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## Model for the flow of a yield fluid through a porous medium

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**Résumé.** — L'étude de l'écoulement d'un fluide newtonien dans une matrice poreuse et la dispersion de marqueurs au cours de l'écoulement a fait l'objet de nombreux travaux depuis Taylor et Saffman. Les caractéristiques de l'écoulement d'un fluide non newtonien sont moins connues. On se propose de généraliser à un fluide de Bingham l'approche développée par Saffman. Le milieu poreux est supposé statistiquement homogène et isotrope. La dispersion est complètement décrite par les coefficients transverse et longitudinal. Les pores sont représentés par des capillaires rigides monodispersés dont la distribution angulaire est isotrope. On montre, dans l'approximation de champ moyen utilisée, que le gradient de pression satisfait encore à l'équation de Laplace. L'existence d'un seuil d'écoulement a pour conséquence l'émergence d'un angle critique au-delà duquel les pores sont exclus de l'écoulement. La dispersion latérale suit une loi de Gauss : le coefficient de dispersion diminue lorsque la contrainte seuil du fluide augmente. La dispersion longitudinale suit une loi anormale de diffusion : le théorème de la limite centrale ne s'applique plus. Le coefficient de dispersion dépend d'un temps de coupure arbitraire. Pour un temps de coupure donné, la compétition entre la dispersion spatiale et temporelle a été mise en évidence. Finalement, on démontre que la distribution en taille des pores d'un milieu poreux peut être obtenue à partir des caractéristiques de l'écoulement d'un fluide à seuil.

**Abstract.** — The study of the flow of a Newtonian fluid in a porous medium and the dispersion of markers has been the subject of numerous works after Taylor and Saffman. Characteristics of the flow of a non-Newtonian fluid have not been investigated as much. In this paper we attempt to generalize the Saffman modelling to the case of a Bingham fluid. The porous medium is assumed to be statistically homogeneous and isotropic. Dispersion is completely described by a lateral and a longitudinal dispersion coefficient. Pores are represented by monodispersed capillary tubes with an isotropic angular distribution. In the used mean field approximation, we show that the pressure gradient still satisfies the Laplace equation. The emergence of a critical angle in the orientation of the capillary tube is the consequence of the presence of the yield stress. The lateral dispersion stays Gaussian : the lateral dispersion coefficient is found to decrease with the increase of the yield stress. The longitudinal dispersion obeys an anomalous diffusion law. The longitudinal dispersion coefficient depends on an arbitrary cut-off time. For a given cut-off, a competition between spatial and temporal dispersion is evidenced. At the end, we demonstrate that the pore size distribution of a porous medium can be obtained from the characteristics of the flow of a Bingham fluid.

### Introduction.

The modelling of the flow of non-Newtonian fluids through porous media has already been the subject of numerous publications because of the wide range of application areas [1, 2]. Many of these areas take today a greater importance and give a new interest to such a study. For instance the flow of mineral colloidal suspensions or diluted aqueous solutions of polymers, which behave as non-Newtonian fluids, through porous media, is a basic step in a water treatment process. In the biomedical area, the flow of a suspension of biological cells through the micro vascular network is a very important process regarding the exchange of oxygen or macromolecules which occurs at the level. All these phenomena can be modelled by a similar approach.

Experimentally, it has long been known that a non linear relationship between the flow rate and the pressure drop imposed on the porous medium is obtained for non-Newtonian fluids [3]. The apparent permeability is not only a feature of the porous medium but also depends on the flow conditions.

The mathematical analysis was classically based on rather empirical models combining the simple capillary bundle model of the porous medium with the power law model for the rheological behaviour of the fluid [1]. The discrete approach, recently developed by Sorbie [4, 5], seems however more appropriate to describe the flow of a non-Newtonian fluid through a porous medium and it will be developed here, as well.

Other theoretical and experimental investigations have shown in the case of Newtonian fluids the strong correlation between dispersion and the disordered state of the porous medium, in particular the pore size distribution [6, 7].

In this paper, the porous medium is assumed to be completely disordered. We compare the yield fluid flow to that of a Newtonian fluid. For that, we generalize the classical results obtained by Saffman about dispersion to the case of a Bingham fluid by using a mean field approximation [8]. Both lateral and longitudinal dispersion coefficients will be expressed as a function of the yield stress characterising the fluid. Then we show, as we have already suggested in a preliminary study [9], that experimental curves relating the flow rate of a Bingham fluid to the imposed pressure drop can be used to determine the pore size distribution of a porous medium. This new method has also been proposed by other authors [10] and seems very fruitful.

### Modelling.

**SAFFMAN HYPOTHESIS.** — Let us recall the basic assumptions proposed by Saffman to describe dispersion in a porous medium by a random walk procedure [8]. The porous medium is statistically homogeneous and isotropic. It is regarded as a 3D assemblage of randomly oriented straight uniform pores. The disordered state is ascribed to a disorder of orientation of pores. The pores are assumed to be connected with one another at the end, several pores may start or finish at these end points. The average velocity  $U$  is assumed to be constant and unidirectional and to satisfy the Darcy law :

$$\mathbf{U} = - \kappa \cdot \nabla p \quad (1)$$

where  $\kappa$  is the permeability for a Newtonian fluid.

The path of a single fluid particle is decomposed in statistically independent steps. Each step corresponds to the passage through one pore whose random direction is classically defined by the two angles  $\vartheta$  and  $\varphi$  as indicated in figure 1. The hypothesis of independence between

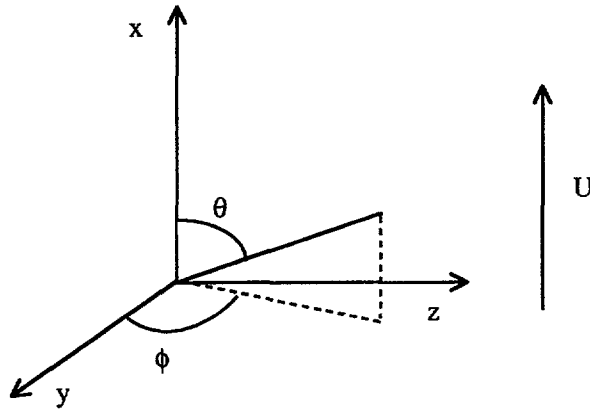


Fig. 1. — Orientation of a capillary tube. The *x* axis refers to the mean direction of the flow. *U* is the average velocity through the porous medium.

successive steps requires that the length of the capillary tubes correspond to the length for which the velocities are decorrelated. The duration  $t_i = t(\vartheta_i)$  of each step *i* depends on the pore direction ; it corresponds to an average convective time through the capillary tube. This time is computed by neglecting the inertia of the fluid. The probability for a fluid particle to choose a given direction is proportional to the flow rate fraction in this direction. The paths of initially neighbouring particles become statistically independent when time is sufficiently large, the dispersion of a marked volume of fluid being obtained from the probability distribution of the displacement of a single fluid particle.

The position  $(X_n, Y_n, Z_n)$  after *n* steps and the transit time  $T_n$  of a fluid particle are computed from the following relationships :

$$X_n = \sum_{i=1}^n n\ell \cos (\vartheta_i) \tag{2a}$$

$$Y_n = \sum_{i=1}^n n\ell \sin (\vartheta_i) \cos (\varphi_i) \tag{2b}$$

$$Z_n = \sum_{i=1}^n n\ell \sin (\vartheta_i) \sin (\varphi_i) \tag{2c}$$

$$T_n = \sum_{i=1}^n n\ell t(\vartheta_i). \tag{2d}$$

CASE OF A BINGHAM FLUID. — Let us consider a Bingham fluid where the rheological behaviour is characterised by a yield value  $\tau_c$  and an asymptotic viscosity  $\mu_\infty$  related by the rheological law :

$$\tau = \tau_c + \mu_\infty \gamma . \tag{3}$$

Below the yield stress, the fluid behaves like a solid. For such a fluid, the average velocity  $\bar{v}(\tau_c)$  through a cylindrical pore subjected to a pressure gradient  $\nabla p$  can be written as :

$$\bar{v}(\nabla p, \tau_c) = \frac{a^2 \nabla p}{8 \mu} \left[ 1 - \frac{4}{3} \frac{\tau_c}{a \nabla p} + \frac{1}{3} \left( \frac{2 \tau_c}{a \nabla p} \right)^4 \right] . \tag{4}$$

By extension, the relation between the average velocity  $U(\tau_c)$  through the porous medium and the pressure gradient is non linear :

$$\mathbf{U}(\tau_c) = -\kappa (\nabla p) \cdot \nabla p \quad (5)$$

where the permeability  $\kappa$  depends upon the pressure gradient.

The basic question is how the pressure gradient is distributed over the porous medium in the case of a non linear fluid ? In the mean field approximation, the pressure gradient is uniformly distributed through the porous medium. In the following part we will show that this assumption is justified as soon as the average velocity and the pressure gradient obey the typical relationship above (Eq. (5)).

In a system of Cartesian coordinates in three dimensions we note :

$$\frac{\partial p}{\partial x} = p_x \quad \frac{\partial p}{\partial y} = p_y \quad \frac{\partial p}{\partial z} = p_z \quad (6a)$$

$$\frac{\partial^2 p}{\partial x^2} = p_{xx} \quad \frac{\partial^2 p}{\partial y^2} = p_{yy} \quad \frac{\partial^2 p}{\partial z^2} = p_{zz} \quad (6b)$$

$$P' = \sqrt{p_x^2 + p_y^2 + p_z^2} \quad (6c)$$

with these notations, the relation (5) associated with the continuity equation for an incompressible fluid, leads to :

$$\begin{aligned} \kappa (P') \cdot (p_{xx} + p_{yy} + p_{zz}) + \\ + \sqrt{P'} \cdot \frac{\partial \kappa}{\partial P'} \cdot [p_x^2 p_{xx} + p_y^2 p_{yy} + p_z^2 p_{zz} + 2 p_x p_y p_{xy} + 2 p_y p_z p_{yz} + 2 p_z p_x p_{zx}] = 0. \end{aligned} \quad (7)$$

It can be easily shown that the Laplace solutions which satisfy the condition :

$$p_x^2 = p_y^2 = p_z^2 = p_0'^2 \quad (8)$$

are solutions of the equation (7).

The pressure fluctuations arise from the existence of the local non linear relationship between the average velocity and the pressure gradient. Experimental evidence of these pressure fluctuations is carried out by direct observations of the diffusion front of a Bingham fluid into another one.

The regular solution  $P = \alpha \cdot x$  which corresponds to the mean field approximation is also the solution of equation (7). As a consequence, there is an excluded volume effect : a solid core entirely fills the volume of all capillary tubes whose direction  $\vartheta$  is greater than a critical value  $\vartheta_c$  defined by :

$$\cos \vartheta_c = \frac{2 \tau_c}{a(\Delta p/L)} \quad (9)$$

where  $L$  is the length of the porous medium. The mean velocity  $\bar{v}(\vartheta ; \vartheta_c)$  inside the capillary  $\vartheta$  can be easily expressed as a function of  $\vartheta$  and  $\vartheta_c$  from the classical Reiner-Buckingham relationship :

$$\bar{v}(\vartheta, \vartheta_c) = \bar{v}_0 \cdot \cos \vartheta \left[ 1 - \frac{4 \cos \vartheta_c}{3 \cos \vartheta} + \frac{1 \cos^4 \vartheta_c}{3 \cos^4 \vartheta} \right] \quad (10)$$

where  $\bar{v}_0$  is the mean velocity of a Newtonian fluid of viscosity  $\mu_\infty$  inside a horizontal capillary

tube subjected to a pressure gradient  $\Delta p/L$ . Neglecting the inertia of the fluid, the mean velocity  $\bar{v}_0$  is determined by the Hagen-Poiseuille relationship :

$$\bar{v}_0 = \frac{a^2}{8 \mu_\infty} \frac{\Delta p}{L}. \quad (11)$$

By using the two above relationships and after straightforward calculations we obtain the probability  $f(\vartheta; \vartheta_c) d\vartheta$  and  $f(\varphi) d\varphi$  for a particle at step  $i$  to choose the direction  $\vartheta + d\vartheta$  and  $\varphi + d\varphi$  :

$$f(\varphi) d\varphi = \frac{1}{2\pi} d\varphi \quad (12)$$

$$f(\vartheta; \vartheta_c) = \frac{\bar{v}(\vartheta; \vartheta_c) \sin(\vartheta) d\vartheta}{\int_0^{\vartheta_c} \bar{v}(\vartheta; \vartheta_c) \sin(\vartheta) d\vartheta} \quad (13)$$

$$f(\vartheta; \vartheta_c) = \frac{\cos(\vartheta) \left[ 1 - \frac{4}{3} \frac{\cos(\vartheta_c)}{\cos(\vartheta)} + \frac{1}{3} \left( \frac{\cos(\vartheta_c)}{\cos(\vartheta)} \right)^4 \right] \sin(\vartheta) d\vartheta}{\frac{1}{2} \sin^2 \vartheta_c + \frac{3}{2} \cos^2 \vartheta_c - \frac{4}{3} \cos \vartheta_c - \frac{1}{6} \cos^4 \vartheta_c} \quad (14)$$

### Discussion and results.

AVERAGE VELOCITY. — The porous medium being characterized by one size of randomly oriented pores, the average velocity  $U_c(\vartheta_c)$  in the  $x$  direction is given by :

$$U_c(\vartheta_c) = \int_0^{\vartheta_c} \bar{v}(\vartheta; \vartheta_c) \cos(\vartheta) \frac{\sin \vartheta}{(1 - \cos \vartheta_c)} d\vartheta = \frac{1 - 2 \cos \vartheta_c + 2 \cos^3 \vartheta_c - \cos^4 \vartheta_c}{1 - \cos \vartheta_c} \quad (15)$$

The influence of the imposed pressure drop on the adimensional average velocity  $U_c/U$  for a given value of the yield stress is evidenced in figure 2, where  $U$  is the average velocity of a Newtonian fluid with the viscosity  $\mu_\infty$  under the same pressure conditions. This curve clearly shows the expected threshold pressure drop  $(\Delta P/L)_c$  below which the average velocity is zero. This threshold is directly derived from the yield stress of the fluid :

$$\left( \frac{\Delta P}{L} \right)_c = \frac{\tau_c}{a} \quad (16)$$

When the pressure is increased above this threshold, some capillary tubes previously closed are open ; and both the average velocity and the apparent permeability of the medium increase. For large enough pressure gradients, we obtain again a linear relationship between the flow rate and the pressure gradient. The corresponding permeability tends to the permeability defined by the Darcy law for a Newtonian fluid with the asymptotic viscosity  $\mu_\infty$ .

THE LATERAL AND LONGITUDINAL DISPERSION. — The next part is concerned with the determination of the asymptotic values of the longitudinal and lateral dispersion coefficients obtained at large time. In accordance with Saffman, calculations proceed in successive steps. First the statistical properties of the displacement of a single particle after  $n$  steps must be established. Then the statistical properties of the time after  $n$  steps are expressed as a function

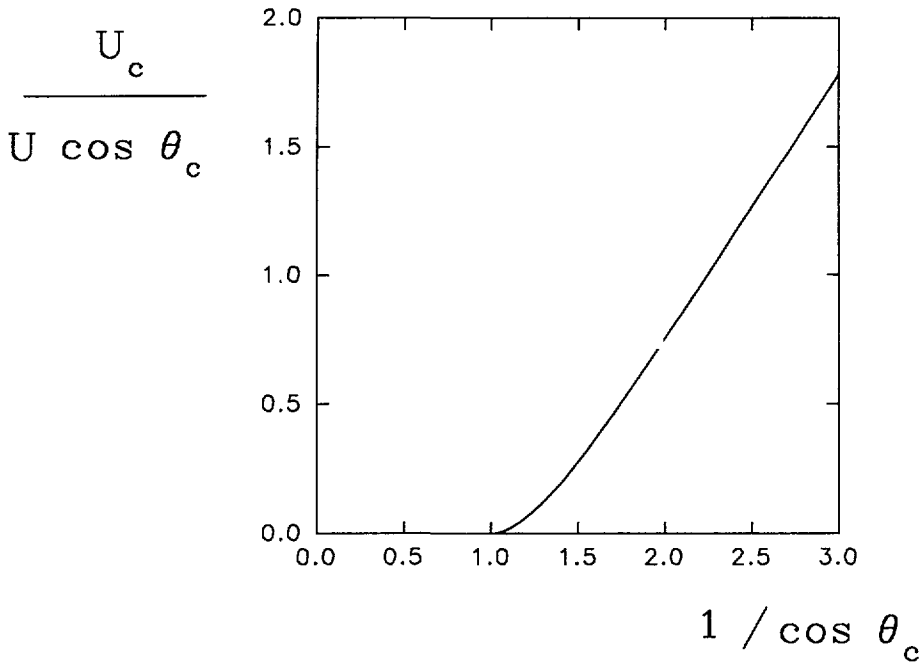


Fig. 2. — Plot of the dimensionless average velocity  $U_c/U$  from the equation (15) against the imposed pressure gradient  $\nabla P$  for a fixed yield stress. This curve shows the existence of a threshold pressure gradient below which the mean velocity approaches zero. At a high pressure gradient the Darcy law is satisfied.

of an arbitrary cut-off time. The average number of steps after a large given time is estimated. By this way, the lateral dispersion coefficient  $D_{\perp}$  can be expressed as a function of the critical angle. The longitudinal dispersion coefficient  $D_{\parallel}$  also depends on the cut-off time.

*Statistical properties of the displacement after n steps.* — The three components  $X_n, Y_n, Z_n$  of the position of a particle after  $n$  steps are the sum of  $n$  independent variables. The average value and the variance are classically expressed as a function of the critical angle  $\vartheta_c$

$$\bar{X}_n(\vartheta_c) = n\ell \int_0^{\vartheta_c} \cos \vartheta \cdot f(\vartheta ; \vartheta_c) d\vartheta \tag{17}$$

$$\overline{(X_n - \bar{X}_n)^2} = n\ell^2 \int_0^{\vartheta_c} (\cos \vartheta - \bar{X}_n)^2 \cdot f(\vartheta ; \vartheta_c) d\vartheta = n\ell^2 \sigma_x^2(\vartheta_c) \tag{18}$$

(other components as in appendix A). Calculations are lengthy but straightforward ; analytical results are given at the end of this paper (see appendix A).

In accordance with Saffman’s analysis, we define dimensionless random variables :

$$x_n = \frac{X_n - \bar{X}_n}{\ell \sqrt{n}} \quad y_n = \frac{Y_n}{\ell \sqrt{n}} \quad z_n = \frac{Z_n}{\ell \sqrt{n}}. \tag{19}$$

Then  $x_n, y_n, z_n$  have zero mean and variances  $\sigma_x^2(\vartheta_c), \sigma_y^2(\vartheta_c)$  and  $\sigma_z^2(\vartheta_c)$  respectively. Because of the symmetry, the covariances are zero. It can be established from the central limit theorem that the probability distributions of these random variables are asymptotically normal and statistically independent as  $n \rightarrow \infty$ .

*Statistical properties of the time after n steps.* — Consider now the average time  $\bar{T}_n(\vartheta_c)$  and the variance  $\sigma_T^2(\vartheta_c)$  after a given number of  $n$  steps. The variance is divergent, so both quantities are computed as a function of a cut-off  $t_0$  such as :

$$\begin{aligned} t_i &= \bar{t}(\vartheta_i) & \text{if } t_i \leq t_0 \\ t_i &= t_0 & \text{otherwise.} \end{aligned} \tag{20}$$

From this assumption  $\bar{T}_n(\vartheta_c)$  and  $\sigma_T^2(\vartheta_c)$  are expressed as :

$$\bar{T}_n(\vartheta_c; \vartheta_0) = n \int_0^{\vartheta_0} t(\vartheta; \vartheta_c) f(\vartheta; \vartheta_c) d\vartheta + n \int_{\vartheta_0}^{\vartheta_c} t_0 f(\vartheta, \vartheta_c) d\vartheta \tag{21}$$

and

$$\begin{aligned} \overline{(T_n - \bar{T}_n)^2} &= n \int_0^{\vartheta_0} (t(\vartheta, \vartheta_c) - \bar{t}(\vartheta_c, t_0))^2 f(\vartheta; \vartheta_c) d\vartheta - \\ &\quad - n \int_{\vartheta_0}^{\vartheta_c} (t_0 - \bar{t}(\vartheta_c, t_0))^2 f(\vartheta; \vartheta_c) d\vartheta = \frac{n\ell^2}{U^2} \sigma_T^2(\vartheta_c, t_0) \end{aligned} \tag{22}$$

where  $\vartheta_0$  and  $t_0$  are related by :

$$\frac{1}{t_0} = U \cos \vartheta_0 \left( 3 - \frac{4 \cos \vartheta_c}{\cos \vartheta_0} + \frac{\cos^4 \vartheta_c}{\cos^4 \vartheta_0} \right). \tag{23}$$

In the case of a Newtonian fluid, this limit time was correlated by Saffman to the molecular diffusion. The physical meaning of  $t_0$  for a Bingham fluid is more complicated. Results are discussed as a function of this arbitrarily theoretical threshold. The average time is

$$\bar{t}(\vartheta_c, t_0) = \frac{\ell(1 - \cos \vartheta_0)}{3 N_c U} + \frac{t_0}{N_c} A \tag{24}$$

and the variance :

$$\sigma_T^2(\vartheta_c, \vartheta_0) = (B + C + D + E)/N_c \tag{25}$$

where the coefficients  $A, B, C, D, E$  and  $N_c$  are given in appendix B.

The variation of the variance  $\sigma_T^2(\vartheta_c)$  with the limit time  $t_0$  is represented in figure 3 for a set of values of  $\vartheta_c$ . The time variance diverges as  $t_0 \rightarrow \infty$ . The dimensionless random variable :

$$t_n = \frac{U T_n - \bar{T}_n}{\ell \sqrt{n}} \tag{26}$$

has a zero mean and variance  $\sigma_T^2(\vartheta_c)$  but it is not normally distributed as it will be more precisely discussed later. The covariance  $\sigma_{XT}(\vartheta_c)$  is written for a given limit time  $t_0$ .

$$\begin{aligned} \overline{x_n t_n}(\vartheta_c, t_0) &= \frac{U}{\ell} \int_0^{\vartheta_0} (\cos \vartheta - \bar{x})(t_0 - \bar{t}(\vartheta_c, t_0)) f(\vartheta, \vartheta_c) d\vartheta + \\ &\quad + \frac{U}{\ell} \int_{\vartheta_0}^{\vartheta_c} (\cos \vartheta - \bar{x})(t(\vartheta, \vartheta_c) - \bar{t}(\vartheta_c, t_0)) f(\vartheta, \vartheta_c) d\vartheta = \sigma_{XT}(\vartheta_c, t_0). \end{aligned} \tag{27}$$



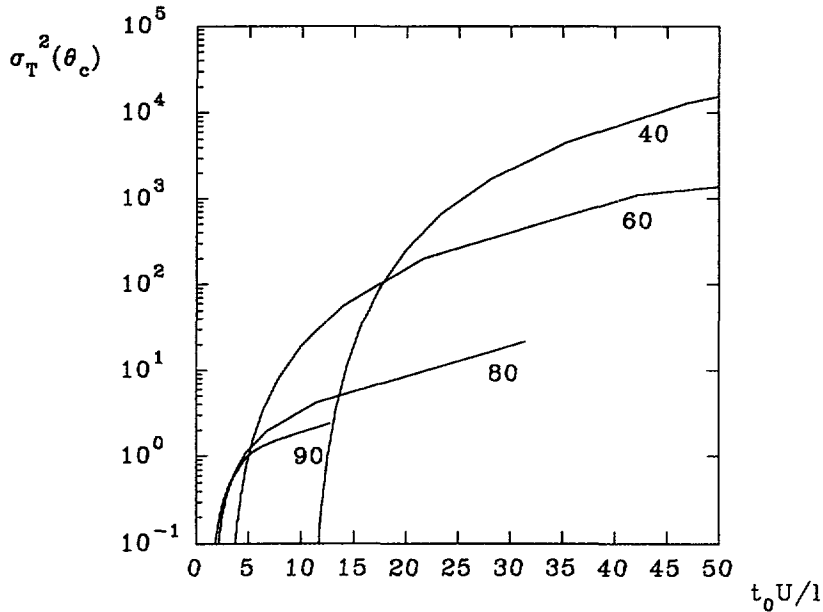


Fig. 3. — Variation of the temporal variance defined in equation (22) with the dimensionless cut-off time  $t_0 U/\ell$  for different values of the critical angle  $\vartheta_c$  as indicated on the plot;  $\vartheta_c$  is such as  $\cos \vartheta_c = 2 \tau_c/a \nabla P$  with  $\tau_c$  is the yield stress of the fluid,  $a$  the radius of the pores and  $\nabla P$  the pressure gradient. At a given cut-off time, the variance increases with the ratio  $2 \tau_c/a \nabla P$  and opposes the decrease of the spatial dispersion introduced in equation (18).

The result is given at the end ; because of the symmetry, other covariances are zero.

The remaining step consists in calculating the probability distribution of the number of steps after a large given time  $T$ . The calculation bears no difficulty ; it is not reported here.

*The lateral dispersion.* — As Saffman, the  $Y$  component of the displacement after a large time  $T$  in the  $y$  direction is approximated from equation (19) by :

$$Y(\vartheta_c) = \ell \sqrt{n}(\vartheta_c) \cdot y_{\bar{n}}(\vartheta_c). \quad (28)$$

$Y$  is normally distributed and the lateral dispersion coefficient  $D_{\perp}$  is given by :

$$D_{\perp}(\vartheta_c) = \frac{1}{2T} \bar{Y}^2(\vartheta_c) = \frac{3 U \ell}{4(1 - \cos \vartheta_c)} \times \\ \times \left[ 0.25 \sin^4 \vartheta_c + \cos^4 \vartheta_c \left( \frac{1}{3} \ln \cos \vartheta_c - \frac{11}{8} \right) + 1.5 \cos^2 \vartheta_c - \frac{8}{9} \cos \vartheta_c \right]. \quad (29)$$

The decrease of the dimensionless lateral dispersion coefficient  $D_{\perp}/U\ell$  with the ratio  $2 \tau_c/a \nabla P$  is shown in figure 4. This decrease reflects the existence of the excluded volume effect due to the yield stress of the fluid. This coefficient does not depend on the cut-off  $t_0$ . Dispersion in the  $z$  direction is governed by the same law.

*The longitudinal dispersion.* — In the same way, the  $X$  component of the displacement after a large given time  $T$  is approximated from equation (19) by :

$$X(\vartheta_c) = \ell \sqrt{n}(\vartheta_c) \cdot x_{\bar{n}}(\vartheta_c) + \bar{X}_n(\vartheta_c). \quad (30)$$

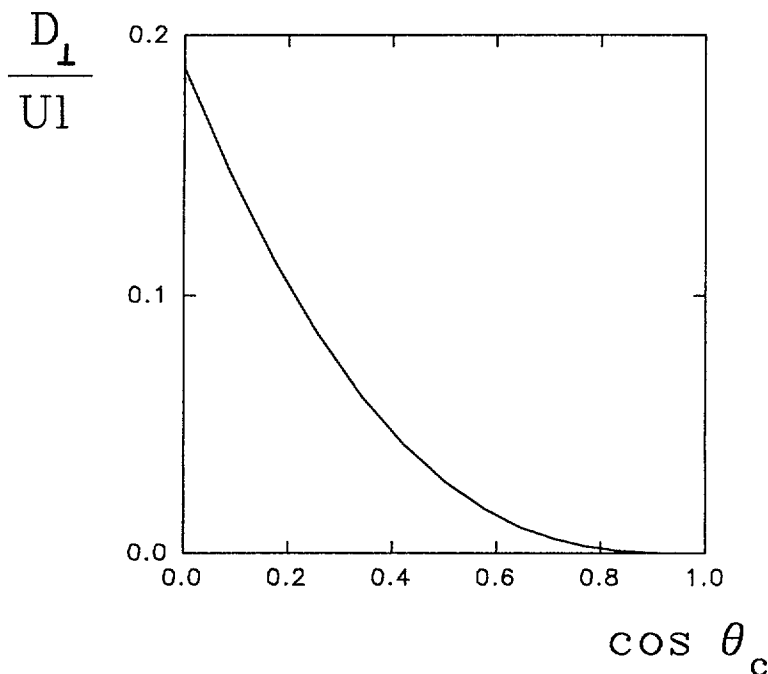


Fig. 4. — Variation of the dimensionless lateral dispersion coefficient  $D_{\perp}/U\ell$  with the ratio  $2 \tau_c/a \nabla P$  ( $\tau_c$  is the yield stress of the fluid,  $a$  the radius of the pores and  $\nabla P$  the pressure gradient). The coefficient  $D_{\perp}/U\ell$  is computed from equation (29). The existence of the yield stress decreases the lateral dispersion.

The longitudinal dispersion coefficient  $D_{\parallel}$  is expressed as a function of the critical angle  $\vartheta_c$  and the cut-off time  $t_0$ .

$$D_{\parallel}(\vartheta_c; t_0) = \frac{1}{2T} \overline{(X - U_c T)^2} = \frac{3N_c}{2(1 - \cos \vartheta_c)} \left[ \sigma_x^2(\vartheta_c) - 2 \frac{U_c}{U} \sigma_{xT}(\vartheta_c; t_0) + \left( \frac{U_c}{U} \right)^2 \sigma_T^2(\vartheta_c; t_0) \right]. \quad (31)$$

When the limit time is fixed, the dimensionless longitudinal coefficient  $D_{\parallel}/U\ell$  exhibit a maximum for a given value of  $\vartheta_c$  as can be seen in figure 5. This unexpected result arises from the competition between the spatial and temporal dispersions represented by the variances  $\sigma_x^2(\vartheta_c)$  and  $\sigma_T^2(\vartheta_c)$ , respectively.

*An anomalous longitudinal diffusion law.* — The dispersion laws of markers in porous medium are usually obtained by applying the central limit theorem. This theorem establishes that if the transit time  $T$  in the porous medium for a particle becomes much larger than the duration  $t_i$  of any step, then the probability distribution of transit time  $T$  for many particles should be normal. By applying the ergodic principle, the spatial distribution of a set of particles should also be normal [11]. More precisely, a Gaussian distribution of transit time  $T$  is expected if the probability  $f(t)dt$  for a particle to stay a long time inside a capillary tube decreases faster than  $1/t^3$  [6]. We have represented the probability functions  $f(t, \vartheta_c)dt$  computed from equations (10), (11) and (14), in figure 6 for different values of the critical angle  $\vartheta_c$ . In the case of a Newtonian fluid ( $\vartheta_c = 90^\circ$ ) Saffman introduced a duration limit

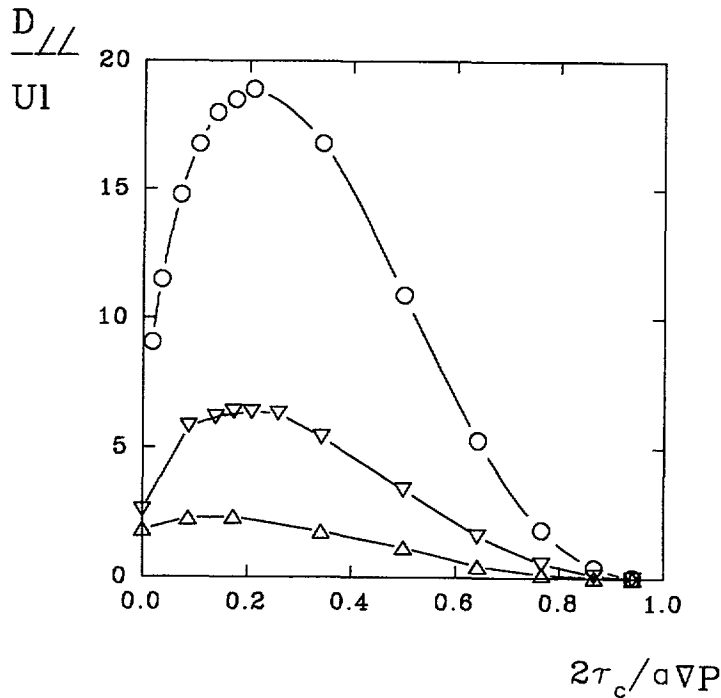


Fig. 5. — Plot of the dimensionless longitudinal dispersion coefficient  $D_1/UL$  computed from equation (31) against the ratio  $2\tau_c/a\Delta P$  where with  $\tau_c$  is the yield stress of the fluid,  $a$  the radius of the pores and  $\Delta P$  the pressure gradient. Each curve corresponds to a particular cut-off time value  $t_0/UL$ : ( $\Delta$ ): 100, ( $\nabla$ ): 1 000, ( $\circ$ ): 10 000.

correlated to the molecular diffusion ; calculations thus lead to a normal diffusion process. An anomalous effect is however expected as soon as molecular diffusion does not significantly occur [12]. In the case of a Bingham fluid, an anomalous diffusion law is always expected and especially at high values of the yield stress or at low pressure gradients. The residence time distributions, obtained from numerical simulations, actually show a long tail of dispersion [12].

**DETERMINATION OF THE PORE SIZE DISTRIBUTION OF A POROUS MEDIUM.** — We pointed out above that the flow of a yield fluid through a porous medium obeys a non-linear relationship between the average velocity and the pressure drop (Eq. (5)). In particular the flow through the porous medium only occurs above a critical value of the pressure gradient. For a 3D network composed of randomly oriented and monodispersed pores of radius  $a$ , the threshold value refers to the yield stress of the fluid (cf. Eq. (16)).

We feel that there exists a strong correlation between the flow characteristics described by the permeability  $\kappa(\Delta P)$  and the structure of the porous medium (orientation and size of pores). Considering now a pore size distribution  $f(a)$ : the flow rate inside a capillary tube both depends on its orientation and its radius seen here as a random variable (cf. Eqs. (10) and (11)). An increase in the pressure drop imposed on the porous medium produces the opening of all capillary tubes for which the radius  $a_i$  and the angle  $\vartheta_i$  satisfy the condition :

$$a_i \cos \vartheta_i \leq \frac{2\tau_c}{\Delta P/L}. \quad (32)$$

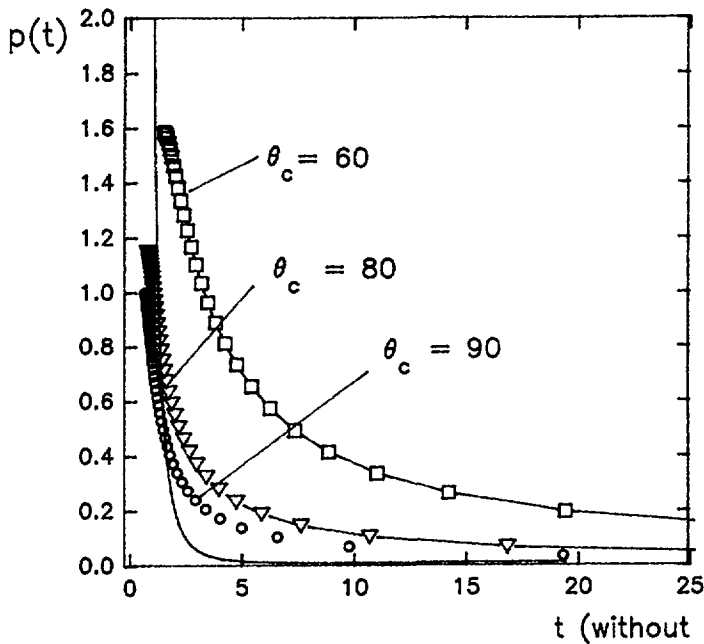


Fig. 6. — Theoretical probability functions  $p(t) dt$  for a particle to stay in a capillary tube for a time lying between  $t$  and  $t + dt$ . Each curve corresponds to a particular value of the critical angle  $\vartheta_c$  as indicated on the plot ( $\cos \vartheta_c = 2 \tau_c/a \nabla P$  with  $\tau_c$  is the yield stress of the fluid,  $a$  the radius of the pores and  $\nabla P$  the pressure gradient). These curves are compared to the reference curve (solid line)  $1/t^3$ . For a Bingham fluid, dispersion is expected to obey an anomalous diffusion law.

A critical value  $\vartheta_c(a_i)$  and consequently an average velocity  $U_c[\vartheta_c(a_i)]$  defined in equation (15) are associated with each radius  $a_i$ . If no correlation between the orientation  $\vartheta_i$  and the radius  $a_i$  of the capillary tube is assumed, the flow  $Q(\nabla P)$  in the  $x$  direction through the porous medium can be expressed as a function of the pore size distribution  $f(a)$  :

$$Q(\nabla P) = \int_{a_0}^{\infty} f(a) \left[ \int_0^{\vartheta_c(a)} q(\nabla P, a, \vartheta) \cos \vartheta \frac{\sin \vartheta}{(1 - \cos \vartheta_c)} d\vartheta \right] da \tag{33}$$

where  $a_0 = 2 \tau_c/\nabla P$  is the size of the smallest tube for which the flow occurs and  $q(\nabla P, a, \vartheta) = \pi a^2 \bar{v}(\nabla P, a, \vartheta)$ , the elementary flow rate through a pore.

The pore size distribution can be obtained from the experimental determination of the flow curve  $Q(\nabla P)$  by using a differential operator :

$$f(a) = \frac{6 \mu \nabla p^2}{11 \pi \tau_c a^4} \left[ 12 \frac{\partial^2}{\partial \nabla p^2} + 8 \nabla p \frac{\partial^3}{\partial \nabla p^3} + \nabla p^2 \frac{\partial^4}{\partial \nabla p^4} \right] Q(\nabla p) . \tag{34}$$

The above relationship corresponds to the generalization of the expression proposed by Ambari *et al.* in the simpler case of a one dimensional porous medium [10].

**Conclusion.**

This paper dealt with the modelling of the flow of a yield fluid through a porous medium. We proposed a model consisting of a generalization of the analysis developed by Saffman for Newtonian fluids. The porous medium was assumed to be statistically homogeneous and

isotropic so dispersion can be defined by lateral and longitudinal coefficients. In the used mean field approximation, we showed that the pressure gradient still satisfies the Laplace equation. When the porous medium is characterized by one size of randomly oriented pores, there exists a critical angle below which pores are excluded from the flow : indeed the local pressure gradient is not large enough compared to the yield stress of the fluid. Flow through the porous medium only occurs above a threshold of the pressure drop.

Dispersion of marked particles was then described by a random walk. The paths of particles were decomposed into many independent steps corresponding to the passage through one pore. The duration of each step was the average convective time in the given direction. The expressions of the asymptotic lateral and longitudinal dispersion coefficients were obtained from the statistical properties of the displacement and time of a single particle after a large given number of steps. The lateral dispersion is Gaussian and decreases when the yield stress of the fluid increases. As a consequence of the existence of the yield stress, the elementary time distribution in pores was broader compared to that of a Newtonian fluid. Then the central limit theorem cannot be applied : longitudinal dispersion obeys an anomalous diffusion law. The longitudinal dispersion coefficient depends on an arbitrary cut-off time. For a given cut-off, a competition between spatial and temporal dispersion was found.

At the end, we pointed out the strong correlation between the structure of the porous medium (orientation and size distribution of pores) and the feature of the flow of a Bingham fluid. A differential operator was found to determine the pore size distribution from the relationship pressure/flow rate.

This paper was concerned with asymptotic dispersion. We performed moreover numerical simulations, by using Monte Carlo techniques, to investigate the transient regime preceding the asymptotic regime described here. Simulations also provide an interesting tool for studying the correlation between dispersion and the disordered state of the porous medium.

## Appendix A.

The components of the average displacement of a single particle after  $n$  steps are :

$$\bar{X}_n(\vartheta_c) = \frac{n\ell}{3N_c} [1 - 2 \sin^2 \vartheta_c \cos \vartheta_c - \cos^4 \vartheta_c] \quad (\text{A.1})$$

$$\bar{Y}_n(\vartheta_c) = n\ell \int_0^{\vartheta_c} \sin \vartheta \cos \varphi \cdot f(\vartheta ; \vartheta_c) d\vartheta = 0 \quad (\text{A.2})$$

$$\bar{Z}_n(\vartheta_c) = n\ell \int_0^{\vartheta_c} \sin \vartheta \sin \varphi \cdot f(\vartheta ; \vartheta_c) d\vartheta = 0. \quad (\text{A.3})$$

The variances are given by :

$$\begin{aligned} \overline{(X_n - \bar{X}_n)^2} = \frac{n\ell^2}{N_c} \left[ \left( \frac{7}{36} - \frac{1}{3} \ln(\cos \vartheta_c) + \frac{2}{3} \bar{X}_n - \frac{1}{6} \bar{X}_n^2 \right) \cos^4 \vartheta_c - \frac{4}{3} \bar{X}_n \cos^3 \vartheta_c + \right. \\ \left. + \bar{X}_n^2 \cos^2 \vartheta_c - \frac{4}{3} \left( \bar{X}_n^2 - \bar{X}_n + \frac{1}{3} \right) \cos \vartheta_c + \frac{1}{2} \bar{X}_n^2 - \frac{2}{3} \bar{X}_n + \frac{1}{4} \right] \quad (\text{A.4}) \end{aligned}$$

$$\overline{(Y_n)^2} = n\ell^2 \int_0^{2\pi} \int_0^{\vartheta_c} (\sin \vartheta \cos \varphi)^2 \cdot f(\vartheta ; \vartheta_c) d\vartheta \cdot f(\varphi) d\varphi = n\ell^2 \sigma_Y^2(\vartheta_c) \quad (\text{A.5})$$

$$\overline{(Z_n)^2} = n\ell^2 \int_0^{2\pi} \int_0^{\vartheta_c} (\sin \vartheta \sin \varphi)^2 \cdot f(\vartheta ; \vartheta_c) d\vartheta \cdot f(\varphi) d\varphi = n\ell^2 \sigma_Z^2(\vartheta_c) \quad (\text{A.6})$$

$$\bar{Y}_n^2 = \bar{Z}_n^2 = \frac{n\ell^2}{2N_c} \left[ \frac{1}{4} \sin^4 \vartheta_c + \cos^4 \vartheta_c \left( \frac{1}{3} \ln(\cos \vartheta_c) \right) + \frac{3}{2} \cos^2 \vartheta_c - \frac{8}{9} \cos \vartheta_c \right]. \quad (\text{A.7})$$

**Appendix B.**

The coefficients occurring in the statistical properties of the time after a large given number of steps  $n$  are respectively :

$$A = 0.5 + \cos^2 \vartheta_c - 0.5 \sin^2 \vartheta_0 - \frac{4}{3} \cos \vartheta_c \cos \vartheta_0 - \frac{\cos^4 \vartheta_c}{6 \cos^2 \vartheta_0} \tag{B.1}$$

$$B = -\frac{5}{54} \ln \left( \frac{s_0 - 1}{s_1 - 1} \right) + \frac{1}{18} \left( \frac{s_1 - s_0}{(s_0 - 1)(s_1 - 1)} \right) - \frac{1}{108} \ln \left( \frac{3 s_0^2 + 2 s_0 + 1}{3 s_1^2 + 2 s_1 + 1} \right) - \frac{1}{54 \sqrt{2}} \left( \operatorname{arctg} \frac{\sqrt{3} S_0}{2} - \operatorname{arctg} \frac{\sqrt{3} S_1}{2} \right) \tag{B.2}$$

$$C = \frac{2 \bar{t}}{3} (\cos \vartheta_0 - 1) \tag{B.3}$$

$$D = \bar{t}^2 \left( 0.5 \sin^2 \vartheta_0 + \frac{4}{3} \cos \vartheta_c \cos \vartheta_0 + \frac{\cos^4 \vartheta_c}{\cos^2 \vartheta_0} - \frac{4}{3} \cos \vartheta_c - \frac{\cos^4 \vartheta_c}{6} \right) \tag{B.4}$$

$$E = (t_0 - \bar{t})^2 \left( 0.5 + \cos^2 \vartheta_c - 0.5 \sin^2 \vartheta_0 - \frac{4}{3} \cos \vartheta_c \cos \vartheta_0 - \frac{\cos^4 \vartheta_c}{6 \cos^2 \vartheta_0} \right) \tag{B.5}$$

with

$$\begin{aligned} s_0 &= \frac{\cos \vartheta_0}{\cos \vartheta_c} & s_1 &= \frac{1}{\cos \vartheta_c} \\ S_0 &= \sqrt{3} \cdot s_0 + \frac{1}{\sqrt{3}} & S_1 &= \sqrt{3} \cdot s_1 + \frac{1}{\sqrt{3}} \end{aligned} \tag{B.6}$$

$$N_c = 0.5 \sin^2 \vartheta_c + \frac{4}{3} \cos^2 \vartheta_c + \frac{1}{6} \cos^2 \vartheta_c \sin^2 \vartheta_c - \frac{4}{3} \cos \vartheta_c. \tag{B.7}$$

The covariance  $\sigma_{XT}(\vartheta_c, t_0)$  is :

$$\begin{aligned} \sigma_{XT}(\vartheta_c, t_0) &= \frac{1}{6} \sin^2 \vartheta_0 + \frac{\bar{x}}{3} \cdot [\cos \vartheta_0 - 1] + \\ &+ \frac{\bar{t}}{3} \cdot \left[ \cos^3 \vartheta_0 - 2 \cos \vartheta_c \cos^2 \vartheta_0 - \cos^4 \vartheta_c \left( \frac{1}{\cos \vartheta_0} - 1 \right) - 1 + 2 \cos \vartheta_c \right] \\ &+ \bar{x} \bar{t} \cdot \left[ \frac{1}{2} \sin^2 \vartheta_0 + \frac{4}{3} \cos \vartheta_c (\cos \vartheta_0 - 1) + \frac{\cos^4 \vartheta_c}{6} \left( \frac{1}{\cos^2 \vartheta_0} - 1 \right) \right] \\ &+ \frac{(t_0 - \bar{t})}{3} \cdot \left[ \cos^3 \vartheta_0 - 2 \cos \vartheta_c \cos^2 \vartheta_0 + 2 \cos^3 \vartheta_c - \frac{\cos^4 \vartheta_c}{\cos \vartheta_0} \right] \\ &- \bar{x} (t_0 - \bar{t}) \left[ \cos^2 \vartheta_c - \frac{1}{2} \sin^2 \vartheta_0 - \frac{4}{3} \cos \vartheta_c \cos \vartheta_0 - \frac{\cos^4 \vartheta_c}{6 \cos^2 \vartheta_0} + 0.5 \right]. \end{aligned} \tag{B.8}$$

## References

- [1] SAVINS J. G., *Ind. Eng. Chem.* **61** (1969) 18.
- [2] WU Y. S., PRUESS K. and WITHERSPOON P. A., Society of Petroleum Engineers (1990).
- [3] MARSHALL R. J. and METZNER A. B., *Ind. Eng. Chem. Fundam.* **6** (1967) 393.
- [4] SORBIE K. S., CLIFFORD P. J. and JONES E. R. W., *J. Colloid Interface Sci.* **139** (1989) 508.
- [5] SORBIE K. S., *J. Colloid Interface Sci.* **139** (1990) 299.
- [6] BOUCHAUD J. P., GEORGES A. and LE DOUSSAL P., *J. Phys. A* **48** (1987) 1855.
- [7] BACRI J. C., RAKOTOMALALA N. and SALIN D., *Physical Rev. Lett.* **58** (1987) 2035.
- [8] SAFFMAN P. G., *J. Fluid Mech.* **6** (1959) 321.
- [9] CHAPLAIN V. and MILLS P., Journées de Carry Le Rouet (mai 1990).
- [10] AMBARI A., BENHAMOU M., ROUX S. and GUYON E., *C. R. Acad. Sci. Paris, série II* **311** (1990) 1291.
- [11] BEAR J., Dynamics of fluids in porous media (Elsevier, New York, 1971).
- [12] CHAPLAIN V. and MILLS P., *Sci. Total Environ.* **123/124** (1992) 451-457.