



HAL
open science

Topological Models of 2D Fractal Cellular Structures

Gérard Le Caër, Renaud Delannay

► **To cite this version:**

Gérard Le Caër, Renaud Delannay. Topological Models of 2D Fractal Cellular Structures. Journal de Physique I, 1995, 5 (11), pp.1417-1429. 10.1051/jp1:1995207 . jpa-00247147

HAL Id: jpa-00247147

<https://hal.science/jpa-00247147>

Submitted on 4 Feb 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification
 Physics Abstracts
 05.50+q — 05.90+m

Topological Models of 2D Fractal Cellular Structures

G. Le Caër⁽¹⁾ and R. Delannay⁽²⁾

⁽¹⁾ Laboratoire de Science et Génie des Matériaux Métalliques(*), Ecole des Mines, F-54042 Nancy Cedex, France

⁽²⁾ Laboratoire d'Energétique et de Mécanique Théorique et Appliquée(**), Ecole des Mines, F-54042 Nancy Cedex, France

(Received 5 January 1995, revised 7 June 1995, accepted 8 August 1995)

Résumé. — Dans des structures cellulaires 2D à sommets trivalents qui remplissent l'espace et dans lesquelles une cellule partage au plus un côté avec toute autre cellule et aucun avec elle-même, la proportion maximum admissible de cellules à trois côtés est obtenue par une décoration de tous les sommets d'une structure initiale quelconque par des cellules à trois côtés. Des structures cellulaires "fractales" 2D sont ainsi engendrées si le processus précédent est répété à l'infini. D'autres méthodes de constructions de structures fractales sont également décrites. La distribution de probabilité $P(n)$ du nombre n de côtés des cellules ainsi que des corrélations de paires sont étudiées pour une structure cellulaire fractale construite à partir du tamis de Sierpinski. Au total, la répartition des cellules dans des structures cellulaires 2D avec $n \geq 3$ et $P(3) \neq 0$ évolue de manière régulière lorsque le désordre topologique, commodément représenté par la variance μ_2 de $P(n)$, s'accroît. Les fortes corrélations qui existent entre les cellules, en particulier dans les structures naturelles ($\mu_2 \lesssim 5$) diminuent progressivement quand μ_2 augmente, la répartition des cellules étant proche d'une répartition aléatoire pour $\mu_2 \sim 12$. Enfin les structures évolueraient vers des structures fractales, pour lesquelles μ_2 est infini, mais cette dernière transition reste encore à caractériser.

Abstract. — In space-filling 2D cellular structures with trivalent vertices and in which each cell is constrained to share at most one side with any cell and no side with itself, the maximum fraction of three-sided cells is produced by a decoration of vertices of any initial structure by three-sided cells. Fractal cellular structures are obtained if the latter decoration process is iterated indefinitely. Other methods of constructions of fractal structures are also described. The probability distribution $P(n)$ of the number n of cell sides and some two-cell topological properties of a 2D fractal cellular structure constructed from the triangular Sierpinski gasket are investigated. On the whole, the repartition of cells in 2D structures with $n \geq 3$ and $P(3) \neq 0$ evolve regularly when topological disorder, conveniently measured by the variance μ_2 of $P(n)$, increases. The strong correlations which exist among cells, in particular in natural structures ($\mu_2 \lesssim 5$), decrease progressively when μ_2 increases, a cell repartition close to a random one being reached for $\mu_2 \sim 12$. We argue that the structures finally evolve to fractal structures (for which μ_2 is infinite) but we have not characterized the latter transition.

(*) associé au C.N.R.S., U.R.A. 159

(**) associé au C.N.R.S., U.R.A. 875

1. Introduction

The ubiquity of space-filling random cellular patterns (2D or 3D) is one of their most fascinating characteristic. They are indeed encountered in numerous scientific fields from atomic to astronomic scale [1, 2], for example in biology (epidermal tissues), materials science (polycrystals and their 2D sections), fluid mechanics (2D and 3D soap froths), geography (administrative divisions) and even in astronomy (supercluster of galaxies). Their omnipresence explains the importance of a deep understanding of their scale independent properties and of their evolution due for instance to coarsens or to fragmentation (mitosis, etc..). Recent thorough investigations have been devoted to the characterization of the topological properties of cellular patterns, more particularly in 2D. Cells in natural disordered 2D structures have, in general, at least three sides and a rather regular geometric shape with trivalent cell vertices which thus belong to three cells.

The most evident topological one-cell characteristic is the probability distribution $P(n)$ of the number n of edges of cells. The average number of sides $\langle n \rangle = 6$ is a consequence of Euler's relation in 2D for trivalent vertices [1, 2]. The information conveyed in $P(n)$ is further condensed into some of its moments, first into its variance, $\mu_2 = \langle n^2 \rangle - 36$, which is one of the most convenient measure of topological disorder in random structures. Two-cell correlations are characterized by related quantities:

- 1) $M_k(n)$ and A_{kn} : a n -sided cell (n -cell) has on average $M_k(n)$ neighbours with k sides. The two-cell correlation A_{kn} , defined in equation (1) [3, 4], is related to the probability P_{kn} that a k -cell and a n -cell are neighbours:

$$A_{kn} = \frac{M_k(n)}{P(k)} = A_{nk} = \frac{6P_{kn}}{P(k)P(n)} \quad (1)$$

Both $M_k(n)$ [5] and A_{kn} help to appreciate the deviations of the arrangement of cells from that of uncorrelated cell distributions.

- 2) $nm(n)$: the mean total number of edges of cells adjacent to n -cells:

$$nm(n) = \sum_{k=3}^{\infty} kM_k(n) = \sum_{k=3}^{\infty} kP(k)A_{kn} = \langle kA_{kn} \rangle \quad (2)$$

The Weaire sum rule establishes the identity of $\langle nm(n) \rangle$ and of $\langle n^2 \rangle = \mu_2 + 36$ [1, 4]. The variance μ_2 plays a major role in "universal" or "quasi-universal" laws which have been found to hold in 2D structures:

- a) The "quasi-universal" relation between μ_2 and $P(6)$ [6] which can be expressed as: $\mu_2 P(6)^2 = 0.150 \pm 0.014$ for $0.1 \lesssim P(6) \lesssim 0.7$ [4, 7].
- b) The Aboav-Weaire law:

$$nm(n) = (6 - a)n + 6a + \mu_2 \quad (3)$$

which depends on a sole parameter "a" which is of the order of 1 in natural structures.

- c) A "quasi-universal" decrease of $\rho = a/\mu_2$ with μ_2 has been recently reported for 2D structures with overall homogeneous cell sizes and $P(3) \neq 0$ [8]. Negative values of ρ are obtained for large disorders with an asymptote $\rho = -1/6$ which corresponds to uncorrelated arrangements of cells.

The variance μ_2 varies from ≈ 0 to at most ≈ 5 in natural structures [8]. The studies of structures with disorder much larger than 5 rely entirely upon model and computer simulations. The largest disorder reached till now is $\mu_2 = 12.847$ [8]. It is observed for a 2D structure produced by a random fragmentation process. Some of the aforementioned simulated structures may even be considered as "pathological". Their unnatural characteristics are worth being investigated because they help to look at natural structures from other angles. The question of the existence of 2D structures still random with larger and larger topological disorder μ_2 may in particular be asked. The latter question may be of interest in relation to 2D quantum gravity [9–11] as numerical simulations of multiple Ising or Potts models on dynamical random graphs with trivalent vertices yield structures with large μ_2 (~ 12 [9]) or with the largest reported values of $P(3)$ ($\approx 1/3$, [10]). It is trivial to exhibit distributions $P(n)$ with infinite variances but we require furthermore the definition of ways of constructing the associated cellular structures. The purpose of the present paper is to describe methods to generate 2D cellular structures with infinite variances and to investigate some topological properties of one of them. A scheme of the evolution of 2D structures when μ_2 increases is also discussed.

2. Two Construction Methods

In the following all initial (Section 2.1) or "mother" (Section 2.2) structures on the one hand and all decorated (Section 2.1) or "daughter" (Section 2.2) structures on the other hand will be named respectively as S and D followed by some symbols.

2.1. VERTEX DECORATION. — We start from any structure S_0 , although "usual" structures with $\mu_2 < 5$ will in general be considered, characterized by a probability distribution $P^{(0)}(n)$ and by two-cell correlations $A_{kn}^{(0)}$. We replace every vertex of S_0 by a 3-sided cell. All vertices of the transformed structure are in turn decorated with 3-sided cells and we keep on iterating this vertex decoration process (Fig. 1). After i iterations, the edge distribution is $P^{(i)}(n)$ and the correlations are $A_{kn}^{(i)}$ and $nm^{(i)}(n)$. The number of sides of the cells of the initial structure are doubled and the cell weights are divided by 3 at every iteration. The distribution $P^{(i+1)}(n)$ at iteration $i + 1$ is deduced from $P^{(i)}(n)$ by:

$$\begin{cases} P^{(i+1)}(3) = \frac{2}{3} \\ P^{(i+1)}(2k) = \frac{1}{3}P^{(i)}(k), k = 3, \dots, \infty \\ \mu_2^{(i+1)} = 18 + \frac{4}{3}\mu_2^{(i)} \end{cases} \quad (4)$$

As proven in Section 4, the vertex decoration process, even when applied only once ($i = 1$), yields the maximum admissible value of $P(3)$ in space-filling cellular structures whose vertices are trivalent if moreover each cell is constrained to share at most one side with any cell and no side with itself. The variance μ_2 steadily increases with the number of iterations. After one iteration, it is already larger than the variance $\mu_2 = 12.847$ [8] reported in the introduction. If a "typical" random structure with $\mu_2 < 5$ is used as an initial structure, the product $\mu_2^{(1)}P^{(1)}(6)^2$ will be much smaller than the quasi-universal value 0.150 ± 0.014 as $P^{(0)}(3)$ is much less than 0.24 in such structures. For the fragmented structure with stationary topological properties, $\mu_2 = 12.847$ and $P^{(0)}(3) = 0.19052$, the product $\mu_2^{(1)}P^{(1)}(6)^2 = 0.142$ ($\mu_2^{(1)} = 35.129$) is by accident still consistent with the quasi-universal value. More generally, the value of the product $\mu_2^{(i)}P^{(i)}(6)^2$ is much larger than 0.150, being even larger than 2 after two iterations at most, whatever S_0 . It reveals already the peculiar character of the transformed structure. Equation (4) shows that the memory of the initial structure is lost very rapidly. Every cell of S_0 is progressively transformed into a cell with a larger and larger number of sides which

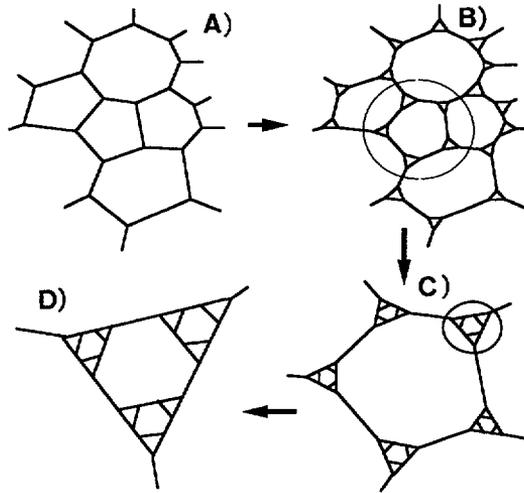


Fig. 1. — Some steps of the transformation of an initial 2D cellular structure S_0 by a repeated decoration of its vertices by triangular cells: A) S_0 ; B) first iteration; C) second iteration shown only for the circled cells of structure B; D) third iteration shown only for the circled cells of structure C. A 2D cellular structure D_{sg} (Fig. 2b) appears progressively at every vertex of the initial structure.

becomes infinite in the limiting structure. The limiting structure $D_{v\infty}$ produced by the vertex decoration method (Fig. 1) may be described in some way as a decoration of every vertex of S_0 by the “Sierpinski cellular structure”, noted here D_{sg} , shown in Figure 1D and in the right part of Figure 2b. A structure with a unique limiting distribution $P(n)$ is thus obtained. The associated probability distribution $P(n)$ is different from zero only for 3-sided cells and for even-sided cells $n = 3 \cdot 2^q$, $q = 1, \dots, \infty$. The power-law decrease of $P(n)$ is:

$$\begin{cases} P(n = 3 \cdot 2^q) = \frac{2}{3^{q+1}} = \frac{\beta}{n^\tau} \\ \tau = \frac{\text{Log } 3}{\text{Log } 2} = 1.58496 \dots, \beta = 2 \cdot 3^{\tau-1} = 3.80301 \dots, q = 0, \dots, \infty. \end{cases} \quad (5)$$

The variance of the limiting distribution is infinite:

$$\mu_2 = 6 \sum_{q=0}^{\infty} \left(\frac{4}{3}\right)^q - 36 \quad (6)$$

The exponent τ in equation (5) is the fractal dimension of the triangular Sierpinski gasket (Fig. 2b, [12–14]). We focus in the following on the limiting structure D_{sg} which springs up from any single trivalent vertex of S_0 by an infinite repetition of vertex decorations (Fig. 1D). A trivalent vertex (Fig. 1A) is transformed in that way into a triangular cell after one iteration (Fig. 1B). The latter triangular cell gives birth in turn to a six-sided central cell and to three triangular cells (Fig. 1C) and so on as shown in Figure 1D after three iterations. The distribution $P(n)$ associated with the structure D_{sg} is also given by equation (5). The relation between the limiting structure D_{sg} and the Sierpinski gasket (Fig. 2b) is explained in the next subsection. It is identical with the 2D fractal foam structure considered by Weaire and Kermodé [15], Weaire and Rivier [1] and Herdtle and Aref [16]. The latter authors were mainly interested by the evolution of such fractal foams governed by von Neumann’s law and their

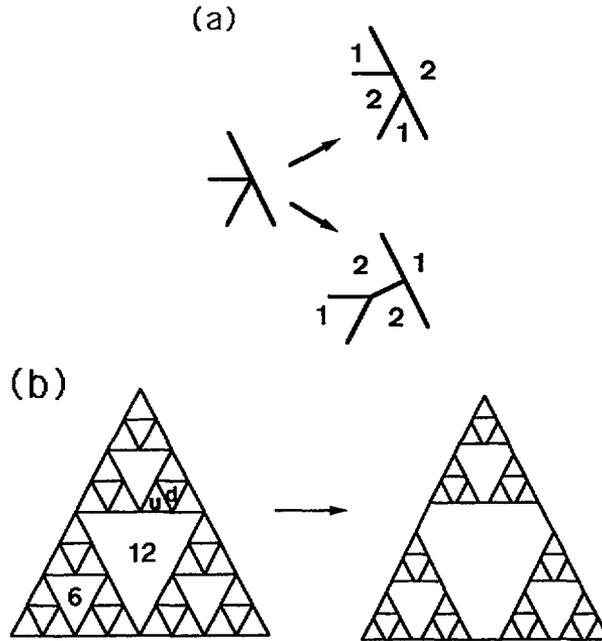


Fig. 2. — a) The two possible stable configurations (states (1212) and (2121)) obtained by adding a side at a tetravalent vertex; b) the 2D cellular structure D_{sg} (right part) associated with a Sierpinski gasket (left part) when all up-triangles (u) contain three state components equal to 1. Down-triangles (d) contain k state components equal to 2, where $k = 3 \cdot 2^q$ ($q = 0, \dots, \infty$) depends on the iteration at which the considered d-triangles have been generated. Three-sided cells (right part) are therefore associated with up-triangles while cells with $n = 3 \cdot 2^{q+1}$ sides are associated with all remaining down-triangles (two down-triangles with $k = 6, 12$ respectively are shown in the left part).

topological correlations were not investigated. Structure D_{sg} also plays an important role in some dynamical planar trivalent graphs considered in the problem of 2D quantum gravity [11]. It is finally of interest in a modern version of an old puzzle known as the tower of Hanoi puzzle [17].

2.2. REMOVAL OF TOPOLOGICAL INSTABILITIES OF “MOTHER” FRACTALS. — The D_{sg} structure and other “fractal” cellular structures can also be constructed iteratively by a second method already used to associate topological models of 2D cellular structures with “mother” lattices with topologically unstable sites [18, 19]. All vertices of a “mother” structure which belong to more than three polygons are topologically unstable. Vertices which belong to four polygons are of particular interest in the present problem and will be solely considered. The splitting of any tetravalent vertex into two trivalent vertices when it is replaced by a segment reflects the unstable nature of such vertices (Fig. 2a). A general method for removing the latter degeneracy for lattices with z -valent vertices ($z \geq 4$) and for constructing trivalent topological models of 2D cellular structures has been described by Le Caër [18, 19] (see also [3, 4]) In the case of tetravalent vertices, the replacement of a vertex by a segment produces two possible stable configurations (Fig. 2a), called states, which are characterized by two 4-dimensional vectors C_k ($k = 1, 2$), whose components C_{kj} ($j = 1, \dots, 4$) are either 1 or 2, namely $C_1 = (1212)$ and $C_2 = (2121)$. As the rule which allows removal of the degeneracy at any vertex does not create

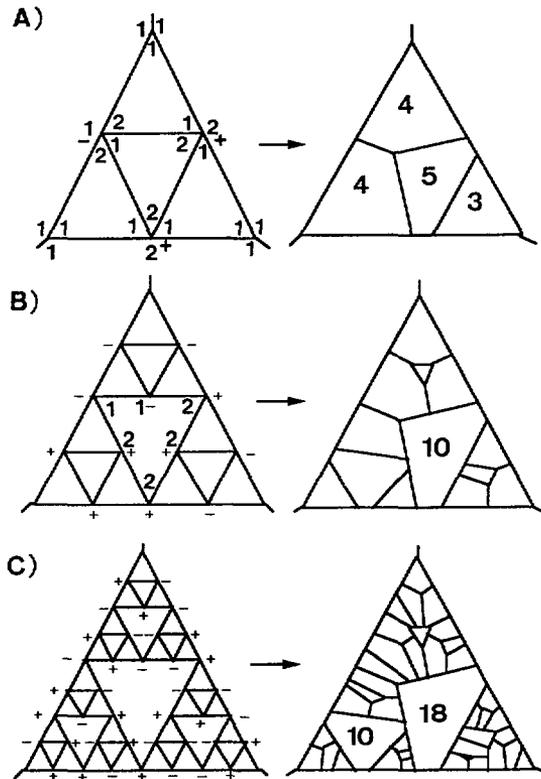


Fig. 3. — Daughter structures D_k (right part) with A) $k=1$, B) $k=2$, C) $k=3$ are associated with mother structures S_k (left part) by a degeneracy removal at every tetravalent vertex of S_k . A “+” state (resp. “-” state) at a tetravalent vertex of S_k is a state for which a component 1 (resp. 2) is found inside each of the two up-triangles at the considered vertex. The number of sides of a cell of D_k is the sum of the state components which are contained in the associated triangle of S_k . When going from S_k to S_{k+1} , states remain here unchanged at all vertices shared by S_k and S_{k+1} . Only cells with more than 9 sides have been indicated in B and C.

or annihilate cells, a cell of the topological cellular model is associated with every polygon of the mother structure. Every state component is the number of vertices that polygon number j will have at the considered unstable vertex in the final stable arrangement (Fig. 3A). The number of sides of a cell in the final configuration is thus the sum of the components C_{k_j} which are inside the subject polygon of the mother structure (Fig. 3).

We consider now the outer equilateral triangle of Figure 3A, named here up-triangle, which has trivalent vertices. Triangles which are related to the outer triangle by a dilatation are also named up-triangles (noted u, Fig. 2b). Triangles which coincide with the outer triangle after a dilatation and a reflection are named down-triangles (noted d, Fig. 2b). A 3-dimensional vector (111) with identical state components is consequently placed at every trivalent vertex of the outer triangle (Fig. 3A). The “mother” structure S_k is constructed iteratively: at iteration $k=1$, the outer triangle is divided into four smaller equilateral triangles (three up-triangles and one central down-triangle, Fig. 3A). The construction process is iterated: every up-triangle at iteration $k-1$ is divided into three up-triangles and one down-triangle. The limiting mother structure S_∞ defined in that way is a Sierpinski fractal. We define a “+” state (resp. “-” state)

at a tetravalent vertex as a state for which a component 1 (rsp. 2) is found inside each of the two up-triangles at the considered vertex (Fig. 3A). States are distributed at random, with a probability p of finding a “+” state, on all tetravalent vertices which have been created at iteration k (nine at iteration $k = 2, \dots, 3^k$ at iteration k). In that way, states are progressively placed on all $V = (3^{k+1} - 3)/2$ tetravalent vertices which exist at iteration k and a “daughter” cellular structure with trivalent vertices D_k (right part of Fig. 3) can be associated with a mother structure S_k (left part of Fig. 3). A different method, particularly useful if states are distributed in a correlated way, consists in distributing states on all vertices and not only on those just created at iteration k . There are 2^V possible daughter structures associated with all possible repartition of states on the vertices of S_k . In the case of a random distribution of states, the statistical weight of a given structure is the same for the two previous methods. In all cases, once states are distributed on the vertices of S_k , a cell of the “daughter” structure is associated with every up-triangle and with every down-triangle: its number of sides is given by the sum of all state components which are located inside the corresponding “mother” cell (Fig. 3). The topological properties of D_k may therefore be obtained. A given down-triangle contains a number of components $m = 3 \cdot 2^q$ where q ($0, \dots, \infty$) depends on the iteration at which the considered d-triangle has been generated (Fig. 2b). The “older” the d-triangle, the larger is m . The outer triangle contains $(3^{k+1} - 1)/2$ u- and d-triangles at generation k : 3^k u-triangles and 3^{k-q-1} d-triangles containing $m = 3 \cdot 2^q$ state components, $q = 0, \dots, k - 1$. For infinite k , the relative populations are $2/3$ for up-triangles and $2/3^{q+2}$ ($q = 0, \dots, \infty$) for down-triangles. The degeneracy removal at any vertex of the mother fractal is performed at random. The perfect limiting 2D cellular structure D_{sg} , whose characteristics are given by equations (5) and (6) is generated for $p = 1$ (Fig. 2b). More generally, a unique distribution $P(n)$ is obtained for a given p when k increases indefinitely:

$$\begin{cases} P(3) = \frac{2}{3}p^3 + \frac{2}{9}(1-p)^3, & P(4) = \frac{2}{3}p(1-p)(1+2p) \\ P(5) = \frac{2}{3}p(1-p)(3-2p) \\ n = 3 \cdot 2^q, q = 1, \dots, \infty \\ P(n) = \frac{\beta}{n^\tau} [p^{n/2} + \frac{1}{3}(1-p)^n] + \frac{2}{3}(1-p)^3 \delta_{n6} \\ P(n+k) = \frac{\beta}{3n^\tau} C_n^k p^k (1-p)^{n-k} \quad 0 < k < n \end{cases} \quad (7)$$

The variance of $P(n)$ is infinite whatever p as it is also for any distribution of states, random or not, on the vertices of the initial fractal. For $p = 0.5$, the distribution $P(n)$ is multimodal (Fig. 4) with an infinite number of modes of decreasing weights which are related to the variation of C_n^k with k for a fixed value of n .

A third construction method (Appendix A of [19]) related to the second one is worth being mentioned in connection with random triangulations and random surfaces [20]. A dual structure S_k^* (Fig. 5A for $k = 1$) of the mother structure S_k of Figure 3 can be constructed at every iteration k by connecting a point (named here center) inside every u-triangle (rsp. d-) to all centers of the d-triangles (rsp. u-) which share one side with it. The centers of “boundary” triangles which share one side with one of the three outer neighbouring cells defined by the largest triangle are all connected to the corresponding cell center (Fig. 5A). In that way, the dual structure S_k^* consists only of quadrilaterals and of three triangles associated with the three trivalent vertices of the largest triangle. The construction method is based on Euler’s diagonal triangulation (T) of every quadrilateral of the dual structure S_k^* by one of the two diagonals chosen at random with probability p . The structure S_k^* is thus transformed into a random triangulation TS_k^* whose dual is a structure D_k constructed by the previous method (Fig. 5B for $k = 1$).

The following generalizations of the two generation methods of fractal cellular structures described previously can be proposed:

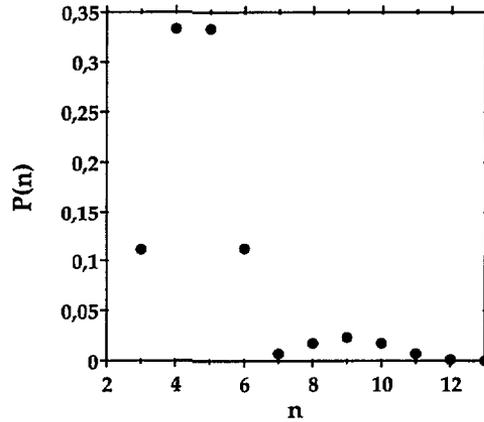


Fig. 4. — Distribution $P(n)$ of the number n of edges of cells for a 2D cellular structure $D_\infty(0.5)$ associated with a random distribution of states ($p = 0.5$, Eq. (7)) on the vertices of a Sierpinski gasket.

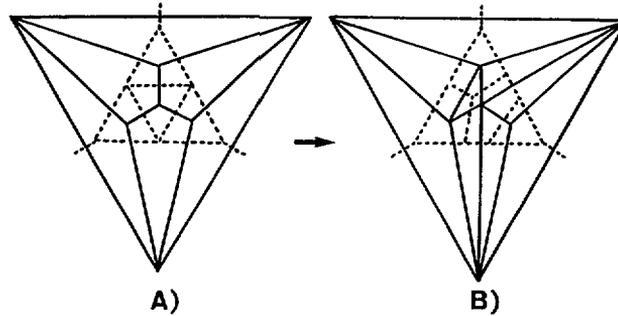


Fig. 5. — A) The dual structure S_1^* of the structure S_1 (dotted lines) of Figure 3A. B) Euler's diagonal triangulation (T) of every quadrilateral of S_1^* by one diagonal transforms it into a triangulation TS_1^* whose dual (dotted lines) is here the structure D_1 of Figure 3A.

- 1) The iterated vertex decoration by triangles can be performed either at random with a probability p_d or in some correlated way on every vertex of an initial structure, assumed here to have a finite μ_2 . A transition from structures with finite μ_2 to structures with infinite μ_2 may thus conceivably take place for a critical probability p_c if stationary topological properties result from such a process whatever p_d . The Sierpinski structure Dsg is recovered for $p_d = 1$. The random decoration of vertices is also a way of increasing the proportion of three-sided cells in a structure with the constraint that neighbouring cells share at most one side, that is with $A_{33} = 0$, etc.
- 2) The degeneracy removal can be extended to z -valent vertices of various "mother" fractals as done previously for periodic lattices [4, 18, 19]. It can be performed either at random or in a correlated way [3, 21].

3. Average Two-Cell Correlations in the D_{sg} Structure

In the D_{sg} structure (right part of Fig. 2b), a 3-cell has no 3-cell as a neighbours: $A_{33} = 0$. Consequently a n -cell has no n -sided neighbouring cell ($A_{nn} = 0$): starting from 3-cells at iteration i and decorating all vertices, we see that all 6-cells at iteration $i + 1$ are generated from the 3-cells living at the previous iteration and have thus no neighbouring 6-cells, etc.. Any n -sided cell ($n \geq 6$) has $n/2$ three-sided neighbouring cells. Consequently (Eq. (1)):

$$A_{3n}^{(i)} = \frac{3n}{4} \quad (n \geq 6) \tag{8}$$

whatever i . Equations (2), (4) and (8) yield:

$$3m^{(i)}(3) = \frac{3\mu_2^{(i)}}{4} + \frac{45}{2} \tag{9}$$

Similarly a n -sided cell ($n = 3 \cdot 2^q$ $q \geq 1$) has $M_{2k}(n)$ $2k$ -sided neighbouring cells with:

$$M_{2k}^{(i+1)}(n) = M_k^{(i)}\left(\frac{n}{2}\right) \text{ for } n = 3 \cdot 2^q, \quad q \geq 1 \tag{10}$$

Thus (Eq. (1)) ($n \geq 6$):

$$nm^{(i)}(n) = nm^{(i-1)}\left(\frac{n}{2}\right) + \frac{3n}{2} \tag{11}$$

It is immediately verified that equations (9) and (11) are consistent with the Weaire sum rule ($\langle nm^{(i)}(n) \rangle = \mu_2^{(i)} + 36$). We need to obtain $nm^{(i)}(n)$ as a function of $\mu_2^{(i)}$ to compare the two-cell correlations $nm(n)$ with the Aboav-Weaire law (Eq. (3)). This is easily done after observing that n -cells, $n = 3 \cdot 2^q$, living at generation i originate from 3-cells living at generation $i - q$. To derive the desired relation, it suffices therefore to express $\mu_2^{(i-q)}$ as a function of $\mu_2^{(i)}$. From:

$$\mu_2^{(i-q)} = \left(\frac{3}{4}\right)^q (\mu_2^{(i)} + 54) - 54 \tag{12}$$

and:

$$nm^{(i)}(n) = nm^{(i-q)}(3) + \frac{3nq}{2} \text{ for } n = 3 \cdot 2^q, \quad q = 0, \dots, i \tag{13}$$

we finally obtain for $i \geq q$:

$$nm^{(i)}(n) = \frac{2n^{\tau-1}}{2\beta} \left[\mu_2^{(i)} + 54 \right] + n \left[\frac{3 \text{Log}(n)}{2 \text{Log } 2} - \frac{3\tau}{2} - 6 \right], \quad n = 3 \cdot 2^q, \quad q = 0, \dots, i \tag{14}$$

As $\langle 3nq/2 \rangle = 18$, the average over n of the right term of equation (14) ($3nq/2 - 6n$) is -18 while the average of the left term is $\mu_2^{(i)} + 54$: the Weaire sum rule is verified as required. Relation (14) bears some resemblance to the Aboav-Weaire law as it is comprised of a term which includes μ_2 and of a term almost linear in n with a slowly varying $n \text{Log}(n)$ term. Moreover, in usual non-fractal cellular structures, the fractal dimension is $\tau = 1$ and the left term would indeed become independent on n . The exponent of n in the left factor, $\tau - 1$, suggests that the latter term shows a relation to sections of cell sides by lines thrown at random on the structure.

4. Discussion and Conclusion

The variances μ_2 of $P(n)$ are infinite for the aforementioned fractal cellular structures. It does not mean that such fractal structures are necessarily more disordered than structures with finite μ_2 . The randomness of $D_{v\infty}$ (Section 2.1) is indeed quite similar to the randomness of the initial structure S_0 even though S_0 may have a finite μ_2 . As discussed previously, structure $D_{v\infty}$ may be described as a structure S_0 whose vertices are decorated by D_{sg} structures. The algorithmic information content or the complexity of an object S is the amount of information necessary to describe S sufficiently precisely for it to be constructed [22]. The complexities of S_0 and of $D_{v\infty}$ do not differ much as the algorithmic content of the perfect structure D_{sg} is clearly small. The random fractal structures constructed by the method described in Section 2.2 are in that sense more "complex" than D_{sg} . A convenient measure of topological disorder in structures with infinite variance remain still to be defined.

The $P(n)$ distributions of cellular structures generated by computer simulations cannot have infinite variances. Large $P(3)$, μ_2 and $\mu_2 P(6)^2$ values may constitute signs of an underlying fractal structure. Probabilities of three-sided cells as large as $1/3$ are for instance reported for multiple Potts models on dynamical random graphs with trivalent vertices [10]. Supplementary indications may be obtained from the variations of $P(n)$ and of μ_2 when all three-sided cells are replaced by trivalent vertices and when the latter transformation of the structure is iteratively applied.

Although fractal cellular structures form a new family of structures, we argue below that they constitute an inevitable end for 2D structures when topological disorder becomes larger and larger. The following question may finally be asked for the class of homogeneous structures with finite variances: what is the most disordered structure still consistent with the quasi-universal relation $\mu_2 P(6)^2 = \text{constant}$ of the order of 0.15 which exhibits a smooth and unimodal distribution $P(n)$?

Although we have not solved the latter problem, we propose below a reasonable guess of the evolution of 2D structures when μ_2 increases. A natural way of increasing μ_2 more and more in a structure with a smooth and unimodal $P(n)$ and with $P(3) \neq 0$ is to keep on increasing $P(3)$. It is therefore important to determine the maximum acceptable value of $P(3)$ for any kind of space-filling structure whose vertices are trivalent if moreover each cell is constrained to share at most one side with any cell and no side with itself. The maximum number of 3-cells in a structure is obtained when every cell has a maximum number of three-sided neighbours. As a 3-cell cannot share a side with another 3-cell, the maximum number of neighbouring 3-cells of a n -cell with $n > 3$ is $\left[\frac{n}{2} \right]$ for $n \geq 5$ but only 1 for $n = 4$, where $[x]$ is the integer part of x . Relation (1) yields:

$$M_3(3) = 0, \quad M_3(4) = 1 = A_{34}P(3), \quad M_3(n) = \left[\frac{n}{2} \right] = A_{3n}P(3) \quad n \geq 5 \quad (15)$$

As obviously $\langle A_{3n} \rangle = 3$ [2-5], the value of $P(3)$ in such structures is obtained from:

$$\langle A_{3n} \rangle = 3 = \frac{1}{P(3)} \left[\sum_{n=4}^{\infty} \frac{n}{2} P(n) - \frac{1}{2} \left\{ 2P(4) + \sum_{k=2}^{\infty} P(2k+1) \right\} \right] \quad (16)$$

Finally:

$$P(3) = \frac{2}{3} - \frac{1}{9} \left\{ 2P(4) + \sum_{k=2}^{\infty} P(2k+1) \right\} \quad (17)$$

Relation (17) proves that the maximum value, $P(3) = 2/3$, is only reached in structures with $P(4) = P(2k+1) = 0$ ($k \geq 2$) that is in the decorated structures described in Section 2.1.

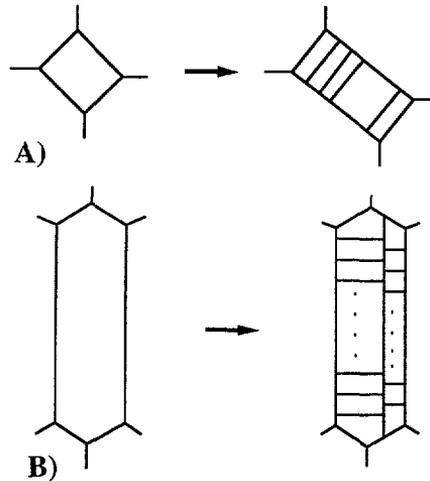


Fig. 6. — A) The transformation of a 4-cell into a 4-4-4-4 chain. B) The increase of the number of 5-cells by fragmentation of an initial 6-cell.

We notice that a distribution which maximizes the entropy $-\sum_{n=3}^{\infty} P(n) \text{Log}(P(n))$ subject to the sole constraints: $\langle n \rangle = 6$ and $\langle (n - 6)^2 \rangle = \mu_2$ is a discretized and truncated Gauss distribution [2, 5, 7] which exists only for $\mu_2 \leq 12$. In the latter case, the maximum value, $P(3) = 0.25$, is reached for $\mu_2 = 12$. A smooth distribution $P(n)$, which exhibits a mode at $n = 3$ or at $n = 4$ and which decreases too rapidly with n cannot indeed reach large values of μ_2 . By a lever effect, the constraint $\langle n \rangle = 6$ prevents $P(3)$ (and $P(4)$) from being too large. This explains the change from an exponential to a power law variation of $P(n)$ which is observed for large values of μ_2 [8]. Such a power law decrease also holds on the average as expected (Eqs. (5) and (7)) for the fractal structures discussed here. The previous arguments already suggest that 2D structures are unavoidably driven to fractal structures when μ_2 increases indefinitely. Three- and four-sided cells play a major role in the expected evolution because of the constraints acting on cells. Simple drawings immediately show that the repartition of 3-cells and of 4-cells are correlated as the latter cells can form at best isolated 3-4 pairs. In particular, any “molecular” chain 3-4-4-...-3 is forbidden whatever its length [4].

Other scenarios, although somewhat “pathological”, may be proposed to increase μ_2 indefinitely: the proportion of 4-cells or the proportion of 5-cells may be made as close to 1 as desired by transforming 4-cells into chains of 4-cells of any length (Fig. 6A) or by fragmenting iteratively 6-cells mostly into 5-cells respectively (Fig. 6B) etc... In the latter structures, the distributions $P(n)$ do not exhibit the general characteristics of usual $P(n)$ with finite variances: they are neither smooth nor unimodal. Fractal structures are also produced from an initial structure by various other transformations which involve for instance 5-cells.

A fruitful connection may be established between the problem raised in the present discussion and the problem of site percolation on fully triangulated planar graphs [23-25]. The critical percolation probability of fully triangulated graphs is $p_c = 1/2$ according to a conjecture of Sykes and Essam [23]. This conjecture does not hold for all graphs (see for instance [24]) but it is valid for statistically homogeneous and isotropic fully triangulated graphs [25]. Counter examples with $p_c = 1$ were indeed constructed [24]. If the sites of the dual triangulated structure of a given cellular structure which belong to three triangles are colored, percolation

will never occur as the constraints acting on cells would otherwise be violated. It would be nevertheless possible to define a kind of measure of the closeness D_p of a given structure to percolation by determining the extra concentration of randomly colored sites which results in percolation (D_p is for instance $1/2$ for $\mu_2 = 0$). We intuitively see that the smaller is D_p or the larger is $P(3)$, in particular the closer $P(3)$ is to $1/2$, the more difficult it will be to build homogeneous repartition of cells. The transformation of homogeneous triangulated graphs into triangulated graphs with much larger values of p_c will allow to increase more easily the number of 3-cells and of 4-cells of the dual structure. In this respect, it is worth observing that the dual triangulations which are associated with chains of 4-4 - - - 4 cells (Fig. 6A) are precisely the building blocks of one of the counterexample given by Wierman for which $p_c = 1$ (Fig. 3 of Ref. [24]).

The overall evolution of "usual" space-filling 2D cellular structures with $n \geq 3$ (and $P(3) \neq 0$), which is driven in fine by the constraints that we have imposed on cells, may finally be summarized in the following schematic way when μ_2 increases. Correlations in the repartition of cells ($\mu_2 \lesssim 5$) tend first to decrease as shown by the ratio a/μ_2 which decreases and becomes close to the ratio $-1/6$ (Sec. 1) expected for an uncorrelated distribution of cells. The evolution to a distribution as random as permitted by the constraints seems to be the rule for values of μ_2 typically larger than about 10. If μ_2 steadily increases, a transformation to more pathological structures with multimodal $P(n)$ and fractal characteristics is ultimately unavoidable. The latter evolution, has not been characterized in detail although some possible transformation paths (for instance local bursts of nucleation of 3-cells at vertices, chains of 4-cells,..) have been described. The problem of a transition from non-fractal to fractal cellular structures, a question first asked by Rivier (personal communication), is worth being investigated using the random vertex decoration method defined in the present paper.

Further studies are needed to know if equation (14) is a call for an extension of the Aboav-Weaire law which would include fractal cellular structures or if its form with terms in n and in $n^{\tau-1}$ is only valid for the particular structure under consideration.

Acknowledgments

We wish to thank N. Rivier (Université de Strasbourg) for useful discussions and suggestions. We also thank D. Johnston (Heriot-Watt University) for bringing the recent paper of M.G. Harris and J.F. Wheeler to our attention after the first submission of the present work and M.G. Harris (University of Oxford) for pointing out useful references on the tower of Hanoi puzzle, in particular the article of Stewart.

References

- [1] Weaire D. and Rivier N., *Contemp. Phys.* **25** (1984) 59.
- [2] Rivier N., Disorder and Granular Media, Chapter 3, D. Bideau and A. Hansen, Eds. (North-Holland, 1993) p. 55-102.
- [3] Delannay R., Le Caër G. and Khatun M., *J. Phys. A* **25** (1992) 6193.
- [4] Le Caër G. and Delannay R., *J. Phys. A* **26** (1993) 3931.
- [5] Peshkin M. A., Strandburg K. J. and Rivier N., *Phys. Rev. Lett.* **67** (1991) 1803.
- [6] Lemaître J., Gervois A., Bideau D., Troadec J. P. and Ammi M., *C.R. Acad. Sci. Paris* **315** (1992) 35.

- [7] Rivier N., From Statistical Mechanics to Statistical Inference and Back, P. Grassberger and J.P. Nadal, Eds. (Kluwer, 1994) p. 77-93.
- [8] Delannay R. and Le Caër G., *Phys. Rev. Lett.* **73** (1994) 1553.
- [9] Catterall S. M. and Kogut J. B., *Phys. Lett. B* **292** (1992) 277.
- [10] Baillie C. F. and Johnston D. A., *Phys. Lett. B* **286** (1992) 44.
- [11] Harris M. G. and Wheeler J. F., *Nucl. Phys. B* **427** (1994) 111.
- [12] Mandelbrot B. B., *The fractal geometry of nature* (W.H. Freeman, 1983).
- [13] Feder J., *Fractals* (Plenum Press, 1988).
- [14] Peitgen H. O., Jürgens H. and Saupe D., *Chaos and Fractals New Frontiers of Science* (Springer-Verlag, 1992).
- [15] Weaire D. and Kermode J.P., *Phil. Mag. B* **47** (1983) L29.
- [16] Herdtle T. and Aref H., *Phil. Mag. Lett.* **64** (1991) 335.
- [17] Stewart I., *Pour La Science* **142** (1989) 102.
- [18] Le Caër G., *J. Phys. A* **24** (1991) 1307.
- [19] Le Caër G., *J. Phys. A* **24** (1991) 4655.
- [20] Wheeler J. F., *J. Phys. A* **27** (1994) 3323.
- [21] Delannay R., Ito N., Khatun M. and Le Caër G., *Physica A* (1994) in the press.
- [22] Chaitin G. J., *The Maximum Entropy Formalism*, R. D. Levine and M. Tribus, Eds. (MIT Press, 1979) pp 477-98.
- [23] Sykes M.F. and Essam J.W., *J. Math. Phys.* **5** (1964) 1117.
- [24] Wierman J.C., *J. Phys. A* **17** (1984) 637.
- [25] Yuge Y. and Hori M., *J. Phys. A* **19** (1986) 3665.