

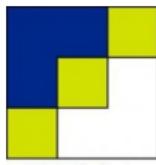
Systèmes paraboliques dégénérés intervenant en mécanique des fluides et en médecine: analyse mathématique et numérique

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Soutenance de thèse en vue de l'obtention du grade de docteur en mathématiques de
l'École Centrale de Nantes sous la direction de Mazen Saad

25 septembre 2014



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Mathématiques
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Leray

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Centrale
Nantes

Outline

1 Introduction

- Pattern Formation
- Chemotaxis Model

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2 Pattern Formation for the Isotropic Chemotaxis Model

- Necessary Conditions
- Bifurcations
- Numerical Simulation

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3 CVFE Scheme for Anisotropic Chemotaxis Model

- Space Discretizations
- CVFE Discretization

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4 Modified CVFE Scheme for the Anisotropic Chemotaxis Model

- Modified CVFE Scheme
- Discrete Properties
- Numerical results

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⑤ Study of a Nonlinear Degenerate Parabolic Equation

- Main results

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Patterns are everywhere



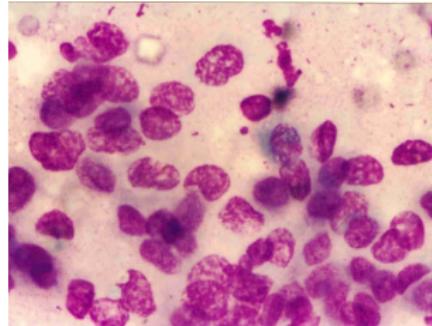
Animal skin: stripes



Sand dunes



Animal skin: spots



Tumor cells

Turing Principle



Alan Turing

- A kinetic system of chemicals, **stable**, in the absence of diffusion (ODE), becomes **unstable** in the presence of diffusion (PDE).

Reaction-Diffusion System

$$\begin{cases} \partial_t u = D_u \nabla^2 u + f(u, v) & \text{in } Q_T, \\ \partial_t v = D_v \nabla^2 v + g(u, v) & \text{in } Q_T. \end{cases}$$

D_u diffusion $f(u, v)$ reaction

u: activator v: inhibitor



A. Turing, The chemical basis of morphogenesis, *Biological Sciences* 237 37-72, 1952.

Turing Principle



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Turing Principle



Alan Turing

- A kinetic system of chemicals, **stable**, in the absence of diffusion (ODE), becomes **unstable** in the presence of diffusion (PDE).
- **Stabilizing** reaction kinetics + diffusion → instability!
- Diffusion driven pattern formation (nowadays: **Turing patterns**).
- **Counter intuitive**: Diffusion was/is thought of having **stabilizing** effect.



A. Turing, The chemical basis of morphogenesis, *Biological Sciences* 237 37-72, 1952.

Full Chemotaxis Model

Full Chemotaxis Model

Consider Ω an open bounded subset of \mathbb{R}^2 ($Q_{t_f} = \Omega \times (0, t_f)$)

Anisotropic Keller-Segel model

$$\begin{cases} \partial_t u - \operatorname{div}(\Lambda(x) a(u) \nabla u - \Lambda(x) \chi(u) \nabla v) = f(u) & \text{in } Q_{t_f} \\ \partial_t v - \operatorname{div}(D(x) \nabla v) = \nu u - \delta v & \text{in } Q_{t_f} \end{cases}$$

- u : cell density $a(u)$: the cell density diffusion
- v : chemoattractant concentration
- $\Lambda(x)$ and $D(x)$: anisotropic diffusion tensors
- $\chi(u)$: cells sensitivity to chemicals
- $f(u)$: cell density proliferation
- ν and δ : chemoattractant production and death rate



E.F Keller and L.A Segel, Model for chemotaxis, *Journal of Theoretical Biology* vol. 30, pp. 225–234, 1971.

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We impose the homogeneous Neumann boundary conditions:

$$(\Lambda(x) a(u) \nabla u - \Lambda(x) \chi(u) \nabla v) \cdot \mathbf{n} = 0 \quad (D(x) \nabla v) \cdot \mathbf{n} = 0$$

Initial conditions:

$$0 \leq u(0, x) = u_0(x) \leq 1, \quad v(0, x) = v_0(x) \geq 0 \quad \forall x \in \Omega$$



E.F Keller and L.A Segel, Model for chemotaxis, *Journal of Theoretical Biology* vol. 30, pp. 225–234, 1971.

Main Assumptions

We consider the main assumptions made about the system

- (A1) a is a nonnegative continuous function on $[0, 1]$ such that $a(0) = a(1) = 0$.
- (A2) χ is a nonnegative continuous function on $[0, 1]$ such that $\chi(0) = \chi(1) = 0$.
- (A3) Λ and D are two bounded, positive symmetric tensors on Ω
 $0 < T_- |\mathbf{w}|^2 \leq \langle T(\mathbf{x})\mathbf{w}, \mathbf{w} \rangle \leq T_+ |\mathbf{w}|^2 < \infty$, $T = \Lambda$ or D .
- (A4) The cell density proliferation f is a continuous function such that $f(0) \geq 0$ and $f(1) \leq 0$.
- (A5) The initial conditions u_0 and v_0 are two measurable functions in $L^2(\Omega)$.

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5 Study of a Nonlinear Degenerate Parabolic Equation

Pattern Formation Analysis

We aim to study the pattern formation for a chemotaxis model

- Determine the necessary conditions (**Turing conditions**)
- Influence of the **bifurcation parameters** on the generation of spatial patterns
- Apply an appropriate **numerical scheme** to capture the patterns

Turing Instability I

In the absence of diffusion:

Kinetic system (ODE)

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \begin{pmatrix} f(u) \\ g(u, v) \end{pmatrix}$$

Requirement

The spatial homogeneous steady state (u_s, v_s) be linearly **stable**.

Linearization [J.D.Murray]:

$$\mathbf{w}_t = A\mathbf{w}, \quad \mathbf{w} = \begin{pmatrix} u - u_s \\ v - v_s \end{pmatrix}, \quad A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(u_s, v_s)}$$

The conditions for which (u_s, v_s) is linearly **stable** are:

$$\text{Tr}(A) = f_u + g_v < 0, \quad \text{Det}(A) = f_u g_v - f_v g_u > 0$$



J.D. Murray, Mathematical biology II: spatial models and biomedical applications, Springer, 2003.

Asymptotic Expansion

Add diffusion (EDP) and examine small perturbations of the form

$$u = u_s + \varepsilon \tilde{u}(x, t), \quad v = v_s + \varepsilon \tilde{v}(x, t), \quad \varepsilon \ll 1.$$

Asymptotic Expansion

Add diffusion (EDP) and examine small perturbations of the form

$$u = u_s + \varepsilon \tilde{u}(x, t), \quad v = v_s + \varepsilon \tilde{v}(x, t), \quad \varepsilon \ll 1.$$

The full chemotaxis model converts into the linearized system

$$\begin{cases} \partial_t \tilde{u} = d_1 \overbrace{a(u_s)}^{\xi_s} \Delta \tilde{u} - \overbrace{\chi(u_s)}^{\psi_s} \Delta \tilde{v} + \tilde{u} f_u + \tilde{v} f_v, \\ \partial_t \tilde{v} = d_2 \Delta \tilde{v} + \tilde{u} g_u + \tilde{v} g_v. \end{cases}$$

- * From the asymptotic expansion, we transform into constants the coefficients of each the diffusion term and the cross-diffusion term.
- * In the sequel, we drop the tildes for convenience.

Turing Instability II

Question

Can (u_s, v_s) transform into an **unstable** steady state?

Linearization about the steady state

$$\partial_t \mathbf{w} = A\mathbf{w} + \mathcal{D}\Delta\mathbf{w}, \quad \mathcal{D} = \begin{pmatrix} d_1\xi_s & \psi_s \\ 0 & d_2 \end{pmatrix}.$$

Set $\mathbf{W}(x)$ as a solution of

$$-\Delta\mathbf{W} = k^2\mathbf{W}, \quad (\mathbf{n} \cdot \nabla) \mathbf{W} = 0 \text{ for } x \text{ in } \partial\Omega,$$

We search for solutions of the form

$$\mathbf{w}(x, t) = \sum_k c_k e^{\lambda t} \mathbf{W}_k(x)$$

Turing Instability II

We obtain

$$\lambda \mathbf{W}_k = A\mathbf{W}_k - \mathcal{D}k^2\mathbf{W}_k = (A - \mathcal{D}k^2)\mathbf{W}_k.$$

Turing Instability II

We obtain

$$\lambda \mathbf{W}_k = A \mathbf{W}_k - \mathcal{D} k^2 \mathbf{W}_k = (A - \mathcal{D} k^2) \mathbf{W}_k.$$

The characteristic polynomial associated with this system is
(k : wave number)

$$|\lambda \mathcal{I} - A + \mathcal{D} k^2| = 0, \quad \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, the eigenvalues satisfy

$$\lambda_1 + \lambda_2 = -(\xi_s d_1 + d_2) k^2 + \text{Tr}(A) < 0,$$

$$\lambda_1 \lambda_2 = \xi_s d_1 d_2 k^4 + (\psi_s g_u - d_2 f_u - \xi d_1 g_v) k^2 + \text{Det}(A).$$

Turing Conditions

The conditions for which we can expect the pattern formation are

$$\text{Tr}(A) = f_u + g_v < 0,$$

$$\text{Det}(A) = f_u g_v - f_v g_u > 0,$$

$$(\psi_s g_u - d_2 f_u - \xi_s d_1 g_v) < 0,$$

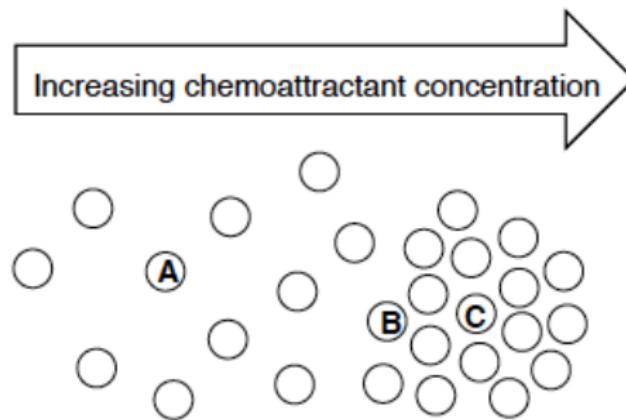
$$(\psi_s g_u - d_2 f_u - \xi_s d_1 g_v)^2 - 4\xi_s d_1 d_2 \text{Det}(A) > 0.$$

Volume Filling Chemotaxis Model

Isotropic Keller-Segel model

$$\begin{cases} \partial_t u - \operatorname{div}(d_1 a(u) \nabla u - \chi(u) \nabla v) = f(u) & \text{in } Q_{t_f} \\ \partial_t v - \operatorname{div}(d_2 \nabla v) = \nu u - \delta v & \text{in } Q_{t_f} \end{cases}$$

Volume-Filling Effect:



Volume Filling Chemotaxis Model

Isotropic Keller-Segel model

$$\begin{cases} \partial_t u - \operatorname{div}(d_1 a(u) \nabla u - \chi(u) \nabla v) = f(u) & \text{in } Q_{t_f} \\ \partial_t v - \operatorname{div}(d_2 \nabla v) = \nu u - \delta v & \text{in } Q_{t_f} \end{cases}$$

For the given functions:

$$q(u) = 1 - \left(\frac{u}{\bar{u}}\right)^\gamma, \quad a(u) = (q(u) - q'(u)u), \quad \chi(u) = \chi u q(u)$$

$$f(u) = \mu u \left(1 - \frac{u}{u_c}\right)$$

Volume Filling Chemotaxis Model

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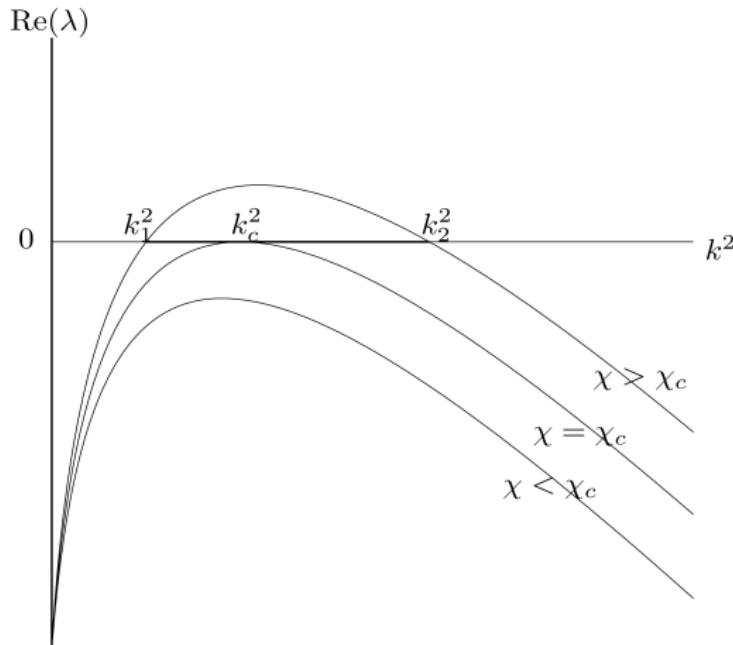
The necessary condition for the pattern formation

$$\mu d_2 + \delta a(u_s) - \chi \nu u_s q(u_s) < -2 \sqrt{d_2 \mu \delta a(u_s)}$$

Bifurcation with Chemotaxis Sensitivity χ

The bifurcation value is

$$\chi_c = \frac{2\sqrt{d_2\mu\delta D(u_s)} + \mu d_2 + \delta D(u_s)}{\nu u_s q(u_s)}$$



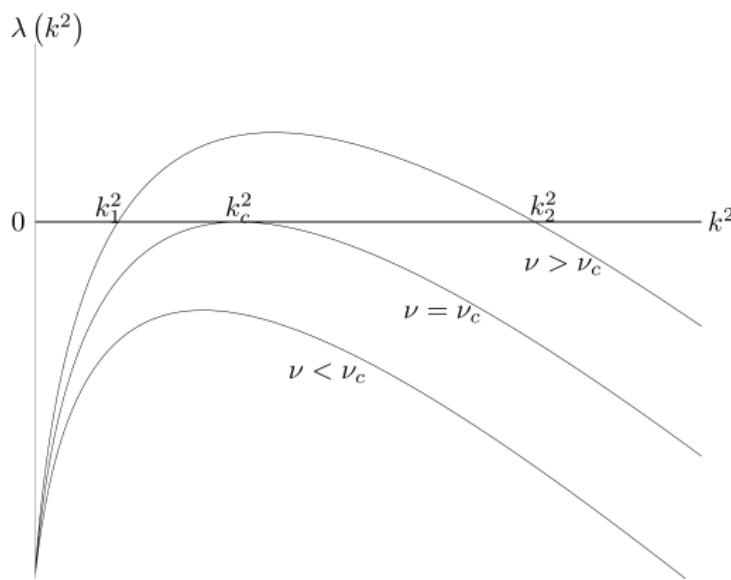
Dispersion relation
as the parameter χ
passes through the
bifurcation value.

$$\begin{aligned}\gamma &= 1 \\ d_1 &= 0.1 \\ d_2 &= 1 \\ u_c &= 2 \\ \bar{u} &= 4 \\ \mu &= 4 \\ \nu &= 5 \\ \delta &= 10\end{aligned}$$

Bifurcation with Growth Rate ν

The bifurcation value is

$$\nu_c = \frac{2\sqrt{d_2\mu\delta D(u_s)} + \mu d_2 + \delta D(u_s)}{\chi u_s q(u_s)}$$



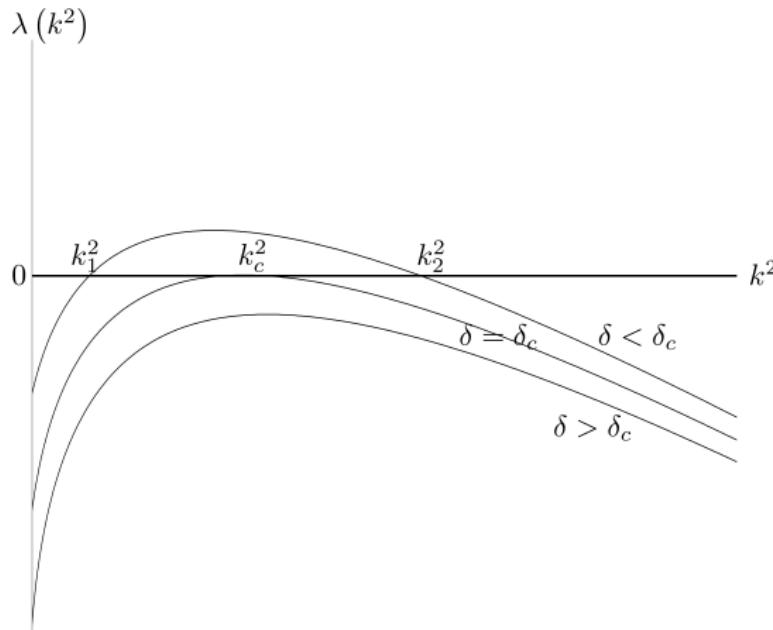
Dispersion relation
as the parameter ν
passes through the
bifurcation value.

$$\begin{aligned}\gamma &= 4 \\ d_1 &= 0.1 \\ d_2 &= 1 \\ u_c &= 2 \\ \bar{u} &= 4 \\ \mu &= 4 \\ \chi &= 0.5 \\ \delta &= 10\end{aligned}$$

Bifurcation with Decay Rate δ

The critical value for δ is

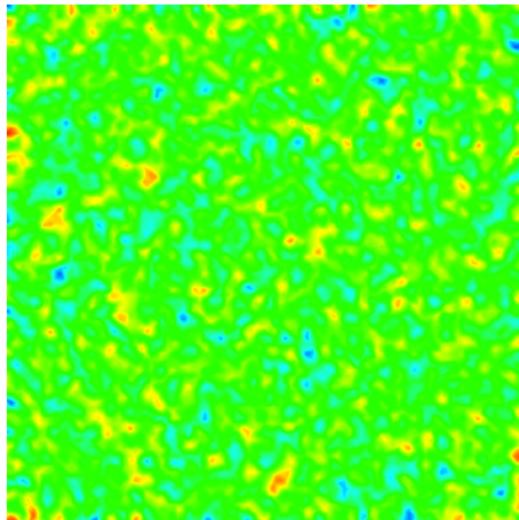
$$\delta_c = \frac{(\sqrt{\nu\chi u_s q(u_s)} - \sqrt{d_2 \mu})^2}{d_1 D(u_s)}$$



Dispersion relation
as the parameter δ
passes through the
bifurcation value.

$$\begin{aligned}\gamma &= 2 \\ d_1 &= 0.1 \\ d_2 &= 1 \\ u_c &= 2 \\ \bar{u} &= 4 \\ \mu &= 10 \\ \nu &= 20 \\ \chi &= 0.5\end{aligned}$$

Initial Conditions



Random perturbation around u_s .

$$u(0, x, y) = u_s + \varepsilon \text{rand}(0, 1)$$

$$u_s = 0.25 \quad \varepsilon = 0.01$$

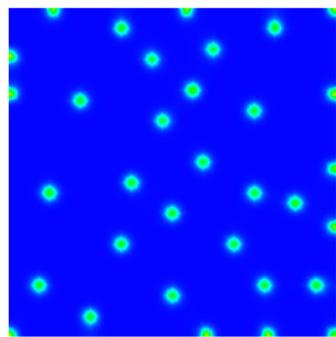
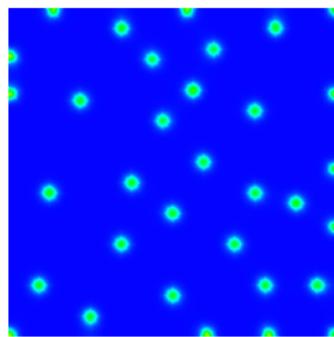
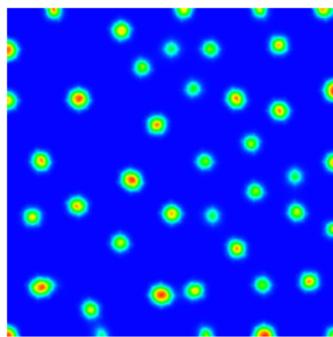
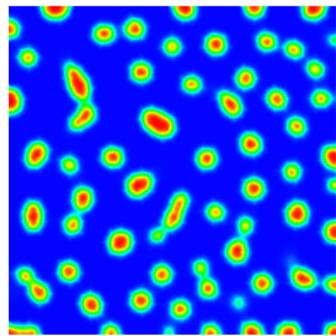
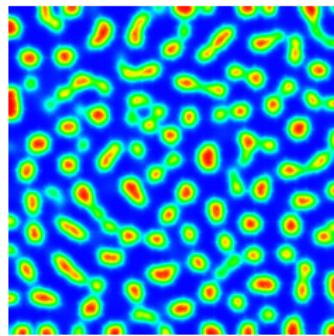
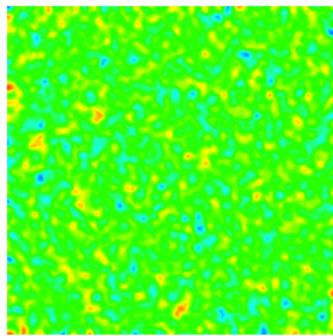


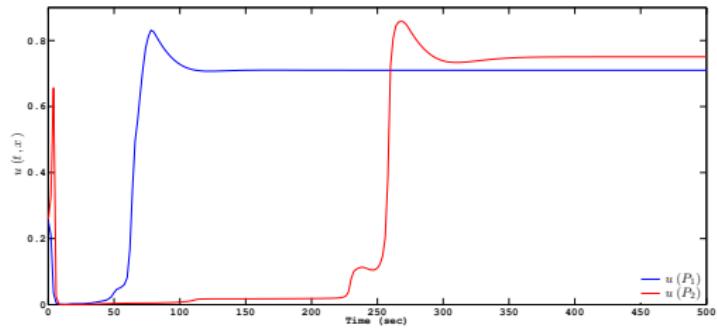
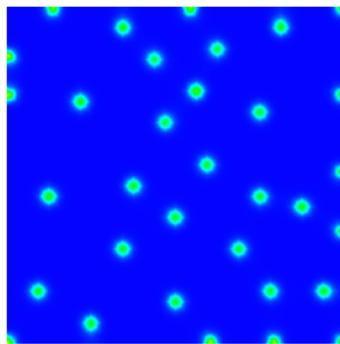
Stationary condition for v .

$$v = v_s = 0.25, \Delta t = 5 \times 10^{-3}$$

$$\Omega = (0, 10) \times (0, 10)$$

Evolution of Cell Density



Evolution of Cell Density at P_1 and P_2 

$$P_1 : (1.25, 4.35)$$

$$u(P_1) = 0.7641, \quad t \geq 150s$$

$$P_2 : (7.5, 6.9)$$

$$u(P_2) = 0.7292, \quad t \geq 400s$$

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Nonlinear CVFE Scheme

We aim to construct a **robust and consistent** scheme for the **anisotropic** chemotaxis model

- Preserves the physical properties of the unknowns
- Converges towards a weak solution of the model

Nonlinear CVFE Scheme

Anisotropic Keller-Segel model

$$\begin{cases} \partial_t u - \operatorname{div}(\Lambda(x) \mathbf{a}(u) \nabla u - \Lambda(x) \chi(u) \nabla v) = f(u) & \text{in } Q_{t_f} \\ \partial_t v - \operatorname{div}(D(x) \nabla v) = \nu u - \delta v & \text{in } Q_{t_f} \end{cases}$$

Problem!

- Degeneracy of the diffusion term

Nonlinear CVFE Scheme

Anisotropic Keller-Segel model

$$\begin{cases} \partial_t u - \operatorname{div}(\Lambda(\mathbf{x}) a(u) \nabla u - \Lambda(\mathbf{x}) \chi(u) \nabla v) = f(u) & \text{in } Q_{t_f} \\ \partial_t v - \operatorname{div}(D(\mathbf{x}) \nabla v) = \nu u - \delta v & \text{in } Q_{t_f} \end{cases}$$

Problem!

- Degeneracy of the diffusion term
- Presence of anisotropic diffusion tensors

Nonlinear CVFE Scheme

Anisotropic Keller-Segel model

$$\begin{aligned} \partial_t u - \underbrace{\operatorname{div}(\Lambda(x) a(u) \nabla u)}_{\text{FE Method}} + \underbrace{\operatorname{div}(\Lambda(x) \chi(u) \nabla v)}_{\text{FV Method}} &= f(u) && \text{in } Q_{t_f} \\ \partial_t v - \underbrace{\operatorname{div}(D(x) \nabla v)}_{\text{FE Method}} &= \nu u - \delta v && \text{in } Q_{t_f} \end{aligned}$$

Problem!

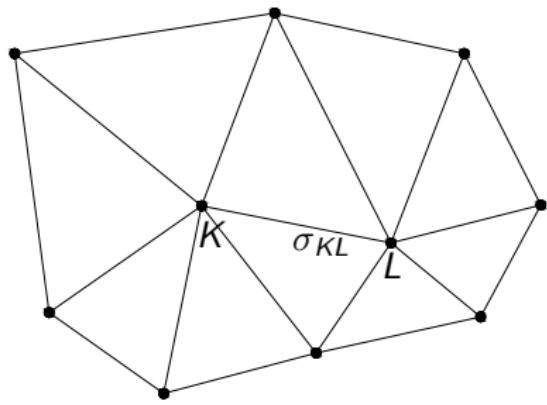
- Degeneracy of the diffusion term
- Presence of anisotropic diffusion tensors

Solution:

- Apply a nonlinear CVFE scheme

Primal Mesh

We perform a uniform **conform triangulation** \mathcal{T}



Primal triangular mesh \mathcal{T}

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}$$

- x_T the barycenter of T
- \mathcal{V} the set of the vertices of \mathcal{T}
- σ_{KL} the line segment joining two vertices K and L
- $h := \text{size}(\mathcal{T}) = \max_{T \in \mathcal{T}} \text{diam}(T)$

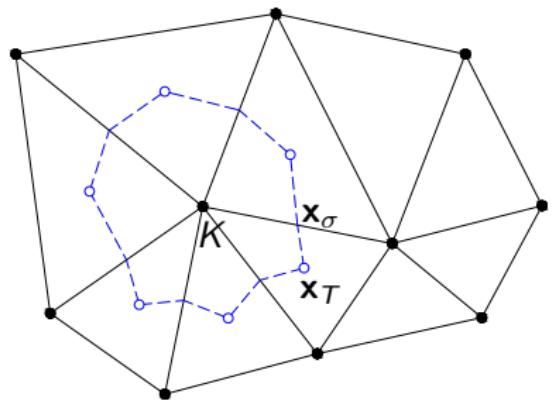
Regularity assumption

$$\max_{T \in \mathcal{T}} \frac{\text{diam}(T)}{\rho_T} \leq \theta_T$$

where ρ_T is the diameter of the largest ball inscribed in the triangle T

Barycentric Dual Mesh

Construction of the **barycentric dual mesh** \mathcal{M} :

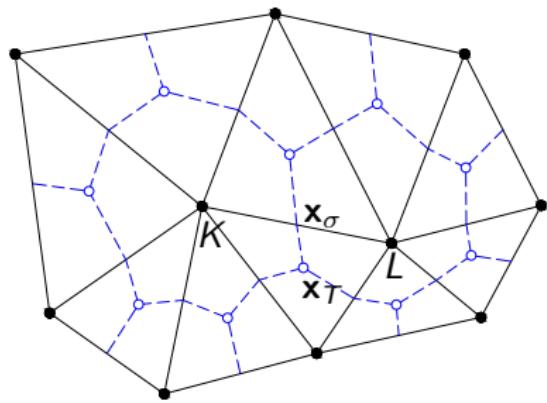


- \mathcal{T}_K the set of triangles having K as a vertex
- \mathcal{E}_K the set of edges of \mathcal{T} having K as an extremity

We construct the dual control volume ω_K around the vertex K by connecting the barycenters of each $T \in \mathcal{T}_K$ with the barycenters x_σ of each $\sigma \in \mathcal{E}_K$.
We denote by m_K the measure of ω_K .

Barycentric Dual Mesh

Construction of the **the barycentric dual mesh \mathcal{M}** :



barycentric dual mesh \mathcal{M}

- \mathcal{T}_K the set of triangles having K as a vertex
- \mathcal{E}_K the set of edges of \mathcal{T} having K as an extremity

We construct the dual control volume ω_K around the vertex K by connecting the barycenters of each $T \in \mathcal{T}_K$ with the barycenters x_σ of each $\sigma \in \mathcal{E}_K$.

We note that

$$\overline{\Omega} = \bigcup_{K \in \mathcal{V}} \overline{\omega_K}$$

Discrete FE-FV Spaces

- We denote by $\mathcal{H}_{\mathcal{T}}$ the finite element space

$$\mathcal{H}_{\mathcal{T}} = \{\phi \in \mathcal{C}^0(\Omega); \phi|_T \in \mathbb{P}_1(\mathbb{R}), \forall T \in \mathcal{T}\} \subset H^1(\Omega)$$

with $(\varphi_K)_{K \in \mathcal{V}}$ the canonical basis ($\varphi_K(x_L) = \delta_{KL}$).

- We also denote by $\mathcal{X}_{\mathcal{M}}$ the finite volume space

$$\mathcal{X}_{\mathcal{M}} = \{\phi \in L^\infty(\Omega); \phi|_{\omega_K} \in \overline{\mathbb{R}} \text{ is constant}, \forall K \in \mathcal{V}\}$$

- We take a uniform time discretization and define the discrete sets

$$\mathcal{H}_{\mathcal{T}, \Delta t} = \{\phi \in L^\infty(Q_{t_f}); \phi(\cdot, t) = \phi(\cdot, t_{n+1}) \in \mathcal{H}_{\mathcal{T}}, \forall t \in (t_n, t_{n+1}]\},$$

$$\mathcal{X}_{\mathcal{M}, \Delta t} = \{\phi \in L^\infty(Q_{t_f}); \phi(\cdot, t) = \phi(\cdot, t_{n+1}) \in \mathcal{X}_{\mathcal{M}}, \forall t \in (t_n, t_{n+1}]\}.$$

CVFE Discretization: Diffusive Term (Isotropic Case)

Integrating the diffusive term over $\omega_K \times (t_n, t_{n+1})$

$$-\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) a(u) \nabla u) d\mathbf{x} dt = - \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} a(u) \Lambda \nabla u \cdot \eta_{\sigma} d\sigma dt$$

In the case of an **isotropic** diffusion tensor ($\Lambda(\mathbf{x}) = d_1 I_2(\mathbf{x})$)

Finite Volume Approximation

$$-\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K d_1 \operatorname{div}(\nabla A(u)) d\mathbf{x} dt \approx d_1 \sum_{L \in N(K)} \frac{|\sigma_{KL}|}{d_{KL}} (A(u_L^{n+1}) - A(u_K^{n+1}))$$

$$A(u) = \int_0^u a(s) ds \Rightarrow \operatorname{div}(\nabla A(u)) = \operatorname{div}(a(u) \nabla u)$$

 B. Andreianov, M. Bendahmane, and M. Saad, Finite volume methods for degenerate chemotaxis model, 2011.

CVFE Discretization: Diffusive Term (Isotropic Case)

Integrating the diffusive term over $\omega_K \times (t_n, t_{n+1})$

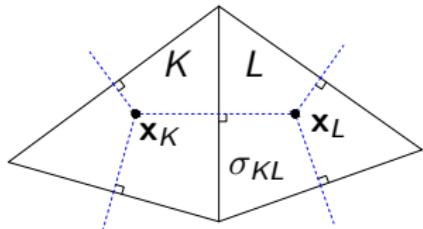
$$-\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) a(u) \nabla u) d\mathbf{x} dt = - \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} a(u) \Lambda \nabla u \cdot \eta_{\sigma} d\sigma dt$$

In the case of an **isotropic** diffusion tensor ($\Lambda(\mathbf{x}) = d_1 I_2(\mathbf{x})$)

Finite Volume Approximation

$$-\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K d_1 \operatorname{div}(\nabla A(u)) d\mathbf{x} dt \approx d_1 \sum_{L \in N(K)} \frac{|\sigma_{KL}|}{d_{KL}} (A(u_L^{n+1}) - A(u_K^{n+1}))$$

The mesh is required to satisfy the **orthogonality condition**



Taylor's formula gives:

$$\nabla A(u) \cdot \eta_{KL} = \frac{A(u_L) - A(u_K)}{d_{KL}} + \mathcal{O}(h)$$

CVFE Discretization: Diffusive Term (Anisotropic Case)

In the case of an **anisotropic** diffusion tensor

Finite Volume Scheme:

- Loss of consistency of the diffusive flux \Rightarrow Finite Volume scheme does not permit to handle anisotropic diffusion problems.

A **possible solution** is to approximate the diffusive flux using the **finite element scheme**

- Simple discretization of a diffusion term with full tensor
- No restrictions on the mesh
- Well known method and widely used in numerical simulations

CVFE Discretization: Diffusive Term (Anisotropic Case)

Using the [finite element approximation](#), then

$$-\Lambda \nabla A(u) \cdot \eta_\sigma \approx \Lambda_{KL}(A(u_K) - A(u_L)), \quad \Lambda_{KL} = \int_{\Omega} \Lambda(\mathbf{x}) \nabla \varphi_K(\mathbf{x}) \cdot \nabla \varphi_L(\mathbf{x}) d\mathbf{x}$$

Finite Volume finite element approximation

$$-\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) a(u) \nabla u) d\mathbf{x} dt \approx \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} (A(u_K^{n+1}) - A(u_L^{n+1}))$$

CVFE Discretization: Convective Term (Anisotropic case)

Integrating the convective term over $\omega_K \times (t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt = \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} \chi(u) \Lambda(\mathbf{x}) \nabla v \cdot \eta_{\sigma} d\sigma dt$$

We consider a **numerical flux function** $G(a, b, c)$ satisfying

- **Monotony:** $G(\cdot, b, c)$ is nondecreasing and $G(a, \cdot, c)$ is nonincreasing
- **Consistency:** $G(a, a, c) = \chi(a)c$
- **Conservation:** $G(a, b, c) = -G(b, a, c)$
- **Boundedness:** $|G(a, b, c)| \leq C(|a| + |b|) |c|$

The approximation of the convective term is

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt \approx - \sum_{\sigma_{KL} \in \mathcal{E}_K} G(u_K^{n+1}, u_L^{n+1}; dV_{KL}^{n+1})$$

Nonlinear CVFE Scheme

In summary, we propose the following **nonlinear CVFE scheme**:

$$u_{\mathcal{M}}^0(\mathbf{x}) = u_K^0 = \frac{1}{m_K} \int_{\omega_K} u_0(\mathbf{y}) d\mathbf{y}, \quad v_{\mathcal{M}}^0(\mathbf{x}) = v_K^0 = \frac{1}{m_K} \int_{\omega_K} v_0(\mathbf{y}) d\mathbf{y}$$

and for all $n \in \{0, \dots, N\}$

$$\begin{aligned} m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \left(A(u_K^{n+1}) - A(u_L^{n+1}) \right) \\ - \sum_{\sigma_{KL} \in \mathcal{E}_K} G(u_K^{n+1}, u_L^{n+1}; dV_{KL}^{n+1}) = m_K f(u_K^{n+1}) \\ m_K \frac{v_K^{n+1} - v_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} D_{KL} (v_K^{n+1} - v_L^{n+1}) = m_K (\alpha u_K^n - \beta v_K^{n+1}) \end{aligned}$$

Nonlinear CVFE Scheme

Theorem

Assume that $\Lambda_{KL} \geq 0$, $\forall K, L \in \mathcal{V}$ then

- The discrete maximum principle is guaranteed
- The schemes converges towards a weak solution to the continuous problem

A **sufficient** condition for the positivity of the transmissibility coefficients

- Λ is isotropic
- All the triangles of the primal mesh have only acute angles



M. Ibrahim and M. Saad, On the efficacy of a control volume finite element method for the capture of patterns for a volume-filling chemotaxis model, 2014.

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Anisotropic Diffusive Term

To approximate the flux $\mathcal{F}(u) = -a(u) \Lambda \nabla u \cdot \eta_\sigma$, we use the **diffusion fluxes** provided by the **finite element method** as well as the **Godunov scheme**.

At an interface σ_{KL} , we have

- $-\Lambda \nabla u \cdot \eta_\sigma \approx \Lambda_{KL} (u_K - u_L)$
- $\mathcal{F}_{KL}(u_K, u_L) = \begin{cases} \min_{u \in [u_K, u_L]} \Lambda_{KL} (u_K - u_L) a(u), & \text{if } u_K \leq u_L, \\ \max_{u \in [u_L, u_K]} \Lambda_{KL} (u_K - u_L) a(u), & \text{if } u_L < u_K, \end{cases}$

where $\Lambda_{KL} = \int_{\Omega} \Lambda(x) \nabla \varphi_K(x) \cdot \nabla \varphi_L(x) dx = \Lambda_{LK}$, $\forall K, L \in \mathcal{V}$

CVFE Discretization: Diffusive Term

In a simplified manner, we get the discretization of the diffusive term

$$-\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) a(u) \nabla u) d\mathbf{x} dt \approx \Delta t \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} a_{KL}^{n+1} (u_K^{n+1} - u_L^{n+1})$$

where

$$a_{KL}^{n+1} = \begin{cases} \max_{s \in I_{KL}^{n+1}} a(s) & \text{if } \Lambda_{KL} \geq 0 \\ \min_{s \in I_{KL}^{n+1}} a(s) & \text{if } \Lambda_{KL} < 0 \end{cases}$$

and

$$I_{KL}^{n+1} = [\min(u_K^{n+1}, u_L^{n+1}), \max(u_K^{n+1}, u_L^{n+1})]$$

CVFE Discretization: Convective Term

Integrating the convective term over $\omega_K \times (t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt = \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} \chi(u) \Lambda(\mathbf{x}) \nabla v \cdot \eta_{\sigma} d\sigma dt$$

We consider a **numerical flux function** $G(a, b, c)$ satisfying

- **Monotony:** $G(\cdot, b, c)$ is nondecreasing and $G(a, \cdot, c)$ is nonincreasing
- **Consistency:** $G(a, a, c) = \chi(a)c$
- **Conservation:** $G(a, b, c) = -G(b, a, c)$
- **Boundedness:** $|G(a, b, c)| \leq C(|a| + |b|)|c|$

Under these properties, we obtain the **maximum principle**. However, we cannot derive **energy estimates** necessary to prove the convergence of the scheme.

CVFE Discretization: Convective Term

Integrating the convective term over $\omega_K \times (t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt = \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} \chi(u) \Lambda(\mathbf{x}) \nabla v \cdot \eta_{\sigma} d\sigma dt$$

To obtain the **energy estimates**, we add an additional assumption on χ

- There exists a nonnegative continuous function μ on $[0, 1]$ such that

$$\chi(u) = \mu(u) \times a(u) \text{ and } \mu(0) = \mu(1) = 0$$

Under this assumption, we choose a **numerical flux function** to approximate the function μ and a **Godunov scheme** to approximate the resulting flux.

CVFE Discretization: Convective Term

Integrating the convective term over $\omega_K \times (t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt = \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} \chi(u) \Lambda(\mathbf{x}) \nabla v \cdot \eta_{\sigma} d\sigma dt$$

We get the discretization of the **convective term**

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt \approx$$

Godunov flux

$$- \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \underbrace{\mu_{KL}^{n+1}}_{\text{upwind scheme}} \overbrace{a_{KL}^{n+1}}^{} (v_K^{n+1} - v_L^{n+1})$$

CVFE Discretization: Convective Term

Integrating the convective term over $\omega_K \times (t_n, t_{n+1})$

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt = \sum_{\sigma \subset \partial \omega_K} \int_{t_n}^{t_{n+1}} \int_{\sigma} \chi(u) \Lambda(\mathbf{x}) \nabla v \cdot \eta_{\sigma} d\sigma dt$$

We get the discretization of the **convective term**

$$\int_{t_n}^{t_{n+1}} \int_{\omega_K} \operatorname{div}(\Lambda(\mathbf{x}) \chi(u) \nabla v) d\mathbf{x} dt \approx - \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1} (v_K^{n+1} - v_L^{n+1})$$

An example on the **numerical flux function** for the approximation of μ :

$$\mu_{KL}^{n+1} = \begin{cases} \mu_{\downarrow}(u_K^{n+1}) + \mu_{\uparrow}(u_L^{n+1}), & \text{if } \Lambda_{KL}(v_K^{n+1} - v_L^{n+1}) \geq 0 \\ \mu_{\uparrow}(u_K^{n+1}) + \mu_{\downarrow}(u_L^{n+1}), & \text{if } \Lambda_{KL}(v_K^{n+1} - v_L^{n+1}) < 0 \end{cases}$$

Modified CVFE Discretization: First Equation

In summary, we get the discretization of the first equation:

For all $n \in \{0, \dots, N\}$

$$\begin{aligned} m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} a_{KL}^{n+1} (u_K^{n+1} - u_L^{n+1}) \\ - \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1} (v_K^{n+1} - v_L^{n+1}) = m_K f(u_K^{n+1}) \end{aligned}$$

Modified CVFE Discretization: Second Equation

The second equation of the Keller-Segel model

$$\partial_t v - \operatorname{div} (D(x) \nabla v) = \nu u - \delta v \quad \text{in } Q_{t_f}$$

We define the functions

$$\eta(v) = \max(0, \min(v, 1)), \quad p(v) = \int_1^v \frac{1}{\eta(s)} ds$$

$$m_K \frac{v_K^{n+1} - v_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} D_{KL} \eta_{KL}^{n+1} (p(v_K^{n+1}) - p(v_L^{n+1})) = m_K (\nu u_K^n - \delta v_K^{n+1})$$

where

$$\eta_{KL}^{n+1} = \begin{cases} \max_{s \in I_{KL}^{n+1}} \eta(s) & \text{if } D_{KL} \geq 0 \\ \min_{s \in I_{KL}^{n+1}} \eta(s) & \text{if } D_{KL} < 0 \end{cases}$$

Modified CVFE Scheme

In summary, we propose the following **nonlinear modified CVFE scheme**:

$$u_{\mathcal{M}}^0(\mathbf{x}) = u_K^0 = \frac{1}{m_K} \int_{\omega_K} u_0(\mathbf{y}) d\mathbf{y}, \quad v_{\mathcal{M}}^0(\mathbf{x}) = v_K^0 = \frac{1}{m_K} \int_{\omega_K} v_0(\mathbf{y}) d\mathbf{y}$$

and for all $n \in \{0, \dots, N\}$

$$\begin{aligned} m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} a_{KL}^{n+1} (u_K^{n+1} - u_L^{n+1}) \\ - \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1} (v_K^{n+1} - v_L^{n+1}) = m_K f(u_K^{n+1}) \\ m_K \frac{v_K^{n+1} - v_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} D_{KL} \eta_{KL}^{n+1} (p(v_K^{n+1}) - p(v_L^{n+1})) = m_K (\nu u_K^n - \delta v_K^{n+1}) \end{aligned}$$

Discrete Maximum Principle

Proposition (Maximum principle)

Let $u_M^0 \in [0, 1]$ and $v_M^0 \geq 0$. Then

$$0 \leq u_K^n \leq 1 \text{ and } v_K^n \geq 0 \quad \forall K \in \mathcal{V}, \forall n \in \{0, \dots, N+1\}$$

A Priori Estimates

We define the functions

$$\Gamma(v) = \int_1^v p(s) ds, \quad \phi(v) = \int_0^v \frac{1}{\sqrt{\eta(s)}} ds$$

Proposition (Estimate on $v_{M,\Delta t}$)

For all $n \geq 0$, there exists a constant $C > 0$ independent of h

$$\sum_{K \in \mathcal{V}} m_K \Gamma(v_K^{n+1}) + \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} D_{KL} (\phi(v_K^{n+1}) - \phi(v_L^{n+1}))^2 \leq C$$

From the definition of ϕ , we get

$$\iint_{Q_{tf}} \Lambda \nabla v_{\mathcal{T}, \Delta t} \cdot \nabla v_{\mathcal{T}, \Delta t} d\mathbf{x} dt = \sum_{n=0}^N \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} \Lambda_{KL} (v_K^{n+1} - v_L^{n+1})^2 \leq C.$$

A Priori Estimates

We define the Lipschitz nondecreasing function

$$\xi(u) := \int_0^u \sqrt{a(s)} \, ds, \quad \forall u \in \mathbb{R}.$$

Proposition (Estimate on $u_{\mathcal{M}, \Delta t}$)

For all $n \geq 0$, there exists a constant $C > 0$ independent of h

$$\begin{aligned} & \sum_{K \in \mathcal{V}} m_K (u_K^{n+1})^2 + \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} \Lambda_{KL} (\xi(u_K^{n+1}) - \xi(u_L^{n+1}))^2 \\ & \leq \sum_{K \in \mathcal{V}} m_K (u_K^{n+1})^2 + \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} (u_K^{n+1} - u_L^{n+1})^2 \leq C. \end{aligned}$$

Convergence Towards a Weak Solution

Theorem (Convergence)

The sequence $(u_{\mathcal{M}_m, \Delta t_m}, v_{\mathcal{M}_m, \Delta t_m})_m$ converges to (u, v) as $m \rightarrow \infty$:

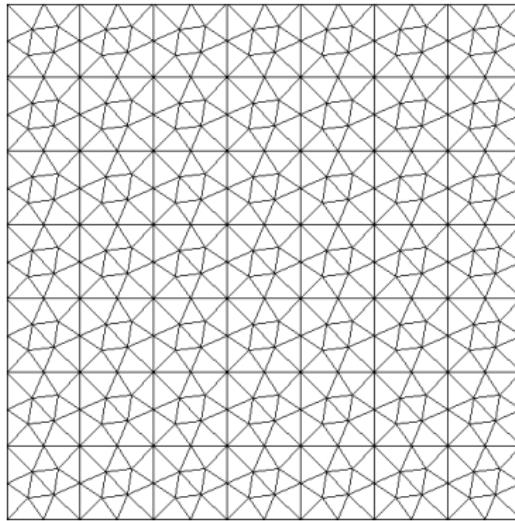
$$0 \leq u \leq 1, \quad v \geq 0 \quad \text{a.e. in } Q_{t_f}$$

$$\xi(u), v \in L^2(0, t_f; H^1(\Omega))$$

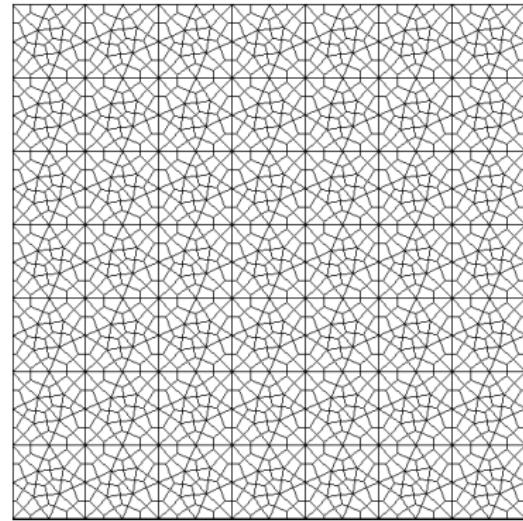
such that for all $\varphi, \psi \in \mathcal{D}(\overline{\Omega} \times [0, t_f])$

$$\begin{aligned} & - \int_{\Omega} u_0(\mathbf{x}) \varphi(\mathbf{x}, 0) d\mathbf{x} - \iint_{Q_{t_f}} u \partial_t \varphi d\mathbf{x} dt + \iint_{Q_{t_f}} a(u) \nabla u \cdot \nabla \varphi d\mathbf{x} dt \\ & \quad - \iint_{Q_{t_f}} \Lambda(\mathbf{x}) \chi(u) \nabla v \cdot \nabla \varphi d\mathbf{x} dt = \iint_{Q_{t_f}} f(u) \varphi(\mathbf{x}, t) d\mathbf{x} dt, \\ & - \int_{\Omega} v_0(\mathbf{x}) \psi(\mathbf{x}, 0) d\mathbf{x} - \iint_{Q_{t_f}} v \partial_t \psi d\mathbf{x} dt \\ & \quad + \iint_{Q_{t_f}} D(\mathbf{x}) \nabla v \cdot \nabla \psi d\mathbf{x} dt = \iint_{Q_{t_f}} g(u, v) \psi d\mathbf{x} dt. \end{aligned}$$

Primal and Dual Mesh (Isotropic case)



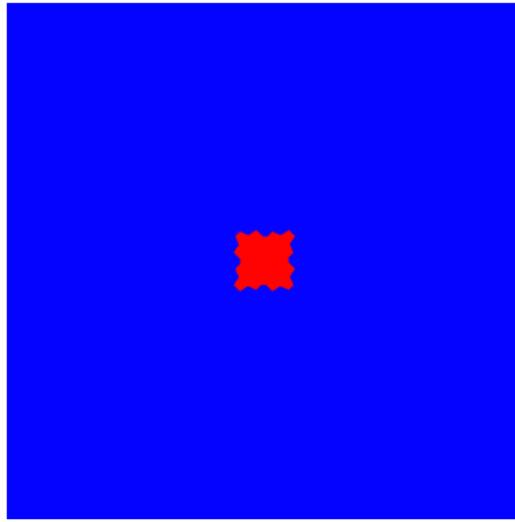
Primal mesh consisting of **14 336** triangles with acute angles.



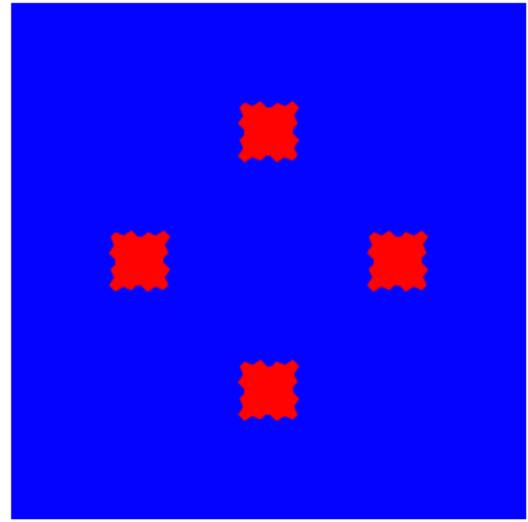
Barycentric dual mesh consisting of **7 297** dual control volumes.

The mesh satisfies the orthogonality condition.

Initial Conditions (Isotropic case)

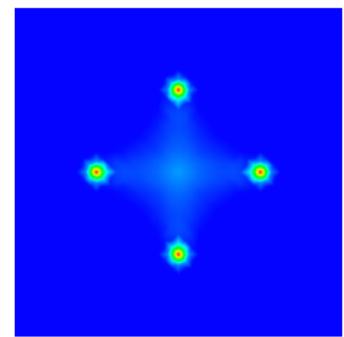
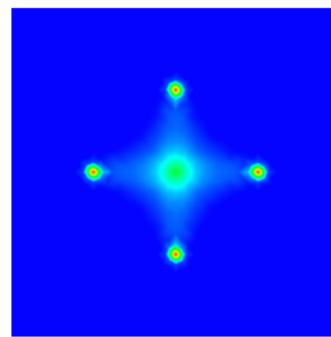
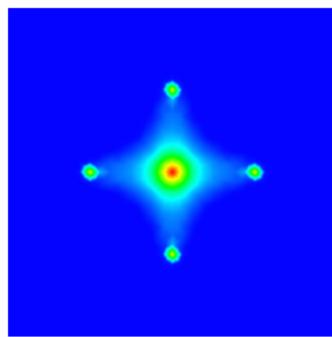
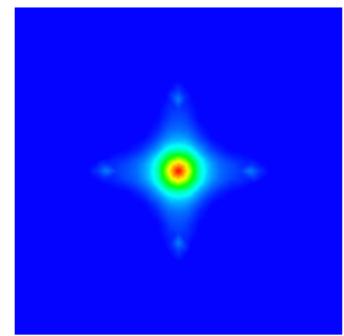
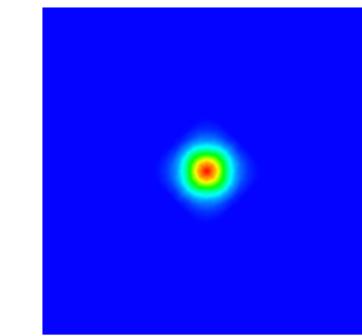
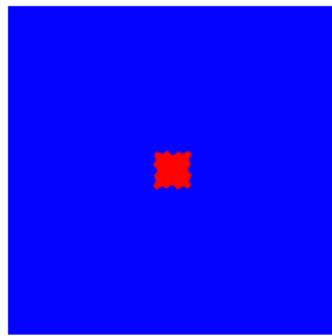


$u(0, \mathbf{x}, \mathbf{y}) = 1$ into the region where
 $(\mathbf{x}, \mathbf{y}) \in [0.45, 0.55] \times [0.45, 0.55]$
 $\chi(u) = 0.05 (u(1 - u))^2$, $f \equiv 0$
 $a(u) = u(1 - u)$.



$v(0, \mathbf{x}, \mathbf{y}) = 5$ into the four regions
of the domain as above.
 $\nu = 0.01$, $\delta = 0.005$, $d_1 = 0.0005$
 $d_2 = 0.0001$, $\Delta t = 0.002$

Evolution of Cell Density (Isotropic case)

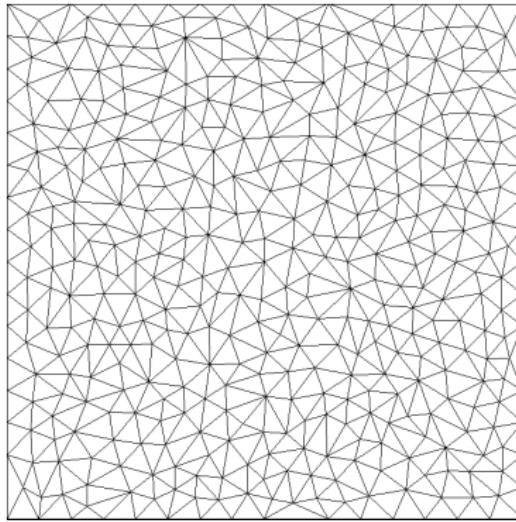


Primal and Dual Mesh (Anisotropic Case)

The anisotropic tensors are given by

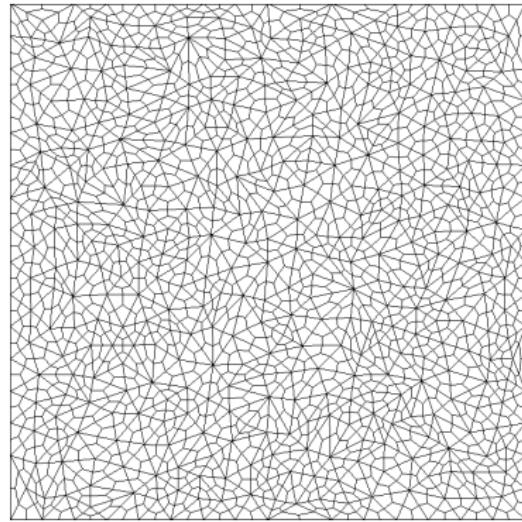
$$\Lambda = d_1 \begin{pmatrix} 7 & 2 \\ 2 & 10 \end{pmatrix}, \quad D(x) = d_2 \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Primal and Dual Mesh (Anisotropic Case)



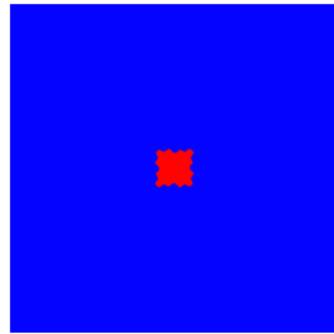
Primal mesh consisting of **5193** triangles with obtuse angles.

The mesh does not satisfy the orthogonality condition.

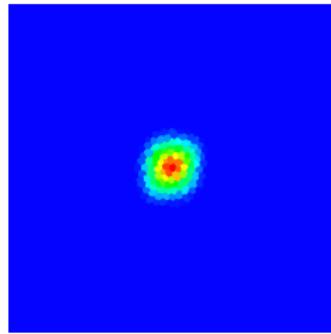


Barycentric dual mesh consisting of **2665** dual control volumes.

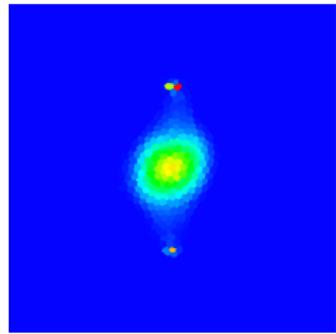
Evolution of Cell Density (Anisotropic Case)



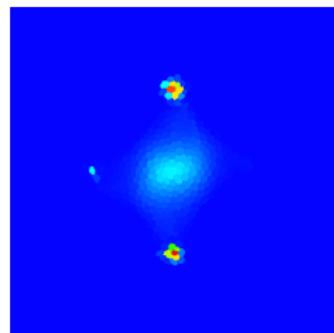
$$0 \leq u \leq 1$$



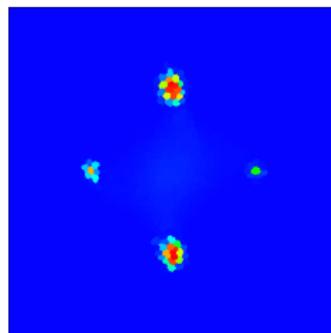
$$0 \leq u \leq 0.772$$



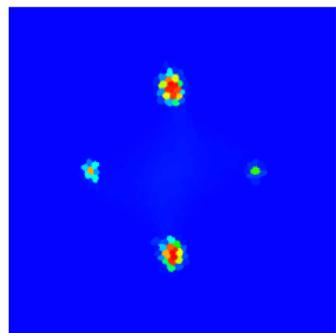
$$0 \leq u \leq 0.535$$



$$0 \leq u \leq 0.97$$



$$0 \leq u \leq 0.983$$



$$0 \leq u \leq 0.981$$

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Chemotaxis/Compressible Two Phase Flow Model

Consider Ω an open bounded subset of \mathbb{R}^d ($Q_{t_f} = (0, t_f) \times \Omega$)

Degenerate Parabolic Equation

$$\partial_t u - \operatorname{div}(a(u)\nabla u - f(u)\mathbf{V}) - g(u)\operatorname{div}(\mathbf{V}) + \gamma a(u)\nabla u \cdot \tilde{\mathbf{V}} = 0, \quad \text{in } Q_{t_f}$$

Chemotaxis/Compressible Two Phase Flow Model

Consider Ω an open bounded subset of \mathbb{R}^d ($Q_{t_f} = (0, t_f) \times \Omega$)

Degenerate Parabolic Equation

$$\partial_t u - \operatorname{div}(a(u)\nabla u - f(u)\mathbf{V}) - g(u)\operatorname{div}(\mathbf{V}) + \gamma a(u)\nabla u \cdot \tilde{\mathbf{V}} = 0, \quad \text{in } Q_{t_f}$$

- In the case where $g(u) \equiv 0$, then we obtain an equation modeling a **chemotaxis-fluid process**
 - u : cell density
 - $a(u)$: diffusivity of the cell density
 - $f(u)$: cells sensitivity to chemicals
 - \mathbf{V} : gradient concentration of chemicals
 - $\tilde{\mathbf{V}}$: constant fluid velocity
 - γ : positive parameter

Chemotaxis/Compressible Two Phase Flow Model

Consider Ω an open bounded subset of \mathbb{R}^d ($Q_{t_f} = (0, t_f) \times \Omega$)

Degenerate Parabolic Equation

$$\partial_t u - \operatorname{div}(a(u)\nabla u - f(u)\mathbf{V}) - g(u)\operatorname{div}(\mathbf{V}) + \gamma a(u)\nabla u \cdot \tilde{\mathbf{V}} = 0, \quad \text{in } Q_{t_f}$$

- In the case where $g(u) \neq 0$ and $\tilde{\mathbf{V}} = \mathbf{V}$, then we obtain an equation modeling a **compressible two-phase flow** in porous media
 - u : saturation
 - $a(u)$: mobility term
 - $f(u)$: fractional flow
 - \mathbf{V} : velocity of the flow
 - γ : compressibility factor

Chemotaxis/Compressible Two Phase Flow Model

Consider Ω an open bounded subset of \mathbb{R}^d ($Q_{t_f} = (0, t_f) \times \Omega$)

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We impose the homogeneous Dirichlet boundary condition:

$$u(\mathbf{x}, t) = 0, \quad \text{on } \Sigma_{t_f} = \partial\Omega \times (0, t_f)$$

Initial condition:

$$0 \leq u(\mathbf{x}, 0) = u_0(\mathbf{x}) \leq 1 \quad \forall \mathbf{x} \in \Omega$$

In the sequel, we assume that $\gamma = 1$ and $\tilde{\mathbf{V}} = \mathbf{V}$.

Main Assumptions

We consider the main assumptions

- (A1) \mathbf{V} is a measurable function lying into $(L^\infty(\Omega))^d$ such that
 $\mathbf{V} \cdot \mathbf{n} \leq 0$.
- (A2) f is a differentiable function in $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$.
- (A3) $g \in \mathcal{C}^1([0, 1])$ such that $g(0) = 0$ and $g(1) = 1$.
In addition, $g'(u) \geq C_{g'} > 0$, $\forall u \in [0, 1]$.
- (A4) The mobility $a \in \mathcal{C}^1([0, 1])$ verifies:
 $a(u) > 0$, $\forall 0 < u < 1$ and it degenerates at 0 or 1.

Degeneracy at $u = 1$, $a(1) = 0$

Case where $a(1) = 0$ and $a(0) > 0$

We assume that, there exists $0 < r_2 \leq 2$, $u_* < 1$, m_1 and $M_2 > 0$ such that: $m_1(1 - u)^{r_2} \leq a(u) \leq M_2(1 - u)^{r_2}$, $\forall u_* \leq u < 1$.

Degeneracy at $u = 1$, $a(1) = 0$

Case where $a(1) = 0$ and $a(0) > 0$

Theorem (Classical weak solution)

$$0 \leq u(\mathbf{x}, t) \leq 1, \quad \forall (\mathbf{x}, t) \in Q_{t_f},$$

$$u \in L^2(0, t_f; H_0^1(\Omega)) \cap C^0([0, t_f]; L^2(\Omega)),$$

$$\partial_t u \in L^2(0, t_f; H^{-1}(\Omega)),$$

and such that: $\forall \varphi \in L^2(0, t_f; H_0^1(\Omega))$

$$\begin{aligned} & \int_0^{t_f} \langle \partial_t u, \varphi \rangle dt + \int_{Q_{t_f}} a(u) \nabla u \cdot \nabla \varphi d\mathbf{x} dt - \int_{Q_{t_f}} f(u) \mathbf{V} \cdot \nabla \varphi d\mathbf{x} dt \\ & + \int_{Q_{t_f}} g(u) \mathbf{V} \cdot \nabla \varphi d\mathbf{x} dt + \int_{Q_{t_f}} g'(u) \mathbf{V} \cdot \nabla u \varphi d\mathbf{x} dt \\ & + \int_{Q_{t_f}} a(u) \mathbf{V} \cdot \nabla u \varphi d\mathbf{x} dt = 0. \end{aligned}$$

Degeneracy at $u = 0$, $a(0) = 0$ and $a(1) > 0$

Consider a regular test function φ then

$$\begin{aligned} \langle \partial_t u, \varphi \rangle + \int_{\Omega} \nabla A(u) \cdot \nabla \varphi d\mathbf{x} - \int_{\Omega} f(u) \mathbf{V} \cdot \nabla \varphi d\mathbf{x} + \int_{\Omega} \mathbf{V} \cdot \nabla g(u) \varphi d\mathbf{x} \\ + \int_{\Omega} \mathbf{V} \cdot \nabla \varphi g(u) d\mathbf{x} + \int_{\Omega} \nabla A(u) \cdot \mathbf{V} \varphi d\mathbf{x} = 0 \end{aligned}$$

Energy Estimates

We obtain $\nabla A(u) \in L^2(\Omega)$ but $\nabla g(u) \notin L^2(\Omega)$

Idea: Replace φ by $\beta(u)\varphi$ as a test function

Intention: Choose the function $\beta(u)$ such that

$$\int_{\Omega} \mathbf{V} \cdot \nabla(g(u)\beta(u))\varphi$$

has a sens

Degeneracy at $u = 0$, $a(0) = 0$ and $a(1) > 0$

Case where $a(0) = 0$ and $a(1) > 0$

We assume that, there exists r_1 , m_1 and $M_2 > 0$ such that:

$$m_1 r_1 u^{r_1-1} \leq a'(u) \leq M_2 r_1 u^{r_1-1}$$

Degeneracy at $u = 0$, $a(0) = 0$ and $a(1) > 0$

Case where $a(0) = 0$ and $a(1) > 0$

We assume that, there exists r_1 , m_1 and $M_2 > 0$ such that:

$$m_1 r_1 u^{r_1-1} \leq a'(u) \leq M_2 r_1 u^{r_1-1}$$

We define the continuous functions β_θ and h_θ , $\theta > 0$ by

$$\beta_\theta(u) = u^{r-1+\theta}, \quad h_\theta = \int_0^u \beta_\theta(\tau) d\tau$$

Theorem (Weighted weak solution)

$$0 \leq u(x, t) \leq 1, \quad \forall (x, t) \in Q_{t_f},$$

$$h_\theta(u) \in L^2(0, t_f; H_0^1(\Omega)),$$

$$\sqrt{\beta'(u)} \nabla u \in (L^2(Q_{t_f}))^d,$$

Degeneracy at $u = 0$, $a(0) = 0$ and $a(1) > 0$

Weighted weak formulation

$\forall \chi \in C^1([0, T); H_0^1(\Omega))$ with $\chi(\cdot, T) = 0$ and $\chi \geq 0$.

$$\begin{aligned} & - \int_{Q_{t_f}} h_\theta(u) \partial_t \chi \, d\mathbf{x} \, dt - \int_{\Omega} h_\theta(u_0) \chi(\mathbf{x}, 0) \, d\mathbf{x} + \int_{Q_{t_f}} a(u) \nabla u \cdot \nabla (\beta_\theta(u) \chi) \, d\mathbf{x} \, dt \\ & + \int_{Q_{t_f}} a(u) \mathbf{V} \cdot \nabla u \beta_\theta(u) \chi \, d\mathbf{x} \, dt - \int_{Q_{t_f}} (f(u) - g(u)) \mathbf{V} \cdot \nabla (\beta_\theta(u) \chi) \, d\mathbf{x} \, dt \\ & + \int_{Q_{t_f}} g'(u) \mathbf{V} \cdot \nabla u \beta_\theta(u) \chi \, d\mathbf{x} \, dt \leq 0, \end{aligned}$$

Conclusion & Perspectives

Conclusion

- We studied the pattern formation for a nonlinear chemotaxis model
- We constructed a nonlinear scheme for an anisotropic chemotaxis model
- We studied the existence of weak solutions to a nonlinear degenerate parabolic equation

Perspectives

- Study the order of convergence of the modified nonlinear CVFE scheme
- Obtain weak formulations instead of the weighted formulations for the nonlinear degenerate parabolic equation

**Thank you for your attention!
Questions and comments are
welcome!**

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