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Vectorial Cellular Automaton for the Stress in Granular Media

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Abstract. — We introduce a simple cellular automaton to model the force distribution in a gravity-loaded granular medium. Two test cases are considered: the sand heap, and the silo geometry. The model gives results consistent with established observations. For the heap, the normal stress on the base plane displays a pronounced depression underneath the apex. In the silo geometry, the model reproduces the correct scaling for the approach to a stationary (depth-independent) stress state.

Résumé. — Nous introduisons un modèle simple d'automate cellulaire pour rendre compte des contraintes dans un milieu granulaire soumis à la gravité. Deux géométries test sont considérées: un simple tas et un silo. Les résultats du modèle sont cohérents avec les caractéristiques connues de ces deux situations. En particulier, pour le tas, la contrainte normale sur le plan de base montre une dépression prononcée sous le sommet. Dans la géométrie du silo, le modèle rend compte correctement de l'approche vers une répartition de contrainte indépendante de la profondeur.

1. Introduction

The static propagation of forces in a granular system submitted to an arbitrary load is till today an unsolved problem. Continuum approaches have been proposed in soil mechanics based on elasto-plastic behavior laws to compute stress and strain, but they have serious problems in reproducing some simple experimental facts like the shape of stagnation zones in silos or the pressure distributions below a heap of sand.

Recently Liu *et al.* [1] introduced a cellular automaton to model the stress distribution in granular media. Although they were successful at reproducing the statistical distribution of interparticle forces, the model suffers from some limitations essentially due to the scalar nature of the force analog they introduced. When such a model is applied to the computation of stresses in a silo geometry, one finds that the characteristic distance ξ after which a stationary stress distribution is reached scales as $\xi \propto L^2$ where L is the silo width. This is a consequence of the

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diffusive nature of the spreading of the lateral force and contradicts experimental observations where the scaling is found to be linear $\xi \propto L$. Stated in more general terms, it is essential to restore the Galilean invariance to faithfully account for the stress distribution.

This model has been extended to the vector nature of forces by de Gennes [2]. In the latter model, a point force exerted on top of a semi infinite domain gives rise to forces propagating inside the medium along a cone with a second order blurring due to a diffusion process induced by the geometrical disorder quite comparable to the previous model. Again, this model is not Galilean invariant. Indeed, one expects that the forces localised along the cone should also undergo a similar splitting into a secondary cone whose axis lies on the surface of the primary one.

The spirit of these discrete approaches is to compute the forces by sweeping the system only once, row after row. Thus it appears difficult to introduce a complete Galilean invariance where forces would have to propagate also upward after a few scatterings. In systems which are only subjected to gravity it is, however, likely that the major principal stress is to first order *vertical* so that it might be sufficient to propagate the forces downward from the free surface. We emphasize that this is a restriction to the applicability of such an approach. We will mention in the course of this article possible extensions to more general loadings. The main interest of the single-sweep choice is its simplicity and its efficiency. As most cellular automaton models, the model is not aimed at providing a realistic picture of the stress distribution locally, but rather to include at this basic level enough ingredients to restore a physically meaningful continuum limit, once details of the model are averaged over enough configuration or large enough scale. The basic ingredients which are here included in the model are 1) the balance equation, 2) the selection of an isostatic subnetwork of active contacts, such that contact selection and balance are enough to determine the forces in the medium, 3) the positivity of all contact forces, and 4) a rule for selecting contacts such that some texture could be developed in the medium according to the local stress. The latter aspect is a key feature of real granular media, which is included in our simple approach.

We introduce a variant of this cellular automaton. In contrast to the model of Liu *et al.*, it deals with vector forces. It shares with the two previous models the “single sweep” property which, as discussed above, is a rather unphysical requirement which prevents a complete Galilean invariance of the model. Thus we will only consider here gravity-loaded systems. We will consider two main applications of this model. The first one is the stress distribution in a sand heap, and the second one is the silo geometry.

The first case (heap) has known a recent upsurge of interest [3–5] motivated by the careful experimental study of Šmíd and Novosad [6] who discovered the intriguing presence of a pronounced dip in the normal stress on the basis of the heap, right under the apex. This surprising observation appears difficult to be reproduced in most models. They rather give rise to a uniform or quasi-uniform stress distribution on the basis of the heap [1–3, 7, 8]. Hemmingsson [9] presented a cellular automaton model which succeeded to reproduce the stress dip. Some continuum approaches can account for this effect through the introduction of either some assumptions concerning the microscopic texture of the medium [4] or a postulate on the local orientation of the principal axis of the stress tensor [5]. These remain to be justified more firmly, or tested on more general geometries.

The second test case (silo geometry) is a well documented subject which has received a considerable attention in the past. Moreover, as mentioned above, it also allows to study systematically the role of the parameter of the model.

2. Model

The spirit of the cellular automaton model is to compute row by row the force distribution in the granular medium represented by a lattice, propagating the information downward. The lattice must have a coordination number strictly larger than four in order to offer enough freedom after the balance equation is imposed at each node. Thus the natural choice is to deal with a triangular lattice. Figure 1 shows two successive rows. The sites of the lattice are the grains, while the bonds represent potential contacts between grains. Each site i in a layer receives from its contacts with the upper layer some compressive forces f_i and g_i as shown in Figure 2. The aim is to compute the forces in the two horizontal bonds and the two downward bonds. The latter are used as the input for the next row.

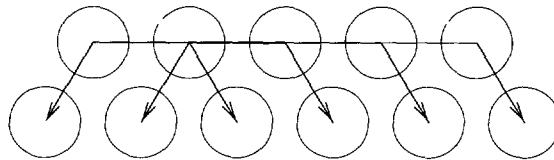


Fig. 1. — Between to broken horizontal bonds, there is one kink site, here the second from the left, that can have two active bonds to the layer below.

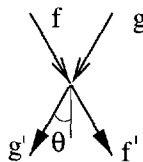


Fig. 2. — Notation of directions of the forces.

The disorder of the medium which is inherent to all granular media will be introduced by cutting out bonds in the network. We introduce the probability p that a horizontal bond is missing. The first step is to distribute randomly with no spatial correlation the missing horizontal bonds in a row. This is done once and for all, and will not change in the course of the computation.

The missing horizontal bonds divide the top row into clusters of grains in contact. Figure 1 shows such a cluster which constitutes the basic unit used in the force distribution. Each cluster is independent from other clusters. Let us call i_l and i_r the leftmost and rightmost site of one cluster. We choose randomly a particular site i_c , called “kink site”, with a uniform probability in the interval $[i_l; i_r]$. We only transmit forces to the left for sites $i < i_c$ and to the right for $i > i_c$. Using the notations of Figure 2 we impose

$$\begin{cases} f'_i = 0 & \text{for } i_l \leq i < i_c \\ g'_i = 0 & \text{for } i_c < i \leq i_r \end{cases} \quad (1)$$

The site i_c is the only one in the cluster allowed to have non-zero downward forces in both directions. Once the “active” bonds have been selected the forces are calculated using the

balance equation at each site.

$$\begin{cases} g'_i = f_i + g_i + \frac{w}{\cos(\theta)} & \text{for } i_l \leq i < i_c \\ f'_i = f_i + g_i + \frac{w}{\cos(\theta)} & \text{for } i_c < i \leq i_r \end{cases} \quad (2)$$

where w is the weight of the grain. See Figure 2 for the definition of θ . We also compute the horizontal force starting from both ends and stopping at i_c . Denoting by $h_{i,i+1}$ the horizontal compressive force transmitted through the horizontal bond between i and $i+1$, we have

$$\begin{cases} h_{i,i+1} = h_{i-1,i} + 2 \sin(\theta) f_i + w \tan(\theta) & \text{for } i_l \leq i < i_c \\ h_{i-1,i} = h_{i,i+1} + 2 \sin(\theta) g_i + w \tan(\theta) & \text{for } i_c < i \leq i_r \end{cases} \quad (3)$$

Finally only the forces f'_{i_c} and g'_{i_c} remain to be determined. Again force balance suffices to compute them

$$\begin{cases} f'_{i_c} = f_{i_c} + \frac{1}{2 \sin(\theta)} (h_{i_c-1,i_c} - h_{i_c,i_c+1}) + \frac{w}{2 \cos(\theta)} \\ g'_{i_c} = g_{i_c} + \frac{1}{2 \sin(\theta)} (h_{i_c,i_c+1} - h_{i_c-1,i_c}) + \frac{w}{2 \cos(\theta)} \end{cases} \quad (4)$$

After these steps the force distribution fulfills the equilibrium conditions at each site. It is, however, not granted that f'_{i_c} and g'_{i_c} are both positive, *i.e.* compressive, in contrast to all other bonds which always carry a positive force. If both are positive, then we accept the force distribution just calculated and turn to the next cluster. If not we have to choose another i_c and for that we use that at most one of the two forces can be negative since their sum must be positive (as can be established recursively). Let us first consider the case when f'_{i_c} is negative. Since a contact cannot be under traction, we set $f' = 0$ and if we accept momentarily the value of all other forces, the particle at site i_c will have to be moved to the left. Thus the right neighbor of i_c may have the possibility to form a contact with its left-downward neighbor. The creation of the new contact and opening of the former one corresponds exactly to the configuration obtained if $i_c + 1$ would have been chosen instead of i_c as kink site in the cluster. Thus we take $i_c + 1$ as new kink site and recompute the forces. It can be shown that one will never encounter a case where the kink site would have to go outside the cluster. If i_c is the rightmost site i_r , then necessarily $f' > 0$. Analogously if g'_{i_c} is negative the kink site is moved to the left, and the forces are recomputed. The procedure is repeated until all forces are compressive. Such a solution always exists. In fact several different sites could be acceptable locations for kink sites. By our procedure one of these is randomly selected.

The presence of downward bonds has been chosen in such a way that each cluster is isostatic, *i.e.* all forces can be computed simply by the use of the local balance of forces at each grain. Thus we have avoided the introduction of a particular constitutive law at the contact level. This is expected to be a valid approximation for rigid (*i.e.* undeformable) grains. Moreover, we have also imposed that the forces are aligned with the contact orientation, so that we assumed a vanishing coefficient of friction at each contact. Different choices are evidently legitimate, but they would require the introduction of additional parameters and presumably rules to account for the force distribution. We thus stick here to the simplest version where p is the only free parameter which quantifies the degree of disorder in the medium.

Before turning to the application of this model, let us emphasize a few of its properties:

- All contact forces are positive, an essential feature sometimes neglected in discrete models.

- The determination of the contact network and the force computation are performed simultaneously. This makes the transcription to the continuum limit more difficult. To make this point clearer, we may consider the case of a strong force being applied to one specific site of a cluster. This site attracts the kink site and thus favors the direct propagation of the force rather than its scattering by an arch.
- It naturally implements a *diffusive* type of behavior when the forces are evenly distributed, (*i.e.* a force can be “scattered” when it encounters a missing bond) and a *wave* behavior when the forces are strongly focussed. The “Peclet” number introduced in Bouchaud *et al.* [3] — a dimensionless quantity which compares the direct propagation of forces over a fixed distance and their scattering — thus appears to depend on the deviatoric part of the stress tensor.
- It is not Galilean invariant even in the case of a regular triangular lattice, because the status of horizontal bonds is different from that of the downward bonds. A better model would allow for the closing and opening of all contacts in a similar manner, but such a model remains to be formulated. As compared with other discrete cellular automaton models, it takes into account properly the vector nature of forces.
- Finally, as a minor detail we note that if we deal with f , g and $j = h/\sin(\theta)$ and set the force unit such that $w/2 \cos(\theta) = 1$, then all forces are integer numbers and the numerical implementation becomes very efficient, with the reservation that the force computation at the level of one cluster cannot be directly parallelised.
- There is some arbitrariness in the selection of the lines (here horizontal rows) along which the lattice is swept through. In our case this choice is justified by the fact that gravity induces a major principal stress which to first order is a vertical compression. However, as we will see in the following applications, the actual mean stress field is not so simple. One could think of building an iterative procedure where the successive boundaries on which forces are computed are adjusted along the minor principal stress. This scheme might be important for cases in which the principal directions of the stress rotate significantly in space. We have not yet developed such a scheme.

3. Heap

In cold regions one can see in the spring that when a snow heap melts it sometimes leaves behind a ring of ice on the ground before completely disappearing. This is an effect of the compactification and caking of snow due to the stress distribution inside the heap. Since the snow in the center melts before the snow in this ring, it seems that the stress has a maximum at some distance away from the center, and has a local minimum at the center. We will not dwell upon the granular properties of snow, and note that the effect might partially be due also to the dynamics of the melting process; it is, however, a nice example to keep in mind as we continue.

Careful experiments on the stress distribution in granular materials have been performed by Šmíd and Novosad [6]. They poured granular material, quartz sand and granulated fertilizer NPK-1 respectively, on a 2×2 square meters rigid steel plate equipped with pressure cells. These cells could simultaneously measure the normal pressure and the shear stress. Opposite to what one might think in the first place and in agreement with the observations on melting snow, a local minimum, and not a maximum is found below the apex of the heap. (The same effect was earlier hinted at by Saul, see [10], but not systematically investigated.) The effect looks more or less the same for all materials used.

It is clear that the phenomenon is the result of a redistribution of stresses. We apply our model to a heap with a prescribed angle of repose (angle between the free boundary of the heap and the horizontal direction), chosen to be 32° in most computations. The angle θ (independent

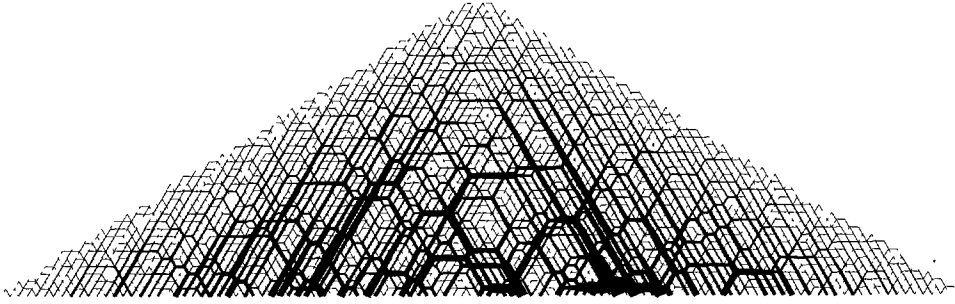


Fig. 3. — An example of stresses inside a heap of granular material, resulting from one single run. The breaking probability p is 0.5.

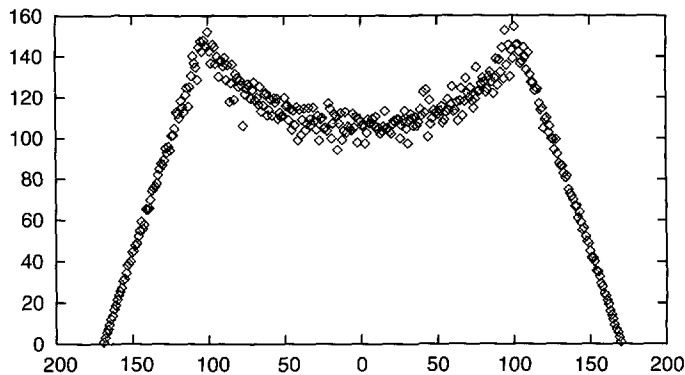


Fig. 4. — The rescaled stresses at the bottom of the heap, averaged over many samples. Breaking parameter p is 0.5, and the angle of repose is 32° .

from the angle of repose), is chosen to be 30° in most cases. In Figure 3 the forces inside the simulated heap are shown. The larger the force, the thicker the line. The scale is linear, but was normalized separately for the horizontal forces, to make the figure (here only shown as an illustration) more readable.

We clearly observe in Figure 3 different regions in which the forces behave differently. Close to the center of the heap, the average stress is nearly an isotropic pressure (*cf.* Fig. 4). The presence of randomness induces the formation of cellular patterns around which the larger forces concentrate. These hexagons are reminiscent of comparable patterns observed through photoelasticity of a regular array of parallel cylinders [11], molecular dynamics simulations [12] or other numerical techniques [13]. As one moves away from the center there is a narrow region where the forces are roughly parallel to the underlying axis of the triangular lattice. In this region, the stress has a strong deviatoric component, and as mentioned above, the forces are essentially convected along these directions. Finally, close to the free surface the forces vanish, as expected.

In order to draw a connection to recently proposed theoretical modeling, we have computed the spatial average of the stresses inside the heap. At each lattice point, we can compute the flux of force transmitted through any infinitesimal surface. The latter flux is naturally the stress tensor. Moreover, although the individual forces are aligned with the principal direction of the triangular lattice, the stress tensor is not constrained in the same way. The principal stresses

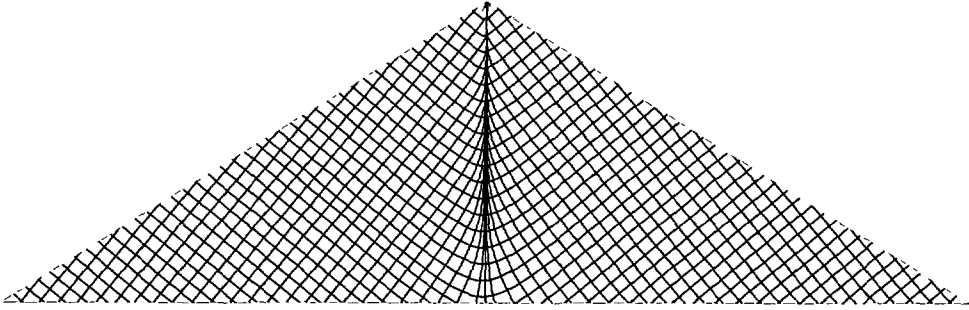


Fig. 5. — Principal stress curves obtained from a configuration average of the forces in the heap geometry.

in particular can assume any orientation. Evidently, as in all cellular automata approach, although the individual forces are far away from any realistic description of the medium, the ensemble average of the force flux, *i.e.* the average stress tensor is expected to have a well-defined physical meaning, and is the quantity which can be compared to experimental, or other theoretical results. We show in Figure 5 the direction of the principal stress. We have drawn two families of orthogonal curves which are at each point tangent to the principal stress directions (eigenvectors of the stress tensor). In reference [5], a postulate was introduced, stating that the major principal stress formed a fixed angle with the vertical direction. We observe in the figure that this postulate gives a fair account of the observed stress distribution in the wings of the heap, and fails only close to the axis. Correspondingly, the normal stress obtained on the basal plane is also in good qualitative agreement with their result. Straight lines of force propagation were also “predicted” by Edwards and Mounfield [4] for a textured heap made of anisometric particles.

In Figure 4, we present the stresses exerted at the basal plane averaged over many configurations, for heaps with a height of 100 rows. The normal stress σ_{zz} increases linearly from the tail of the heap up to a radius which corresponds to the projection of the apex along the principal axis of the triangular lattice. Finally, for smaller radii, one observes a marked dip in the normal stress distribution with a depression of up to about 1/6 of its maximum value. We checked that this result only depends on p and the heap angle (see below), and not on the heap height, if all distances and stresses are rescaled accordingly. The mean shape is in good qualitative agreement with the observed experimental curve obtained by Šmíd and Novosad [6]. It is difficult to go beyond this qualitative comparison. On one hand, our model is two dimensional and only aimed at describing rigid frictionless particles. On the other hand, the underlying lattice used in the model is artificially regular. One consequence of this is the occurrence of the sharp maximum of the stress seen in Figure 4.

The dependence on p , the probability of breaking the horizontal bonds, and the heap angle α is shown in Figure 6. The ratio of the vertical stress below the center and the maximal vertical stress is plotted as a function of p for three different values of α . We note that for $p = 0$, the dip in the normal stress will be as deep as the maximal height, and we reproduce the result of Edwards and Mounfield [4] who obtained a vanishing normal stress just under the apex of the heap. We also checked that the vertical stress field has the same double tent shape as in their model. (The depression for $p = 0$ is not exactly one because each data point in Figure 6 is an average over neighboring sites, and the weight of a grain on the bottom layer always contributes to the vertical stress field.) In the other extreme case, $p = 1$, there is no dip at all.

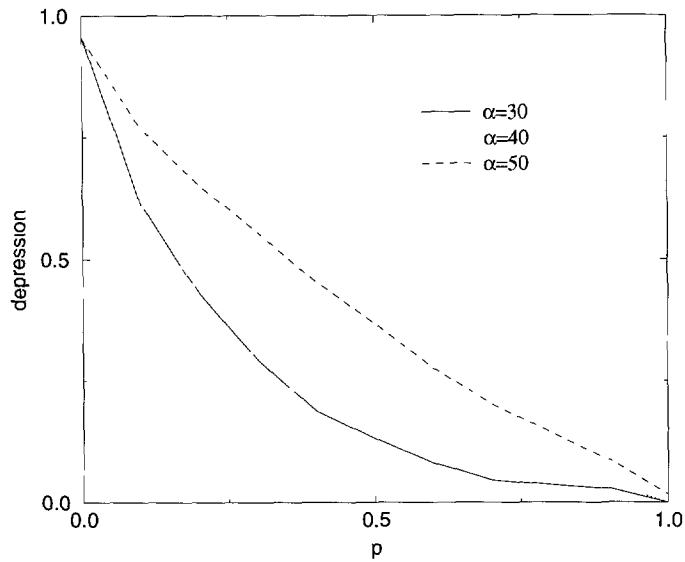


Fig. 6. — The depression as a fraction of the maximal normal stress. The dependence on the parameter p is shown for three different slopes, 30, 40 and 50 degrees respectively.

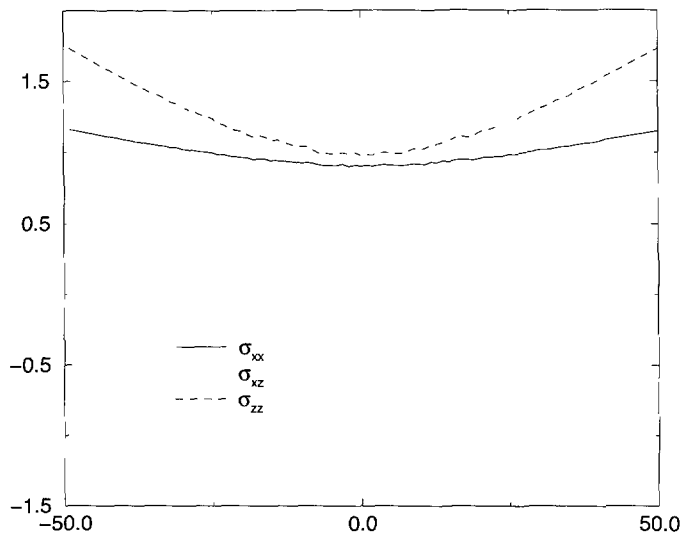


Fig. 7. — Stresses σ_{zz} , σ_{xz} and σ_{xx} as functions of the lateral position in the silo. The width of the silo was 100 sites, and the plot is an average over 500 samples.

The vertical stress increases linearly to become constant for smaller radii. This corresponds to the (quasi-) uniform stress distributions found in other models [1-3, 7, 8]. We note that both these extremes represent completely ordered systems, and we do not expect that they are very realistic descriptions of the stress distribution inside the medium. A certain degree of disorder is closer to reality.

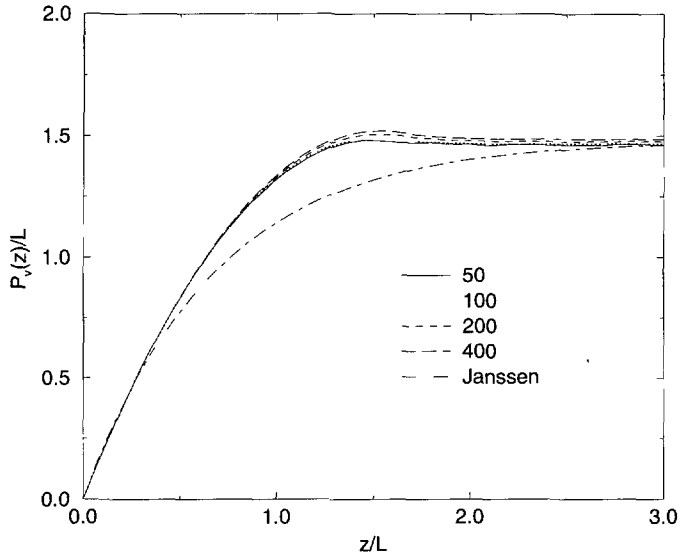


Fig. 8. — The rescaled mean vertical stress $P_v(z)/L$ as a function of the relative depth z/L for systems a width L of 50, 100, 200 and 400 sites, respectively. All data points collapse onto a single curve. The probability p for breaking horizontal bonds is 0.5. For comparison, a fit of Janssen's law, equation (5), is plotted.

4. Silo

In 1895, Janssen [14] proposed a model to calculate the pressure inside a bin of granular material as a function of the depth. This often-quoted result is based on four assumptions:

- 1) the vertical pressure P_v is constant in the horizontal plane;
- 2) the density ρ of the material is constant over all depths;
- 3) the wall friction f is fully mobilized;
- 4) the horizontal pressure P_h is proportional to the vertical pressure P_v where $K = P_h/P_v$ is constant in space.

By using these assumptions in a force balancing calculation, Janssen arrives at an equation that in 2 dimensions describes the vertical pressure as a function of the depth z (see *e.g.* Ref. [15] for a recent use of this result for analysing experimental results):

$$P_v(z) = \frac{\rho g L}{2Kf} \left(1 - e^{(-2Kfz)/L} \right) \quad (5)$$

where L is the width of the silo.

This description is a one-dimensional phenomenological approach, which can be justified more solidly in a continuum modeling. Let us note that the various assumptions have very different statuses. The first one is generally not valid, but it is unessential to the result. The second assumption may simply result from the incompressibility of the material. The third one is a natural assumption if only a radial loading (like filling under influence of gravity alone) is considered. Finally, the fourth assumption should be the result of the constitutive behavior of the material.

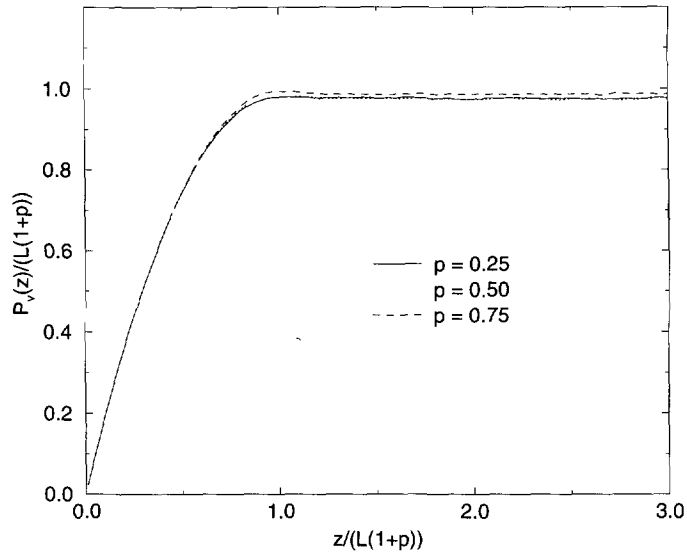


Fig. 9. — The rescaled mean vertical stress $P_v(z)/(L(1+p))$ as a function of the relative depth $z/(L(1+p))$ for p values of 0.25, 0.5 and 0.75 respectively. Again, all data points collapse onto a single curve. The width L is 50 sites.

Using scalar models, one finds a partial agreement with the above Janssen's law. The approach to a stationary state is indeed an exponential law, $P_v(\infty) - P_v(z) \propto e^{-z/\xi}$ but the characteristic length ξ does not depend linearly on L but rather quadratically $\xi \propto L^2$ due to the diffusive nature of the stress distribution in these models.

Using our model inside a box with fixed walls we simulated a silo for different values of the parameters. In contrast to the heap case, we must consider the walls in the silo geometry. A force that reaches the wall will simply be cancelled, *i.e.* it will meet an equally large, but oppositely directed force from the wall. Since two consecutive layers are shifted one half lattice spacing in the horizontal direction, the contact with the wall will be alternatively at the left and the right side. This means that at one layer, the force g' at the leftmost site reaches the left wall where it meets an equally strong force in the opposite direction. At the next layer, the force f' at the rightmost lattice site will meet the right wall, *etc.* In Figure 8 the result for the mean vertical stress can be seen as a function of the depth for different widths. For comparison, a fit of Janssen's law is plotted. The stress increases, in the beginning almost linearly, and reaches a constant as in Janssen's law. Both the value of the constant stress and the depth at which it is reached depend linearly on the width of the silo, just as in equation (5). In the intermediate regime, however, our data cannot be described by Janssen's simple exponential saturation. The scaling with p is seen in Figure 9, where the vertical stress and the depth both are rescaled with $L(1+p)$. Data points for different values on p all collapse onto one single curve. From the figures, we conclude that for our model, the characteristic length scale $L/(2Kf)$ scales like $\xi \propto L(1+p)$.

The average stresses σ_{xx} , σ_{xz} and σ_{zz} in the cross section are shown in Figure 7. We do observe that the vertical component of the stress σ_{zz} increases towards the edge of the pile, consistently with the fact that the load is redirected toward the walls. The horizontal normal stress σ_{xx} follows the same trend although less markedly. The shear stress σ_{xz} is observed to be very close to linear along the cross section.

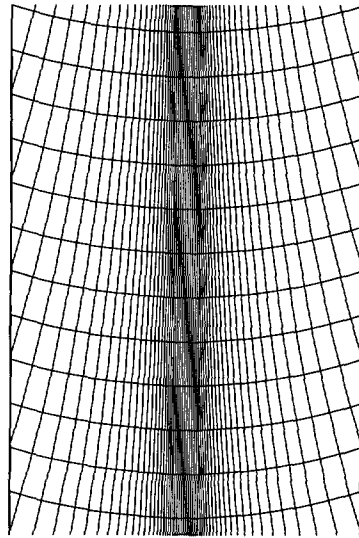


Fig. 10. — Principal stress curves in the silo geometry in the stationary stress state (deep below the free surface). In contrast to the heap geometry, no region can be accurately described with a uniform tilt of principal directions with respect to the vertical.

Finally, we show in Figure 10 the principal stress curves in the stationary domain, from the average stress estimated over many samples. This figure shows that the postulate introduced in reference [5] (*i.e.* constant angle between the major principal stress and the vertical direction), does not apply to this case, in contrast to the heap geometry. Thus, this postulate appears fragile and a more solid foundation is needed to determine its scope of applicability.

5. Conclusion

We have introduced a simple cellular automaton which takes into account the vector character of interparticular forces, and the non-linearity due to contact unilaterality. We have tested our model in two simple geometries, and obtained results which compare favorably with the literature.

It appears that a key feature to reproduce the observed dip in the heap geometry is the possibility of having a simple propagating mode for strongly deviatoric stress regions such as expected parallel to the free surface together with a diffusive behavior in the central core which scatters away the stresses which are finally trapped in the wings of the heap. This effect is introduced either as a postulate in the continuum approach of Wittmer *et al.* [5], or in the phenomenological approach of Bouchaud *et al.* [3]. It is also included through the cellular automaton rules of the model introduced in Hemmingsson [9] and in the present model.

The wave-like propagation is also an essential feature to recover the correct scaling of the stress shielding in a silo, otherwise a diffusive length scale may become dominant as for instance in the Liu model [1].

Acknowledgments

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