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## Drift instabilities of cellular patterns

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**Résumé.** — Nous décrivons des mécanismes élémentaires qui induisent une dérive à vitesse constante d'une structure cellulaire. Nous considérons soit une structure spatiale statique, soit une onde stationnaire. Dans les deux cas nous montrons comment « l'instabilité de dérive » est liée à la brisure de l'invariance par réflexion d'espace. Lorsque le mode le plus instable a un nombre d'onde fini, « l'instabilité de dérive » peut être précédée par une instabilité oscillatoire qui engendre une modulation spatiotemporelle de la longueur d'onde de la structure initiale. Nous interprétons divers résultats expérimentaux récents dans le cadre du modèle que nous proposons.

**Abstract.** — We report basic mechanisms that generate a secondary drift instability of a stationary cellular pattern or a standing wave pattern. In both cases, we show how the « drift bifurcation » is associated with a space-reflection symmetry-breaking. When the most unstable mode comes in at finite wavenumber instead of zero wavenumber, the drift instability can be pre-empted by an oscillatory instability of the basic pattern wavenumber. We interpret different recent experimental observations within this framework.

### 1. Introduction.

Drift instabilities of cellular patterns have been widely observed in various experimental situations. Convection in binary fluid mixtures [1], or Couette flow between two horizontal coaxial cylinders with a partially filled gap [2], display transitions from stationary to traveling rolls. As clearly noticed in reference [2], the traveling rolls are tilted and the direction of the propagation is determined by this asymmetry. The traveling-roll state is either homogeneous in space, or there exist domains of inclined rolls with opposite tilt and thus opposite propagation direction. Similar results have been found recently in a film draining experiment [3].

Drift instabilities have been also observed in directional crystal growth experiments. Above the onset of the Mullins-Sekerka instability of liquid crystals, « solitary modes » propagating along the interface have been observed [4]. These « solitary modes » consist of domains of stretched asymmetric cells that connect two regions with symmetric cells. Similarly, domains of tilted lamellae moving transversally along the growth front have been observed during

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directional solidification of eutectics [5], and the relationship of the tilt direction to the one of propagation has been also emphasized.

Finally, a drift instability was observed recently for a standing surface wave excited parametrically in a horizontal layer of fluid contained in a thin annulus, submitted to vertical vibrations [6]. It was observed that, as the driving amplitude is increased, the standing wave pattern either begins to move at a constant speed in one direction, or undergoes an oscillatory instability that corresponds to a compression mode of the periodic structure (i.e. a wavenumber modulation in space and time).

On the theoretical side, it is interesting to note that a secondary bifurcation that transforms a stationary structure into a traveling one has been predicted by Malomed and Tribelsky before the experimental results quoted above [7]. They used a Galerkin approximation for model equations of the Kuramoto-Sivashinsky type, and pointed out that the drift instability arises in that case from the coupling between the spatial phase of the basic structure with the second harmonic generation. Recently this bifurcation was understood in a more general way from symmetry considerations [8]. Finally, a drift instability has been observed by numerical integration of the Kuramoto-Sivashinsky equation [9].

This paper is organized as follows. We first study a simple model of a drift instability of stationary patterns, that shows the basic structure of this bifurcation. We then propose a model inspired by the mechanism described by Malomed and Tribelsky (see Sect. 3). In section 4 we show that a drift instability of a standing wave can be understood as a secondary bifurcation described by the evolution equations for the amplitudes of the right and left propagating waves. In section 5, we consider the stability of a homogeneous drifting pattern to space dependent perturbations. Finally we show that when the drift bifurcation occurs at finite wavenumber, we get an oscillatory instability of the basic pattern wavenumber, observed in convection experiments [10] and for parametrically excited surface waves [6]. In all cases, we show that the basic mechanism consists of the coupling between the spatial phase  $\phi$  of the primary pattern, with the order parameter  $V$  associated with the space-reflection broken-symmetry,  $\phi_t \propto V$ , and we identify the physical significance of  $V$  for the various examples considered. The notation must be considered independently in each section, except for sections 5 and 6

## 2. A naive model.

We first consider a stationary pattern-forming bifurcation and thus write the field  $\mathbf{u}(x, t)$  that describes the physical system,

$$\mathbf{u}(x, t) = [A(t) e^{ikx} + \text{c.c.}] \mathbf{u}_k + h.o.t \quad (1)$$

where  $A(t)$  is the slowly varying amplitude of the most-unstable wavenumber  $k$  that gives rise to the stationary periodic pattern.  $\mathbf{u}(x, t)$  stands for the velocity and temperature fields in a convection experiment or for the interface position in the reference frame moving at the front velocity in a crystal growth experiment.  $A(t)$  obeys to leading order a Landau equation,

$$A_t = \mu A - |A|^2 A \quad (2)$$

where  $\mu$  is the distance from criticality; we have considered a supercritical bifurcation and simplified equation (2) by appropriate scaling of amplitude.

A periodic pattern, drifting at velocity  $V$  along the  $x$ -axis, is described by a similar equation in the reference frame moving at velocity  $V$ . We go back to the laboratory reference frame *via*

the transformation  $x \rightarrow x - Vt$ , or equivalently for  $A$ ,  $A \rightarrow A \exp(iVkt)$ . Thus, equation (2) becomes,

$$A_t = \mu A - |A|^2 A - ikAV. \quad (3)$$

We must at this stage give an evolution equation for  $V$ . We consider the model,

$$V_t = V(-\nu + |A|^2) - V^3. \quad (4)$$

The following considerations have been taken into account. The space reflection symmetry,  $x \rightarrow -x$ , implies the invariance  $V \rightarrow -V$ .  $V$  is linearly damped ( $\nu > 0$ ) in order to get a stationary primary bifurcation. The term  $-V(\nu - |A|^2)$  is the leading order expansion of the resonant term  $-Vf(A, \bar{A})$ , compatible with the  $x$ -translation invariance which implies the invariance under  $A \rightarrow A \exp i\phi$ . This model is too simple because a coupling of the form  $V|A|^2$  does not exist in general (it is for instance forbidden if the system is Galilean invariant), but it allows one to understand simply the structure of the bifurcation from the stationary pattern to the drifting one that occurs when  $\mu = \nu$ .

For  $\mu > 0$  equations (3), (4) have stationary solutions,  $A = A_0$ ,  $V = 0$ , with  $A_0^2 = \mu$ , that represent periodic patterns of wavenumber  $k$ . We write

$$A = [A_0 + r(t)] \exp i[\phi(t)], \quad V = V(t)$$

and get from (3), (4),  $r_t = -2\mu r + h.o.t$ , thus  $r(t)$  is damped and

$$\phi_t = -kV \quad (5a)$$

$$V_t = (\mu - \nu)V - V^3 + h.o.t \quad (5b)$$

The basic pattern bifurcates first for  $\mu = \nu$ , and begins to drift with a velocity  $\pm \sqrt{\mu - \nu}$ . The structure of the «drift bifurcation» is as follows. Its order parameter  $V$  undergoes a  $x \rightarrow -x$  symmetry-breaking pitchfork bifurcation, and the coupling with the neutral mode  $\phi$  due to translational invariance in space, generates the drift.

This bifurcation structure was found in reference [7] where  $V$  corresponds to the basic pattern second harmonic amplitude. More generally it was recently pointed out on symmetry considerations, translational invariance in space ( $\phi \rightarrow \phi + constant$ ) and space reflection symmetry ( $x \rightarrow -x$ ,  $\phi \rightarrow -\phi$ ,  $V \rightarrow -V$ ), that when a stationary pattern undergoes a secondary pitchfork bifurcation that breaks the reflection symmetry, the coupling of the pattern spatial phase is generically of the form given by equations (5), and generates a drift instability [8]. Thus, on symmetry considerations, equations (5) can be considered as the leading order amplitude equations that describe the «drift bifurcation».

We should emphasize that  $V$  does not need to be a velocity field in order to generate the drift, which occurs because of the form of the coupling with the spatial phase given by equation (5a).  $V$  is the order parameter of the «drift bifurcation», associated with the  $x \rightarrow -x$  broken symmetry. The only constraint on  $V$  is thus the symmetry constraint, ( $x \rightarrow -x$ ,  $\phi \rightarrow -\phi$ ,  $V \rightarrow -V$ ). Different types of symmetry breaking bifurcation can be of course considered for  $V$ . In particular, interesting pattern dynamics can occur if  $V$  bifurcates at a finite wavenumber instead of zero wavenumber (see Sect. 6). In realistic problems, one needs to find the nature of the order parameter  $V$  of the «drift bifurcation». A general model is given in the next section for the transition from stationary to propagating patterns. In section 4, this problem is solved for parametrically excited standing waves undergoing a drift instability.

### 3. A drift instability of stationary patterns.

In their Galerkin approximation of a model equation, Malomed and Tribelsky found that the drift instability occurs when the second harmonic of the basic pattern is not linearly damped strongly enough. We thus consider a situation where two modes  $k$  and  $2k$  interact resonantly,

$$\mathbf{u}(x, t) = [A(x, t) e^{ikx} + \text{c.c.}] \mathbf{u}_k + [B(x, t) e^{2ikx} + \text{c.c.}] \mathbf{u}_{2k} + \dots \quad (6)$$

From symmetry argument (translational invariance in space), the evolution equations for  $A$  and  $B$  read, to third order,

$$A_t = \mu A - \bar{A}B - \alpha |A|^2 A - \beta |B|^2 A \quad (7a)$$

$$B_t = \nu B + \varepsilon A^2 - \gamma |A|^2 B - \delta |B|^2 B \quad (7b)$$

The quadratic coupling terms describe the resonant interaction between the modes  $k$  and  $2k$ . Their coefficients can be taken equal to  $\varepsilon = \pm 1$  by appropriate scaling of the amplitudes; the coefficient of  $\bar{A}B$  can be taken equal to  $-1$ , making the transformation  $\mathbf{u} \rightarrow -\mathbf{u}$ , if necessary. Positive values of  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  ensure global stability. The bifurcation diagram of equations (7) have been studied by several authors in the context of resonant wave interaction [11, 12, 13] or as a model of spatial period doubling bifurcation [14]. We thus refer to these papers for the mathematical aspects and discuss equations (7) in the restricted context of the « drift bifurcation ».

Writing

$$A = R e^{i\phi}, \quad B = S e^{i\theta}, \quad \Sigma = 2\phi - \theta,$$

we get from (7)

$$R_t = (\mu - \alpha R^2 - \beta S^2) R - RS \cos \Sigma \quad (8a)$$

$$S_t = (\nu - \gamma R^2 - \delta S^2) S + \varepsilon R^2 \cos \Sigma \quad (8b)$$

$$\Sigma_t = \left( 2S - \varepsilon \frac{R^2}{S} \right) \sin \Sigma \quad (8c)$$

$$\phi_t = S \sin \Sigma. \quad (8d)$$

In the context of our study we must take  $\nu < 0$  (the second harmonic is linearly damped) and increase the bifurcation parameter  $\mu$ . When  $\mu$  becomes positive, the null state bifurcates to an orbit of stable stationary patterns related to each other by space translation.

$$R = R_0 \neq 0, \quad S = S_0 \neq 0, \quad \Sigma = \Sigma_0 = 0, \quad \text{and } \phi \text{ arbitrary.}$$

A cellular pattern drifting with a constant velocity, corresponds to

$$R_t = 0, \quad S_t = 0, \quad \Sigma_t = 0, \quad \phi_t = \text{constant} \neq 0.$$

This implies  $2S - \varepsilon R^2/S = 0$ , and thus  $\varepsilon = 1$ . So the coefficients of the quadratic terms must have opposite signs in order to observe the drift instability. Note that this means that the second harmonic does not enhance the stationary instability near onset; indeed, for  $\mu \approx 0$  and  $\nu < 0$ ,  $B$  follows adiabatically  $A$  ( $B \propto A^2$ ), and the quadratic term of equation (7a) contributes to saturate the primary instability. The stationary pattern is destabilized when  $2S_0 - R_0^2/S_0$  vanishes as  $\mu$  is increased. This happens if the condition  $1 + \nu(2\gamma + \delta) > 0$  is satisfied, which corresponds to the condition that the second harmonic is not strongly damped

( $|\nu|$  not too large). The system of equations (8a), (8b), (8c) then undergoes a supercritical pitchfork bifurcation. The two bifurcated stationary states are such that  $R_T^2 = 2 S_T^2$ ,  $\Sigma = \pm \Sigma_T \neq 0$ . Above the instability onset,  $\phi$  increases linearly in time according to equation (8d). As noted earlier, this state represents traveling waves.

The bifurcation from the stationary pattern to the traveling one has all the characteristics of the « drift bifurcation » described in section 2. Its order parameter,  $\Sigma = 2\phi - \theta$ , undergoes a pitchfork bifurcation that breaks the basic pattern reflection symmetry. The coupling with the basic pattern spatial phase  $\phi$  induces the drift motion according to equation (8d), and the direction of propagation is determined by the sign of  $\Sigma$ . Thus the mechanism described by Malomed and Tribelsky [7] for their model equations appears to be a general one. Although the drift bifurcations observed in the experiments of reference [1, 2] near codimension-two bifurcations are not described by this mechanism, we expect that our  $k - 2k$  interaction mechanism is relevant for the experiments of references [3, 4, 17].

#### 4. The drift instability of a parametrically excited standing wave.

We have observed recently a drift instability of a standing surface wave, generated by parametric excitation in a horizontal layer of fluid contained in a thin annulus, submitted to vertical vibrations [6]. It was observed that, as the driving amplitude is increased, the standing wave pattern either begins to move at a constant speed in one direction, or undergoes an oscillatory instability that corresponds to a wavenumber modulation in space and time. We first describe the drift bifurcation and discuss the oscillatory instability in section 6.

Close to the onset of instability, we write the surface deformation in the form

$$\xi(x, t) = A \exp i(\omega t - kx) + B \exp i(\omega t + kx) + c.c. + h.o.t., \quad (9)$$

where  $A$  and  $B$  are the slowly varying amplitudes of the right and left waves at frequency  $\omega = \omega_e/2$ , where  $\omega_e$  is the external driving frequency. The equations for  $A$  and  $B$  are at leading order [6],

$$A_t + cA_x = (-\lambda + i\nu)A + \mu \bar{B} + \alpha A_{xx} + (\beta |A|^2 + \gamma |B|^2)A \quad (10a)$$

$$B_t - cB_x = (-\lambda + i\nu)B + \mu \bar{A} + \alpha B_{xx} + (\beta |B|^2 + \gamma |A|^2)B, \quad (10b)$$

where  $\lambda$  is the dissipation ( $\lambda > 0$ ),  $\nu$  corresponds to the detuning between the surface wave frequency  $\omega_0$  and  $\omega_e/2$ ,  $\mu$  is proportional to the external forcing amplitude. The imaginary parts of  $\beta$  and  $\gamma$  describe the nonlinear frequency variation of the wave as a function of the amplitude, whereas the real parts correspond to nonlinear dissipation. The real part of  $c$ ,  $c_r$ , is the group velocity. Note that the imaginary part  $c_i$  is non-zero here, contrary to the case of waves generated by a Hopf bifurcation in an infinite system, and describes as  $\alpha_r$  does, wavenumber dependent dissipation.  $\alpha_i$  corresponds to dispersion.

When  $\mu \geq 0$ , a standing wave regime is observed. To analyze its stability we write,

$$A = \exp(S + R) \exp i(\Theta + \Phi) \quad \text{and} \quad B = \exp(S - R) \exp i(\Theta - \Phi),$$

and get from equations (10) for spatially homogeneous waves

$$S_t = -\lambda + \cosh 2R [\mu \cos 2\Theta + (\beta_r + \gamma_r) \exp 2S] \quad (11a)$$

$$\Theta_t = \nu + \cosh 2R [-\mu \sin 2\Theta + (\beta_i + \gamma_i) \exp 2S] \quad (11b)$$

$$R_t = \sinh 2R [-\mu \cos 2\Theta + (\beta_r - \gamma_r) \exp 2S] \quad (11c)$$

$$\Phi_t = \sinh 2R [\mu \sin 2\Theta + (\beta_i - \gamma_i) \exp 2S] \quad (11d)$$

$\Theta$  and  $\Phi$  are respectively the temporal and spatial phases of the pattern. Note that the equation for  $\Phi$  decouples, because of the translation invariance of the system in space. The standing-wave solutions correspond to  $(S_0, \Theta_0, R = 0)$ . Their stability with respect to spatially homogeneous perturbations is simple to investigate from equations (11). We assume  $\beta_r + \gamma_r < 0$  and the detuning small enough ( $|\nu| < \lambda |\beta_r/\beta_i|$ ). Then, perturbations in  $S$  and  $\Theta$  are damped. Perturbations in  $R$  and  $\Phi$  obey the equations,

$$\Phi_t = 2[\nu + 2\beta_i \exp 2S_0] R + h.o.t. \quad (12a)$$

$$R_t = 2[-\lambda + 2\beta_r \exp 2S_0] R + h.o.t. \quad (12b)$$

When the standing wave pattern amplitude  $\exp S_0$  is small,  $R$  is damped and the standing wave pattern is stable. As the driving amplitude is increased,  $S_0$  increases and  $R$  becomes unstable for  $\exp 2S_0 = \lambda/2\beta_r$ , provided that  $\beta_r > 0$ . A non zero value of  $R$  breaks the  $x \rightarrow -x$  symmetry (see Eq (9) and the expressions of  $A$  and  $B$  versus  $S, R, \Theta, \Phi$ ). Thus  $R$  is the order parameter of the « drift bifurcation » for this standing wave problem. The coupling with the spatial phase  $\Phi$  generates the drift (Eq (12a)). The structure of the « drift bifurcation » is again similar to that described in section 2 (Eqs. (5)). However, higher order terms in equations (12) show that the « drift bifurcation » is subcritical in this case. One can easily check this by noting that the drifting solution of equations (11),  $S_t = R_t = \Theta_t = 0$ ,  $\Phi_t \neq 0$ , exists for  $\exp 2S_0 < \lambda/2\beta_r$ , i.e. only before the onset of the drift bifurcation. But additional terms of the form  $|A|^4 A$ ,  $|B|^4 B$ , can stabilize the drifting solution, and even make the drift bifurcation supercritical.

### 5. Instability of the homogeneous drifting pattern.

The experiments of references [3, 4] usually do not display stable homogeneous drifting patterns, but often localized drifting regions. This was described by assuming that the parity breaking bifurcation is subcritical [8]. We show here that even if the bifurcation is supercritical, the homogeneous drifting solution is generally unstable in the long wavelength limit. To wit, we generalize equations (5) to take into account space-dependent perturbations:

$$\phi_t = V \quad (13a)$$

$$V_t = \lambda V - V^3 + a\phi_{xx} + bV_{xx} - c\phi_{xxx} - dV_{xxx} + fV\phi_x + gVV_x + h\phi_x\phi_{xx} + h.o.t. \quad (13b)$$

Equations (13) are a gradient expansion, the form of which is given by symmetry arguments, translational invariance in space ( $\phi \rightarrow \phi + \text{constant}$ ), and space reflection symmetry ( $x \rightarrow -x$ ,  $\phi \rightarrow -\phi$ ,  $V \rightarrow -V$ ). Higher-order terms in equation (13a) can always be removed via a nonlinear transformation [15]. (The coefficient  $-k$  of Eq (5a) has been scaled in  $V$ ). If the coefficients,  $a, b, c, d$ , are positive, the  $V = 0$  solution first bifurcates when  $\lambda$  vanishes and becomes positive. The homogeneous drifting pattern,  $V_0 = \pm\sqrt{\lambda}$ ,  $\phi_0 = V_0 t$ , bifurcates supercritically, its stability to inhomogeneous disturbances of the form  $\exp(\eta t + iqx)$ , is governed by the dispersion relation,

$$\eta^2 + (2\lambda - iqgV_0 + bq^2)\eta - iqfV_0 + aq^2 = 0(q^2), \quad (14)$$

that shows that the term  $fV\phi_x$  can destabilize the homogeneous pattern independently of the sign of  $f$ , thus leading to spatially inhomogeneous patterns without assuming a subcritical bifurcation for  $V$ .

The observation of such a pattern in the vicinity of a drift bifurcation was recently reported [19] in the experiment described in reference [3], where an oil film is drained between two cylinders. When one cylinder is at rest, say the outer, a stationary cellular pattern of the oil meniscus is created above a critical rotation velocity of the inner cylinder. A slight counter-rotating motion of the outer cylinder then generates a drift of this pattern, that consists of many drifting domains moving erratically. The homogeneous drift is stabilized only for larger counter-rotation velocities [19]. Another experimental observation that can be understood with equations (13) is the stationary pattern wavenumber selection with increasing rotation velocity of the inner cylinder, when the outer cylinder is at rest. It is observed that an abrupt increase of the inner cylinder velocity leads to a pattern wavenumber modification by nucleation of a transient drifting domain. As noted in reference [8] this drifting domain generates a phase gradient, say  $q$ , that leads to a new periodic pattern of wavenumber  $k + q$ . Indeed,  $V = 0$ ,  $\phi = qx$ , is a particular solution of equations (13), for which the damping rate of perturbations in  $V$  is  $\lambda + fq$ . Consequently the drift of this new pattern is inhibited if  $fq < 0$ , for  $|q| > \lambda/f$ . The new periodic pattern thus remains stationary because of wavenumber modification (see Fig 1). Within the framework of the model of section 3, this stabilization mechanism is associated to the increase of the second harmonic damping rate when the pattern wavenumber is increased (Fig. 1).

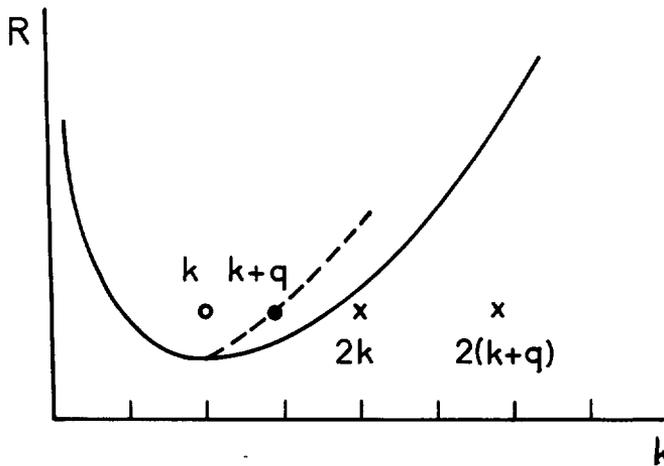


Fig. 1 — Sketch of the wavenumber selection mechanism when the control parameter  $R$  is abruptly increased above the marginal stability curve (solid line), the pattern of wavenumber  $k$  (○) becomes unstable to the drift bifurcation because its second harmonic is not sufficiently damped. However, the homogeneous drifting solution is unstable and the propagation of drifting inclusions increases the wavenumber to  $k + q$  (●), thus stabilizing a new static pattern because the damping rate of the second harmonic increases. The dashed curve represents the selected wavenumber of the static cellular pattern. (Note that in the model equations (7), the linear growth rates  $\mu$  and  $\nu$  of the modes  $k$  and  $2k$  are not independent)

**6. Oscillatory phase modulation of periodic patterns.**

As said above, an oscillatory phase modulation of periodic patterns has been recently observed as a secondary instability of convection rolls [10] or surface waves [6]. After this instability onset, the position of the rolls (or of the wavecrests respectively) is modulated in space and time by a standing wave. In the surface wave experiment [6], this oscillatory

instability was observed close to the « drift bifurcation » in the experimental parameter space. The numerical integration of equations (10) has shown that this oscillatory instability corresponds to a standing wave modulation of the basic pattern spatial and temporal phases, in agreement with the experimental observations [6] (see Fig. 2).

We show that the coupling that generates the « drift bifurcation » is also a possible mechanism to describe phase modulation of periodic patterns, if the order parameter  $V$  is destabilized at a finite wavenumber. We consider equations (13), that govern the space dependent perturbations of the basic periodic pattern, with  $\lambda \simeq -\lambda_0$ ,  $\lambda_0 > 0$ . Thus, the standing pattern is stable with respect to homogeneous perturbations. The growth rate of a perturbation of the form  $\exp(\eta t + i q x)$ , is governed by the dispersion relation,

$$\eta^2 + \eta(-\lambda + b q^2 + d q^4) + a q^2 + c q^4 = o(q^4). \quad (15)$$

Stability at short wavelength implies  $d > 0$ . For  $a > 0$  and  $b < 0$  an oscillatory instability occurs first for  $\lambda_0 = b^2/4d$ , with a finite wavenumber  $q_0 = (\lambda_0/d)^{1/4}$ , and a frequency at onset  $\omega_0 = (a q_0^2 + c q_0^4)^{1/2}$ , provided that  $a + c q_0^2 > 0$ . When  $\lambda_0$  is small, i.e. close to the « drift

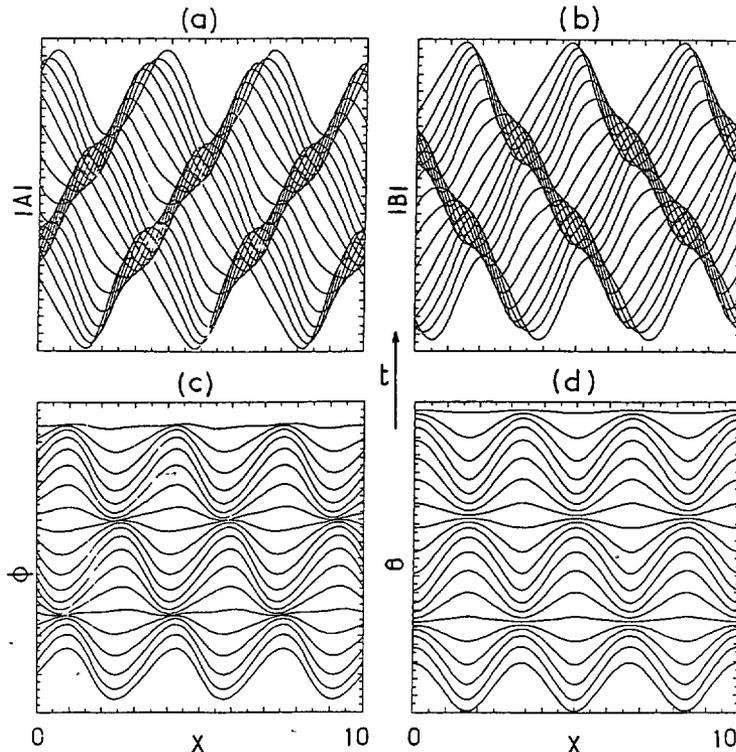


Fig. 2 — Numerical observation of the oscillatory mode. The control parameters used in the numerical simulation of the amplitude equations (10) are  $\lambda = 2.5$ ,  $\nu = 3$ ,  $\mu = 4$ ,  $\alpha_r = 1$ ,  $\alpha_i = -1$ ,  $\beta_r = 0.1$ ,  $\beta_i = -1$ ,  $\gamma_r = -0.2$ ,  $\gamma_i = -0.1$ ,  $c_r = 3.5$ ,  $c_i = 2.5$ ,  $L = 10$ . Temporal evolution of the modulii profiles  $|A(x, t)|$  and  $|B(x, t)|$  [respectively a) and b)], of the spatial phase  $\Phi(x, t) = \frac{1}{2}(\arg A - \arg B)$  [c)] and of the temporal phase  $\Theta(x, t) = \frac{1}{2}(\arg A + \arg B)$  [d)]. Twenty one successive profiles are displayed on the same figure with an arbitrary upward shift, for a time interval  $t \in [0, 1]$

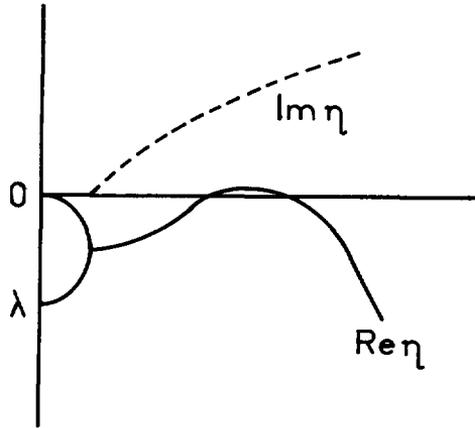


Fig. 3 — Perturbation growthrate as a function of the wavenumber of the oscillatory instability ( $a > 0, b < 0$ )

bifurcation»,  $q_0$  is small and thus we expect the gradient expansion to be justified. The corresponding growthrate is displayed on figure 3; it shows that the oscillatory instability results from the interaction of the neutral mode, because of the translation invariance in space, with the slightly damped, reflection symmetry-breaking mode associated with the « drift bifurcation » before its onset value. An instability leading to a stationary modulation of the basic pattern wavelength can occur for  $a < 0$  and  $b > 0$ . A convective pattern with rolls of irregular length has been observed experimentally, but the relevance of this mechanism in that case remains to be checked [18].

In the vicinity of the oscillatory instability onset, we write

$$\phi(x, t) = A(t) \exp i(\omega_0 t - q_0 x) + B(t) \exp i(\omega_0 t + q_0 x) + c.c. + C(t) + h.o.t. \quad (16)$$

The fields  $A(t)$  and  $B(t)$  describe the slowly varying complex amplitudes of the waves that propagate to the right and to the left, whereas the real field  $C(t)$  takes into account the existence of marginal modes for  $q \rightarrow 0$ .

Taking only into account the leading order non-linearity,  $fV\phi_x$  we obtain with standard asymptotic methods (see Appendix),

$$\dot{A} = \frac{(\lambda_0 - |\lambda|)}{2} A + (\alpha |A|^2 + \beta |B|^2) A \quad (17a)$$

$$\dot{B} = \frac{(\lambda_0 - |\lambda|)}{2} B + (\beta |A|^2 + \alpha |B|^2) B \quad (17b)$$

$$\dot{C} = -\frac{2f\omega_0 q_0}{\lambda_0} (|A|^2 - |B|^2), \quad (17c)$$

with

$$\alpha = \frac{f^2 q_0^2}{\lambda_0} \left( 1 + \frac{1}{9 - 6ic/d\omega_0} \right), \quad \beta = -\frac{f^2 q_0^2}{\lambda_0} \quad (18)$$

Thus, the leading order nonlinearity does not saturate the oscillatory instability. In the long wavelength limit, the effect of other nonlinear terms is of higher order, and the instability is

subcritical. If we discard terms which depend explicitly on  $V$ , the leading order term is  $h\phi_x\phi_{xx}$ . We obtain  $C = 0$  and similar equations for  $A$  and  $B$  with,

$$\alpha = -\frac{h^2 q_0^6}{3\lambda_0(3\omega_0 - 2ic/d)}, \quad \beta = -i\frac{dh^2 q_0^6}{3c\lambda_0}. \quad (19)$$

The bifurcation is supercritical and standing waves ( $|A| = |B|$ ) are the stable post-bifurcation state, whatever the sign of  $h$ . This corresponds to the experimental observations [6, 10]. However, we have no general argument to discard the terms  $V\phi_x$ ,  $VV_x$  and  $V^3$  in equation (13b).

We have thus shown in this paper that a variety of recent experimental observations of periodic pattern secondary instabilities, can be understood in a simple framework: the coupling of the neutral mode associated with translational invariance in space, with a reflection symmetry-breaking bifurcation. Note that a similar singularity, with two zero eigenvalues at a secondary instability onset, occurs for the oscillatory instability of convection rolls and leads to traveling waves that propagate along their axis, although the underlying physical reasons are different [15]. Let us finally mention that the secondary instabilities described here obviously fit in the general classification, proposed recently on the basis of symmetry arguments [16], but the present approach gives simple physical mechanisms that do generate these secondary instabilities.

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#### Appendix : Derivation of the amplitude equation (17) from equations (13).

Taking into account only the nonlinear term  $f\phi_t\phi_x$ , equations (13) read

$$\mathcal{L}\phi = f\phi_t\phi_x,$$

where

$$\mathcal{L}\phi = \phi_{tt} - \lambda\phi_t - a\phi_{xx} - b\phi_{xxt} + c\phi_{xxxx} + d\phi_{xxxxt}$$

The dispersion relation is

$$\mathcal{L}(\eta, iq) = 0,$$

where

$$\mathcal{L}(\eta, iq) = \eta^2 + \eta(\lambda + bq^2 + dq^4) + aq^2 + cq^4$$

for  $a > 0$ ,  $b < 0$  and  $d > 0$ , an oscillatory instability occurs when  $\lambda = -\lambda_0 = -b^2/4d$ , with a finite wavenumber  $q_0 = (\lambda_0/d)^{1/4}$  and a frequency at onset  $\omega_0 = (aq_0^2 + cq_0^4)^{1/2}$  provided that  $a + cq_0^2 > 0$ .

Close to the instability onset, following standard asymptotic methods [21], we expand

$$\begin{aligned} \lambda &= -\lambda_0 + \varepsilon^2 \lambda_2 + \dots \\ \phi &= \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \dots, \end{aligned}$$

and introduce the slow time scales  $T_1 = \varepsilon t$ ,  $T_2 = \varepsilon^2 t$ , ..

To leading order we have  $\mathcal{L}_0 \phi^{(1)} = 0$  ( $\mathcal{L}_0 = \mathcal{L}(\lambda = -\lambda_0)$ ), that implies

$$\begin{aligned} \phi^{(1)}(x, t, T_1, T_2, \dots) = & \tilde{A}(T_1, T_2, \dots) \exp i(\omega_0 t - q_0 x) + \\ & + \tilde{B}(T_1, T_2, \dots) \exp i(\omega_0 t + q_0 x) + \tilde{C}(T_1, T_2, \dots) + \text{c.c.} \end{aligned}$$

We get at the next order

$$\mathcal{L}_0 \phi^{(2)} = f \phi_t^{(1)} \phi_x^{(1)} + \frac{\partial \tilde{C}}{\partial T_1}$$

The solvability condition gives the evolution equation for  $\tilde{C}$

$$\frac{\partial \tilde{C}}{\partial T_1} = -\frac{2f\omega_0 q_0}{\lambda_0} (|\tilde{A}|^2 - |\tilde{B}|^2).$$

Then,

$$\phi^{(2)} = \frac{f\omega_0 q_0}{\mathcal{L}_0(2i\omega_0, 2iq_0)} [\tilde{A}^2 \exp 2i(\omega_0 t - q_0 x) + \tilde{B}^2 \exp 2i(\omega_0 t + q_0 x)] + \text{c.c.}$$

We get at the next order

$$\begin{aligned} \mathcal{L}_0 \phi^{(3)} = & f[\phi_t^{(1)} \phi_x^{(2)} + \phi_t^{(2)} \phi_x^{(1)}] \\ & + \left[ i\omega_0 \lambda_2 \tilde{A} - 2i\omega_0 \frac{\partial \tilde{A}}{\partial T_2} + iq_0 f \tilde{A} \frac{\partial \tilde{C}}{\partial T_1} \right] \exp i(\omega_0 t - q_0 x) \\ & + \left[ i\omega_0 \lambda_2 \tilde{B} - 2i\omega_0 \frac{\partial \tilde{B}}{\partial T_2} - iq_0 f \tilde{B} \frac{\partial \tilde{C}}{\partial T_1} \right] \exp i(\omega_0 t + q_0 x) + \text{c.c.} \end{aligned}$$

The solvability conditions gives

$$\begin{aligned} \frac{\partial \tilde{A}}{\partial T_2} &= \frac{\lambda_2}{2} \tilde{A} + (\alpha |\tilde{A}|^2 + \beta |\tilde{B}|^2) \tilde{A} \\ \frac{\partial \tilde{B}}{\partial T_2} &= \frac{\lambda_2}{2} \tilde{B} + (\beta |\tilde{A}|^2 + \alpha |\tilde{B}|^2) \tilde{B} \end{aligned} \tag{17b}$$

with  $\alpha$  and  $\beta$  given by equation (18). After rescaling to original variables,  $A = \varepsilon \tilde{A}$ ,  $B = \varepsilon \tilde{B}$  and  $C = \varepsilon \tilde{C}$ , we get the amplitude equation (17). The same method can be used when considering the other nonlinear terms of (13).

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