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Objective Determination of Branching Angles in Fractals

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Abstract. — We propose an objective pattern recognition procedure to determine branching angles in 2-dimensional fractals. Recognition is performed by correlating the object with V-shaped identifiers. We define a quantitative criterion to prove the existence of characteristic branching angles in such figures. The whole procedure is tested on a mathematical (snowflake) fractal and then applied to a few physical examples. A neat characteristic angle is found in dendritic aggregates, in agreement with visual impression. However, the procedure does not find any dominant branching angle in basic Witten and Sander aggregates.

1. Introduction

From a qualitative point of view, fractals are highly ramified structures featuring structural details and branch splitting whatever the scale. A more rigorous statement is that the mass distribution in fractals exhibits a dilation invariance [1]. Mass scaling is a direct consequence of this property and is characterized by a set of fractal dimensions [2].

Figure 1, a to c, shows three examples of 2-dimensional fractals. Figure 1a is a so called snowflake (or Vicsek’s) [3] fractal, which is built through a simple mathematical construction rule. Figure 1b shows a classical Witten-Sander (WS) [4] aggregate, which is a computer simulation of a diffusion limited growth process. Here the elementary growth step involves a random walker (a particle undergoing Brownian motion) which ultimately collides the already existing cluster at some site on its boundary. The rule is that the particle remains bound to the cluster at this site. In other words a lattice site contiguous to the cluster boundary is occupied as soon as it is visited by a random walker. Figure 1c shows a “noise reduced” WS aggregate (NRWS). The rule here is that for the same site to be occupied it has to be visited \( n \) times \( (n \gg 1) \). In the example shown, \( n = 3000 \). This rule obviously favors sites with large growth probability. This leads to typical dendritic shapes.

WS aggregates are realizations of the most general process of diffusion limited aggregation (DLA). These structures have attracted a considerable interest in the recent years because many growth phenomena in very different fields (electro-chemistry, fluid mechanics, bacteria proliferation...) pertain to DLA through the same first principles (the so-called Stefan problem

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Fig. 1. — Examples of 2d fractals. a) Snowflake. b) Regular Witten and Sander aggregate. c) Noise reduced Witten and Sander aggregate. d) Modified snowflake. The snowflake has 5 generations. The WS and NRWS aggregates were built by an off-lattice procedure, and contain 25000 and 9000 particles, respectively. The ordinary snowflake (a) is generated from an initial square, whose sides are cut in $p = 3$ equal parts. This gives $p^2 = 9$ smaller squares, from which only $q = 5$ ones (the central and the corner ones) are kept. The same procedure is repeated at infinitum, which gives a self-similar figure whose fractal dimension is equal to $\ln q / \ln p = 1.464\ldots$. The modified snowflake is built in a similar way, with $p = 5$ and $q = 13$. This gives $d_F = 1.59\ldots$

This problem, although it can be very simply stated (a Laplace equation with a moving boundary) has no analytical solution and, up to now, has resisted all attempts of numerical resolution. WS aggregates are solutions of Stephan’s problem obtained by simulation.

Considerable effort has been devoted to characterizing the structure of these solutions, with the ultimate goal of elaborating a theory in the form of a construction rule (as for mathematical fractals). Such a theory should of course reproduce all known morphological characteristics of the aggregates. Reproducing the value of the fractal dimension ($d_F \sim 1.60$) [5,6] is not enough: this is evident from Figure 1d. There is shown a mathematical fractal, which was generated using a construction rule similar to that of the snowflake (see caption of the figure for details). This figure has a fractal dimension $d_F = \ln 13 / \ln 5 \approx 1.6$. It is obvious that WS aggregates and the structure shown in Figure 1d have completely different morphologies, although they have nearly equal fractal dimensions.

Diffusion limited growth appears everywhere directed, a property which is equivalent to
stating that ramification (or branching) angles are smaller than 90°. The distribution of branching angles is a basic morphological property of a fractal figure. This property is different from mass scaling in that it addresses the anisotropy of fractal mass distribution.

In Figure 1a, there is an obvious dominant angle equal to 90°, which is the direct consequence of the snowflake’s construction rule. Only very few attempts have been made to measure angles in WS aggregates, probably because of the difficulty of defining angles in a way which is both unambiguous and objective. A look at Figure 1b is sufficient to convince oneself of the difficulty of the problem. Ossadnik [7] used a nonlinear filtering procedure to identify branch stems in WS aggregates and calculated angles between stems from their inertia tensors. Results turn out dependent on the branch “order” (see Ref. [7] for a definition of the order), and range between 85 and 38°. Recently, Arnéodo et al. [8,9] have used wavelet transform (see Section 3 of this paper for a definition and Figure 4d for illustration) and by visual inspection identified branches and measured angles between them. The angle distribution found using this procedure turn out to be roughly independent of the wavelet size (i.e. the resolution in the image) and has a maximum at \( \theta \approx 36° \). This finding, together with the uncovering of Fibonacci sequences [9] in the ramification of the aggregates, supports the conjecture that there is a hidden five-fold symmetry [10] in WS aggregates.

In this paper, we propose an objective procedure to define and determine characteristic angles in 2-dimensional fractal figures. At the present stage, the procedure is fully numerical, but it is based on classical methods in Optical Pattern Recognition [11]. The paper is organized as follows: in Section 2, we give the principles of the procedure, then we test it on a model situation and on the mathematical snowflake. As we will see, the branching points and the characteristic angles in these figures are correctly identified. The same recognition procedure is then applied to the WS and NRWS aggregates. In Section 3, we elaborate a modified version of the procedure, namely a “Wavelet Transform” version, which amounts to reducing the resolution in the object by bandpass filtering. As we will explain, a dominant angle is found in the NRWS aggregate whose value is not affected by resolution. Conversely, angle values found in the WS aggregate are not robust with respect to a resolution change or statistical fluctuations. These results are summarized and commented in the final Section 4.

2. Bare Procedure

We start from the simple idea of correlating a characteristic shape, that we will name an “identifier”, with the object to be investigated. We calculate:

\[
C(M, l, \theta, \Omega) = \rho(M) * V(M, l, \theta, \Omega)
\]

Here \( \rho(M) \) is the mass density of the object at point \( M \), and the symbol * stands for correlation product. In the practical situation of a finite size (or finite number of generations) fractal, \( \rho(M) \) is a binary function defined on a 512 × 512 square lattice, which is equal to 1 at any occupied site and equal to 0 otherwise.

\( V \) is the identifier. It is just a simple \( V \) (see Fig. 2a) made of two one-pixel-thick arms of length \( l \). \( \theta \) is the aperture angle of the \( V \) and \( \Omega \) defines its orientation relatively to a fixed direction. Because of discretization, the number of pixels in the identifier (its mass) depends on \( \Omega \). This artifact leads to systematic errors which we correct for by dividing the bare correlation product (Eq. (1)) by \( \int V^2(M) dM \). This definition of the normalizing factor is general and applies whatever the definition of \( V(M) \). In the simple case of a binary function, this is just the mass of the identifier.

Numerical calculations are performed on an Alliant VFX40 computer. For a given value of \( \Omega \), the function \( C(M, l, \theta, \Omega) \) is computed via the Fourier transforms of \( \rho \) and \( V \) (for this
Fig. 2. — Illustration of the branching angles determination procedure using a simple V-shaped identifier (a) on a test sample (b) made of a collection of disordered different Vs. For clarity and definition, the identifier in (a) has been magnified 5 times. Apertures in (b) are 20°, 40°, 60°, 85°, 90°, 100° and 120°, with 7 Vs in each case. The small wiggles on most of the V arms are due to discretization on the 512 × 512 square lattice (see text). (c) Contrast histogram obtained with a 60° identifier. For instance, \( p(\gamma_0, l, 60°) \) is the number of sites in Figure 2b where the contrast \( \gamma \) is equal to \( \gamma_0 \) within a ±\( \Delta \gamma \) margin. \( \Delta \gamma \) is chosen equal to \( \gamma_m/200 \), where \( \gamma_m \) is the maximum contrast in the image. (d) Different moments \( (n=5,10,15) \) of the contrast histograms versus identifier aperture. For each order, the maximum of \( E^{(n)}(l,0) \) has been normalized to 100.

For this purpose, we used a classical fast Fourier transform (FFT) algorithm. For each \( \theta \), we collect 360 such functions, for \( \Omega \) ranging from 0 to 360° with a 1° step. This demands about 1 hour of computing time. We calculate the correlation product at each point for all orientations of the identifier and define the contrast:

\[
\gamma(M,l,\theta) = \text{Max}|C|^2 - \text{Min}|C|^2 ,
\]

for \( 0 \leq \Omega \leq 2\pi \). The relevance of this quantity is understandable by looking at the test sample shown in Figure 2b. This sample is a collection of more or less entangled Vs with 7 different values of the aperture angle. Clearly \( C \) will take on large values whenever the identifier is brought in coincidence with a V of aperture \( \theta \) in the figure. A “branching point” in the figure is defined as a position of the identified vortex where \( C \) varies very much as a function of
Ω, namely where γ takes a “large” value. Here, quotation marks mean that we need a quantitative criterion to decide how large γ has to be. This is the subject of the following discussion.

In Figure 2d, it is obvious that “branching points” are just the V vortices. These high contrast points (sites) represent a very small mass fraction in the system. Most of the points have small contrast. This means that in a histogram of all contrasts (the set of γ values for all the occupied sites in the image) corresponding to a particular identifier, say \( p(\gamma, l, \theta) \), the branching points participate only in the histogram tail. This is clearly visible in Figure 2c, where 60° vortices give rise to a small bump in the tail of the \( p(\gamma, l, 60^\circ) \) graph. The large bump near \( \gamma = 0 \) is due to the many points in the figure which have low contrasts.

Our goal is to find the dominant branching angles in the figure. One way to do so is to count the number of points having a contrast higher than some threshold value \( \overline{\gamma} \) for each identifier aperture (θ) and then to plot this number versus θ. This amounts to weighing each histogram tail as a function of θ. However this procedure has the drawback of introducing an arbitrary parameter, \( \overline{\gamma} \), in the final result. A slightly different but smarter way is to calculate a high order moment of \( p \),

\[
E_n(l, \theta) = \int \gamma^n p(\gamma, l, \theta) d\gamma,
\]

and to plot \( E_n(l, \theta) \) versus θ. This is shown in Figure 2d, for different values of \( n \). As the moment order increases, the characteristic V angles 20, 40, 60, 85, 90, 100 and 120°, progressively show up as pronounced peaks. As \( n \) increases, the width of each peak progressively decreases down to about 1°, which is the practical limit set by discretization.

We will take the existence of these peaks for large \( n \) as our operational definition of dominant branching angles in the picture.

In Figure 2d, the large order moments tend to an asymptotic limit, which turn out to be the exact distribution of branching angles in the test figure. The existence of this asymptotic limit means that the result of our analysis can be stated independently of the value of \( n \), which in turn means that the way in which the tails of the histograms are “weighed” is of marginal importance.

Note that the coincidence of \( E_n(l, \theta) \) with the distribution of branching angles (in the \( n \to +\infty \) limit), which is visible in the test example, may not be a general property. Our point is just that if the distribution has a few well defined peaks, these will clearly appear in \( E_n(l, \theta) \) for sufficiently large \( n \).

We now proceed to applying our procedure to the fractals shown in Figure 1. Looking at Figure 1a, it is obvious that the number of resonances (or branching points) will depend on the value of the identifier size \( l \). Choosing the value of \( l \) amounts to selecting a number of generations, say \( N \), in the figure. \( N = 1, 2, 3, \ldots \) correspond to \( L/l \) equal to 1, 3, 9, . . . , i.e. \( 3^N \) (\( L \) is the overall size of the figure). From the snowflake’s construction rule [3], we expect the number of identified branching points to scale roughly as \( 5^x \), with \( x \) given by \( 3^x = L/l \). By “roughly”, we mean that this result is exact only for integer values of \( x \). The results shown in Figure 3a and 5b correspond to \( l \) equal to \( L/20 \).

This choice is rather arbitrary. We just choose the value of \( l \) far from the large and small cut-off lengths in the figure, i.e. in a size domain where all the figures are dilation invariant (this is the basic property of fractals). Thus we expect the conclusion of the analysis — in terms of dominant branching angles — to be independent of the value of the identifier size. We checked this by trying different values of \( l \), as we will explain.

Our results are displayed in Figure 3a to c. In the first case (a), \( E^{(10)}(\theta) \) and \( E^{(15)}(\theta) \) feature a very neat peak at 90°, which is the result one expects from the snowflake construction rule. This may be taken as a further test of the method. The \( p(\gamma) \) histogram for \( \theta = 90^\circ \) features a
Fig. 3. — Moments $E^{(n)}(l, \theta)$ of the contrast histogram as a function of identifier aperture angle. a) Mathematical snowflake. b) WS aggregate. c) Noise reduced WS aggregate. In (a), the different moments shown ($n = 5, 10, 15$) have all a maximum at $\theta = 90^\circ$.

distinct bump at high contrasts as in Figure 2c. This bump is made of 25 branching points. If the size of the identifier is reduced to $L/40$, we still find a unique maximum in $E^{(15)}(\theta)$ at $\theta = 90^\circ$ and 125 branching points. Notice that the ratio $25/125 = 1/5$ is equal to $3^{-d_F}$, with $d_F = \ln 5/\ln 3$, as expected. The other fairly clear result is with the noise reduced aggregate (Fig. 3c) where a dominant angle near $70^\circ$ progressively shows up as $n$ increases. This result is obtained with $l = L/20$ (where $L$ is the largest linear size of the object). The result with $l = L/40$ (not shown in the figure) is quite similar.

The bare WS aggregate (Fig. 1b) is obviously a more complex case. Several peaks show up in the $E^{(n)}(l, \theta)$ diagrams (Fig. 3b), but with no clear asymptotic convergence as $n$ increases. Notice that Figure 1b is just one realization of the statistical WS aggregate. We repeated the angle analysis on 10 different realizations. Each of them resulted in a multitude of peaks, but at positions depending on the particular realization. In other words, the peaks shown in Figure 3b are not stable with respect to the statistical fluctuations. As we will see in the following paragraph, these are also not robust with respect to the resolution set in the analysis procedure.

3. Wavelet Transform Version

At this stage we just blindly applied the V-identifier procedure to fractals. We found a quantitative (numerical) criterion that yields angle values, not a geometrical definition of angles in a complex figure. In fact there is no way of correctly defining angles in fractals, because of the existence (by definition) of structural details at all scales. One way of circumventing this difficulty is to observe the fractal object at finite spatial resolution, in other words to smear out all details smaller than an arbitrary length. This can be achieved by low-pass filtering, more generally by band-pass filtering.

In Figure 4 the mass densities of the different objects were convoluted by a Mexican hat, a function equal to the second derivative of a Gaussian function:

$$W_a(r) = \frac{1}{a^2} (2 - \frac{r^2}{a^2}) \exp(-\frac{r^2}{a^2}) ,$$

(4)
Fig. 4. — a) Scheme of the Mexican hat wavelet. b) to f): Isotropic wavelet components of the test sample (b), the snowflake (c), the WS aggregate (d), the noise reduced WS aggregate (e). (f) is the V-shaped identifier. The wavelet size parameter $a$ was chosen equal to 5 lattice sizes, which is about $1/8$ of the V-identifier arm length. The quantities represented in b) to f) are positive values, with a 16-level coding from white (0) to black (max $|T_a|$). For clarity, (a) and (f) are magnified views ($\times 10$ and $\times 5$, respectively).

where $r$ is the 2-dimensional coordinate. The parameter $a$ defines the width of the function (the full width at half maximum is equal to $1.33a$). The Fourier transform of $W_a(r)$ reads:

$$\tilde{W}_a(q) = |aq|^2 \exp(-\frac{1}{2}a^2q^2) ,$$

where $q$ is the 2-dimensional spatial frequency. Clearly, convolution by $W_a(r)$ acts as a band-pass filter (with smooth edges). In the context of signal analysis, the Mexican hat is known as a very simple example of a wavelet. This wavelet is "isotropic", because $W_a(r)$ depends only on the modulus of $r$. The convolution product

$$T_a(r) = \rho(r) \otimes W_a(r)$$

may be called an isotropic wavelet component of $\rho$. The whole set of wavelet components of $\rho$, for $a$ ranging from 0 to infinity, is the so-called Wavelet Transform (WT) of $\rho$.

Isotropic WT of 1d and 2d fractals has recently been the subject of considerable research [12,13] and shown to be a powerful tool to visualize structural hierarchy and to characterize
mass scaling \(^{(1)}\). There is a great flexibility in the choice of a particular function as a wavelet. Here the Mexican hat was selected for reasons of smoothness and numerical simplicity \(^{(2)}\). Figure 4f shows an isotropic wavelet component of the V-shaped identifier, which we will denote \(V_a\):

\[
V_a = V \otimes W_a
\]

\(V_a\) itself is an anisotropic wavelet (it depends on both the modulus and the direction of \(r\)), which we used as the angle identifier to analyse \(T_a\) in the different cases (Fig. 4b, 4c, 4d, 4e). We choose \(a/l = 1/8\) in this study. Here the choice of \(a\) fixes the resolution in the image to be processed. \(l\) has to be larger than this resolution limit for the identifier to be resolved as a V-shape. Notice that

\[
C_a = T_a \otimes W_a = \left[ \rho \otimes W_a^{(2)} \right] \otimes V
\]

where \(W_a^{(2)} = W_a \otimes W_a\) is just another isotropic wavelet whose size is still on the order of \(a\). Equation (9) means that the same \(V\) as before is used to investigate a finite resolution version of the image. This choice makes the procedure closest to the visual one used in reference \([8]\).

In the same spirit as in Section 2, we define in each case a contrast \(\gamma_a\), a contrast histogram \(\rho_a\) and the corresponding moments \(E_a^{(n)}\). The results are displayed in Figure 5a to 5d, for \(l = L/20\).

Notice that the results obtained with the test sample (Fig. 5a) and with the snowflake (Fig. 5b) are very similar to those shown in Figure 2d and 3a. Peak positions and widths are identical. There are small fluctuations between the peak heights in Figure 5a which are not present in Figure 2d. These fluctuations are due to the fact that there are different length scales in the test sample. For instance the branches of small angle (20°) Vs are close enough for their wavelet components to interfere. This implies that the wavelet versions of Vs of different aperture angles are qualitatively different. A complication of the same nature occurs between neighboring Vs, which also interfere at finite resolution.

These interferences do not greatly affect the behavior of the high order moments in the simple figures, namely the test sample, the snowflake and the noise reduced aggregate (Fig. 5d). In the latter case, we recover the dominant angle around 70°, which makes this result robust versus a resolution change. For the snowflake and NRWS aggregate, we also checked for self-similarity, as we did with the bare procedure. The results with \(l = L/20\) and \(l = L/40\) are essentially the same in terms of characteristic angles. The proliferation of branching points in the snowflake goes as \(l^{-d}\), as expected.

The situation is very different with the basic WS aggregate, because the peaks found in Figure 5c \((l = L/20)\) are uncorrelated to those found with the bare procedure (Fig. 3b). The same is true with \(l = L/40\). The set of angle values is not robust with respect to a change in the size of the identifier. As a practical conclusion, we state that our procedure identifies no significant angle in regular WS aggregates.

\(^{(1)}\) Studying self similarity in a signal requires eliminating any regular (not self similar) background from the signal. This task is achieved by bandpass, not by lowpass filtering. See for instance \([14]\).

\(^{(2)}\) On the other hand, the optical WT version which was developed in our laboratory \([15,16]\) is based on true (sharp edges) bandpass filtering, because this made the optical hardware simplest.
We defined an objective procedure to find significant angles in 2-dimensional pictures. The procedure is based on a classical pattern recognition by means of a V-shaped identifier, followed by calculation of high order moments in the contrast distribution. The procedure can be applied directly to the picture, or to an isotropic wavelet component of it.

The method was successfully tested on simple examples, such as a collection of simple V-shapes or a mathematical snowflake, and shown to produce angle values in full agreement with the expected ones.

We then applied the procedure to physical fractals with no a priori knowledge of the branching angles therein. In a noise reduced WS aggregate we found a conspicuous signal at $\theta \sim 70^\circ$, which was moreover found independent (robust) of the particular version of the procedure.

Conversely, we found no definite robust signal in the basic WS aggregate. This result should...
not be taken as a failure to detect a weak signal. Our statement that there is no dominant angle in WS aggregate is a significant result, which is to be understood in the context of pattern recognition.

This however does not mean that we definitely rule out the idea of a dominant (36°) [17] angle in such aggregates. In fact, a look at Figure 4 shows that the isotropic WT components of the fractal figure may be viewed as a collection of spots, say, “atoms”. In the snowflake, these atoms are obviously ordered in a square lattice. In the noise reduced aggregate they are closely packed in well defined rows. There is no such a well defined local order in the basic WS aggregate and accordingly no obvious way of defining local directions in the picture. These local directions are what one may term “branches”, and the collection of these branches is the fractal skeleton, at the scale of the wavelet used. We believe that the recognition of significant angles in a non elementary fractal first demands elaborating a skeleton, which is a highly nonlinear operation and then applying the recognition procedure to the skeleton. We are now currently investigating different ways of building a skeleton: this may be simply the collection of crestlines in one WT component or in a more sophisticated version be deduced from the proliferation of “atoms” when the wavelet scale is decreased.

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