + 2^d \sum_{n=n_0}^{\infty} e^{2\rho\beta^n} \mathcal{P}_0\left[\#\left(\xi^0 \cap C_{n-1}\right) = 0\right] \\
 &\leq c_{30} + 2^d \sum_{n=n_0}^{\infty} e^{(2\rho\beta - c_{31})\beta^{(n-1)d}}.
 \end{aligned}$$

The last series converges if ρ is small enough and (4.11.1) is proved.

In the same way, thanks to Fact 4.32, one can write:

$$\begin{aligned}
 \mathcal{E}_0\left[e^{\rho \deg_{\text{DT}}(\xi^0)(0)}\right] &= \sum_{n=0}^{\infty} \mathcal{E}_0\left[e^{\rho \deg_{\text{DT}}(\xi^0)(0)} \mathbf{1}_{\mathcal{A}_n}\right] \\
 &\leq \mathcal{E}_0\left[e^{\rho \#\left(\xi^0 \cap \Gamma^0\right)}\right] + \sum_{n=1}^{\infty} \mathcal{E}_0\left[e^{\rho \#\left(\xi^0 \cap \Gamma^n\right)} \mathbf{1}_{\mathcal{A}_n}\right] \\
 &\leq \mathcal{E}_0\left[e^{\rho \#\left(\xi^0 \cap \Gamma^0\right)}\right] + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} e^{\rho k} \mathcal{P}_0\left[\{\#\left(\xi^0 \cap \Gamma^n\right) = k\} \cap \mathcal{A}_n\right] \\
 &\leq \mathcal{E}_0\left[e^{\rho \#\left(\xi^0 \cap \Gamma^0\right)}\right] \\
 &\quad + 2^d \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} e^{\rho k} \mathcal{P}_0\left[\{\#\left(\xi^0 \cap \Gamma^n\right) = k\} \cap \{\#\left(\xi^0 \cap \Phi_1^{n-1}\right) = 0\}\right] \\
 &\leq \mathcal{E}_0\left[e^{\rho \#\left(\xi^0 \cap \Gamma^0\right)}\right] \\
 &\quad + 2^d \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} e^{\rho k} \mathcal{P}_0\left[\{\#\left(\xi^0 \cap \Gamma^n \setminus \Phi_1^{n-1}\right) = k\} \cap \{\#\left(\xi^0 \cap \Phi_1^{n-1}\right) = 0\}\right].
 \end{aligned}$$

Now, there exists a constant $c_{32} > 0$ such that, if n is large enough Φ_1^{n-1} contains a cube C_{n-1} of side $c_{32}\beta^{n-1}$ satisfying $d(C_{n-1}, \Gamma^n \setminus \Phi_1^{n-1}) > m$. Thanks to the m -dependence

assumption on the point process and to **(EM)**, it follows that:

$$\begin{aligned}
 & \mathcal{E}_0 \left[e^{\rho \deg_{\text{DT}}(\xi^0)(0)} \right] \\
 & \leq c_{33} + 2^d \sum_{n=n_0}^{\infty} \sum_{k=0}^{\infty} e^{\rho k} \mathcal{P}_0 \left[\#(\xi^0 \cap \Gamma^n \setminus \Phi_1^{n-1}) = k \right] \mathcal{P}_0 \left[\#(\xi^0 \cap C_{n-1}) = 0 \right] \\
 & \leq c_{33} + 2^d \sum_{n=n_0}^{\infty} \mathcal{E}_0 \left[e^{\rho \#(\xi^0 \cap [-2\beta^n, 2\beta^n]^d)} \right] \mathcal{P}_0 \left[\#(\xi^0 \cap C_{n-1}) = 0 \right].
 \end{aligned}$$

Finally, with **(V')** and **(EM)**, we obtain:

$$\mathcal{E}_0 \left[e^{\rho \deg_{\text{DT}}(\xi^0)(0)} \right] \leq c_{34} + 2^d c_5 \sum_{n=n_1}^{\infty} e^{(2^d f(\rho) \beta^d - c_{35}) \beta^{(n-1)d}}.$$

This concludes the proof since $f(\rho)$ goes to 0 with ρ by assumption. □

Perspectives

Dans cette dernière partie, nous présentons quelques problèmes ouverts et des pistes de travail dans le cadre des marches aléatoires sur des triangulations de Delaunay engendrées par des processus ponctuels. Bien entendu, il fait sens de considérer les mêmes questions sur d'autres graphes aléatoires.

Bornes sur la résistance efficace en volume fini. Considérons la triangulation de Delaunay $DT(\xi)$ d'une réalisation ξ d'un processus ponctuel dans \mathbb{R}^2 et notons, pour $n \in \mathbb{N}^*$, $R_n(\xi)$ la résistance efficace entre le point $x_0 \in \xi$ le plus proche de l'origine et $\xi \cap (\mathbb{R}^2 \setminus [-n, n]^2)$ dans $DT(\xi)$ vu comme un réseau électrique dans lequel chaque arête a une conductance 1. Le nombre moyen de retours en x_0 de la marche simple sur $DT(\xi)$ avant de quitter pour la première fois la fenêtre $[-n, n]^2$ est alors donné par $\deg_{DT(\xi)}(x_0)R_n(\xi)$. Il serait donc intéressant d'obtenir un équivalent asymptotique de $R_n(\xi)$ pour presque tout ξ . Rappelons que le critère de récurrence du Chapitre 2 a été appliqué, dans le cas de la triangulation de Delaunay, avec $L(i)$ de l'ordre de $\sqrt{\log i}$ et $N(i)$ de l'ordre de $i\sqrt{\log i}$. Ainsi, on obtient, en reprenant la preuve de ce critère, une minoration de $R_n(\xi)$ de l'ordre de $\log \log n$ ce qui ne semble pas être l'ordre correct. Lorsque le processus ponctuel sous-jacent est un processus ponctuel de Poisson, on peut améliorer un peu cette minoration en utilisant le résultat sur le *stabbing number* de la mosaïque de Poisson-Delaunay démontré dans [PR12, §4.3] et en choisissant $N(i)$ de l'ordre de i . Dans [CFG09], Caputo, Faggionato et Gaudillièr ont obtenu des minoration et majoration de la résistance efficace en volume fini coïncidant dans presque tous les cas pour les modèles qu'ils étudiaient. Leurs arguments sont constructifs au sens où ils n'utilisent pas directement de résultats connus sur \mathbb{Z}^d . Cependant, la méthode semble assez spécifique au cas du graphe complet et ne s'adapte pas facilement à notre cadre.

Relâcher l'hypothèse de rang de dépendance fini pour les processus ponctuels. Chaque fois que nous avons eu recours au « processus des bonnes boîtes » et demandé que celui-ci domine stochastiquement un processus de percolation sur-critique, nous n'avons pu fournir que des exemples de processus ponctuels ayant un rang de dépendance fini. Il est pourtant crédible que nos résultats soient encore satisfaits par des processus ponctuels ayant des interactions à longue portée non triviales mais dont les corrélations décroissent suffisamment vite. On voudrait, par exemple, pouvoir appliquer nos résultats à des processus de Cox, de Gibbs ou déterminantaux. L'énoncé du Théorème 0.0 de [LSS97] ne se limite

pas au cas du rang de dépendance fini et une idée serait d'exploiter pleinement celui-ci. Néanmoins, nous n'avons pas pu trouver d'exemple de processus ponctuel satisfaisant les hypothèses de nos résultats mais pas la condition de rang de dépendance fini.

Comme nous venons de le mentionner et même si nous ne savons pas le démontrer pour le moment, on peut conjecturer que nos résultats sont valables pour des processus déterminantaux stationnaires. Intuitivement, la répulsivité de ces processus ponctuels rend la structure des graphes que nous considérons plus régulière. La validité de nos résultats est plus incertaine pour des processus ponctuels attractifs tels que les processus permanantaux. On peut se demander si, pour un ordre convenable sur les processus ponctuels, il existe une transition entre un régime pour lequel les résultats de récurrence/transience et les principes d'invariance sont valables et un régime où ils ne le sont plus. Notons que, dans thèse [Dha10], Dhandapani a défini des ordres sur les mesures aléatoires et les a utilisés pour comparer différents modèles de percolation continue.

Estimées précises du noyau de la chaleur. Comme il a déjà été signalé, Ferrari, Grisi et Groisman [FGG12] ont suggéré d'établir des bornes précises sur le noyau de la chaleur des marches sur les triangulations de Delaunay aléatoires, similaires à celles obtenues par Barlow [Bar04] dans le cadre des marches sur l'amas infini de percolation, pour obtenir un principe d'invariance presque sûr en dimension plus grande que 3. Il s'agirait de montrer que, pour presque tout ξ , il existe des variables aléatoires réelles T_x , $x \in \xi$ et des constantes c_1, \dots, c_4 telles que, pour tous $x, y \in \xi$, si $t \geq T_x$, on a :

$$c_1 \frac{e^{-c_2 \frac{\|x-y\|^2}{t}}}{t^{\frac{d}{2}}} \leq \frac{p_t^\xi(x, y)}{\deg_{DT(\xi)}(y)} \leq c_3 \frac{e^{-c_4 \frac{\|x-y\|^2}{t}}}{t^{\frac{d}{2}}},$$

où $p_t^\xi(x, y)$ désigne la probabilité pour que la marche à temps continu et vitesse constante sur $DT(\xi)$ partant de x se trouve en y au temps t . Bien que nous n'ayons pas eu besoin de démontrer de telles bornes pour établir nos principes d'invariance *quenched*, la question reste intrinsèquement intéressante, ouverte, et pourrait faire l'objet d'un prochain travail. Ceci nécessiterait en particulier une meilleure compréhension des inégalités isopérimétriques.

Sites favoris. Considérons la marche simple à temps discret $(X_n^\xi)_{n \in \mathbb{N}}$ sur $DT(\xi)$ partant du germe de la cellule de Voronoï contenant l'origine et notons $\mathfrak{l}_\xi(x, n)$ le temps local du site $x \in \xi$ pour cette marche jusqu'au temps n , c'est-à-dire la quantité :

$$\mathfrak{l}_\xi(x, n) := \# \{0 \leq i \leq n : X_i^\xi = x\}.$$

On définit l'ensemble des sites préférés de la marche par :

$$\mathfrak{F}_\xi(n) := \# \left\{ x \in \xi : \mathfrak{l}_\xi(x, n) = \max_{y \in \xi} \mathfrak{l}_\xi(y, n) \right\}.$$

De nombreuses questions concernant cet ensemble peuvent être posées : quelle est sa cardinalité ? s'échappe-t-il à l'infini ? si oui, à quelle vitesse ? Les cellules de Voronoï le composant ont-elles des caractéristiques géométriques particulières ? ont-elles un grand nombre de faces ? Bien que ces questions soient faciles à formuler, elles ne sont généralement pas faciles à résoudre même dans le cas de la marche simple symétrique sur \mathbb{Z} (voir l'article de survol de Shi et Tóth [ST00] et les références s'y trouvant).

Piéger les marches. Nous avons démontré que les marches simples sur les triangulations de Delaunay aléatoires admettent des limites d'échelles diffusives. Si l'on souhaite observer des comportements sous-diffusifs, il faut considérer le cas des marches à vitesses variables et ralentir considérablement celles-ci. Deux solutions s'offrent à nous : soit donner à chaque site x un poids p_x fonction à croissance rapide du volume de la cellule de Voronoï dont il est le germe, soit considérer des processus ponctuels marqués, chaque site x ayant un poids encore noté p_x . On pourrait alors considérer l'analogie exact des marches aléatoires en pièges aléatoires dans \mathbb{Z}^d , c'est-à-dire les marches dont les taux de sauts sont donnés par :

$$C(x, y) := \frac{(p_y)^a}{(p_x)^{1-a}} \mathbf{1}_{x \sim y \text{ dans DT}(\xi)}, \quad x, y \in \xi, a \in [0, 1].$$

Rappelons que, pour définir les marches aléatoires en pièges aléatoires dans \mathbb{Z}^d , on se donne une mesure de probabilité \mathbb{P} sur l'ensemble Ω des familles $\omega = \{t_x : x \in \mathbb{Z}^d\}$ de réels strictement positifs indicés par les points de \mathbb{Z}^d , stationnaire et ergodique. Un site de \mathbb{Z}^d est alors appelé *piège* et le nombre t_x est sa *profondeur*. Un nombre $a \in [0, 1]$ étant fixé, les *marches aléatoires en pièges aléatoires* sont les marches associées à la fonction de conductance :

$$C(x, y) := \frac{(t_y)^a}{(t_x)^{1-a}} \mathbf{1}_{x \sim y}.$$

Lorsque les t_x , $x \in \mathbb{Z}^d$ sont i.i.d. et tels que $t_0 \geq 1$ et qu'il existe $\alpha \in]0, 1[$ tel que :

$$\mathbb{P}[t_0 \geq y] \underset{+\infty}{\sim} y^{-\alpha},$$

il est possible de montrer que le processus est sous-diffusif et d'identifier la limite d'échelle dans certains cas ($d = 1$ et $a = 0$ [FIN02], $d = 1$ et a quelconque [BAČ05], $d = 2$ et $a = 0$ [BAČM06, BAČ07], $d \geq 3$ et a quelconque [BČ11a] avec une correction apportée dans [BČ11b], $d \geq 5$ et a quelconque [Mou11]).

Scènes aléatoires. L'étude des *marches aléatoires en scènes aléatoires* dans \mathbb{Z}^2 a débuté à la fin des années 1980. Pour construire ce modèle, on tire une variable aléatoire réelle m_x en chaque site de \mathbb{Z}^2 , puis on considère la somme :

$$S_n := \sum_{i=0}^{n-1} m_{X_i},$$

où $(X_i)_{i \in \mathbb{N}}$ est la marche simple partant de l'origine dans \mathbb{Z}^2 . Dans [Bol89], Bolthausen a démontré que si les marques attachées à chaque site sont i.i.d. de moyenne nulle et de variance positive et finie, $(n \log n)^{-\frac{1}{2}} S_n$ satisfait un TCL fonctionnel. En utilisant les marques p_x du paragraphe précédent, on pourrait étendre la notion de marches aléatoires en scènes aléatoires hors du cadre du réseau carré \mathbb{Z}^2 . On ne verrait plus les marques p_x comme des pièges mais au contraire des récompenses (ou des durées de visite) que la marche collecterait durant son parcours. Le but serait alors d'étudier :

$$S_n(\xi) := \sum_{i=0}^{n-1} p_{X_i^\xi},$$

où $(X_i^\xi)_{i \in \mathbb{N}}$ est la marche simple, à temps discret, sur $DT(\xi)$. Lorsque l'on pense aux p_x , distribués aléatoirement, comme à la durée de visite du site x , le modèle représente un milieu dans lequel plusieurs phases coexistent (un alliage de matériaux par exemple). L'étude des temps de sortie de la marche est relié à la détermination de conductivités équivalentes dans cet alliage et pose par extension la question de l'homogénéisation du milieu et l'étude de trajectoires browniennes de coefficient de diffusion variable.

Variations autour de l'Agrégation Limitée par Diffusion Interne. L'Agrégation Limitée par Diffusion Interne (IDLA) est un modèle de croissance dans lequel des ensembles aléatoires sont construits récursivement à l'aide de marches au hasard. Nous proposons ici une variation autour de ce modèle obtenue en ajoutant au début de sa construction une source d'aléa. Considérons la triangulation de Delaunay engendrée par ξ^0 tiré selon la mesure de Palm d'un processus ponctuel. À l'étape 0, l'agrégat A_0 n'est constitué que de la cellule de Voronoï de l'origine. Puis, à chaque étape n , nous démarrons une nouvelle marche aléatoire simple depuis l'origine dans $DT(\xi^0)$ que nous arrêtons dès que celle-ci est sortie de A_{n-1} . La cellule de Voronoï du dernier site visité par la marche à l'étape n est alors ajoutée à A_{n-1} pour former A_n .

Pour le modèle d'IDLA dans \mathbb{Z}^d , Lawler, Bramson et Griffeath [LBG92] ont montré que la forme est proche d'une boule. Les fluctuations ont ensuite été étudiées par Lawler [Law95], puis Jerison, Levine et Sheffield [JLS12] et par Asselah et Gaudillière [AG13a, AG13b, AG14].

Les premières questions que nous nous poserions pour notre modèle spécifique seraient : quelle est la forme limite obtenue après un changement d'échelle adéquat ? si le processus ponctuel sous-jacent n'est pas isotrope, la forme est-elle différente d'une boule ? peut-on maîtriser les fluctuations du processus renormalisé autour de cette forme limite ?

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