



A Formal Solution for Wave Propagation in Rectangular Waveguide with an Inserted Nonlinear Dielectric Slab

Salvatore Caorsi, Andrea Massa, Matteo Pastorino

► To cite this version:

Salvatore Caorsi, Andrea Massa, Matteo Pastorino. A Formal Solution for Wave Propagation in Rectangular Waveguide with an Inserted Nonlinear Dielectric Slab. *Journal de Physique III*, 1995, 5 (1), pp.43-57. 10.1051/jp3:1995109 . jpa-00249294

HAL Id: jpa-00249294

<https://hal.science/jpa-00249294>

Submitted on 4 Feb 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification

Physics Abstracts

41.10H — 03.40K — 42.20G — 42.65B

A Formal Solution for Wave Propagation in Rectangular Waveguide with an Inserted Nonlinear Dielectric Slab

Salvatore Caorsi, Andrea Massa and Matteo Pastorino

Department of Biophysical and Electronic Engineering, University of Genoa, Via all'Opera Pia, 11A, I-16145 Genova, Italy

(Received 24 March 1994, revised 25 July 1994, accepted 20 October 1994)

Résumé. — Dans cet article nous présentons une solution formelle en série pour déterminer la distribution du champ électromagnétique dans un guide d'onde rectangulaire dans lequel une plaque diélectrique non linéaire a été insérée. La solution est développée en termes de la fonction de Green pour les guides d'ondes rectangulaires dont les parois sont parfaitement conductrices. Nous supposons que le guide d'onde est excité en mode TE_{10} à la fréquence f_0 . La méthode tient compte de la génération de modes supérieurs évoluant à la fréquence f_0 et à des fréquences différentes. Nous supposons aussi que la plaque non linéaire est isotropique et non homogène (même sans champ appliqué). L'article montre comment la solution du problème (c'est-à-dire le calcul des coefficients de la série) peut être réduite à la solution d'un système d'équations intégrales couplées. La solution formelle est dérivée pour un guide d'onde infini et un guide d'onde court-circuité.

Abstract. — A formal series solution is presented for the electromagnetic field distribution inside a rectangular waveguide in which a nonlinear dielectric slab has been inserted. The solution is developed in terms of the Green function for rectangular waveguides with perfectly conductive walls. We assume the waveguide to be excited in the TE_{10} mode at a frequency f_0 . The approach takes into account the higher-order mode generation at f_0 and at different frequencies. We also assume the nonlinear slab to be isotropic and inhomogeneous (even without any impressed field). The paper shows how the problem solution (i.e., the computation of the series coefficients) can be reduced to the solution of a system of coupled integral equations. The formal solution is derived for two cases: an infinite waveguide and a short-circuited waveguide.

1. Introduction

This paper deals with nonlinear electromagnetic wave propagation in a guiding structure. In recent years, wave propagation in nonlinear media has been extensively investigated. There exists a vast literature on this topic; we refer the reader, for example, to some works [1-4] and to the references cited therein. Considerable efforts have been devoted to studying partial differential equations for nonlinear propagation, to defining interesting phenomena, like soliton formation and decay [5], and shock waves [6], and to describing a large number of applications for which it is impossible to provide a complete list of references. Analytical,

variational and approximate methods have been devised and numerical techniques are becoming increasingly important. In the past, some numerical results in the field of nonlinear wave propagation have constituted a strong stimulus for further theoretical studies of very great interest.

However, nonlinear wave propagation is in general considered for propagation media of infinite extent, both in free space and inside guiding structures (conductive and dielectric waveguides, planar structures, etc.). Very few works have addressed the interaction of electromagnetic waves with nonlinear bodies of limited dimensions, the scattering from which had to be evaluated. In this paper, we discuss the case of a bounded nonlinear slab inserted in a rectangular waveguide. We assume the waveguide to be excited in its fundamental mode, the TE_{10} mode, at a frequency f_0 . The slab interfaces with air through cylindrical surfaces whose axes are parallel to the E -field polarization. The dielectric permittivity of the slab is assumed to be dependent on the internal electric field. In addition, we assume that the operator that links the nonlinear dielectric permittivity to the electric field vector is such as not to modify the scalar nature of the permittivity (consequently, it does not produce depolarization in the wave field). Moreover, the slab is inhomogeneous not only due to its nonlinearity, but even when no e.m. field is applied (i.e., the linear part of the relative dielectric permittivity itself is inhomogeneous). The remaining part of the waveguide is empty (or filled with a linear homogeneous dielectric). Nonmagnetic materials are assumed for all media and the conductive walls of the waveguide are assumed to be perfect conductors. In the following section, we develop a formal series solution for the electromagnetic waves inside the waveguide in terms of the Green function for rectangular waveguides. This solution takes into account the generation of direct and reflected higher-order modes at frequency f_0 and at harmonic frequencies. Then, we describe how the use of a specified nonlinear operator results in a numerical problem solution in which all the coefficients of the series expansion for the electromagnetic fields in the various waveguide regions are given as the solution of a system of nonlinear coupled integral equations. The case of a particular nonlinearity (whose highest order is proportional to the power) is detailed. Finally, the possibility of solving numerically the resulting system of integral equations is discussed. In the paper, the mathematical formulation of the approach is presented for the cases of an infinite waveguide and of a semi-infinite waveguide closed by a short circuit and loaded with the described nonlinear slab.

2. The Electromagnetic Field Problem

Let us consider Figure 1, in which three regions are shown. For these regions, the following mathematical relations for the electromagnetic quantities hold.

2.1. REGIONS I AND III. — Under the assumptions made in the Introduction, in region I ($z \leq z_1(x)$), the total time-dependent electric field vector $\mathbf{e}^{(1)}(\mathbf{r}, t)$ given by the sum of the incident and the reflected fields) is polarized along the y axis and is independent of the y coordinate : $\mathbf{e}^{(1)}(\mathbf{r}, t) = e_v^{(1)}(x, z, t) \mathbf{y}$.

As is well-known, $e_v^{(1)}(x, z, t)$ can be expressed as a Fourier series with a fundamental pulsation $\omega_0 = 2\pi f_0$: $e_v^{(1)}(x, z, t) = \sum_{n=-\infty}^{+\infty} E_{v_n}^{(1)}(x, z) e^{jn\omega_0 t}$ where $E_{v_n}^{(1)}(x, z)$, which satisfies :

$$\nabla_t^2 E_{v_n}^{(1)}(x, z) + \omega_n^2 \mu_0 \varepsilon_1 E_{v_n}^{(1)}(x, z) = 0 \quad (\omega_n = n\omega_0) \quad (1)$$

is given by :

$$E_{v_n}^{(1)}(x, z) = \sum_{m=1}^{+\infty} \sin \frac{m\pi}{a} x [h_{mn} e^{-\gamma_{mn}^{(1)} z} + A_{mn} e^{+\gamma_{mn}^{(1)} z}] \quad (2)$$

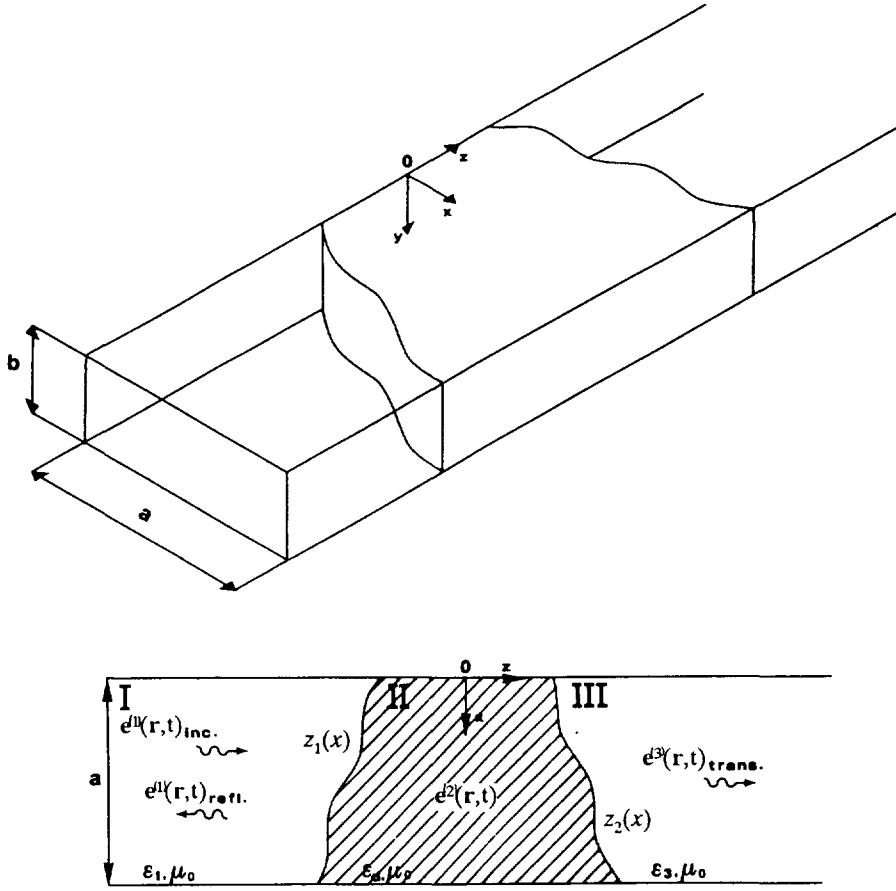


Fig. 1. — Rectangular waveguide and nonlinear dielectric slab.

where $h_{mn} = 0$ for $m \neq 1$ and $n \neq 1$, $h_{mn} = 1$ for $m = 1$ and $n = 1$ (we assume a unit amplitude for the fundamental incident mode), and

$$\gamma_{mn}^{(1)} = \sqrt{\left[\omega_n^2 \mu_0 \varepsilon_1 - \left(\frac{m\pi}{a} \right)^2 \right]}. \quad (3)$$

The corresponding vector component of the magnetic field is related to the electric field by the Maxwell equation and is given by :

$$\mathbf{H}_n^{(1)}(x, z) = \frac{1}{j \omega_n \mu_0} \left(\frac{\partial}{\partial z} E_n^{(1)}(x, z) \mathbf{x} - \frac{\partial}{\partial x} E_n^{(1)}(x, z) \mathbf{z} \right) \quad (4)$$

and its x component can be expressed as :

$$H_{x_n}^{(1)}(x, z) = \sum_{m=1}^{+\infty} -Y_{mn}^{(1)} \sin \frac{m\pi}{a} x [h_{mn} e^{-j\gamma_{mn}^{(1)} z} - A_{mn} e^{+j\gamma_{mn}^{(1)} z}] \quad (5)$$

where $Y_{mn}^{(1)}$ is the wave admittance equal to $\frac{\gamma_{mn}^{(1)}}{\omega_n \mu_0}$. In region III ($z \geq z_2(x)$), the total electric field vector $\mathbf{e}^{(3)}(\mathbf{r}, t)$ (given by the transmitted field) is still polarized in the y direction : $\mathbf{e}^{(3)}(\mathbf{r}, t) = e_y^{(3)}(x, z, t) \mathbf{y}$. By analogy to the field in region I, we can expand $e_y^{(3)}(x, z, t)$ in Fourier series whose n -th term satisfies :

$$\nabla_t^2 E_{y_n}^{(3)}(x, z) + \omega_n^2 \mu_0 \varepsilon_3 E_{y_n}^{(3)}(x, z, t) = 0 \quad (6)$$

and is given by :

$$E_{y_n}^{(3)}(x, z) = \sum_{m=1}^{+\infty} D_{mn} \sin \frac{m\pi}{a} x e^{-j\gamma_{mn}^{(3)} z} \quad (7)$$

where

$$\gamma_{mn}^{(3)} = \sqrt{\left[\omega_n^2 \mu_0 \varepsilon_3 - \left(\frac{m\pi}{a} \right)^2 \right]}. \quad (8)$$

The magnetic field vector is given by a relation corresponding to (4) (valid for region I), and its x component can be expressed as :

$$H_{x_n}^{(3)}(x, z) = \sum_{m=1}^{+\infty} -Y_{mn}^{(3)} D_{mn} \sin \frac{m\pi}{a} x e^{-j\gamma_{mn}^{(3)} z} \quad (9)$$

where

$$Y_{mn}^{(3)} = \frac{\gamma_{mn}^{(3)}}{\omega_n \mu_0}$$

2.2. REGION II. — Under the hypothesis that the total electric field, $\mathbf{e}^{(2)}(\mathbf{r}, t)$, is still y -polarized ($\mathbf{e}^{(2)}(\mathbf{r}, t) = e_y^{(2)}(x, z, t) \mathbf{y}$), in region II ($z_1(x) \leq z \leq z_2(x)$) (i.e., the nonlinear slab), the following homogeneous wave equation holds :

$$\nabla_t^2 e_y^{(2)}(x, z, t) - \mu_0 \frac{\partial^2}{\partial t^2} \varepsilon_2(x, z, t) e_y^{(2)}(x, z, t) = 0 \quad (10)$$

where $\varepsilon_2(x, z, t)$ is given by :

$$\varepsilon_2(x, z, t) = \varepsilon_0 [\varepsilon_{21}(x, z) + \varepsilon_{22} O \{ \mathbf{e}^{(2)}(x, z, t) \}] \quad (11)$$

where $\varepsilon_{21}(x, z)$ is the linear part of the relative dielectric permittivity, which can be inhomogeneous itself (when no field is applied) ; $O \{ \mathbf{e}^{(2)}(x, z, t) \}$ is a nonlinear operator and ε_{22} is a constant parameter. The nonlinear operator is assumed to fulfil the constraint of not modifying the scalar nature of the dielectric permittivity (isotropic medium). We also assume $O \{ \mathbf{e}^{(2)}(x, z, t) \}$ to be a time-periodic function. Then, we can write : $e_y^{(2)}(x, z, t) = \sum_{n=-\infty}^{+\infty} E_{y_n}^{(2)}(x, z) e^{jn\omega_0 t}$ and $O \{ \mathbf{e}^{(2)}(x, z, t) \} = \sum_{n=-\infty}^{+\infty} O_n(x, z) e^{jn\omega_0 t}$. The product of these quantities can also be written in Fourier series as :

$$\sum_{n=-\infty}^{+\infty} E_{y_n}^{(2)}(x, z) e^{jn\omega_0 t} \sum_{m=-\infty}^{+\infty} O_m(x, z) e^{jm\omega_0 t} = \sum_{h=-\infty}^{+\infty} \Psi_h(x, z) e^{jh\omega_0 t} \quad (12)$$

where:

$$\Psi_h(x, z) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} E_{v_n}^{(2)}(x, z) O_m(x, z) \delta_{mn}^h \quad (13)$$

where $\delta_{mn}^h = 1$, if $m + n = h$, and $\delta_{mn}^h = 0$, otherwise. Now, for each frequency $\omega_n = n\omega_0$, relation (10) can be rewritten as :

$$\nabla_t^2 E_{v_n}^{(2)}(x, z) + \omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{21}(x, z) E_{v_n}^{(2)}(x, z) + \omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{22} \Psi_n(x, z) = 0 \quad (14)$$

and the magnetic field vector becomes :

$$\mathbf{H}_n^{(2)}(x, z) = \frac{1}{j\omega_n \mu_0} \left(\frac{\partial}{\partial z} E_{v_n}^{(2)}(x, z) \mathbf{x} - \frac{\partial}{\partial x} E_{v_n}^{(2)}(x, z) \mathbf{z} \right). \quad (15)$$

The boundary conditions for region II are :

$$\mathbf{n} \times \mathbf{e}^{(2)}(x, z, t) \Big|_{z=z_1(x)} = \mathbf{n} \times \mathbf{e}^{(1)}(x, z, t) \Big|_{z=z_1(x)} \quad (16)$$

$$\mathbf{n} \times \mathbf{e}^{(2)}(x, z, t) \Big|_{z=z_2(x)} = \mathbf{n} \times \mathbf{e}^{(3)}(x, z, t) \Big|_{z=z_2(x)} \quad (17)$$

$$\mathbf{n} \times \mathbf{h}^{(2)}(x, z, t) \Big|_{z=z_1(x)} = \mathbf{n} \times \mathbf{h}^{(1)}(x, z, t) \Big|_{z=z_1(x)} \quad (18)$$

$$\mathbf{n} \times \mathbf{h}^{(2)}(x, z, t) \Big|_{z=z_2(x)} = \mathbf{n} \times \mathbf{h}^{(3)}(x, z, t) \Big|_{z=z_2(x)}. \quad (19)$$

For each n , the formal solution of equation (14) can be obtained by considering the term $-\omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{22} \Psi_n(x, z)$ as an equivalent source term, according to the equivalence principle for electromagnetic fields [7] :

$$\nabla_t^2 E_{v_n}^{(2)}(x, z) + \omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{21}(x, z) E_{v_n}^{(2)}(x, z) = -\omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{22} \Psi_n(x, z). \quad (20)$$

It should be noted that the terms $\Psi_n(x, z)$ generally depend on the mixing of all the field components, $E_{v_m}^{(2)}(x, z)$, for any m . Under the above conditions, we express the field solution as : $E_{v_n}^{(2)} = E_{v_n}^{(2)(0)} + E_{v_n}^{(2)(p)}$ where $E_{v_n}^{(2)(0)}$ denotes the solution of the homogeneous wave equation (corresponding to (20)) in region II (with $\varepsilon(x, z) = \varepsilon_0 \varepsilon_{21}(x, z)$) and $E_{v_n}^{(2)(p)}$ is a particular solution (dependent on $\Psi_n(x, z)$) that can be expressed in terms of the Green function for rectangular waveguides :

$$E_{v_n}^{(2)(p)} = \omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{22} \int_0^a \int_{z_1(x)}^{z_2(x)} \Psi_n(x', z') G_n(x, z/x', z') dx' dz' \quad (21)$$

where $G_n(x, z/x', z')$ is given by [8] :

$$G_n(x, z/x', z') = - \sum_{m=1}^{+\infty} \frac{1}{\gamma_{mn}^{(2)}} \sin \frac{m\pi}{a} x \sin \frac{m\pi}{a} x' e^{-j\gamma_{mn}^{(2)} |z - z'|} \quad (22)$$

where $\gamma_{mn}^{(2)} = \sqrt{\left[\omega_n^2 \mu_0 \varepsilon_0 \varepsilon_{21} - \left(\frac{m\pi}{a} \right)^2 \right]}$. It should be stressed that, unlike in the linear case, this is just a formal solution, as in relation (21), for any n , $E_{v_n}^{(2)(p)}$ depends (through

$\Psi_n(x, z)$ on all the field components, $E_{y_j}^{(2)}$ (total field), for any j , including $j = n$. Now, since the solution for $E_{y_n}^{(2)(0)}$ is given by :

$$E_{y_n}^{(2)(0)}(x, z) = \sum_{m=1}^{+\infty} \sin \frac{m\pi}{a} x [B_{mn}^{(0)} e^{-j\gamma_{mn}^{(2)} z} + C_{mn}^{(0)} e^{+j\gamma_{mn}^{(2)} z}] \quad (23)$$

the electric field in region II can be formally expressed as :

$$E_{y_n}^{(2)}(x, z) = \sum_{m=1}^{+\infty} \sin \frac{m\pi}{a} x \{ [B_{mn}^{(0)} + B_{mn}(z)] e^{-j\gamma_{mn}^{(2)} z} + [C_{mn}^{(0)} + C_{mn}(z)] e^{+j\gamma_{mn}^{(2)} z} \} \quad (24)$$

where, of course, $B_{mn}(z)$ and $C_{mn}(z)$ are unknown coefficients still dependent on $E_{y_n}^{(2)}(x, z)$ and, in general, on all the other field components. Such coefficients can be defined as :

$$B_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_{z_1(x)}^z \Psi_n(x', z') \sin \frac{m\pi}{a} x' e^{+j\gamma_{mn}^{(2)} z'} dx' dz' \quad (25)$$

$$C_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_z^{z_2(x)} \Psi_n(x', z') \sin \frac{m\pi}{a} x' e^{+j\gamma_{mn}^{(2)} z'} dx' dz' \quad (26)$$

Moreover, the x components of the magnetic field are given by :

$$H_{x_n}^{(2)}(x, z) = \sum_{m=1}^{+\infty} -Y_{mn}^{(2)} \sin \frac{m\pi}{a} x \left\{ [B_{mn}^{(0)} + B_{mn}(z)] e^{-j\gamma_{mn}^{(2)} z} + j \frac{1}{\gamma_{mn}^2} \frac{\delta}{\delta z} B_{mn}(z) - [C_{mn}^{(0)} + C_{mn}(z)] e^{+j\gamma_{mn}^{(2)} z} - j \frac{1}{\gamma_{mn}^2} \frac{\delta}{\delta z} C_{mn}(z) \right\} \quad (27)$$

where

$$Y_{mn}^{(2)} = \frac{\gamma_{mn}^{(2)}}{\omega_n \mu_0} \quad (28)$$

From relation (24), one can deduce again that the harmonic components depend on the others *via* the nonlinear coefficients $\Psi_n(x, z)$. These coefficients are responsible for the mixing of modes (at f_0 and at different frequencies), which, in a linear medium, propagate without interactions (if somewhere generated). They can be derived directly from relation (13), once the nonlinear operator $O \{e^{(2)}(x, z, t)\}$ in (11) has been defined.

3. Parallelepipedal Dielectric Slab

In order to show how to handle equations (24-28), we consider the simple case in which the inserted nonlinear slab is a parallelepiped. We set : $z_1(x) = 0$ and $z_2(x) = z_0$ (Fig. 2a). The coupling coefficients, $B_{mn}(z)$ and $C_{mn}(z)$ of the nonlinear slab become :

$$B_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_0^z \Psi_n(x', z') \sin \frac{m\pi}{a} x' e^{+j\gamma_{mn}^{(2)} z'} dx' dz' \quad (29)$$

$$C_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_z^{z_0} \Psi_n(x', z') \sin \frac{m\pi}{a} x' e^{+j\gamma_{mn}^{(2)} z'} dx' dz' \quad (30)$$

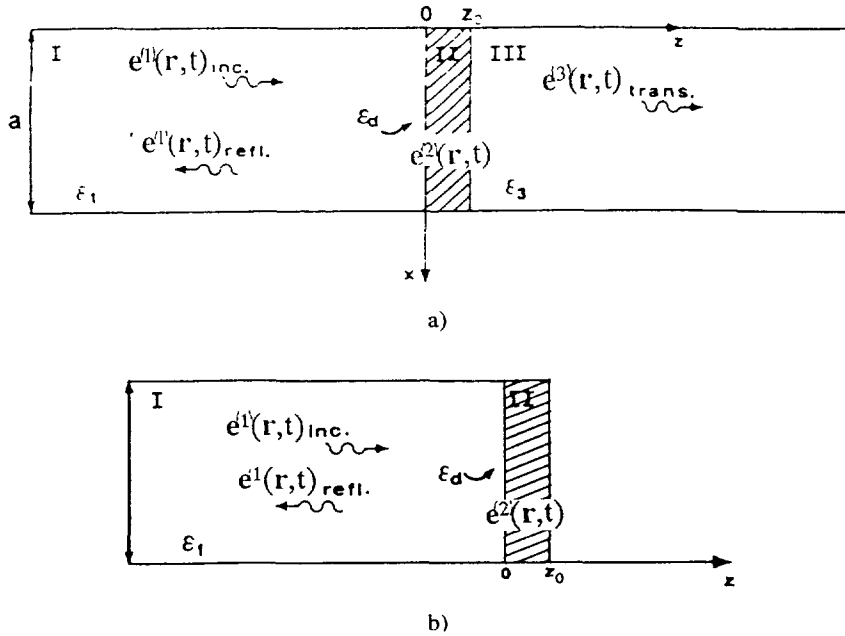


Fig. 2. — a) Infinite waveguide containing a nonlinear slab parallelepipedal in shape. b) Waveguide terminated by an ideal short circuit.

Two cases are described. In the first case, the waveguide is assumed to be infinite (it is of course equivalent to a waveguide terminated by a matched load (at all frequencies) ; in the second case, we assume the waveguide to be short-circuited (i.e., terminated by a perfectly conductive wall).

3.1. INFINITE WAVEGUIDE. — By imposing, at the interfaces $z_1(x) = 0$ and $z_2(x) = z_0$, the boundary conditions, we obtain the explicit dependence of each linear coefficient A_{mn} , $B_{mn}^{(0)}$, $C_{mn}^{(0)}$, and D_{mn} (relations (2), (7), and (23)) on the nonlinear coefficients $B_{mn}(z)$ and $C_{mn}(z)$ (all still unknown), calculated for $z_1(x) = 0$ and $z_2(x) = z_0$. In particular, for each n and m , a system of equations is obtained by applying the boundary conditions. Considering the linear coefficients of this system of equations as unknowns and the nonlinear coefficients as unknown terms, for each n and m , the system turns out to be made up of four equations and four unknowns. The explicit solution of these systems (e.g., by the Cramer method) gives the following relations (the related trivial steps are omitted) :

$$A_{11} = \frac{2 \gamma_{11}^2}{\det [S_{11}]} \gamma_{11}^{(1)} [C_{11}(0) (\gamma_{11}^{(1)} + \gamma_{11}^{(2)}) e^{j\gamma_{11}^{(2)} z_0} + B_{11}(z_0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{-j\gamma_{11}^{(2)} z_0} - (\gamma_{11}^{(1)} - \gamma_{11}^{(2)}) (e^{-j\gamma_{11}^{(2)} z_0} - e^{j\gamma_{11}^{(2)} z_0})] \quad (31)$$

$$B_{11}^{(0)} = \frac{1}{\det [S_{11}]} e^{-j\gamma_{11}^{(1)} z_0} \{ C_{11}(0) [(\gamma_{11}^{(2)})^2 - (\gamma_{11}^{(1)})^2] e^{j\gamma_{11}^{(2)} z_0} + B_{11}(z_0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)})^2 e^{-j\gamma_{11}^{(2)} z_0} - 2 \gamma_{11}^{(1)} (\gamma_{11}^{(2)} + \gamma_{11}^{(1)}) e^{j\gamma_{11}^{(2)} z_0} \} \quad (32)$$

$$C_{11}^{(0)} = \frac{1}{\det [S_{11}]} e^{-J \gamma_{11}^{(1)} z_0} \left\{ B_{11}(0) [(\gamma_{11}^{(1)})^2 - (\gamma_{11}^{(2)})^2] e^{-J \gamma_{11}^{(2)} z_0} + C_{11}(z_0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)})^2 e^{-J \gamma_{11}^{(2)} z_0} + 2 \gamma_{11}^{(1)} (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{-J \gamma_{11}^{(2)} z_0} \right\} \quad (33)$$

$$D_{11} = -\frac{2}{\det [S_{11}]} \gamma_{11}^{(2)} [B_{11}(0) (\gamma_{11}^{(1)} + \gamma_{11}^{(2)}) + C_{11}(z_0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) + 2 \gamma_{11}^{(1)}] \quad (34)$$

$$A_{mn} = \frac{2 \gamma_{mn}^{(2)}}{\det [S_{mn}]} e^{-J \gamma_{mn}^{(1)} z_0} [C_{mn}(0) (\gamma_{mn}^{(1)} + \gamma_{mn}^{(2)}) e^{J \gamma_{mn}^{(2)} z_0} + B_{mn}(z_0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)}) e^{-J \gamma_{mn}^{(2)} z_0} \quad (35)$$

$$B_{mn}^{(0)} = \frac{1}{\det [S_{mn}]} e^{-J \gamma_{mn}^{(1)} z_0} \left\{ C_{mn}(0) [(\gamma_{mn}^{(2)})^2 - (\gamma_{mn}^{(1)})^2] e^{J \gamma_{mn}^{(2)} z_0} + B_{mn}(z_0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)})^2 e^{-J \gamma_{mn}^{(2)} z_0} \right\} \quad (36)$$

$$C_{mn}^{(0)} = \frac{1}{\det [S_{mn}]} e^{-J \gamma_{mn}^{(1)} z_0} \left\{ B_{mn}(0) [(\gamma_{mn}^{(2)})^2 - (\gamma_{mn}^{(1)})^2] e^{J \gamma_{mn}^{(2)} z_0} + C_{mn}(z_0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)})^2 e^{-J \gamma_{mn}^{(2)} z_0} \right\} \quad (37)$$

$$D_{mn} = -\frac{2}{\det [S_{mn}]} \gamma_{mn}^{(2)} [B_{mn}(0) (\gamma_{mn}^{(1)} + \gamma_{mn}^{(2)}) + C_{mn}(z_0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)})] \quad (38)$$

where $\det [S_{11}]$ and $\det [S_{mn}]$ are given by :

$$\det [S_{11}] = e^{-J \gamma_{11}^{(1)} z_0} [(\gamma_{11}^{(1)} + \gamma_{11}^{(2)})^2 e^{J \gamma_{11}^{(2)} z_0} - (\gamma_{11}^{(1)} - \gamma_{11}^{(2)})^2 e^{-J \gamma_{11}^{(2)} z_0}] \quad (39)$$

$$\det [S_{mn}] = e^{-J \gamma_{mn}^{(1)} z_0} [(\gamma_{mn}^{(1)} + \gamma_{mn}^{(2)})^2 e^{J \gamma_{mn}^{(2)} z_0} - (\gamma_{mn}^{(1)} - \gamma_{mn}^{(2)})^2 e^{-J \gamma_{mn}^{(2)} z_0}] \quad (40)$$

3.2. SHORT-CIRCUITED WAVEGUIDE. — In the case where the waveguide is terminated by a short circuit, we assume $z \leq z_0$, with the additional boundary conditions (Fig. 2b) :

$$\mathbf{n} \times \mathbf{e}^{(2)}(x, z, t) = \mathbf{0} \quad \text{for } z = z_0. \quad (41)$$

As in the previous case, the application of the boundary conditions gives the relations between the linear and nonlinear coefficients :

$$A_{11} = -\frac{2 \gamma_{11}^{(2)}}{\det [S_{11}]} \gamma_{11}^{(1)} [C_{11}(0) \gamma_{11}^{(2)} e^{J \gamma_{11}^{(2)} z_0} - B_{11}(z_0) \gamma_{11}^{(2)} e^{-J \gamma_{11}^{(2)} z_0} - (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{J \gamma_{11}^{(2)} z_0} - (\gamma_{11}^{(2)} + \gamma_{11}^{(1)}) e^{J \gamma_{11}^{(2)} z_0}] \quad (42)$$

$$B_{11}^{(0)} = \frac{1}{\det [S_{11}]} [C_{11}(0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{J \gamma_{11}^{(2)} z_0} + B_{11}(z_0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{-J \gamma_{11}^{(2)} z_0} + 2 \gamma_{11}^{(1)} e^{J \gamma_{11}^{(2)} z_0}] \quad (43)$$

$$C_{11}^{(0)} = -\frac{1}{\det [S_{11}]} [B_{11}(z_0) (\gamma_{11}^{(2)} + \gamma_{11}^{(1)}) e^{-J \gamma_{11}^{(2)} z_0} + C_{11}(0) (\gamma_{11}^{(2)} - \gamma_{11}^{(1)})^2 e^{-J \gamma_{11}^{(2)} z_0} + 2 \gamma_{11}^{(1)} e^{-J \gamma_{11}^{(2)} z_0}] \quad (44)$$

$$A_{mn} = -\frac{2 \gamma_{mn}^{(2)}}{\det [S_{mn}]} [C_{mn}(0) \gamma_{mn}^{(2)} e^{J \gamma_{mn}^{(2)} z_0} - B_{mn}(z_0) e^{-J \gamma_{mn}^{(2)} z_0}] \quad (45)$$

$$B_{mn}^{(0)} = \frac{1}{\det [S_{mn}]} [C_{mn}(0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)}) e^{J \gamma_{mn}^{(2)} z_0} + B_{mn}(z_0) (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)}) e^{-J \gamma_{mn}^{(2)} z_0}] \quad (46)$$

$$C_{mn}^{(0)} = \frac{1}{\det [S_{mn}]} [B_{mn}(z_0) (\gamma_{mn}^{(2)} + \gamma_{mn}^{(1)}) e^{-J \gamma_{mn}^{(2)} z_0} - C_{mn}(0) (\gamma_{mn}^{(1)} - \gamma_{mn}^{(2)}) e^{-J \gamma_{mn}^{(2)} z_0}] \quad (47)$$

where

$$\det [S_{11}] = (\gamma_{11}^{(1)} + \gamma_{11}^{(2)}) e^{J \gamma_{11}^{(2)} z_0} + (\gamma_{11}^{(2)} - \gamma_{11}^{(1)}) e^{-J \gamma_{11}^{(2)} z_0} \quad (48)$$

$$\det [S_{mn}] = (\gamma_{mn}^{(1)} + \gamma_{mn}^{(2)}) e^{J \gamma_{mn}^{(2)} z_0} + (\gamma_{mn}^{(2)} - \gamma_{mn}^{(1)}) e^{-J \gamma_{mn}^{(2)} z_0} \quad (49)$$

4. Formal Series Solutions

The formal series solutions can now be completed in both cases. For the infinite waveguide, relations (31-38) express the linear coefficients A_{mn} , $B_{mn}^{(0)}$, $C_{mn}^{(0)}$, D_{mn} as functions of the unknown nonlinear coefficients $B_{mn}(z)$ and $C_{mn}(z)$, calculated for $z_1(x) = 0$ and $z_2(x) = z_0$. By substituting relations (31-38) into (24), we can express $E_{v_n}^{(2)}(x, z)$ only as a function of $B_{mn}(0)$, $B_{mn}(z_0)$, $C_{mn}(0)$, and $C_{mn}(z_0)$. If the nonlinear operator $O \{e^{(2)}(\mathbf{r}, t)\}$ is specified, from relation (13) we derive the term $\Psi_n(x, z)$ as a function of $B_{mn}(z)$ and $C_{mn}(z)$. Finally, substitution into (25) and (26) reduces the problem to the solution of the following system of nonlinear integral equations :

$$B_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_0^z \Psi_n(B_{mn}(z'), C_{mn}(z'), x', z') \times \\ \times \sin \frac{m\pi}{a} x' e^{+J \gamma_{mn}^{(2)} z'} dx' dz' \quad (50)$$

$$C_{mn}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{mn}^{(2)})^{-1} \varepsilon_{22} \int_0^a \int_z^{z_0} \Psi_n(B_{mn}(z'), C_{mn}(z'), x', z') \times \\ \times \sin \frac{m\pi}{a} x' e^{+J \gamma_{mn}^{(2)} z'} dx' dz' \quad (51)$$

where the dependence of $\Psi_n(x, z)$ on $B_{mn}(z)$ and $C_{mn}(z)$ is indicated.

For the short-circuited waveguide, the same results can be obtained by considering relations (42-47), by substituting into (24), and by using (25) and (26). Relations formally equal to (50) and (51) can then be derived.

5. Choice of the Nonlinearity

In this section, under the assumptions made in Section 1, we consider a particular choice for the nonlinear operator (Eq. (11)), which is assumed to be given by :

$$O \{e^{(2)}(x, z, t)\} = \theta_2 (e_{v_n}^{(2)}(x, z, t))^2 + \theta_1 e_{v_n}^{(2)}(x, z, t) \quad (52)$$

where θ_1 and θ_2 are known constants. Relation (52) corresponds to a nonlinearity for the relative dielectric permittivity truncated at the second-order term (dependence on the field

power) :

$$\varepsilon_2(x, z, t) = \varepsilon_0 \left\{ \varepsilon_{21}(x, z) + \varepsilon_{22} [\theta_2 (e_y^{(2)}(x, z, t))^2 + \theta_1 e_v^{(2)}(x, z, t)] \right\}. \quad (53)$$

At this point, we can give expressions for the nonlinear terms $\Psi_n(B_{mn}(z'), C_{mn}(z'), x', z')$ included in (50) and (51). To simplify the notation, we assume the simple case in which $M = 1$ and $N = 2$ (more complex cases would complicate the formalism). Under all the assumptions made and on the basis of (13) and (52), relations (50) and (51) can be easily made explicit by direct substitution and, after performing a direct integration with respect to the variable x' , we obtain :

$$B_{11}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{11}^{(2)})^{-1} \varepsilon_{22} \int_0^z a [3/2 \theta_2 \xi_1(z') |\xi_2(z')|^2 + 9/8 \theta_2 \xi_1(z') |\xi_1(z')|^2 - 8/3 \theta_1 \xi_1^*(z') \xi_2(z')] e^{+j\gamma_{11}^{(2)} z'} dz' \quad (54)$$

$$B_{12}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{12}^{(2)})^{-1} \varepsilon_{22} \int_0^z a [3/2 \theta_2 \xi_2(z') |\xi_1(z')|^2 + 3/4 \theta_2 \xi_2(z') |\xi_2(z')|^2 + 4/3 \theta_1 \xi_1(z')^2] e^{+j\gamma_{12}^{(2)} z'} dz' \quad (55)$$

$$C_{11}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{11}^{(2)})^{-1} \varepsilon_{22} \int_0^z a [3/2 \theta_2 \xi_1(z') |\xi_2(z')|^2 + 9/8 \theta_2 \xi_1(z') |\xi_1(z')|^2 + 8/3 \theta_1 \xi_1^*(z') \xi_2(z')] e^{-j\gamma_{11}^{(2)} z'} dz' \quad (56)$$

$$C_{12}(z) = -\omega_0^2 \mu_0 \varepsilon_0 (\gamma_{12}^{(2)})^{-1} \varepsilon_{22} \int_0^z a [3/2 \theta_2 \xi_2(z') |\xi_1(z')|^2 + 3/4 \theta_2 \xi_2(z') |\xi_2(z')|^2 + 4/3 \theta_1 \xi_1(z')^2] e^{-j\gamma_{12}^{(2)} z'} dz' \quad (57)$$

where:

$$\xi_1(z) = \frac{E_{y1}^{(2)}(x, z)}{\sin \frac{\pi}{a} x} \quad (58)$$

$$\xi_2(z) = \frac{E_{v2}^{(2)}(x, z)}{\sin \frac{\pi}{a} x} \quad (59)$$

where $E_{y1}^{(2)}(x, z)$ and $E_{v2}^{(2)}(x, z)$ are given by (24).

6. Numerical Results and Discussion

In both cases considered in Sections 3.1 and 3.2, respectively, the system solutions allow in principle the determination of the coefficients $B_{mn}(z)$ and $C_{mn}(z)$. Since no analytical solutions can be derived for the sets of nonlinear integral equations (50, 51), a numerical procedure must be applied. A preliminary example is now given. We considered a parallelepipedal dielectric slab inserted in an infinite waveguide (Eqs. in Sect. 3.1). We assumed $z_0 = 0.6 \times 10^{-2}$ m, and the transversal dimensions of the waveguide (excited in the TE₁₀ mode) were $a = 2.286 \times 10^{-2}$ m, $b = a/2$. The slab exhibited a nonlinearity given by (52), and the operating frequency (fundamental frequency) was chosen equal to 10 GHz.

Under the simplifying assumptions made in Section 5, the system of equations to be solved was given by relations (54-57). The involved integrals were partitioned along the z axis under the assumption of uniform partition steps. In particular, we considered $P = 6$ partitions, for which $\Delta z = z_0/P = 0.1 \times 10^{-2}$ m. Then, we assumed the unknown coefficients to be constant in each partition. This allowed equations (54-57) to be reduced to a nonlinear system of 24 algebraic equations with 24 unknowns; the system can be written in compact form as :

$$\begin{aligned} & \underline{E} \{B_{11}(1), B_{11}(2), \dots, B_{11}(P), B_{12}(1), B_{12}(2), \dots, B_{12}(P), \\ & C_{11}(1), C_{11}(2), \dots, C_{11}(P), C_{12}(1), C_{12}(2), \dots, C_{11}(P)\} = 0 \quad (60) \end{aligned}$$

where $B_{ij}(p)$ and $C_{ij}(p)$ denote the unknown values of the coefficients $B_{ij}(z)$ and $C_{ij}(z)$ inside the p -th partition, characterized by $(p-1)\Delta z \leq z \leq p\Delta z$. In order to obtain a solution for this preliminary example, Wolfe's method [9] was used, which constitutes a generalization of the secant method. This method is an iterative method that required that 25 starting arrays of 24 elements (randomly generated) be fixed. The computation was considered completed when, at a k^* iteration, the residual square norm of relation (60) was less than 10^{-4} . However, in general, if an accurate field computation is required and a large number of modes and many frequency components are assumed, systems analogous to (54-57) can be obtained, which turn out to consist of many equations with many unknowns. Algorithms for global optimization, like simulated annealing and genetic algorithms (which are currently of growing interest to the scientific community), seem particularly suited to solving such complex problems with many possible local minima [10, 11].

The results of the above example are given in Figures 3-7 : we assumed $\theta_2 = 0.01$ and θ_1 was made to vary between 0 and 0.1. Moreover, we assumed $\varepsilon_{21}(x, z) = \varepsilon_{21} = 3$ (higher values of the linear part of the dielectric permittivity tend to mask the effect of the nonlinearity). In particular, Figure 3 gives the amplitude values of the coefficients of the reflected waves in region I, A_{11} and A_{12} , for different values of θ_1 . Figures 4-6 illustrate the

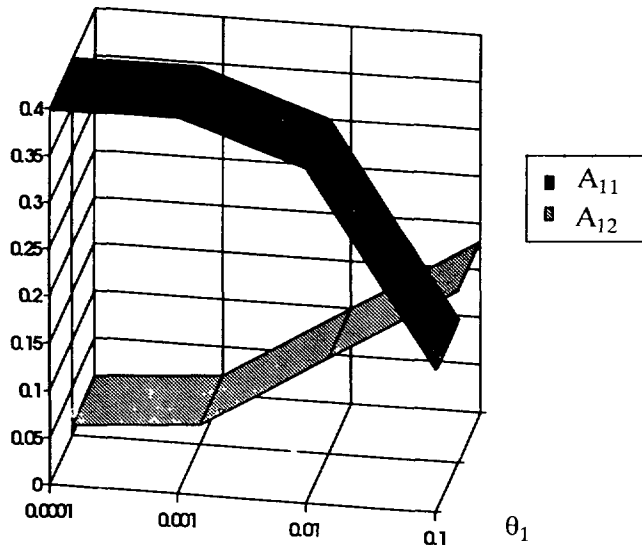


Fig. 3. — Amplitudes of the coefficients of the reflected waves in region I : a) A_{11} , b) A_{12} , for some values of the nonlinear parameter θ_1 .

field distribution inside the nonlinear slab (region II). Figure 4 gives the values of the coefficients B_{11} and B_{12} versus θ_1 , computed for each partition p , $p = 1, \dots, 6$. Figure 5 gives the analogous values for the coefficients C_{11} and C_{12} . Finally, Figure 6 gives the amplitudes of the total electric fields $E_{v_1}^{(2)}(x, z)$ and $E_{v_2}^{(2)}(x, z)$, computed by the obtained nonlinear

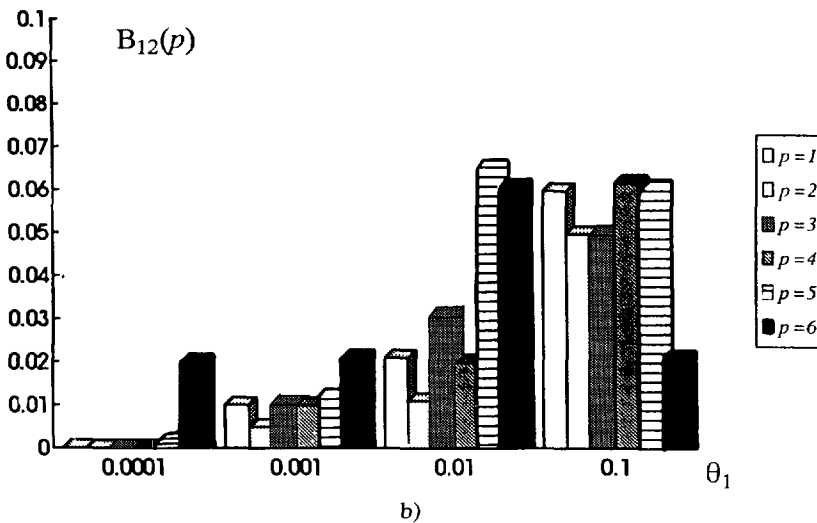
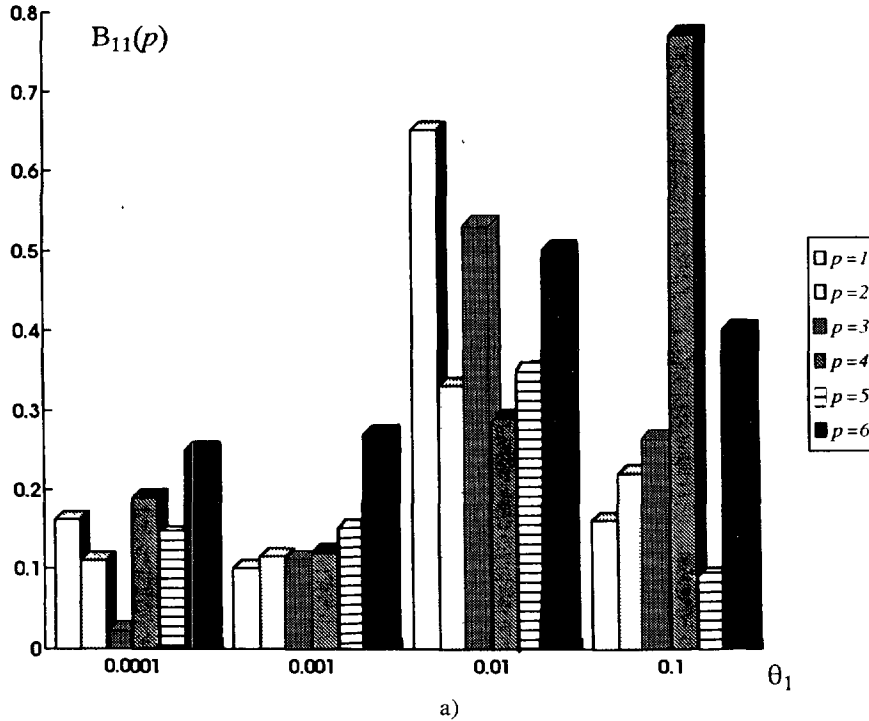
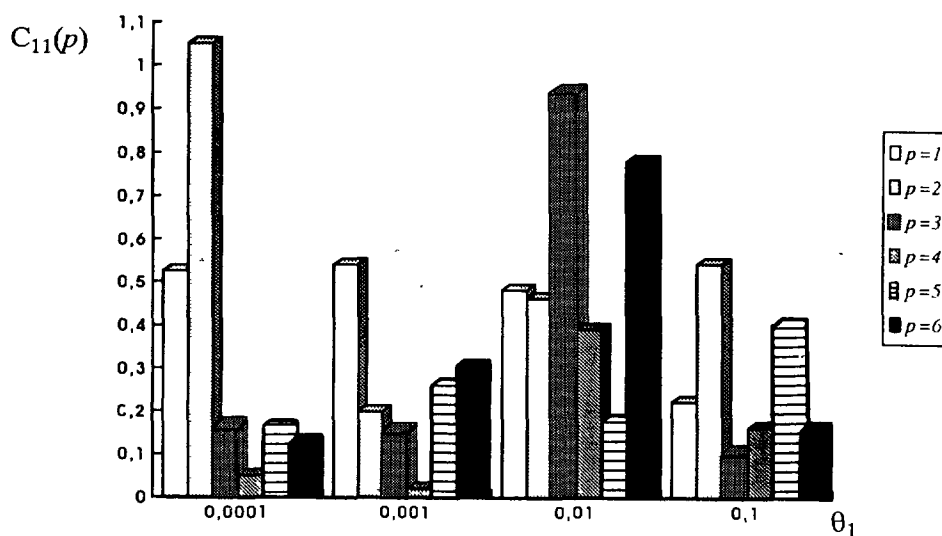
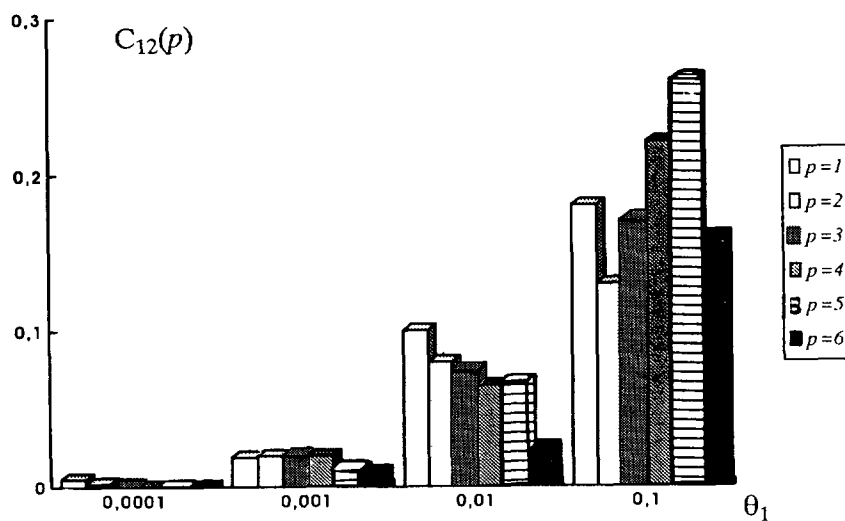


Fig. 4. — Amplitudes of the nonlinear coefficients of the direct waves computed in the partitions of region II ($p = 1, \dots, P$; $P = 6$): a) B_{11} , b) B_{12} , for some values of the nonlinear parameter θ_1 .



a)



b)

Fig. 5. — Amplitudes of the nonlinear coefficients of the direct waves computed in the partitions of region II ($p = 1, \dots, P$; $P = 6$): a) C_{11} , b) C_{12} , for some values of the nonlinear parameter θ_1 .

coefficients, for $x = a/2$ and for a nonlinear parameter θ_1 equal to 0.1. From (2), it follows that the field is continuous across the interface $z = 0$, as required by (16). The same conclusion holds for $z = z_0$.

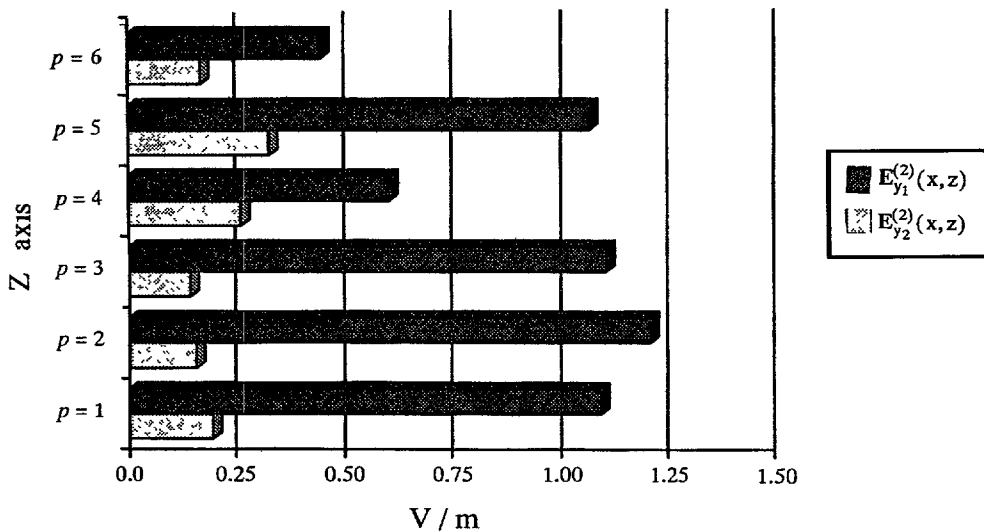


Fig. 6. — Amplitudes of the total electric fields, $E_{y1}^{(2)}(x, z)$ and $E_{y2}^{(2)}(x, z)$, computed in the partitions of region II ($p = 1, \dots, P, P = 6$).

7. Conclusions

In this paper, we have proposed a formal series solution for the wave propagation in a rectangular waveguide (excited in its dominant mode) containing a nonlinear dielectric slab of finite size. The solution is based on an integral-equation formalism in terms of the Green function for rectangular waveguides. We have shown that, if the nonlinear operator is expressed in its explicit form, determining the coefficients of the series solution for the electric field distribution inside all the waveguide regions reduces to solving a system of nonlinear integral equations. We have described the formal solution by assuming a slab of specific shape and a particular nonlinearity whose highest term was power dependent. Two cases have been considered : an infinite waveguide and a short-circuited waveguide. In both cases, the series solutions have taken into account the mode (transmitted and reflected) generation and the harmonic production in the various waveguide regions.

In the paper, emphasis has been placed on the mathematical formulation of the approach for the particular configurations considered and on the reduction of the problem to one suitable for a numerical treatment. However, a preliminary numerical example has been developed and the obtained results reported. The authors are currently starting a wider numerical analysis, which is necessary for both an evaluation of the effectiveness of the proposed approach and a physical interpretation of the solutions achieved.

References

- [1] Broer L. J. F., Wave propagation in nonlinear media, *ZAMP* **16** (1965) 18-26.
- [2] Varley E. and Cumberbatch E., Nonlinear theory of wave-front propagation, *J. Inst. Math. Appl.* **1** (1965) 101-112.

- [3] Whitham G. B., *Linear and Nonlinear Waves* (New York, Wiley-Interscience, 1974).
- [4] Karpman V. I., *Nonlinear Waves in Dispersive Media* (New York, Pergamon Press, 1975).
- [5] Miyagi M. and Nishida S., TM-type soliton in nonlinear self-focusing media, *Proc. IEEE* (1974) 1284-1285.
- [6] Katayev I. G., *Electromagnetic Shock Waves* (London, Iliffe, 1966).
- [7] Harrington R. F., *Time-Harmonic Electromagnetic Fields* (New York, McGraw Hill, 1961).
- [8] Morse P. M. and Feshbach H., *Methods of theoretical Physics* (New York, McGraw-Hill, 1953).
- [9] Wolfe P., The Secant method for Simultaneous Nonlinear Equations, *Commun. ACM* **12** (1959) 12-13.
- [10] Kirkpatrick S., Gelatt C. D. and Vecchi M. P., Optimization by simulated annealing, *Science* **220** (1983) 671-680.
- [11] Goldberg D. E., *Genetic Algorithms in Search, Optimization and Machine Learning* (Wesley, Addison, 1992).