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Dietmar Weinmann, Jean-Louis Pichard, Yoseph Imry. Thouless Numbers for Few-Particle Systems with Disorder and Interactions. *Journal de Physique I*, 1997, 7 (12), pp.1559-1581. 10.1051/jp1:1997100 . jpa-00247472

HAL Id: jpa-00247472

<https://hal.science/jpa-00247472>

Submitted on 4 Feb 2008

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Thouless Numbers for Few-Particle Systems with Disorder and Interactions

Dietmar Weinmann ^(1,2), Jean-Louis Pichard ^(1,*) and Yoseph Imry ⁽³⁾

⁽¹⁾ CEA, Service de Physique de l'État Condensé, Centre d'Études de Saclay, 91191 Gif-sur-Yvette Cedex, France

⁽²⁾ Institut für Physik, Universität Augsburg, Memminger Strasse 6, 86135 Augsburg, Germany

⁽³⁾ Weizmann Institute of Science, Department of Condensed Matter Physics, 76100 Rehovot, Israel

(Received 9 May 1997, revised 17 July 1997, accepted 5 August 1997)

PACS.72.15.-v – Electronic conduction in metals and alloys

PACS.73.20.-r – Surface and interface electron states

Abstract. — Considering N spinless Fermions in a random potential, we study how a short range pairwise interaction delocalizes the N -body states in the basis of the one-particle Slater determinants, and the spectral rigidity of the N -body spectrum. The maximum number g_N of consecutive levels exhibiting the universal Wigner-Dyson rigidity (the Thouless number) is given as a function of the strength U of the interaction for the bulk of the spectrum. In the dilute limit, one finds two thresholds U_{c1} and U_{c2} . When $U < U_{c1}$, there is a perturbative mixing between a few Slater determinants (Rabi oscillations) and $g_N \propto |U|^P < 1$, where $P = N/2$ (even N) or $(N + 1)/2$ (odd N). When $U = U_{c1}$, the matrix element of a Slater determinant to the “first generation” of determinants directly coupled to it by the interaction is of the order of the level spacing of the latter determinants, $g_N \approx 1$ and the level spacing distribution exhibits a crossover from Poisson to Wigner, related to the crossover between weak perturbative mixing and effective golden-rule decay. Moreover, we show that the same U_{c1} signifies also the breakdown of the perturbation theory in U . For $U_{c1} < U < U_{c2}$, the states are extended over the energetically nearby Slater determinants with a non-ergodic hierarchical structure related to the sparse form of the Hamiltonian. Above a second threshold U_{c2} , the sparsity becomes irrelevant, and the states are extended more or less ergodically over g_N consecutive Slater determinants. A self-consistent argument gives $g_N \propto U^{N/(N-1)}$. We compare our predictions to a numerical study of three spinless Fermions in a disordered cubic lattice. Implications for the interaction-induced N -particle delocalization in real space are discussed. The applicability of Fermi's golden rule for decay in this dilute gas of “real” particles is compared with the one characterizing a finite-density Fermi gas. The latter is related to the recently suggested Anderson transition in Fock space.

(*) Author for correspondence (e-mail: pichard@spec.saclay.cea.fr)

1. Introduction

For non-interacting electrons the Thouless energy E_c has proven to be a very relevant energy scale for several physical properties. The related "Thouless number" $g_1 = E_c/\Delta_1$, where Δ_1 is the single-particle level spacing at the Fermi energy, plays an important role. For the disordered case, $E_c = \hbar D/L^2$. D being the diffusion constant of the electrons and L the relevant sample length. In this case, the Thouless number g_1 is equal [1] to the dimensionless conductance, *i.e.* the conductance in units of e^2/\hbar . This important relationship is at the basis of the scaling theory of localization [2], which has been quite successful in describing transport in disordered metals for non-interacting electrons.

E_c is also an important energy scale for the spectral correlations of diffusive non-interacting particles in a random potential. It was found by Altshuler and Shklovskii [3] that the usual random-matrix correlations [4] of the density of states at different energies E and E' hold only when the relative energy $|E - E'|$ is smaller than E_c . This means that one has only g_1 consecutive one-particle levels which exhibit the universal Wigner-Dyson rigidity. For $|E - E'| \gtrsim E_c$ a novel spectral correlation function was obtained which depends on the dimensionality and the diffusion constant. This new dependence and the crossover associated with it follow rather easily [5] from a semiclassical theory for the spectral correlations due to Berry [6].

When electron-electron interactions are introduced [7], a single-electron (or hole) excitation with an energy ϵ acquires a finite width $\Gamma_{sp}(\epsilon)$ ⁽¹⁾. This width obviously increases with ϵ . It has been calculated for an *isolated* metallic ($g_1 \gg 1$) compact quantum dot in reference [12], where it turns out that at the Thouless energy E_c this width becomes comparable to Δ_1 and the single-quasi-particle excitations can no longer be resolved. Thus the number of single-particle levels that can be resolved is of the order of the Thouless number g_1 . This result which agrees with the experimental findings of reference [13] is universal and does not depend on material parameters, nor on the dimensionality of the dot.

The problem of interacting particles in a random potential is of great fundamental interest. In particular, the suggestion [14] that some states of two interacting particles (TIP) in a random potential may be less strongly localized than each particle separately has recently caused much interest [15–23]. This idea can be understood within the scaling theory based on the Thouless picture [15]. According to this, the delocalization of the particular TIP states from one block of size L_1 (the one-particle localization length) to the neighboring block follows from their having an interaction dependent Thouless number $g_2 \sim E_{c2}/\Delta_2$ much larger than that of the single-particle states g_1 . Here, Δ_2 is the two-particle level separation at the given energy and the corresponding Thouless energy E_{c2} was first identified [15] to the interaction dependent decay rate between neighboring blocks. As in the single-particle case, the multiple role played by the Thouless energy, as discussed above, immediately suggested that this TIP Thouless parameter will also be relevant to the level correlation problem. This is based on the general qualitative picture. While the TIP-spectrum without interaction contains only hidden one-particle correlations appearing on energy scales larger than Δ_1 and is close to uncorrelated levels on lower energy scales, the interaction re-establishes the universal Wigner-Dyson rigidity up to the energy $E_{c2} \equiv E_U$ which depends on the strength U of the interaction. In the *localized*

⁽¹⁾ The width of the single-quasi-particle excitation exists only when the energy of the excitation is high enough, namely above a threshold ϵ^* , and the golden-rule formulation for the decay is valid. An analysis of this question either by using [8, 9] the formulation of reference [10], or *via* a correspondence with localization on a Cayley tree [11], shows that this crossover energy ϵ^* is of the order of $\sqrt{E_c \Delta_1} \sim \Delta_1 \sqrt{g_1}$. Note that since $\sqrt{E_c \Delta_1} \ll E_c$, the level width is well defined and valid on the scale E_c and for a large window below it. In reference [11] further extremely interesting results were obtained to which we shall return later.

regime ($L > L_1$), this was formally described by a nonlinear σ -model for the TIP problem, as presented in reference [19]. The latter gives a theoretical foundation for the scaling picture for TIP on equal footing to that for non-interacting particles. In the *metallic* regime ($L < L_1$), a study of the TIP-level statistics [18] confirmed that E_U gives also the characteristic energy scale up to which the TIP-spectrum exhibits the universal Wigner-Dyson rigidity. This was qualitatively explained by mapping [18] the TIP-Hamiltonian onto a Gaussian matrix model with preferential basis.

We see that the scaling properties for interacting particles can thus be studied *via* the spectral correlations of their levels *in the metallic regime*. This is an extremely useful observation. The metallic regime is easier to study both analytically, where reliable methods exist, and numerically. In the latter case, the necessity to go to very large system sizes larger than the localization length with weak disorder in low dimensions is eliminated thereby. Since the study of two interacting particles is only the first step towards the treatment of a more realistic many-body system, it is highly desirable to increase the number of particles. Even a modest program of going from two to three, four and larger numbers of interacting particles can be best accomplished by analyzing the Thouless parameters in the metallic regime for rather small system sizes. This is the strategy we adopt in this paper.

It was mentioned before (footnote 1) that when a state is coupled to a quasi-continuum, the golden rule expression for its width starts to be valid only when the coupling is strong enough, or the density of the final states is high enough. The crossover between perturbative mixing (Rabi regime) and effective decay in fact occurs when the typical matrix element of the coupling becomes larger than the mean level spacing of the accessible states [8, 9, 18]. An equivalent condition is that the golden-rule width be larger than the final level spacing. This very general crossover, which becomes a phase transition in the appropriate “thermodynamic limit” is the essence of delocalization in the usual Thouless scaling theory for a single particle. It applies to two-particle delocalization [15] and should likewise describe delocalization for N particles. The Hilbert-space transition found in reference [11] is another example. In this case one gets a proper transition by the hierarchical coupling to higher and higher numbers of quasi-particle excitations. In the work presented here, as in reference [18], this transition is observed numerically as a function of the interaction strength U . When U is weak, it can couple only a few very close quasi-degenerate states and leads at most to Rabi-type oscillations between adjacent levels. When U is larger than a certain threshold U_{c1} (or at larger excitation energy), many non-interacting states are coupled and Fermi’s golden rule describes the spreading width of a non-interacting state over the (quasi-continuum) of other non-interacting states which are nearby in energy. U_{c1} is also the crossover interaction between Poisson and Wigner-Dyson statistics for the spectral fluctuations. We show that the same U_{c1} signifies also the breakdown of the perturbation theory in U . Above a higher threshold U_{c2} , the states are ergodically mixed and g_N is suggested to increase like $U^{N/(N-1)}$

Most of this paper will be concerned with the three-particle problem. Simple analytical arguments will be presented for the behavior of the spectral correlations in a small diffusive quantum dot, and compared to a numerical study. Thus, we work in this paper in the metallic regime and do not *directly* study the delocalization in real space for stronger disorder, when the one-particle states are localized. However, some remarks will be eventually made on the implication of this picture to interaction-induced delocalization in real space and on its generalization to quasi-particle excitations in a degenerate metallic Fermi system. In particular, the basic delocalization mechanism discussed in the original locator expansion of Anderson was the divergence of perturbation theory around the initial localized states. It will be shown that a seemingly analogous divergence can be identified in the perturbation theory in the interaction, around the noninteracting states. Here, this divergence signifies (as is also true

in the Anderson localization case) the onset of Wigner-Dyson correlations in the full spectrum, where the basis of noninteracting eigenstates becomes well-mixed due to the interactions. A similar process appears in the recent work [11] of Altshuler *et al.*, using an analogous expansion for the quasiparticle excitations in a degenerate Fermi system, decaying by emitting electron-hole pairs. In the three cases of the Anderson delocalization in real space and the delocalization processes found due to interactions in the Hilbert space of wavefunctions, the basic condition for the transition is very similar. It demands that the matrix element of a state to the “first generation” of states directly coupled to it by the interaction, be comparable to the level spacing of the latter states.

2. N -Body Hamiltonian in the Fock Basis

In the presence of interactions, it is convenient to consider the N -body system in a certain Fock basis. Since we use this terminology in a slightly unusual way, let us make precise what we mean by Fock basis. We consider the one-particle states which take into account exactly the kinetic energy, the random electrostatic potential seen by the electron, the chaotic or the integrable dynamics yielded by the boundaries in a ballistic billiard *etc.*, and we use the exact one-particle states to build up the Slater determinants which we refer to as the Fock states. Therefore, by Fock basis we just mean the eigenbasis of the N -body Hilbert space in which the system Hamiltonian is diagonal at $U = 0$.

Moreover, we do not focus on the low excitation energies, (*i.e.* on the restricted space available from the Fermi vacuum by successive applications of quasi-particle creation operators) but rather to higher energies in the bulk of the N -body spectrum. Therefore, in contrast to reference [11], the parameter in our study is not the excitation energy of an extra quasi-particle above the Fermi sea, but the strength U of the interaction, at a given total energy chosen close to the band center of the N -body spectrum. Another important difference between this study and the problem considered in reference [11] is that we have in mind the *dilute limit* ⁽²⁾, where the number of “real” particles is arbitrary, but nevertheless of zero density. Therefore, we have not in this study a Fermi vacuum from which an arbitrary large number of quasi-particles can be indefinitely created.

In this Fock basis, the Hamiltonian with interaction is a random matrix with preferential basis. For the sake of simplicity, we assume that the one-particle states are more or less uniformly extended inside the sample (no one-particle localization); *i.e.* the Hamiltonian without interaction \mathcal{H}_0 is a sum of one-particle Hamiltonians which can be described by a random matrix being statistically invariant under the orthogonal transformation $O(M)$. $M = L^d$ is the number of considered sites for a sample size L . \mathcal{H}_0 , which contains the kinetic and potential energy of the particles, is combined with a two-body interaction of the form

$$\mathcal{H}_{\text{int}} = \frac{1}{2} \sum_{ijkl} \langle ij | \hat{U} | kl \rangle c_i^\dagger c_j^\dagger c_l c_k, \quad (1)$$

where $|i\rangle$ denotes the Wannier function localized at the i th site, c_i/c_i^\dagger destroys/creates a particle on site i , and

$$\hat{U} = \sum_{i < j} |ij\rangle u_{ij} \langle ij| \quad (2)$$

⁽²⁾ This dilute limit strictly means that $N/M \rightarrow 0$ when $M \rightarrow \infty$, where $M = L^d$ is the number of sites in a tight-binding model. This should be distinguished from the finite density limit where N/M is a constant and the Fermi system is degenerate at zero temperature

is a local interaction of strength measured by a parameter U . We will give estimates assuming on-site interaction in the following, but the interaction should be obviously extended to nearest neighbors in the case of spinless Fermions. We write the Hamiltonian in the basis of the $n_{\text{tot}} = M!/(N!(M-N)!)$ Slater determinants (Fock states), which are antisymmetrized products $|A_n\rangle = |\alpha_1, \dots, \alpha_N\rangle$ ($n = 1, \dots, n_{\text{tot}}$) of one-particle eigenfunctions $|\alpha_j\rangle$. \mathcal{H}_0 is a diagonal matrix with the different possible sums $\sum_{I=1}^N \epsilon_{\alpha_I}$ of one-particle energies as entries. The interaction term \mathcal{H}_{int} yields a matrix with entries

$$\langle \alpha'_1 \dots \alpha'_N | \mathcal{H}_{\text{int}} | \alpha_1 \dots \alpha_N \rangle = \sum_{IJ} \left(\prod_{i \neq IJ} \delta_{\alpha'_i \alpha_i} \right) Q_{\alpha'_I \alpha'_J \alpha_I \alpha_J} \quad (3)$$

where

$$Q_{\alpha'_I \alpha'_J \alpha_I \alpha_J} \equiv \sum_{p, p'=1}^M \psi_{\alpha'_I}^*(p') \psi_{\alpha'_J}^*(p') u_{pp'} \psi_{\alpha_I}(p) \psi_{\alpha_J}(p), \quad (4)$$

$\psi_{\alpha_I}(p)$ denoting the amplitude of the wave-function in the one-particle eigenstate $|\alpha_I\rangle$ at the site p .

The existence of the interaction yields two effects that we consider separately. The diagonal matrix elements of \mathcal{H}_{int} shift the location of the N -body levels, an effect which is predominant for small system size and large strength of the interaction, and which can yield an important re-arrangement of the spectrum (see Section 8.4). This situation is shortly described in the following subsection, and has been extensively discussed by Kamimura [24], in the case of Anderson insulators with a very small localization length. The off-diagonal matrix elements of \mathcal{H}_{int} give rise to hopping among certain Fock states, and thus to delocalization in the Fock basis. In this study, we mainly focus on the description of this interaction induced delocalization in the Fock basis, in the limit where the second effect dominates the first. This delocalization in the Fock basis is a generic effect of the interaction which should not be confused with delocalization in real space. It is only when the one-particle states are themselves localized in real space that Hilbert space delocalization may result in delocalization in real space. A recent analysis ⁽³⁾ [25] of the sensitivity of the energy levels to a change of boundary conditions has stressed this difference.

2.1. DIAGONAL MATRIX ELEMENTS OF THE INTERACTION AND LARGE U -LIMIT. — For very large U and small system size L , the previously defined Fock basis is no longer appropriate. It is more instructive to consider the Fock basis built of the on-site orbitals, and not of the one-particle eigenstates. The kinetic energy, and not the interaction, can then be treated perturbatively. In this basis the N -particle states without kinetic terms can be classified according to the number of next-neighbor configurations, for a next-neighbor interaction. This limit will be discussed in more detail in Section 8.4, where numerical results show that at $U \approx 15$, the spectrum of three spinless Fermions is split into three separated bands, with a density of states approximately given by the sum of three Gaussians centered at $E = 0, U,$

⁽³⁾In reference [25], it is also shown that the relation between the spectral rigidity and the level curvature is not direct for the N body problem. For $N = 1$, the original definition of the Thouless number, in terms of the curvature of the energy levels, coincides with our definition, based on the spectral rigidity. For $N = 2$, these two definitions are not the same in the metallic regime, but the (spectral) Thouless number can be expressed in terms of another (topological) curvature, assuming that distinct Aharonov-bohm fluxes can be associated to the distinct particles. For $N \geq 3$, the study of those different curvatures is postponed to a further study.

and $2U$. The weights of those three bands are directly related to the number of next-neighbor configurations in the on-site Fock basis states.

However, for $U \approx 1$ we are far from seeing interaction induced gaps in the spectrum. We then assume that the diagonal matrix elements of the Hamiltonian in the Fock basis built of the one-particle eigenstates, are mainly dominated by the one-particle contributions, and that the effects coming from the interaction can be neglected for those elements. Therefore, we consider only the delocalization in this basis, which results from the *off-diagonal* terms.

2.2. OFF-DIAGONAL MATRIX ELEMENTS OF THE INTERACTION. — For $N \geq 3$, one can see from equation (3) that there are only non-zero matrix elements between Fock states having $N-2$ one-particle states in common. This means that the Hamiltonian (1) is a *sparse* matrix in this Fock basis, with a strongly preferential basis. This property was not present in the former studies [14, 15, 18] for $N = 2$, and a straightforward generalization of the former results to N particles would require N -body interactions. The two-body form of the interaction introduces specific problems for $N \geq 3$, which has been recently discussed in references [26, 27]. Moreover, there is a large degeneracy of these non-zero terms. For instance, when $N = 3$, all the elements $\langle \alpha_1 \alpha_2 \beta | \mathcal{H}_{\text{int}} | \alpha'_1 \alpha'_2 \beta' \rangle = Q_{\alpha_1 \alpha_2 \alpha'_1 \alpha'_2} \delta_{\beta \beta'}$ are the same for all of the one-particle states $\beta = \beta'$.

The form of the distribution of the degenerate non-zero off-diagonal terms (Eq. (4)) is by itself a non-trivial one-particle problem. If the underlying classical one-particle dynamics is diffusive, as in a disordered metal, ballistic chaotic, as in a billiard, or integrable, one gets different estimates [11, 28] for the magnitude of the interaction matrix elements. For simplicity we will use the very rough approximation of uncorrelated one-particle wave-functions, with amplitude of the order $1/\sqrt{M}$ on each site with a random sign. This corresponds to a one-particle Hamiltonian being statistically invariant under orthogonal transformations ($O(M)$ invariance assumed in standard Random Matrix Theory). As pointed out in references [11, 28], this evaluation of the interaction matrix elements only reproduces ⁽⁴⁾ the zero wave mode contribution of a diffusion process. This gives $Q_{\alpha'_1 \alpha'_2 \alpha_1 \alpha_2} \approx \pm Q_{\text{typ}} \approx \pm U/M^{3/2}$ for on-site interactions. We use this approximation for the simplified theoretical picture that we present before the numerical study. This is because we want to compare our predictions to simulations on disordered systems with too small sizes to have one-particle diffusion. The more detailed description of the off-diagonal matrix element will modify the quantitative dependence as a function of the system parameters, but will not change the general scheme of the effect of the interaction on the spectral correlations.

Moreover, we will neglect the energy dependence of the N -particle density ρ_N , taking $\rho_N^{-1} = \Delta_N \approx B_1/M^N$ Δ_N denotes the typical N -particle level spacing in the bulk of the N -particle spectrum and B_1 is the bandwidth (one-particle kinetic energy scale).

3. Spread Width over States Directly Coupled by the Interaction

For $N = 2$, the full Hamiltonian can be modeled [18] by a Gaussian ensemble of random matrices with a preferential basis [31]. The structure of the projections $C_{\beta n} \equiv \langle \Psi_\beta | A_n \rangle$ of the many-body eigenfunctions $|\Psi_\beta\rangle$ (labeled by $\beta = 1, \dots, n_{\text{tot}}$) onto the Fock states $|A_n\rangle$ is well

⁽⁴⁾ We note two important modifications to our rough approximation for the matrix elements. For energy transfer smaller than the Thouless energy E_c , its magnitude is enhanced to the order of $\pm \Delta_1/g_1$ when $g_1 \geq 1$. In one dimension ($g_1 \leq 1$), it was pointed out by Ponomarev and Silvestrov [29] that there are important modifications of the matrix elements. More precisely, as recently shown in reference [30], when the one particle states are localized, the fluctuations of the interaction matrix elements are so large that the effective density of directly coupled Fock states becomes multifractal.

described by the Breit-Wigner form [18, 21]

$$\langle |C_{\beta n}|^2 \rangle = \Delta_2 \frac{\Gamma_2}{2\pi[(E_\beta - E_n)^2 + \Gamma_2^2/4]}, \tag{5}$$

where the brackets denote ensemble averaging, and the spread width

$$\Gamma_2 = 2\pi \langle Q^2 \rangle \rho_2 \approx 2\pi \frac{U^2}{M^3} \frac{M^2}{B_1} \tag{6}$$

increases with the interaction according to Fermi's golden rule. This means that (for $\Gamma_2 > \Delta_2$) an eigenfunction $|\Psi_\beta\rangle = \sum_n C_{\beta n} |A_n\rangle$ has significant projections on typically Γ_2/Δ_2 Fock states.

For $N \geq 3$, this can be generalized to a spreading width $\Gamma_N^{(d)} \propto \langle Q^2 \rangle \rho_N^{(d)}$ where $\rho_N^{(d)}$ is the density of the N -body Fock states *directly* coupled by the interaction. For spinless Fermions, one has $n_{\text{tot}} = M!/(N!(M-N)!)$ Fock states and the number of Fock states directly coupled by the two-body interaction is $n_{\text{eff}}^{(d)} = N(M-N) + N(N-1)(M-N)(M-N-1)/4$. In the dilute limit $N \ll M$, one finds $n_{\text{tot}} \propto M^N/N!$ while $n_{\text{eff}}^{(d)} \propto M^2$ is much smaller than n_{tot} . Assuming uniform densities, this means that the effective level spacing $\Delta_N^{(d)}$ between Fock states directly coupled by the interaction is of order $\Delta_2^{\text{eff}} \approx B_1/n_{\text{eff}}^{(d)}$. For very few particles, $\Delta_2^{\text{eff}} \approx \Delta_2$, but we emphasize that this approximate relation becomes uncorrect if N is large, mainly in the finite density case where $N \propto M$.

A quantity closely related to the local density of states is the participation ratio $R = \langle \sum_{n=1}^{n_{\text{tot}}} |C_{\beta n}|^4 \rangle^{-1}$, which gives the number of Fock states mixed by the interaction. Using the structure (5) of the eigenfunctions at $\Gamma_2 > \Delta_2$, one can get the estimate $R \sim \pi \Gamma_N^{(d)} / \Delta_N^{(d)} \approx 2\pi^2 U^2 (M^3 B_1^2)^{-1} (n_{\text{eff}}^{(d)})^2$. Therefore we expect to find $R \propto U^2$, since the contribution of the states directly coupled by the interaction will dominate for small U .

Therefore, the first observable effect of the interaction will be the broadening of a Fock state over $\Gamma_N^{(d)} / \Delta_N^{(d)}$ other Fock states separated by a characteristic scale $\Delta_N^{(d)} \approx B_1/n_{\text{eff}}^{(d)} \approx \Delta_2^{\text{eff}}$. This spreading width is proportional to U^2 , but does *not* characterize the coupling of the original Fock State to the N -body spectrum. In this spread width, there are many Fock states (of a density $\rho_3 = 1/\Delta_3$ for $N = 3$) which are not directly coupled to the original Fock state at this order in U . This is the major difference between $N = 2$ and $N \geq 3$: For $N = 2$, the width Γ_2 which characterizes the local density of interacting states in the Fock basis is directly related to the spectral statistics: the energy scale E_U up to which the spectrum exhibits the universal Wigner-Dyson rigidity is given [18] by this spread width Γ_2 , provided $\Gamma_2 > \Delta_2$. For $N \geq 3$, even when $\Gamma_N^{(d)} > \Delta_N$, the levels separated by Δ_N are not necessarily coupled and can be statistically independent. For the level repulsion at the scale Δ_N , the spreading width $\Gamma_N^{(d)}$ does not provide a relevant energy scale. The N -particle Thouless number g_N is not given by $\Gamma_N^{(d)} / \Delta_N^{(d)}$, when $N \geq 3$.

In the following, we discuss what should provide this relevant energy scale for the spectral statistics of *consecutive* N -body levels, with a density $\rho_N = 1/\Delta_N$, and thus the relevant g_N . We consider first the case $N = 3$. The generalization to an arbitrary number N of particles is straightforward, as far as we are in the dilute limit.

4. Perturbative Regime ($U \leq U_{c1}$)

The spectrum without interaction contains only one-particle correlations appearing on energy scales larger than Δ_1 and is close to uncorrelated levels on lower energy scales. The interaction

re-establishes [20] the universal Wigner-Dyson rigidity up to an energy E_U which depends on the strength of the interaction. When U is weak, it can couple only quasi-degenerate states and leads at most to Rabi-oscillations between adjacent levels.

For $N = 2$, $E_U \propto |U|$, while one finds $E_U = \Gamma_2 \propto U^2$ at larger U when $g_2 \equiv \Gamma_2/\Delta_2 > 1$, *i.e.* when many Fock states are coupled and Fermi's golden rule applies. This can be understood [18] from the following arguments. For very weak U ($\Gamma_2 < \Delta_2$) only the coupling between two Fock states with a separation $\leq \Delta_2$ is relevant. This restricts the problem to the analysis of a solvable 2×2 random matrix, with diagonal terms typically much larger than the off-diagonal coupling term. A model with independent Gaussian entries, the variance of the diagonal entries being much larger than these of the off-diagonal term, was exactly solved in reference [31]. Since a 2×2 real symmetric matrix can be diagonalized by a rotation of angle θ , it is easy to write the probability distribution in terms of the two eigenvalues and of θ . Integrating over θ gives the joint probability distribution of the two eigenvalues. One finds for this 2×2 matrix model that E_U is given by the absolute value (r.m.s) of the off-diagonal term. It was interpreted in terms of Rabi oscillations between two Fock states at typically Δ_2 away from each other in energy, and the range E_U of the level repulsion was identified ⁽⁵⁾ with this Rabi frequency. For energy separation $\epsilon < E_U$, the consecutive levels repel each other as in standard random matrix theory, while their fluctuations are uncorrelated for $\epsilon > E_U$.

For $N = 3$, we denote by $|A_n\rangle = |\alpha_1\alpha_2\alpha_3\rangle$ the three-particle Fock states, of energy $E_n = \epsilon_{\alpha_1} + \epsilon_{\alpha_2} + \epsilon_{\alpha_3}$, and we consider two energetically nearby Fock states $|A_n\rangle$ and $|A_{n'}\rangle$: *i.e.* with $E_n - E_{n'} \approx \Delta_3$, the three-particle level spacing. For a weak interaction, as pointed out by Shepelyansky and Sushkov [27], the *effective* matrix element U_3^{eff} of the interaction between those two consecutive Fock states can be estimated using perturbation theory. It is only in *second order* that one gets a non-zero contribution resulting from terms like

$$\sum_{n''} \frac{\langle A_n | U_{12} | A_{n''} \rangle \langle A_{n''} | U_{23} | A_{n'} \rangle}{E_n - E_{n''}}, \quad (7)$$

where particle 1 interacts with particle 2, then particle 2 with particle 3. Since we have a two-body interaction ($\langle A_n | U_{12} | A_{n''} \rangle = \langle \alpha_1\alpha_2 | U_{12} | \alpha_1''\alpha_2'' \rangle \delta_{\alpha_3\alpha_3''}$), the summation over n'' is reduced to a sum over the single-particle quantum number α_2'' . This sum is of the order of its largest term, *i.e.* of a term with an energy denominator of order Δ_1 , the one-particle level spacing, and not Δ_3 . This eventually gives for the effective matrix element which couples two consecutive Fock states a magnitude of order

$$U_3^{\text{eff}} \approx \pm \frac{U_{\text{typ}}^2}{\Delta_1} \approx \pm \frac{U^2}{M^3 \Delta_1} \quad (8)$$

if one takes for U_{typ} our simple estimate $Q_{\text{typ}} = \pm U/M^{3/2}$

Therefore, in this perturbative regime, Fock states at an energy $\Delta_3^{(d)}$ from each other are coupled by a matrix element $U_3^{(d)} \approx U_{\text{typ}} \approx \pm U/M^{3/2}$ while Fock states at Δ_3 from each other are only coupled by $U_3^{\text{eff}} \propto \pm U_{\text{typ}}^2/\Delta_1 \approx \pm U^2/(M^3 \Delta_1)$. From this, we draw two main conclusions for the three-particle problem that we extend to an arbitrary number N of particles.

4.1. HIERARCHICAL STRUCTURE IN THE FOCK BASIS. — The states in the Fock basis have a very particular hierarchical structure, as sketched in Figure 1. A Fock state is broadened over a density $\rho_3^{(d)} = 1/\Delta_3^{(d)} \approx 1/\Delta_2^{\text{eff}}$ of neighboring Fock states. This broadening has

⁽⁵⁾ A related effect is known [32–34] for the one-particle problem, where the Thouless sensitivity to boundary conditions is proportional to the square root of the Landauer conductance when $g_1 < 1$ and to the Landauer conductance when $g_1 > 1$.

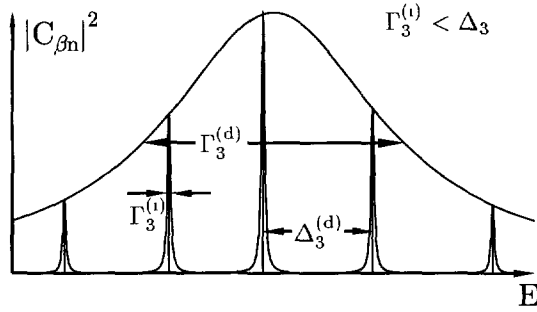


Fig. 1. — Structure of the eigenfunction at $U < U_{c1}$ in the Fock basis.

a Breit-Wigner form characterized by a width $\Gamma_3^{(d)} \approx U^2 / (M^3 \Delta_3^{(d)})$. The projections over the neighboring Fock states at $\Delta_3^{(d)}$ away from each other are themselves broadened over a density $\rho_3 = \Delta_3^{-1}$, with a Breit-Wigner shape characterized by a width $\Gamma_3^{(i)} \approx (U_3^{\text{eff}})^2 \Delta_3^{-1}$. In the perturbative regime, $\Gamma_3^{(i)} \leq \Delta_3$ when $U < U_{c1}$.

4.2. RABI FREQUENCY AND WIGNER-DYSON RIGIDITY. — When U is so small that the broadening $\Gamma_3^{(i)}$ is smaller than Δ_3 , the levels are essentially uncorrelated. However, level repulsion occurs for energy scales smaller than $E_U \approx |U_3^{\text{eff}}| < \Delta_3$. The reason for this is a straightforward generalization of the argument given for $N = 2$: The energy level correlations come from the very small coupling terms between Fock states which are nearest neighbors in energy. This reduces the complicated problem of a very large random matrix with a few non-zero small off-diagonal terms to the solvable problem of a 2×2 matrix with an *effective* off-diagonal term of magnitude U_3^{eff} . Rabi-oscillations between the two coupled diagonal terms occur. Their frequency, of order $|U_3^{\text{eff}}|$, characterizes also the scale E_U below which the universal level repulsion occurs. For $N = 2$, this gives $E_U \propto |U|$ (direct coupling), while $E_U \approx |U_3^{\text{eff}}| \propto |U^2|$ for $N = 3$.

Let us consider now the case where $N = 4$. As for the three particle case, it is sufficient to go to the second order in perturbation theory for having a non zero effective matrix element coupling two consecutive Fock states. One has:

$$U_4^{\text{eff}} \approx \sum_{n''} \frac{\langle A_n | U_{12} | A_{n''} \rangle \langle A_{n''} | U_{34} | A_{n'} \rangle}{E_n - E_{n''}}, \tag{9}$$

where the $|A_n\rangle$ are now the Fock states for 4 particles. The difference with the case where $N = 3$ is that the summation over n'' is now totally suppressed. This yields a smallest possible denominator $E_n - E_{n''}$ of order B_1 and not Δ_1 . One finds $U_4^{\text{eff}} \approx \pm U_{\text{typ}}^2 / B_1$. Similarly, one finds $U_5^{\text{eff}} \approx \pm U_{\text{typ}}^3 / (B_1 \Delta_1)$, $U_6^{\text{eff}} \approx \pm U_{\text{typ}}^3 / (B_1^2)$, . . . and the general expression is given by $U_N^{\text{eff}} \approx \pm B_1 (U_{\text{typ}} / B_1)^P$ for an even number N of particles with $P = N/2$, and $U_N^{\text{eff}} \approx \pm (U_{\text{typ}} / B_1)^P (B_1^2 / \Delta_1)$ for an odd number N with $P = (N + 1)/2$. This gives us the energy range $E_U \approx |U_N^{\text{eff}}|$ of the level repulsion for weak interaction and arbitrary N .

5. Crossover From Poisson to Wigner: $U = U_{c1}$.

When $\Gamma_3^{(i)} \approx (U_3^{\text{eff}})^2 / \Delta_3 \approx \Delta_3$, the Wigner-Dyson rigidity is established on the scale Δ_3 for the three particle case. This defines the first interaction threshold U_{c1} where a sharp cross-over

from Poisson to Wigner should be observed in the distribution $P(S)$ of the spacings between consecutive energy levels. For an arbitrary number N of particles, the first threshold U_{c1} is given by the condition

$$g_N \approx \frac{|U_N^{\text{eff}}|}{\Delta_N} \approx 1. \quad (10)$$

Using the estimate for U_N^{eff} given by the first non-zero order in the perturbative expansion in U , and assuming that N is small enough to have $\Delta_2 \approx \Delta_2^{\text{eff}}$, (*i.e.* neglecting a factor of order $1/N^2$), one finds the general relation

$$g_N \approx g_2^P, \quad (11)$$

where $P = N/2$ when N is even and $P = (N+1)/2$ when N is odd. Therefore, the interaction U_{c1} where the two-particle Thouless number g_2 , given by $(U/B_1)^2 M$ in our estimate, is of order one (*i.e.* $U_{\text{typ}} \approx \Delta_2^{\text{eff}}$), does not signify a Poisson-Wigner cross-over for the two-particle case only, but also for the N -body spectrum, as far as we stay in the dilute limit. We note that our general relation ⁽⁶⁾ implies $g_3 \approx g_2^2$, in agreement with reference [27].

U_{c1} can also be understood from the parametric motion of the energy levels when U increases. Let us consider again the case where $N = 3$. Level repulsion starts to be efficient when a certain relative characteristic energy change $\Sigma_R^{(3)}(U)$ of the levels due to the interaction is of the order of the mean level spacing. The displacement resulting from the diagonal matrix elements of the interaction induces a parallel motion of the levels and is therefore irrelevant. For the real part of the self-energy $\Sigma_R(U)$ of a given Fock state, perturbation theory gives different terms involving loops starting from the considered Fock State and visiting one, two, three and more Fock states being directly coupled among each other by the two-body interaction before returning to the starting point. The term of order U^2 (loop visiting a single Fock state) cannot be relevant since it involves an energy denominator of order Δ_2^{eff} , and not Δ_3 . The first term involving loops visiting Fock states at Δ_3 away from the unperturbed level is only given by the term of order U^4 (loop starting from the Fock state and visiting three different Fock states before return). For the real part, this gives terms like

$$\sum_{n'n''n'''} \frac{\langle A_n | U_{12} | A_{n'} \rangle \langle A_{n'} | U_{23} | A_{n''} \rangle \langle A_{n''} | U_{23} | A_{n'''} \rangle \langle A_{n'''} | U_{12} | A_n \rangle}{(E_n - E_{n'}) (E_n - E_{n''}) (E_n - E_{n'''})}. \quad (12)$$

These sums are of the order of the term having the smallest possible denominator, *e.g.* $(E_n - E_{n''}) \approx \Delta_3$, which fixes the state $|A_{n''}\rangle$ and thus suppresses the summation over n'' . Then, the two-body character of the interaction yields for the energy differences $E_n - E_{n'}$ and $E_n - E_{n'''}$, with E_n and $E_{n''}$ fixed, a smallest possible value of order Δ_1 . One can see that $\Sigma_R^{(3)}(U) \approx \Delta_3$ precisely when $g_3 = \Gamma_3^{(i)}/\Delta_3 \approx 1$, for $U = U_{c1}$. It is straightforward to check that it is also at $U = U_{c1}$ that the term of order U^4 of the perturbative expansion of the imaginary part of the self-energy is of order Δ_3 . Moreover, this argument can be easily extended to an arbitrary number N of particles, with a conclusion in agreement with these previously presented in order to obtain U_{c1} .

⁽⁶⁾ In equation (10), the Thouless numbers are defined by the energy scale below which level repulsion occurs, in units of the mean level spacing Δ_N . This definition, valid in the perturbative regime only, differs from the definition used in reference [27] ($g_N \equiv \Gamma_N^{(i)}/\Delta_N \approx (U_N^{\text{eff}}/\Delta_N)^2$). However, this subtlety (see the previous footnote) does not matter for equation (11).

6. Breakdown of the Perturbation Theory and Non-Ergodic Wigner-Dyson

Regime: $U_{c1} < U < U_{c2}$

The argument that we propose to characterize the relevant Thouless numbers g_N is reminiscent of a locator expansion [35,36] “à la Anderson”, where the self-energy of the Fock states in the presence of the hopping terms yielded by the interaction is evaluated using perturbation theory. For one-particle localization, the breakdown of this perturbative expansion in the basis built of the site orbitals was related to a metal-insulator transition due to delocalization in real space. One knows from the scaling theory too that this transition occurs for $g_1 \approx 1$. This is a closely related consideration which has led the authors of reference [11] to conjecture that interactions should give an Anderson transition in Fock space, for a critical value ϵ^* of the excitation energy of the extra quasi-particle injected above the Fermi sea. This led us to determine up to what maximum value of U the relevant Thouless numbers g_N can be given by perturbation theory. If one follows Shepelyansky and Sushkov [27], who assume that perturbation theory remains valid when $g_3 \geq 1$ for $N = 3$, the relation $g_3 \approx g_2^2$ gives $g_3 = \Gamma_3^{(i)}/\Delta_3 \propto U^4$ above U_{c1} . However, in our numerical study (see Sect. 8) we observe above U_{c1} a $|U|$ -increase of g_3 , following the perturbative U^2 -increase (Rabi regime). Moreover, all the quantities calculated for three spinless Fermions (local density of states, participation ratio, spectral rigidity) have not given any trace of a U^4 -proportional behavior when $U > U_{c1}$.

This leads us to suspect that perturbation theory can only be used up to U_{c1} , where level repulsion is established at the scale Δ_3 . Above this threshold, the relevant Γ_3 does not coincide with the perturbative estimate $\Gamma_3^{(i)}$. This breakdown of the perturbation theory above U_{c1} , when $U_{typ} > \Delta_2^{eff}$, can be shown if one evaluates the effective matrix element coupling nearby Fock states at higher orders in U . Let us present the argument for $N = 4$. The coupling term of order U^2 was found to be $U_4^{eff} \approx \pm U_{typ}^2/B_1$. This corresponds to a process where particle 1 interacts with particle 2, then particle 3 with particle 4. A term of order U^3 is given for instance if particle 3 interacts with particle 4 once more.

$$U_4^{eff}(\text{order } 3) \approx \sum_{n'',n'''} \frac{\langle A_n | U_{12} | A_{n''} \rangle \langle A_{n''} | U_{34} | A_{n'''} \rangle \langle A_{n'''} | U_{34} | A_{n'} \rangle}{(E_n - E_{n''})(E_n - E_{n'''})} . \tag{13}$$

The smallest energy denominator is the product of two energy differences. The first one is of order B_1 , but one has extra degrees of freedom for the choice of two one-particle quantum numbers of $|A_{n'''}\rangle$ since particle 3 interacts with particle 4 two times. This reduces the second energy difference to a smallest value of order Δ_2^{eff} , and gives $U_4^{eff}(\text{order } 3) \approx U_4^{eff}(U_{typ}/\Delta_2^{eff})$. This means that the terms in U^2 and in U^3 of the perturbative expansion are of the same order for $U_{typ} \approx \Delta_2^{eff}$. The generalization is straightforward: For arbitrary N , one finds

$$U_N^{eff}(\text{order } p) \approx U_N^{eff} \left(\frac{U_{typ}}{\Delta_2^{eff}} \right)^{p-P} \tag{14}$$

for the terms of order $p > P$, where P is the order giving U_N^{eff} . This indicates that the sum to all orders in U does not converge above U_{c1} . Similar considerations, used here to evaluate to all orders the effective matrix element coupling Fock states which are nearby in energy, can be given for the self-energy. From this emerges the general result that the perturbative sum in U has terms with similar magnitude to all orders when $U_{typ} \approx \Delta_2^{eff}$. One can then conclude that perturbation theory breaks down at the Poisson-Wigner cross-over in the N -body spectrum. If one compares our conclusions with the ones presented in reference [27], we also find that the threshold U_{c1} corresponds to $g_2 \approx 1$ for very few particles. However, in contrast

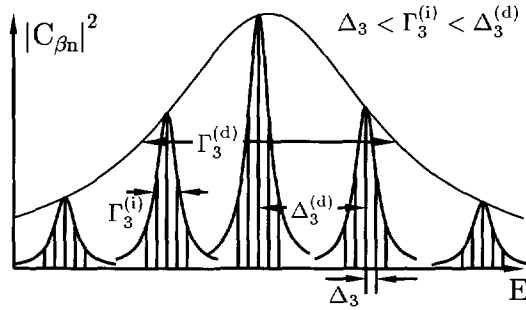


Fig. 2. — Structure of the eigenfunction at $U_{c1} < U < U_{c2}$ in the Fock basis

to the conclusions of reference [27], we have shown that the relation $g_3 \approx g_2^2$ for $N = 3$ cannot be extended for $g_2 > 1$ and holds only when $g_2 \leq 1$. For finite N in a finite size system, the spectral statistics exhibits a cross-over from Poisson to Wigner. When N increases, this cross-over should become sharper and sharper to eventually give a real transition. It is natural that such a transition is accompanied by a breakdown of perturbation theory of the self-energy of the Fock states, for $U_{typ} \approx \Delta_2^{eff}$, as it happens in the locator expansion of the self-energy for the one-particle problem at $g_1 \approx 1$. In both cases the condition is that the matrix element of a state to the “first generation” of states directly coupled to it by the interaction, becomes comparable to the level spacing of the latter states. A similar picture and delocalization condition applies also to the analogous transition in the Hilbert space of different numbers of excited quasiparticles in a degenerate Fermi system, suggested [11] recently by Altshuler *et al.*

As mentioned above, g_3 does not increase as U^4 when $g_3 \gtrsim 1$ (see Fig. 9). We are however not able to explain the observed linear increase for $U > U_{c1}$. Nevertheless, one can say that the interacting states for $U > U_{c1}$ should not be ergodic in the energy window where they are broadened, but still have a structure as sketched in Figure 2. When $\Gamma_3 \equiv g_3 \Delta_3$ is much smaller than $\Delta_3^{(d)} \approx \Delta_2^{eff}$, there are still many Fock states inside the energy width $\Gamma_3^{(d)}$ where an interacting state has essentially a zero projection. This will disappear only at a second threshold U_{c2} , characterized by the condition: $\Gamma_3(U_{c2}) = \Delta_3^{(d)}$

7. Ergodic Wigner-Dyson Regime and Self-consistent Theory: $U > U_{c2}$

We present here a conjecture for the regime of rather large interactions, when a Fock state is well coupled by the interaction to the three-body spectrum. We consider again the case

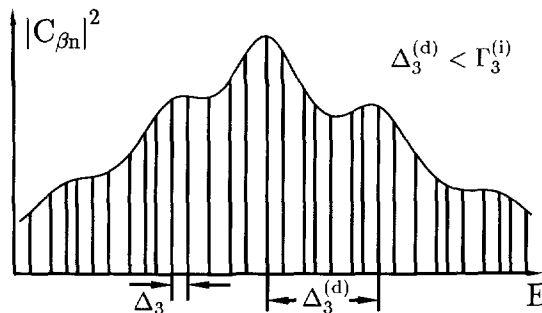


Fig. 3. — Structure of the eigenfunction for $U > U_{c2}$ in the Fock basis

where $N = 3$ and a sufficiently strong interaction for having $\Gamma_3 \geq \Delta_3^{(d)}$, but nevertheless small enough for not being in the large U -limit dominated by the diagonal terms of the interaction. For $U > U_{c2}$, one can assume that the interacting states are not unambiguously related to the previous hierarchy of Fock states, but are closer to random mixtures of g_3 consecutive Fock states, each of them contributing with a projection of random sign and of typical amplitude of order $1/\sqrt{g_3}$. In other words, one has a simpler case where all the Fock states in an energy window Γ_3 are now well coupled, and remain decoupled from the other Fock states outside this window. This is what we mean by “ergodic Wigner-Dyson regime”, where the sparsity of the 3-body Hamiltonian yielded by the pairwise character of the interaction becomes essentially irrelevant. A self-consistent evaluation of g_3 becomes possible if one assumes that an interacting state $|\Psi_\beta\rangle$ has the following structure in the Fock basis:

$$|\Psi_\beta\rangle = \sum_{\alpha_1\alpha_2\alpha_3} C_{\alpha_1\alpha_2\alpha_3}^\beta |\alpha_1\alpha_2\alpha_3\rangle. \tag{15}$$

For $N = 3$, we calculate the interaction matrix element $Q_{\beta'\beta}$ between two states $|\Psi_\beta\rangle$ and $|\Psi_{\beta'}\rangle$ at nearby energies, each of them being superpositions (with amplitude $C_{\alpha_1\alpha_2\alpha_3}^\beta \approx 1/\sqrt{g_3}$) of $g_3 = \Gamma_3/\Delta_3$ Fock states ($\Delta_3 = B_1/n_{\text{tot}}$) for $L \leq L_1$:

$$Q_{\beta'\beta} = \sum_{\substack{\alpha'_1\alpha'_2\alpha'_3 \\ \alpha_1\alpha_2\alpha_3}} C_{\alpha'_1\alpha'_2\alpha'_3}^{\beta'} C_{\alpha_1\alpha_2\alpha_3}^\beta \langle \alpha'_1\alpha'_2\alpha'_3 | \mathcal{H}_{\text{int}} | \alpha_1\alpha_2\alpha_3 \rangle \tag{16}$$

where the sums have to run over g_3 basis states. This means that each of the α -summations runs over $g_3^{1/3}$ values. As can be seen from (3), the matrix elements between the Fock states contain three terms of the form $Q_{\alpha'_I\alpha'_J\alpha'_K\alpha_I\alpha_J\alpha_K} \delta_{\alpha'_K\alpha_K}$ where $\{I, J, K\}$ are the different cyclic permutations of $\{1, 2, 3\}$. The Kronecker- δ reduces the relevant summations occurring in (16) to 5 summations, each of them running over roughly $g_3^{1/3}$ values. Thus, $Q_{\beta'\beta}$ consists of a sum of $3g_3^{5/3}$ terms of typical size Q_{typ}/g_3 with random sign, which yields the result

$$\langle |Q_{\beta'\beta}|^2 \rangle \approx 3g_3^{5/3} \frac{Q_{\text{typ}}^2}{g_3^2} = \frac{3U^2}{M^3g_3^{1/3}}. \tag{17}$$

Plugging this into a Fermi golden rule evaluation of the spread width

$$\Gamma_3 = \frac{2\pi \langle |Q_{\beta'\beta}|^2 \rangle}{\Delta_3}, \tag{18}$$

one obtains

$$g_3 = \frac{\Gamma_3}{\Delta_3} = \frac{(6\pi)^{3/4}}{M^{9/4}} \left(n_{\text{tot}} \frac{U}{B_1} \right)^{3/2} \tag{19}$$

At $U > U_{c2}$, we therefore expect the decay width of the eigenfunctions and the participation ratio R to increase like $U^{3/2}$

For N spinless Fermions, this ergodic Wigner Dyson regime is characterized by $g_N \propto U^{N/(N-1)}$, as it can be seen from a straightforward generalization of the self-consistent argument presented above for $N = 3$.

8. Numerical Study of Three Spinless Fermions

In order to illustrate our theory, we have performed a numerical study of the many-body eigenstates and eigenenergies for three spinless Fermions in a disordered cubic lattice.

8.1. NUMERICAL MODEL AND CHARACTERISTIC SCALES. — For the numerical simulations, we use a three-dimensional tight binding model on a cubic lattice containing $3 \times 3 \times 3$ sites. The disorder and the hopping terms are described by the usual Anderson Hamiltonian with on-site potentials drawn from a rectangular distribution of width $2W$ with $W = 2$ and nearest neighbor hopping terms $V \equiv 1$ which set the energy scale. We use rigid boundary conditions in all three directions. In addition, we use a two-body interaction of the form (1) with $u_{ij} = U$ when the sites i and j are nearest neighbors on the lattice and $u_{ij} = 0$ otherwise.

In such a cube, there are $M = 27$ one-particle states with a typical density of $\rho_1 = 1/\Delta_1 \approx 4$ in the center of the band. For spinless Fermions, this leads to $M_2 = M(M-1)/2 = 351$ two-particle states and $n_{\text{tot}} = M_3 = M(M-1)(M-2)/3! = 2925$ three-particle states. For $U \approx 1$ and in the center of the band, the density of the three-particle levels is about $\rho_3 = 1/\Delta_3 \approx 270$, while the density of two-particle levels amounts to $\rho_2 = 1/\Delta_2 \approx 40$. The density of three-particle Slater determinants directly coupled to a given state by the interaction is larger: with the number $n_{\text{eff}}^{(d)} = N(M-N) + N(N-1)(M-N)(M-N-1)/4 = 900$ of non-zero off-diagonal interaction matrix elements in a line of \mathcal{H}_{int} , one finds $\rho_3^{(d)} = 1/\Delta_3^{(d)} \equiv 1/\Delta_2^{\text{eff}} \approx n_{\text{eff}}^{(d)}/(n_{\text{tot}}\Delta_3) \approx 83$.

For the analysis of the numerical results, we will slightly improve our estimates of the interaction matrix elements, taking into account that the interaction is not strictly on-site, but of range 1, since a particle on a given site can interact with another one when the latter is on one of the adjacent sites. The matrix element coupling two states of the Fock basis (4) contains a double sum over the sites p and p' of the lattice. There are non-zero contributions to the sum whenever site p' is a next neighbor of site p on the lattice. In the cube we consider, there is one site which has 6 next neighbors (NN), 6 sites with 5 NN, 12 sites with 4 NN and 8 sites with 3 NN. The mean number of next neighbors is $Z = 4$ and the sums in (4) run over a total number of ZM terms. Assuming the statistical invariance of the one-particle Hamiltonian under orthogonal transformations, this yields a typical size of the off-diagonal interaction matrix elements $Q_{\text{typ}} \approx \pm U\sqrt{Z}/M^{3/2} = \pm 0.014U$.

In the same way, one finds $Q_{\alpha_1\alpha_2\alpha_1\alpha_2} \approx 3ZU/M \approx 0.45U$ for the diagonal terms of the interaction, which lead to a shift of the diagonal elements of the Hamiltonian and conserve the sign of the interaction. The factor of three is due to the combinatorial factor which counts the number of different pairs out of three particles. For not too large U we neglect them as compared to the fluctuations of the diagonal elements (which are of the order of the band-width $B_1 \approx 10$) for $U = 0$.

This must be carefully taken into account when estimating statistical properties of these matrices. In the Fermi golden rule formula for the spread width of the levels (6) one must introduce the effective level spacing $\Delta_3^{(d)}$ of the directly available levels and finds the expression

$$\Gamma_3^{(d)} = 2\pi\langle Q^2 \rangle / \Delta_2^{\text{eff}} \approx 2\pi U^2 Z M^{-3} \rho_3^{(d)} \approx 0.11U^2. \quad (20)$$

The effective interaction between basis states which are coupled by second order processes only, can therefore be estimated to be

$$U_3^{\text{eff}} \approx \pm \frac{\sqrt{3}ZU^2}{M^3\Delta_1} \approx 0.002U^2, \quad (21)$$

leading to the spread width

$$\Gamma_3^{(i)} \approx 2\pi \frac{|U_3^{\text{eff}}|^2}{\Delta_3} \approx 0.0067U^4 \quad (22)$$

in the perturbative regime where $\Gamma_3^{(i)} < \Delta_3$. From these estimates, we get the first threshold $U_{c1} \approx 0.85$ for which $U_{\text{typ}} \approx \Delta_2^{\text{eff}}$.

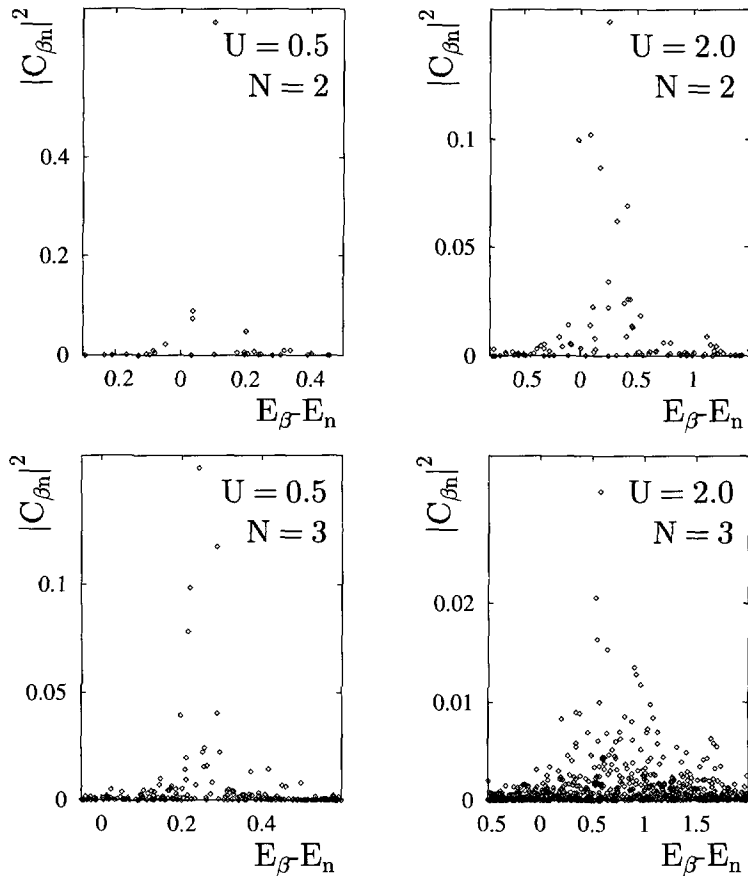


Fig. 4. — The projections $|C_{\beta n}|^2$ of a typical eigenstate $|\Psi_{\beta}\rangle$ ($E_{\beta} \approx 0$) at interaction strength $U = 0.5$ (left) and $U = 2.0$ (right) on the basis of Fock states $|A_n\rangle$. The data are for $N = 2$ (upper) and $N = 3$ (lower) spinless Fermions in a cube of $3 \times 3 \times 3$ sites with interaction U between next neighbors and on-site disorder $W = 2$.

8.2. STRUCTURE OF THE WAVE FUNCTIONS. — We first concentrate on the structure of the eigenstates in the Fock basis. Examples are shown in Figure 4. In each of the pictures, only one eigenstate $|\Psi_{\beta}\rangle$ at an energy $E_{\beta} \approx 0$ is shown. Each point represents the overlap $|C_{\beta n}|^2 = |\langle \Psi_{\beta} | A_n \rangle|^2$ with a Fock state $|A_n\rangle$ and is plotted as a function of the energy difference $E_{\beta} - E_n$ between the eigenstate and the Fock state.

It can be seen that in the case of two particles, almost all of the Fock states which are in a certain energy range around the energy of the unperturbed eigenstate have a non-negligible overlap with it. For three particles, however, many very small values of the projections onto Fock states occur, even when they are quite close in energy. However, it is difficult to observe the hierarchical structure of the three-particle states because the scales Δ_3 and Δ_2^{eff} differ by a factor of 3 only in our case and because of the statistical fluctuations.

From these overlap matrix elements, taking into account several different realizations, we have computed the local density of interacting states in the Fock basis (Wigner strength function). In spite of the fact that there are many very small values in the individual overlap matrix elements, the average is well described by the Breit-Wigner form (5). Its spread width

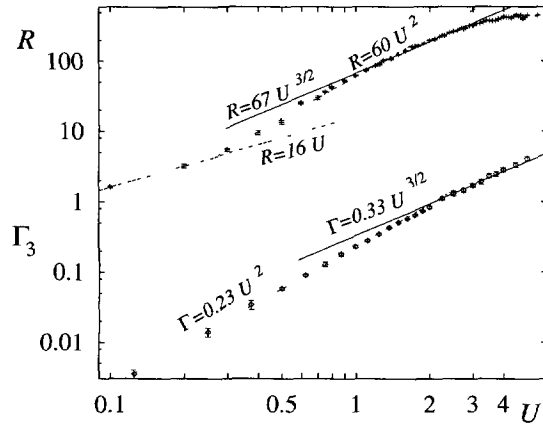


Fig. 5. — Upper curve: participation ratio R for 3 spinless Fermions in a $3 \times 3 \times 3$ disordered lattice ($V = 1, W = 2$) with rigid boundary conditions and nearest neighbor interaction U . The dashed, dotted and solid line represent power law fits in the different regimes yielding $R = 16U$, $60U^2$, and $67U^{3/2}$, respectively. Lower curve: spread width Γ_3 characterizing the local density of states of 3 spinless interacting Fermions in the Fock basis. The dotted and solid line represent power law fits in the different regimes yielding $\Gamma_3 = 0.23U^2$, and $0.33U^{3/2}$, respectively.

Γ_3 is shown in Figure 5 (lower data points) as a function of U . Therefore by Γ_3 here, we mean the total spread width extracted from the average local density of interacting states in the Fock basis, and not the partial spread widths $\Gamma_3^{(d)}$ and $\Gamma_3^{(i)}$ introduced previously. Γ_3 behaves quadratically down to rather low interaction values while $\Gamma_3 \propto U^{3/2}$ above $U \approx 1.5$.

For weak interaction, we are in the regime where the hierarchical structure of the eigenstates should be important. The spread width is then dominated by the spread width $\Gamma_3^{(d)}$. Our estimates presented above give $\Gamma_3^{(d)} \approx 0.11U^2$ which is the correct order of magnitude.

From our theoretical considerations, we expect to obtain a regime in which the wave-functions are ergodic and the sparseness of the Hamiltonian irrelevant when $U > U_{c2} \approx 1.8$. Taking into account the refined estimates of this section, the spread width is expected to be given by the self-consistent expression

$$\Gamma_3 = \Delta_3 \left(\frac{6\pi ZU^2}{M^3 \Delta_3^2} \right)^{3/4} \approx 0.25U^{3/2} \quad (23)$$

In the numerical data, one observes indeed a transition at $U \sim 2$ to a regime in which $\Gamma_3 \propto U^{3/2}$ with a prefactor whose order of magnitude coincides again with the expected value.

The upper points in Figure 5 show the participation ratio $R = \langle \sum_{n=1}^{n_{\text{tot}}} |C_{\beta n}|^4 \rangle^{-1}$. The behavior of R is quite similar to the one of the spread width Γ_3 . The participation ratio, which goes to $R = 1$ at $U = 0$, increases proportionally to the square of the interaction in the regime $0.25 < U < 1.5$ and, as Γ_3 , exhibits the signature of the ergodic regime for $U > 1.5$. When calculating the participation ratio R , one has also to take into account, that not all of the states which are in the available energy interval can participate. Again, one has to replace the level spacing Δ_3 by Δ_2^{eff} to obtain $R \approx \pi \Gamma_3 / \Delta_2^{\text{eff}} \approx 2\pi^2 U^2 Z (M^3 (\Delta_2^{\text{eff}})^2)^{-1} \approx 28U^2$. In the ergodic regime, the self-consistent estimate presented above gives $R \approx 212U^{3/2}$. However, at low U , a difference arises since $R = 1$ and $\Gamma_3 = 0$ at $U = 0$.

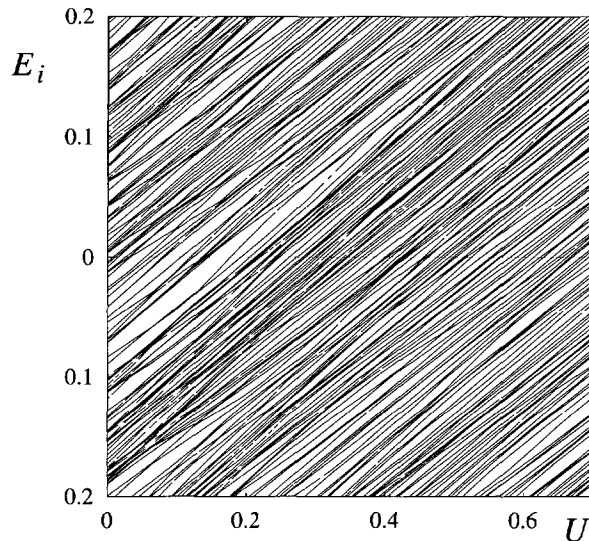


Fig. 6. — Parametric dependence of the three-particle spectrum as a function of U , around the band center. The change in spectral correlation can be seen. For $U < 0.2$, many levels seem to cross (weak level repulsion), while for larger U , the crossings are avoided (strong level repulsion).

The ratio $R/\Gamma_3 \approx 260$ in the quadratic regime is much smaller than the one expected from a democratic participation of the Fock states according to (5), which yielded $R/\Gamma_3 = \pi/\Delta_3 \approx 850$. This is a consequence of the fact that individual states can show strong fluctuations around (5), thereby lowering the participation ratio. Furthermore, the sparse structure of the Hamiltonian, and the resulting hierarchical structure of the eigenfunctions reduces the number of participating basis states as has been seen in Figure 4.

8.3. SPECTRAL STATISTICS. — The evolution of some energy levels in the center of the band as a function of U is shown in Figure 6. First of all, the positive slope of all of the levels is visible. This is due to the diagonal matrix elements of the interaction which lead to a shift in energy of the order of $0.45U$ as expected from the typical size of these elements.

But we can also observe changes in the statistical behavior of the spectrum. At low U , there are strong fluctuations in the level spacing while the spectrum becomes more rigid around $U \approx 0.4$. This happens when neighboring states in energy (which usually are not directly coupled by the interaction) become correlated.

This Poisson-Wigner transition can be studied more systematically by calculating the *local* fluctuation properties of the spectrum. The first quantity we look at is the level spacing distribution $P(s)$ in the center of the many-body spectrum around $E = 0$. $P(s)$ for three particles is shown for a few values of U in Figure 7 (left). Δ is the corresponding mean level spacing (Δ_2 or Δ_3). At low U , $P(s)$ is quite close to the Poisson limit of uncorrelated levels, while it tends towards the universal Wigner result (GOE) at stronger U . This transition is described in a quantitative way by the integral

$$I := \int_{1/2}^2 ds P(s), \quad (24)$$

which is shown in the right hand side picture of Figure 7. The transition is much more abrupt for $N = 3$ than for $N = 2$. This agrees with the expected U -dependence of the Rabi frequency

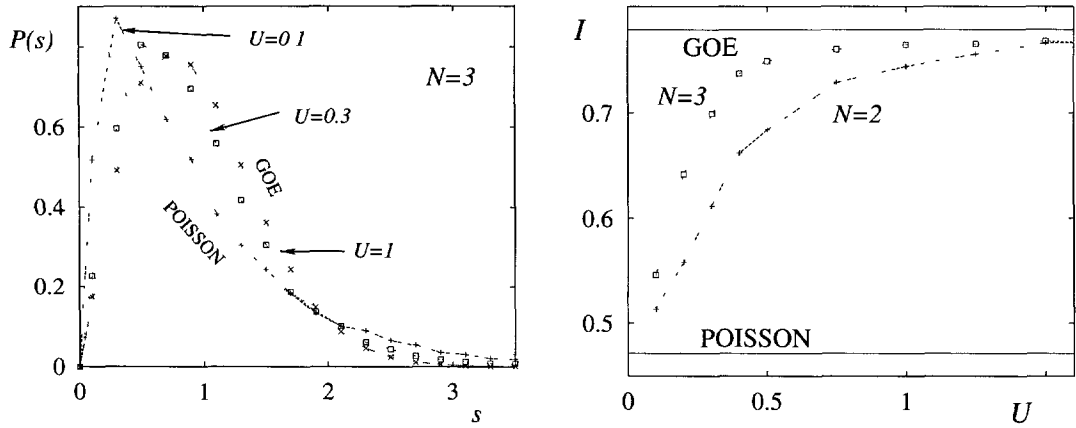


Fig. 7. — Left: level spacing distribution $P(s)$ for three spinless Fermions at $U = 0.1$, $U = 0.3$ and at $U = 1$. For comparison, the Poisson and GOE distributions are plotted also. Right: the integrated level spacing distribution I as a function of the interaction for two and three particles.

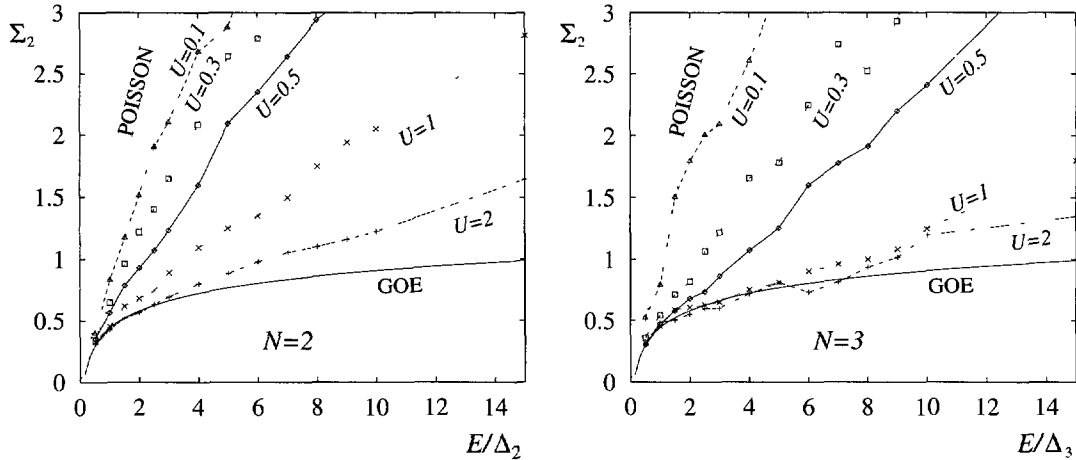


Fig. 8. — $\Sigma_2(E)$ for the spectrum of two (left) and three (right) spinless Fermions in a $3 \times 3 \times 3$ disordered lattice ($V = 1, W = 2$) for different values of U .

($|U|$ for $N = 2$ and $|U|^2$ for $N = 3$) which characterizes the range of the level repulsion for weak U .

The energy scale E_U characterizing the universal spectral Wigner-Dyson rigidity can be extracted from the variance $\Sigma_2(E) = \langle N(E)^2 \rangle - \langle N(E) \rangle^2$ of the number of energy levels in an interval of width E . Comparing the behavior of the spectrum to the GOE-behavior, one can identify this energy E_U [18] (up to which the GOE-rigidity can be observed and above which one can see significant deviations). $\Sigma_2(E)$ is shown in Figure 8, near the band center. One can observe, for two as well as for three particles, the Wigner-Dyson rigidity up to the energy scale E_U where the spectrum becomes less rigid.

For $N \geq 3$ particles, one can see that the characteristic energy E_U does not coincide with the spread width Γ_3 of the eigenfunctions (as it is the case for two particles when $\Gamma_2 > \Delta_2$ [18]).

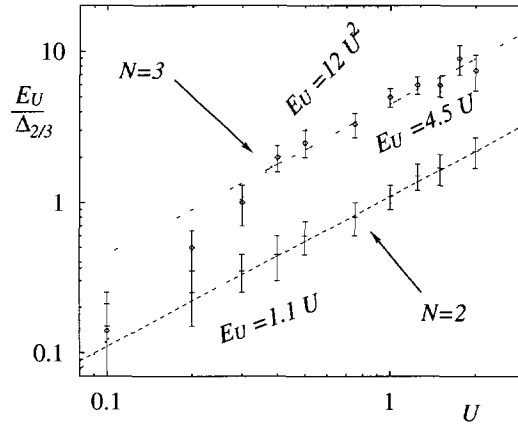


Fig. 9. — The characteristic energy E_U up to which the 2/3-particle spectrum exhibits the universal Wigner-Dyson rigidity.

For $N = 3$, the crossover from the Poissonian behavior of uncorrelated levels to the universal behavior of the GOE is sharper. As can be seen in Figure 9, E_U increases as U^2 for weak interaction. This corresponds to Rabi-oscillations due to second order coupling between nearby Fock states. This $|U^2|$ -increase when $E_U < \Delta_3$ for $N = 3$ is the analog of the $|U|$ -increase observed for $N = 2$ when $E_U < \Delta_2$. Above U_{c1} , E_U seems to linearly increase as a function of U .

8.4. LARGE U -LIMIT. — For large U and small M , it is instructive to consider the Fock basis built of the on-site orbitals. In this basis the 3-particle states, and the role of U , can be classified according to the number of next-neighbor configurations. For a $3 \times 3 \times 3$ cubic lattice, we have 1746 out of the 2925 basis states where no next neighbor pairs occur and the Fock state is not shifted when the interaction increases. For 1008 basis states, there is one pair of particles nearby and the energy of the Fock states is $E(U = 0) + U$. For the remaining 171 Fock states the particles are clustered such that their energy increases like $E(U = 0) + 2U$. Since the non-diagonal elements in this representation are only due to one-particle kinetic energy which does not depend on U , the different shifts of the Fock states lead to a splitting of the band into three parts, 1746 states around $E = 0$, 1008 states around $E = U$ and 171 states around $E = 2U$.

This can be seen for $U = 15$ in Figure 10, where the integrated density of states $\text{IDOS} = \int_{-\infty}^E dE' \rho(E')$ is plotted for one realization of the disorder and different values of the interactions. The density of states can be fitted by a sum of three Gaussians centered at $E = 0$, U , and $2U$ with weights corresponding to the above mentioned numbers of occurrence of these shifts when $U \approx 15$. For $U \approx 1$, the interaction induced gaps in the spectrum are totally removed by the one-particle kinetic contributions.

8.5. EVIDENCE FOR THE EXISTENCE OF TWO THRESHOLDS. — The numerical results for the three-particle spectrum clearly exhibit two different characteristic interaction strengths. First, the local fluctuations of the level spacing changes from Poisson to GOE when $U \approx 0.3$, as it can be seen from Figure 7. This is consistent with the results obtained for Σ_2 where $E_U \approx \Delta_3$ when $U \approx 0.3$. Thus, adjacent energy levels are correlated when $U > 0.3$. This gives $U_{c1} \approx 0.3$. Below U_{c1} , E_U increases as U^2 in agreement with Rabi-oscillations

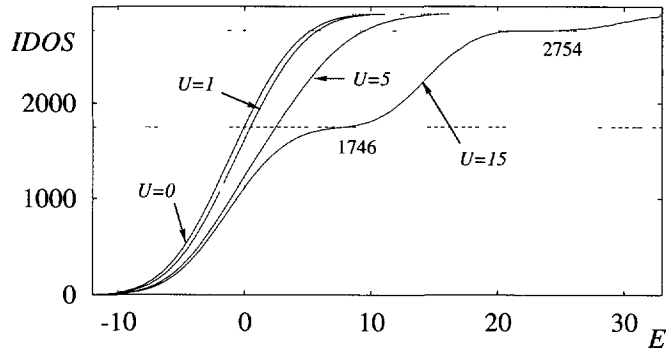


Fig. 10. — The integrated density of the three-particle states for different values of the interaction.

in the perturbative regime. Above $U = U_{c1}$, $g_3 = E_U/\Delta_3 > 1$ and the perturbation theory breaks down. An indication for this is that $E_U \propto U$ and not $E_U \propto \Gamma_3^{(i)} \propto U^4$, as implied by the perturbation theory. Though we have no explanation for this linear behavior, we emphasize that this absence of a U^4 -behavior gives a strong hint that the range of validity of perturbation theory is limited to $U_{\text{typ}} \leq \Delta_2^{\text{eff}}$.

Note that nothing striking is observed in the behavior of the total spread width Γ_3 of the states at $U = U_{c1}$. There seems to be a crossover from $R \propto U$ to $R \propto U^2$ in the participation ratio R , but since $R = 1$ at $U = 0$, it might also be the signature of a saturation which is not connected to a characteristic energy scale but which becomes irrelevant when $R \gg 1$.

However, there is something interesting happening at $U \approx 1.5$ where both, Γ_3 and R undergo a transition from a regime where they increase as U^2 to a regime where they increase as $U^{3/2}$. This is clear evidence for a change in the structure of the eigenstates as it is expected from our theoretical considerations at $U = U_{c2}$. At this strength of the interaction, we expect the indirect spread width $\Gamma_3^{(i)}$ to be of the order of Δ_2^{eff} . The problem is now to estimate Γ_3 when $U > U_{c1}$ since the perturbation theory does not work. If one uses the numerical result for $E_U \approx \Gamma_3^i$ in this non-ergodic Wigner-Dyson regime, one gets for three particles $U_{c2} - U_{c1} \approx 0.8$ since $\Delta_2^{\text{eff}} \approx 3.3\Delta_3$.

In summary, we find a clear evidence of the existence of two distinct thresholds U_{c1} and U_{c2} , as well as a strong indication that the self-energy of an individual Fock state cannot be evaluated by perturbation theory for $g_N \geq 1$.

9. Implication for Quantum Localization in Real Space

Very recently, scaling-type concepts have been applied to two particles with a local interaction. When Shepelyansky [14] had pointed out that in insulating systems certain two-particle wave-functions could be delocalized with respect to the one-particle states, Imry extended the Thouless Block-scaling picture [15] and introduced a "two-particle conductance" $g_2 = \Gamma_U/\Delta_2$, where Γ_U is the decay rate of the states in boxes of size L_1 due to the interactive coupling to other boxes. This Γ_U is identical to the Γ_2 characterizing the spectral fluctuations for a given block. Δ_2 is the mean spacing of the two-particle spectrum in a block. Using Fermi's golden rule for the estimate of Γ_U , this yields a pair localization length $L_2 \propto U^2 L_1^2$, in agreement with the results of Shepelyansky. The existence of this delocalization effect has been confirmed in numerical studies [16, 17, 22].

While in this paper we have considered the metallic phase, away from the localization transition, our treatment may form the basis for finding the localization length for an N -particle system (where $N \geq 3$). This localization length too is increased for large enough interactions. The scaling theory prescription for obtaining this localization length is straightforward, in principle. One has to increase the system size, L , and watch the L -dependence of both the *effective* spacing Δ_N^{eff} of the N -particle levels which are re-organized by the interaction when $L > L_1$ and the energy E_U above which the Wigner-Dyson rigidity does not apply for this subset of levels. The L at which $E_U/\Delta_N^{\text{eff}}$ becomes of order unity is the N -particle localization length. The parametric dependence embodied in our estimate of g_3 (g_N) can be used for this purpose. We expect the delocalization to become even stronger for $N \geq 3$. It is tempting to suggest that such effects are at the origin of the recent observations [37, 38] of a two-dimensional metallic phase driven by the interactions in Si-MOSFET.

10. Quasi-Particle Lifetime and Localization Transition in the Fock-Space

We conclude by stressing the analogies and the differences between this study and the recently proposed [11] approach to quasi-particle lifetime in an isolated system. In these two studies, two characteristic energies are identified and a transition appears when their ratio is of order unity. This is the transition from weak perturbative mixing to the golden-rule decay, as mentioned in the introduction. This is also the threshold where the perturbation theory in U or ϵ breaks down. In this sense, the two studies use very similar concepts, but the considered characteristic scales are not the same. This is because two different situations were considered (quasi-particles in a finite-density Fermi gas, *versus* a dilute gas of “real” particles). It may be argued, as in reference [15], that the difference between these two systems is mainly in the counting of the densities of excited states, and that in principle both could be treated by similar scaling considerations.

We have mainly discussed what is the ratio (Thouless number) which controls the transition from Poisson to Wigner in the bulk of the N -body spectrum. The threshold U_{c1} corresponds to $U_{\text{typ}} \approx \Delta_2^{\text{eff}}$. Therefore, for very few particles where $\Delta_2^{\text{eff}} \approx \Delta_2$, and $g_n \approx g_2 \approx 1$ at U_{c1} , this first threshold is eventually related to the ratio g_2 of the two-particle decay width Γ_2 over the two particle spacing Δ_2 . This is not due to the fact that Δ_N is not a relevant energy scale, but because this is the contribution of order $P = N/2$ or $(N+1)/2$ (depending on the parity of N) in the perturbation theory of the decay width Γ_N which matters for the spectral fluctuations at the scale Δ_N . Therefore, as previously proposed by Shepelyansky and Sushkov, we just need to have the two-particle levels well coupled by the interaction in order to have the same thing for the N -body levels. In this case the conditions for the establishment of Wigner-Dyson rigidity and to have an effective decay are similar and both are $U_{\text{typ}} \sim \Delta_2^{\text{eff}}$.

For the lifetime of quasi-particles in a zero-dimensional Fermi system, the relevant ratio is made from the two different scales: the decay width of a single quasi-particle $\Gamma_{\text{sp}}(\epsilon)$ (resulting from the disintegration of a single quasi-particle into two quasi-electrons and a hole) and the accessible three quasi-particle level spacing $\Delta_3(\epsilon)$. In this connection, a quasi-particle is considered at an excitation energy ϵ above the Fermi energy of an isolated system. The corresponding Fermi vacuum is assumed to always provide a new electron-hole pair at each interaction process, such that the relevant decay width is not the one characterizing the decay of a certain Slater determinant to those with the same quasi-particle content, but to those with a quasi-particle content increased by a new electron-hole pair. It should be emphasized that this does not correspond to the dilute limit discussed in our work, but to a limit of finite density of particles, such that the Fermi vacuum can be considered as an unlimited reservoir of particle-hole excitations. $\Gamma_{\text{sp}}(\epsilon) \approx U_{\text{typ}}^2/\Delta_3(\epsilon) \approx \Delta_3(\epsilon)$ defines the (second) excitation

threshold ϵ^* in reference [11]. The threshold is obtained when the typical magnitude of the interaction matrix element U_{typ} is of order $\Delta_3(\epsilon)$, unlike the value Δ_2^{eff} which was relevant in our dilute limit. In reference [11], it is suggested that the two problems of level statistics and golden rule decay are unrelated: that delocalization in Fock space does not mean that the spectrum should have Wigner-Dyson statistics. For a Cayley tree, this disagrees with the results of recent supersymmetric calculations [40] using a non-linear sigma model formulation. For the “dilute” case of a small number of particles in a large volume, it follows from our work that these two properties are very intimately related.

In this work, the transition ⁽⁷⁾ from an uncorrelated spectrum to a fully ergodic one seems to occur in two stages: U_{c1} and U_{c2} . We do not have a qualitative understanding of the behavior in the intermediate regime. Whether this regime will survive with increasing number of particles, and what is the precise relationship ⁽⁸⁾ with reference [11] are questions which deserve further investigation.

Acknowledgments

We are grateful to Yuval Gefen, Alex Kamenev and Dima Shepelyansky for instructive discussions. Dietmar Weinmann acknowledges the financial support of the European HCM program. Research at the Weizmann Institute was supported by the Fund for Basic Research administered by the Academy of Sciences of Israel and by the German-Israeli Foundation (GIF), Jerusalem.

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⁽⁷⁾ This study is clearly restricted to disordered systems, where the one body states are chaotic. In clean systems, the structure of the interaction in the Fock basis built out from plane waves is very different (see for instance Refs. [29,30]). However, disorder is not necessary for having chaotic N -body states and Wigner-Dyson rigidity. When the one body problem is integrable, the condition suggested by many numerical studies [39] for having chaos is the nonintegrability of the strongly correlated fermion system.

⁽⁸⁾ After completion of this manuscript, we have noticed related works [41,42] where the relevance of our first threshold U_{c1} for the onset of chaos is studied, particularly in the case of low energy quasi-particles excited above the Fermi sea.

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