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## Replica Fourier Transforms on Ultrametric Trees, and Block-Diagonalizing Multi-Replica Matrices

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**Abstract.** — The analysis of objects living on ultrametric trees, in particular the block-diagonalization of 4-replica matrices  $M^{\alpha\beta;\gamma\delta}$ , is shown to be dramatically simplified through the introduction of properly chosen operations on those objects. These are the Replica Fourier Transforms on ultrametric trees. Those transformations are defined and used in the present work.

**Résumé.** — On montre que l'analyse d'objets vivant sur un arbre ultramétrique, en particulier, la diagonalisation par blocs d'une matrice  $M^{\alpha\beta;\gamma\delta}$  dépendant de 4-répliques, se simplifie de façon dramatique si l'on introduit les opérations appropriées sur ces objets. Ce sont les Transformées de Fourier de Répliques sur un arbre ultramétrique. Ces transformations sont définies et utilisées dans le présent travail.

Spin glasses and their typical glassy phases appear to be present in a wide spectrum of domains [1–3]. The high level of complexity inherent to their structure (field  $\phi_{\alpha\beta}(\mathbf{x})$  depending upon two replicas, propagators  $G^{\alpha\beta;\gamma\delta}(\mathbf{x}, \mathbf{y})$  depending upon four of them) has prevented so far a systematic study of the glassy phase with the standard tools of field theory and in particular the renormalization group where objects with 6 (or 8) replicas would be needed. One recent step in the process of founding a field theory has been the spelling out of a Dyson like equation [4, 5] relating the propagators  $G$  and their inverses  $M$  (mass operator plus kinetic terms), a triviality in standard field theories. To be more precise, the obtained relationship is not directly between  $G$  and  $M$ , but between what was termed the kernel ( $F$ ) of  $G$  and the kernel ( $K$ ) of  $M$ . What makes the algebra complicated, is that one cannot directly work with replicas (whose number  $n$  is to be set to 0). Instead, one is led to use, for the observables, a replica symmetry broken representation (RSB) usually inferred from mean field studies (be it with  $R = 1$ , one step RSB or  $R = \infty$  as in Parisi's [7] ansatz). As a result, the inversion process of the 4-replica matrix  $M^{\alpha\beta;\gamma\delta}$ , becomes a highly non trivial exercise with results fairly involved for  $R = \infty$  [4], and even much more so for a generic  $R$  [5].

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Here we would like to show that with the appropriate choice of transformations, the previously obtained results, considerably simplify, and in some sense, become almost transparent.

To that effect we introduce the notion of (i) Replica Fourier Transform (RFT) and (ii) RFT on a (ultrametric) tree. With those definitions, given in Section 1, we show that the relationship between (matrix) functions and their kernels is nothing but a Replica Fourier Transformation (a double transform for the Replicon component and a single one for the longitudinal anomalous component). This is done in Sections 2 and 3. Further the same RFT, used now on the eigenvalue equations immediately (block-)diagonalizes them (Sect. 4). Used on the equation  $GM = 1$ , the RFT yields Dyson's equation relating the corresponding kernels of  $M$  and  $G$  (Sect. 5).

It must be mentioned at the outset that this work was prompted by the yet unpublished results of Parisi and Sourlas [6] where a Fourier transform on  $p$ -adic numbers is introduced <sup>(1)</sup>, which diagonalizes the Replicon sector components. The RFT used here has the flexibility that also allows for a block diagonalization of the Longitudinal Anomalous sector.

## 1. Replica Fourier Transform

It is assumed, as in Parisi [7], that the 2-replica object  $\langle \phi_{\alpha\beta}(\mathbf{x}) \rangle \equiv q_{\alpha\beta} = q_t$  only depends upon the overlap (or codistance  $\alpha \cap \beta = t$ ). Likewise a  $m$ -replica object will depend upon  $m - 1$  independent overlaps. The RFT [9], which is a discretized version of the algebra introduced by Mezard and Parisi [10] defines the RFT transform  $\hat{A}$  of  $A$  via

$$\hat{A}_k = \sum_{t=k}^{R+1} p_t (A_t - A_{t-1}) \quad (1)$$

Here the  $p_t$ 's are the sizes of the Parisi boxes,  $p_0 \equiv n$  the replica number,  $p_{R+1} \equiv 1$ . Note that  $\alpha \cap \beta = t$  means that  $\alpha$  and  $\beta$  belong to the same Parisi box of size  $p_t$ , but to two distinct boxes of size  $p_{t+1}$ , *i.e.* they belong to the (relative) multiplicity

$$\delta_t = p_t - p_{t+1} \quad (2)$$

$$\delta_{R+1} = p_{R+1} \quad (3)$$

Objects with indices out of the range of definition (here  $(0, R + 1)$ ) are taken as null. Conversely one has

$$A_j = \sum_{k=0}^j \frac{1}{p_k} (\hat{A}_k - \hat{A}_{k+1}) \quad (4)$$

and the associated (relative) multiplicity in what will be termed below "resolution" (or pseudo-momentum) space,

$$\bar{\delta}_k = \frac{1}{p_k} - \frac{1}{p_{k-1}} \quad (5)$$

$$\bar{\delta}_0 = \frac{1}{p_0} \quad (6)$$

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<sup>(1)</sup> See also B. Grossman, reference [8].

Characteristically, the convolution

$$\sum_{\gamma=1}^n A_{\alpha\gamma} B_{\gamma\beta} = C_{\alpha\beta} \quad (7)$$

becomes, after RFT

$$\widehat{A}_k \widehat{B}_k = \widehat{C}_k \quad (8)$$

If we take

$$C_{\alpha\beta} = \delta_{\alpha\beta} \quad (9)$$

we have

$$\widehat{C}_k = 1 \quad (10)$$

$$\widehat{A}_k = 1/\widehat{B}_k \quad (11)$$

This convolution property conveniently allows to write for example,

$$\text{tr}(A)^s = n \sum_{k=0}^{R+1} \bar{\delta}_k \left( \widehat{A}_k \right)^s. \quad (12)$$

to be compared with

$$\sum_{\alpha\beta} (A_{\alpha\beta})^s = n \sum_{j=0}^{R+1} \delta_j (A)_j^s. \quad (13)$$

One is often using a multiplicity  $\mu(k)$ , related to the (relative) multiplicity  $\bar{\delta}_k$  introduced here by

$$\mu(k) = n \bar{\delta}_k \quad (14)$$

$$\mu(0) = n \frac{1}{p_0} = 1 \quad (15)$$

## 2. RFT on a Tree

Despite their usefulness [9], the above introduced objects are not yet tailored to fit the complexity of situations arising in the study of functions of more than two replicas. We need to extend the RFT definition to the case where the two replicas (*i.e.* the overlap) summed over, move on a (ultrametric) tree in the presence of other *passive* overlaps. The simplest example is the 3-replica function (see below Sect. 4)

$$f^{\alpha\beta;\mu} \equiv f_t^r \quad (16)$$

where we have a (fixed) overlap  $r = \alpha \cap \beta$ , and the *lower index*,  $t$  is the *cross-overlap*  $t$

$$\begin{aligned} \max(\alpha \cap \mu; \beta \cap \mu) &= t \\ \alpha \cap \beta &= r \end{aligned} \quad (17)$$

As we move  $\mu$  *i.e.*  $t$  along the 3-replica tree, we now have the successive multiplicities

$$\begin{aligned} p_t - p_{t+1} & t < r \\ p_t - 2p_{t+1} & t = r \\ 2(p_t - p_{t+1}) & t > r \end{aligned} \quad (18)$$

Indeed, in the case  $t = r$ , there are two boxes of size  $r + 1$  excluded for  $\mu$  (the box  $p_{r+1}$  of  $\alpha$ , and the one of  $\beta$ ). In the case  $t > r$  there are two branches of the tree available for  $\mu$  ( $\alpha \cap \mu = t$ ,  $\beta \cap \mu = r$  or  $\alpha \cap \mu = r$ ,  $\beta \cap \mu = t$ ). It is thus appropriate to define effective boxes in the presence of a passive, direct overlap  $r$ , viz.  $p_t^{(r)}$  with

$$\begin{aligned} p_t^{(r)} &= p_t & t \leq r \\ p_t^{(r)} &= 2p_t & r < t \end{aligned} \quad (19)$$

and the associated RFT on the 3-replica tree

$$\widehat{A}_k^{(r)} = \sum_{t=k}^{R+1} p_t^{(r)} \left( A_t^{(r)} - A_{t-1}^{(r)} \right) \quad (20)$$

$$A_j^{(r)} = \sum_{t=0}^j \frac{1}{p_t^{(r)}} \left( \widehat{A}_t^{(r)} - \widehat{A}_{t+1}^{(r)} \right) \quad (21)$$

The RFT involved in this paper will always concern *cross-overlaps*, i.e. *lower indices*. In the case we are on a 4-replica tree with two passive, direct overlaps, e.g. for  $r < s$  we shall use  $p_t^{(r,s)}$  (see Sects. 3.2, 4.1) with

$$\begin{aligned} p_t^{(r,s)} &= p_t & t \leq r \\ p_t^{(r,s)} &= 2p_t & r < t \leq s \\ p_t^{(r,s)} &= 4p_t & r < s < t \end{aligned} \quad (22)$$

and obvious extensions for  $r = s$  or when more passive overlaps are involved.

### 3. Kernels as RFT on a Tree

Consider the 4-replica matrix  $M^{\alpha\beta;\gamma\delta}$  which we choose to parametrize as follows:

- (i) on the *Replicon-like* configurations of the 4-replica tree: i.e.  $\alpha \cap \beta \equiv \gamma \cap \delta = r$

$$M^{\alpha\beta;\gamma\delta} = M_{u,v}^{r;r} \quad (23)$$

with the lower indices

$$\max(\alpha \cap \gamma, \alpha \cap \delta) = u,$$

$$u, v \geq r + 1$$

$$\max(\beta \cap \gamma, \beta \cap \delta) = v$$

- (ii) The other configurations of the 4-replica tree belong exclusively to the so called *Longitudinal-Anomalous* (LA) component

$$M^{\alpha\beta;\gamma\delta} = M_t^{r;s} \equiv {}_A M_t^{r;s} \quad (24)$$

where  $\max(\alpha \cap \gamma, \alpha \cap \delta, \beta \cap \gamma, \beta \cap \delta) = t$ , and where it may happen, accidentally, that  $r = s$ .

The upper indices take values  $0, 1, \dots, R$  (for the special problem considered here,  $R + 1 \equiv (\alpha \cap \alpha)$  is excluded). Lower indices take values  $0, 1, \dots, R + 1$ . These two sets of variables will also be referred as direct overlaps and cross-overlaps, respectively.

We now show, that the “kernels” defined in reference [4,5], in terms of which it was possible to invert the 4-replica matrix  $M^{\alpha\beta;\gamma\delta}$ , are nothing but appropriate RFT’s on *lower indices* (cross-overlaps).

3.1. THE REPLICON COMPONENT  ${}_R M$ . — It was recognized quite early [11, 12] that  ${}_R M_{u;v}^{r;r}$  (or the corresponding component of the inverse *i.e.* of the propagator  ${}_R G_{u;v}^{r;r}$ ) was obtained *via* some double transform of a more elementary object, the “kernel”  $K_{k;l}^{r;r}$  (or  $F_{k;l}^{r;r}$  for the propagator). The kernel, identified with the Replicon eigenvalue  $\lambda(r; k, l)$  (see below), was indeed explicitly written as [13]

$$\begin{aligned} \text{kinetic term} + \lambda(r; k, l) &\equiv K_{k;l}^{r;r} = \\ &= \sum_{u=k}^{R+1} p_u \sum_{v=l}^{R+1} p_v (M_{u;v}^{r;r} - M_{u-1;v}^{r;r} - M_{u;v-1}^{r;r} + M_{u-1;v-1}^{r;r}), \quad k, l \geq r + 1 \end{aligned} \quad (25)$$

which we recognize now as a double RFT. Note that

- (i) we have written  $M$  instead of  ${}_R M$ . We shall see below why this does not make any difference;
- (ii) we have not used the RFT on the tree (with passive  $r$ ) since here  $k, l \geq r + 1$ , rendering its use trivial (and pedantic).

The direct relationship is then obtained by inverting the double RFT,

$${}_R M_{u,v}^{r;r} = \sum_{k=r+1}^u \frac{1}{p_k} \sum_{l=r+1}^v \frac{1}{p_l} (K_{k;l}^{r;r} - K_{k+1;l}^{r;r} - K_{k,l+1}^{r;r} + K_{k+1,l+1}^{r;r}) \quad (26)$$

This was, unknowingly, the type of relationship between  ${}_R G_{u;v}^{r;r}$  and  $F_{k;l}^{r;r}$  that first came out in the unravelling of the bare propagators (for the Parisi limit,  $R = \infty$ ) As already mentioned, analogous results have also been recently obtained, through the use of  $p$ -adic theory, by Parisi and Sourlas [6].

3.2. THE LA-COMPONENT  ${}_A M$ . — With the above defined RFT on the tree (22) we can now write

$$M_t^{r,s} = \sum_{k=0}^t \frac{1}{p_k^{(r;s)}} [K_k^{r;s} - K_{k+1}^{r;s}] \quad (27)$$

$$K_k^{r,s} = \sum_{t=k}^{R+1} p_t^{(r;s)} [M_t^{r;s} - M_{t-1}^{r;s}] \quad (28)$$

Corresponding equations may be written relating the propagator  $G_t^{r,s}$  to its kernel, obtained by RFT,  $F_k^{r;s}$ . Equation (28) generates the LA kernel  $K_k^{r;s}$ , including the limiting value  $r = s$ . Conversely, knowing the kernel everywhere, we can generate the mass operator  ${}_A M$  for all values of  $r, s$ .

In the Replicon configurations of the 4-replica tree we have two lower indices  ${}_A M_{u;v}^{r;r}$  and therefore

$${}_A M_{u;v}^{r;r} = \sum_{k=0}^{\max(u,v)} \frac{1}{p_k^{(r,u,v)}} (K_k^{r;r} - K_{k+1}^{r;r}) \quad (29)$$

Note that  $p_k^{(r,u,v)} \equiv p_k^{(r,\min(u,v))}$  in the interval bounded by  $\max(u, v)$ . From (29), it follows that

$${}_A M_{u;v}^{r;r} = {}_A M_u^{r;r} + {}_A M_v^{r;r} - {}_A M_r^{r;r}, \quad u, v \geq r + 1 \quad (30)$$

where  ${}_A M_t^{r;r}$  is given by (27) taken at  $r = s$ . This relationship explains why on the right hand side of (25) one may use, indifferently,  $M$  or  ${}_R M$ : indeed  ${}_A M_{u:v}^{r;r}$  where  $u, v \geq r + 1$ , is shown in (30) to depend upon a *single* <sup>(2)</sup> lower index at a time, it is thus projected out under a *double* RFT.

The equations given above for the kernels (25,28) can be taken as *defining the kernels* once the primary functions  $M$  are known (*e.g.* by loop expansion). It is shown below that the same objects then block-diagonalize the eigenvalue equations (Sect. 4) and the Dyson equation (Sect. 5).

#### 4. Eigenvalue Equations

We now turn to the eigenvalue equations for the matrix  $M^{\alpha\beta;\gamma\delta}$ . The three eigenvector classes, as introduced by de Almeida and Thouless [14], write  $f^{\alpha\beta}$ ,  $f^{\alpha\beta,\mu}$ ,  $f^{\alpha\beta,\mu\nu}$  for the L, A and R sectors respectively.

##### 4.1. THE L-SECTOR

$$\frac{1}{2} \sum_{\gamma\delta} M^{\alpha\beta;\gamma\delta} f^{\gamma\delta} = \lambda_L f^{\alpha\beta}, \quad (31)$$

here the sum is over all the configurations of the 4-replica tree. Spelling it out we get

$$\sum_{s=0}^R \left\{ \delta_{r;s}^{\text{Kr}} \sum_{u=r+1}^{R+1} \delta_u \sum_{v=r+1}^{R+1} \delta_v M_{u:v}^{r;r} + \frac{\delta_s}{2} \sum_{t=0}^{R+1} \delta_t^{(r;s)} M_t^{r;s} \right\} f^s = \lambda f^r \quad (32)$$

where the  $\delta_{r;s}^{\text{Kr}}$  term is the contribution of the Replicon-like configurations of the 4-replica tree.

In terms of RFT's, this writes

$$\sum_{s=0}^R \left\{ \delta_{r;s}^{\text{Kr}} K_{r+1,r+1}^{r,r} + \frac{1}{2} K_0^{r;s} \delta_s \right\} f^s = \lambda_L f^r \quad (33)$$

Here we have used

$$\delta_t^{(r;s)} \equiv p_t^{(r;s)} - p_{t+1}^{(r;s)} \quad (34)$$

and  $p_t^{(r;s)}$  as in (22). We have also used the fact that, for a sum carried over the *full* definition interval  $(0, R + 1)$  or when in the R-sector over  $(r + 1, R + 1)$ , one has

$$\sum_b^{R+1} \delta_t A_t = \sum_b^{R+1} p_t (A_t - A_{t-1}) = \widehat{A}_b \quad (35)$$

*i.e.* yielding the RFT at its lower bound  $b = 0$  or  $b = r + 1$  respectively. Hence in equation (32) the sum over  $\delta_u, \delta_v$  is a double RFT. Therefore it projects out  ${}_A M_{u:v}^{r;r}$  and yields the RFT at its lower bound value  $b = r + 1$  for  ${}_R M_{u:v}^{r;r}$ . For the single  $\delta_t^{(r;s)}$  sum, we recover the single RFT of  $M_t^{r;s}$  at its lower bound  $b = 0$ .

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<sup>(2)</sup>Note that if one wishes to attach a second *lower* index to  $G_t^{r;s}$  (*e.g.* for the sake of faithfully representing a propagator by one pair of lines) it has to be  $\min(r, s, t)$ .

## 4.2. THE A-SECTOR

$$\frac{1}{2} \sum_{\gamma\delta} M^{\alpha\beta;\gamma\delta} f^{\gamma\delta;\mu} = \lambda_A f^{\alpha\beta;\mu} \quad (36)$$

The idea is now the following: just like before we obtained simple relationships by taking RFT over lower indices, *i.e.* cross-overlaps, we now take one more step and do RFT on the convolution product of cross-overlaps. More precisely, we have the concatenations

$$\overset{\bullet}{\alpha}\beta;\overset{\bullet}{\gamma}\delta \quad \overset{\bullet\bullet}{\gamma}\delta;\overset{\bullet\bullet}{\mu} \quad \text{and} \quad \overset{\bullet}{\alpha}\beta;\overset{\bullet}{\mu}$$

or

$$\alpha\overset{\bullet}{\beta};\overset{\bullet}{\gamma}\delta \quad \overset{\bullet\bullet}{\gamma}\delta;\overset{\bullet\bullet}{\mu} \quad \text{and} \quad \alpha\overset{\bullet}{\beta};\overset{\bullet}{\mu}$$

and two other possibilities depending on the configurations of the 5-replica tree. The replica pairs with alike dots are those whose (cross-)overlap is to be Replica Fourier transformed. We sum first over the 5-replicas, at  $\alpha \cap \beta = r$ ,  $\gamma \cap \delta = s$ , fixed and passive, to extract the  $k$  RFT component (as in going from (7) to (8) ) and then we sum over  $s$  to obtain

$$\sum_{s=0}^R \left\{ \delta_{r;s}^{Kr} K_{k;r+1}^{r,r} + \frac{1}{4} K_k^{r;s} \delta_s^{(k-1)} \right\} \widehat{f}_k^s = \lambda_A \widehat{f}_k^r. \quad (37)$$

Note that

- (i) the summation over the concatenated lower indices yields the product  $K_k f_k$  (as in (8)).
- (ii) in the Replicon subspace, there is a *second* lower index which is *freely* summed over its complete range  $(r+1, R+1)$  hence yielding the  $b = r+1$  component of  $K^{r;r}$  (as in (35))
- (iii) the  $s$  upper index summation, at fixed (RFT) cross-overlap  $k$ , comes out as  $\delta_s^{(k-1)}$
- (iv) the “monochromatic” RFT  $\widehat{f}_k^r$  corresponds to an eigenvector (see (19-21))

$$f_t^r = \begin{cases} 0 & t < k-1 \\ -\frac{1}{p_{k-1}^{(r)}} \widehat{f}_k^r & t = k-1 \\ \left( \frac{1}{p_k^{(r)}} - \frac{1}{p_{k-1}^{(r)}} \right) \widehat{f}_k^r & t > k-1 \end{cases} \quad (38)$$

that is *null* for the cross-overlaps smaller than the *resolution*  $k$ , and *independent* of the *cross-overlap* when larger than  $k$ .

- (v) The L case identifies with  $k = 0$  and

$$f^r = \frac{1}{p_0} \widehat{f}_0^r \quad (39)$$

- (vi) The full multiplicity associated with a given  $\widehat{f}_k$  is

$$\mu(k) = n \bar{\delta}_k = n \left( \frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \quad (40)$$



## 4.3. THE R-SECTOR

$$\frac{1}{2} \sum_{\gamma\delta} M^{\alpha\beta;\gamma\delta} f^{\gamma\delta,\mu\nu} = \lambda_R f^{\alpha\beta;\mu\nu} \quad \alpha \cap \beta \equiv \gamma \cap \delta \quad (41)$$

If  $\mu \cap \nu \neq \alpha \cap \beta$ , then there is a single (independent) cross-overlap, *i.e.* those eigenvectors belong to the LA subspace. We thus have necessarily

$$\alpha \cap \beta \equiv \gamma \cap \delta \equiv \mu \cap \nu \quad (42)$$

and we now have a *double* set of concatenations, *e.g.*

$$\overset{\bullet}{\alpha}\beta; \overset{\bullet}{\gamma}\delta \quad \overset{\bullet\bullet}{\gamma}\delta; \overset{\bullet\bullet}{\mu}\nu \quad \text{and} \quad \overset{\bullet}{\alpha}\beta; \overset{\bullet}{\mu}\nu$$

where one set is exhibited and the other is its complement. Altogether four double sets depending on configurations of the 6-replica tree (restricted to its Replicon sector through (41)). The double RFT associated (block)-diagonalizes the eigenvalue equations (41) (the blocks are here of dimension  $1 \times 1$ , instead of  $(R+1) \times (R+1)$  in the LA sector):

$$K_{k,l}^{r;r} f_{kl}^r = \lambda_R f_{kl}^r \quad (43)$$

$$K_{k;l}^{r;r} \equiv \text{kinetic term} + \lambda(r; k, l), \quad k, l \geq r+1 \quad (44)$$

The associated multiplicities can be written in term of  $\bar{\delta}_k$  the (relative) multiplicity in pseudo-momentum or resolution space, with the proviso that, at the lower bound  $b$  of the interval of definition, we have

$$\bar{\delta}_b = \frac{1}{p_b} \quad (45)$$

$b = 0$  as for (6), for the LA sector, and  $b = r+1$  for the R sector of direct overlap  $r$ . Again, using  $p_k^{(r)}$  instead of  $p_k$  would amount to some trivial change in the  $\bar{\delta}_k$  since  $k, l \geq r+1$ . One finds

$$\mu(r; k, l) = \frac{n}{2} \bar{\delta}_k \bar{\delta}_l \delta_r(k, l) \quad (46)$$

where  $\delta_r(k, l) = p_r - (\alpha + 1)p_{r+1}$  and  $\alpha = 0, 1, 2$  is the occupation of boxes  $p_{r+1}$  by the cross-overlaps  $k, l$ . For further use, this is conveniently separated as

$$\mu \equiv \mu_{\text{reg}} + \mu_{\text{sing}} \quad (47)$$

$$\mu_{\text{reg}}(r; k, l) = \frac{n}{2} \bar{\delta}_k \bar{\delta}_l \delta_r \quad (48)$$

$$\mu_{\text{sing}}(r; k, l) = \begin{cases} 0 & k, l > r+1 \\ -\frac{n}{2} \bar{\delta}_l & k = r+1, l > r+1 \\ -n \sum_{s=0}^{r+1} \bar{\delta}_s & k, l = r+1 \end{cases} \quad (49)$$

The total degeneracy becomes

$$\sum_{r=0}^R \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \mu_{\text{reg}}(r; k, l) = \frac{n(n-1)}{2}$$

$$\sum_{r=0}^R \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \mu_{\text{sing}}(r; k, l) = -n(R+1) \tag{50}$$

and for the LA sector, from the  $(R+1) \times (R+1)$  blocks

$$(R+1) \sum_{k=0}^{R+1} \mu(k) = +n(R+1) \tag{51}$$

yielding back the appropriate count.

**5. Block-Diagonalization of the Dyson Equation**

Whatever has been found above for the matrix  $M$  and the associated kernels  $K$  ( $K_{k;l}^{r;r}$  in the R-sector,  $K_k^{r;s}$  in the LA one) can be repeated for the inverse matrix  $G$  (the propagator matrix) and the associated kernels  $F$ . Expressing their relationship *via*

$$\sum_{\gamma\delta} M^{\alpha\beta;\gamma\delta} G^{\gamma\delta;\mu\nu} = \delta_{\alpha\beta;\mu\nu}^{\text{Kr}} \tag{52}$$

one is now able to block-diagonalize this matrix equation. To do this, one uses

- (i) a double RFT in the Replicon sector yielding

$$K_{k;l}^{r;r} F_{k;l}^{r;r} = 1 \tag{53}$$

a result which could also be obtained *via*  $p$ -adic theory according to Parisi and Sourlas [6].

- (ii) a single RFT in the LA sector, *viz*

$$\sum_{t=0}^R \left( \delta_{r;t}^{\text{Kr}} \Lambda_k(r) + \frac{1}{4} K_k^{r;t} \delta_t^{(k-1)} \right) \left( \delta_{t;s}^{\text{Kr}} \frac{1}{\Lambda_k(t)} + \frac{1}{4} F_k^{t,s} \delta_s^{(k-1)} \right) = \delta_{r;s}^{\text{Kr}} \tag{54}$$

Here

$$\Lambda_k(r) = \begin{cases} K_{r+1;r+1}^{r;r} & \text{if } k \leq r+1 \\ K_{r+1;k}^{r;r} & \text{if } k > r+1 \end{cases}$$

and in (54) we have used (53) to obtain  $1/\Lambda_k(r)$ .

After division by  $\delta_s^{(k-1)}/4$  one gets the Dyson's equation, relating the kernels of inverses

$$F_k^{r;s} = -\frac{1}{\Lambda_k(r)} K_k^{r;s} \frac{1}{\Lambda_k(s)} - \sum_{t=0}^R \frac{1}{\Lambda_k(r)} K_k^{r;t} \frac{\delta_t^{(k-1)}}{4} F_k^{t;s} \tag{55}$$

or

$$\bar{F}_k^{r,s} = K_k^{r;s} - \sum_{t=0}^R K_k^{r;t} \frac{\delta_t^{(k-1)}}{4\Lambda_k(t)} \bar{F}_k^{t;s} \tag{56}$$

if

$$\bar{F}_k^{r,s} \equiv -\Lambda_k(r) F_k^{r,s} \Lambda_k(s) \tag{57}$$

Note that if one writes out explicitly  $\text{tr } MG$ , one has to sum over multiplicities, and an *exact* cancellation occurs then between the terms in

$$\mu_{\text{sing}}(r; k, l) K_{k;l}^{r;r} F_{k;l}^{r,r}$$

coming from the R-sector, and the terms

$$\mu(k) K_{k;r+1}^{r,r} F_{k;r+1}^{r,r}$$

originating from the LA-sector.

## 6. Conclusion

Assuming that *via* perturbation expansion we have computed  $M$  up to some loop order then, to obtain the corresponding  $G$  one would do the following:

- (i) compute the kernels associated with  $M$ :  $K_{k,l}^{r;r}$  *via* (25) and  $K_k^{r;s}$  *via* (28)
- (ii) obtain the corresponding kernel associated with  $G$ , the inverse of  $M$ , *viz*  $F_{k;l}^{r;r} = 1/K_{k,l}^{r;r}$  *via* (53) and  $F_k^{r;s}$  *via* a solution of (55). This last step has been shown to be analytically feasible [4] if, as it turns out to be at the zero loop level,  $K_k^{r;s} = K_k(\min r, s)$
- (iii) knowing  $F_{k;l}^{r;r}$  one obtains  ${}_R G_{u,v}^{r,r}$  *via* (26, transposed for  $F, G$ ). The knowledge of  $F_k^{r;s}$  gives  ${}_A G_{u,v}^{r,r}$ ,  $u, v \geq r+1$ , *via* (29, transposed for  $F, G$ ) and the other components of  ${}_A G_t^{r;s}$  *via* (27, transposed for  $F, G$ ).

The sum over multiplicities in “resolution” space generated by the inverse RFT directly constructs  $\mu_{\text{reg}}(r; k, l)$ . The cancellations described just above, that occur between R contributions with  $\mu_{\text{sing}}$  weight and the (Replicon-like) contributions arising as diagonal terms in the block-diagonalized LA sector are already taken care of since, *via* the RFT on the tree, we directly (block) diagonalize the eigenvalue equation and the Dyson equation.

To conclude, let us emphasize that the “conservation law” <sup>(3)</sup> on pseudo-momentum ( $k$ , or  $k$  and  $\ell$  in the double RTF) which allows for the “mass-operator” diagonalization (just like the ordinary Fourier Transform under translational invariance) should play a central role in the derivation of the much wanted “Feynman Rules” for the field theory of the spin glass.

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<sup>(3)</sup> On the ultrametric tree! *I.e.* resulting in the pseudo-momentum equality for a 2-point function, but ultrametric inequalities for a  $p$ -point function,  $p \geq 3$ .

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