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Benard-Marangoni thermoconvective instability in presence of a temperature-dependent viscosity

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Résumé. — On propose une analyse linéaire des instabilités thermoconvectives en présence de gravité et de tension superficielle dans des fluides dont la viscosité varie avec la température. La couche de fluide d'extension horizontale infinie, limitée par deux surfaces planes est initialement au repos et est soumise à un gradient de température vertical. On considère deux dépendances différentes de la viscosité par rapport à la température : une loi linéaire et une loi exponentielle. Des résultats numériques ont été obtenus pour toutes valeurs — positives et négatives — des nombres de Marangoni et de Rayleigh. Le rôle stabilisateur ou déstabilisateur d'une viscosité fonction de la température est lié au signe de sa variation avec la température. Les résultats obtenus pour le glycérol et l'huile de silicone montrent que cette dépendance en la température exerce une influence significative au niveau de l'initiation de l'instabilité, tout spécialement dans des couches très minces. Une analyse numérique des résultats indique que le principe d'échange de stabilité reste satisfait dans un large domaine de variation des paramètres.

Abstract. — A linear analysis of surface-tension and gravity driven instabilities in fluids with a temperature-dependent viscosity is proposed. The fluid layer is initially at rest, of infinite horizontal extent, limited by two flat boundaries and subject to a vertical temperature gradient. Two different dependences of the viscosity with respect to the temperature are considered : a linear and an exponential law. Numerical results have been obtained for any value — either positive or negative — of the Marangoni and the Rayleigh numbers. The stabilizing or destabilizing role of a temperature-dependent viscosity depends on the sign of its variation with temperature. The results obtained for glycerol and silicone oils show that this temperature-dependence has a significant influence on the onset of instability, specially in very thin layers. A numerical analysis indicates that the principle of exchange of stability holds true within a wide range of variation of the parameters.

1. Introduction.

This paper is concerned with thermoconvective instability in a horizontal monolayer of fluid, initially at rest and heated from below. The fluid is isotropic, extends horizontally to infinity

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and is bounded by two flat boundaries. The lower boundary is rigid but the upper one is free and submitted to a surface tension varying linearly with temperature.

This problem was first studied by Pearson [1] who neglected buoyancy effects and by Nield [2] who included gravity forces in his analysis. Both authors took for granted the Boussinesq approximation which implies, in particular, that the viscosity is temperature-independent. However, it is well known that for a given class of fluids, as for instance silicone oils or glycerine, viscosity may vary significantly with temperature. Palm [3] has studied the effect of a temperature-dependent viscosity on the onset of convection in Bénard's problem. Palm considered the particular case of a fluid layer between two free boundary surfaces with a not very realistic cosine law for the temperature dependence of the viscosity μ . Stengel *et al.* [4] compared Palm's results with these arising from the selection of an exponential viscosity law. The same problem was solved by Busse and Frick [5] for low viscous fluids with a linear dependence of the viscosity on the temperature.

Our objective in this article is twofold. First to use a viscosity law $\mu(T)$ more realistic than that proposed by Palm and second, to extend the analyses by Palm, Stengel *et al.*, Busse and Frick by including surface-tension driven instabilities. In more precise terms, we shall determine by means of a linear analysis the effect of a temperature-dependent viscosity on the critical instability threshold. Two different viscosity laws for $\mu(T)$ have been selected : a linear and an exponential law and their range of applicability will be discussed. In contrast with some of the aforementioned works [1-3], exchange of stability is not assumed *a priori*.

2. Mathematical description.

2.1 EVOLUTION EQUATIONS. — Consider a thin layer of an incompressible Newtonian fluid with thickness d and infinite horizontal extent. A Cartesian reference frame with origin O at the bottom of the layer and axis \underline{e}_3 pointing vertically in the opposite direction of gravity is selected. The thermohydrodynamic equations are :

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1)$$

$$\rho_0 \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} - \rho_0 [1 - \alpha(T - T_0)] g \delta_{i3} + \frac{\partial}{\partial x_j} \left[\mu(T) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \quad (2)$$

$$\rho_0 c \frac{dT}{dt} = K \frac{\partial^2 T}{\partial x_i \partial x_i} \quad (3)$$

Einstein's summation convention on repeated indices is used, u_i is the velocity, P the pressure, T the temperature, $\mu(T)$ the dynamic viscosity, c the specific heat, K is the supposed constant thermal conductivity while

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

stands for the material time derivative. The density $\rho(T)$ satisfies the state equation :

$$\rho = \rho_0 [1 - \alpha(T - T_0)]$$

where α is the positive coefficient of thermal expansion and T_0 an arbitrary reference temperature, for example the temperature at the bottom of the fluid layer. By formulating (1)-(3), Boussinesq's approximation is partially used. Indeed equations (1)-(3) satisfy the following assumptions, which are typical of Boussinesquian fluids :

- i) density is everywhere constant except in the gravity term of equation (2) ;
- ii) viscous dissipation is neglected in the energy balance (3). Only the restriction of constant viscosity is relaxed.

It was shown [6] that approximations i) and ii) are justified under the conditions

$$\varepsilon_1 \ll 1 \quad \varepsilon_2 \ll 1$$

where ε_1 and ε_2 are defined by

$$\varepsilon_1 = \alpha \Delta T, \quad \varepsilon_2 = \frac{k_0^2}{\Delta T d^2 c}$$

with $k_0 (= K/\rho_0 c)$ the heat diffusivity and $\Delta T (= T_0 - T_1)$, the temperature difference between the lower and upper surfaces.

2.2 LINEARIZATION. — In the reference state, the layer is at rest and heat propagates by conduction only ; the velocity, temperature and pressure fields are given respectively by

$$u_i^{\text{ref}} = 0; \quad T_{\text{ref}} = T_0 - \beta x_3; \quad \frac{dp_{\text{ref}}}{dx_3} = -\rho_{\text{ref}} g \left(\beta = \frac{\Delta T}{d} \right).$$

At the first order approximation in the infinitesimally small perturbations $\{u'_i, T', P'\}$, equations (1)-(3) write as

$$\text{Pr}^{-1} \frac{\partial}{\partial \tau} \left(\frac{\partial^2 V_3}{\partial X_i^2} \right) = \text{Ra} \left(\frac{\partial^2 \theta}{\partial X_1^2} + \frac{\partial^2 \theta}{\partial X_2^2} \right) - (D^2 \nu_a) \frac{\partial^2 V_3}{\partial X_i^2} + 2(D^2 \nu_a) \frac{\partial^2 V_3}{\partial X_3^2} + 2(D \nu_a) \frac{\partial^3 V_3}{\partial X_i^2 \partial X_3} + \nu_a \frac{\partial^4 V_3}{\partial X_i^2 \partial X_j^2} \quad (4)$$

$$\frac{\partial \theta}{\partial \tau} - V_3 = \frac{\partial^2 \theta}{\partial X_i^2}. \quad (5)$$

In (4), the pressure field has been eliminated by application of the curl operator on (2) and dimensionless quantities have been introduced in (4) and (5). The following scaling has been used for the space coordinate, time, velocity and temperature respectively

$$X_i = \frac{x_i}{d}, \quad \tau = t \frac{k_0}{d^2}; \quad V_i = u'_i \frac{d}{k_0}; \quad \theta = \frac{T'}{\beta d}$$

operator D stands for $D = d/dX_3$ while Pr and Ra are the Prandtl and Rayleigh numbers respectively :

$$\text{Pr} = \frac{\nu_0}{k_0}, \quad \text{Ra} = \frac{\rho_0 g \alpha \beta d^4}{k_0 \mu_0} \quad (6)$$

with μ_0 the dynamic viscosity calculated at $T = T_0$.

The Rayleigh number is given a univocal definition since it is expressed in terms of the constant reference viscosity at the temperature T_0 . The undefined quantity ν_a appearing in (4) is given by

$$\nu_a = \frac{\mu_{\text{ref}}}{\mu_0} \quad (7)$$

with $\mu_{\text{ref}} = \mu(T_{\text{ref}})$.

According to the normal mode technique, one seeks solutions of the form

$$\begin{bmatrix} V_3 \\ \theta \end{bmatrix} = \begin{bmatrix} W(X_3) \\ \Theta(X_3) \end{bmatrix} \exp(i\mathbf{k} \cdot \mathbf{X} + s\tau) \quad (8)$$

with

$$s = \sigma + i\omega; \quad \mathbf{k} = (k_1, k_2); \quad \mathbf{X} = (X_1, X_2); \quad k = \|\mathbf{k}\|.$$

Introducing (8) into (4)-(5) yields

$$s \text{Pr}^{-1} (D^2 - k^2) W = -k^2 \text{Ra} \Theta + (D^2 \nu_a)(D^2 + k^2) W + 2(D\nu_a) D(D^2 - k^2) W + \nu_a (D^2 - k^2)^2 W \quad (9)$$

$$(D^2 - k^2 - s) \Theta = -W. \quad (10)$$

2.3 BOUNDARY CONDITIONS. — The upper surface of the fluid is assumed to be free, undeformable, but submitted to a surface tension ξ depending linearly on the temperature according to

$$\xi(T) = \xi(T_0) + \frac{\partial \xi}{\partial T} (T - T_0)$$

with T_0 an arbitrary reference temperature. If it is assumed that there is no interaction between the fluid and the external environment, the corresponding boundary condition is :

$$\frac{\partial \xi}{\partial x_i} = \mu \frac{\partial u_i}{\partial x_3} \quad \text{at } x_3 = d \quad (i = 1, 2). \quad (11)$$

After substitution of expression (8) in (11) and linearization, one obtains in dimensionless notation

$$D^2 W = -k^2 \text{Ma} \frac{\mu(T_0)}{\mu(T_1)} \Theta \quad \text{at } X_3 = 1 \quad (12)$$

wherein Ma is the Marangoni number defined by

$$\text{Ma} = - \frac{(\partial \xi / \partial T) \beta d^2}{k_0 \mu_0} \quad (13)$$

with μ_0 taken at the temperature T_0 of the lower surface of the fluid. It may be surprising to refer the viscosity in expression (13) of the Marangoni number to the temperature of the lower boundary as condition (12) is applicable at the upper boundary. The definition (13) has been used because it allows for a clearer and easier physical interpretation. Condition (12) must be coupled with

$$W = 0 \quad \text{at } X_3 = 1, \quad (14)$$

expressing the non-deformability condition on the free surface. The lower boundary is supposed to be rigid so that :

$$W = DW = 0 \quad \text{at } X_3 = 0. \quad (15)$$

The kinematic boundary conditions (12), (14) and (15) must be completed by the following thermal boundary conditions.

$$\begin{aligned}\Theta &= 0 & \text{at } X_3 = 0 \\ D\Theta &= -h\Theta & \text{at } X_3 = 1\end{aligned}\quad (16)$$

h is the heat transfer coefficient at the interface, the so-named Biot number. The lower wall is perfectly heat conducting, while the upper boundary allows for a heat transfer between the fluid and the external environment.

For convenience, the six relevant boundary conditions (12), (14)-(16) will be written in terms of the temperature field Θ only; this results in

$$\begin{aligned}\text{at } X_3 = 0 : \quad \Theta &= 0 \\ D^2 \Theta &= (k^2 + s) \Theta \\ D^3 \Theta &= (k^2 + s) D\Theta\end{aligned}\quad (17)$$

$$\begin{aligned}\text{at } X_3 = 1 : \quad D\Theta + h\Theta &= 0 \\ D^2 \Theta &= (k^2 + s) \Theta \\ D^4 \Theta &= [(k^2 + s)^2 + k^2 \text{Ma}] \Theta.\end{aligned}\quad (18)$$

2.4 CONSTITUTIVE LAWS FOR THE VISCOSITY. — Two different constitutive laws will be considered in this work. The most current attitude is to assume that the dynamic viscosity μ obeys a linear law of the form :

$$\mu(T) = \mu_0 [1 - \gamma(T - T_0)] \quad (19)$$

γ is a positive constant for a liquid and negative for a gas. Another common attitude is to take μ varying exponentially with the temperature :

$$\mu(T) = \mu_0 e^{-\gamma(T - T_0)} \quad (\gamma > 0). \quad (20)$$

This constitutive equation has proven well suited for highly viscous liquids. In both expressions (19) and (20), γ measures the viscosity variation with temperature and is given by

$$\gamma = - \left. \frac{1}{\mu_0} \frac{d\mu}{dT} \right|_{T=T_0} \quad (21)$$

Although some more complex empirical expressions for $\mu(T)$ are proposed in the literature, our analysis is limited to the two above simple laws on account of their generality and their analytical simplicity. The linear and exponential laws involve only two parameters, μ_0 and γ , and fit quite well experimental data on restricted temperature ranges ($\Delta T < 30$ K) for a wide class of gases and liquids (see Tab. I).

It follows from (7) that within the linear approximation, one may write the non-dimensional viscosity ν_a as

$$\nu_a = 1 + LX_3 \quad (22)$$

for the linear dependence, while for the exponential law

$$\nu_a = e^{LX_3} \quad (23)$$

the quantity L stands for $L = \gamma \Delta T$.

Table I. — Values of γ calculated with a least square method from equation (20) for liquids and from equation (19) for gases. Data on viscosity were taken from [4, 12-15].

		Dynamic viscosity μ at 25 °C (10^{-3} kg.m $^{-1}$.s $^{-1}$)	Gamma γ (K $^{-1}$)	Temperature range (°C)
Water	H ₂ O	0.8904	0.025	4-30
Acetic acid	C ₂ H ₄ O ₂	1.155	0.0153	20-50
Silicone oil	M200	168.6	0.0209	0-25
Glycerol	C ₃ H ₈ O ₃	938.7	0.1005	0-25
Air		18.515	- 0.0019	20-50
Carbon dioxide	CO ₂	15.05	- 0.0026	20-50

The non-dimensional factor L is a quantity proportional to γ but also to the temperature jump between the boundary surfaces; for a constant viscosity, one has $L = 0$. Consequently, L expresses the temperature dependence of the viscosity. Furthermore, L can be related to the ratio of the viscosities taken respectively at T_1 and T_0 ; indeed, by defining

$$r = \frac{\mu_1}{\mu_0} \quad (24)$$

it follows from (19) and (20) that

$$r = 1 + L^l, \quad (25)$$

$$\ln r = L^e \quad (26)$$

where the superscripts l and e refer to the linear or the exponential law respectively. It is evident that within a given temperature range $[T_0, T_1]$ the value of γ calculated either from (19) or (20) will give different results. Like Busse and Frick [5], we found it interesting to introduce the ratio r because it is a typical quantity reflecting the importance of the viscosity variation independently of the selected viscosity law. According to (25), L^l measures the relative variation of μ_1 with respect to μ_0 ; values of L^l smaller or equal to -1 are not allowed as they imply a negative or a zero viscosity μ_1 . The quantity L^e is an expression of the variation of μ between both boundaries and can be given positive or negative real values.

2.5 EXCHANGE OF STABILITY. — Exchange of stability has been shown to be valid for the problem of pure Marangoni convection by Vidal and Acrivos [7]. A numerical proof has been given by Takashima [8] for the more general case of Bénard-Marangoni driven instability. However both demonstrations rest on the hypothesis of constant viscosity. By following Pellew and Southwell's reasoning [9], it is easy to show analytically that the principle of exchange of stability holds for fluid layers with a temperature-dependent viscosity under the condition that

$$D^2 \nu_a > 0$$

and for boundary surfaces which are either both perfectly heat conducting ($\Theta = 0$) or perfectly insulating ($D\Theta = 0$), it is moreover required that both surfaces are either rigid ($DW = 0$) or free without surface tension effects ($D^2W = 0$). Under more general conditions, it is impossible to prove exchange of stability by an analytical method.

However, the Marangoni number can be expressed numerically as a function of the parameters Ra, k, L, h, Pr, ω and as a result contains a real and an imaginary part. The problem amounts to determine the ω -value for which the imaginary part of Ma vanishes. We have investigated the following range of parameters

$$0.5 \leq k \leq 8; \quad -10^4 \leq Ra \leq 4\,000; \quad 10^{-2} \leq Pr < \infty; \quad -0.1 \leq r \leq 10;$$

$$0 \leq \omega \leq 10^4; \quad h = 0, 1, 10^4$$

and have found that $\omega = 0$ is the single value of ω corresponding to $\text{Im}[Ma] = 0$. It can thus be concluded that, at least within the range of parameters examined in this work, the possibility of overstability must be excluded. As a consequence, we shall consider the marginal state as a steady state defined by a zero stability number ($s = 0$).

Setting $s = 0$ in (9, 10) and in the boundary conditions (17, 18), leads to a sixth order eigenvalue problem. For a further purpose, let us establish the differential equation governing the behaviour of the temperature field $\Theta(X_3)$: it is given by

$$[\nu_a D^6 + 2(D\nu_a) D^5 + (D^2 \nu_a - 3k^2 \nu_a) D^4 - 4k^2(D\nu_a) D^3 + 3k^4 \nu_a D^2 +$$

$$+ 2k^4(D\nu_a) D + k^2 Ra - \nu_a k^6 - (D^2 \nu_a) k^4] \Theta = 0. \quad (27)$$

It is observed that this relation contains variable coefficients whose explicit expression depends on the specific dependence of the viscosity with respect to the temperature as well as on its derivatives up to second order. The eigenvalue problem, obtained with the five dimensionless numbers

$$\{k, Ra, Ma, L, h\} \quad (28)$$

has been solved numerically from a representation of the profile $\Theta(X_3)$ by means of a power series in X_3 . The method has preliminary been checked by solving the pure Bénard problems treated by Busse *et al.* [5] and Stengel *et al.* [4] for the viscosity laws (19) and (20) respectively. Excellent agreement has been reached. Moreover, for $L = 0$, one recovers the results of Nield [2] while for $Ra = 0$, one finds the same results as Lebon and Cloot [10] who supposed a linear viscosity law.

3. Results and discussion.

Stationary solutions for the homogeneous problem defined by (17, 18) and (27) are only found for some particular values of the parameters (28). By fixing h, L and Ra , it is possible to represent the eigenvalues Ma as a function of the wavenumber k . In all cases, the marginal stability curve is a unimodal function and presents a minimum, say Ma^c , at which the static layer becomes unstable with respect to infinitesimal perturbations.

First, the upper surface is assumed to be perfectly insulated ($h = 0$). This corresponds to the most unstable situation because the whole thermal energy communicated to the system remains enclosed inside the fluid layer. Figure 1 shows neutral stability curves Ma versus k for various values of r at $Ra = 50$. The behaviour of the critical Marangoni number Ma^c indicates that stability increases significantly with r . This conclusion holds for both viscosity laws, linear and exponential. The value $r = 1$ (i.e. $L^1 = L^e = 0$) refers to a fluid whose viscosity is constant.

The corresponding stability curve will be taken as a reference to show the deviations of the results by taking a variable viscosity. Obviously, when r approaches unity, the stability curves for the linear and exponential viscosity laws tend to coincide ; this result is not surprising since the law (19) is a first order approximation of the law (20) for small absolute values of γ . It is worth noticing that the more important is the variation of viscosity with temperature, the larger is the divergence between both families of curves. This justifies the interest of introducing an exponential law for highly viscous liquids.

In most thermo-convective problems, it is supposed that the Rayleigh and the Marangoni numbers are both positive. Let us relax this assumption and allow Ra and Ma to take negative values as well. Negative Ra 's are obtained either by heating a fluid layer from above or, by heating from below systems with a negative-coefficient of thermal expansion, like water between 0° and $4^\circ C$. Negative Ma numbers are representative of fluid layers heated from above, with a negative rate of change of the surface tension with the temperature ($\frac{\partial \xi}{\partial T} < 0$). Negative Ma 's may also be found in systems heated from below with a positive surface tension gradient ($\frac{\partial \xi}{\partial T} > 0$). Such behaviour is rather unusual but may be found in some alloys, molten salts, long chain alcohols or liquid crystals [11]. Here we confine our attention to the more frequent situations for which $\frac{\partial \xi}{\partial T} < 0$.

In figure 2 we report the critical values (Ra^c, Ma^c) whatever their sign ; the quadrants are designated anticlockwise with Roman numbers.

Quadrant I ($Ra > 0, Ma > 0$) : it corresponds to a fluid layer with $\alpha > 0$ and heated from below or to a fluid adhering to a hot ceiling with $\alpha < 0$. In both cases one has $T_0 > T_1$.

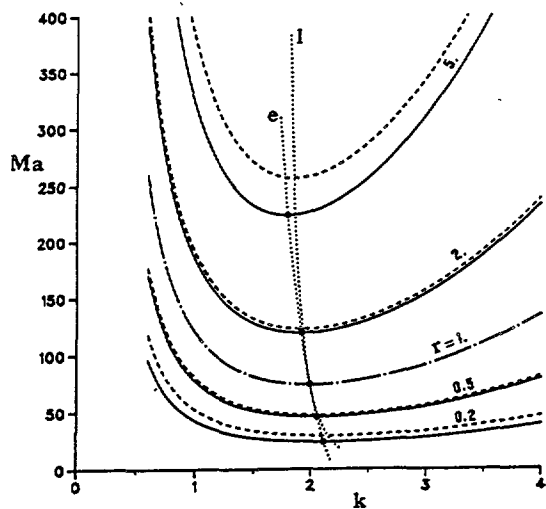


Fig. 1.

Fig. 1. — Marginal stability curves Ma versus k for the exponential and linear viscosity laws ($Ra = 50, h = 0$). The solid line refers to the exponential viscosity law and the dashed line to the linear law. The chain dots correspond to a constant viscosity ($r = 1$). A dotted line joins the minimum of each curve pertaining to the same family ($e =$ exponential law ; $l =$ linear law).

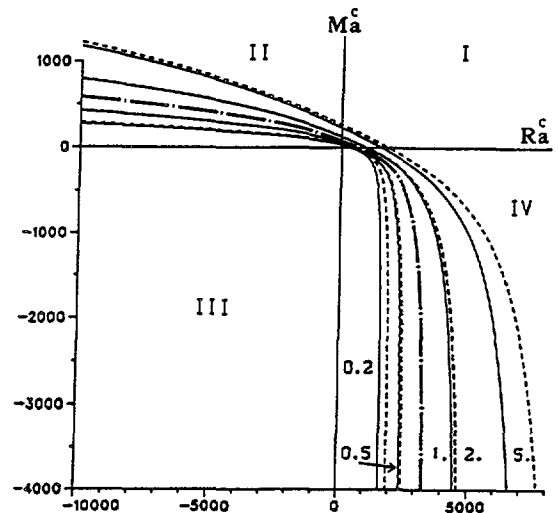


Fig. 2.

Fig. 2. — Stability curves in the $Ra^c - Ma^c$ plane for $r = 0.2 ; 0.5 ; 1 ; 2 ; 5$, and Biot number $h = 0$. The drawing convention is the same as in figure 1.

Quadrant II ($Ra < 0, Ma > 0$) : the description is similar to the previous one with the sign of the coefficient of volume expansion reverted.

Quadrant IV ($Ra > 0, Ma < 0$) : it describes a fluid with a negative volume expansion coefficient ($\alpha < 0$) heated from above or a liquid adhering to a cold ceiling with $\alpha > 0$. In this quadrant, the sign of L is opposite to its sign in the quadrants I and II.

Quadrant III corresponds to $Ra < 0$ and $Ma < 0$ but is less interesting in the present context because, as will be shown, it is characteristic of situations unconditionally stable.

In the case of fluids whose surface tension increases with temperature ($\frac{\partial \xi}{\partial T} > 0$), the above conclusions remain true at the condition to reverse the direction of heating and the direction of the gravity force.

Figure 2 shows the locus of points (Ra^c, Ma^c) calculated for $r = 0.2, 0.5, 1, 2, 5$ respectively. Each curve is related to a given value of r and separates the plane in two areas : the stability domain stands below the curve and the instability domain above the curve ; solid lines refer to the exponential law for the viscosity, dashed lines to the linear law. The critical points have the form of a beam of curved lines varying continuously with r in each quadrant independently of the constitutive law. Observe that the coupling between buoyancy and surface-tension effects is particularly tight in quadrants II and IV : a small variation of Ma (respectively Ra) results in a drastic variation of Ra (respectively Ma). A detailed view of quadrant I is supplied by figure 3. As a matter of fact, the effects of a variable viscosity are less important in the first quadrant than in the others. The departure from the results obtained for a constant viscosity is the largest when Ma^c or Ra^c tends to minus infinity. The behaviour of the critical threshold for a gas ($\gamma < 0$) is seen to be the opposite of that of a liquid ($\gamma > 0$).

The above results can be interpreted as follows.

Although the value of γ can be obtained from data tables, this is no longer true for L (or r) because it depends on the critical value of ΔT which has to be determined in each individual experiment. Moreover, this ΔT -value appears in both expressions of the Rayleigh and Marangoni numbers. Eliminating ΔT between Ra and Ma yields the equation of a straight line passing through the origin of the plane (Ra, Ma), namely

$$Ma = \frac{-\frac{\partial \xi}{\partial T}}{\rho_0 g \alpha d^2} Ra . \tag{29}$$

The slope of this line is fixed after that the depth of the layer and all the thermophysical properties of the fluid are known. Thus, when ΔT grows from zero, one progresses along a path defined by (29) following the trajectory 0, 1, 2, . . shown in figure 4.

If the viscosity of the fluid is constant, say μ_0 , then the determination of the critical ΔT^c results from the intersection of (29) with the stability curve corresponding to $r = 1$ (see point marked C in Fig. 4). However, for a variable viscosity, at each increase of ΔT , the value of L increases by an equal amount in absolute value, and a different stability curve is found for each representative points 1, 2, . . as shown in figure 4. When ΔT increases, one deviates more and more from the curve $r = 1$ and the critical ΔT^c is given by the intersection of the straight line and the moving stability curve (F). The process sketched in figure 4 remains valid in any quadrant. The length of the segment OF is a measure of ΔT^c in the plane (Ra^c, Ma^c), recalling that $\sqrt{Ra^c^2 + Ma^c^2}$ is proportional to ΔT^c . A tentative physical interpretation is the following. For the sake of brevity, we limit our analysis to the first quadrant which corresponds to the most usual situation. Let ΔT_0^c be the critical temperature jump sufficient to initiate the instability in a fluid of constant viscosity

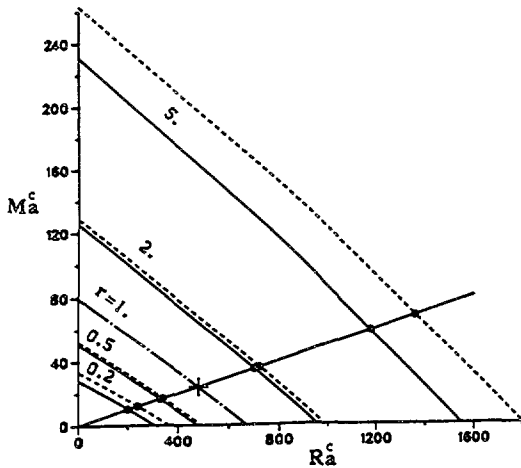


Fig. 3.

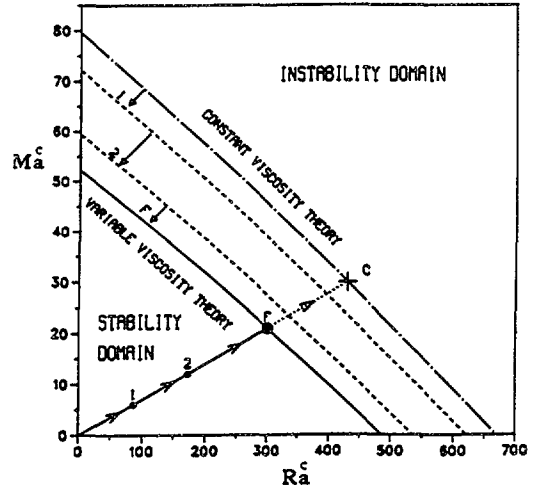


Fig. 4.

Fig. 3. — A detailed view of quadrant I from figure 2.

Fig. 4. — A sketch of the determination of the critical temperature difference ΔT^c . For a given fluid with a fixed depth d , this determination is direct for a constant viscosity (point C). With a variable viscosity, the determination is iterative since r changes its value when progressing along the straightline 0, 1, 2, F.

$\mu(T_0)$. Take now into account the temperature dependence of the viscosity and let $\mu(T_1)$ be the viscosity for $T_1 (< T_0)$, the temperature at the upper boundary. In a liquid layer (respectively a gas layer), the viscosity is larger (smaller) at the top than at the bottom where the temperature is larger and maintained at the constant value T_0 . Thus, globally, the whole layer will offer a greater (worse) resistance to convection than in the constant viscosity case. Therefore the critical temperature difference ΔT^c is expected to be larger (smaller) than ΔT_0^c . This is confirmed by figure 3 where the length of any straight segment, drawn from the origin up to its intersection with a stability curve characterized by r greater (less) than 1, is always larger (smaller) than the length corresponding to the curve $r = 1$. The same argument holds in any other quadrant.

The above choice of $\mu(T_0)$ as reference viscosity has been introduced to allow for a simple physical interpretation of the effect of a temperature-dependent viscosity on the critical instability threshold. In experiments, the relevant quantity is the critical temperature difference ΔT^c rather than the Rayleigh and the Marangoni numbers which are not directly measurable. With $\mu(T_0)$ as reference viscosity, the value of ΔT^c is directly obtained by measuring the distance OF of figure 4. Indeed the norm

$$\sqrt{Ra^c + Ma^c} \tag{30}$$

is related to ΔT^c through the simplest possible relationship : i.e., a linear and homogeneous function of ΔT^c , independent of r . Now by selecting the reference viscosity at mid-depth of the fluid,

$$\mu_{ref} = \mu \left(\frac{T_0 + T_1}{2} \right)$$

together with the linear viscosity law (19), one has

$$\mu_{\text{ref}} = \mu \left(T_0 + \frac{\Delta T^c}{2} \right) = \mu \left(T_0 + \frac{r-1}{\gamma} \right) \tag{31}$$

and the reference viscosity turns out to be a function of r . Consequently, the expression of ΔT^c will be given by

$$\Delta T^c = \frac{\mu_{\text{ref}} k_0}{d \sqrt{\left(\frac{\partial \xi}{\partial T} \right)^2 + (\rho_0 g \alpha d^2)^2}} \sqrt{\text{Ra}^c{}^2 + \text{Ma}^c{}^2} \tag{32}$$

from which is seen that ΔT^c is not completely determined from the mere observation of the distance OF in figure 4 but requires an extra computation for μ_{ref} in terms of r . Clearly with the choice μ_{ref} at mid-depth the conclusions about stability thresholds are only valid for one specific fluid, the one defined by the r -value appearing in (31). It is clear that the above mentioned problem is circumvented by selecting

$$\mu_{\text{ref}} = \mu |_{X_3=0} = \mu(T_0)$$

independently of the thermophysical properties of the fluid layer, with T_0 the temperature at the perfectly heat conducting boundary.

In most problems dealing with Rayleigh-Bénard convection, the reference viscosity is taken at mid-depth of the fluid layer (Stengel *et al.* [4], Busse *et al.* [5]). Such a choice is justified because these problems are generally characterized by a geometrical symmetry with respect to the mid-plane : this property does however no longer hold in Bénard-Marangoni instability problems. Nevertheless, for the sake of completeness, we have represented the marginal stability curves corresponding to the first quadrant when the reference viscosity in both Ra and Ma numbers is taken at mid-depth of the layer (see Fig. 5). Clearly the situation is a little bit more confusing than in the corresponding graphs of figure 3 where the reference viscosity is taken at the bottom plate. Note that the shift of the stability curves as a function of r is less sensitive than in figure 3. For example, in the case of the exponential viscosity law (20), with $h = 0$ and $\text{Ra} = 0$, the relative increase

$$\frac{\text{Ma}^c(r = 5) - \text{Ma}^c(r = 1)}{\text{Ma}^c(r = 1)}$$

when r is varying from 1 to 5 is 30 % if $\mu_{\text{ref}} = \mu |_{X_3=1/2}$ versus 190 % if $\mu_{\text{ref}} = \mu |_{X_3=0}$. Likewise, the relative increase of Ra^c with $\text{Ma} = 0$ and $h = 0$, is respectively 3.4 % and 131 %. It should be mentioned that for a layer enclosed between a rigid bottom surface and an upper free boundary (with $\text{Ma} = 0$), one recovers the Ra^c -values found by Stengel *et al.* [4] with an exponential viscosity-law.

Let us now examine the effect of a temperature-dependent viscosity on the critical wavenumber for situations described in quadrant I. In figure 6, we have represented the critical wavenumber k_c as a function of the angle of the straight line (29) and the $\text{Ma} = 0$ axis. A zero slope corresponds to a fluid without surface tension effect while an angle of 90° refers to a fluid layer in a microgravity environment. It is shown that for r -values larger than 1 — which is representative of a liquid — the critical wavenumber decreases, or equivalently the critical wave-length increases, when r is increasing. The reciprocal behaviour is observed for gases ($r < 1$) : k^c increases when r is decreasing. Since the upper free boundary of the liquid layer is cooler than the lower one, the viscosity at the upper surface is larger than for a constant

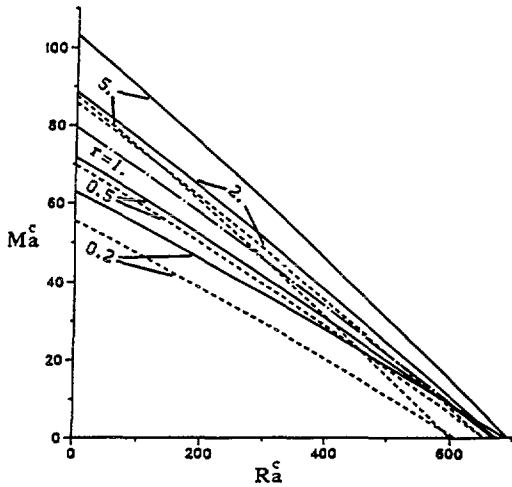


Fig. 5.

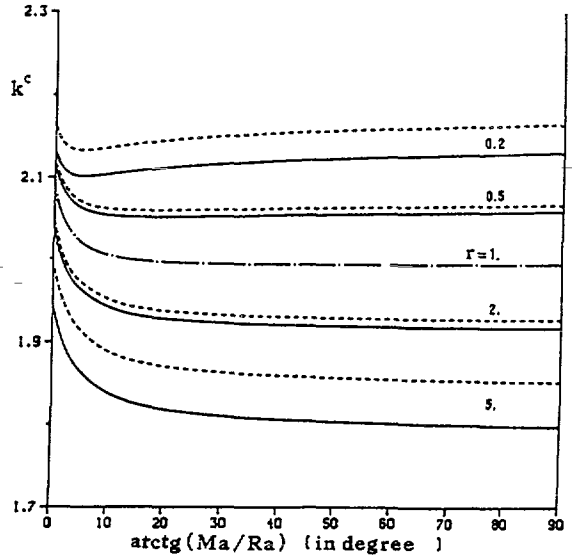


Fig. 6.

Fig. 5. — Stability curves in the $Ra^c - Ma^c$ plane for $r = 0.2; 0.5; 1; 2; 5$ and Biot number $h = 0$. Ra^c and Ma^c are defined with μ_{ref} taken at mid-depth.

Fig. 6. — Critical wavenumbers k^c versus $\arctg(Ma/Ra)$, the angle (in degrees) of the slope of the straight line (25) with the $Ma = 0$ axis.

viscosity. Therefore, it offers a greater cohesion and when convection is initiated, a larger number of fluid particles are involved by the convective motion. This results in a larger size of the convective cells and hence, a smaller critical wave number. For a gas layer, the converse conclusion will hold.

Figure 7 represents the effect of the Biot number on the marginal stability curves; normalized coordinates (Ra^c/Ra^* , Ma^c/Ma^*) are used with Ra^* the critical Rayleigh number

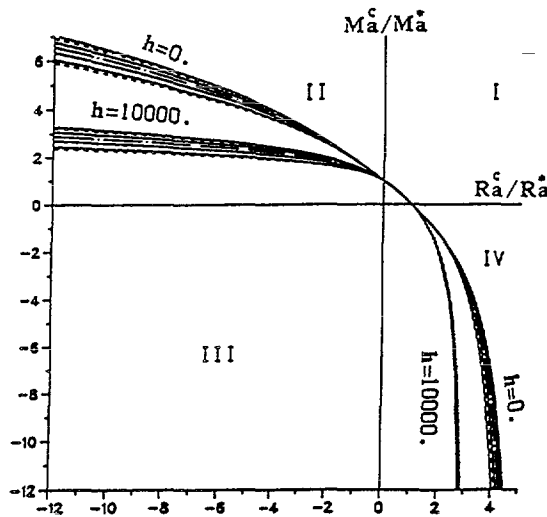


Fig. 7. — Critical stability curves in normalized co-ordinates $Ra^c/Ra^* \cdot Ma^c/Ma^*$ for both viscosity laws and $h = 0, h = 10\ 000$.

when $Ma = 0$ and Ma^* the critical Marangoni number when $Ra = 0$. In the first quadrant, it is observed that by increasing h from 0 to 10^4 , one modifies only slightly the general shape of the stability curve. Nevertheless, the thermal property of the interface is seen to play a non-negligible influence on the onset of convection. The system becomes more and more stable as h is increased (Ma^* grows linearly with h when h tends to infinity, see Nield [2]). This result is expected as more energy must be transferred to the fluid to set in motion when more and more heat is dissipated from the upper surface. Therefore the correction on the critical temperature drop owing to the viscosity variation increases with h .

4. Final comments.

It was shown that in the Bénard-Marangoni problem, the variation of the viscosity with temperature has a deep influence on the determination of the instability threshold. To summarize the previous results, we have represented in figures 8 and 9 the critical temperature difference ΔT^c in a glycerol and a silicone oil M200 layer respectively, taking into account a variable viscosity, for different depths of the layer. The relative difference on ΔT^c computed with a variable viscosity compared to the case of a constant viscosity is plotted on the same graphs. Thermophysical data for glycerol are taken from Stengel *et al.* [4] and Cardin *et al.* [12]. Data for the silicone oil are those given by Bühler *et al.* [14] and Dupont *et al.* [15]. T_0 is kept at 25 °C in both cases. It is seen that the difference is particularly important at small depths. For glycerol, the relative difference may reach 70 % for a thickness of 8 mm while for the silicone oil the difference is only 0.4 % for $d = 1$ mm, the Biot number $h = 0$ in both cases. However the relative difference becomes significant as the heat conductivity of the silicone free surface increases. Clearly, a temperature-dependent viscosity plays a stabilizing role at least for highly viscous liquids.

Exponential and linear laws exhibit qualitatively the same effect on the instability threshold. Nield's results were recovered in the limiting case $r = 1$ which corresponds to a constant

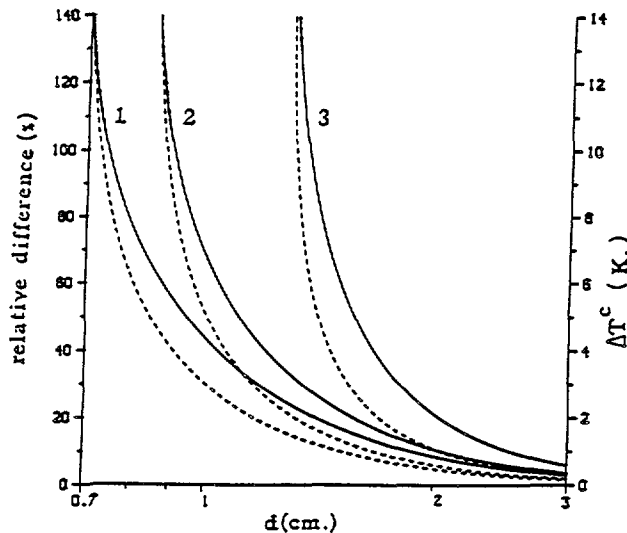


Fig. 8. — Critical temperature difference ΔT^c (in K) for various depths d (in cm) of a glycerol layer, under conditions of quadrant I, with $h = 0$; 1; 10^4 (curves 1, 2, 3 respectively). The solid lines correspond to an exponential viscosity law. The dashed lines give the relative difference in % between our results and those obtained from Nield's theory [2].

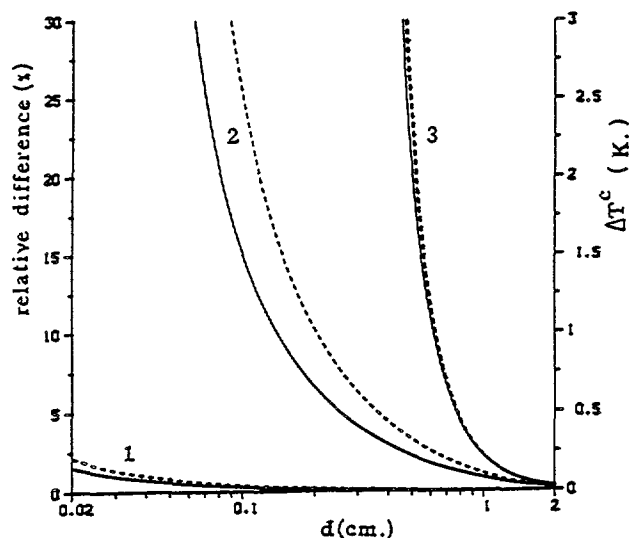


Fig. 9. — Critical temperature difference ΔT^c (in K) for various depths (in cm) of a silicone oil M200 layer, under conditions of quadrant I, with $h = 0 ; 10^2 ; 10^4$ (curves 1, 2, 3 respectively). The drawing convention is the same as in figure 8.

viscosity. Furthermore, Nield's approximate straight line $\frac{Ra^c}{R^*} + \frac{Ma^c}{M^*} = 1$ remains a good approximation in quadrant I whatever the constitutive expression of the viscosity law or the value of h .

The present work complements earlier contributions by Palm [3], Stengel *et al.* [4], Busse *et al.* [5] who limited their analyses to the pure Bénard buoyancy driven-instability problem. This paper also generalizes the contribution of Lebon and Clout [10] who used a linear law for $\mu(T)$ and neglected gravity effects.

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