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Random Field Ising Model: Dimensional Reduction or Spin-Glass Phase?

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Abstract. — The stability of the random field Ising model (RFIM) against spin glass (SG) fluctuations, as investigated by Mézard and Young, is naturally expressed via Legendre transforms, stability being then associated with the non-negativeness of eigenvalues of the inverse of a generalized SG susceptibility matrix. It is found that the signal for the occurrence of the SG transition will manifest itself in free-energy *fluctuations* only, and not in the free energy itself. Eigenvalues of the inverse SG susceptibility matrix are then investigated by the Rayleigh Ritz method which provides an upper bound. Coming from the paramagnetic phase on the Curie line, one is able to use a virial-like relationship generated by scaling the single unit length ($D < 6$; in higher dimension a new length sets in, the inverse momentum cut off). Instability towards a SG phase being probed on pairs of *distinct* replicas, it follows that, despite the repulsive coupling of the RFIM the effective pair coupling is *attractive* (at least for small values of the parameter $g\bar{\Delta}$, g the coupling and $\bar{\Delta}$ the effective random field fluctuation). As a result, “bound states” associated with replica pairs (negative eigenvalues) provide the instability signature. *Away from the Curie line*, the attraction is damped out till the SG transition line is reached and paramagnetism restored. In $D < 6$, the SG transition always precedes the ferromagnetic one, thus the domain in dimension where standard dimensional reduction would apply (on the Curie line) shrinks to zero.

After nearly twenty years of intense activity, there is yet no consensus on the critical behavior of random field systems (for recent reviews see [1, 2]).

A blatant contradiction arose when a calculation to all orders in perturbations [3, 4], later supported by a non perturbative approach [5], established dimensional reduction (between the RFIM in dimension D and the pure Ising system in $D - \theta$) both for hyperscaling relationships between critical exponents and for the exponents themselves as a function of D . With $\theta = 2$ this was predicting a lower critical dimension $D_\ell = 3$ for the existence of a ferromagnetic phase, in contradiction with an early Imry-Ma [6] argument predicting $D_\ell = 2$ (later supported by rigorous work of Imbrie [7], proving the existence of ferromagnetism in $D = 3$).

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Despite the fact that $D_\ell = 2$ is now widely accepted, there remains the question of down to which dimension are the resummed perturbation results valid, and what happens below that dimension.

Meanwhile several groups have proposed the existence of a glassy phase sector in the Δ, T plane (Δ is the width of the random field gaussian distribution and T the temperature) out of numerical studies [8–12] or from analytical work, extending to random field systems [13–16] the techniques of replica symmetry breaking [17] (RSB). In particular Mézard and Young [14] have used as a control parameter the number of components m (as in Bray [18] self consistent screening approximation) and written out explicit self consistent equations for the exponents.

Here, we follow the most straightforward approach to analyze properties of the RFIM, i.e., we mimick what is done in the pure system to describe the paramagnetic and condensed (ferromagnetic) phases.

We know since the work of Yvon [19] that the appropriate way to have access to the condensed phase is to replace the expansion in the local field H_i , by one in the local magnetization M_i , through a Legendre transform. The Jacobian of the transform $\det(\partial M_i/\partial H_j)$ vanishes at the transition (with the lowest eigenvalue of the matrix $\partial M_i/\partial H_j$), displaying the non-equivalence of the H_i and the M_i expansions.

Likewise here we consider the RFIM described by an effective hamiltonian with an external field Δ and perform the appropriate Legendre transform to the conjugate observable. Again the lowest eigenvalue of the jacobian matrix yields the locus of the singularities of the associated susceptibility, here the SG susceptibility, i.e., the line of the SG transition.

In Sections 1-2, we recall the perturbation expansion and effective hamiltonian for the RFIM. In Section 3, the Legendre transform is effected yielding stationarity conditions and eigenvalue equations for the SG transition. In Section 4 we study the transition and show that it manifests itself in the free-energy fluctuation (as contrasted with the standard SG). Section 5 is devoted to a study of the phase diagram using the Rayleigh-Ritz variational method [20] and we conclude in Section 6, where our results are summarized.

1. The Pure Ising System

Consider the pure Ising hamiltonian, in its soft spin version,

$$\mathcal{H} = \frac{1}{2} \sum_p (t_0 + p^2) \varphi(p)\varphi(-p) + \frac{g}{4!} \sum_j \varphi_j^4 - \sum_j H_j \varphi_j \quad (1.1)$$

and

$$W \{H_j\} = \ln \int D\varphi \exp -\mathcal{H}\{\varphi\} \equiv \ln Z \{H_j\} \quad (1.2)$$

One may describe the system by expanding in H ; or via a Legendre transform

$$W \{H_j\} = -\Gamma \{M_j\} + \sum_j H_j M_j \quad (1.3)$$

where the magnetization M_i is

$$M_j = \frac{\partial W}{\partial H_j} = \langle \varphi_j \rangle \quad (1.4)$$

and the Legendre transform Γ satisfies

$$H_j = \frac{\partial \Gamma}{\partial M_j}, \quad (1.5)$$

by expanding in M . The bracket in (1.4) stands for “thermal” average

$$M_j = \int D\varphi \varphi_j e^{-\mathcal{H}\{\varphi\}} / Z \tag{1.6}$$

The Jacobian of the transformation is the determinant of the inverse susceptibility matrix

$$(\chi^{-1})_{ij} = \frac{\partial^2 \Gamma}{\partial M_i \partial M_j} \tag{1.7}$$

and when a zero eigenvalue occurs it signals the inequivalence of the two expansions and the occurrence of a transition. In Fourier transform $\chi^{-1}(q)$, and, e.g., in zero momentum for a standard system, χ^{-1} vanishes in zero field, at T_c the Curie point

$$\chi^{-1}(q = 0; T_c) = 0, \tag{1.8}$$

below which $M \neq 0$ even for $H = 0$.

This well-known description of the paramagnetic to ferromagnetic transition we would like now to extend to the random field system.

2. The Random Field Ising System: Perturbation, a Reminder [21]

Let us consider now H_i to be a quenched random field with a pure Gaussian probability distribution, i.e.,

$$\overline{H_i} = 0 \tag{2.1}$$

$$\overline{H_i H_j} = \delta_{ij} \Delta \tag{2.2}$$

where the bar stands for probability average.

2.1. DIRECT AVERAGING. — One may compute the H expansion of *observables* and then perform the Wick average of the H 's on each term of the expansion. In the *paramagnetic* phase, to keep things simple, one obtains

$$\overline{W\{H\}} = \sum \text{connected graphs with all pairs of } H\text{'s} \\ \text{coalesced as in (2.2).} \tag{2.3}$$

$$G(i; j) \equiv \overline{\langle \varphi_i \varphi_j \rangle - \langle \varphi_i \rangle \langle \varphi_j \rangle} = \sum \text{connected graphs, rooted at } i \text{ and } j, \\ \text{with all pairs of } H\text{'s coalesced.} \tag{2.4}$$

$$C(i; j) \equiv \overline{\langle \varphi_i \rangle \langle \varphi_j \rangle} = \sum \text{graphs made of two } \textit{disconnected} \text{ pieces, respectively rooted} \\ \text{at } i \text{ and } j \text{ with all pairs of } H\text{'s coalesced.} \tag{2.5}$$

The coalescence of all pairs of H 's transforms the two disconnected pieces into one single *field-connected* graph.

2.2. AVERAGING VIA THE EFFECTIVE HAMILTONIAN. — The above results can be recovered using the replica trick, that is, computing

$$\overline{Z^n} = \overline{(\exp W\{H\})^n} = \exp \left\{ n \overline{W} + \frac{n^2}{2} \left[\overline{W^2} - \left(\overline{W} \right)^2 \right] + \dots \right\} \quad (2.6)$$

where one then recovers the averaged free energy

$$-F \equiv \overline{W} = \left(\overline{Z^n} - 1 \right) \Big|_{n \rightarrow 0} \quad (2.7)$$

but also its successive fluctuation cumulants.

The effective Hamiltonian is now⁽¹⁾,

$$\begin{aligned} \mathcal{H}_n = & \sum_{\alpha} \left\{ \frac{1}{2} \sum_p (t_0 + p^2) \varphi^{\alpha}(p) \varphi^{\alpha}(-p) + \frac{g}{4!} \sum_j (\varphi_j^{\alpha})^4 \right\} \\ & - \frac{1}{2} \sum_{\alpha, \beta} \sum_p \Delta \varphi^{\alpha}(p) \varphi^{\beta}(-p) \end{aligned} \quad (2.8)$$

The propagator becomes a matrix G with components

$$G_{\alpha\beta} = \langle \varphi_{\alpha} \varphi_{\beta} \rangle_n - \langle \varphi_{\alpha} \rangle_n \langle \varphi_{\beta} \rangle_n \quad (2.9)$$

where the replica-thermal average is shown as $\langle \rangle_n$. In the paramagnetic phase (no magnetization) $\langle \varphi_{\alpha} \rangle_n = 0$. The bare propagator $G_{\alpha\beta}^0$ is the inverse of the matrix $(p^2 + t_0) \delta_{\alpha\beta} - \Delta$, i.e., with $[G^0]^{-1} = p^2 + t_0$

$$G_{\alpha\beta}^0 = G^0 \delta_{\alpha\beta} + \frac{G^0 \Delta G^0}{1 - n \Delta G^0} \quad (2.10)$$

The first term is the *connected* (bare) propagator, the last is the *field-connected* (bare) propagator (suppressing the H -coalescence into Δ 's it falls into several disconnected pieces). In general we write

$$G \equiv G_{\alpha\beta} \equiv G_{\alpha} \delta_{\alpha\beta} + C_{\alpha\beta} \quad (2.11)$$

Of course, in the paramagnetic region there is no explicit replica dependence (in a RSB phase $C_{\alpha\beta}$ however depends [13–16] upon the α, β overlap).

The observables calculated by direct averaging as in Paragraph 2.1, are recovered via

$$\begin{aligned} G &= G_{\alpha} \Big|_{n \rightarrow 0} \\ C &= C_{\alpha\beta} \Big|_{n \rightarrow 0} \end{aligned} \quad (2.12)$$

Under RSB, one can relate [14, 15] them by,

$$C = \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} C_{\alpha\beta} \Big|_{n \rightarrow 0} \quad (2.13)$$

⁽¹⁾ For a non Gaussian probability law (e.g., a bimodal one) terms in $\frac{g'}{4!} \sum_j \left(\sum_{\alpha} \varphi_j^{\alpha} \right)^4$ are also generated. See below.

3. Legendre Transform

To keep things simple we work from the paramagnetic phase, i.e., with $\langle \varphi_\alpha \rangle_n = 0$. We can then forget about the first Legendre transform that takes H into $\langle \varphi_\alpha \rangle_n$ and concentrate on the second transform [22–24] taking $\Delta_{\alpha\beta}(p)$ into $\langle \varphi_\alpha(p) \varphi_\beta(-p) \rangle_n$. We treat here Δ as a source which we extend to values $\Delta \equiv \Delta_{\alpha\beta}(p)$ for the purpose of generating appropriate observables, with in the end $\Delta_{\alpha\beta}(p) \longrightarrow \Delta$. Together with $W_n \equiv \ln \left(\overline{Z^n} \right)$ we introduce

$$W_n\{\Delta\} = -\Gamma_n\{G\} + \frac{1}{2} \text{tr} \Delta G \tag{3.1}$$

with

$$G_{\alpha\beta}(p) = \frac{\partial W_n}{\partial \Delta_{\alpha\beta}(p)} \equiv G_\alpha(p) \delta_{\alpha\beta} + C_{\alpha\beta}(p) \tag{3.2}$$

and

$$\Delta_{\alpha\beta}(p) = \frac{\partial \Gamma_n}{\partial G_{\alpha\beta}(p)} \tag{3.3}$$

instead of (1.4,5).

The Γ_n functional is itself given by

$$-\Gamma_n\{G\} = \frac{1}{2} \text{tr} \ln G - \frac{1}{2} \text{tr} [G^0]^{-1} G + \mathcal{K}^{(1)}\{G\} \tag{3.4}$$

that is exhibiting components,

$$\begin{aligned} -\Gamma_n\{G_\alpha; C_{\alpha\beta}\} &= \frac{1}{2} \sum_\alpha \ln G_\alpha + \frac{1}{2} \text{tr}_{\alpha,\beta} \ln (\delta_{\alpha\beta} + G_\alpha^{-1} C_{\alpha\beta}) \\ &- \frac{1}{2} \sum_\alpha [G^0]^{-1} [G_\alpha + C_{\alpha\alpha}] + \sum_{s=1} \mathcal{K}_s^{(1)}\{G_\alpha; C_{\alpha\beta}\} \end{aligned} \tag{3.5}$$

Here $\mathcal{K}^{(1)}$ is the 1-irreducible functional built with φ_α^4 vertex and $G_{\alpha\beta}$ lines (i.e., such that by cutting off two such lines, whether connected ($G_\alpha \delta_{\alpha\beta}$) or field connected ($C_{\alpha\beta}$) the representative graph does not fall into two disconnected pieces). The subscript s in $\mathcal{K}_s^{(1)}$ is the number of free replica indices, after account of the $\delta_{\alpha\beta}$ constraints of the connected propagators.

3.1. STATIONARITY CONDITION. — We consider separately, stationarity with respect to off-diagonal and diagonal components.

Off-diagonal component $\delta/\delta C_{\alpha\beta}$:

$$[\mathbb{1} + G^{-1}C]_{\alpha\beta}^{-1} G_\beta^{-1} + \Delta_{\alpha\beta} + \sum_{s=2} \frac{\delta \mathcal{K}_s^{(1)}}{\delta C_{\alpha\beta}} = 0 \tag{3.6}$$

In the paramagnetic phase, $C_{\alpha\beta} \longrightarrow C$ and, in the above equation, only $s = 2$ contributes

$$G^{-1}CG^{-1} = \Delta + \frac{\delta \mathcal{K}_2^{(1)}}{\delta C} \{G; C\} \tag{3.7}$$

Diagonal component $\delta/\delta G_\alpha \equiv \delta/C_{\alpha\alpha}$:

The equation obtained is more subtle to interpret because it contains *both* connected and field-connected graphs and hence provides *two* equations. In Appendix A it is shown that one equation is the Dyson equation for G_α

$$G_\alpha^{-1} - [G^0]^{-1} + \sum_{s=1} \left[\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \right]_{\text{conn}} = 0 \quad (3.8)$$

where, in the paramagnetic phase, only $s = 1$ contributes. The other equation is the corresponding equation for $C_{\alpha\alpha}$

$$- \left[[\mathbb{1} + G^{-1}C]^{-1} G^{-1}C \right]_{\alpha\alpha} G_\alpha^{-1} + \Delta_{\alpha\alpha} + \sum_{s=2} \left[\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \right]_{f\text{-conn}} = 0 \quad (3.9)$$

Both equations (3.6) and (3.9) can then be rewritten as (Appendix A)

$$- \left[[\mathbb{1} + G^{-1}C]^{-1} G^{-1}CG^{-1} \right]_{\alpha\beta} + \Delta_{\alpha\beta} + \sum_{s=2} \frac{\delta \mathcal{K}_s^{(1)}}{\delta C_{\alpha\beta}} = 0 \quad (3.10)$$

an equation valid for $\alpha \neq \beta$ and $\alpha = \beta$. This seemingly formal result has the consequence that, contrary to what happens in the standard SG

$$C(x = 1 - \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} C(1) \quad (3.11)$$

showing that there is *no jump* in a RSB phase as the $\alpha \cap \beta$ overlap x is taken to be exactly equal to one⁽²⁾.

3.2. SECOND-DERIVATIVE MATRIX. — We have

$$\mathcal{M}_{\alpha\beta;\gamma\delta}(p;p') = \frac{\delta \Gamma_n \{G\}}{\partial C_{\alpha\beta}(p) \partial C_{\gamma\delta}(p')} \quad (3.12)$$

a matrix in p, p' and in replica pairs $\alpha\beta, \gamma\delta$. The structure in replica pair space has been analyzed by de Almeida and Thouless [25] (for the paramagnetic region and in the absence of diagonal components of G). Here again the dangerous sector is the replicon one with the matrix

$$\lambda_R(p;p') = M_1 - 2M_2 + M_3 \equiv \mathcal{M}_R(p;p') \quad (3.13)$$

where

$$\begin{aligned} M_1 &= \mathcal{M}_{\alpha\beta;\alpha\beta}(p;p') \\ M_2 &= \mathcal{M}_{\alpha\beta;\alpha\gamma}(p;p') = \mathcal{M}_{\alpha\beta;\gamma\beta}(p;p') \\ M_3 &= \mathcal{M}_{\alpha\beta;\gamma\delta}(p;p') \end{aligned} \quad (3.14)$$

The above expression simplifies greatly if one recognizes compensations occurring between the M components. These compensations are handily taken care of as follows.

⁽²⁾ Note that for consistency, the extension $\Delta \rightarrow \Delta_{\alpha\beta}$, introduced here, has to satisfy a relationship analog to (3.11).

Consider the first functional derivative

$$\Delta_{\alpha\beta} = \frac{\delta\Gamma_n}{\delta C_{\alpha\beta}} \tag{3.15}$$

$$\Delta_{\alpha\alpha} = \frac{\delta\Gamma_n}{\delta C_{\alpha\alpha}} = \frac{\delta\Gamma_n}{\delta G_\alpha} \tag{3.16}$$

Contributing graphs are such that the ends α, β in $\Delta_{\alpha\beta}$ are necessarily field-connected, whereas in $\Delta_{\alpha\alpha}$ the ends may be connected (contributing to the equation for G_α^{-1}) or field-connected (contributing to the equation for $C_{\alpha\alpha}$). Upon a second derivative, consider the connectedness of the new end points to the pair of initial end points. These may be connected or field-connected. The structure of eigenvalues (3.11-14) is such that, one recovers

$$\lambda_R(p; p') = M_1|_{\text{conn}} \equiv \mathcal{M}_{\alpha\beta; \alpha\beta}(p; p')|_{\text{conn}} \tag{3.17}$$

where the index conn. stands for the connectedness between the right and left pairs.

All the graphs with field-connexions between the left and right pairs compensate each other⁽³⁾ to only leave (3.17). From the explicit form of Γ_n one gets, in exact form

$$\lambda_R(p; p') = G_\alpha^{-1}(p)G_\beta^{-1}(p')\delta_{p+p'; 0} - \sum_{s=2} \frac{\delta^2 \mathcal{K}_s^{(1)}}{\delta C_{\alpha\beta}(p)\delta C_{\alpha\beta}(p')} \Big|_{\text{conn}} \tag{3.18}$$

where only $s = 2$ contributes in the $n \rightarrow 0$ limit.

Note that λ_R starts with an attractive coupling making it a candidate to come out with a null eigenvalue.

4. Δ -Susceptibility and the SG Transition

We are now in a situation that bears some analogy with the SG in field. In the paramagnetic region we have $C_{\alpha\beta} = C$, the analog (now space-dependent) of the SG order parameter $q_{\alpha\beta} = q$. As one crosses the line, defined by the vanishing of the lowest eigenvalue of $\lambda_R(p; p')$, playing the role of the Almeida-Thouless line, to avoid negative eigenvalues one has to break replica-symmetry [13-16] and write $C_{\alpha\beta} = C(x)$ where $x = \alpha \cap \beta$ is the overlap of the replica pair (in the Parisi [17] sense).

Just like in the pure system the vanishing of the jacobian signals the occurrence of a singularity in the H -susceptibility, here the vanishing of the lowest λ_R eigenvalue signals a singularity in the Δ -susceptibility.

$$\mathcal{G}_{\alpha\beta; \gamma\delta}(p; p') \equiv \frac{\partial^2 W_n}{\partial \Delta_{\alpha\beta}(p)\partial \Delta_{\gamma\delta}(p')} \tag{4.1}$$

Indeed in its replicon sector, we have

$$\mathcal{G}_R(p; p') = \mathcal{G}_1(p; p') - 2\mathcal{G}_2(p; p') + \mathcal{G}_3(p; p') \tag{4.2}$$

which is just the inverse of $\mathcal{M}_R(p; p')$:

$$\mathcal{G}_R(p; p') = [\mathcal{M}_R]^{-1}(p; p') \tag{4.3}$$

⁽³⁾In particular the $\sum_j \left(\sum_\alpha \varphi_j^\alpha \right)^4$ terms generated by a non Gaussian probability law, do not contribute to (3.17).

that is

$$\mathcal{G}_R(p; p') = G^2(p) \left[\delta_{p+p';0} + \sum_{p''} \mathcal{M}_{\alpha\beta; \alpha\beta}(p; p'') \right] \Big|_{\text{conn}} \mathcal{G}_R(p''; p') \tag{4.4}$$

One may also directly write out the standard SG susceptibility

$$\chi_{\text{SG}}(r_1 - r_2) = \overline{[\langle \varphi(r_1) \varphi(r_2) \rangle - \langle \varphi(r_1) \rangle \langle \varphi(r_2) \rangle]^2} \tag{4.5}$$

which is related to \mathcal{G}_R by

$$\sum_{p, p'} \mathcal{G}_R(p; p') = \sum_{1,2} \chi_{\text{SG}}(r_1 - r_2) \tag{4.6}$$

It is striking to see that in the random field system, SG singularities are confined to 2-replica contributions, i.e., looking back at (2.6) into $-F_2 \equiv \overline{W^2} - \left(\overline{W}\right)^2$ the free energy *fluctuation* and not the free energy itself.

However, as soon as we are in a RSB phase, free energy fluctuations $F_2, F_3 \dots$ are no longer of order n^2, n^3 respectively but all become proportional to n and thus contribute in a finite way to the free energy.

Let us see that effect on a simple example. Let us consider the lowest order contribution to $\mathcal{K}_2^{(1)}$:

$$\sum_{1,2} \sum_{\alpha, \beta} C_{\alpha\beta}^4(r_1 - r_2) \tag{4.7}$$

We have

$$\begin{aligned} \sum_{\alpha, \beta} C_{\alpha\beta}^4 &= \sum_{\alpha} \left[\sum_{\beta \neq \alpha} C_{\alpha\beta}^4 + C_{\alpha\alpha}^4 \right] \\ &= n \left[- \int_0^{1-\epsilon} dx C^4(x) + C^4(1) \right] \\ &= n \int_0^1 x dx \frac{d}{dx} C^4(x) \end{aligned} \tag{4.8}$$

where one has used (3.11). Hence the $O(n^2)$ term is now $O(n)$ and one understands the vanishing of that contribution in the RS limit with the vanishing of the derivative $dC(x)/dx$. In general the total number of derivatives (with respect to overlaps $x, y \dots$) is equal to the number of replicas involves minus one ($s - 1$ in $\mathcal{K}_s^{(1)}$). This is an unusual example where the topology of the graphs contributing to the free energy strongly depends on the phase one is into.

5. Phase Diagram

To investigate, in the plane (Δ, T) what line is defined by the occurrence of a null eigenvalue, we first consider the Curie line. On that line the propagators are massless and we write them as follows,

$$C(p) = \frac{1}{p^{2-\eta}} c \left(\frac{\Delta^\omega}{p^\theta} \right) \tag{5.1}$$

Here we have $\theta = 2 - (\bar{\eta} - \eta)$, $\omega > 0$ ($\omega = 1$ in the mean field limit), and $c(y) = 0$ if $y = 0$ (the pure Ising case), $c(y) = y$ if $y \rightarrow \infty$ to recover a behavior in $p^{-4+\bar{\eta}}$. Hence we may take

$$C(p) = \frac{\Delta^\omega}{p^{4-\bar{\eta}}} \equiv \frac{\bar{\Delta}}{p^{4-\bar{\eta}}} \tag{5.2}$$

As for the connected propagator we take first, for simplicity

$$G(p) = \frac{1}{p^{2-\eta}} \tag{5.3}$$

noting that in the crossover region to the pure system, (5.3) will have to be modified.

(i) *Lowest order in $\bar{\Delta}$: on the Curie line*

To lowest order the eigenvalue equation reads,

$$p^{4-2\eta} f_\lambda(\vec{p}) - (g\bar{\Delta})^2 \int \frac{d^D p}{(2\pi)^D} C_2(q) f_\lambda(\mathbf{p} - \mathbf{q}) = \lambda f_\lambda(\mathbf{p}) \tag{5.4}$$

with $f_\lambda(\mathbf{p})$ the eigenvector with λ eigenvalue and

$$C_2(q) = \int \frac{d^D s}{(2\pi)^D} \frac{1}{s^{4-\bar{\eta}} (\mathbf{s} + \mathbf{q})^{4-\bar{\eta}}} \equiv \frac{c}{q^{8-D-2\bar{\eta}}}$$

$$c = \frac{1}{(4\pi)^{D/2}} \Gamma((8-D-2\bar{\eta})/2) \frac{\Gamma^2((D-4+\bar{\eta})/2)}{\Gamma(D-4+\bar{\eta})} \tag{5.5}$$

Here $C_2(q)$ can also be interpreted as the first cumulant contribution of a random temperature term.

To overcome the difficulty of solving the above integral equation (or in Fourier transform, the ‘‘Schrödinger’’ equation with an ‘‘almost quartic’’ kinetic term) we resort to the Rayleigh-Ritz variational approach that provides an *upper bound*, by writing

$$\lambda_R \equiv \lambda = \int \frac{d^D p}{(2\pi)^D} p^{4-2\eta} |f(\mathbf{p})|^2 - (g\bar{\Delta})^2 \int \frac{d^D q}{(2\pi)^D} C_2(q) \phi(\mathbf{q}) \tag{5.6}$$

$$\phi(q) = \int \frac{d^D p}{(2\pi)^D} f^*(\mathbf{p}) f(\mathbf{p} - \mathbf{q}) \tag{5.7}$$

Here $f(\mathbf{p})$ a *normalized* trial wave function (i.e., $\phi(0) = 1$) whose *parameters* are to be determined *variationally*.

Since we are looking for the *lowest* eigenvalue (‘‘zero-energy bound state’’) we take $f(\mathbf{p}) = f(p)$ and real. We can now *scale out* the unit length R .

$$\lambda = \frac{a}{R^{4-2\eta}} - \frac{(g\bar{\Delta})^2 c}{R^{2(D-4+\bar{\eta})}} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^{8-D-2\bar{\eta}}} \phi(q)$$

$$\equiv \frac{a}{R^{4-2\eta}} - (g\bar{\Delta})^2 \frac{b}{R^{2(D-4+\bar{\eta})}} \tag{5.8}$$

Writing the stationarity condition with respect to R , one gets

$$0 = \frac{(2-\eta)a}{(R^2)^{2-\eta}} - \frac{(D-4+\bar{\eta})(g\bar{\Delta})^2 b}{(R^2)^{D-4+\bar{\eta}}} \tag{5.9}$$

Hence solving from equations (5.8,9), we obtain finally

$$\lambda = (D - 6 + \bar{\eta} + \eta) \left[\frac{a}{(D - 4 + \bar{\eta}) (R^2)^{2-\eta}} \right] = (D - 6 + \bar{\eta} + \eta) \left[\frac{(g\bar{\Delta})^2}{(2 - \eta) (R^2)^{D-4+\bar{\eta}}} \right] \quad (5.10)$$

with the length scale (i.e., correlation length)

$$\frac{1}{R^2} = \left[\frac{D - 4 + \bar{\eta}}{2 - \eta} \frac{b}{a} (g\bar{\Delta})^2 \right]^{1/(6-D-\eta-\bar{\eta})} \quad (5.11)$$

As we rest on the Curie line we see that the eigenvalue *upper bound* remains *negative* for all $\bar{\Delta}$ (except $\bar{\Delta}^* = 0$ where $\lambda = 0$, a limit upon which we return below).

It follows that, within the interval of dimension where the (ultraviolet) cut off does not spoil the length scaling, the paramagnetic to ferromagnetic transition is superceded by a paramagnetic to SG transition, provided it is meaningful to keep only the lowest order contribution to the $\mathcal{M}_R(p; p')$ kernel.

The boundaries of the dimension interval are

$$D_u = 6 - \eta(D_u) - \bar{\eta}(D_u) = 6$$

and

$$D_\ell = 4 - \bar{\eta}(D_\ell)$$

Given that the standard dimensional reduction (and the associated $\eta = \bar{\eta}$ result) is no more applicable, one is entitled to take $\bar{\eta} = 2\eta$ which is correct [26–28] near $D = 2$. The lower critical dimension $D_\ell = 2$ then obtained by using $\theta = 2 - (\bar{\eta} - \eta)$ in the vicinity of $D = 2$, that is $\eta = 1$ for $D = 2$.

Thus, *modulo* the (inessential) changes that will be introduced below for a treatment of the cross over region, what this very simple calculation is telling us reduces to the following: the results obtained by perturbation to all orders (dimensional reduction with $\theta = 2$, and $\bar{\eta} = \eta$) are superceded by the occurrence of the SG transition which originates in the *attraction* existing between pairs of distinct replicas. In contradistinction, and *a contrario*, for “animals” (i.e., branched polymers) whose effective Lagrangean is alike the RFIM one but with a pure imaginary coupling [29–31], the attraction becomes a *repulsion*, and in a random field, dimensional reduction is indeed correct [30].

(ii) To test the robustness of the above result, one may follow Mézard and Young [14] in adopting Bray’s [18] approach, i.e., use a screened interaction for an m -component system and work consistently to a given $\frac{1}{m}$ order.

Equation (5.6) is now replaced by

$$\begin{aligned} \lambda &= \int \frac{d^D p}{(2\pi)^D} p^{4-2\eta} f^2(p) - \frac{(g\bar{\Delta})^2}{m} \int \frac{d^D q}{(2\pi)^D} S^2(q) C_2(q) \phi(q) \\ &- \frac{(g\bar{\Delta})^2}{m} \int \frac{d^D q}{(2\pi)^D} S^2(q) \left[\int \frac{d^D p}{(2\pi)^D} f(p) \frac{1}{(\mathbf{p} + \mathbf{q})^{4-\bar{\eta}}} \right]^2 - O\left(\frac{1}{m^2}\right) \end{aligned} \quad (5.12)$$

with $C_2(q)$, $\phi(q)$ as of (5.5,7) and

$$S(q) = \mu^{6-D-\eta-\bar{\eta}} [1 - \Theta(\mu q_0 - q)] + \left(\frac{q}{q_0}\right)^{6-D-\eta-\bar{\eta}} \Theta(\mu q_0 - q) \quad (5.13)$$

$$q_0^{6-D-\eta-\bar{\eta}} = \frac{g\bar{\Delta}}{(4\pi)^{D/2}} \Gamma((6-D-\eta-\bar{\eta})/2) \times \frac{\Gamma((D-4+\bar{\eta})/2) \Gamma((D-2+\eta)/2)}{\Gamma(D-((6-\bar{\eta}-\eta)/2)) \Gamma((2-\eta)/2) \Gamma((4-\bar{\eta})/2)} \quad (5.14)$$

Here μ can either be unity, or chosen to take the best account of screening. Again scaling out the unit length yields

$$\lambda = \frac{a}{R^{4-2\eta}} - \frac{(g\bar{\Delta})^2}{m} \frac{1}{R^{2(D-4+\bar{\eta})}} \times \left\{ \int_0^{\mu q_0 R} \frac{d^D q}{(2\pi)^D} \left(\frac{q}{q_0 R}\right)^{2(6-D-\bar{\eta}-\eta)} A(q) + \int_{\mu q_0 R}^\infty \frac{d^D q}{(2\pi)^D} \mu^{2(6-D-\bar{\eta}-\eta)D} A(q) \right\} - O\left(\frac{1}{m^2}\right) \quad (5.15)$$

$$A(q) = C_2(q)\phi(q) + \left[\int \frac{d^D p}{(2\pi)^D} f(p) \frac{1}{(\mathbf{p} + \mathbf{q})^{4-\bar{\eta}}} \right]^2 \quad (5.16)$$

Note that the R -derivative with respect to integration boundaries does not contribute. The stationarity condition upon R yields then

$$O = (2-\eta) \left[\frac{a}{R^{4-2\eta}} - \frac{(g\bar{\Delta})^2}{m} \frac{1}{R^{2(D-4+\bar{\eta})}} \int_0^{\mu q_0 R} \frac{d^D q}{(2\pi)^D} \left(\frac{q}{q_0 R}\right)^{2(6-D-\bar{\eta}-\eta)} A(q) \right] - \frac{(g\bar{\Delta})^2}{m} \frac{(D-4+\bar{\eta})}{R^{2(D-4+\bar{\eta})}} \int_{\mu q_0 R}^\infty \frac{d^D q}{(2\pi)^D} \mu^{2(6-D-\bar{\eta}-\eta)} A(q) . \quad (5.17)$$

leading again to the Rayleigh-Ritz approximation of the eigenvalue,

$$\lambda = \frac{(D-6+\bar{\eta}+\eta)}{2-\eta} \frac{(g\bar{\Delta})^2}{m} \frac{1}{R^{2(D-4+\bar{\eta})}} \int_{\mu q_0 R}^\infty \frac{d^D q}{(2\pi)^D} \mu^{2(6-D-\bar{\eta}-\eta)} A(q) . \quad (5.18)$$

This expression remains negative for $D < 6$, confirming the result obtained in (i).

(iii) So far we have shown instability along the Curie line when keeping the terms in $\frac{1}{m} (g_{\text{scr}}\bar{\Delta})^2$

One would obtain analogous qualitative behavior for $\left[\frac{1}{m} (g_{\text{scr}}\bar{\Delta})^2\right]^2$ terms. The first repulsive

contribution only occurs as $\frac{g_{\text{scr}}}{m} \bar{\Delta} \left[\frac{(g_{\text{scr}}\bar{\Delta})^2}{m}\right]^2$

(iv) *Lowest order in $\bar{\Delta}$: cross over region*

If we want to use $G(p)$ throughout the crossover to the pure limit $\bar{\Delta} = 0$, one should replace (5.3) by

$$G(p) = \frac{\bar{\Delta}^{(\eta_p - \eta)/\theta}}{p^{2-\eta}} g\left(\frac{\bar{\Delta}^{1/\theta}}{p}\right) \tag{5.19}$$

with $g(x) \sim x^{\eta - \eta_p}$ as $x \rightarrow 0$ and $g(x) \sim C$ as $x \rightarrow \infty$, the subscript p referring to the pure limit, or alternatively, by

$$G(p) = \frac{1}{p^{2-\eta_p}} \tilde{g}\left(\frac{\bar{\Delta}^{1/\theta}}{p}\right) \tag{5.20}$$

with $\tilde{g}(0) = 1$ and $\tilde{g}(x) \sim 1/x^{\eta - \eta_p}$ as $x \rightarrow \infty$.

Now equation (5.8) is replaced by

$$\lambda = \frac{\bar{\Delta}^{\frac{2}{\theta}(\eta - \eta_p)}}{R^{4-2\eta}} \int \frac{d^D p}{(2\pi)^D} p^{4-2\eta} f^2(p) g^{-2}\left(R\bar{\Delta}^{1/\theta}/p\right) - \frac{\bar{\Delta}^2 b}{R^{2(D-4+\bar{\eta})}} \tag{5.21}$$

and with the stationarity condition

$$O = \frac{2 - \eta}{D - 4 + \bar{\eta}} \cdot \frac{\bar{\Delta}^{\frac{2}{\theta}(\eta - \eta_p)}}{R^{4-2\eta}} \cdot \left[\int \frac{d^D p}{(2\pi)^D} p^{4-2\eta} f^2(p) g^{-2}(y/p) \left[1 + \frac{2}{2 - \eta} \frac{y}{p} \frac{\dot{g}(y/p)}{g(y/p)} \right] \right] - \frac{\bar{\Delta}^2 b}{R^{2(D-4+\bar{\eta})}} \tag{5.22}$$

where $y \equiv R\bar{\Delta}^{1/\theta}$, thus yielding, with obvious notations,

$$\lambda = \frac{D - 6 + \bar{\eta} + \eta}{D - 4 + \bar{\eta}} \frac{\bar{\Delta}^{\frac{2}{\theta}(\eta - \eta_p)}}{R^{4-2\eta}} \left[\langle g^{-2} \rangle + \frac{2}{6 - D - \bar{\eta} - \eta} \left\langle \frac{y}{p} \dot{g} g^{-3} \right\rangle \right] \tag{5.23}$$

A sufficient condition to keep λ negative is to have $g(x)$ be a monotonously increasing function, a very natural property given the above limiting values for $x = 0, x \rightarrow \infty$.

The correlation length R is now given by (5.22) and one verifies that when D is between D_l and D_u , λ vanishes with $\bar{\Delta}$.

(v) *Lowest order in $\bar{\Delta}$: away from the Curie line*

As one is departing from the Curie line, the propagators become massive. In equation (5.6), e.g., the “kinetic” contribution will be increased slightly but the “potential” one will be sharply decreased, the mass playing the role of an infrared cutoff. Hence the eigenvalue will increase and become null at some point, on the SG transition line. To obtain that line we need the scaling functions for the propagators which are now significantly more complex, since they depend upon two variables $x = \bar{\Delta}^{1/\theta}/p \equiv \Delta_0/p$ and $y = \delta T^{\nu_p}/p$ where $\delta T = |T - T_c(\bar{\Delta})|$ is the distance, for a given $\bar{\Delta}$ to the Curie line. If one is willing to become more speculative, one may use a typical scaling form, restituting the appropriate behavior in all limits, as

$$C^{-1}(p) = \frac{p^{4-\bar{\eta}}}{\Delta_0^{2-\bar{\eta}+\eta_p}} + \frac{(\delta T)^{\bar{\gamma}}}{\Delta_0^{(\bar{\gamma}-\nu_p)/\nu_p}} = p^{2-\eta_p} \left\{ \frac{1}{x^{2-\bar{\eta}+\eta_p}} + x^{2-\eta_p} \left(\frac{y}{x}\right)^{\bar{\gamma}/\nu_p} \right\} \tag{5.24}$$

where $\bar{\gamma} = \nu(4 - \bar{\eta})$, and

$$\begin{aligned}
 G^{-1}(p) &= p^{2-\eta_p} + p^{2-\eta} \Delta_0^{\eta-\eta_p} + \frac{\delta T^\gamma}{\delta T^{\gamma-\gamma_p} + \Delta_0^{(\gamma-\gamma_p)/\nu_p}} \\
 &= p^{2-\eta_p} \left\{ 1 + x^{\eta-\eta_p} + y^{2-\eta_p} \frac{(y/x)^{(\gamma-\gamma_p)/\nu_p}}{1 + (y/x)^{(\gamma-\gamma_p)/\nu_p}} \right\} \tag{5.25}
 \end{aligned}$$

Leaving out a complicated discussion to be dealt with separately, let us just consider the vicinity of the upper critical dimension, in the simplest case where screening is left out.

Note first that the Curie line becomes now the locus of a SG/Ferro-SG transition. However the G propagator being weakly dependent upon the SG order parameter, we shall assume it unchanged near $D = 6$, that is given by

$$T - T_c + a\Delta \simeq 0 \tag{5.26}$$

where a is positive.

Let us compute the SG transition line for $D = 6 - \epsilon$, at vanishing values of δT and Δ . Proceeding in the same manner as in (i) and (iv) we get to leading order

$$\delta T \sim b\Delta^{4/\epsilon} \tag{5.27}$$

where b is positive and vanishes with ϵ . We thus have⁽⁴⁾

$$T - T_c \sim -a\Delta + b\Delta^{4/\epsilon}, \tag{5.28}$$

i.e., the SG transition line starts *tangent* to and remains *very close* to the Curie line for small ϵ .

As D decreases, the SG domain gets wider but too little is known about the behaviour of the Curie line itself to decide whether there is reentrance (i.e., whether the Δ exponent of the b term in (5.28) can become smaller than the one of the a term). Besides we have not taken care of screening which plays an important role when $D < 6$.

6. Conclusion

We have shown that for small enough values of $g\bar{\Delta}$, one obtains a negative upper bound for the eigenvalues of the inverse spin glass susceptibility, as it occurs in the free-energy fluctuation.

This enforces the occurrence of a SG phase and replica symmetry breaking [13-16]. It enforces it for all D between the upper ($D_u = 6$) and lower ($D_l = 2$) dimensions. Thus the general properties obtained via perturbation to all orders [2-4], for $D < 6$ (i.e., dimensional reduction with $\theta = 2$) have no domain of application on the Curie line, if the above results are not reversed for higher values of $g_{scr}\bar{\Delta}$. Indeed, they are then superceded by the SG transition for any D , $2 < D < 6$.

Whether this entails that the appropriate description is via a SG order parameter as in Mézard-Young remains to be seen

(i) although the above result is unlikely to be reversed, one would like to render foolproof the above derivation (by extending it to all orders in $g\bar{\Delta}$).

⁽⁴⁾Note that the same reasoning may be applied when approaching the Curie line from the low temperature side, i.e., from the ferromagnetic phase to determine a Ferro/Ferro SG transition line, as in reference [32].

(ii) this being given, it would also be desirable to see whether the above SG transition is not superceded itself by 3-replicas “zero energy bound states”, 4-replicas, etc. (as contrasted with the 2-replica studied above).

Or put another way, just as we have seen above, that the SG transition associated to the order parameter $\langle \varphi_\alpha \varphi_\beta \rangle$ can be interpreted as governed by an effective random temperature (i.e., mass, with a $C_2(q) \sim 1/q^{(8-D-2\bar{\eta})}$ correlation), likewise one may ask whether higher order parameters, e.g., $\langle \varphi_\alpha \varphi_\beta \varphi_\gamma \rangle$, etc., associated with higher random couplings [33] will not become relevant and spoil the above results.

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Appendix A

Here we show that the stationarity condition on the Γ_n functional (3.4,5) yields equations (3.8-10) with the implication (3.11).

Consider the contributions associated with a given graph to $\mathcal{K}^{(1)}\{\mathbf{G}\}$.

Then take a given choice for the \mathbf{G} components (with a constraint as in $G_\alpha \delta_{\alpha\beta}$ or no constraint as in $C_{\alpha\beta}$). The contributions of $\mathcal{K}^{(1)}$ are then associated with $\mathcal{K}_1^{(1)}, \mathcal{K}_2^{(1)}$,

Consider now the first functional derivative

$$\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \tag{A.1}$$

and the connectedness with respect to the input-output pair of lines (opened by the functional derivation). Those two lines can be connected or field-connected (if suppressing the Δ -coalescence the input-output lines fall apart) and one can always write

$$\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} = \left[\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \right]_{\text{conn}} + \left[\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \right]_{\text{field-conn}} \tag{A.2}$$

With (A.2) one can separate out the stationarity condition with respect to G_α (or $C_{\alpha\alpha}$) as (3.8) and (3.9).

By inspection, one then writes

$$\left[\frac{\delta \mathcal{K}_s^{(1)}}{\delta G_\alpha} \right]_{\text{field-conn}} = \left. \frac{\delta \mathcal{K}_{s+1}^{(1)}}{\delta C_{\alpha\beta}} \right|_{\alpha=\beta}$$

Noting that any $\delta/\delta C_{\alpha\beta}$ contribution is by definition field-connected. Hence one obtains (3.10) and the result that the equation for $C_{\alpha\alpha}$ (3.9) is obtained by working with a pair of distinct replicas, and letting (after functional derivation) $\beta \rightarrow \alpha$, implying (3.11).

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