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Rheology and Structure of Concentrated Suspensions of Hard Spheres. Shear Induced Particle Migration

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Résumé. — A partir d'une estimation de la dissipation visqueuse, nous avons précédemment évalué la viscosité apparente de cisaillement des suspensions homogènes de sphères dures monodisperses et non Browniennes dans les systèmes dilués ou concentrés. Nous utilisons la dépendance de la viscosité de cisaillement avec la distance moyenne entre les particules pour déterminer les composantes d'un tenseur local de contraintes associé à l'amas transitoire de particules au dessus de la fraction volumique critique de percolation. Nous développons pour cela un modèle basé sur les forces de lubrification s'exerçant au cours des collisions entre les particules afin de coupler le tenseur local de contraintes associé aux particules et le tenseur des contraintes dans la suspension assimilée à un milieu effectif. Dans le cas des écoulements non uniformes présentant des zones de faible cisaillement, il est nécessaire de considérer un tenseur non local des contraintes car les forces peuvent alors se transmettre directement à travers l'amas de particules sur une longueur de corrélation supérieure au diamètre des particules. Cette approche montre que la migration des particules dans un écoulement est induite par les variations spatiales de la contrainte tangentielle. Les prévisions du modèle relatives aux profils de concentrations en particules dans un écoulement de Couette ou dans un écoulement de Poiseuille plan sont en bon accord avec les données expérimentales de la littérature.

Abstract. — The apparent shear viscosity, in the non-Brownian limit, for a homogeneous suspension of monodispersed hard spheres in systems ranging from dilute to concentrated was previously established. From an estimation of the viscous dissipation. We use the inter-particle distance dependence of the shear viscosity for determining the components of a local stress tensor associated with the transient network of particles for the volume fraction above the percolation threshold. For this purpose, we develop a model based on lubrication forces between colliding particles for coupling the particle stress tensor to the stress tensor of the suspension considered as an effective medium. In the case of non-uniform flows with low shear rate regions, it is necessary to introduce a non-local stress tensor since the stress can be directly transmitted by the network of particles over a correlation length larger than the particle diameter. This approach shows

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that particle migration in a flow is induced by the spatial variation of the tangential shear stress. Predictions of the model for the equilibrium concentration profiles in Couette flow or in plane Poiseuille flow are found to be in good agreement with available experimental data

1. Introduction

Computing the effective shear viscosity of a suspension is a well known and long standing problem in fluid mechanics. The basic problem for suspensions is how to compute the momentum transfer using the microstructural mechanics. More than half a century passed between the early work of Einstein presented as his thesis [1], and the work of Batchelor and Green [2]. Batchelor and Green extended Einstein's formula to second order in sphere volume fraction ϕ :

$$\mu(\phi) = \mu_0(1 + 2.5\phi + k\phi^2) \quad (1)$$

where μ_0 is the viscosity of the suspending liquid medium. The k coefficient depends on both the two-particle hydrodynamic interactions and the suspension microstructure through the pair distribution function. By making plausible assumptions about particle collision in a well mixed suspension of hard spheres, Batchelor and Green found that $k \approx 5.2$ [2].

For concentrated suspensions of non-Brownian hard spheres in a viscous flow, one has to be able to compute the many-body interactions and determine the microstructure configuration. Two important approaches were developed in the last few years: a calculation by Beenakker [3] based on the method of Mazur and Van Saarloos, and numerical simulations based on Stokesian dynamics by Bossis and Brady [4].

In the first Section, we recall the expression of the apparent shear-viscosity in the non-Brownian limit of a homogeneous suspension of monodispersed spheres for systems ranging from dilute to concentrated [5,6]. A collection of monodispersed spheres suspended in a viscous liquid constitutes a single scaled system. A mean approach was then used for determining the viscous dissipation in the suspending fluid. The basis for this calculation is the relationship between the characteristic mesoscopic scale of the system (the mean distance between particles) and the volume fraction which was derived from statistical arguments.

In the second Section, starting from a homogeneous suspension in a flow, we examine how the shear induced particle migration may lead to strongly inhomogeneous concentration profiles. Recently, Leighton and Acrivos [7] proposed a mechanism for particle migration in non-homogeneous shear flows. Leighton and Acrivos assume that a driving force perpendicular to shear planes is resulting from both the effects of spatially varying interparticle interaction and effective viscosity.

A modified version was later proposed by Phillips *et al.* [8] who derived a diffusion equation for describing the time dependence evolution of particle concentration profiles. This approach is based upon the effects of irreversible two-body interactions. The equilibrium concentration profile obtained for a given flow depends on an experimentally determined adjustable parameter.

In this work, we introduce a stress tensor associated with the transient network of particles for volume fractions above a percolation threshold. The stress tensor components are derived from the previously established relationship relating the apparent viscosity to the mean distance between particles. Finally, we deduce the diffusive flux of particles in the direction transverse to the mean flow and the steady state concentration profile without any adjustable parameter.

2. Apparent Shear Viscosity of Suspensions of Hard Spheres

We consider an elementary volume containing fluid (density ρ_f) and hard particles (density ρ_p) in purely hydrodynamic interactions. The volume averaged velocity field \mathbf{V} is

$$\mathbf{V} = \phi \mathbf{V}_p + (1 - \phi) \mathbf{V}_f$$

where \mathbf{V}_p and \mathbf{V}_f are the volume-averaged velocities of particles and fluid, respectively, and ϕ the particle volume fraction. The suspension is incompressible

$$\text{div} \mathbf{V} = 0.$$

The velocity field is governed by a transport equation for the momentum

$$\langle \rho \rangle d_t \mathbf{V} = -\text{grad } p + \text{div} \boldsymbol{\tau} + \mathbf{F} \quad (2)$$

where p is the pressure, $\langle \rho \rangle = \phi \rho_p + (1 - \phi) \rho_f$ is the mean density of the suspension considered as an effective continuous medium, d_t is the substantial derivative and \mathbf{F} the external force per unit volume. In a stationary flow at low Reynolds number, the left term in equation (2) can be neglected. In a simple shear flow, the relationship between the viscous stress tensor and the rate of strain tensor further reduces to

$$\boldsymbol{\tau} = \mu(\phi, \gamma^0) \boldsymbol{\gamma}^0 \quad (3)$$

where $\mu(\phi, \gamma^0)$ is the apparent shear viscosity of the suspension and γ^0 is the shear rate.

In a previous study [5, 6], a relationship between the effective shear viscosity $\mu(\phi, \gamma^0)$ and the mean distance $d(\phi)$ between particles (radius a) was established. The viscous dissipation can be expressed as the square of the ratio of the mean velocity variation δv_f of fluid between particles (scaling as $\gamma^0 a$) and the characteristic length d of the flow between particles

$$d_t D = \mu(\phi) \gamma^{02} = \mu_0 \left\langle \frac{\gamma^0 a}{d} \right\rangle^2 (1 - \phi). \quad (4)$$

In the concentrated regime ($\phi > 0.2$), the apparent shear viscosity is further well fitted by the following expression (5) proposed in our previous papers [5, 6] :

$$\mu(\phi) = \mu_0 \frac{(1 - \phi)}{(1 - \phi/\phi^*)^2} \quad (5)$$

where ϕ^* is the critical volume fraction relative to the transition threshold between the fluid state and the solid state

The random dense packing of monodispersed hard spheres is $\phi^* \approx 0.637$ and corresponds to an isotropic microstructure around the zero shear rate limit. In the high shear rate limit, the shear forces between particles make the pair distribution function anisotropic along the main strain axes [7]. The flow-induced anisotropic microstructures (deformation and orientation of transient clusters in a shear flow) and the maximum packing fraction then closely depend on the shear rate. In a Couette flow, the maximal packing fraction allowing the flow of suspension is $\phi^* \approx 0.67$.

3. Shear Induced Particle Migration

3.1. TRANSVERSE DIFFUSIVE FLUX. — Following a conjecture proposed by de Gennes [8], we expect a transient infinite cluster above a critical volume fraction corresponding to a percolation threshold ($\phi_{\text{per}} \approx 0.2$).

In the concentrated regime ($\phi > \phi_c > \beta\phi_{\text{per}}$ with $\beta > 1$) the strong lubrication interactions between particles thus result in the transmission of viscous stresses through a transient connected network of spheres. Under uniform shearing conditions, the life time of the infinite cluster being much shorter than the characteristic flow time $1/\gamma^0$, the apparent shear viscosity and the derivative $\partial\mu/\partial\phi$ are smooth functions of the volume fraction. Nevertheless, a discontinuity could appear above the percolation threshold for the diffusive flux dependence. Hence, above ϕ_c , we consider a stress tensor σ_p associated with interacting particles and we introduce two velocity fields as in [9, 10]. The first velocity field \mathbf{V} describes the motion of particles and fluid as an incompressible effective medium and the second one \mathbf{V}_p gives the average local velocity of the neutrally buoyant particles. At low Reynolds number, the vector momentum equation for both the effective medium and the set of particles (radius a) can be written as:

$$0 = -\text{grad } p + \text{div } \boldsymbol{\tau} \quad \text{with} \quad \text{div } \mathbf{V} = 0 \quad (6a)$$

$$0 = -\phi \text{ grad } p + \text{div } \boldsymbol{\sigma}_p - \phi\mu_0\mu^* \frac{\mathbf{V}_p - \mathbf{V}}{a^2} \quad (6b)$$

where the last term is the external friction force exerted on particles (mean velocity \mathbf{V}_p) by the suspension considered as an effective medium with a mean velocity \mathbf{V} . The dimensionless function $\mu^*(\phi)$ takes into account the multibody interactions.

Since two-particle hydrodynamic interactions are known to be reversible in Stoke's flow, we need a source of irreversibility to produce a net lateral displacement of particles *via* two-body collisions. Particle roughness was proposed by Leighton and Acrivos [11] as a candidate for the irreversibility source but, as pointed in [12], the close contact of particles also may lead to *chaotic* trajectories and irreversible interactions during two-body collisions. In a transient cluster, irreversible interactions between particles should give rise to a transverse normal stress similar to that which occurs in the static "arching" of grains in contact. In a bounded flow, each particle of the solid phase described by equation (6b) experiences transverse forces either transmitted from the wall through the connected network over a correlation length ξ [13] or due to the shear stress $\boldsymbol{\tau}$ in the effective medium described by equation (6a). Both, the shear rate and the flow geometry may influence the correlation length ξ . In the case of a uniform shear field, the length ξ is about the mean distance between two particles. On the other hand, the flattened velocity profile in the core region of a Poiseuille flow increases the correlation length which then scales as the distance to the channel center. In the very dilute regime and in the absence of irreversible processes, the time average transverse force is zero.

We propose here a model based on lubrication forces for relating the stress tensor $\boldsymbol{\sigma}_p$ and the mean viscous stress tensor $\boldsymbol{\tau}$ in the suspension. The diffusive fluxes of particles result from the mean energy received by colliding particles in the shear flow. During a collision, the microflows between particles give rise to lubrication drag forces along and perpendicular to the shear planes. The hydrodynamic drag between colliding spheres scales as the shear stress component along the shearing surfaces.

The time-averaged lubrication drag force \mathbf{F}_h experienced by a particle (radius a) during a collision is then proportional to the shear stress force due to the effective medium surrounding the particles and to the inverse of the dimensionless distance (a/d) between near particles [14]:

$$\mathbf{F}_h \approx \tau a^2 \frac{a}{d} \quad \text{for } a/a \ll 1 \quad (7)$$

The mean shear stress σ_p resulting from irreversible particle collisions can be obtained by multiplying the time-averaged lubrication force \mathbf{F}_h by the mean number $N \approx \phi/a^2$ of particles per unit surface in a shear plane:

$$\sigma_p \approx N \mathbf{F}_h \approx \tau \phi \frac{a}{d} \quad (8)$$

Equation (4) established in the first part of this article gives a further relationship between the effective shear viscosity and the space-averaged distance between particles:

$$\mu(\phi, d) \approx \mu_0(1 - \phi) \left(\frac{a}{d}\right)^2 \quad (9)$$

Substituting equation (9) into equation (8), we then obtain an estimate for the mean stress tensor in the concentrated regime:

$$\sigma_p \approx \tau \left(\frac{\mu}{\mu_0}\right)^{1/2} \frac{\phi}{(1 - \phi)^{1/2}} \quad (10)$$

In the diluted regime, we may no longer use the effective medium concept for deriving the mean stress tensor which then scales as $\sigma_p \approx \phi^3 \tau$ [11].

The y component of the vector momentum equation (6b) along a direction perpendicular to the shearing planes in a creeping flow is:

$$0 = -\phi \mu_0 \mu^*(\phi) \frac{V_p}{a^2} + \partial_y \sigma_{yy} + \partial_x \sigma_{xy} + \partial_z \sigma_{zy} \quad (11)$$

The last two terms of equation (11) vanish since the components of the stress tensor are constant along the shearing planes.

We finally obtain for the transverse diffusive flux of particles:

$$J = \phi V_p = -\frac{a^2}{\mu_0 \mu^*} \frac{\partial}{\partial y} \left[\frac{\phi \mu_r^{1/2} \tau}{(1 - \phi)^{1/2}} \right] \quad (12)$$

where $\mu_r(\phi)$ is the relative effective shear viscosity.

It follows from equation (12) that the transverse diffusive flux of particles is not dependent upon the viscosity of the free suspending fluid in agreement with the experimental results from *Abbot et al.* [15]. The diffusive flux dependence on the square of particle radius would lead to strong segregation effects in flows of highly polydispersed concentrated suspensions or granular media considered as fluids.

3.2. STEADY STATE CONCENTRATION PROFILE. — The diffusion equation in a Eulerian reference frame is written as

$$D_t \phi + \text{div } \mathbf{J} = 0$$

where $D_t = \partial_t + \mathbf{v}$ is the material time derivative. For viscometric flows, the term $\mathbf{v} \cdot \text{grad}$ is clearly zero and the diffusion equation then takes the following form:

$$\partial_t \phi + \text{div } \mathbf{J} = 0$$

Under steady state conditions in viscometric flows, the condition for equilibrium concentration profiles implies zero transverse diffusive flux everywhere in the flow:

$$J = -\frac{a^2}{\mu_0\mu^*} \frac{\partial}{\partial y} \left[\frac{\phi\mu_r^{1/2}\tau}{(1-\phi)^{1/2}} \right] = 0 \quad (13)$$

Integrating this equation over the space variable y with an integration constant fixed by the boundary conditions then gives the equilibrium concentration profile over the flow cross section:

$$\frac{\phi}{(1-\phi)^{1/2}}\mu_r^{1/2}\tau = \frac{\phi_w}{(1-\phi_w)^{1/2}}\mu_{rw}^{1/2}\tau_w \quad (14)$$

where the subscript "w" refers to the wall.

As a remark, we note that in flow between plane parallel plates with relative motion, no migration is predicted by the model as observed experimentally. On the other hand, in Couette flow or in Poiseuille flow, the space variation of the shear stress along the y direction perpendicular to the shear planes induces migration.

3.3 COUETTE FLOW — For a Couette flow, the θ component of the vector momentum equation in cylindrical coordinates yields:

$$\frac{1}{r^2} \partial_r (r^2 \tau_{r\theta}) = 0 \quad (15)$$

Integrating equation (15) over the space variable r gives:

$$\tau_{r\theta} = \frac{C}{r^2} \quad (16)$$

Substituting the expression of the shear stress into equation (14), we obtain:

$$\frac{\phi(r)}{(1-\phi(r))^{1/2}}\mu_r^{1/2}\frac{C}{r^2} = \frac{\phi_w}{(1-\phi_w)^{1/2}}\mu_{rw}^{1/2}\frac{C}{R_w^2} \quad (17)$$

where the subscript "w" refers to the outer cylinder ($r = R_w$).

Equation (17) together with expression (5) of the effective shear viscosity gives the particle volume fraction $\phi(r)$ over the flow cross section:

$$\frac{\phi(r)}{(1-\phi(r)/\phi^*)} \left(\frac{R_w}{r} \right)^2 = \frac{\phi_w}{(1-\phi_w/\phi^*)} = A \quad (18)$$

We finally obtain

$$\frac{\phi(r)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \left(\frac{R_w}{r} \right)^2} \quad (19)$$

for $R_i \leq r \leq R_w$ and $\phi > \phi_c$ with:

$$A = \frac{\phi_w}{(1-\phi_w/\phi^*)}, \quad \int_{R_i}^{R_w} \phi(r)rdr = \phi \frac{R_w^2 - R_i^2}{2} \quad \text{and} \quad \phi_w = \phi(R_w)$$

where the subscript "i" refers to the inner cylinder.

This expression is the same as that recently proposed by Phillips *et al.* [16]. However, these authors introduced two mutually compensating incorrect terms in their theoretical formulation.

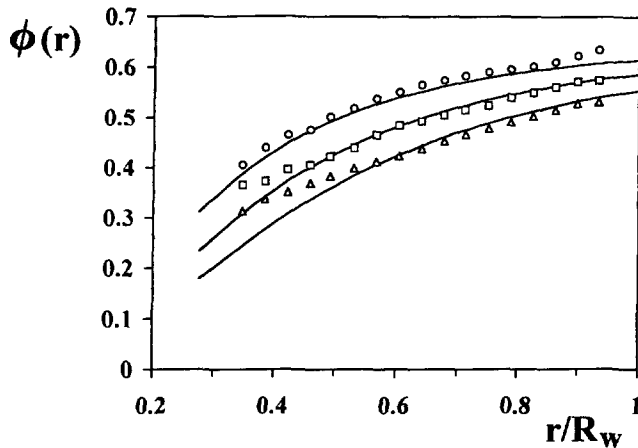


Fig. 1. — Equilibrium concentration profile $\phi(r)$ along an axis perpendicular to the velocity field for suspensions of 675 μm particles in a Couette flow (inner radius $R_i = 0.64$ cm and outer radius $R_w = 2.38$ cm) Experimental data from Phillips *et al.* [16] for particle volume fraction $\phi = 0.45$ (Δ), $\phi = 0.50$ (\square) and $\phi = 0.55$ (\circ) The solid line is obtained from equation (19) with $\phi^* = 0.67$ using the average volume fraction ϕ as input of the model

First, the shear stress component along the shearing planes does not appear explicitly in the diffusion equation, but as the product of the shear viscosity by the shear rate. Secondly, the expression of the shear stress component given in reference [16] ($\tau_{r\theta} = \mu \partial_r V_\theta$ instead of $\tau_{r\theta} = \mu \partial_r v_\theta - \mu v_\theta / r$) is not correct, especially in the wide-gap Couette device used in reference [16].

The theory is compared in Figure 1 to the data in [16] using the average volume fraction ϕ as input of the model. The relation (19) well describes particle migration in the high concentration regime. For less concentrated suspensions, the predictions deviate from the experimental data since the transverse diffusive flux is noticeably decreased in regions close to the inner cylinder where the local volume fraction derived from equation (19) may be less than the percolation threshold (Fig. 1). However, the predicted transition for the stress tensor σ_p in the vicinity of the percolation threshold is clearly supported by the experimental results.

3.4. POISEUILLE FLOW. — For a Poiseuille flow, the z component of the vector momentum equation in cylindrical coordinates yields

$$\frac{1}{r} \partial_r (r \tau_{rz}) = d_z p = p \tag{20}$$

where $d_z p$ is the z component of the pressure gradient driving the flow.

Integrating equation (20) over the space variable r gives.

$$\tau_{rz} = \frac{Pr}{2} \tag{21}$$

In cylindrical coordinates, the transverse diffusive flux of particles over the flow cross section is then given by

$$J_r = \phi v_p = - \frac{a^2}{\mu_0 \mu^*} \frac{1}{r} \left[\frac{\partial}{\partial r} (r \sigma_{rr}) - \sigma_{\theta\theta} / r \right] \tag{22}$$

with

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{\phi}{(1-\phi)^{1/2}} \mu_r^{1/2} \tau_{rz}$$

Integrating the steady state condition $J_r = 0$ over the space variable r with an integration constant fixed by the boundary conditions gives the equilibrium concentration profile over the flow cross section

$$\frac{\phi(r)}{(1-\phi(r))^{1/2}} \mu_r^{1/2} \frac{Pr}{2} = \frac{\phi_w}{(1-\phi_w)^{1/2}} \mu_{rw}^{1/2} \frac{PR_w}{2} \quad (23)$$

where the subscript "W" refers to the outer wall of the channel ($r = R_w$ with $\phi(R_w)$).

Substituting equation (5) for the relative shear viscosity into equation (23), we then obtain the equilibrium particle volume fraction $\phi(r)$ in a Poiseuille flow

$$\frac{\phi(r)}{(1-\phi(r)/\phi^*)} \frac{\tau}{R_w} = \frac{\phi_w}{(1-\phi_w/\phi^*)} = A \quad (24)$$

which finally gives:

$$\frac{\phi(r)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \frac{\tau}{R_w}} \quad (25)$$

for $0 \leq r \leq R_w$ and $\phi > \phi_c$ with

$$A = \frac{\phi_w}{(1-\phi_w/\phi^*)}, \quad \int_0^{R_w} \phi(r) r dr = \phi \frac{R_w^2}{2} \quad \text{and} \quad \phi_w = \phi(R_w).$$

The volume fraction profile reaches a maximum ϕ^* at $r = 0$ which reflects particle migration toward the centre of the channel (Fig. 2). We point out that the packing value is reached after an infinite time since both the shear rate and the transverse diffusive flux decrease to zero near the channel axis $r = 0$.

Expression (25) is similar to the relation obtained through a semi-empirical approach by Phillips *et al.* [16]. However, the concentration profile exhibits a sharp cusp near the flow centre (Fig. 2) which shows the limitations of the present model in the low shear rate regions of non-uniform flows. A sharp cup also appears in the case of a plane Poiseuille flow between two infinite parallel plates with a separation distance $2H$:

$$\frac{\phi(y)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \frac{y}{H}} \quad (26)$$

for $0 \leq y \leq H$ and $\phi > \phi_c$ with

$$A = \frac{\phi_w}{(1-\phi_w/\phi^*)}, \quad \int_0^H \phi(y) dy = \phi H \quad \text{and} \quad \phi_w = \phi(H).$$

A Non-Local Stress Tensor

As noted earlier, we expect the formation of a transient infinite cluster above the percolation threshold $\phi_{\text{per}} \approx 0.2$ [8]. In the concentrated regime, the strong lubrication interactions between particles thus result in the transmission of stress through a transient connected network of spheres over a correlation length ξ .

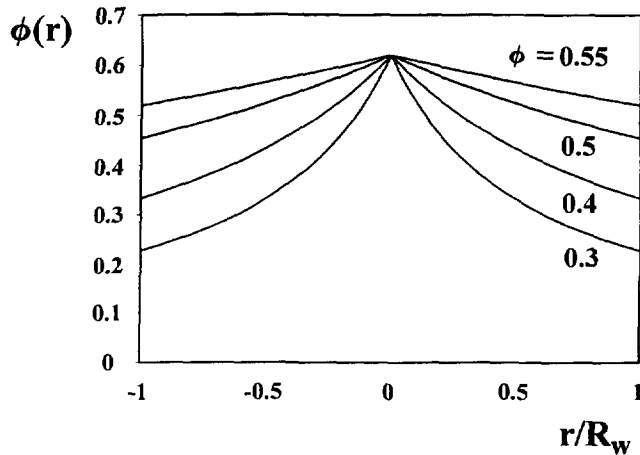


Fig. 2. — Theoretical equilibrium concentration profiles $\phi(r)$ along a radial axis perpendicular to the velocity field in a Poiseuille flow for various mean particle volume fraction in the channel with diameter R_w . The solid line is obtained from equation (25) with $\phi^* = 0.62$

The volume fraction dependence of the shear viscosity in a Couette flow displays no slope discontinuity since the life time of the transient structure over a length ξ remains short compared to the eddy diffusion time over the same length. On the other hand, the shear rate at the centre plane in a planar Poiseuille flow is always zero and therefore the life time of a cluster can be longer than the eddy diffusion time. Hence, above ϕ_{per} , we consider a non local stress tensor σ_p^* in the centre region of the flow.

For a plane Poiseuille flow, the z component of the vector momentum equation in Cartesian coordinates yields

$$\partial_y(\tau_{yz}) = d_z p = p \tag{27}$$

where $d_z p$ is the z component of the pressure gradient driving the flow between the two infinite planes $y = \pm H$. Integrating equation (27) over the y space variable gives

$$\tau_{yz} = P y \tag{28}$$

In a previous section, we determined the stress tensor σ_p associated with particles by assuming that particle collisions in each shear plane isotropically distribute over space a part of the mean tangential shear stress. This process induces a normal shear stress σ_{yy} . The non local normal stress σ_p^* is obtained by integrating over a correlation length y the shear force transmitted along the chain of particles normally to the shear plane:

$$\sigma_{yy}^* \approx -a \frac{\phi}{1 - \phi/\phi^*} \int_0^{y/a} \tau_{yz} d(y/a) = -\frac{\phi}{1 - \phi/\phi^*} \frac{P y^2}{2a}. \tag{29}$$

We then deduce the transverse diffusive flux

$$J_r = \phi v_p = -\frac{a^2}{\mu_0 \mu^*} \frac{\partial}{\partial y} (\sigma_{yy}^*) \tag{30}$$

Integrating the steady state condition $J_r = 0$ over the space variable y with an integration constant fixed by the boundary conditions, we find

$$\frac{\phi(y)}{(1 - \phi(y)/\phi^*)} \left(\frac{y}{H}\right)^\alpha = \frac{\phi_W}{(1 - \phi_W/\phi^*)} = A \quad (31)$$

with $\alpha = 2$ for $0 \leq y \leq \xi$ and $\alpha = 1$ for $\xi \leq y \leq H$.

Then, we obtain an estimate for the equilibrium concentration profile which exhibits no cusp in the centre region of the flow

$$\frac{\phi(y)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \left(\frac{y}{H}\right)^\alpha} \quad (32)$$

for $0 \leq y \leq H$ and $\phi > \phi_c$ with

$$A = \frac{\phi_W}{(1 - \phi_W/\phi^*)}, \quad \int_0^H \phi(y) dy = \phi H \quad \text{and} \quad \phi_W = \phi(H).$$

Recent experimental results by Koh *et al.* [12] concerning the flow of concentrated suspensions ($\phi = 0.3$) in a rectangular channel indeed show concentrations $\phi(y = 0) \approx 0.62$ near the centre plane close to the predicted random packing volume fraction $\phi^* \approx 0.62$ and a strong blunting of the velocity profile. As a matter of fact, the velocity profile becomes blunted from the parabolic for particle volume fraction $\phi > 0.2$ above the percolation threshold [17]. Furthermore, the relatively large slip velocity between particles and fluid for this particular experimental study shows that the correlation length ξ is of the same order as the channel gap. Under the assumption $\xi = H$, we thus compare in Figure 3 the experimental data from Koh *et al.* [12] to the theoretical concentration profile using the average volume fraction ϕ as input of the model

$$\frac{\phi(y)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \left(\frac{y}{H}\right)^2} \quad (33)$$

for $0 \leq y \leq H$ and $\phi > \phi_c$ with:

$$A = \frac{\phi_W}{(1 - \phi_W/\phi^*)}, \quad \int_0^H \phi(y) dy = \phi H \quad \text{and} \quad \phi_W = \phi(H).$$

It is noteworthy that equation (33) derived from a non-local stress tensor provides a good description of particle migration in planar Poiseuille flow without any adjustable parameter.

In the case of a Poiseuille flow in circular cross section pipe, the diffusive flux expressed in cylindrical coordinates obeys a similar relation without cusp near the centre axis $r = 0$ of the channel (Fig 4):

$$\frac{\phi(r)}{\phi^*} = \frac{1}{1 + \frac{\phi^*}{A} \left(\frac{r}{R_W}\right)^2} \quad (34)$$

for $0 \leq r \leq R_W$ and $\phi > \phi_c$ with

$$A = \frac{\phi_W}{(1 - \phi_W/\phi^*)}, \quad \int_0^{R_W} \phi(r) r dr = \phi \frac{R_W^2}{2} \quad \text{and} \quad \phi_W = \phi(R_W).$$

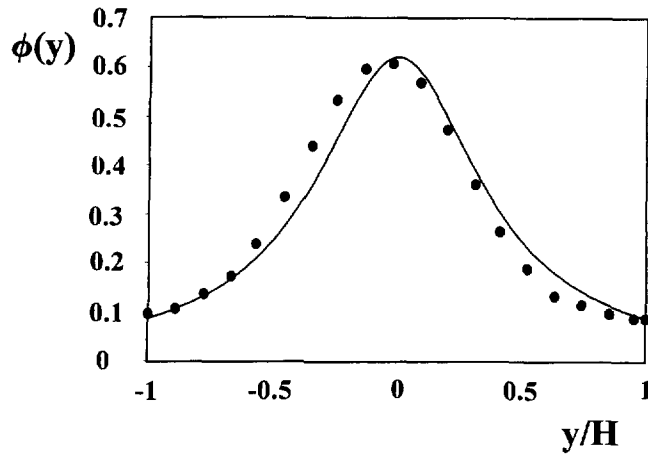


Fig. 3. — Comparison of the theoretical concentration profile $\phi(y)$ along an axis perpendicular to the velocity field in a plane Poiseuille flow with the experimental results from Koh *et al* [1] for a particle volume fraction $\phi = 0.3$ (●). The solid line is obtained from equation (33) with $\phi^* = 0.62$ using the experimental average volume fraction ϕ as input of the model.

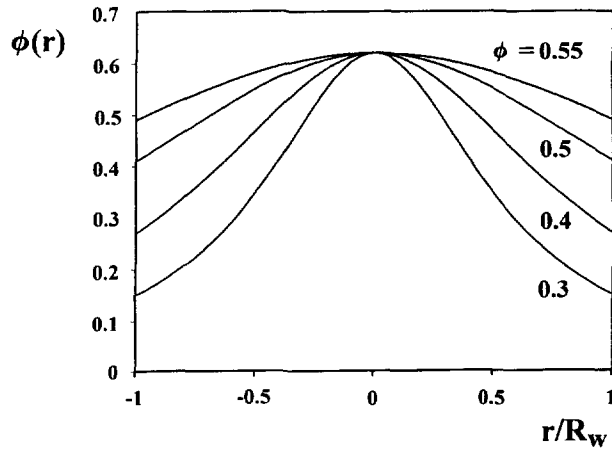


Fig. 4. — Theoretical equilibrium concentration profiles $\phi(r)$ along a radial axis perpendicular to the velocity field in a Poiseuille flow of circular cross section for various mean particle volume fraction in the channel with diameter R_w . The solid line is obtained from equation (34) with $\phi^* = 0.62$.

Conclusion

A relationship between the effective shear viscosity and the mean distance between particles was established in a previous paper. The viscosity dependence $\mu(d)$ was used to determine the components of a stress tensor associated with the transient network of particles for volume fraction above a percolation threshold. For this purpose, we assume that particle collisions in each shear plane isotropically distributes over space a part of the mean tangential shear stress. The particle stress tensor was coupled to the stress tensor of the suspension considered as an

effective medium. This coupling induces a transverse diffusive flux proportional to the divergence of the particle stress tensor. This model shows that migration is initiated by the spatial variation of the tangential shear stress, instead of the rate of strain. Comparison between the theoretical concentration profiles and experimental data further suggests that particle migration mainly occurs above a volume fraction threshold. In fact, a non-zero transverse diffusive flux exists even for low volume fractions [11]. Of immediate importance for future work, is the detailed study of concentration profiles for less concentrated suspensions in the vicinity of the expected percolation threshold.

In the case of a Poiseuille flow, it appears that near the centre region of the channel, the stress is directly transmitted by the network of particles over a correlation length larger than particle diameter. In the case of non-uniform flows, it is then necessary to introduce a non-local stress tensor for describing particle migration in low shear rate regions of the flow.

To sum up, particle migration can be described through a local particle stress tensor related to lubrication forces only if the correlation length of the flow is of the same order as the particle diameter over the whole flow cross section. On the other hand, for non-uniform flows with correlations over large scales, we have to consider a non-local stress tensor depending on the flow geometry.

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