



HAL
open science

An h-formulation for the computation of magnetostatic fields. Implementation by combining a finite element method and a boundary element method

B. Bandelier, F. Rioux-Damidaou, Christian Daveau

► To cite this version:

B. Bandelier, F. Rioux-Damidaou, Christian Daveau. An h-formulation for the computation of magnetostatic fields. Implementation by combining a finite element method and a boundary element method. Journal de Physique III, 1993, 3 (5), pp.995-1004. 10.1051/jp3:1993177 . jpa-00248977

HAL Id: jpa-00248977

<https://hal.science/jpa-00248977v1>

Submitted on 4 Feb 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification
Physics Abstracts
41.90 — 02.60

An h -formulation for the computation of magnetostatic fields. Implementation by combining a finite element method and a boundary element method.

B. Bandelier ^(1,2), C. Daveau ⁽²⁾ and F. Rioux-Damidau ⁽¹⁾

⁽¹⁾ Institut d'Electronique Fondamentale, Universités Paris VI et XI, URA CNRS 022, Bât 220, 91405 Orsay, France

⁽²⁾ Laboratoire de Modèles de Physique Mathématique, Université François Rabelais, Parc Grandmont, 37200 Tours, France

(Received 27 November 1993, revised 1 February 1993, accepted 8 February 1993)

Abstract. — A new formulation of magnetostatics is given: it uses the magnetic field h as variable and a penalty technique. For its discretization, a finite element method inside the magnetic materials is combined with a boundary integral method which describes the exterior domain. Numerical tests are presented. The value to be chosen for the penalty parameter and a criterion of validity of the computation are given.

1. Introduction.

When modelling eddy currents problems, the condition $\operatorname{div} b = 0$ is generally satisfied as a consequence of Faraday's law: $\operatorname{curl} e = -\partial b / \partial t$. It has not to be explicitly written.

The situation is different with magnetostatic problems. The two relations $\operatorname{curl} h = j$ and $\operatorname{div} b = 0$ with $b = \mu(|h|) h$ are to be satisfied together and this question is usually solved by introducing scalar or vector potentials [1-3]. The advantage of a formulation working directly with the magnetic field is to avoid the loss of precision due to differentiation. Such a formulation was proposed by Kikuchi for the cases where the permeability μ was constant and where boundary conditions were known [4]. An analog mixed method using the magnetic induction and the scalar potential as unknown was established by Stenberg and Trouvé. It uses also boundary conditions [5]. On the other hand, Bossavit studied the possibility of applying to magnetostatics the h -formulation of magnetodynamics in \mathbb{R}^3 by making the frequency tend to zero [6].

We present here a variational formulation using h as variable and suited to nonlinear materials. It is obtained from a mixed problem on which a penalty is applied. Moreover, we do not prescribe arbitrary boundary conditions on the boundary of the chosen computation

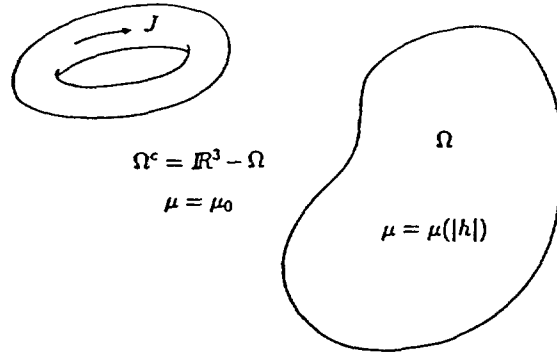


Fig.1. — The system.

domain, but we describe the problem of the infinite exterior domain by an integral method. For the numerical computation, the field is discretized with nodal variables inside the magnetic materials and with edge variables on the boundary.

2. Variational formulation.

We consider a finite domain Ω with magnetic materials, Γ is its boundary and Ω^c is its complement in \mathbb{R}^3 . Ω is simply connected.

The permeability $\mu(x, y, z)$ depends only on the point (x, y, z) and not on $|h|$. The nonlinearity of the materials can be taken into account with iterations. The current density $j(x, y, z)$ does not depend on the time; its bounded support lies in Ω^c ($j = 0$ in Ω) and its value is known. This current creates a source field h^s in the whole space when Ω is absent, such that $curl h^s = j$ and $div h^s = 0$. The total magnetic field h verifies $curl h = j$ or $curl(h - h^s) = 0$ in \mathbb{R}^3 .

The following functional spaces are introduced:

$$H_0(div^0, \Omega) = \{h \in \{L^2(\Omega)\}^3, div h = 0, h \cdot n = 0 \text{ on } \Gamma\}$$

$$H(curl, \mathbb{R}^3) = \{h \in \{L^2(\mathbb{R}^3)\}^3, curl h \in \{L^2(\mathbb{R}^3)\}^3\}$$

$$H(curl^0, \mathbb{R}^3) = \{h \in \{L^2(\mathbb{R}^3)\}^3, curl h = 0\}$$

$$H_0 = \{h \in H(curl, \mathbb{R}^3), curl h = 0 \text{ in } \Omega^c\}$$

$$H_j = \{h \in H(curl, \mathbb{R}^3), curl h = j \text{ in } \Omega^c\}$$

The space H_0 is an Hilbert space for the scalar product:

$$(h, h') = \int_{\mathbb{R}^3} h \cdot h' + \int_{\Omega} curl h \cdot curl h'$$

2.1 FORMULATION IN $H(curl^0, \mathbb{R}^3)$. — Let us now introduce the reaction magnetic field $\tilde{h} = h - h^s$. With these notations, the equations of the magnetostatic problem are:

$$div[\mu(\tilde{h} + h^s)] = 0 \text{ in } \mathbb{R}^3 \tag{1}$$

$$curl \tilde{h} = 0 \text{ in } \mathbb{R}^3 \tag{2}$$

where h^s is given (or obtained from j by Biot and Savart's law).

If \tilde{h} is looked for in the space $H(\text{curl}^0, \mathbb{R}^3)$, it is a strong solution of (2). We can write a weak form of (1):

$$\int_{\mathbb{R}^3} \text{div}(\mu\tilde{h}) \varphi' = - \int_{\mathbb{R}^3} \text{div}(\mu h^s) \varphi' \quad \forall \varphi' \in W^1(\mathbb{R}^3) \tag{3}$$

where $W^1(\mathbb{R}^3)$ is the closure of $C_0^\infty(\mathbb{R}^3)$ with norm $\|\text{grad}\varphi\|_{L^2}$.

After integration by parts in (3), it follows:

$$\int_{\mathbb{R}^3} \mu\tilde{h} \cdot \text{grad} \varphi' = - \int_{\mathbb{R}^3} \mu h^s \cdot \text{grad} \varphi' \quad \forall \varphi' \in W^1(\mathbb{R}^3)$$

So, taking into account that $H(\text{curl}^0, \mathbb{R}^3) = \text{grad} W^1(\mathbb{R}^3)$, \tilde{h} is solution of the following problem:

Problem 1: given $h^s \in \{L^2(\mathbb{R}^3)\}^3$, find $\tilde{h} \in H(\text{curl}^0, \mathbb{R}^3)$ such that:

$$\int_{\mathbb{R}^3} \mu\tilde{h} \cdot h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H(\text{curl}^0, \mathbb{R}^3)$$

Conversely, if \tilde{h} is solution of problem 1, it can be shown easily that it satisfies equations (1) and (2).

Existence and uniqueness of the solution of problem 1 can be classically proved. We have obviously, as $\mu \geq \mu_0$:

$$\int_{\mathbb{R}^3} \mu\tilde{h} \cdot \tilde{h} \geq \min(\mu) \|\tilde{h}\|_{H(\text{rot}^0, \mathbb{R}^3)}^2$$

The coerciveness condition is thus satisfied. Lax-Milgram's theorem can be applied [7], since $h^s \in \{L^2(\mathbb{R}^3)\}^3$ and the mapping

$$h' \longrightarrow \int_{\mathbb{R}^3} \mu h^s \cdot h'$$

is linear and continuous.

Another way to show that is to remark that the solution \tilde{h} of problem 1 is the only element of $H(\text{curl}^0, \mathbb{R}^3)$ which minimizes the quadratic coenergy functional

$$J(h') = \frac{1}{2} \int_{\mathbb{R}^3} \mu(h^s + h')^2$$

This solution exists in $H(\text{curl}^0, \mathbb{R}^3)$ which is a closed subspace of $\{L^2(\mathbb{R}^3)\}^3$ [8].

2.2 MIXED FORMULATION. — However, we want to keep the magnetic field alone as unknown, and it is not easy to build curl-free vector fields without introducing any scalar potential. To avoid this, we shall write a weak form of (2). From (1), we deduce that there exists a vector potential $a \in \{W^1(\mathbb{R}^3)\}^3$ such that:

$$\mu(\tilde{h} + h^s) = \text{curl} a \tag{4}$$

where a is unique if the gauge $\text{div} a = 0$ is chosen. Multiplying (4) by a test function $h' \in H_0$, integrating over \mathbb{R}^3 , and then integrating by parts, it becomes:

$$\int_{\mathbb{R}^3} \mu\tilde{h} \cdot h' - \int_{\Omega} a \cdot \text{curl} h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H_0 \tag{5}$$

(2) can also be transformed into:

$$\int_{\Omega} \text{curl } \tilde{h} \cdot a' = 0 \quad \forall a' \in H_0(\text{div}^0, \Omega) \tag{6}$$

Thus, we obtain the new following problem:

Problem 2: given j or $h^s \in \{L^2(\mathbb{R}^3)\}^3$, find $(\tilde{h}, a) \in H_0 \times H_0(\text{div}^0, \Omega)$, solution of (5) and (6).

Problem 2 is a classical mixed formulation [7].

Conversely, let $(\tilde{h}, a) \in H_0 \times H_0(\text{div}^0, \Omega)$ be a solution of problem 2. Ω being simply connected, any vector u of $\{L^2(\Omega)\}^3$ can be written as

$$u = a + \text{grad}\varphi$$

where $a \in H_0(\text{div}^0, \Omega)$. As $\tilde{h} \in H_0$, then

$$\int_{\Omega} \text{curl } \tilde{h} \cdot \text{grad}\varphi = 0$$

So, we can replace (6) by

$$\int_{\Omega} \text{curl } \tilde{h} \cdot u' = 0 \quad \forall u' \in \{L^2(\Omega)\}^3$$

Therefore if (6) holds, then $\text{curl } \tilde{h} = 0$ in Ω . But $\tilde{h} \in H_0$. Thus $\text{curl } \tilde{h} = 0$ in \mathbb{R}^3 .

The vector potential a , solution of problem 2, is defined only in Ω . But it can be extended to Ω^c by setting in Ω^c $\text{curl } a = \mu_0(\tilde{h} + h^s)$ and $\text{div } a = 0$.

The second integral of (5) can be taken in \mathbb{R}^3 since $\text{curl } h' = 0$ in Ω^c . Integrating by parts, we obtain

$$\int_{\mathbb{R}^3} \mu \tilde{h} \cdot h' - \int_{\mathbb{R}^3} \text{curl } a \cdot h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H_0$$

Integrating again by parts and taking h' in $H(\text{curl}^0, \mathbb{R}^3)$, it follows

$$\int_{\mathbb{R}^3} \mu \tilde{h} \cdot h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H(\text{curl}^0, \mathbb{R}^3)$$

Thus, \tilde{h} is solution of problem 1.

Problem 1 and problem 2 are then equivalent. The vector potential a , solution of problem 2, is unique as it is looked for in $H_0(\text{div}^0, \Omega)$.

3. Penalty formulation.

In order to solve more easily problem 2 and to reduce the number of unknowns, we shall perturb it [4, 7]. Let us call $\tau > 0$ a small parameter and replace (6) by:

$$\int_{\Omega} \text{curl } \tilde{h}_{\tau} \cdot a' + \tau \int_{\Omega} a_{\tau} \cdot a' = 0 \quad \forall a' \in H_0(\text{div}^0, \Omega) \tag{7}$$

This equation gives $a_{\tau} = -\frac{1}{\tau} \text{curl } \tilde{h}_{\tau}$ in Ω , which on putting back in (5), permits us to write what we shall call:

Problem 3: for j or $h^s \in \{L^2(\mathbb{R}^3)\}^3$ given, find $\tilde{h}_\tau \in H_0$ such that:

$$\int_{\mathbb{R}^3} \mu \tilde{h}_\tau \cdot h' + \frac{1}{\tau} \int_{\Omega} \text{curl } \tilde{h}_\tau \cdot \text{curl } h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H_0 \tag{8}$$

Existence and uniqueness of problem 3 can be proved by Lax-Milgram's theorem since the bilinear form

$$a(h, h') = \int_{\mathbb{R}^3} \mu h \cdot h' + \frac{1}{\tau} \int_{\Omega} \text{curl } h \cdot \text{curl } h'$$

is coercive on H_0 , and since the right hand of (8) is a linear continuous functional.

It can be shown that the solution \tilde{h}_τ of problem 3 tends to the solution of problem 2 when τ tends to 0 [7]. In problem 3, the unknown a_τ appears no more but its value can be determined, if desired.

One can note the analogy between the formulation (8) and the one implemented in TRIFOU [9] for magnetodynamics, when the magnetic fields and the currents vary sinusoidally at the frequency ω and when the domain Ω has a resistivity ρ :

$$i\omega \int_{\mathbb{R}^3} \mu \tilde{h} \cdot h' + \rho \int_{\Omega} \text{curl } \tilde{h} \cdot \text{curl } h' = - \int_{\mathbb{R}^3} \mu h^s \cdot h' \quad \forall h' \in H_0 \tag{9}$$

This formulation (9) has been applied to some magnetostatics computations by letting ω tend to zero [6]. We observe that $\rho/i\omega$ of (9) corresponds to $1/\tau$ of (8). But the parameter τ and the magnetic fields of (8) are real whereas $\rho/i\omega$ and h of (9) are complex.

In (8), the quantity $1/\tau\mu_0$ has the dimension of the square of a length and we so let us set:

$$\frac{1}{\tau\mu_0} = p^2 \tag{10}$$

As $\mu = \mu_0 \mu_r$, equation (8) then rewrites as:

$$\int_{\mathbb{R}^3} \mu_r \tilde{h}_\tau \cdot h' + p^2 \int_{\Omega} \text{curl } \tilde{h}_\tau \cdot \text{curl } h' = - \int_{\mathbb{R}^3} \mu_r h^s \cdot h' \quad \forall h' \in H_0 \tag{11}$$

and its solution has to be obtained for p large enough, i.e. for $p \gg R$ where R is a characteristic dimension of Ω .

4. Discretization: FEM and BIM.

The integration in (11) over $\Omega^c = \mathbb{R}^3 - \Omega$ can be transformed into a boundary integral if we put $\tilde{h} = \text{grad } \varphi$ in Ω^c which is simply connected, with $\Delta\varphi = 0$ [10]. Moreover, we introduce the total field $h_\tau = \tilde{h}_\tau + h^s$ in Ω . We thus obtain:

$$\int_{\Omega} \mu_r h_\tau \cdot h' + p^2 \int_{\Omega} \text{curl } h_\tau \cdot \text{curl } h' + \int_{\Gamma} \left(\frac{\partial \varphi_\tau}{\partial n} + h^s \cdot n \right) \varphi' = 0 \tag{12}$$

$$\forall h' \in H_0, h' = \text{grad } \varphi' \text{ in } \Omega^c$$

where n is the inward unit vector to Γ . The unknowns are the total field h_τ in Ω and the reaction scalar potential φ_τ on Γ . For the discretization of (11) we associate a finite element

method (FEM) in Ω and a boundary integral method on Γ (BIM) [9].

We implemented this method as in the CELFI code [11]. Ω is meshed into tetrahedra and, in each of these, a linear representation of the magnetic field is taken. Several kinds of variables are combined.

In a tetrahedron without any vertex on Γ , the field is written as:

$$h_\tau = \sum_{n=1}^4 h_n \lambda_n$$

where λ_n is the barycentric coordinate associated with the vertex n and h_n a nodal vectorial variable.

In the tetrahedra which own vertices on the boundary Γ , the field is expressed with nodal variables and edge variables (two per edge)[12]. These edge variables are themselves expressed as functions of φ_τ when the edges are located on Γ .

To evaluate the boundary integral in (12), we compute an exterior stiffness matrix [13] which relies $\partial\varphi/\partial n$ to φ . We take a quadratic representation of the potential φ_τ in the triangles of Γ , which is consistent with the linear representation of h_τ .

The degrees of freedom are then: the nodal magnetic fields inside Ω , edge variables on internal edges with one or two vertices on Γ , scalar potential on the vertices and on the middle of the edges of Γ .

With this representation, we satisfy the continuity of the magnetic field inside an homogeneous material and of its tangential component at the interface between two materials.

As we take a nodal permeability μ , the global matrix obtained from Galerkin's method is, in general, not symmetric.

If the scalar potential, instead of the magnetic field, is chosen as variable inside the material as it is often done, it is necessary to take a quadratic representation of φ_τ in each tetrahedron to obtain a linear approximation of h . The number of variables is then, inside the material, $N_v + N_e$ (number of vertices + number of edges) $\sim 7 N_v$ instead of $3 N_v$ with h as variable. Moreover each node of interpolation of scalar potential is connected with vertex nodes and middle of edge nodes. So, to ensure a linear approximation of the field the matrix of the system has more elements and the CPU time would be greater than with h variables.

5. Numerical test.

In order to test the method, we applied it to a sphere of constant permeability $\mu = \mu_0 \mu_r$ and of radius R , embedded into a constant magnetic field h^0 . Two meshes were used. The first one, S1, concerned the whole sphere. There were 360 tetrahedra in it and 120 triangles on Γ . The second one, S2, concerned one eighth of the sphere and had 702 tetrahedra and 230 triangles on Γ . The computation was done by using symmetry properties and corresponded to a whole sphere of 5608 tetrahedra.

As the analytical solution h_{ana} of this problem is known, we can calculate the rms error:

$$\left[\frac{1}{V} \int_{\Omega} \left(\frac{h_{num} - h_{ana}}{h_{ana}} \right)^2 dv \right]^{\frac{1}{2}}$$

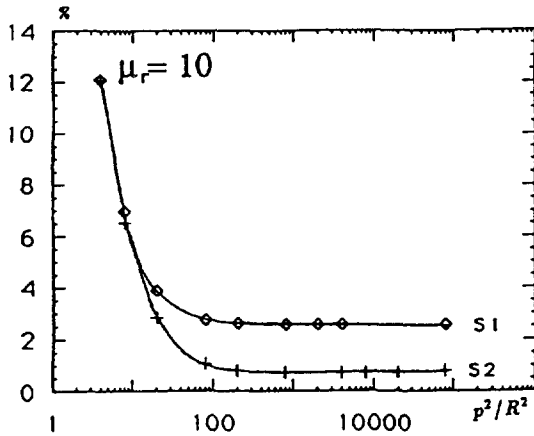


Fig. 2

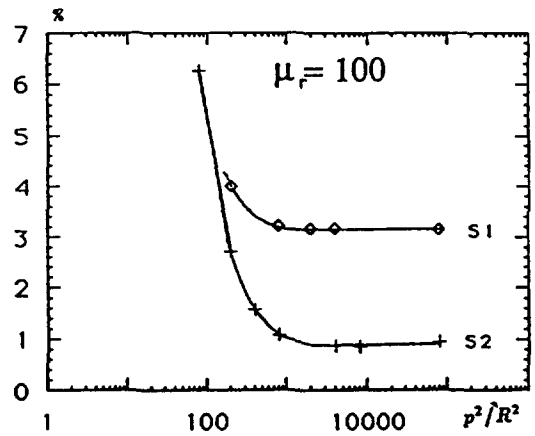


Fig. 3

Fig.2 — The error on the magnetic field for $\mu_r = 10$.

Fig.3 — The error on the magnetic field for $\mu_r = 100$.

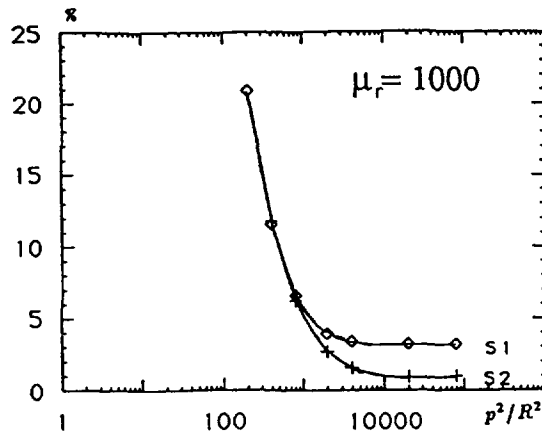


Fig.4. — The error on the magnetic field for $\mu_r = 1000$.

The figures 2, 3, 4 give this error, for $\mu_r = 10, 100, 1000$, as a function of the dimensionless parameter p^2/R^2 . We see that:

- as well for S1 than for S2, the error diverges when p tends to zero. It becomes not too big when $p^2/R^2 \sim \mu_r$ and it reaches its minimum and remains constant for $p^2/R^2 \geq 10 \mu_r$.
- the ratio between the errors for a great p with S1 and S2 does not depend on μ_r and is about 3.5.

Equation (12) shows that the greater p^2 , the larger the relative weight of the error on $\text{curl } h_r$

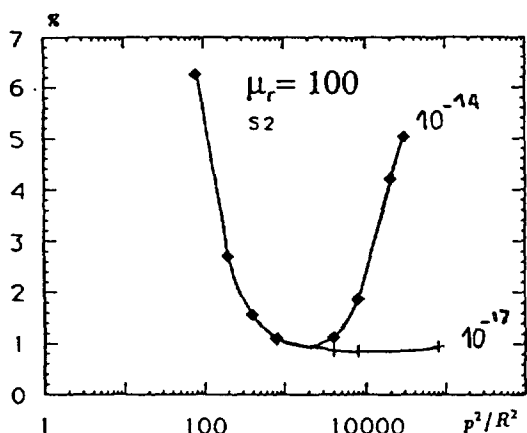


Fig. 5

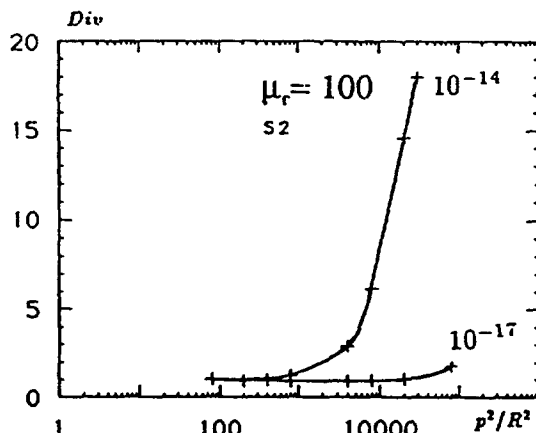


Fig. 6

Fig.5 — The error on the magnetic field for two values of the criterion of the conjugate gradient method.

Fig.6 — The value of *Div* (in arbitrary units) for two values of the convergence criterion ϵ of the conjugate gradient method.

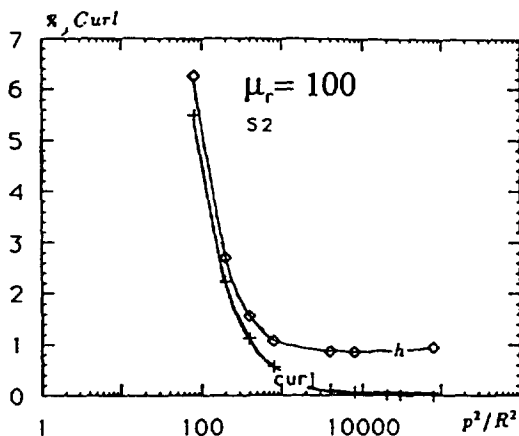


Fig.7. — The value of *Curl* (in arbitrary units- *Curl* is the same for $\epsilon = 10^{-14}$ and $\epsilon = 10^{-17}$) and the error on the magnetic field *h* for $\epsilon = 10^{-17}$

in the total error. For high values of p^2 , one has to pay attention to the precision of the solutions of the linear system $Ax = b$ obtained from Galerkin's method. The solutions are here computed with a conjugate gradient method. The iterations are stopped when the ratio $(\frac{Ax - b}{b})^2$ falls below a given ϵ . We observe that the error on the magnetic field is very sensitive to the value of ϵ (see Fig. 5). In fact, when we start from a value of p^2 giving a

minimal error and when p^2 is again increased, the error does not always remain constant but increases again if ε is not small enough.

We then calculated

$$Div = \left[\frac{1}{V} \int_{\Omega} (\operatorname{div} \mu h)^2 \right]^{1/2}$$

$$Curl = \left[\frac{1}{V} \int_{\Omega} |\operatorname{curl} h|^2 \right]^{1/2}$$

We see (Fig. 6) that *Div* is generally constant and small. But, for a poor choice of ε , the error on the magnetic field and the quantity *Div* begin to increase simultaneously when p^2/R^2 increases. At the same time, *Curl* remains always very small (Fig. 7).

The characteristics of *Curl* are: it does not depend too much on ε , it is important only for $p^2/R^2 \leq \mu_r$ like the error on the field, it continuously decreases with p^2/R^2 .

The divergence, but not the curl, can thus be used to test the validity of the computation. It has to remain constant when p^2 is varied around the choice defined above.

The figures 2, 3, 4 are drawn with $\varepsilon = 10^{-17}$

6. Conclusion.

A formulation of the magnetostatics in terms of h has been established. The use of h variables instead of φ variables reduces the number of elements of the matrix of the linear system if we want to keep the same order of approximation of the field. Therefore, at first sight, it is interesting.

This formulation includes a penalty term. The value to be given to the dimensionless parameter p^2/R^2 , where R is a typical dimension of the device, has been clarified. It must be at least 10 times the relative permeability μ_r . A good test of the validity of the computation is the value of the divergence, which must remain constant when the parameter is slightly varied. These conclusions have to be clarified when μ depends on h . It will be done in the near future.

Acknowledgment.

We thank Alain Bossavit for his fruitful remarks.

References

- [1] Preis K., Bardi I., Biro O., Magele C., Renhart W., Richter K. R., Vrisk G., Numerical analysis of 3D magnetostatic fields, *IEEE Trans. Mag.* **27** (1991) 3798-3803.
- [2] Coulomb J. L., Finite element three dimensional magnetic field computation, *IEEE Trans. Mag.* **17**(1981) 3241-3246.
- [3] Simkin J., Trowbridge C. W., On the use of the total scalar potential for field problems in electromagnetics, *Int. J. Numer. Meth. Eng.* **14** (1979) 434-440.
- [4] Kikuchi F., Mixed formulations for finite element analysis of magnetostatic and electrostatic problems, *Jpn J. Appl. Math.* **6** (1989) 209-221.
- [5] Stenberg R. and Trouvé P., A Mixed Finite Element Method for the Magnetic Field. Proc. of the Fifth. Int. Symp. Numer. Meth. Engng., vol 2, R. Gruber, J. Périaux, and R. P. Shaw Eds. (Computational Mechanics and Springer-Verlag, 1989) pp. 137-141.
- [6] Bossavit A., TRIFOU et la magnétostatique, *Bull. DER. EDF- Série C* No 2 (1986) 101-109.
- [7] Girault V., Raviart P. A., Finite element methods for Navier-Stokes equations (Springer-Verlag, Berlin - Heidelberg - New York - Tokyo, 1986).

- [8] Cessenat M., Exemples en électromagnétisme et en physique quantique, Analyse mathématique et calcul numérique pour les sciences et les techniques, R. Dautray and J.L. Lions Eds. vol 5 (Masson, 1988), chap IX, pp 235-273.
- [9] Bossavit A., Vérité J. C., The Trifou code: solving the 3D eddy currents problems by using h as state variable, *IEEE Trans. Mag.* **19** (1983) 2465-2470.
- [10] Bossavit A., A theoretical introduction to Trifou, Actes du colloque Modelec (Ed. Pluralis, 1984) pp. 81-96.
- [11] Bandelier B., Rioux-Damidau F., 3D modelling of electromagnetic fields: advantages and disadvantages of nodal or edge elements, *Int. J. Comp. Appl. Tech.*, to be published.
- [12] Mur G., de Hoop A., A finite element method for computing three-dimensional electromagnetic fields in inhomogeneous media, *IEEE Trans. Mag.* **21** (1985) 2188-2191
- [13] Vérité J. C., Trifou : un code de calcul tridimensionnel des courants de Foucault, *Bull. DER. EDF- Série C* No 2 (1983) 79-92.