A Gradient-Enhanced Damage Approach to Fracture

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Abstract: A gradient-enhanced damage is formulated in which the Laplacian of the internal variable which memorises the damage enters the damage loading function in addition to the conventional dependence on this internal variable itself. The computational setting for this higher-order continuum model is fully elaborated. Its embedment in thermodynamics is discussed and some remarks are made on the additional boundary conditions that arise. Finally, the special case is discussed for which there is a linear relation between the damage variable and the internal variable that memorises the damage evolution.

1. INTRODUCTION

Damage mechanics theories can be used to describe degradation and failure of structural materials and components. In its simplest form, it degrades the elastic properties, in particular Young's modulus with the accumulation of damage. Often, and this is the approach that will also be used here, the damage is linked to the existing strain level [1].

A particular difficulty that is inherent in damage mechanics theories, and in fact in all continuum mechanics models that try to capture degradation and failure of materials, is the observation that at a certain threshold level of loading the governing differential equations locally lose ellipticity (or hyperbolicity if dynamic loading conditions are considered). Consequently, the boundary or initial value problem becomes ill-posed [2] and analytical or numerical solutions become meaningless.

A host of solutions has been suggested to remedy this deficiency of the standard continuum approach. For high-speed phenomena the inclusion of the inherent rate dependence of a material seems natural [3-7]. For granular materials a revival of the Cosserat continuum seems meaningful. For this case micromechanical foundations for applying such a theory exist [8] and numerical approaches have been elaborated that can be implemented in standard finite element codes in a straightforward fashion [9,10]. For cracking in concrete and ceramics, and for describing void growth in metals nonlocal theories either in an integral format or in a differential format seem most appropriate. Within the context of a simple elastic-damaging material a fully nonlocal theory has been proposed by Pijaudier-Cabot and Bazant [11,12]. Aifantis [13-15], Coleman and Hodgdon [16], Schreyer and Chen [17], Lasry and Belytschko [18] and Vardoulakis and his colleagues [19,20] have proposed gradient theories in a plasticity-based format, while, motivated by the work of Mühlhaus and Aifantis [21], de Borst and his colleagues [22-24] have derived algorithms for finite element implementations of a gradient-enhanced plasticity theory. Recently, a gradient theory has been proposed within a damage mechanics framework [25]. A different formalism, in which a nonstandard equivalent strain was introduced, was advocated by Peerlings [26] and successfully implemented in a finite element code.

In this contribution a gradient-enhanced damage theory will be elaborated that is quite similar to the one proposed in Reference [25]. First, some remarks are made about the possible thermodynamic restrictions that can apply to such a theory. Then, the theory is elaborated and the finite element equations are presented in detail. Finally, a special case is considered in which there exists a linear relation between the damage variable and the history parameter that measures the maximum damage experienced by the material.

2. THEORY AND CONSEQUENCES FROM THERMODYNAMICS

We consider an elastic-damaging material for which the free or Helmholtz energy Ψ depends on the strain tensor ε_{ij}, a scalar-valued damage variable ω and its spatial gradient $\omega_{,i}$, the comma denoting differentiation with respect to x_i:

$$\Psi = \Psi(\varepsilon_{ij}, \omega, \omega_{,i}) . \tag{1}$$

The damage variable ω has an initial value 0 for a complete intact material and attains the value 1 for complete loss of coherence. In consideration of eq. (1) the differential change of the free energy can be written as

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial \Psi}{\partial \omega} \dot{\omega} + \frac{\partial \Psi}{\partial \omega_{,i}} \dot{\omega}_{,i} . \tag{2}$$

We now substitute this identity into the Clausius-Duhem inequality for isothermal processes:

$$- \rho \dot{\Psi} + \sigma_{ij} \dot{\varepsilon}_{ij} \geq 0 , \tag{3}$$

where ρ is the mass density and σ_{ij} is the stress tensor. The result reads:

$$\left(- \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}} + \sigma_{ij} \right) \dot{\varepsilon}_{ij} - \rho \frac{\partial \Psi}{\partial \omega} \dot{\omega} - \rho \frac{\partial \Psi}{\partial \omega_{,i}} \dot{\omega}_{,i} \geq 0 \tag{4}$$

and by standard thermodynamic arguments we obtain

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ij}} , \tag{5}$$

the thermodynamic force conjugate to the damage variable ω,

$$Y = - \rho \frac{\partial \Psi}{\partial \omega} \tag{6}$$

and the thermodynamic force conjugate to the gradient of the damage variable,

$$h_i = - \rho \frac{\partial \Psi}{\partial \omega_{,i}} . \tag{7}$$

Similar relations were obtained in Reference [25].
In the remainder of this study we consider a particular elastic-damage type relation, where the free energy is given by

$$\rho \Psi = \frac{1}{2} (1 - \omega) \varepsilon_{ij} D_{ijkl} \varepsilon_{kl} + \frac{1}{2} c \omega_{,i} \omega_{,i} , \tag{8}$$

with D_{ijkl} the fourth-order tensor that contains the virgin elastic moduli and c a gradient constant, which is a measure for the influence of the gradient terms in eq. (8). Note that the gradient terms act as a singular perturbation of the standard free energy function, which depends only on the strain ε_{ij} and the damage variable ω [25]. Now, relations (5)-(7) take the form:

$$\sigma_{ij} = (1 - \omega) D_{ijkl} \varepsilon_{kl} , \tag{9}$$

$$Y = \frac{1}{2} \varepsilon_{ij} D_{ijkl} \varepsilon_{kl} , \tag{10}$$

and the damage flux vector,

$$h_i = - c \omega_{,i} . \tag{11}$$

Upon substitution of eqs (5), (10) and (11) into the Clausius-Duhem identity (4), we obtain the standard

result that the dissipation rate $\dot{\phi}$ should be non-negative:

$$\dot{\phi} = \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl}\dot{\omega} - c\omega_{,i}\dot{\omega}_{,i} \geq 0 \ . \tag{12}$$

The requirement of a non-negative dissipation rate leads to the result that $\dot{\omega} \geq 0$ in absence of gradients, since the thermodynamic force Y conjugate to the damage variable ω is a quadratic form for the given choice (8) of the free energy. However, the presence of gradients relaxes the requirement $\dot{\omega} \geq 0$, so that we can observe a temporal decrease of damage during the process.
Next, we introduce a damage loading function,

$$f = \tilde{\varepsilon} - \bar{\kappa} \ , \tag{13}$$

such that during progressive damage evolution, the identity $f = 0$ holds, else $f < 0$. In eq. (13) $\tilde{\varepsilon}$ is an equivalent strain, which is a function of the strain tensor, $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon_{ij})$, for instance,

$$\tilde{\varepsilon} = Y = \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} \ . \tag{14}$$

The nonlocal history variable $\bar{\kappa}$ measures the largest value that has been attained by $\tilde{\varepsilon}$. Accordingly, it is a non-decreasing function and grows only when $f = 0$. The requirements on the damage loading function f and the history variable $\bar{\kappa}$ can be described conveniently by the Kuhn-Tucker relations:

$$f \leq 0 \ , \quad \dot{\bar{\kappa}} \geq 0 \ , \quad f\dot{\bar{\kappa}} = 0 \ . \tag{15}$$

An example of $\bar{\kappa}$ is the full nonlocal expression,

$$\bar{\kappa} = \frac{1}{V_r}\int\limits_V g(x_i + s_i)\kappa(s_i)\mathrm{d}s_i \ , \tag{16}$$

with g a weighting function [11,12] and κ the local history parameter. Herein we shall adopt a gradient approximation of (16), that can be obtained by developing ω into a Taylor series, e.g. [21]. For an isotropic, infinite medium and truncating after the second term we then have

$$\bar{\kappa} = \kappa + c\nabla^2\kappa \ , \tag{17}$$

where the gradient constant c depends on the weighting function and on the dimension of the medium. In principle, eq. (13) could now be replaced by

$$f = \tilde{\varepsilon} - \kappa - c\nabla^2\kappa \ , \tag{18}$$

but we will consider κ as an independent variable in the finite element implementation to be discussed next, and therefore we shall retain the form (13) for the damage loading function, where $\bar{\kappa}$ is given by (17).
We finally define a nonlocal damage variable $\bar{\omega}$ which is a function of the nonlocal history variable $\bar{\kappa}$,

$$\bar{\omega} = \bar{\omega}(\bar{\kappa}) \ , \tag{19}$$

in a fashion similar to the conventional dependence of the local damage variable ω on the local history variable κ:

$$\omega = \omega(\kappa) \ . \tag{20}$$

Using eq. (19), the Kuhn-Tucker conditions can also be written as:

$$f \leq 0 \ , \quad \dot{\bar{\omega}} \geq 0 \ , \quad f\dot{\bar{\omega}} = 0 \ . \tag{21}$$

3. FINITE ELEMENT DESCRIPTION

For the finite element implementation we consider the equilibrium condition at iteration $n + 1$:

$$\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma}_{n+1} + \rho\mathbf{g} = \mathbf{0} , \tag{22}$$

with $\mathbf{g}$ the gravity acceleration vector and $\mathbf{L}$ an operator matrix which, for the three-dimensional case, reads:

$$\mathbf{L}^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial \cdot}{\partial x} & 0 & 0 & \dfrac{\partial \cdot}{\partial y} & 0 & \dfrac{\partial \cdot}{\partial z} \\[2ex] 0 & \dfrac{\partial \cdot}{\partial y} & 0 & \dfrac{\partial \cdot}{\partial x} & \dfrac{\partial \cdot}{\partial z} & 0 \\[2ex] 0 & 0 & \dfrac{\partial \cdot}{\partial z} & 0 & \dfrac{\partial \cdot}{\partial y} & \dfrac{\partial \cdot}{\partial x} \end{bmatrix} . \tag{23}$$

Multiplying the equilibrium equation with $\delta\mathbf{u}$, where $\mathbf{u}$ is the continuous displacement field vector and the symbol δ denotes the variation of a quantity, and integrating over the entire volume occupied by the body, one obtains the corresponding weak form:

$$\int_V \delta\mathbf{u}^{\mathrm{T}}(\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma}_{n+1} + \rho\mathbf{g})\mathrm{d}V = 0 . \tag{24}$$

Similarly, the weak form of the Helmholtz equation for the distribution of the local history parameter κ, eq. (17), can be derived as:

$$\int_V \delta\dot{\kappa}(\kappa_{n+1} + c\nabla^2\kappa_{n+1} - \bar{\kappa}_{n+1})\mathrm{d}V = 0 . \tag{25}$$

We now introduce the decompositions

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathrm{d}\boldsymbol{\sigma} , \tag{26}$$

and

$$\kappa_{n+1} = \kappa_n + \mathrm{d}\kappa , \tag{27}$$

for the stress $\boldsymbol{\sigma}$ and the history parameter κ, respectively. The d-symbol signifies the iterative improvement of a quantity between two successive iterations. With aid of these decompositions and applying the divergence theorem to eqs (24)-(25) one obtains

$$\int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\mathrm{d}\boldsymbol{\sigma}\mathrm{d}V = \int_V \rho\delta\mathbf{u}^{\mathrm{T}}\mathbf{g}\mathrm{d}V + \int_S \delta\mathbf{u}^{\mathrm{T}}\mathbf{t}\mathrm{d}S - \int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\boldsymbol{\sigma}_n\mathrm{d}V , \tag{28}$$

where $\mathbf{t}$ is the boundary-traction vector and

$$\int_V (\delta\kappa\,\mathrm{d}\kappa - c\nabla\delta\kappa \cdot \nabla\mathrm{d}\kappa)\mathrm{d}V - \int_V \delta\kappa\,\mathrm{d}\bar{\kappa}\mathrm{d}V = \int_V \delta\kappa\,\bar{\kappa}_n\mathrm{d}V - \int_V (\delta\kappa\,\kappa_n - c\nabla\delta\kappa \cdot \nabla\kappa_n)\mathrm{d}V . \tag{29}$$

where the non-standard natural boundary condition

$$\mathbf{n}^{\mathrm{T}}\nabla\kappa = 0 \tag{30}$$

has been adopted, $\mathbf{n}$ being the outward normal to the body surface. Since κ can be directly related to the damage variable ω this condition can be interpreted as no damage flux through the boundary of the body. Finally we interpolate displacements $\mathbf{u}$ and the history parameter κ as

$$\mathbf{u} = \mathbf{Ha} \tag{31}$$

and

$$\kappa = \tilde{\mathbf{H}}\mathbf{k} \tag{32}$$

with $\mathbf{a}$ and $\mathbf{k}$ the vectors that contain the nodal values of $\mathbf{u}$ and κ, respectively. $\mathbf{H}$ and $\tilde{\mathbf{H}}$ contain the interpolation polynomials of $\mathbf{u}$ and κ, respectively. The gradient of κ is then computed as

$$\nabla\kappa = \tilde{\mathbf{B}}\mathbf{k} \quad , \quad \tilde{\mathbf{B}} = \nabla \cdot \tilde{\mathbf{H}} . \tag{33}$$

Substitution of eqs (31)-(33) into eqs (28) and (29) and using the fact that the ensuing relations must hold for any admissible $\delta\mathbf{u}$ and $\delta\kappa$ then yields

$$\int_V \mathbf{B}^{\mathrm{T}}\mathrm{d}\sigma\mathrm{d}V = \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int},n} , \tag{34}$$

where $\mathbf{B} = \mathbf{L}\mathbf{H}$, and

$$\mathbf{f}_{\mathrm{ext}} = \int_V \rho\delta\mathbf{u}^{\mathrm{T}}\mathbf{g}\mathrm{d}V + \int_S \delta\mathbf{u}^{\mathrm{T}}\mathbf{t}\mathrm{d}S \tag{35}$$

$$\mathbf{f}_{\mathrm{int},n} = \int_V \mathbf{B}^{\mathrm{T}}\sigma_n\mathrm{d}V , \tag{36}$$

and the identity

$$\mathbf{K}_{kk}\mathrm{d}\mathbf{k} - \int_V \tilde{\mathbf{H}}^{\mathrm{T}}\mathrm{d}\bar{\kappa}\mathrm{d}V = \int_V \tilde{\mathbf{H}}^{\mathrm{T}}\bar{\kappa}_n\mathrm{d}V - \mathbf{K}_{kk}\mathbf{k}_n , \tag{37}$$

where

$$\mathbf{K}_{kk} = \int_V (\tilde{\mathbf{H}}^{\mathrm{T}}\tilde{\mathbf{H}} - c\tilde{\mathbf{B}}^{\mathrm{T}}\tilde{\mathbf{B}})\mathrm{d}V . \tag{38}$$

We now adopt the standard elastic-damage stress-strain relation of eq. (9) and cast it into an incremental format:

$$\mathrm{d}\sigma = (1 - \omega_n)\mathbf{D}\mathrm{d}\varepsilon - \mathbf{D}\varepsilon_n\mathrm{d}\omega \tag{39}$$

Restricting the treatment to small displacement gradients we introduce the linear kinematic relation

$$\varepsilon = \mathbf{L}\mathbf{u} , \tag{40}$$

with $\mathbf{L}$ as defined in the preceding, or using eq. (31) and $\mathbf{B} = \mathbf{L}\mathbf{H}$,

$$\varepsilon = \mathbf{B}\mathbf{a} . \tag{41}$$

Furthermore, the local damage ω is a function of the local history parameter κ: $\omega = \omega(\kappa)$, so that

$$\mathrm{d}\omega = \frac{\partial\omega}{\partial\kappa} \mathrm{d}\kappa . \tag{42}$$

Taking also into account eqs (32) and (41), eq. (39) is elaborated as:

$$\mathrm{d}\sigma = (1 - \omega_n)\mathbf{D}\mathrm{d}\varepsilon - \mathbf{D}\varepsilon_n \frac{\partial\omega}{\partial\kappa} \tilde{\mathbf{H}}\mathrm{d}\mathbf{k} . \tag{43}$$

To complete the formulation the iterative improvement of the non-standard history parameter $\mathrm{d}\bar{\kappa}$ is elaborated using eqs (40) as

$$\mathrm{d}\bar{\kappa} = \frac{\partial\bar{\kappa}}{\partial\bar{\varepsilon}} \frac{\partial\bar{\varepsilon}}{\partial\varepsilon} \mathbf{B}\mathrm{d}\mathbf{a} , \tag{44}$$

where $\partial\bar{\kappa}/\partial\bar{\varepsilon} = 1$ for loading and $\partial\bar{\kappa}/\partial\bar{\varepsilon} = 0$ otherwise. Inserting the above expressions for $\mathrm{d}\sigma$ and $\mathrm{d}\bar{\kappa}$ into

eqs (34) and (37) yields the following set of equations that describe the incremental process in the discretised gradient-enhanced elastic-damaging continuum:

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ak} \\ \mathbf{K}_{ka} & \mathbf{K}_{kk} \end{bmatrix} \begin{bmatrix} d\mathbf{a} \\ d\mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{ext} - \mathbf{f}_{int,n} \\ \mathbf{f}_{k,n} - \mathbf{K}_{kk}\mathbf{k}_n \end{bmatrix} \tag{45}$$

where $\mathbf{f}_{ext}$, $\mathbf{f}_{int,n}$ and $\mathbf{K}_{kk}$ are defined in accordance with eqs (35)-(38) and

$$\mathbf{K}_{aa} = \int_V (1 - \omega_n)\mathbf{B}^T\mathbf{D}\mathbf{B}dV \ , \tag{46}$$

$$\mathbf{K}_{ak} = -\int_V \mathbf{B}^T\mathbf{D}\varepsilon_n \frac{\partial \omega}{\partial \kappa} \tilde{\mathbf{H}}dV \ , \tag{47}$$

$$\mathbf{K}_{ka} = -\int_V \frac{\partial \bar{\kappa}}{\partial \tilde{\varepsilon}} \tilde{\mathbf{H}}^T \frac{\partial \tilde{\varepsilon}}{\partial \varepsilon} \mathbf{B}dV \tag{48}$$

and

$$\mathbf{f}_{k,n} = \int_V \tilde{\mathbf{H}}^T \bar{\kappa}_n dV \ . \tag{49}$$

After solution of the set (45), we proceed as follows:

1. Update $\mathbf{a}$ and $\mathbf{k}$ at the nodal points:

$$\mathbf{a}_{n+1} = \mathbf{a}_n + d\mathbf{a} \ ,$$

$$\mathbf{k}_{n+1} = \mathbf{k}_n + d\mathbf{k} \ .$$

2. Compute in the integration points:

strains: $\varepsilon_{n+1} = \mathbf{B}\mathbf{a}_{n+1}$,

equivalent strain: $\tilde{\varepsilon}_{n+1} = \tilde{\varepsilon}(\varepsilon_{n+1})$,

damage loading function: $f = \tilde{\varepsilon}_{n+1} - \bar{\kappa}_n$,

If $f \geq 0$: $\bar{\kappa}_{n+1} = \tilde{\varepsilon}_{n+1}$,

else: $\bar{\kappa}_{n+1} = \bar{\kappa}_n$,

Interpolate: $\kappa_{n+1} = \tilde{\mathbf{H}}\mathbf{k}_{n+1}$,

Compute: $\omega_{n+1} = \omega(\kappa_{n+1})$,

Compute: $\sigma_{n+1} = (1 - \omega_{n+1})\mathbf{D}\varepsilon_{n+1}$.

3. Update the internal forces:

$$\mathbf{f}_{int,n+1} = \int_V \mathbf{B}^T\sigma_{n+1}dV \ ,$$

$$\mathbf{f}_{k,n+1} = \int\limits_V \tilde{\mathbf{H}}^{\mathrm{T}} \bar{\kappa}_{n+1} \mathrm{d}V \, ,$$

as well as the new tangential stiffness matrix, and enter a new iteration if the process has not yet converged.

It is noted that because the basic variables are differentiated only once in the above expressions, a simple C^0-continuity of the interpolation polynomials suffices. Of course, the displacements should be interpolated one order higher than the history variable in order to avoid stress oscillations, cf. the Babuska-Brezzi condition for mixed finite elements in incompressible solids.

4. LINEAR DEPENDENCE ON DAMAGE VARIABLE

We shall now consider a special case of the above theory, namely for which there exists a linear relation between the local history parameter κ and the local damage variable ω:

$$\kappa = \kappa_0 + M\omega \, , \tag{50}$$

with κ_0 representing a threshold damage below which there is no damage growth, and M a constant. In addition, we assume that the equivalent strain is given by the thermodynamic force conjugate to the damage variable with the free energy as defined in eq. (8). Then,

$$\tilde{\varepsilon} = Y = \frac{1}{2} \varepsilon^{\mathrm{T}} \mathbf{D} \varepsilon \, ,$$

cf. eq. (14). As a direct consequence of this choice we have

$$\frac{\partial \tilde{\varepsilon}}{\partial \varepsilon} = \varepsilon^{\mathrm{T}} \mathbf{D} \, ,$$

which symmetrises the set of equations (45).

Returning to the linear relation (50) we observe that by virtue of the nonlocal generalisation (16)

$$\bar{\kappa} = \kappa_0 + M\bar{\omega} \, , \tag{51}$$

since by definition

$$\bar{\omega} = \frac{1}{V_r} \int\limits_V g(x_i + s_i)\omega(s_i)\mathrm{d}s_i \, . \tag{52}$$

Furthermore, we have

$$\frac{\partial \omega}{\partial \kappa} = \frac{1}{M} \tag{53}$$

and in view of eq. (51)

$$\frac{\partial \bar{\kappa}}{\partial \tilde{\varepsilon}} = M \frac{\partial \bar{\omega}}{\partial Y} \, , \tag{54}$$

where now $\partial \bar{\omega}/\partial Y = 1/M$ during loading, otherwise 0. Finally, eq. (50) must also hold at the nodes, so that

$$\mathbf{k}_n = \mathbf{k}_0 + M\boldsymbol{\omega} \tag{55}$$

and

$$\mathrm{d}\mathbf{k} = M\mathrm{d}\boldsymbol{\omega} \, , \tag{56}$$

with $\boldsymbol{\omega}$ the vector that assembles the nodal values of the damage ω, which is now being interpolated instead of the history parameter κ. Substitution of the latter two identities in eqs (45) and dividing the second equation of (45) by M yields

$$
\begin{bmatrix} \mathbf{K}_{aa} & M\mathbf{K}_{ak} \\ M^{-1}\mathbf{K}_{ka} & \mathbf{K}_{kk} \end{bmatrix} \begin{bmatrix} \mathrm{d}a \\ \mathrm{d}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int},n} \\ \int\limits_V \tilde{\mathbf{H}}^T (M^{-1}\kappa_0 + \bar{\omega}_n)\mathrm{d}V - M^{-1}\mathbf{K}_{kk}\mathbf{k}_0 - \mathbf{K}_{kk}\boldsymbol{\omega}_n \end{bmatrix}.
\tag{57}
$$

Defining

$$
\mathbf{K}_{a\omega} = -\int\limits_V \mathbf{B}^T \mathbf{D}\boldsymbol{\varepsilon}_n \tilde{\mathbf{H}}\mathrm{d}V ,
\tag{58}
$$

$$
\mathbf{K}_{\omega a} = -\int\limits_V \frac{\partial \bar{\omega}}{\partial Y} \tilde{\mathbf{H}}^T \boldsymbol{\varepsilon}^T \mathbf{D}\mathbf{B}\mathrm{d}V
\tag{59}
$$

and considering eqs (53) and (54) one then obtains instead of eqs (57):

$$
\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{a\omega} \\ \mathbf{K}_{\omega a} & \mathbf{K}_{\omega\omega} \end{bmatrix} \begin{bmatrix} \mathrm{d}a \\ \mathrm{d}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int}} \\ \int\limits_V \tilde{\mathbf{H}}^T (M^{-1}\kappa_0 + \bar{\omega}_n)\mathrm{d}V - M^{-1}\mathbf{K}_{\omega\omega}\mathbf{k}_0 - \mathbf{K}_{\omega\omega}\boldsymbol{\omega}_n \end{bmatrix},
\tag{60}
$$

where $\mathbf{K}_{kk} = \mathbf{K}_{\omega\omega}$. Observing that $\mathbf{k}_0$ is a constant vector, $\mathbf{k}_0 = \kappa_0 \mathbf{i}$, where $\mathbf{i}^T = [1,1,\ldots\ldots,1]$, we have that $\mathbf{K}_{\omega\omega}\mathbf{k}_0 = \int\limits_V \tilde{\mathbf{H}}^T \kappa_0 \mathrm{d}V$, and eqs (60) reduce to

$$
\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{a\omega} \\ \mathbf{K}_{\omega a} & \mathbf{K}_{\omega\omega} \end{bmatrix} \begin{bmatrix} \mathrm{d}a \\ \mathrm{d}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int},n} \\ \mathbf{f}_{\omega,n} - \mathbf{K}_{\omega\omega}\boldsymbol{\omega}_n \end{bmatrix}.
\tag{61}
$$

where

$$
\mathbf{f}_{\omega,n} = \int\limits_V \tilde{\mathbf{H}}^T \bar{\omega}_n \mathrm{d}V .
\tag{62}
$$

The ensuing update algorithm now marginally changes, as follows:

1. Update $\mathbf{a}$ and $\boldsymbol{\omega}$ at the nodal points:

$$
\mathbf{a}_{n+1} = \mathbf{a}_n + \mathrm{d}a ,
$$

$$
\boldsymbol{\omega}_{n+1} = \boldsymbol{\omega}_n + \mathrm{d}\boldsymbol{\omega} .
$$

2. Compute in the integration points:

strains: $\boldsymbol{\varepsilon}_{n+1} = \mathbf{B}\mathbf{a}_{n+1}$,

equivalent strain: $Y_{n+1} = \dfrac{1}{2} \boldsymbol{\varepsilon}_{n+1}^T \mathbf{D}\boldsymbol{\varepsilon}_{n+1}$,

damage loading function: $f = Y_{n+1} - Y_0 - M\bar{\omega}_n \ \ (Y_0 = \kappa_0)$,

If $f \geq 0$: $\bar{\omega}_{n+1} = M^{-1}(Y_{n+1} + Y_0)$,

else: $\bar{\omega}_{n+1} = \bar{\omega}_n$,

Interpolate: $\omega_{n+1} = \tilde{\mathbf{H}}\boldsymbol{\omega}_{n+1}$,

Compute: $\sigma_{n+1} = (1 - \omega_{n+1})\mathbf{D}\varepsilon_{n+1}$.

3. Update the internal forces:

$$\mathbf{f}_{\text{int},n+1} = \int_V \mathbf{B}^{\mathrm{T}}\sigma_{n+1}\mathrm{d}V \ ,$$

$$\mathbf{f}_{\omega,n+1} = \int_V \tilde{\mathbf{H}}^{\mathrm{T}}\bar{\omega}_{n+1}\mathrm{d}V \ ,$$

as well as the new tangential stiffness matrix, and enter a new iteration if the process has not yet converged.

The present case can be used to simulate void growth in ductile metals, including plasticity-like effects. Consider the uniaxial case. The requirement that during damage evolution the loading function equals zero then gives

$$\omega = \frac{E(\varepsilon^2 - \varepsilon_0^2)}{2M} \quad , \quad \varepsilon_0 = \sqrt{2Y_0/E} \ , \tag{63}$$

E being Young's modulus. The uniaxial stress-strain relation $\sigma = (1 - \omega)E\varepsilon$ can then be elaborated as

$$\sigma = (1 - E(\varepsilon^2 - \varepsilon_0^2)/(2M))E\varepsilon \ . \tag{64}$$

This is a cubic relation, which starts at a strain level $\varepsilon_0 = \sqrt{2Y_0/E}$ (until this strain level we have purely elastic behaviour), exhibits an ascending branch to mimic plasticity-like effects, and after passing the peak stress level shows a rapid decrease to a zero stress level at $\varepsilon_u = \sqrt{\varepsilon_0 + 2M/E}$. For $\varepsilon > \varepsilon_u$ the load-carrying capacity has vanished.

5. NUMERICAL EXAMPLES

We shall now illustrate the characteristics of the theory and the performance of the algorithm by a simple one-dimensional example of a bar loaded in pure tension. The length of the bar is $L = 100$ mm with a 10% reduced cross-section of the centre 10 mm of the bar.

The computations have been carried out for the case that we have a linear relation between the history variable κ and the damage parameter ω, so that the cubic stress-strain relation (64) holds. A Young's modulus $E = 200,000$ MPa has been adopted, while the parameters for the damage law read: $Y_0 = 0.4$ MPa, $M = 25$ MPa and $c = 3$ mm^2 unless stated otherwise.

Results have been obtained for meshes consisting of 60, 120 and 240 one-dimensional elements with a quadratic interpolation of the displacements and a linear interpolation of the damage. The load-displacement curves for these three discretisations are depicted in Figure 1 and show a rapid convergence towards physically meaningful solution in the sense that a finite energy dissipation is obtained. The return to the origin is a property of the theory, which can only be remedied by making the gradient parameter c a function of the damage evolution. Of particular interest are the damage and strain evolutions along the bar which have been plotted in Figures 2 and 3. A narrowing of the localisation zone is observed upon continued loading. This narrowing is more pronounced than for an averaging of the equivalent strain $\bar{\varepsilon}$, cf. Peerlings et al. [26]. Finally, the effect of the gradient parameter c is shown in Figure 4 and 5. A larger value of c evidently leads to a higher peak load and more energy dissipation as well as to a broadening of the localisation zone. For the smallest value of $c = 0.2$ mm^2 a non-smooth strain profile is observed, which is due to the discretisation which is too coarse for this relatively narrow localisation zone.

6. CONCLUDING REMARKS

A gradient-enhanced damage theory has been elaborated which features the Laplacian of the history parameter in addition to the history parameter itself. A full finite element algorithm has been given and the particular case of a linear relation between the history parameter and the damage variable has been discussed.

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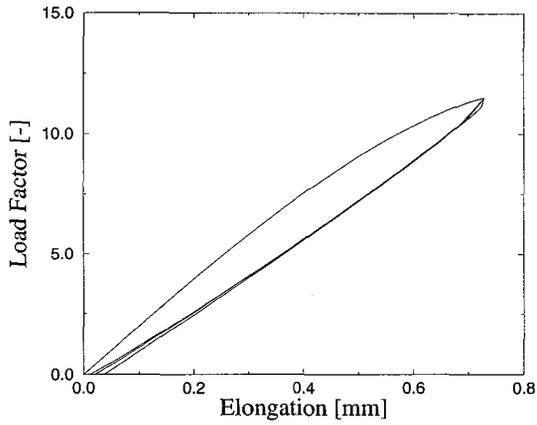
Figure 1. Load versus end displacement of the bar for three different discretisations.
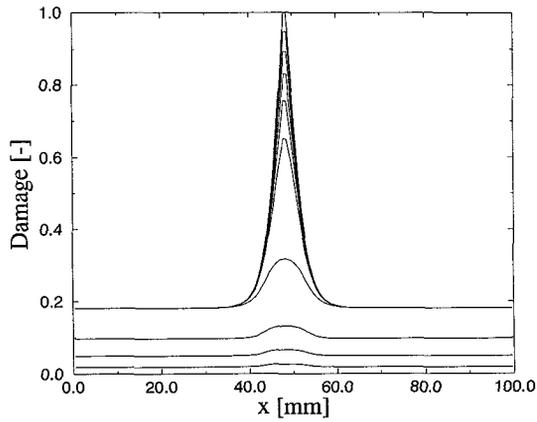

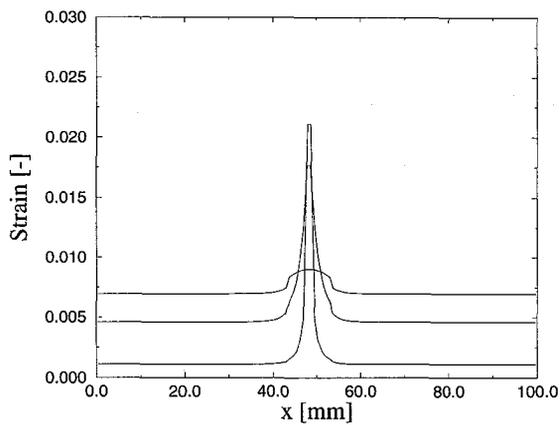Figure 2. Damage evolution along the bar for progressive loading.

Figure 3. Strain evolution along the bar for progressive loading.
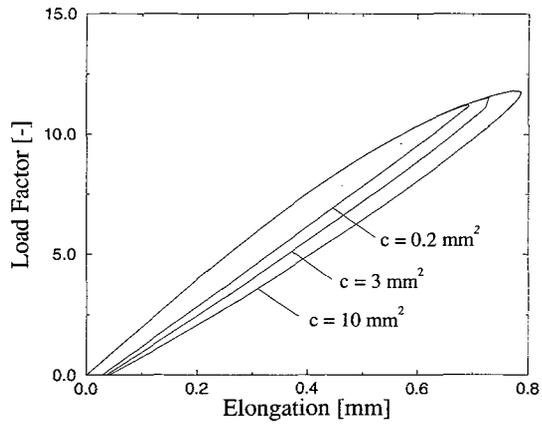
Figure 4. Load versus end displacement of the bar for different values of the gradient parameter.
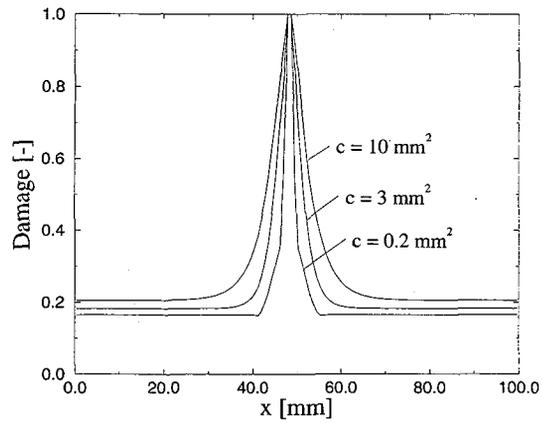

Figure 5. Strain distribution along the bar for different values of the gradient parameter.