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Algebraic Dynamical Approach to the $\mathfrak{su}(1,1) \oplus \mathfrak{h}(3)$ Dynamical System: A Generalized Harmonic Oscillator in an External Field

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Abstract. — The exact solution and the invariant Cartan operator of the linear nonautonomous system with $\mathfrak{su}(1,1) \oplus \mathfrak{h}(3)$ dynamical algebra are obtained by using the method of algebraic dynamics. A novel, indirect quantum-classical correspondence of the solutions has been found. It has been shown that algebraic dynamics can be generalized from the linear dynamical system with $\mathfrak{su}(1,1)$ Lie algebra to a more complicated system with $\mathfrak{su}(1,1) \oplus \mathfrak{h}(3)$ Lie algebra. Nonadiabatic Berry's phase is also calculated.

In reference [1], Wei and Norman proved the theorem that the time evolution operator of a quantum system whose Hamiltonian is a time-dependent linear combination of the generators of finite Lie algebra can be expressed as a finite product of exponentials of the generators multiplied by suitable scalar functions. Based on this theorem, a method called algebraic dynamics [2–4] has been proposed in the study of linear nonautonomous systems with semi-simple Lie algebras. The essential ingredients of the method are gauge transformation and time-dependent dynamical symmetry. The properties of algebraic dynamics are: (a) by means of gauge transformation, the problem of a nonautonomous system can be solved under different gauges which correspond to different representations. By a proper choice of gauge, a time-dependent dynamical symmetry can be converted into a stationary dynamical symmetry; (b) by virtue of this method, the number of parameters needed for solving the quantum equations of motion (in terms of these parameters, the gauge transformation can be constructed) can be reduced to the minimum (which is equal to the order of the algebra minus the number of its Cartan operators); (c) within the framework of algebraic dynamics, the quantum-classical correspondence of the solutions is readily found; (d) besides, the nontrivial time-dependent dynamical symmetries of a linear nonautonomous quantum system appear automatically in the process of solving the Schrödinger equation of the system. The method has been applied to problems of the dynamical systems possessing $\mathfrak{su}(1,1)$ [3] and $\mathfrak{su}(2)$ [4] algebraic structures and proved to be useful.

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To explore the applicability of the approach of algebraic dynamics, in this paper we shall consider a more complicated quantum system, *i.e.* the dynamical $\text{su}(1,1) \oplus \text{h}(3)$ system which is a linear nonautonomous system with $\text{su}(1,1) \oplus \text{h}(3)$ dynamical algebra. The Hamiltonian of this system can be applied not only to the problem of a generalized harmonic oscillator in an external field [5–8], but also to the study of the laser-plasma scattering [9]. Many related problems have been studied extensively in literature [10–15]. Dattoli *et al.* [16] have employed Lie algebraic methods to get the solutions of the linear partial differential equations generated essentially by $\text{su}(2)$ and $\text{su}(1,1)$ groups. Lo [17] has obtained the propagator of the general driven time-dependent harmonic oscillator by using the Lie-algebraic technique. Combescure [18] has studied the correspondences between a variety of physical systems with time-dependent quadratic Hamiltonians and the harmonic oscillator by applying the scaling properties of the time-dependent Schrödinger equation. Cho and Kim [19] have discussed the connection between the integrals of classical equations of motion and the Lewis-Riesenfeld invariant for a time-dependent harmonic oscillator. There have also been many other related works, for example, Leach *et al.* have studied the invariants and their associated symmetry group generators for the time-dependent harmonic and anharmonic oscillators for classical mechanical case using Noether's theorem and linear canonical transformation method [20–24].

In this paper, we shall give the exact solution of the linear nonautonomous quantum system with $\text{su}(1,1) \oplus \text{h}(3)$ dynamical algebra within the framework of algebraic dynamics [2–4], and address the complete set of commuting (simultaneous) invariants of the quantum system studied.

Consider a dynamical system whose Hamiltonian is a linear combination of $\text{su}(1,1) \oplus \text{h}(3)$ generators

$$\begin{aligned}\hat{H}(t) &= X_+(t)\hat{k}_+ + iX_0(t)\hat{k}_0 + X_-(t)\hat{k}_- + X_1(t)\hat{k}_1 + X_2(t)\hat{k}_2 + X(t) \\ &= \frac{1}{2}X_+(t)\hat{p}^2 + \frac{1}{4}X_0(t)(\hat{p}\hat{q} + \hat{q}\hat{p}) + \frac{1}{2}X_-(t)\hat{q}^2 + X_1(t)\hat{p} + X_2(t)\hat{q} + X(t)\end{aligned}\quad (1)$$

where $X_\nu(t)$ ($\nu = +, 0, -, 1, 2$) and $X(t)$ are non-singular and real functions of time. A direct realization of the Hamiltonian (1) is the time-dependent harmonic oscillator in an external field. The operators \hat{k}_+, \hat{k}_0 and \hat{k}_- form a $\text{su}(1,1)$ algebra

$$\hat{k}_+ = \frac{1}{2}\hat{p}^2, \quad \hat{k}_0 = -\frac{i}{4}(\hat{p}\hat{q} + \hat{q}\hat{p}), \quad \hat{k}_- = \frac{1}{2}\hat{q}^2, \quad (2a)$$

$$[\hat{k}_+, \hat{k}_-] = 2\hat{k}_0, \quad [\hat{k}_0, \hat{k}_\pm] = \pm\hat{k}_\pm; \quad (2b)$$

while $\hat{k}_1 = \hat{p}$, $\hat{k}_2 = \hat{q}$, and 1 constitute the Heisenberg $\text{h}(3)$ algebra, which satisfy the following commutation rule

$$[\hat{k}_1, \hat{k}_2] = -i. \quad (3)$$

The commutation relations between the two sets of operators are

$$\begin{aligned}[\hat{k}_+, \hat{k}_1] &= 0, \quad [\hat{k}_0, \hat{k}_1] = \frac{1}{2}\hat{k}_1, \quad [\hat{k}_-, \hat{k}_1] = i\hat{k}_2, \\ [\hat{k}_+, \hat{k}_2] &= -i\hat{k}_1, \quad [\hat{k}_0, \hat{k}_2] = -\frac{1}{2}\hat{k}_2, \quad [\hat{k}_-, \hat{k}_2] = 0.\end{aligned}\quad (4)$$

The time evolution of the system is determined by the following Schrödinger equation (assume $\hbar = 1$)

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle. \quad (5)$$

To solve equation (5), a gauge transformation $\hat{U}_g(t)$ may be introduced [2, 3]

$$\hat{U}_g(t) = \exp[iv(t)] \exp[iv_2(t)\hat{k}_2] \exp[-iv_1(t)\hat{k}_1] \exp[iv_-(t)\hat{k}_-] \exp[v_0(t)\hat{k}_0], \quad (6)$$

where $v_\nu(t)$ ($\nu = 0, -, 1, 2$) and $v(t)$ are time-dependent and real parameters. Within the framework of algebraic dynamics, the transformation parameters $v_\nu(t)$ and $v(t)$ may be given any real initial values. For convenience, we let $\hat{U}_g(t=0) = 1$, namely

$$v_0(0) = v_-(0) = v_1(0) = v_2(0) = v(0) = 0. \quad (7)$$

After the above gauge transformation, the Schrödinger equation (5) becomes

$$i \frac{\partial}{\partial t} |\bar{\Psi}(t)\rangle = \hat{\bar{H}}(t) |\bar{\Psi}(t)\rangle, \quad (8)$$

where the gauged Hamiltonian $\hat{\bar{H}}(t)$ and the gauged wave function $|\bar{\Psi}(t)\rangle$ are defined, respectively, as

$$\hat{\bar{H}}(t) = \hat{U}_g^{-1} \hat{H}(t) \hat{U}_g - i \hat{U}_g^{-1} \frac{\partial \hat{U}_g}{\partial t}, \quad (9)$$

$$|\bar{\Psi}(t)\rangle = \hat{U}_g^{-1} |\Psi(t)\rangle. \quad (10)$$

Since the generators \hat{k}_+ , \hat{k}_- , \hat{k}_1 and \hat{k}_2 are Hermitian and \hat{k}_0 anti-Hermitian, the Hamiltonian $\hat{H}(t)$ is Hermitian and the transformation $\hat{U}_g(t)$ is unitary for real $v_\nu(t)$ ($\nu = 0, -, 1, 2$) and $v(t)$. The Hermiticity of the Hamiltonian implies the probability conservation of the system.

After some calculation, one obtains the gauged Hamiltonian $\hat{\bar{H}}$

$$\hat{\bar{H}} = Z_+(t)\hat{k}_+ + iZ_0(t)\hat{k}_0 + Z_-(t)\hat{k}_- + Z_1(t)\hat{k}_1 + Z_2(t)\hat{k}_2 + Z(t), \quad (11)$$

where the coefficients are given by

$$Z_+(t) = \exp[-v_0(t)]X_+(t), \quad (12a)$$

$$Z_0(t) = -\dot{v}_0(t) + X_0(t) + 2X_+(t)v_-(t), \quad (12b)$$

$$Z_-(t) = \exp[v_0(t)][\dot{v}_-(t) + X_-(t) + X_0(t)v_-(t) + X_+(t)v_-^2(t)], \quad (12c)$$

$$Z_1(t) = \exp\left[-\frac{v_0(t)}{2}\right]\left[-\dot{v}_1(t) + X_1(t) + X_+(t)v_2(t) + \frac{1}{2}X_0(t)v_1(t)\right], \quad (12d)$$

$$Z_2(t) = \exp\left[\frac{v_0(t)}{2}\right]\left\{\left[\dot{v}_2(t) + X_2(t) + X_-(t)v_1(t) + \frac{1}{2}X_0(t)v_2(t)\right] + \left[-\dot{v}_1(t) + X_1(t) + X_+(t)v_2(t) + \frac{1}{2}X_0(t)v_1(t)\right]v_-(t)\right\}, \quad (12e)$$

$$Z(t) = \dot{v}(t) + v_1(t)\dot{v}_2(t) + X(t) + X_2(t)v_1(t) + X_1(t)v_2(t) + \frac{1}{2}X_-(t)v_1^2(t) + \frac{1}{2}X_0(t)v_1(t)v_2(t) + \frac{1}{2}X_+(t)v_2^2(t). \quad (12f)$$

As has been pointed out in references [2–4], one of the merits of algebraic dynamics is that it allows one to convert a time-dependent dynamical symmetry into a stationary dynamical symmetry by a proper choice of gauge. This will largely simplify the quantum problem. The choice of the gauge transformation $\hat{U}_g(t)$ is motivated by the requirement that the gauged

Hamiltonian should consist of the Cartan operators only and all the lowering and raising operators be eliminated. This requirement determines the number of the independent parameter functions to be $(n - r)$ (n is the order of the dynamical algebra, r is its rank). Under the above requirement, there are of course many possibilities to construct $\hat{U}_g(t)$, which correspond to different ways of parametrization in group space and will result in different nonlinear relations between $v_i(t)$ and coefficients $\alpha_\mu(t)$ of the Cartan operator (see below, Eq. (17)). Under a proper gauge, a time-dependent quantum problem can be converted into two separate and simple problems: to solve a stationary eigenvalue problem and to determine the solutions of a set of classical equations of motion. For the system considered in this paper, a suitable gauge is chosen through such a set of $v_\nu(t)$ ($\nu = 0, -, 1, 2$) and $v(t)$ that ⁽¹⁾

$$\frac{Z_-(t)}{Z_+(t)} = \text{const.} = \kappa, \quad Z_0(t) = 0, \quad Z_1(t) = 0, \quad Z_2(t) = 0, \quad Z(t) = 0. \quad (13)$$

From (12a-f) and (13), one has

$$\frac{\exp[2v_0(t)]}{X_+(t)} [\dot{v}_-(t) + X_-(t) + X_0(t)v_-(t) + X_+(t)v_-^2(t)] = \kappa, \quad (14a)$$

$$-\dot{v}_0(t) + X_0(t) + 2X_+(t)v_-(t) = 0, \quad (14b)$$

$$-\dot{v}_1(t) + X_1(t) + X_+(t)v_2(t) + \frac{1}{2}X_0(t)v_1(t) = 0, \quad (14c)$$

$$\dot{v}_2(t) + X_2(t) + X_-(t)v_1(t) + \frac{1}{2}X_0(t)v_2(t) = 0, \quad (14d)$$

$$\begin{aligned} &\dot{v}(t) + v_1(t)\dot{v}_2(t) + X(t) + X_2(t)v_1(t) + X_1(t)v_2(t) \\ &+ \frac{1}{2}X_-(t)v_1^2(t) + \frac{1}{2}X_0(t)v_1(t)v_2(t) + \frac{1}{2}X_+(t)v_2^2(t) = 0. \end{aligned} \quad (14e)$$

Equations (14a-e) are the differential equations satisfied by $v_\nu(t)$. Once the coefficients $X_\nu(t)$ ($\nu = +, 0, -, 1, 2$) and $X(t)$ are specified, the above set of equations (14a-e) with initial condition (7) can be readily solved. Under the special gauge (13), the gauged Hamiltonian $\hat{H}(t)$ becomes

$$\hat{H}(t) = f(t)\hat{I}(0), \quad (15)$$

where

$$\hat{I}(0) = \hat{k}_+ + \kappa\hat{k}_- = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\kappa\hat{q}^2, \quad (16a)$$

$$f(t) = X_+(t)\exp[-v_0(t)]. \quad (16b)$$

The merit of the special choice of gauge (13) is that the Cartan operator \hat{I} does not depend on time explicitly and has the standard form of harmonic oscillator operator. The invariant Cartan operator is

$$\hat{I}(t) = \hat{U}_g\hat{I}(0)\hat{U}_g^{-1} = \alpha_+\hat{k}_+ + \alpha_0\hat{k}_0 + \alpha_-\hat{k}_- + \alpha_1\hat{k}_1 + \alpha_2\hat{k}_2 + \delta, \quad (17)$$

⁽¹⁾ If we consider the gauge transformation

$$\hat{U}_g = \exp[iv(t)] \exp[iv_2(t)\hat{k}_2] \exp[iv_-(t)\hat{k}_-] \exp[v_0(t)\hat{k}_0] \exp[iv_+(t)\hat{k}_+]$$

and choose $Z_+ = Z_0 = Z_- = Z = 0$, $Z_1/Z_2 = i$, then we have $\hat{H} = Z_2(\hat{q} + i\hat{p}) = \sqrt{2}Z_2\hat{a}$, and can get the exact solution of the system in coherent state representation.

where the coefficients $\alpha_\nu(t)$ ($\nu = +, 0, -, 1, 2$) and $\delta(t)$ are given by

$$\alpha_+(t) = \exp[v_0(t)], \quad (18a)$$

$$\alpha_0(t) = -2iv_-(t) \exp[v_0(t)], \quad (18b)$$

$$\alpha_-(t) = v_-^2(t) \exp[v_0(t)] + \kappa \exp[-v_0(t)], \quad (18c)$$

$$\alpha_1(t) = \exp[v_0(t)][-v_2(t) + v_-(t)v_1(t)], \quad (18d)$$

$$\alpha_2(t) = \exp[v_0(t)]v_-(t)v_2(t) - \left\{ v_-^2(t) \exp[v_0(t)] + \kappa \exp[-v_0(t)] \right\} v_1(t), \quad (18e)$$

$$\delta(t) = v_2(t) \exp[v_0(t)] \left[\frac{1}{2} v_2(t) - v_-(t)v_1(t) \right] + \frac{1}{2} v_1^2(t) \left\{ v_-^2(t) \exp[v_0(t)] + \kappa \exp[-v_0(t)] \right\}. \quad (18f)$$

It is easy to prove that $\hat{I}(t)$ is an invariant of the system. With the aid of equations (12a-f), the equations of motion for α_+ , α_0 , α_- , α_1 and α_2 are obtained

$$\dot{\alpha}_+(t) = X_0(t)\alpha_+(t) + iX_+(t)\alpha_0(t), \quad (19a)$$

$$\dot{\alpha}_0(t) = 2iX_-(t)\alpha_+(t) - 2iX_+(t)\alpha_-(t), \quad (19b)$$

$$\dot{\alpha}_-(t) = -X_0(t)\alpha_-(t) - iX_-(t)\alpha_0(t), \quad (19c)$$

$$\dot{\alpha}_1(t) = X_2(t)\alpha_+(t) + i\frac{1}{2}X_1(t)\alpha_0(t) + \frac{1}{2}X_0(t)\alpha_1(t) - X_+(t)\alpha_2(t), \quad (19d)$$

$$\dot{\alpha}_2(t) = -i\frac{1}{2}X_2(t)\alpha_0(t) - X_1(t)\alpha_-(t) + X_-(t)\alpha_1(t) - \frac{1}{2}X_0(t)\alpha_2(t). \quad (19e)$$

Equations (19a-e) can also be obtained from the equation of motion for the invariant operator $\hat{I}(t)$, i.e., $\frac{\partial \hat{I}(t)}{\partial t} + i[\hat{H}(t), \hat{I}(t)] = 0$. In contrast to the nonlinear equations (14a-e) for v_ν , the equations (19a-e) for α_ν lead to the linearization of the quantum equations of motion, which is realized through the nonlinear transformations (18a-f).

Now, we discuss the correspondence between the quantum and classical solutions of the system. The classical algebraic dynamics is

$$\frac{dk_\nu(q, p)}{dt} = \{k_\nu(q, p), H(q, p, t)\}, \quad (\nu = -, 0, +, 1, 2), \quad (20)$$

where the classical Hamiltonian and the classical algebra are given by

$$H(t) = X_+(t)k_+ + iX_0(t)k_0 + X_-(t)k_- + X_1(t)k_1 + X_2(t)k_2 + X(t), \quad (21)$$

$$k_+ = \frac{1}{2}p^2, \quad k_0 = -\frac{i}{2}pq, \quad k_- = \frac{1}{2}q^2, \quad k_1 = p, \quad k_2 = q, \quad (22a)$$

$$\{k_1, k_2\} = -1. \quad (22b)$$

In (22b), $\{k_1, k_2\}$ denotes the Poisson brackets between k_1 and k_2 . The Poisson brackets among other classical quantities can be obtained from (22a, b). From the above equations, the dynamical equations of $k_\nu(q, p)$ ($\nu = +, 0, -, 1, 2$) can be readily obtained

$$\frac{dk_+}{dt} = -X_0(t)k_+ - 2iX_-(t)k_0 - X_2(t)k_1, \quad (23a)$$

$$\frac{dk_0}{dt} = -iX_+(t)k_+ + iX_-(t)k_- - i\frac{1}{2}X_1(t)k_1 + i\frac{1}{2}X_2(t)k_2, \quad (23b)$$

$$\frac{dk_-}{dt} = 2iX_+(t)k_0 + X_0(t)k_- + X_1(t)k_2, \quad (23c)$$

and

$$\frac{dk_1}{dt} = -\frac{1}{2}X_0(t)k_1 - X_-(t)k_2 - X_2(t), \quad (24a)$$

$$\frac{dk_2}{dt} = X_+(t)k_1 + \frac{1}{2}X_0(t)k_2 + X_1(t). \quad (24b)$$

It is easy to check that (23a-c) and (24a, b) are not independent of each other. In fact, one can derive (23a-c) from (24a, b) and (22a). Comparing (24a, b) and (14c, d), we find the following remarkable relation

$$v_1 = k_2 = q, \quad v_2 = k_1 = p, \quad (25)$$

where k_ν ($\nu = 1, 2$) are real solutions of (24a, b). The quantum-classical correspondence relation (25) indicates that v_1 and v_2 are determined by the classical equations of motion of the system [10–15], while the other parameters v_0 and v_- for quantum solution can not be obtained simply from the classical solutions. As a result, for the dynamical system with $\text{su}(1,1) \oplus \text{h}(3)$ algebra, a direct correspondence between α_ν and k_ν is lacking. This is because the structure constant matrices of the invariant operator \hat{I} are not completely Hermitian, which follows naturally from the algebraic structure of $\text{su}(1,1) \oplus \text{h}(3)$. However, as will be seen below, there exists an indirect quantum-classical correspondence. Introducing the following transformation

$$v_1(t) = \sqrt{X_+(t)}Q(t), \quad v_2(t) = \sqrt{X_-(t)}P(t), \quad (26)$$

equations (14c, d) or (24a, b) can be written as

$$\ddot{Q}(t) + \eta_1(t)Q(t) = \mu_1(t), \quad (27a)$$

$$\ddot{P}(t) + \eta_2(t)P(t) = \mu_2(t), \quad (27b)$$

where η_i and μ_i are defined by

$$\eta_1(t) = \left[X_+X_- - \frac{1}{4}X_0^2 - \frac{X_+}{2} \frac{d}{dt} \left(\frac{X_0}{X_+} \right) + \frac{1}{2} \frac{\ddot{X}_+}{X_+} - \frac{3}{4} \left(\frac{\dot{X}_+}{X_+} \right)^2 \right], \quad (28a)$$

$$\eta_2(t) = \left[X_+X_- - \frac{1}{4}X_0^2 + \frac{X_-}{2} \frac{d}{dt} \left(\frac{X_0}{X_-} \right) + \frac{1}{2} \frac{\ddot{X}_-}{X_-} - \frac{3}{4} \left(\frac{\dot{X}_-}{X_-} \right)^2 \right], \quad (28b)$$

and

$$\mu_1(t) = \frac{1}{\sqrt{X_+}} \left[\frac{1}{2}X_0X_1 - X_+X_2 + X_+ \frac{d}{dt} \left(\frac{X_1}{X_+} \right) \right], \quad (29a)$$

$$\mu_2(t) = \frac{1}{\sqrt{X_-}} \left[\frac{1}{2}X_0X_2 - X_-X_1 - X_- \frac{d}{dt} \left(\frac{X_2}{X_-} \right) \right]. \quad (29b)$$

If the parameters $X_\nu(t)$ are periodic functions of time with the same period T , namely $X_\nu(t+T) = X_\nu(t)$, then (27a) and (27b) are called inhomogeneous Hill equations which have wide-spread use in accelerator physics [25]. Once $X_\nu(t)$ are specified, $v_1(t)$ and $v_2(t)$ can be obtained from the real solutions of the inhomogeneous Hill equations (27a, b). Before giving

the exact solution of the Schrödinger equation (5), we first discuss how to determine $v_0(t)$ and $v_-(t)$. Performing the following transformations

$$\alpha_+(t) = \frac{1}{2}\xi(t)\xi^*(t), \quad (30a)$$

$$\alpha_0(t) = -\frac{i}{2}[\xi(t)\zeta^*(t) + \xi^*(t)\zeta(t)], \quad (30b)$$

$$\alpha_-(t) = \frac{1}{2}\zeta(t)\zeta^*(t), \quad (30c)$$

with the aid of the differential equations (19a-c) of α_+ , α_0 and α_- , we get the dynamical equations for $\xi(t)$ and $\zeta(t)$

$$\dot{\xi} = X_+\zeta + \frac{1}{2}X_0\xi, \quad (31a)$$

$$\dot{\zeta} = -\frac{1}{2}X_0\zeta - X_-\xi. \quad (31b)$$

It is directly proved that (19a-c) and (31a, b) are equivalent, and (31a, b) are the homogeneous equations of (24a, b). This is important for establishing an indirect quantum-classical correspondence. Through the transformations

$$\xi(t) = \sqrt{X_+(t)}\Xi(t), \quad \zeta(t) = \sqrt{X_-(t)}\Pi(t), \quad (32)$$

(31a, b) can also be written as

$$\ddot{\Xi}(t) + \eta_1(t)\Xi(t) = 0, \quad (33a)$$

$$\ddot{\Pi}(t) + \eta_2(t)\Pi(t) = 0, \quad (33b)$$

where $\eta_1(t)$ and $\eta_2(t)$ are determined by (28a, b). Obviously, (33a, b) are just the homogeneous equations of (27a, b). It is seen that $v_0(t)$ and $v_-(t)$ can be determined by the complex solution of the homogeneous equations (33a, b)

$$v_0 = \ln \left[\frac{1}{2}|\xi(t)|^2 \right] = \ln \left[\frac{X_+}{2}|\Xi(t)|^2 \right], \quad (34a)$$

$$v_- = \frac{1}{2} \left[\left(\frac{\zeta}{\xi} \right)^* + \left(\frac{\zeta}{\xi} \right) \right] = \frac{1}{2} \sqrt{\frac{X_-}{X_+}} \left[\left(\frac{\Pi}{\Xi} \right)^* + \left(\frac{\Pi}{\Xi} \right) \right]. \quad (34b)$$

As has been pointed out in reference [3], the complex solutions of (33a, b) are necessary. The physical significance of (34a, b) is that a continuous set of classical complex orbits corresponds to just one quantum motion. In brief, the above analysis has established a novel, indirect correspondence between quantum and classical solutions: an infinite and continuous set of classical orbits of the homogeneous equations (31a, b) corresponds to one subset of quantum solutions α_+ , α_0 , and α_- , while the solutions of the classical inhomogeneous equations (24a, b) correspond to the other subset of quantum solutions v_1 and v_2 (α_1 and α_2 are given from (19a-e)). There is a common feature between the above indirect quantum-classical correspondence and the direct one in reference [3] for the $su(1,1)$ dynamical system: an infinite set of classical orbits corresponds to only one quantum solution. For the $su(1,1) \oplus h(3)$ dynamical system, due to the inhomogeneity of the classical equations of motion, there is a one-to-one correspondence between k_1 , k_2 and v_1 , v_2 . However, one has to resort to the homogeneous solutions (33a, b) of the inhomogeneous classical equations of motion (27a, b) in order to establish a complete classical-quantum many-to-one correspondence.

In what follows, we proceed to solve the Schrödinger equation (5) of the system by using algebraic dynamics. First consider the eigenvalue problem of the invariant Cartan operator $\hat{I}(t) = \hat{U}_g \hat{I}(0) \hat{U}_g^{-1}$. Let $|n\rangle$ denote the eigenstate of $\hat{I}(0)$, one has

$$\hat{I}(0)|n\rangle = (n + \frac{1}{2})\omega_0|n\rangle, \quad (35)$$

where $\omega_0^2 = \kappa$. The eigenvalue equation of $\hat{I}(t)$ can be written as

$$\hat{I}(t)\hat{U}_g|n\rangle = (n + \frac{1}{2})\omega_0\hat{U}_g|n\rangle = (n + \frac{1}{2})\omega_0|\phi_n(t)\rangle, \quad (36)$$

where $|\phi_n(t)\rangle = \hat{U}_g|n\rangle$ is the eigenstate of $\hat{I}(t)$ with eigenvalue $(n + \frac{1}{2})\omega_0$. Under the special choice of gauge (13), the gauged Schrödinger equation is

$$i\frac{\partial}{\partial t}|\bar{\Psi}(t)\rangle = f(t)\hat{I}(0)|\bar{\Psi}(t)\rangle, \quad (37)$$

which has the following solution

$$|\bar{\Psi}_n(t)\rangle = \exp[i\Theta_n(t)]|n\rangle, \quad (38)$$

where

$$\Theta_n(t) = -(n + \frac{1}{2})\omega_0 \int_0^t f(\tau) d\tau. \quad (39)$$

$f(t)$ is defined by (16b). In q -representation, the above solution $|\bar{\Psi}_n(t)\rangle$ takes the form

$$\bar{\Psi}_n(t) = N_n(\alpha) \exp[i\Theta_n(t)] \exp[-(\alpha q)^2/2] H_n(\alpha q), \quad (40)$$

where $N_n(\alpha) = [\alpha/\sqrt{\pi}2^n n!]^{1/2}$ is a normalization constant, $\alpha^2 = \omega_0 = \sqrt{\kappa}$, and $H_n(\alpha q)$ is the Hermite polynomial of order n . By virtue of the algebraic relations

$$\exp[v_0 \hat{k}_0] F(q) = \exp(-v_0/4) F(e^{-v_0/2} q), \quad (41a)$$

$$\exp[-iv_1 \hat{p}] F(q) = F(q - v_1), \quad (41b)$$

one obtains the orthonormal diabatic basis, which is an exact solution of the original equation (5)

$$\begin{aligned} \Psi_n(t) &= \hat{U}_g \bar{\Psi}_n(t) \\ &= \exp[i\Theta_n(t)] \phi_n(t) \\ &= N_n(\chi\alpha) \exp\left\{i[\Theta_n(t) + v(t)]\right\} \exp\left\{i\left[\frac{1}{2}v_-(q - v_1)^2 + v_2 q\right]\right\} \\ &\quad \times \exp\left\{-\frac{1}{2}[\chi\alpha(q - v_1)]^2\right\} H_n(\chi\alpha q - \chi\alpha v_1), \end{aligned} \quad (42)$$

where $\chi = \exp[-v_0/2]$ is determined by the classical equations (33a, b). (42) describes a one-dimensional harmonic oscillator in a moving and dilating coordinate $\chi(q - v_1)$ with an extra mean velocity potential $\{\frac{1}{2}v_-(t)[q - v_1(t)]^2 + v_2(t)q\}$ and a time-dependent phase.

$\Psi_n(t)$ provides a time-dependent basis in Hilbert space like a local, moving frame in differential geometry. The diabatic energy levels are

$$\begin{aligned} E_n(t) &= \langle \Psi_n | \hat{H}(t) | \Psi_n \rangle \\ &= \langle \bar{\Psi}_n | \hat{U}_g^{-1} \hat{H} \hat{U}_g | \bar{\Psi}_n \rangle \\ &= \left(n + \frac{1}{2} \right) \left\{ \omega_0 f(t) - \frac{\dot{v}_-(t) \exp[v_0(t)]}{2\alpha^2} \right\} - v_1(t) \dot{v}_2(t) - \dot{v}(t). \end{aligned} \quad (43)$$

An arbitrary solution of the Schrödinger equation (5) can be expanded as

$$\Psi(t) = \sum_n C_n \Psi_n(t) = \sum_n C_n \exp[i\Theta_n(t)] \phi_n(t), \quad (44)$$

where C_n are independent of time and only determined by the initial state of the system, while the dynamical information is completely contained in the diabatic basis $\Psi_n(t)$. The expectation value of the invariant Cartan operator $\hat{I}(t)$ is a constant of motion

$$\langle \Psi(t) | \hat{I}(t) | \Psi(t) \rangle = \sum_n |C_n|^2 \left(n + \frac{1}{2} \right) \omega_0. \quad (45)$$

Now let us calculate Berry's phase for the system studied. Suppose the parameters $X_\nu(t)$ and $X(t)$ are periodic functions of time with the same period T , then the dynamical equations of $v_\nu(t)$ and $v(t)$ will have periodic solutions. After one period of evolution the system returns to its initial state except for acquiring a total phase

$$\phi_t = \Theta_n(T) = -\left(n + \frac{1}{2} \right) \omega_0 \int_0^T f(t) dt. \quad (46)$$

The conventional dynamical phase over one period is

$$\phi_d = \int_0^T E_n(t) dt = \int_0^T \left\{ \left(n + \frac{1}{2} \right) \left[\omega_0 f(t) - \frac{\dot{v}_-(t) \exp[v_0(t)]}{2\alpha^2} \right] - v_1 \dot{v}_2 \right\} dt. \quad (47)$$

Thus the nonadiabatic Berry's phase [26] is given by

$$\phi_B = \phi_t + \phi_d = - \int_0^T \left\{ v_1 \dot{v}_2 + \left(n + \frac{1}{2} \right) \frac{\dot{v}_-(t) \exp[v_0(t)]}{2\alpha^2} \right\} dt. \quad (48)$$

Up to now, we have got the exact solution of the linear nonautonomous system with the $su(1,1) \oplus h(3)$ algebra. It has been shown that the essence of algebraic dynamics is to emphasize the dynamical aspect of an algebra and the algebraic structure of a dynamics. On the one hand, the time evolution is introduced into algebra; on the other hand the algebraic method is introduced into dynamics and makes its solution feasible.

The study in the present paper shows that the basic methods and concepts of algebraic dynamics, such as gauge transformation, invariant operator, time-dependent dynamical symmetry, time-dependent orthonormal representation, and linearization of equation of motion *etc.* can be generalized from the linear nonautonomous systems with $su(1,1)$ Lie algebra to a special system with a more complicated Lie algebra, *i.e.* $su(1,1) \oplus h(3)$ algebra. However, the direct correspondence between quantum and classical solutions is lost. This is because the structure constant matrices related to the dynamical equation of invariant Cartan operator can not be chosen Hermitian. It is interesting that there is still an indirect quantum-classical correspondence for the system studied, although the direct correspondence does not exist: both the homogeneous solutions (33a, b) and the inhomogeneous solutions (27a, b) together correspond to the quantum solution. The physical and mathematical implication of this indirect quantum-classical correspondence is a subject for further investigation.

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