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## Critical Self-Tuning: the Example of Zero Prandtl Number Convection

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**Abstract.** — We present a new type of bifurcation scenario where nonlinear saturation of a stationary instability takes place only because of the competition with an oscillatory one. This is shown on the example of convection at zero Prandtl number between stress-free boundaries. We show with direct numerical simulations that time-dependent wavy rolls are generated at the onset of convection. Using a Galerkin model, we analyze the nonlinear interactions between rolls and waves and find that they maintain the system in the vicinity of the oscillatory instability onset, thus preventing the blow-up of the growing nonlinear roll solution. An interesting feature of this type of dynamics is that the system is self-tuned in the vicinity of a transition point.

The dynamics of thermal convection in the Boussinesq approximation involves two kinds of nonlinearities. The first,  $\mathbf{v} \cdot \nabla \mathbf{v}$ , describes the self interaction of the velocity field  $\mathbf{v}$ ; the second,  $\mathbf{v} \cdot \nabla \theta$ , results from the advection of the temperature disturbance  $\theta$  by the velocity field. The relative importance of these nonlinearities is measured by the Prandtl number,  $P = \nu/\kappa$ , where  $\nu$  is the fluid kinematic viscosity and  $\kappa$  is the heat diffusivity [1]. For  $P$  very large,  $\mathbf{v} \cdot \nabla \mathbf{v}$  can be neglected; for  $P$  very small, which is a limit of interest in the astrophysical context ( $P \sim 10^{-8}$  for convection in stars) or for liquid metals ( $P \sim 10^{-2}$ ), one expects that  $\mathbf{v} \cdot \nabla \theta$  could be neglected [2]. The small  $P$  limit is however subtle and might involve a singular limit problem very similar to the one of infinite Reynolds number in incompressible fluid dynamics. In particular, there exist flow regimes (inertial convection) for which the  $\mathbf{v} \cdot \nabla \mathbf{v}$  nonlinearity is compensated by a pressure gradient and thus is unable to saturate the instability [3, 4]. This problem exists at convection onset with stress-free boundary conditions because the linearly unstable roll modes are exact nonlinear solutions if  $\mathbf{v} \cdot \nabla \theta$  is neglected. Because of the existence of these exact solutions growing in time, it has been considered for a long time that the zero-Prandtl number approximation which amounts to neglect  $\mathbf{v} \cdot \nabla \theta$  was not realistic. However, it has been shown recently by direct numerical simulation of these zero-Prandtl number asymptotic equations, and comparison with the full

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Oberbeck-Boussinesq equations, that this approximation leads to accurate results [5]. We show in this letter, that just above the onset of convection, two-dimensional (2D) rolls saturate by developing a three-dimensional wavy instability. The nonlinear flow regime is thus time-dependent, although the convective instability is a stationary one. We use a simple Galerkin model to describe qualitatively how the growing roll-modes are saturated by transferring energy to three-dimensional (3D) waves. The analysis of the stabilization mechanism shows a new and interesting bifurcation structure: the energy transfer from rolls to waves automatically tunes the roll velocity in order to keep the system in the vicinity of an instability onset, thus preventing blow-up.

We consider a horizontal layer of fluid of thickness  $d$ , heated from below by imposing the temperatures  $T_1$  and  $T_2$  at the bottom and top horizontal conductive plates. We assume that the fluid motion is described by the Oberbeck-Boussinesq equations in the limit of zero Prandtl number, as discussed in [5]:

$$\begin{aligned} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nabla^2 \mathbf{v} + R\theta \mathbf{e}_3 \\ \nabla \cdot \mathbf{v} &= 0 \\ 0 &= v_3 + \nabla^2 \theta. \end{aligned} \quad (1)$$

where  $\mathbf{v}(x, y, z, t) = (v_1, v_2, v_3)$  is the velocity and  $\theta(x, y, z, t)$  is the deviation from the diffusive temperature profile. The units chosen for this dimensionless formulation are  $d$  for length,  $d^2/\nu$  for time and  $(T_1 - T_2)\nu/\kappa$  for temperature.  $R$  is the Rayleigh number, and  $\mathbf{e}_3$  is the vertical unit vector. We impose horizontally periodic boundary conditions, which introduce two fundamental wave numbers  $k$  along  $x$  and  $q$  along  $y$ -axes. The free-slip boundary conditions imply  $\partial_z v_1 = \partial_z v_2 = 0$  and  $v_3 = 0$  at  $z = 0$  and  $z = 1$ .

We perform numerical simulations of this system with a pseudo-spectral code based on a Fourier decomposition in the horizontal, on cosine or sine decomposition in the vertical, and on a poloidal-toroidal representation of the velocity field (see [5] for details). In this letter, we investigate values of  $R$  which are very close to the onset  $R_c = 27\pi^4/4 \approx 658$ . In order to study the very long transients that exist at these values of the control parameter, we have chosen a small spatial resolution of  $8^3$ . We have checked, on test cases, that the same results were obtained with higher resolutions. This low truncation is equivalent to a Galerkin approximation of 64 consistent eigenmodes of the convective instability. For  $k = k_c$ , here  $k_c = \pi/\sqrt{2}$  is the critical wave number at the convection onset,  $q = 0.25 k_c$  and  $R = 1.005 R_c$  (i.e.  $R \approx 660$ ) we observe the limit cycle shown in Figure 1a. In this regime, the 2D convective mode  $v_3(x, y, z, t) = w_{101}(t) \cos(k_c x) \sin(\pi z)$  is oscillating around a mean value, while the vertical vorticity mode  $\omega_3(x, y, z, t) = \zeta_{010}(t) \sin(qy)$  exhibits a standing wave pattern along the roll axis. The frequency of the roll mode  $w_{101}$  is twice that of the vorticity mode. This flow regime traces back to the one generated by the oscillatory instability in small Prandtl number convection [6]. The oscillation is observed at the onset of convection here, whereas it occurs as a secondary instability for  $R - R_c$  finite when  $P$  is finite [7, 8]. We stress the stabilizing effect of the vertical vorticity, since, without it, the amplitude of the 2D convection rolls would grow to infinity as said above. We have observed this standing wave oscillation over thousands of periods, which is reasonably long, at this value of  $R$  close to the onset.

We consider a minimal Galerkin model to describe the saturation mechanism of the growing 2D-rolls in presence of 3D-perturbations. For a model able to treat the oscillatory instability [6, 9], we expand the vertical velocity  $v_3$  and the vertical vorticity  $\omega_3$  in Fourier series,

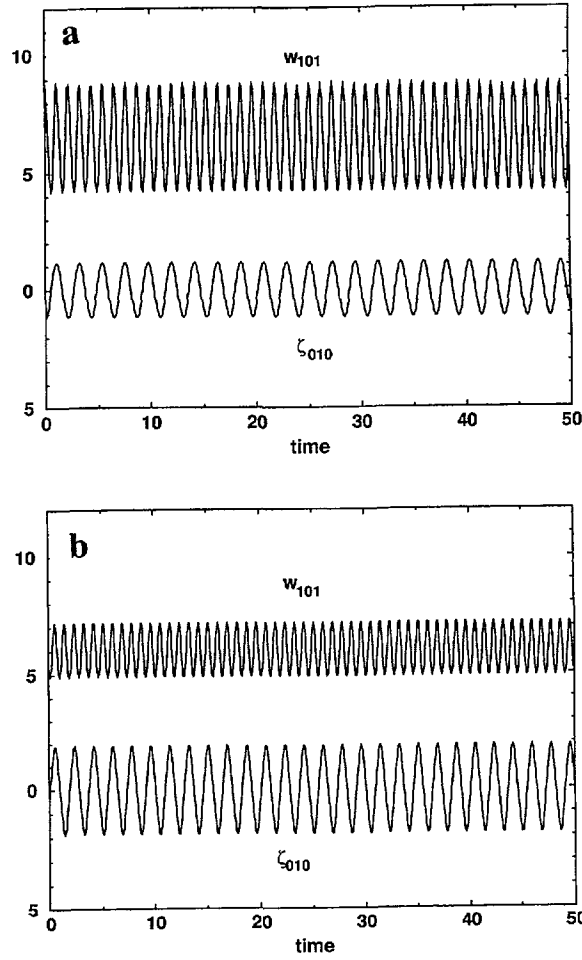


Fig. 1. — a) Oscillatory regime observed with the pseudo-spectral code, at  $q = 0.25 k_c$  and  $r = 1.005$ . The time evolution of the roll mode amplitude  $w_{101}$  and the vertical vorticity mode amplitude  $\zeta_{010}$  are displayed. Time unit is viscous timescale. b) Oscillatory regime observed with the six-mode Galerkin model at  $q = 0.4 k_c$  and  $r = 1.005$ .

compatible with the equation of continuity and with free-slip boundary conditions, as

$$\left\{ \begin{array}{l} v_3(x, y, z, t) = w_{101}(t) \cos(k_c x) \sin(\pi z) + 2w_{111}(t) \sin(k_c x) \cos(qy) \sin(\pi z) + \cdot \\ \omega_3(x, y, z, t) = \zeta_{010}(t) \sin(qy) + 2\pi\zeta_{111}(t) \cos(k_c x) \sin(qy) \cos(\pi z) \\ \quad + 2\pi\zeta_{012}(t) \sin(qy) \cos(2\pi z) + 5\zeta_{210}(t) \cos(2k_c x) \sin(qy) + \cdot \end{array} \right. \quad (2)$$

We insert these expansions in the hydrodynamic equations (1) and project on these modes to

get the following dynamical system:

$$\left\{ \begin{array}{l} \tau \dot{u} = (r-1)u + \alpha^{-1} (a\zeta + c\xi + d\chi)v + b\beta(\zeta + \xi + \gamma\chi)\eta \\ \tau \dot{v} = -(1 + \tau q^2) \left( 1 - \frac{r}{\beta(1 + \tau q^2)^3} \right) v - \alpha^{-1} (\zeta + f\xi + f\gamma\chi)u \\ \tau \dot{\zeta} = -\tau q^2 \zeta + \frac{\beta'}{2} (\alpha v - \eta)u \\ \tau \dot{\eta} = -(1 + \tau q^2)\eta - (\zeta + 3\xi - 5\chi)u \\ \tau \dot{\xi} = -(b + \tau q^2)\xi - \frac{\beta'}{4} (\alpha v - \eta)u \\ \tau \dot{\chi} = -\left(\frac{b}{2} + \tau q^2\right)\chi - \frac{1}{20} (\alpha\beta'v + g\eta)u, \end{array} \right. \quad (3)$$

where  $(u, v, \eta) = -\frac{\pi\tau}{\sqrt{2}} (w_{101}, w_{111}, \zeta_{111})$ ,  $(\zeta, \xi, 2\chi) = -\frac{\tau}{2} (\zeta_{010}, \pi\zeta_{012}, \zeta_{210})$ ,  $r = R/R_c$  is the reduced Rayleigh number,  $\alpha = q/k_c$ ,  $\beta = k_c^2/(k_c^2 + q^2)$ ,  $\beta' = q^2/(k_c^2 + q^2)$ ,  $\gamma = 5q^2/(4k_c^2 + q^2)$ ,  $\tau = 1/(\pi^2 + k_c^2)$ ,  $a = 2\tau[2\pi^2\beta - (\pi^2 - k_c^2)]$ ,  $b = 4\pi^2/(\pi^2 + k_c^2)$ ,  $c = 2\tau[(3\pi^2 - k_c^2) - 2\pi^2\beta']$ ,  $d = 2\gamma\tau[2\pi^2\beta + (\pi^2 - k_c^2)]$ ,  $f = (b - 1 - \tau q^2)/(1 + \tau q^2)$  and  $g = 4(b + 6\tau q^2)/(b + 8\tau q^2)$ .

We perform numerical integration of this six-mode Galerkin model with a standard fourth order Runge-Kutta method. For  $q/k_c$  less than roughly 0.5, and close to the onset of convection, we observe the same oscillatory flow regime as the one obtained above with direct numerical simulations (see Fig. 1b). A quantitative agreement is observed in the vicinity of the onset convection, the validity of the Galerkin model being obviously limited by its truncation at higher values of  $r$ . This limitation of the model is more severe when  $q \rightarrow 0$  ( $r - 1 < O(q^4)$ ).

As said above, the roll mode  $u(t)$  oscillates around its mean value at a frequency  $2\omega$  whereas all the other modes which describe the 3D-waves have zero mean and oscillate at frequency  $\omega$ . The oscillation amplitude  $A$  of the 3D-velocity mode  $v(t)$ , and the one of the roll mode  $u(t)$ ,  $B$ , are displayed in Figure 2 for  $q = 0.4 k_c$ , on one decade in  $r - 1$ . In the vicinity of the instability onset  $A$  scales as  $\sqrt{r - 1}$  whereas  $B$  scales as  $r - 1$ . The mean roll-velocity jumps abruptly to an order one value at the instability onset and slightly decreases when  $r$  is increased.

Let us now understand the saturation mechanism of the 2D rolls by the oscillatory instability. The oscillatory instability corresponds to the generation of 3D-disturbances involving vertical vorticity; it occurs at long wavelength (i. e. small  $q$ ) for a critical value of the roll velocity [6]. For  $r - 1$  and  $q$  small, the Galerkin model involves two types of modes :  $u$ ,  $v$  and  $\zeta$  which are nearly marginal, and  $\eta$ ,  $\xi$  and  $\chi$  which are linearly damped. Adiabatic elimination of the damped modes gives three equations for the marginal modes; keeping terms up to order  $q^2$ , we get

$$\left\{ \begin{array}{l} \dot{u} = \frac{r-1}{\tau}u + k_c \left( \frac{2}{\tau} + \frac{15}{4}q^2 \frac{u^2}{4+3u^2} \right) v\zeta - \frac{15}{16}q^2 uv^2 - \frac{32}{3\tau}q^2 \frac{u}{4+3u^2}\zeta^2 \end{array} \right. \quad (4.1)$$

$$\left\{ \begin{array}{l} \dot{v} = -\lambda v - k_c \left( \frac{1}{\tau} + \frac{15}{8}q^2 \frac{u^2}{4+3u^2} \right) u\zeta + \frac{15}{32}q^2 u^2 v \end{array} \right. \quad (4.2)$$

$$\left\{ \begin{array}{l} \dot{\zeta} = -q^2 \frac{4-3u^2}{4+3u^2}\zeta + \frac{3q^2}{2k_c} uv, \end{array} \right. \quad (4.3)$$

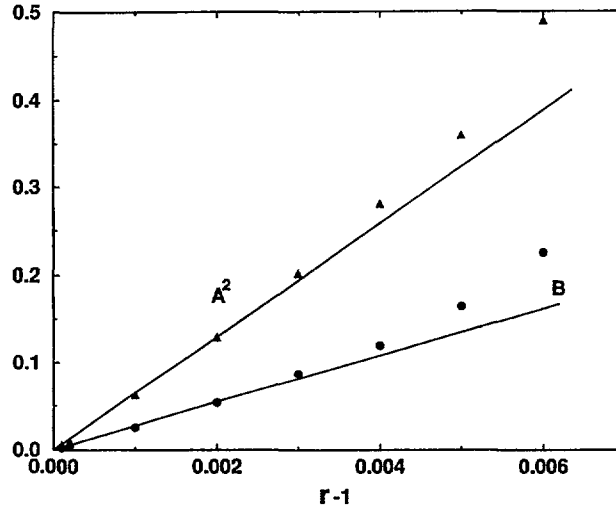


Fig. 2. — Oscillation amplitudes above the onset of convection from the six-mode Galerkin model at  $q = 0.4 k_c$ . *A*: amplitude of the 3D-velocity mode  $w_{111}$ , *B*: amplitude of the oscillatory part of the 2D-roll mode  $w_{101}$ .

with  $-\lambda\tau = r - 1 - 3\tau^2 q^4$ . The model is valid if  $v$  remains linearly damped, thus  $r - 1 < 3\tau^2 q^4$ ; otherwise additional modes should be taken into account.

At the onset of convection only the roll-mode  $u$  begins to grow and would not saturate if  $v$  and  $\zeta$  are not excited. If  $r - 1 \approx 0$ , the growth rate is vanishingly small and we first consider  $u$  nearly constant. The equations for  $v$  and  $\zeta$  can be written up to terms of order  $q^2$

$$\ddot{v} + q^2 \Lambda(u^2) \dot{v} + \frac{3q^2}{2\tau} u^2 v = 0, \tag{5}$$

with  $\Lambda(u^2) = \frac{4-3u^2}{4+3u^2} - \frac{15}{32}u^2$ . Thus  $v$  obeys a Van der Pol equation and the  $v = 0$  solution is unstable for a critical value  $u_0$  such that  $\Lambda(u_0^2) = 0$ , i.e.  $u_0 \approx 0.8$ . For  $u > u_0$ , an oscillation of frequency  $\omega = \sqrt{3/2\tau} u q$  is generated by the roll motion.

We will show on (4) how the transfer of kinetic energy from 2D-rolls to 3D-waves saturates the roll velocity [10]. To wit, we write

$$\begin{aligned} u &= u_0 + \epsilon^2(\bar{u} + \tilde{u}) \\ v &= \epsilon \tilde{v}, \end{aligned}$$

where  $r - 1 = \mu\epsilon^2$ , with  $\mu$  is of order one and  $r - 1 \rightarrow 0$ .  $\tilde{u}$  is the oscillatory part of  $u$  and  $\bar{u}$  is the correction to the mean value  $u_0$  for the onset of the oscillatory instability. The above scalings follow from inspection of equations (4) and are consistent with the results of figure 2. We have to leading order

$$\begin{aligned} \tilde{u} &= B \cos(2\omega t + \phi) \\ \tilde{v} &= A \cos \omega t. \end{aligned}$$

Taking the mean of equation (4.1) gives at leading order  $A^2 = \mu(4 + 3u_0^2)/24\tau^2 q^4$ . Subtracting the mean of (4.1) from (4.1) and solving for  $\tilde{u}$  gives  $B \approx -A^2/2u_0$ . Finally, multiplying (5) by

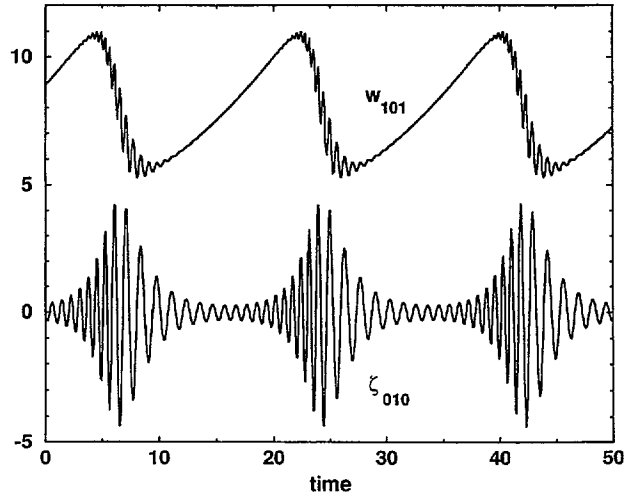


Fig. 3. — Relaxation oscillation regime observed with the six-mode Galerkin model at  $q = 0.6 k_c$  and  $r = 1.004$ . Time unit is viscous timescale.

$\dot{v}$  and averaging on one cycle gives  $\bar{u} \approx -A^2/4u_0$ . The mean value of the 2D-roll velocity thus decreases when  $r$  is increased, as observed by numerical integration of the model. This is how the stabilization mechanism operates: the mean roll-velocity is kept below the critical velocity  $u_0$  for the oscillatory instability onset;  $u$  is larger than  $u_0$  only on one part of the limit cycle since  $B \approx 2\bar{u}$ . This prevents the blow-up of the oscillatory modes. The transfer of kinetic energy from the roll-modes to the 3D-wave modes prevents the blow-up of the roll-mode.

For  $q/k_c$  greater than approximately 0.5, numerical simulations of the Galerkin model shows relaxation oscillations at the onset of convection (see Fig. 3). The period of this relaxation oscillation tends to infinity for  $r \rightarrow 1$ , and decreases when  $r$  increases; the regime then continuously evolves to a harmonic oscillation like the one observed for small  $q$ . Chaotic relaxation oscillations are observed for large  $q$ . These behaviors are also in qualitative agreement with the regimes observed in the direct simulations for any  $q$  up to  $k_c$ .

The interesting new features of the dynamics we have described above, i.e. the abrupt jump to a finite value of the roll velocity at the convective instability onset without any hysteresis, and the self tuning of the system in the vicinity of the oscillatory instability, crucially depend on the existence of a growing linear mode which happens to be also an exact nonlinear solution of the problem.

In the case of rigid boundary conditions, 2D-rolls are no longer an exact growing solution of (1). However, it is known that there exist 2D flow regimes (inertial convection) for which the  $\mathbf{v} \cdot \nabla \mathbf{v}$  nonlinearity is unable to saturate the flow [3, 4]. As pointed out in [8], (1) could be also used to study inertial convection for a 3D flow in the zero  $P$  limit; a manifestation of inertial convection would be either unbounded growth or a subcritical bifurcation at a certain Rayleigh number. This is not observed up to  $R = 128000$  in [5], and the existence of 3D inertial convection is an open question.

Another limitation should be discussed in the case of stress-free boundaries. It is known that the monotonous skewed varicose and the oscillatory skewed varicose instabilities restrict the range of stable two-dimensional rolls, which shrinks to zero when the Prandtl number is decreased below 0.543 [11]. However, these instabilities are strongly inhibited when the

horizontal extension of the fluid layer is not large compared to its height. Consequently, as observed in three-dimensional direct numerical simulations for  $P = 0.025$  with a small aspect ratio of the horizontal periodicity interval, stable steady rolls exist and undergo the oscillatory instability as the Rayleigh number is increased [12].

In conclusion, we have shown that the zero-Prandtl number asymptotics (1) can be used to model 3D-vorticity dynamics at the onset of convection in fluids of small Prandtl number with stress-free boundary conditions and a small aspect ratio geometry. The fact that 2D-rolls are an exact nonlinear growing solution makes the bifurcation structure at the convection onset strongly nonlocal; the roll-mode abruptly jumps to finite amplitude whereas the wave amplitude grows continuously from zero. Energy transfers between rolls and waves are triggered through triad interactions and keep the system in the vicinity of the waves instability onset. We thus expect a similar situation in other problems of fluid dynamics or plasma physics which commonly involve triad interactions as a governing mechanism. An interesting feature of this type of dynamics is that the system is self-tuned in the vicinity of a transition point.

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