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The Temperature of Turbulent Flows

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Abstract. — We show that the random character of turbulent flows allows to define a quantity conserved along the length scales which we call the temperature. The corresponding “canonical” distribution of velocity differences at a given scale is derived on very general grounds. It is compared to the one derived by Dubrulle [1] and She and Waymire [2] from the hypothesis of She and Lévêque [3]. Some consequences are drawn and this temperature is shown to be a rather easily measurable quantity.

Résumé. — Nous montrons que le caractère aléatoire des écoulements turbulents permet de définir une quantité conservée le long des échelles de longueur que nous nommons température. La distribution “canonique” correspondante pour les différences de vitesse à une échelle donnée est tirée d’arguments très généraux. Nous la comparons à celle que Dubrulle [1] et She et Waymire [2] ont tirée des hypothèses de She et Lévêque [3]. Nous examinons quelques autres conséquences et nous montrons que cette température est une quantité aisément mesurable.

1. Introduction

The number of degrees of freedom (NDF) in a turbulent flow is very large. Traditional estimates [4] even give, for a typical atmospheric flow, a NDF comparable to the Avogadro number. It is clear that controlling all of them is an impossible and futile program, due to this number and the sensitivity to initial conditions. Similar situation occurs in thermodynamics. Then we know that a few number of pertinent quantities exist, the thermodynamical parameters, allowing a complete description of all the relevant aspects of the system.

Several attempts have been made to transpose the thermodynamic approach to turbulence [5] but this program, despite some successes, is far to be completed. Recently an experimental work [6] suggested interesting analogies. The scale at which the turbulent velocity field is observed would be a thermodynamic variable, its logarithm being the equivalent of a volume. The variation of quantities with the scale, for a given flow, would be the equivalent of isothermal variations, indicating a correspondence between the temperature and the Reynolds number.

The goal of this paper is to push this analogy. In particular we shall be able to define the temperature and to derive the “canonical” distribution for velocity differences at a given scale. This distribution is shown to be a very simple extension of a recently proposed one [1, 2], on

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completely different grounds. Using some experimental hints, we evaluate the temperature corresponding to usual laboratory flows (Taylor scale based Reynolds number R_λ of order 10^3). We finally suggest an analogy between the infinite Reynolds number limit and a zero temperature critical point.

2. The cascade

For the purpose of the paper being self contained we recall in this paragraph why the velocity statistics at scale r can be obtained from that at large scale by the repetition of an elementary process. As proposed by several authors [7–10], the distribution $P_r(\delta v)$ of longitudinal velocity differences δv at the distance r is given by:

$$P_r(\delta v) = \int G_{rL}(\ln \alpha) \frac{1}{\alpha} P_L\left(\frac{\delta v}{\alpha}\right) d \ln \alpha \quad (1)$$

The physical meaning is as follows. If G_{rL} were a Dirac distribution, the distribution P_r would have the same shape than the distribution at the distance L , within a scaling factor in the velocity amplitudes. This corresponds to the Kolmogorov 41 picture. Equation (1) means that the true distribution is the superposition of such shapes with different velocity scales: $\sigma = \alpha \sigma_L$ where σ_L is a characteristic velocity scale at distance L (e.g. σ_L could be the root mean square of P_L). Each scaling factor α has a probability $G_{rL}(\ln \alpha)$. The physical soundness of such a decomposition has been experimentally demonstrated in reference [11].

Now, as remarked in a previous paper [12], the distance L plays no peculiar role and we can replace it by any scale r_1 , larger than r :

$$P_r(\delta v) = \int G_{rr_1}(\ln \alpha_1) \frac{1}{\alpha_1} P_{r_1}\left(\frac{\delta v}{\alpha_1}\right) d \ln \alpha_1 \quad (2)$$

Using (1):

$$\frac{1}{\alpha_1} P_{r_1}\left(\frac{\delta v}{\alpha_1}\right) = \int G_{r_1L}(\ln \alpha_0) \frac{1}{\alpha_1 \alpha_0} P_L\left(\frac{\delta v}{\alpha_1 \alpha_0}\right) d \ln \alpha_0$$

and:

$$P_r(\delta v) = \int G_{rr_1}(\ln \alpha_1) G_{r_1L}(\ln \alpha_0) \frac{1}{\alpha_1 \alpha_0} P_L\left(\frac{\delta v}{\alpha_1 \alpha_0}\right) d \ln \alpha_1 d \ln \alpha_0 \quad (3)$$

Comparing with (1) we have to identify α and $\alpha_1 \alpha_0$ and:

$$G_{rL}(\ln \alpha) = \int G_{rr_1}(\ln \alpha_1) G_{r_1L}(\ln \alpha - \ln \alpha_1) d \ln \alpha_1 \quad (4)$$

which means that G_{rL} is the convolution of G_{rr_1} and G_{r_1L} :

$$G_{rL} = G_{rr_1} \otimes G_{r_1L}$$

If the transfer process, materialized by the distributions G , is self-similar along the scales, it is possible to find a series of scales r_i

$$r_0 = L > r_1 > \dots > r_n = r \quad (5)$$

such that

$$G_{rL} = G_{rr_{n-1}} \otimes \dots \otimes G_{r_1L} \quad (6)$$

with all distributions $G_{r_i, r_{i-1}}$ equal to the same function H . This series allows to define a function $n(r)$ with $n(r_i) = i$, with increments $\delta n = n(r_i) - n(r_j)$ such that:

$$G_{\delta n} = G_{r_i, r_j} = H^{\otimes \delta n} \quad (7)$$

and

$$G_{\delta n_1} \otimes G_{\delta n_2} = G_{\delta n_1 + \delta n_2} \quad (8)$$

Equation (8) express the fact that the process $G_{\delta n}$ is an infinitely divisible process. Note that the infinite divisibility recently quoted by She and Waymire [2] concerned the distribution of $\ln \varepsilon$ where ε is an ill-defined quantity: local dissipation or local energy transfer rate depending on the authors.

The series of scales r_i materializes a cascade process. At each step i , $i + 1$, the velocity scale goes from σ_i to $\sigma_{i+1} = \alpha_i \sigma_i$ with the probability $H(\ln \alpha_i)$. Note however that the discrete character of this cascade is artificial, and the step size can be arbitrarily chosen: no experimental signature of a particular step size has never been reported.

3. Definition of the Temperature

We shall now derive the shape of G_{rL} on very general grounds, defining the temperature as a quantity independent of the scale when some statistical equilibrium is reached in the cascade.

As remarked above, $G_{rL} \left(\ln \frac{\sigma}{\sigma_L} \right)$ gives the probability to "observe" a characteristic velocity σ at scale r , the value σ_L for the scale L being fixed. As in thermodynamics we shall consider the logarithm of this probability for dealing with additive quantities. This logarithm, as G , depends indeed on the scales r and L , on σ_L and σ . For finite Reynolds number R_e it also depends on the kinematic viscosity of the fluid. Being an undimensional quantity, the logarithm of the probability depends only on undimensional combinations of the above listed variables (Buckingham's theorem [13]). These are three, which can be chosen as: $\frac{L}{\eta}, \frac{r}{\eta}, \frac{\sigma}{\sigma_L}$, where η is the Kolmogorov dissipative scale $\left(\frac{L}{\eta} \approx R_e^{3/4} \right)$:

$$\ln G_{rL} \left(\ln \frac{\sigma}{\sigma_L} \right) = Q \left(\frac{L}{\eta}, \frac{r}{\eta}, \frac{\sigma}{\sigma_L} \right) \quad (9)$$

Note that in the invicid limit ($R_e \rightarrow \infty$), Q should only depend on $\frac{L}{r}$. However this limit can be singular: for a general discussion of the second kind self similarity see Barenblatt [14].

As remarked in Section 2, L should play no peculiar role. We can consider the probability to observe a characteristic velocity σ_2 at scale r_2 , given a value σ_1 at scale r_1 . Its logarithm is

$$\ln G_{r_2 r_1} \left(\ln \frac{\sigma_2}{\sigma_1} \right) = Q \left(\frac{r_1}{\eta}, \frac{r_2}{\eta}, \frac{\sigma_2}{\sigma_1} \right) \quad (10)$$

Let us now consider two adjacent ranges of scales $[r_1, r']$ and $[r', r_2]$, and let us call σ' the characteristic velocity at scale r' . The value of $Q \left(\frac{r_1}{\eta}, \frac{r_2}{\eta}, \frac{\sigma_2}{\sigma_1} \right)$ is dominated by the case where σ' makes maximum the following sum:

$$Q \left(\frac{r_1}{\eta}, \frac{r'}{\eta}, \frac{\sigma'}{\sigma_2} \right) + Q \left(\frac{r'}{\eta}, \frac{r_2}{\eta}, \frac{\sigma_2}{\sigma'} \right)$$

which is the logarithm of the integrand in the convolution equation (4). It corresponds to:

$$\frac{\partial}{\partial \ln \left(\frac{\sigma'}{\sigma_1} \right)} Q \left(\frac{r_1}{\eta}, \frac{r'}{\eta}, \frac{\sigma'}{\sigma_1} \right) = \frac{\partial}{\partial \ln \left(\frac{\sigma_2}{\sigma'} \right)} Q \left(\frac{r'}{\eta}, \frac{r_2}{\eta}, \frac{\sigma_2}{\sigma'} \right) = \frac{1}{T} \quad (11)$$

We introduce the quantity T by analogy with thermodynamics where Q would be the entropy and $\ln \left(\frac{\sigma_2}{\sigma_1} \right)$ the energy. The above reasoning is valid for any adjacent range of scales. Consider thus three successive ones $A = [r_1, r_2]$, $B = [r_2, r_3]$ and $C = [r_3, r_4]$. The quantity T , defined by (11) as the inverse of the derivative of Q , is the same for A and B , and for B and C . It is thus the same for A and C and for any interval of scales, if the considered turbulence is in statistical equilibrium.

If the two intervals are very different in size, say $[L, r + dr]$ and $[r + dr, r]$, the largest one can act as a "thermostat" for the other. Namely $\ln \frac{\sigma_r}{\sigma_{r+dr}}$ can change in large proportions without really changing $\ln \frac{\sigma_{r+dr}}{\sigma_L}$

Then $Q \left(\frac{r + dr}{\eta}, \frac{r}{\eta}, \frac{\sigma_r}{\sigma_{r+dr}} \right)$ can be approximated by its linear part and

$$G_{r \ r+dr}(x) \propto \exp \frac{x}{T} \quad (12)$$

where $x = \ln \frac{\sigma_r}{\sigma_{r+dr}}$

Two important remarks must be made at this stage. We have first to normalize this probability. This asks for a maximum value of x if T is positive, or a minimum value if $T < 0$. Let us concentrate on $T > 0$, the transposition to $T < 0$ being easy, and call x_0 this maximum value of $\ln \frac{\sigma_r}{\sigma_{r+dr}}$. The normalized version of the above probability distribution is

$$\begin{aligned} F(x) &= \frac{1}{T} \exp \frac{(x - x_0)}{T} \quad (x \leq x_0) \\ &= 0 \quad (x > x_0) \end{aligned} \quad (13)$$

The second remark is that, when dr goes to zero, the probability for $\sigma_{r+dr} = \sigma_r$ goes to 1. Thus:

$$\begin{aligned} G_{r \ r+dr}(x) &= \frac{1}{(1+a)} (\delta(x) + a F(x)) \\ &= H(x) \end{aligned}$$

where a goes to zero when dr goes to zero and is proportional to it for sufficiently small dr [15]. Using for $G_{r \ r+dr}$ the notation H , we want to recall, as in Section 2, that we can divide the whole interval $[L, r]$ into n intervals such that:

$$G_{rL} = H^{\otimes n} \quad (14)$$

Going to the Fourier transforms, equation (14) gives:

$$\tilde{G}_{rL} = \tilde{H}^n = \frac{1}{(1+a)^n} (1 + a \tilde{F})^n \cong \exp \{ na(\tilde{F} - 1) \} \quad (15)$$

Clearly a and n depend on the choice of dr , but na should not. We shall use the notation:

$$na = s \quad (16)$$

s is an intrinsic measure of the depth of the cascade. We can expand (15):

$$\tilde{G}_{r_1 L} = e^{-s} \sum \frac{s^i}{i!} \tilde{F}^i$$

which allows to Fourier transform back:

$$G_{rL} = e^{-s} \sum \frac{s^i}{i!} F^{\otimes i} \quad (17)$$

(such a distribution corresponds to a compound Poisson process [16]).

Now the possibilities are twofold. Either $|x_0|$ is much larger than T . Then $F(x)$ can be approximated by a Dirac distribution centred on x_0 , as its width, of order T , can be neglected. Equation (17) corresponds then to a pure Poisson process (the distribution of σ is log Poisson). The situation is that proposed by She and Levêque [3], and the coefficient β they introduce, whose logarithm is the quantum in $\ln \varepsilon$, is simply $\exp 3x_0$. This is what we could call the extreme quantum case [2]: x_0 is the quantum in $\ln \sigma$.

The opposite situation is that of interest for us. This is when $|x_0| \ll T$. Then we can take $x_0 = 0$, and call this case the “thermodynamic” case.

We shall see later that T should go to zero when the Reynolds number goes to infinity. If x_0 , which should be determined by the most singular events, is different from zero and not “temperature” dependent we could observe a cross over from the thermodynamic to the quantum case, when raising the Reynolds number. We consider the intermediate case in the Appendix. Here we focus on the thermodynamic case where:

$$F(x) = \begin{cases} \frac{1}{T} e^{x/T} & x \leq 0 \\ 0 & x > 0 \end{cases}$$

$$\begin{aligned} F^{\otimes n}(x) &= \int \int d^n x_i \delta \left(x - \sum_i x_i \right) \prod_{i=1}^n F(x_i) \\ &= \begin{cases} \frac{x^{n-1}}{(n-1)! T^n} e^{x/T} & x \leq 0 \\ 0 & x > 0 \end{cases} \end{aligned}$$

the δ -distribution ensuring that the sum of x_i is x . Thus:

$$G_{rL}(x) = \begin{cases} e^{-s} \sum_n \frac{s^n x^{n-1}}{n!(n-1)! T^n} e^{x/T} & x \leq 0 \\ 0 & x > 0 \end{cases} \quad (18)$$

4. Some Consequences

Two types of measurements give an insight in the distributions G_{rL} . A series of studies have looked at the variance of $\ln \sigma$ which measure the depth of the cascade and its evolution with r . An other type of studies have been initiated by Ruiz Chavarria *et al.* in order to check the predictions of She and Lévéque and aim to determine the shape of the distribution G_{rL} . In this paragraph we calculate the quantities allowing a comparison with these experimental results.

4.1. THE DEPTH OF THE CASCADE. — The variance of $\ln \frac{\sigma}{\sigma_L}$ is calculated in the Appendix. In the thermodynamic case (Eq. (18)) it gives:

$$\left\langle \left(\delta \ln \frac{\sigma}{\sigma_L} \right)^2 \right\rangle = 2sT^2 \quad (19)$$

The dependence with the scale r is entirely contained in s . The consequence of the variational approach for this quantity is discussed in references [5,9] and the experimental tests are examined in references [6,9,17].

Let us make here a remark. Following the same process than for Q (the logarithm of probability) we can infer that the difference $s(r_1) - s(r_2)$ depends on the three non dimensional parameters: $\frac{r_1}{\eta}$, $\frac{r_2}{\eta}$ and the Reynolds number (or T). But, as remarked after equation (9), in the infinite Reynolds number limit, this difference only depends on the ratio $\frac{r_1}{r_2}$ $s(r_1) - s(r_2) = \delta s \left(\frac{r_1}{r_2} \right)$. From the definition of δs we have the obvious property:

$$\delta s \left(\frac{r_1}{r_2} \right) + \delta s \left(\frac{r_2}{r_3} \right) = \delta s \left(\frac{r_1}{r_3} \right)$$

which implies that s depends linearly on $\ln r$. This gives the Kolmogorov Obukhov law for $\langle (\delta \ln \sigma)^2 \rangle$:

$$\langle (\delta \ln \sigma)^2 \rangle = \mu \ln \frac{L}{r} \quad (20)$$

and this is also true within all the multifractal models which explicitly refer to this infinite Reynolds limit [10].

4.2. THE STRUCTURE FUNCTIONS. — Here we must make the comment that what follows is only an approximation of a more complex situation. The non zero skewness of the distributions P_r of δv introduces a shift for the component $P_L \left(\frac{\delta v \sigma_L}{\sigma} \right)$ of the decomposition, equation (1). The consequences are discussed in [9] and the shift is experimentally observed in [11]. However these consequences concern mainly the odd part of the distribution which allows to neglect them for the symmetric structure function:

$$S_p(r) = \langle |\delta v|^p \rangle$$

already considered by Benzi and coworkers [18]. Neglecting this shift and using equation (1) we have:

$$\begin{aligned} \langle |\delta v|^p \rangle &= \int G_{rL} \left(\ln \frac{\sigma}{\sigma_L} \right) \left(\frac{\sigma}{\sigma_L} \right)^p \, d \ln \sigma \int \left| \frac{\delta v \sigma_L}{\sigma} \right|^p P_L \left(\frac{\delta v \sigma_L}{\sigma} \right) d \frac{\delta v \sigma_L}{\sigma} \\ &= \sigma_L^p B_p \exp \left(-s \frac{pT}{1+pT} \right) \end{aligned} \quad (21)$$

(see appendix)

where $B_p = \frac{1}{\sigma_L^p} \int |x|^p P_L(x) \, dx$ is a non dimensional number independent of T and r .

The consequence, which can be extended to the quantum case, is that various structure functions behave as power laws of each other. For instance:

$$\left\langle \left| \frac{\delta v}{\sigma_L} \right|^p \right\rangle \propto \left(\left\langle \left| \frac{\delta v}{\sigma_L} \right|^3 \right\rangle \right)^{\zeta'_p}$$

with

$$\zeta'_p = \frac{p}{3} \frac{1 + 3T}{1 + pT} \quad (22)$$

This is the extended self similarity recently proposed [18] which is thus equivalent to the infinite divisibility of the $\ln \sigma$ distribution (not the $\ln \varepsilon$ one). We find here that the exponents ζ'_p are T dependent while they are generally considered as universal. However, a recent compilation [19] shows that a large range of T is compatible with the experimentally observed exponents centered around $T = 0.05$.

Another consequence of equation (21) is the linear dependence of $\langle (\delta \ln \sigma)^2 \rangle$ with the logarithm of the structure functions:

$$\begin{aligned} \langle (\delta \ln \sigma)^2 \rangle &= 2sT^2 = \frac{-2(1 + 3T)}{3} T \ln \frac{\langle |\delta v|^3 \rangle}{\sigma_L^3 B_3} \\ &= -\mu' \ln \frac{\langle |\delta v|^3 \rangle}{\sigma_L^3 B_3} \end{aligned} \quad (23)$$

At large scale $\langle \delta v^3 \rangle \propto r$. If we assume the same scaling for $\langle \delta v^3 \rangle$ and $\langle |\delta v|^3 \rangle$ (again an approximation) we have $\mu' \simeq \mu$ (Eq. (20)). Note that μ' is linear in T for small T , which is coherent with the thermodynamical interpretation of a recent experiment [6]. The experimentally observed μ' seem to be slightly Reynolds dependent, with an average value again in agreement with $T \simeq 0.05$ [20, 21].

4.3. THE SHAPE OF G_{rL} . — In order to check the She and Lévêque conjectures, Ruiz Chavarria *et al.* [22] proposed to look at the quantity:

$$y_p = \ln \frac{S_p(r) S_{p-1}(r')}{S_p(r') S_{p-1}(r)} \quad (24)$$

In the frame of the She and Lévêque theory, the graph of y_{p+1} versus y_p is a straight line with a slope smaller than 1 [22]. Using equation (21) gives for the thermodynamic case:

$$y_p = \frac{(s(r') - s(r)) T}{(1 + pT)(1 + (p-1)T)} \quad (25)$$

Developing in powers of T , we have:

$$\frac{y_{p+1} - y_p}{y_p - y_{p-1}} = 1 - 3T + 3(p+1)T^2 \quad (26)$$

Obviously, the graph of y_{p+1} versus y_p is not here a straight line which would correspond to a pure Poisson process. But the variation of the slope only appears on the second order term in T . Comparing (26) with the value proposed by She and Lévêque $\left((2/3)^{1/3} = 0.88 \right)$, we again obtain $T \simeq 0.05$ as an order of magnitude.

4.4. THE REYNOLDS NUMBER DEPENDENCE OF T . — All the above considered quantities are generally assumed universal, while we find a T dependence which implies a Reynolds number dependence. This raises the problem of the Reynolds number dependence of T . For instance, the compilation above mentioned [19] shows that for $100 \lesssim R_\lambda \lesssim 3000$, the error bar on ζ_6 is less than 5% without apparent trend with R_λ :

$$1.67 \lesssim \zeta_6 \lesssim 1.83$$

However T is proportional to the correction to the Kolmogorov 41 theory,

$$0.33 \gtrsim 2 - \zeta_6 \gtrsim 0.17$$

almost a factor 2 of possible variations. From the observation that the shape of the velocity gradient distribution seems independent of R_λ (a point which is always under consideration, see [17, 23]) we infer that sT^2 for the dissipative scale is Reynolds independent and, following (23), that μ' and T are inversely proportional to $\ln R_\lambda$:

$$\frac{1}{T} \propto \ln \frac{R_\lambda}{R^*} \quad (27)$$

If R^* is of order 1, the corresponding variation of T and of $2 - \zeta_6$ is within the error bar. This simply shows that the ζ_p are not the good quantity to measure for evaluating T . The previously mentioned ones are much better.

Let us finish by an amazing remark. The infinite Reynolds number situation has often been compared to the critical point [24] of a second order phase transition. In statistical physics there are several examples of zero temperature critical points (for instance the one-dimensional Ising model). The important thermodynamic quantities (susceptibility,...) then behave as $\exp \frac{\theta}{T}$ where θ is some characteristic temperature. This is exactly the behaviour which is assumed by (27) for R_λ .

5. Conclusion

The main point of this paper is the following : in a developed turbulent flow, the random character of the (energy) cascade process results in the conservation of a quantity along the scales. This quantity is Reynolds dependent and we call it a temperature by analogy with thermodynamics.

We have shown how this temperature determines the distribution of turbulent intensities at each scale. This canonical distribution (again so named by reference to thermodynamics) does not involve the quantization emphasized in a recent work [2]. We claim however that it is coherent with all available experimental data. As for the exponent ζ_p often invoked to characterize developed turbulence, we predict that they are Reynolds dependent, but in a very weak way, which makes them a very poor characterization. Indeed, the best characterization comes from the depth of the cascade, defined as the width of the canonical distribution, and the curvature of the ζ_p , which may be directly evaluated. Both measurements allow an estimation of the temperature. Finally, let us note that the present theory gives results which differ in many ways from generally accepted ideas. These points can be experimentally checked and will provide us new insight in the statistical properties of developed turbulence.

Appendix

When $x_0 \neq 0$, $F(x)$ is given by (13) and

$$F^{\otimes n}(x) = \begin{cases} \frac{(x - nx_0)^{n-1}}{T^n(n-1)!} e^{-\frac{x-nx_0}{T}} & x \leq nx_0 \\ 0 & x > nx_0 \end{cases}$$

Let us note $\langle \rangle_n$ the average with the distribution $F^{\otimes n}$. We have:

$$\langle x \rangle_n = \langle x - nx_0 \rangle_n + nx_0 = n(x_0 + T)$$

$$\begin{aligned} \langle x^2 \rangle_n &= \langle (x - nx_0)^2 \rangle_n + 2nx_0 \langle x - nx_0 \rangle_n + n^2x_0^2 \\ &= n(n+1)T^2 + 2n^2x_0T + n^2x_0^2 \end{aligned}$$

Thus:

$$\langle \ln \frac{\sigma}{\sigma_L} \rangle = e^{-s} \sum \frac{s^n}{n!} n(x_0 + T) = s(x_0 + T)$$

$$\begin{aligned} \left\langle \left(\ln \frac{\sigma}{\sigma_L} \right)^2 \right\rangle &= e^{-s} \sum \frac{s^n}{n!} \left[n(n-1)(T+x_0)^2 + n(2T^2 + 2x_0T + x_0^2) \right] \\ &= s^2(x_0 + T)^2 + s \left[(x_0 + T)^2 + T^2 \right] \end{aligned}$$

and:

$$\langle (\delta \ln \sigma)^2 \rangle = \left\langle \left(\ln \frac{\sigma}{\sigma_L} \right)^2 \right\rangle - \langle \ln \frac{\sigma}{\sigma_L} \rangle^2 = s \left[(x_0 + T)^2 + T^2 \right] \quad (28)$$

The moment $\langle |\delta v|^p \rangle$ are proportional to $\left\langle \left(\frac{\sigma}{\sigma_L} \right)^p \right\rangle = \langle e^{px} \rangle$. We have:

$$\begin{aligned} \langle e^{px} \rangle_n &= \langle e^{p(x-nx_0)} \rangle_n e^{pnx_0} \\ &= \frac{e^{pnx_0}}{(n-1)! T^n} \int (x - nx_0)^{n-1} e^{(p+\frac{1}{T})(x-nx_0)} dx \\ &= \left[\frac{e^{px_0}}{1+pT} \right]^n \end{aligned}$$

and:

$$\left\langle \left(\frac{\sigma}{\sigma_L} \right)^p \right\rangle = e^{-s} \sum \frac{s^n}{n!} \left[\frac{e^{px_0}}{1+pT} \right]^n = \exp -s \left[1 - \frac{e^{px_0}}{1+pT} \right] \quad (29)$$

The consequence is that the various moments $\langle |\delta v|^p \rangle$ behave as power laws of each other. This is the extended self similarity [18] which is here derived from first principles, within the warning quoted at the beginning of Section 4.2.

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$$\zeta'_p = \left(\frac{p}{3}\right) \left(K + \frac{T}{1+pT}\right) / \left(K + \frac{T}{1+3T}\right) \text{ and } \mu = 2T^2 / \left(K + \frac{T}{1+3T}\right)$$
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