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Magnetoconductance of Ballistic Chaotic Quantum Dots: A Brownian Motion Approach For the S -Matrix

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Abstract. — Using the Fokker-Planck equation describing the evolution of the transmission eigenvalues for Dyson's Brownian motion ensemble, we calculate the magnetoconductance of a ballistic chaotic dot in the crossover regime from the orthogonal to the unitary symmetry. The correlation functions of the transmission eigenvalues are expressed in terms of quaternion determinants for arbitrary number N of scattering channels. The corresponding average, variance and autocorrelation function of the magnetoconductance are given as a function of the Brownian motion time t . A microscopic derivation of this S -Brownian motion approach is discussed and t is related to the applied flux. This exactly solvable random matrix model yields the right expression for the suppression of the weak localization corrections in the large N -limit and for small applied fluxes. An appropriate rescaling of t could extend its validity to larger magnetic fluxes for the averages, but not for the correlation functions.

1. Introduction

Ballistic transport through quantum dots of various shapes are the subject of many theoretical and experimental investigations. One of the fundamental issues is to determine in a conductance measurement the signature of "quantum chaos", i.e., of the quantum analog of a classically chaotic dynamics. Usually, such dots are made with two-dimensional semiconductor nanostructures (quantum billiards) connected to two electron reservoirs, and one measures the magnetic field dependence of the conductance at very low temperature [1–3] (quantum coherent transport). The question is to describe the statistical properties of the recorded magneto-fingerprints for different shapes of the dot. We restrict ourselves to fully chaotic systems where we hope to state universal results, leaving aside the integrable systems. The usual theoretical approaches [4, 5] concentrate on microscopic models, using numerical calculations and semiclassical approximations, or more or less straightforward random matrix models, possibly combined with supersymmetric methods. The semiclassical approach requires in the final evaluation the so-called diagonal approximation, which is problematic [5] for the total weak localization correction. Supersymmetry [6, 7] and random matrix model have also been recently used [8] to describe the ensemble averaged behavior of the magneto conductance. In reference [8] the Hamiltonian of the dot was modelled by a Pandey-Mehta random matrix

Hamiltonian [9], but the method used leads to substantial technical problems and the averaged magnetoconductance is expressed in terms of a rather complicated integral expression which can be evaluated by numerical means.

Another random matrix approach for the quantum billiards was used in references [10, 11] and models directly the scattering matrix S of the dot instead of its Hamiltonian. There are reasons [12, 13] to assume that S is suitably described by one of Dyson's classical circular ensembles [14] (COE-CUE-CSE). In references [10, 11], the associated transport properties have been calculated, e.g., for the circular orthogonal ensemble (COE) and for circular unitary ensemble (CUE) corresponding to chaotic dots without or with a large enough applied magnetic field, respectively. We propose a generalization of this approach in terms of a continuous crossover from COE to CUE, that is described by one of Dyson's Brownian motion ensembles [15]. The idea is to consider a Brownian motion with a fictitious time t that leads to a *diffusion* in S -matrix space. In the limit $t \rightarrow \infty$ the probability distribution diffuses to a stationary solution that corresponds in our application to the circular unitary ensemble. Choosing as initial condition for $t = 0$ the circular orthogonal ensemble, we have a model for the magnetoconductance of a chaotic dot, at least as a function of the Brownian motion time t . Before completion of this manuscript, we received a preprint by Jochen Rau [16] where a very qualitative analysis of this model is presented with essentially similar conclusions than our detailed study. One can also mention that this crossover ensemble was already considered in references [17, 18] where the correlation functions for the scattering phase shifts θ_i were calculated. The decisive difference here is that we are interested in the transport properties, i.e., in the statistics of the transmission eigenvalues T_i of tt^\dagger , when S is parametrized by

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} . \quad (1.1)$$

The Brownian motion leads to a Fokker-Planck equation for the probability distribution of the variables T_i which have been first derived in reference [19]. This Fokker-Planck equation can be mapped onto a Schrödinger equation of a one-dimensional system of free Fermions and solved for an arbitrary initial condition. The corresponding propagator was already calculated in reference [19], where the crossover of two decoupled CUEs of dimension N to one CUE of dimension $2N$ was considered. Here we use another initial condition that is just the joint probability distribution for the variables T_i of the COE-case. The time dependent solution of the Fokker-Planck equation is then the joint probability distribution for the crossover ensemble COE \rightarrow CUE.

This paper is subdivided in two main parts (Sections 2 and 3). Section 2 is completely devoted to an exact solution of the Brownian motion model as it stands. We give any k -point correlation function for the T_i at one particular time t . In addition, we also give the most simple correlation functions between two different time values. The technical method to obtain these results is based on a formulation in terms of quaternion determinants [20] and skew orthogonal polynomials. The final results for the correlation functions are given in terms of Legendre Polynomials and can be easily evaluated by numerical means. The averages for the conductance and the autocorrelation of its variance (between different Brownian motion times) are exactly calculated for an arbitrary value of the channel number. All results of Section 2 are exact consequences of the S -Brownian motion ensemble applied to the COE \rightarrow CUE crossover.

It remains to express the Brownian motion time t in terms of a real physical parameter such as the applied magnetic flux Φ . This issue is considered in detail in Section 3. We first discuss the relation of the Pandey-Mehta Hamiltonian [9] with finite disordered systems (in the diffusive regime) or with quantum billiards (in the chaotic ballistic regime). Both cases have already been considered [21–23] and we give a simple treatment that accounts for both

cases and generalizes the known results to the quantum billiard case. In Subsection 2.2 we briefly recall the different magnetic field scales in a quantum billiard and the effects of the Landau quantization for very strong fields. In Subsection 3.3, a microscopic justification ⁽¹⁾ of the Brownian motion ensemble is sketched, assuming a crossover Pandey-Mehta Hamiltonian for H and using a well-known expression of S in terms of H . We obtain an explicit expression that relates the Brownian motion time with the Pandey-Mehta parameter α and therefore with the magnetic flux. Using this translation we find a Gaussian instead of a Lorentzian behavior (as given in Ref. [8]) for the suppression of the weak localization correction to the averaged conductance, when a flux Φ is applied. One finds a certain characteristic flux Φ_C separating two regimes. For a small flux $\Phi \leq \Phi_C$ our result matches precisely that of reference [8]. For a large flux $\Phi \gg \Phi_C$ the discrepancy between the Gaussian and Lorentzian becomes more important. This indicates that the Brownian motion model is not precisely equivalent to the model used in [8] for large times of the Brownian motion. The difficulty to obtain directly from our model the lorentzian shape for the weak localization suppression and the lorentzian *square* behavior for the correlation functions is discussed in conclusion.

2. S-Brownian Motion Ensemble and COE to CUE Crossover

In reference [19] we used the original Dyson Brownian motion ensemble for the scattering matrix to obtain a Fokker-Planck equation for the *transmission eigenvalues* T_i . This is suitable for having the conductance g of a scatterer connected by N -channel contacts to the electron

reservoirs, since $g = \sum_{i=1}^N T_i$. The stationary solution of the Fokker-Planck equation is given by

the transmission eigenvalue distribution of the circular unitary ensemble describing a chaotic dot without time reversal invariance. The chosen initial condition for the Brownian motion is the COE-distribution for the T_i . Concerning the definition of the Brownian motion, the derivation of the Fokker-Planck equation for the transmission eigenvalues and its solution for an arbitrary initial condition, we refer the reader to reference [19]. It is convenient to use the same coordinates $x_i = 1 - 2T_i$ of [19], where the T_i are the eigenvalues of tt^\dagger .

2.1. JOINT PROBABILITY DISTRIBUTION OF THE TRANSMISSION EIGENVALUES. — The joint probability distribution $p(x, t)$ for the Brownian motion ensemble can be expressed [19] as follows

$$p(x, t) = \int d^N \tilde{x} \hat{p}(\tilde{x}) G(\tilde{x}, x; t) \quad , \tag{2.1}$$

where the many-particle Green's function is given by

$$G(\tilde{x}, x; t) = \frac{1}{N!} \rho(\tilde{x})^{-1} \rho(x) \det(g(x_i, \tilde{x}_j; t)) e^{-C_N t} \quad , \tag{2.2}$$

$$\rho(x) = \prod_{i>j}^N (x_i - x_j) \quad , \tag{2.3}$$

$$g(x_i, \tilde{x}_j; t) = \sum_{n=0}^{\infty} \frac{1}{2} (1 + 2n) P_n(x_i) P_n(\tilde{x}_j) e^{-\epsilon_n t} \tag{2.4}$$

⁽¹⁾ We mention that in reference [18] a semiclassical and numerical justification for the validity of the Brownian motion ensemble has been given as far as the scattering phase shifts correlations are concerned.

$P_n(x_i)$ is the Legendre polynomial of degree n , $\epsilon_n = (1 + 2n)^2$ is the eigenvalue of the “one-particle Hamiltonian” [19] and the constant C_N is given by the sum $C_N = -\sum_{n=0}^{N-1} \epsilon_n$. The function $\hat{p}(x) = p(x, 0)$ is the initial condition. In our case, it is given by the joint probability distribution for the orthogonal circular ensemble which in terms of the variables x_j reads as [10, 11]

$$\hat{p}(x) \propto \prod_{i=1}^N \frac{1}{\sqrt{2(1-x_i)}} \prod_{i>j}^N |x_i - x_j| \quad (2.5)$$

The integral in equation (2.1) can be evaluated by the method of integration over alternate variables [24]. As usually in such situations, we consider now the case of an even number N of channels. The complications that arise from the case of odd N will be discussed in Appendix B. Our final results of this Section are valid for all values of N . The result for the joint probability distribution can be expressed in terms of the so-called Pfaffian

$$p(x, t) \propto \prod_{i>j}^N |x_i - x_j| \sqrt{\det(H(x_i, x_j; t))} e^{-C_N t} \quad (2.6)$$

where $H(x_i, x_j; t)$ is an antisymmetric function given by the double integral

$$H(x_1, x_2; t) = \frac{1}{2} \int_{-1}^1 dy_1 \int_{-1}^1 dy_2 \frac{\epsilon(y_2 - y_1)}{\sqrt{(1-y_2)(1-y_1)}} g(x_1, y_1; t) g(x_2, y_2; t) \quad (2.7)$$

where $\epsilon(u) = +1, 0, -1$ according to the cases $u > 0, = 0, < 0$. In the next subsection we will present a formulation in terms of quaternion determinants [25] that enables us to evaluate all types of correlation functions and to get exact expressions for the average conductance and the conductance fluctuations.

We first briefly discuss the implications of equation (2.6) for the “level” repulsion in a more qualitative way. For this discussion the angles φ_j with $x_j = \sin^2 \varphi_j$ are more appropriate. The one-particle Green’s function $g(x_i, x_j; t)$ fulfills in terms of the φ_j a time-dependent Schrödinger equation [19] with a particular potential. In the (extreme) short time limit the effect of this potential can be neglected and the Green’s function can be approximated by the free “diffusion propagator”, i.e.,

$$g(\sin^2(\varphi_i), \sin^2(\varphi_j); t) \simeq \frac{1}{\sqrt{\sin(2\varphi_i) \sin(2\varphi_j) 16\pi t}} \exp\left(-\frac{1}{4t}(\varphi_i - \varphi_j)^2\right) \quad (2.8)$$

Using this expression we can evaluate the integral in equation (2.7) and we obtain for the joint probability distribution for the variables φ_i in the limit $t \rightarrow 0$ the expression

$$\tilde{p}(\varphi, t) \simeq F_1(\varphi) \sqrt{\det\left(\text{erf}\left(\frac{(\varphi_i - \varphi_j)}{\sqrt{4t}}\right)\right)} \quad (2.9)$$

where $F_1(\varphi)$ is the joint probability distribution for the orthogonal case, i.e., for exactly $t = 0$. Pandey and Mehta [9] obtained for the eigenvalue distribution of the random matrix Hamiltonian described by equation (3.1) a very similar expression if the φ_j are identified with the energy levels of E_α and if one relates the symmetry breaking parameters α and t in a suitable way. The expression (2.9) shows that the crossover from COE to CUE has an effect for arbitrarily small values of $t > 0$ if the difference $|\varphi_i - \varphi_j|$ is sufficiently small. Let us first consider

the situation where all φ_i are well separated, i.e., $|\varphi_i - \varphi_j| \gg \sqrt{4t}$. In this case the error function in equation (2.9) can be replaced by $\text{sign}(\varphi_i - \varphi_j)$ and the determinant is just one. The probability distribution then equals the distribution $F_1(\varphi)$ for the COE. On the other hand the determinant that appears in (2.9) is an even function of $\varphi_i - \varphi_j$ and vanishes if one of these differences is zero. If for example two values of the φ_i are very close, e.g., $|\varphi_1 - \varphi_2| \ll \sqrt{4t}$, the square root of the determinant becomes proportional to the difference $|\varphi_1 - \varphi_2|/\sqrt{4t}$ which leads together with the factor $F_1(\varphi)$ to the *quadratic* level repulsion of the unitary case. This behavior means that the COE to CUE crossover for the transmission eigenvalue correlations appears at arbitrarily small values for the time parameter t on scales smaller than $\sqrt{4t}$. The same effect is also observed for the energy levels E_α [9] and for the scattering phase shifts θ_i [17].

2.2. QUATERNION FORMULATION. — In this Subsection, we use the quaternion technique, that was introduced in the theory of random matrices [25], to calculate the m -point correlation functions given by the usual definition

$$R_m(x_1, \dots, x_m, t) = \frac{N!}{(N-m)!} \int_{-1}^1 dx_{m+1} \cdots \int_{-1}^1 dx_N p(x, t) \tag{2.10}$$

We define for two arbitrary functions f, g the antisymmetric scalar product

$$\langle f, g \rangle^{(t)} = \int_{-1}^1 dy_1 \int_{-1}^1 dy_2 H(y_1, y_2; t) f(y_1) g(y_2) \tag{2.11}$$

where $H(y_1, y_2; t)$ is just given by equation (2.7). Suppose that the number N of channels is even and that we have a set of skew orthogonal polynomials $q_n^{(t)}(x)$ of degree $n = 0, \dots, N-1$ with respect to the scalar product (2.11), i.e.,

$$\langle q_n^{(t)}, q_m^{(t)} \rangle^{(t)} = Z_{nm} \tag{2.12}$$

where Z_{nm} is an antisymmetric matrix with $Z_{2n, 2n+1} = -Z_{2n+1, 2n} = 1$ and all other $Z_{nm} = 0$. The superscript means that both the polynomials and the scalar product depend explicitly on the time parameter t . At the moment, we do not consider explicit expressions for the $q_n^{(t)}(x)$ and the following method remains general. In addition, we introduce the functions $Q_n^{(t)}(x)$ via

$$Q_n^{(t)}(y_2) = \int_{-1}^1 dy_1 H(y_1, y_2; t) q_n^{(t)}(y_1) \tag{2.13}$$

which are dual to the $q_n^{(t)}(x)$:

$$\int_{-1}^1 dy Q_n^{(t)}(y) q_m^{(t)}(y) = Z_{nm} \tag{2.14}$$

We define furthermore functions of two variables x, y by

$$K_{F\bar{F}}(x, y; t) = \sum_{n,m=0}^{N-1} F_n^{(t)}(x) Z_{nm} \bar{F}_m^{(t)}(y) \tag{2.15}$$

where F and \bar{F} stand for one of the symbols q or Q . Hence, equation (2.15) defines four different functions K_{qq}, K_{qQ}, K_{Qq} and K_{QQ} . From the definition of the matrix Z_{nm} and

equations (2.13–2.15) we can immediately state the following properties

$$K_{F\bar{F}}(x, y; t) = -K_{\bar{F}F}(y, x; t) \quad , \quad (2.16)$$

$$-\int_{-1}^1 dx K_{qQ}(x, x; t) = \int_{-1}^1 dx K_{Qq}(x, x; t) = N \quad , \quad (2.17)$$

$$\int_{-1}^1 dy K_{Fq}(x, y; t) K_{Q\bar{F}}(y, z) = -\int_{-1}^1 dy K_{FQ}(x, y; t) K_{q\bar{F}}(y, z) = K_{F\bar{F}}(x, z; t) \quad (2.18)$$

and

$$\int_{-1}^1 dy H(y, x; t) K_{qF}(y, z; t) = K_{QF}(x, z; t). \quad (2.19)$$

From now on, we strongly refer to the notations and results of reference [25] concerning quaternion properties. We introduce the 2×2 -matrix (or quaternion) function $\sigma(x, y; t)$ by

$$\sigma(x, y; t) = \begin{pmatrix} -K_{qQ}(x, y; t) & K_{qq}(x, y; t) \\ H(x, y; t) - K_{QQ}(x, y; t) & K_{Qq}(x, y; t) \end{pmatrix} \quad (2.20)$$

that fulfills the three properties

$$\int_{-1}^1 dx \sigma(x, x; t) = N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.21)$$

$$\int_{-1}^1 dy \sigma(x, y; t) \sigma(y, z; t) = \sigma(x, z; t) + \tau \sigma(x, z; t) - \sigma(x, z; t) \tau \quad , \quad (2.22)$$

$$p(x, t) = \text{const.} \cdot \text{QDet}((\sigma(x_i, x_j))_{1 \leq i, j \leq N}) \quad (2.23)$$

where $\tau = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\text{QDet}(\dots)$ is the quaternion-determinant defined in reference [25]. The verification of equations (2.21) and (2.22) is a straightforward calculation applying the properties (2.16-2.19). The proof of equation (2.23) is not so obvious and can be obtained as follows. First, we note that (2.16) and the antisymmetry of $H(x, y; t)$ imply that the quaternion-matrix $\sigma(x_i, x_j; t)$ is self-dual, i.e.,

$$\sigma(x_i, x_j; t) = \bar{\sigma}(x_j, x_i; t) \quad (2.24)$$

where

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (2.25)$$

is the quaternion adjoint in the notation of reference [25]. As in reference [25] we denote with $A(\sigma(x_i, x_j; t))$ the conventional $2N \times 2N$ -matrix that contains the quaternions as 2×2 blocks. From theorem 2 of [25] we get the identity $(\text{QDet}(\sigma(x_i, x_j; t)))^2 = \det(A(\sigma(x_i, x_j; t)))$ that relates the conventional with the quaternion-determinant for self-dual matrices. We consider therefore the matrix

$$\begin{aligned} A(\sigma(x_i, x_j; t)) &= \begin{pmatrix} -K_{qQ}(x_i, x_j; t) & K_{qq}(x_i, x_j; t) \\ H(x_i, x_j; t) - K_{QQ}(x_i, x_j; t) & K_{Qq}(x_i, x_j; t) \end{pmatrix} \\ &= \begin{pmatrix} q(x_i)^T & 0 \\ Q(x_i)^T & 1 \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -Q(x_j) & q(x_j) \\ H(x_i, x_j; t) & 0 \end{pmatrix} \end{aligned} \quad (2.26)$$

Here we have used a quasi block notation where 1 and Z stand for the $N \times N$ unit matrix or the matrix with entries Z_{nm} respectively. $q(x)$ (or $Q(x)$) is a column vector with entries

$q_n^{(t)}(x)$ (or $Q_n^{(t)}(x)$). $q(x)^T$ and $Q(x)^T$ denote the corresponding row vectors. The determinant of (2.26) is immediately calculated

$$\det(A(\sigma(x_i, x_j; t))) = \det \left(q_n^{(t)}(x_i) \right)^2 \det(H(x_i, x_j; t)) \sim (p(x, t))^2 \tag{2.27}$$

where we have expressed the determinant of the skew orthogonal polynomials as a Vandermond determinant. Equation (2.27) completes the proof of the property (2.23).

We can now use (2.21- 2.23) and apply theorem (4) of reference [25]. Then, we find directly the m -point correlation functions as quaternion-determinants

$$R_m(x_1, \dots, x_m, t) = \text{QDet}(\sigma(x_i, x_j; t)_{1 \leq i, j \leq m}) \tag{2.28}$$

In addition, the constant in (2.23) must have the value $1/N!$.

2.3. CONDUCTANCE CORRELATION FUNCTIONS AND AVERAGES. — A more explicit evaluation of equation (2.28) for the m -point correlation functions requires the knowledge of the skew orthogonal polynomials $q_n^{(t)}(x)$ and of their dual functions $Q_n^{(t)}(x)$ given by (2.13). The calculation of Appendix A shows that the skew orthogonal polynomials are not unique and one has to specify a certain choice. Of course, the final results of the last subsection do not depend on the particular choice. A possible choice is

$$\begin{aligned} q_{2n}^{(t)}(x) &= \frac{1}{2}(1 + 4n) P_{2n}(x) e^{\varepsilon_{2n} t} + s_n^{(t)}(x) \quad , \\ q_{2n+1}^{(t)}(x) &= \frac{1}{2}(3 + 4n) P_{2n+1}(x) e^{\varepsilon_{2n+1} t} + s_n^{(t)}(x) \end{aligned} \tag{2.29}$$

where $P_n(x)$ are the Legendre polynomials and

$$s_n^{(t)}(x) = \sum_{m=0}^{2n-1} (-1)^m \frac{1}{2}(1 + 2m) P_m(x) e^{\varepsilon_m t} \tag{2.30}$$

The complete calculation that yields equation (2.29) is done in Appendix A. Equation (2.29) can be inverted with the result

$$\begin{aligned} \frac{1}{2}(1 + 4n) P_{2n}(x) e^{\varepsilon_{2n} t} &= q_{2n}^{(t)}(x) - s_n^{(t)}(x) \quad , \\ \frac{1}{2}(3 + 4n) P_{2n+1}(x) e^{\varepsilon_{2n+1} t} &= q_{2n+1}^{(t)}(x) - s_n^{(t)}(x) \end{aligned} \tag{2.31}$$

where now the sum $s_n^{(t)}(x)$ is expressed in terms of the $q_n^{(t)}(x)$

$$s_n^{(t)}(x) = \sum_{m=0}^{2n-1} (-1)^m q_n^{(t)}(x) \quad . \tag{2.32}$$

We consider now equation (2.31) for $t = 0$ and replace in the definition of $g(x, y; t)$ (compare Eq. (2.4)) one of the Legendre polynomials $P_n(y)$ with the expansion in terms of the $q_n^{(0)}(y)$. Then the integral in equation (2.7) can be done because the $q_n^{(0)}(y)$ are just the skew orthogonal polynomials of the scalar product (A.1). After some algebra with the sums (it is useful to consider these type of operations as matrix multiplications with suitable defined matrices) we arrive at the expansion of the kernel $H(x_1, x_2; t)$

$$H(x_1, x_2; t) = \sum_{k, n=0}^{\infty} e^{-(\varepsilon_n + \varepsilon_k) t} \text{sign}(k - n) P_n(x_1) P_k(x_2) \tag{2.33}$$

Our first application of (2.33) is the calculation of the dual functions $Q_n^{(t)}(x)$ that appear in equation (2.13),

$$\begin{aligned} Q_{2n}^{(t)} &= P_{2n+1}(x) e^{-\varepsilon_{2n+1} t} + S_n^{(t)}(x) \quad , \\ Q_{2n+1}^{(t)} &= -P_{2n}(x) e^{-\varepsilon_{2n} t} + S_n^{(t)}(x) \end{aligned} \tag{2.34}$$

where we have introduced the sum

$$S_n^{(t)}(x) = \sum_{m=2n+2}^{\infty} e^{-\varepsilon_m t} P_m(x) \tag{2.35}$$

We have now to insert the expansions (2.29), (2.33) and (2.34) in equations (2.20) and (2.28) in order to get the m -point correlation functions. In the following, we state the result for the 1- and the 2-point functions. First, we emphasize that this quaternion description, as it is shown in the last Subsection, is valid for an even value of N but the formulae below are valid for the odd N case, too. In Appendix B, we discuss briefly how the complications for odd N can be treated. Second, we give also an expression for the correlation function $R_{1,1}(x, t; y, t + \tau)$ that describes the probability density of finding x at time t and y at time $t + \tau$. Its formal definition is given in reference [19] and we calculated this function using a standard method of functional derivatives described by Mehta [24].

For convenience, we introduce the following three functions

$$\begin{aligned} S_N^{(t_1, t_2)}(x, y) &= \sum_{n=0}^{N-1} \frac{1}{2}(1 + 2n) P_n(x) P_n(y) e^{\varepsilon_n(t_1 - t_2)} \\ &+ \sum_{k=0}^{N-1} (-1)^{N-1-k} \frac{1}{2}(1 + 2k) P_k(x) e^{\varepsilon_k t_1} \sum_{n=N}^{\infty} P_n(y) e^{-\varepsilon_n t_2} \quad , \end{aligned} \tag{2.36}$$

$$D_N^{(t_1, t_2)}(x, y) = \sum_{n, k=0}^{N-1} \text{sign}(n - k) \frac{1}{4}(1 + 2n)(1 + 2k) (-1)^{n+k} P_n(x) P_k(y) e^{\varepsilon_n t_1 + \varepsilon_k t_2} \tag{2.37}$$

$$I_N^{(t_1, t_2)}(x, y) = - \sum_{n, k=N}^{\infty} \text{sign}(n - k) P_n(x) P_k(y) e^{-\varepsilon_n t_1 - \varepsilon_k t_2} \tag{2.38}$$

Then the quaternion function (2.20) becomes

$$\sigma(x, y; t) = \begin{pmatrix} S_N^{(t, t)}(x, y) & D_N^{(t, t)}(x, y) \\ I_N^{(t, t)}(x, y) & S_N^{(t, t)}(y, x) \end{pmatrix} \tag{2.39}$$

and we eventually get for the correlation functions

$$R_1(x; t) = S_N^{(t, t)}(x, x) \quad , \tag{2.40}$$

$$\begin{aligned} R_2(x, y; t) &= R_1(x; t) R_1(y; t) - S_N^{(t, t)}(x, y) S_N^{(t, t)}(y, x) \\ &+ D_N^{(t, t)}(x, y) I_N^{(t, t)}(x, y) \quad , \end{aligned} \tag{2.41}$$

$$\begin{aligned} R_{1,1}(x, t; y, t + \tau) &= R_1(x; t) R_1(y; t + \tau) - S_N^{(t, t+\tau)}(x, y) S_N^{(t+\tau, t)}(y, x) \\ &+ D_N^{(t, t+\tau)}(x, y) I_N^{(t, t+\tau)}(x, y) + g(x, y; \tau) S_N^{(t+\tau, t)}(y, x) \end{aligned} \tag{2.42}$$

For $\tau = 0$ the function (2.42) fulfills the identity

$$R_{1,1}(x, t; y, t) = R_2(x, y; t) + \delta(x - y) R_1(x; t) \tag{2.43}$$

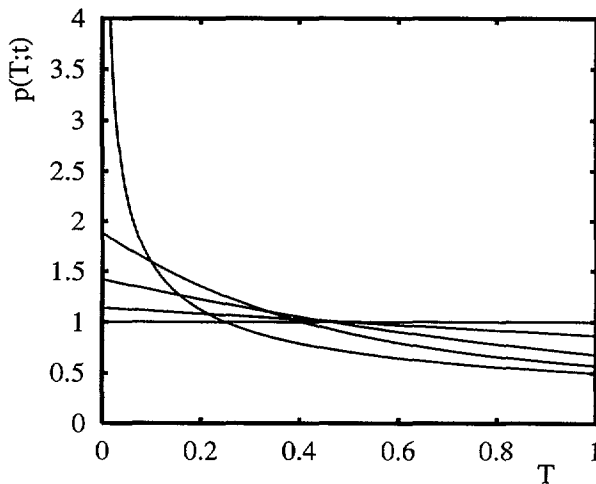


Fig. 1. — Conductance distribution $p(T; t)$ for $N = 1$ (Eq. (2.44)) and $t/t_c = 0, 0.5, 1, 2, \infty$.

where the last term accounts for the selfcorrelation of the eigenvalue x . In the limit $t_1 = t_2 \rightarrow \infty$ in (2.36) only the first sum survives and the product $D_N \cdot I_N$ vanishes. Then equations (2.40-2.41) equal the 1- and 2-point correlation functions of the CUE case that are directly obtained with the Legendre polynomials as orthogonal polynomials [11]. The first corrections to this behavior for $t \gg 1$ are proportional $e^{-(\epsilon_N - \epsilon_{N-1})t} = e^{-8Nt} = e^{-t/t_c}$ where $t_c = 1/(8N)$ is the critical time for the Brownian motion, already given in [19]. Of particular interest is the density for one channel which is just for $N = 1$ the distribution of the conductance $g = T = (1 - x)/2$ in units of $2e^2/h$. The result (2.40) implies

$$p(T; t) = 2R_1(1 - 2T; t) \Big|_{N=1} = \sum_{n=0}^{\infty} P_n(1 - 2T) e^{-4n(n+1)t} \tag{2.44}$$

The limiting cases are $p(T; 0) = 1/(2\sqrt{T})$ and $\lim_{t \rightarrow \infty} p(T; t) = 1$ and correspond to the results of the orthogonal and unitary cases respectively given in references [10,11], which have been found to agree with microscopic numerical results [11]. Figure 1 illustrates the crossover between these limiting cases, where $p(T; t)$ is shown for $t/t_c = 0, 0.5, 1, 2, \infty$.

In the limit $N \rightarrow \infty$, it is possible to get a more explicit expression for the density. It is useful to express the density in terms of the angle variable φ , i.e., to consider $\rho_1(\varphi; t) = 2 \sin(2\varphi) R_1(\cos(2\varphi), t)$. The most important contributions in the second sum of equation (2.36) come from terms with indices near N , i.e., $N - k \ll N$ and $n - N \ll N$. We can therefore linearize $\epsilon_k - \epsilon_n \simeq -8N(n - k)$ and replace the Legendre Polynomials by their asymptotic expression for large n, k . After some smoothing over the strongly oscillating contributions we get the expression

$$\rho_1(\varphi; t) \simeq \frac{2}{\pi} \left(N + \sum_{l=0}^{\infty} e^{-(2l+1)(t/t_c)} \cos(2(2l+1)\varphi) \right) = \frac{2}{\pi} \left(N + \frac{\sinh(t/t_c) \cos(2\varphi)}{\cosh(2t/t_c) - \cos(4\varphi)} \right) \tag{2.45}$$

which simplifies in the limit $t \rightarrow 0$ to

$$\rho(\varphi; 0) = \frac{2N}{\pi} + \frac{1}{4} (\delta_+(\varphi) - \delta_+(\frac{\pi}{2} - \varphi)) \tag{2.46}$$

where $\delta_+(\dots)$ is the half-sided delta function that is only defined for positive arguments. In terms of the transmission eigenvalues T the density $\tilde{R}_1(T; t) = 2R_1(1 - 2T; t)$ reads

$$\tilde{R}_1(T; t) = \frac{1}{\pi} \frac{1}{\sqrt{T(1-T)}} \left(N + \frac{\sinh(t/t_c)(1-2T)}{2(\sinh^2(t/t_2) + 4T(1-T))} \right) \quad (2.47)$$

with the limit

$$\tilde{R}_1(T; 0) = \frac{N}{\pi} \frac{1}{\sqrt{T(1-T)}} + \frac{1}{4} (\delta_+(T) - \delta_+(1-T)) \quad (2.48)$$

We see that the asymptotic density for the variable φ is constant for the leading order in N . Equations (2.45-2.48) contain also the next order, responsible for the so-called weak-localization correction of g . Equation (2.48) confirms the result of reference [10] ⁽²⁾.

Some finite N -results for the density are given in Figures 2 and 3 (density for the transmission eigenvalue T with $N = 5$). Figures 2a-b compare the CUE- and COE-limits with the asymptotic density (cp. Eq. (2.47)). One can find strange that the COE-density at finite N apparently fits better the asymptotic density than the CUE-density. This is due to the finite N -oscillations on the scale of the "level-spacing" which are stronger for the CUE than for the COE. In the asymptotic results (2.47) and (2.48) these oscillations are smoothed out. The crossover density interpolates between the two limits. Since the relative change for $N = 5$ is not so strong as for $N = 1$, in Figure 3 only the difference $\tilde{R}_1(T, t) - \tilde{R}_1(T, \infty)$ is shown for $t/t_c = 0, 0.5, 1, 2, \infty$ on a larger scale. In Figure 4 a representative range for the 2-point correlation function $\tilde{R}_2(T_0, T; t)$ for $N = 5$, $t/t_c = 10^{-6}, 10^{-3}, 10^{-2}, 1, \infty$ and $T_0 = 0.7, 0.6 \leq T \leq 0.8$ is illustrated. One can very well see the effect already discussed at the end of Subsection 2.1 that the COE \rightarrow CUE crossover happens at first for very small differences $|T - T_0|$ if t/t_c is small. All the curves shown in Figures 1-4 are obtained by a direct numerical evaluation of expressions (2.40-2.44) and (2.36-2.38).

The average of the conductance $g = \sum_j T_j = \frac{1}{2} \sum_j (1 - x_j)$ and the autocorrelation of the fluctuation $\delta g = g - \langle g \rangle$ can be expressed in terms of integrals with densities (2.40-2.42). These integrals can be evaluated by the recursion relation of the Legendre polynomials and their orthogonality relation with the result

$$\langle g(t) \rangle = \frac{N}{2} - \frac{N}{2(2N+1)} e^{-8Nt} \quad , \quad (2.49)$$

$$\langle \delta g(t) \delta g(t + \tau) \rangle = e^{-8N\tau} \left\{ \frac{N^2}{4(4N^2 - 1)} + \frac{N(N+1)}{4(2N+1)(2N+3)} e^{-8(2N+1)t} + \frac{N(N-1)}{4(2N-1)(2N+1)} e^{-8(2N-1)t} - \frac{N^2}{4(2N+1)^2} e^{-16Nt} \right\} \quad (2.50)$$

These expressions yield for $\tau = 0, t \rightarrow \infty$, the CUE results

$$\langle g \rangle_{\text{CUE}} = \frac{N}{2} \quad , \quad \langle \delta g^2 \rangle_{\text{CUE}} = \frac{N^2}{4(4N^2 - 1)} \quad (2.51)$$

and for $\tau = t = 0$, the COE expressions

$$\langle g \rangle_{\text{COE}} = \frac{N}{2} - \frac{N}{2(2N+1)} \quad , \quad \langle \delta g^2 \rangle_{\text{COE}} = \frac{N(N+1)^2}{(2N+1)^2(2N+3)} \quad (2.52)$$

⁽²⁾ The delta function at $T = 0$ (or $\varphi = 0$) was not found in [10], since the density was expressed in terms of the variable $\lambda = 1/T - 1$ so that $T = 0$ corresponds to $\lambda \rightarrow \infty$, but this contribution is needed here to conserve the normalization.

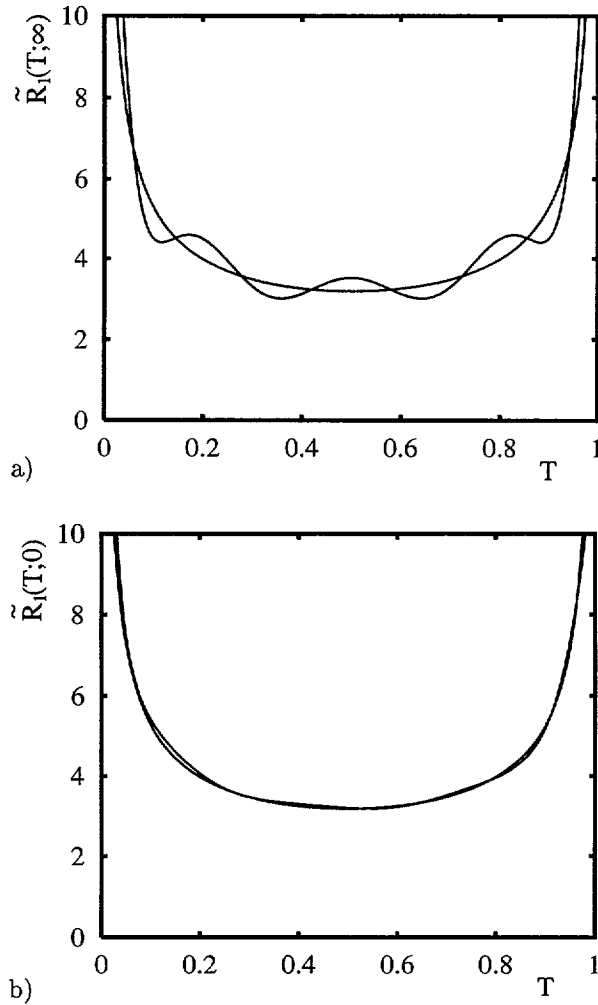


Fig. 2. — Transmission eigenvalue density $\tilde{R}_1(T; t)$ a) for $N = 5$ and $t \rightarrow \infty$ (unitary limit) and the large N -limit $N/(\pi\sqrt{T(1-T)})$ found in reference [10]; b) for $t = 0$ (orthogonal limit).

which are given in reference [11].

In the limit $N \rightarrow \infty$, one has to keep constant the ratio $t/t_c = 8Nt$ in order to get a non-trivial crossover. Equations (2.49) and (2.50) then become up to corrections of order $1/N$

$$\langle g(t) \rangle = \frac{N}{2} - \frac{1}{4} e^{-t/t_c} \quad , \quad (2.53)$$

$$\langle \delta g(t) \delta g(t + \tau) \rangle = \frac{1}{16} e^{-\tau/t_c} (1 + e^{-2t/t_c}) \quad (2.54)$$

The average (2.53) can also be obtained by the asymptotic density (2.45).

Figures 5 and 6 give illustrations of the expressions (2.49) and (2.50) for the average conductance and its variance at $N = 1, 5, \infty$.

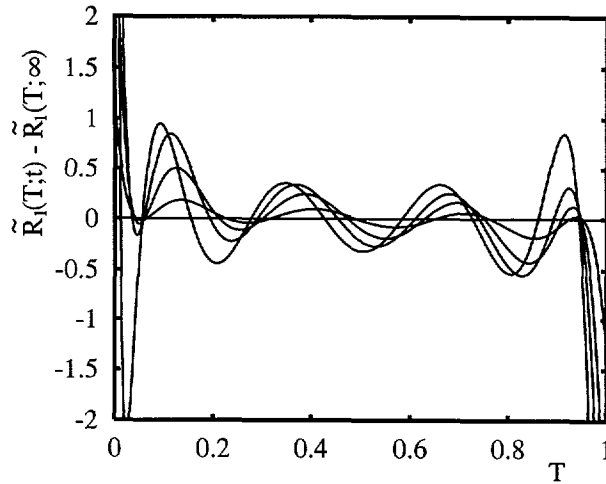


Fig. 3. — Difference $\tilde{R}_1(T;t) - \tilde{R}_1(T;\infty)$ for $N = 5$ and $t/t_c = 0, 0.5, 1, 2, \infty$. The curve with largest deviation corresponds to $t/t_c = 0$.

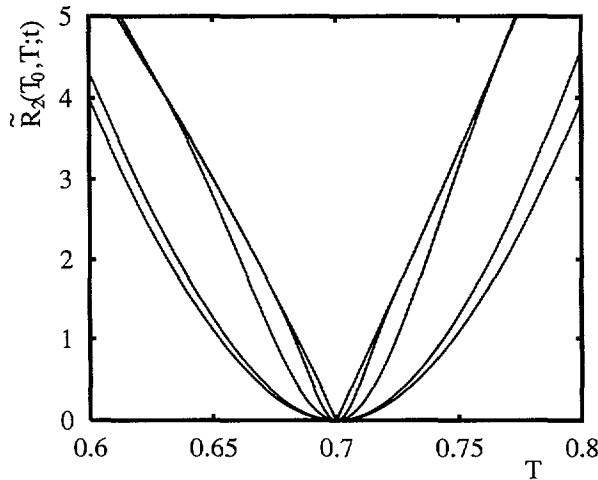


Fig. 4. — 2-point correlation function $\tilde{R}_2(T_0, T; t)$ for $T_0 = 0.7$ and $0.6 \le T \le 0.8$ at $N = 5$ and $t/t_c = 10^{-6}, 10^{-3}, 10^{-2}, 1, \infty$. One can see the beginning of the COE \rightarrow CUE crossover on small scales $|T - T_0|$ when $t/t_c \ll 1$.

3. S-Brownian Motion Time and Magnetic Flux

Till now, we have considered the Brownian motion ensemble COE \rightarrow CUE in terms of the parameter t . We suppose as in references [10, 11] that the scattering matrix of a chaotic quantum dot is well described by a COE or a CUE matrix for zero or sufficiently large magnetic flux, respectively. Then, the obvious question concerning the relation between the Brownian motion time t and the magnetic flux Φ arises. This section is devoted to this question. The consideration of the magnetic flux requires some microscopic model. Usually, such models are treated by numerical or semiclassical means [5, 22, 23]. Here we use another approach that

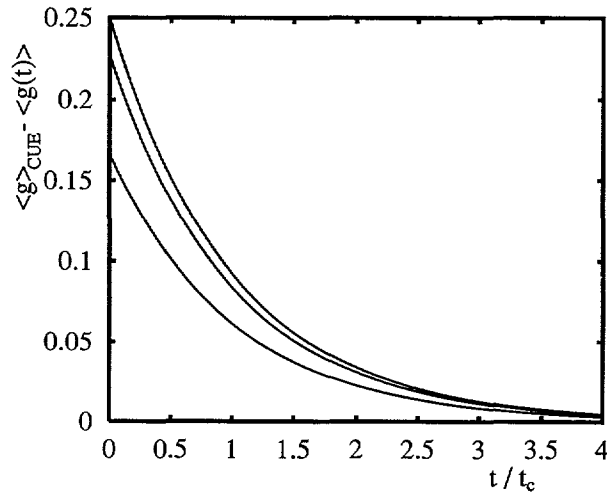


Fig. 5. — The weal-localization suppression $\langle g \rangle_{\text{CUE}} - \langle g(t) \rangle$ for $N = 1, 5, \infty$ as a function of t/t_c .

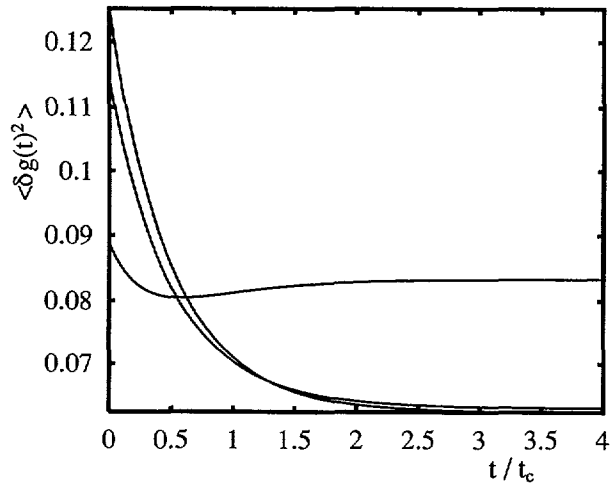


Fig. 6. — Conductance fluctuations $\langle \delta g(t)^2 \rangle$ for $N = 1, 5, \infty$ as a function of t/t_c .

connects S with a Pandey-Mehta Hamiltonian [9] describing the crossover between Gaussian ensembles for the energy levels ($\text{GOE} \rightarrow \text{GUE}$). The crossover parameter of the latter has been related to the magnetic flux for disordered systems [21] and for ballistic cavities [22, 23].

The argumentation of reference [22] relies essentially on numerical calculations and that of reference [23] on a semiclassical approximation. In principle, the relation between the Pandey-Mehta parameter and the magnetic flux is understood due to these references. However, we reconsider this relation in Subsection 3.1, using arguments that account for both the disordered and the ballistic case. The known result for the disordered case [21] is recovered. In the ballistic chaotic case, we find that the relation between the Pandey-Mehta parameter and the magnetic flux is also governed by the quotient of two characteristic energies. In Subsection 3.2 we discuss briefly the typical field scales of a ballistic billiard. In particular, for a very strong magnetic field

the system becomes integrable due to the appearance of Landau-levels. Finally, in Subsection 3.3, we relate the Pandey-Mehta model for the Hamiltonian with the S -Brownian motion ensemble for the scattering matrix. We find that, in order to map the S -Brownian motion model onto this more microscopic model, we need to assume a certain relation between the Brownian motion time and the Pandey-Mehta parameter. This identification gives identical expression for the average quantities in the limit of small fluxes and large number of channels, but does not solve remaining discrepancies for the correlation functions.

3.1. MAGNETIC FLUX AND PANDEY-MEHTA HAMILTONIAN. — The generic random matrix ensemble that describes the crossover from GOE \rightarrow GUE is the Pandey-Mehta Hamiltonian [9]

$$H_\alpha = H_{\text{GOE}}(v^2) + i\alpha A(v^2) \quad (3.1)$$

where H_{GOE} is a GOE matrix [24] (v^2 being the variance of the non diagonal elements) and A is an *antisymmetric matrix* whose independent matrix elements A_{ij} are Gaussian-distributed with variance v^2 . $\alpha \in [0, 1]$ is the crossover parameter. $\alpha = 0$ corresponds to the GOE-case whereas $\alpha = 1$ is the GUE-case.

A typical one-particle Hamiltonian is of the form

$$H = \frac{1}{2m} (\mathbf{p} - e\mathbf{A})^2 + V(\mathbf{r}) \quad (3.2)$$

in $d = 2$ or $d = 3$ space dimensions where $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ is the vector potential in the symmetric gauge. The potential $V(\mathbf{r})$ can model the boundary of a chaotic ballistic cavity or the impurities of a disordered conductor with bulk diffusion, such that the energy levels have Wigner-Dyson GOE-statistics without magnetic field. We decompose H in a real and imaginary part, $H = H_1 + iH_2$ with $H_1 = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) + \frac{e^2}{2m}\mathbf{A}^2$, $H_2 = -\frac{e}{m}\mathbf{p} \cdot \mathbf{A} = -\frac{e}{2m}\mathbf{B} \cdot \mathbf{L}$ and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum. The contribution $\sim \mathbf{A}^2$ in H_1 modifies the potential. For the purpose of this Subsection we restrict ourselves to the case where the magnetic field is sufficiently small and does not change the GOE statistics for $H_1 \simeq H(\mathbf{B} = 0)$. The range of magnetic field at which this behavior is not true corresponds to a strong influence of the Landau Levels and the system can not be considered as "chaotic". A more detailed discussion of this topic and the relevant scales of the magnetic field will be given in the next Subsection.

We consider a certain energy E_F and want to describe H by the Pandey-Mehta Hamiltonian (3.1) in an energy range $E \in [E_F - \Delta E, E_F + \Delta E]$. When $\mathbf{B} = 0$ we assume that $H = H_1$ with $H_{\text{GOE}}(v^2)$. The variance v^2 is determined by the requirement [24] $v = \Delta_0 \sqrt{\mathcal{N}}/\pi$ where Δ_0 ($\ll \Delta E$) is the level spacing of H_1 and \mathcal{N} is the chosen dimension of the GOE ensemble, assumed to be large. We do not discuss here whether the Pandey-Mehta Hamiltonian gives an appropriate description of the microscopic Hamiltonian H at $\mathbf{B} \neq 0$. The numerical and perturbative calculations of reference [21] confirm that the level correlations of a disordered ring in the diffusive region agree with the correlation functions of H_α found in reference [9]. The parameter α is then given by the identification [21]

$$\alpha = \sqrt{\frac{\pi}{\mathcal{N}} \frac{E_T}{\Delta_0}} \left(2\pi \frac{\Phi}{\Phi_0} \right) \quad (3.3)$$

where $E_T = \hbar D/(2\pi R)^2$ ($D =$ diffusion constant, $R =$ ring radius) is the Thouless energy and $\Phi_0 = h/e$ is the flux quantum. The dependence on \mathcal{N} indicates that the GOE to GUE crossover is properly described by the parameter $\lambda = \sqrt{\mathcal{N}}\alpha/\pi$ in the limit $\mathcal{N} \rightarrow \infty$ [9]. The ballistic quantum billiard has been studied by numerical diagonalization [22]. Again the correlation

functions [9] are valid and the parameter α is found to be proportional to a flux Φ applied through a typical area of the system.

We now give an alternative derivation of equation (3.3) based on a crude estimation of the variance of a typical matrix element $(H_2)_{ij}$ in the eigenvector basis of the real part H_1 of H . Let $|i\rangle$ be an eigenvector of H_1 with eigenvalue ε_i . For $\varepsilon_i \simeq E_F$, we consider the expectation value

$$\langle i|L_z(0)L_z(t)|i\rangle = \langle i|L_z e^{iH_1 t/\hbar} L_z e^{-iH_1 t/\hbar}|i\rangle = \sum_j |\langle i|L_z|j\rangle|^2 e^{i(\varepsilon_j - \varepsilon_i)t/\hbar} \quad (3.4)$$

where L_z is the z -component of the angular momentum (assuming that the direction of \mathbf{B} is chosen as the z -axis). $L_z(t)$ is the time dependent angular momentum with a dynamic determined by H_1 . We now average over the disorder or over the random shape of the cavity and assume that the average of the matrix element of the angular momentum is a smooth function of the energy difference $\varepsilon_j - \varepsilon_i$, i.e., $\langle |\langle i|L_z|j\rangle|^2 \rangle_V \simeq l(\varepsilon_j - \varepsilon_i)$. equation (3.4) then becomes

$$\langle L_z(0)L_z(t) \rangle_{Q,V} = \frac{\hbar}{\Delta_0} \int d\omega l(\hbar\omega) R_2\left(\frac{\hbar\omega}{\Delta_0}\right) e^{i\omega t} \quad (3.5)$$

The subscript on the left side denotes a combined expectation value from quantum mechanics and from the random parameters of V . R_2 is just the two-point correlation function in the orthogonal case with the limit $R_2(x) \simeq 1$ if $|x| \gg 1$. The inversion of the Fourier transform gives

$$l(\hbar\omega) R_2\left(\frac{\hbar\omega}{\Delta_0}\right) = \frac{\Delta_0}{2\pi\hbar} \int dt \langle L_z(0)L_z(t) \rangle_{Q,V} e^{-i\omega t} \quad (3.6)$$

The expectation value in the integral vanishes at a time scale $|t| \gg \tau_L$ where τ_L is the correlation time for the angular momentum. Hence, the integral itself vanishes as a function of ω at the scale $|\omega| \gg 1/\tau_L$. From this behavior we can already see the limitations of the GOE description for H_1 . The absolute squared matrix elements of an arbitrary fixed operator with respect to the eigenvectors of a GOE-Hamiltonian always yields a constant value independent of the particular indices $i \neq j$ after the average, since there is no preferential basis. Equation (3.6) implies that for H_1 this is only true for an energy difference $|\varepsilon_j - \varepsilon_i| \ll \hbar/\tau_L =: E_{\text{corr}}$. This limits the size of the energy range that we may consider. Since we are interested by the off-diagonal matrix elements for $\Delta_0 \ll |\varepsilon_j - \varepsilon_i| \ll E_{\text{corr}}$, we take $R_2(\frac{\hbar\omega}{\Delta_0}) \simeq 1$. From equation (3.6) we get for the variance of the imaginary matrix elements $(H_2)_{ij} = -\frac{eB}{2m} \langle i|L_z|j\rangle$ of H

$$\langle |(H_2)_{ij}|^2 \rangle \simeq \frac{\Delta_0}{\pi\hbar} \left(\frac{eB}{2m}\right)^2 \tau_L \langle L_z^2 \rangle_{Q,V} \quad (3.7)$$

where the correlation time τ_L is defined as $\tau_L \langle L_z^2 \rangle_{Q,V} = \frac{1}{2} \int dt \langle L_z(0)L_z(t) \rangle_{Q,V}$. Let $2R$ be the typical diameter of the system and v_F be the Fermi velocity. Furthermore, we introduce a characteristic ballistic energy scale $E_{\text{ball}} = \hbar v_F/(2R)$ and the flux $\Phi = \pi R^2 B$ applied through a circle of radius R . The expectation value for the angular momentum is just $\langle L_z^2 \rangle_{Q,V} = \kappa m^2 v_F^2 R^2$ where κ is a numerical factor characteristic of the shape of the dot which can be evaluated in a semiclassical way, as discussed in Appendix C.

The variance (3.7) now becomes

$$\langle |(H_2)_{ij}|^2 \rangle \simeq \frac{4\kappa}{\pi} \left(\frac{\Phi}{\Phi_0}\right)^2 \frac{\Delta_0 E_{\text{ball}}^2}{E_{\text{corr}}} \quad (3.8)$$

and has to be compared with the variance of the imaginary matrix elements in equation (3.1) that is just $\alpha^2 \Delta_0^2 \mathcal{N} / \pi^2$. This gives for the parameter α

$$\alpha = \sqrt{\frac{4\pi\kappa}{\mathcal{N}} \frac{E_{\text{ball}}}{\sqrt{\Delta_0 E_{\text{corr}}}}} \left(\frac{\Phi}{\Phi_0} \right) \quad (3.9)$$

This expression contains the dimensionless flux and three typical energies $E_{\text{ball}} = \hbar v_F / (2R)$, $E_{\text{corr}} = \hbar / \tau_L$ and the level spacing Δ_0 .

Let us first consider a disordered conductor where the electrons are scattered by many impurities. The angular momentum correlation time is then the elastic scattering time $\tau_L = \tau_{\text{el}}$ and the energy quotient becomes $E_{\text{ball}}^2 / (E_{\text{corr}} \Delta_0) = (2\pi)^2 E_T d / \Delta_0$ where E_T is the Thouless energy ($E_T = \hbar D / (2\pi R)^2$ for a diffusion constant $D = v_F^2 \tau_{\text{el}} / d$). Equation (3.9) confirms the result (3.3) [21] for a ring with $\kappa = 1/d$ (Appendix C).

For a ballistic quantum billiard, τ_L equals the time of flight $\tilde{\kappa} 2R / v_F$ ($\tilde{\kappa}$ is a further numerical factor) through the cavity and $E_{\text{ball}} = \tilde{\kappa} E_{\text{corr}}$. This gives

$$\alpha = \sqrt{\frac{4\pi\kappa\tilde{\kappa}}{\mathcal{N}} \frac{E_{\text{ball}}}{\Delta_0}} \left(\frac{\Phi}{\Phi_0} \right) \quad (3.10)$$

Both, this result and equation (3.3) for the disordered case depend on the energy ratio E_C / Δ_0 where $\tau_C = \hbar / E_C$ is the typical time for the electron to cross the system, either by a direct ballistic or by a diffusive motion.

It is well known [26] that the level statistics of a disordered conductor are well described by the GOE-statistics up to an energy scale E_T . Our derivation yields the energy scale $E_{\text{corr}} = \hbar / \tau_{\text{el}}$. This is due to the fact that we use a criterion based on the matrix elements of the angular momentum. If one estimates in a similar way the matrix elements of the position operator, one recovers the characteristic scale E_T , which matters if we consider only for the level correlations ⁽³⁾.

Such a difference does not matter for a chaotic ballistic dot where those two energies merge to a single characteristic scale $E_{\text{ball}} = \hbar v_F / (2R)$. The energy ratio $E_{\text{ball}} / \Delta_0$ is of the order of $(k_F R)^{d-1} \gg 1$ (k_F is the Fermi momentum, for comparison: $E_F / \Delta_0 \sim (k_F R)^d$). The critical flux for the GOE to GUE crossover (for an energy interval of order Δ_0) is $\Phi_C \sim \Phi_0 / (k_F R)^{(d-1)/2} \ll \Phi_0$. In $d = 2$ dimensions one gets $\sqrt{E_{\text{ball}} / \Delta_0} \sim (k_F R)^{1/2} \sim n^{1/4}$ where n is the number of energy levels below the Fermi energy in the quantum billiard. This scaling behavior was already given in reference [22] and applied in reference [8]. In the numerical simulations of reference [27] the factor $\sqrt{E_{\text{ball}} / \Delta_0}$ is of the order of one because it is rather difficult to increase n to values where this scaling behavior becomes important. The situation changes a bit in three dimensions, where $\sqrt{E_{\text{ball}} / \Delta_0} \sim (k_F R) \sim n^{1/3}$. An estimation of the numerical factor that appears in equation (3.10) is given in Appendix C.

3.2. STRONG AND WEAK MAGNETIC FIELD SCALES IN A QUANTUM BILLIARD. — We now briefly discuss the effects of stronger magnetic fields in a quantum billiard. For simplicity, we consider the two-dimensional case, where a free electron in a magnetic field B has Landau levels with energies $E_n = \hbar \omega_C (n + \frac{1}{2})$. $n = 0, 1, 2, \dots$ is the number of the Landau band and $\omega_C = (eB) / m$ is the cyclotron frequency. The eigenstates are a combination of harmonic oscillator

⁽³⁾ The considerations made here for the disordered case are only valid in the diffusive or metallic regime where the states are not localized. In the localized regime the GOE statistics are no longer valid and, in addition, the expectation value of L_z^2 would depend on the localization length instead of the typical system size.

states (with frequency ω_c) in one direction and plane waves in the orthogonal direction. The magnetic length $l_c = \sqrt{\hbar/(m\omega_c)}$ is the characteristic length of the harmonic oscillator. l_c is the typical width of the states (in the oscillator direction) of the first Landau band with $n = 0$. At an arbitrary Energy E_F the states have a width $\sim \sqrt{E_F/(\hbar\omega_c)}$ $l_c \sim v_F/\omega_c = R_{lam}$ of the order of the Lamor radius.

If in a quantum billiard of typical size R the magnetic field is very high so that $R_{lam} \ll R$, the effect of the Landau Levels becomes very strong. There is a finite number of states of each Landau band (with energies $\leq E_F$) that do not feel the boundary because their typical width R_{lam} is sufficiently small. The states near the boundary (“edge states”) are raised to higher energies. The energy spectrum near E_F does not differ very much from the spectrum of the unbounded plane and cannot be described by a GUE spectrum. The only effects are a (very) small breaking of the degeneracies and the appearance of the edge states. The classical trajectories are circles with radius R_{lam} that do not see the boundary. If the magnetic field is lowered (such that $l_c \ll R \ll R_{lam}$) the fast electrons at the Fermi energy E_F feel very well the boundary and the energy spectrum should be closer to a GUE statistics. But at low energies $E < E_F$ the Landau levels still exist and the corresponding edge states can have energies near E_F . The latter may well complicate a simple GUE picture because the corresponding eigenfunctions have a very small overlap with most of the states at E_F . This effect disappears if the magnetic field becomes smaller such that $R \leq l_c$. Then the real GUE statistics applies.

Taking into account the effects described above and the results of the last Subsection, we distinguish four regions for the magnetic field.

- (i) $\Phi/\Phi_0 \leq \sqrt{\Delta_0/E_{ball}}$, the GOE to GUE crossover regime. The level statistic of this regime is described by the correlation functions of reference [9]. If $\Phi/\Phi_0 \ll \sqrt{\Delta_0/E_{ball}}$, the behavior is near to a GOE but one must take into account that the level repulsion becomes immediately quadratic on a very short energy scale.
- (ii) $\sqrt{\Delta_0/E_{ball}} \ll \Phi/\Phi_0 \leq 1$, the GUE regime. The level statistics are described by a GUE.
- (iii) $1 \ll \Phi/\Phi_0 \ll E_F/E_{ball}$ or $l_c \ll R \ll R_{lam}$, the regime where first Landau level effects appear. The level statistic is roughly GUE with complications due to edge states.
- (iv) $E_F/E_{ball} \ll \Phi/\Phi_0$, or $R_{lam} \ll R$, the Landau level regime. The behavior of the eigenvalues near E_F is very close to the spectrum of the Landau levels and thus regular.

This subdivision of the different magnetic field scales remains essentially valid in three instead of two space dimensions. The behavior of the regions (i) and (ii) is the same. In the regions (iii) and (iv) the influence of the z -component of the momentum becomes important. In the region (iv) every energy eigenvalue is a sum of one Landau level of the xy -plane and a contribution due to the kinetic energy in the z -direction. The Hamiltonian is (roughly) a sum of two regular independent parts. The situation in region (iii) is again rather complicated due to a mixture of “Poisson”- and “GUE”-levels.

Our main concern is the crossover between the regions (i) and (ii), where we suppose that the Hamiltonian is well described by the Pandey-Mehta Hamiltonian (3.1) with a parameter α given by equation (3.10). In the next Subsection we discuss how α can be related with the Brownian motion time t of the COE \rightarrow CUE crossover.

3.3. S-BROWNIAN MOTION TIME AND PANDEY-MEHTA HAMILTONIAN. — The $2N \times 2N$ -scattering matrix S of a physical system that is described by an $\mathcal{N} \times \mathcal{N}$ -random Hamiltonian H can be expressed by

$$S(H) = 1 - 2\pi i W^\dagger G_+ W \quad (3.11)$$

where we have used the abbreviation

$$G_{\pm} = (E - H \pm i\pi W W^{\dagger})^{-1} \quad (3.12)$$

W is a $\mathcal{N} \times 2N$ matrix that describes the coupling between the states of the system with the scattering channels. The expression (3.11) is well justified [7, 28] (and references therein) and is the usual starting point for supersymmetric calculations [7, 8, 28, 29]. It is not very difficult to see that $S(H)$ given by (3.11) is indeed a unitary matrix if H is Hermitian. We take the limit $\mathcal{N} \rightarrow \infty$ but the dimension $2N$ of the scattering matrix may remain finite (as considered in reference [8]). When we consider the limit $N \gg 1$, we nevertheless assume $N \ll \mathcal{N}$. We note $M = 2N$ the dimension of S .

The expression (3.11), relating an appropriate Hamiltonian to S , is what we need for giving a microscopic motivation of the S -Brownian motion model. However, we assume for simplicity a crossover random matrix model for the Hamiltonian, and not a real microscopic model.

Let us assume that the Hamiltonian H contains a typical parameter depending on the magnetic field. A small change of the parameter leads to the replacement $H \rightarrow H + \Delta H$ which gives for S

$$S(H + \Delta H) = S(H) \exp(-i\delta X) \quad (3.13)$$

where δX is a Hermitian matrix. Expanding equation (3.11) up to second order we obtain after some matrix-algebra the expression

$$\begin{aligned} \delta X = & 2\pi \left\{ W^{\dagger} G_{-} \Delta H G_{+} W \right. \\ & \left. + W^{\dagger} G_{-} \Delta H G_{+} (E - H) G_{-} \Delta H G_{+} W \right\} + \mathcal{O}((\Delta H)^3) \end{aligned} \quad (3.14)$$

We now assume that the Hamiltonian can be written in the form

$$H(\gamma) = H_0 + \gamma H_{\text{GUE}}(v^2) \quad (3.15)$$

where H_0 is some (fixed or random) part of the Hamiltonian that may be rather general at the moment. $H_{\text{GUE}}(v^2)$ is a GUE-random matrix (*statistically independent* of H_0) whose elements are independent and normally distributed. The average and variance of $H_{\text{GUE}}(v^2)$ are characterized by

$$\langle H_{\text{GUE}}(v^2) \rangle = 0 \quad , \quad \langle H_{\text{GUE}}(v^2)^{\dagger} A H_{\text{GUE}}(v^2) \rangle = v^2 \text{tr}(A) \mathbf{1}_{\mathcal{N}} \quad (3.16)$$

where A is an arbitrary $\mathcal{N} \times \mathcal{N}$ test-matrix. We consider now an infinitesimal change of the parameter $\gamma \rightarrow \gamma + \delta\gamma$. Since the sum of two normally distributed variables is also normally distributed, we can apply the decomposition

$$(\gamma + \delta\gamma) H_{\text{GUE}}(v^2) \rightarrow \gamma H_{\text{GUE},1}(v^2) + \Delta\gamma H_{\text{GUE},2}(v^2) \quad (3.17)$$

where $H_{\text{GUE},1}$ and $H_{\text{GUE},2}$ are two different and completely independent GUE-matrices both characterized by (3.16). The variances are related through

$$(\gamma + \delta\gamma)^2 = \gamma^2 + (\Delta\gamma)^2 \Rightarrow \Delta\gamma \simeq \sqrt{2\gamma \delta\gamma} \quad (3.18)$$

The replacement (3.17) is exact and does not incorporate any approximation if the translation (3.18) is properly used. We emphasize that this substitution of one set of Gaussian variables with a sum of two independent sets of new Gaussian variables works as far as we are only interested in simple but arbitrary averages of the type $\langle f[S(H(\gamma))] \rangle$ (where f is an arbitrary

function of the scattering matrix) for a given value of γ . In the case of correlations between different γ values, i. e. if we want to calculate averages of the type $\langle f_1[S(H(\gamma_1))] f_2[S(H(\gamma_1 + \gamma_2))] \rangle$, the decomposition (3.17) can not be used because it is not possible to vary γ_2 in this way without changing γ_1 . However, for the case $\gamma_1 = 0$ the dependence on γ_2 can be considered in the same way. Therefore, the following argumentation concerning the relation between the Brownian motion time t and the parameter γ holds for averages $\langle f[S(H(\gamma))] \rangle$ and the particular correlations $\langle f_1[S(H(0))] f_2[S(H(\gamma))] \rangle$.

We identify the small matrix ΔH with $\Delta\gamma H_{\text{GUE},2}$. The hermitian matrix δX given by equation (3.14) is now a random matrix matrix because of $H_{\text{GUE},2}$. We average over this GUE-matrix (denoted by $\langle \dots \rangle_2$) and obtain with the help of (3.16)

$$\langle \delta X \rangle_2 = \frac{1}{2} v^2 (\Delta\gamma)^2 (\text{tr}(G_+) + \text{tr}(G_-)) Q(H) \quad , \quad (3.19)$$

$$\langle \delta X^\dagger \tilde{A} \delta X \rangle_2 = v^2 (\Delta\gamma)^2 \text{tr}(\tilde{A} Q(H)) Q(H) \quad (3.20)$$

where now \tilde{A} is an arbitrary $M \times M$ -test matrix and $Q(H)$ is the Wigner time delay matrix given by

$$Q(H) = 2\pi W^\dagger G_- G_+ W = -i S^\dagger \frac{\partial S}{\partial E} \quad (3.21)$$

We underline that the matrix elements $Q_{i,j}$ give the time delay characterizing the scattering modes of the chaotic cavity. This physical meaning is very useful for a more concrete and intuitive understanding of this somewhat abstract S -Brownian motion ensemble. We have to compare the averages (3.19) and (3.20) with the corresponding averages of the Brownian motion "increment" $\delta X^{(B)}$ [15, 19] given by

$$\langle \delta X^{(B)} \rangle_2 = 0 \quad , \quad (3.22)$$

$$\langle (\delta X^{(B)})^\dagger \tilde{A} \delta X^{(B)} \rangle_2 = D \delta t \text{tr}(\tilde{A}) \mathbf{1}_M \quad (3.23)$$

where δt is the change of the Brownian motion time t and D is the diffusion constant (which is chosen as $D = 4$ in the applications of this paper and in reference [19]). From this we can directly state:

The Brownian motion model is the idealization in which the following two conditions are fulfilled

$$\text{tr}(G_+) + \text{tr}(G_-) = 0 \quad , \quad (3.24)$$

$$Q(H) = q_0(\gamma^2) \mathbf{1}_M \quad (3.25)$$

where $q_0(\gamma^2)$ is a number that may depend on γ^2 but not on the different realizations of H_{GUE} in (3.15). If these conditions are valid the relation between the Brownian motion time and the parameter γ is determined by $D \delta t = v^2 (\Delta\gamma)^2 q_0^2(\gamma^2) \simeq v^2 2\gamma \delta\gamma q_0^2(\gamma^2)$ which results in the translation

$$t = \frac{v^2}{D} \int_0^{\gamma^2} dx q_0^2(x) \quad (3.26)$$

Assuming (3.25) we can evaluate $q_0(\gamma^2)$ as the trace of the matrix $Q(H)$ which is just Wigner's time delay

$$\begin{aligned} M \cdot q_0(\gamma^2) &= \text{tr}(Q(H)) = (-i)\text{tr}(G_+(2\pi i WW^\dagger)G_-) \\ &= (-i)(\text{tr}(G_-) - \text{tr}(G_+)) \end{aligned} \quad (3.27)$$

where the last equality holds due to (3.12). Let us assume (as in reference [29]) that the operator $H + i\pi WW^\dagger$ has complex eigenvalues $E_\mu + i\Sigma_\mu$, $\Sigma_\mu > 0$ such that

$$\text{tr}(G_\pm) = \sum_\mu \frac{E - E_\mu \mp i\Sigma_\mu}{(E - E_\mu)^2 + \Sigma_\mu^2} \simeq \mp i \frac{\pi}{\Delta} \quad (3.28)$$

The second equality is an approximation valid if the typical values of the imaginary parts satisfy $\Sigma_\mu \gg \Delta$ where Δ is the level spacing of H . Then, one can replace the μ -sum by an integral over E_μ . This condition is related to the limit $M \gg 1$ where the conditions (3.24) and (3.25) make sense. Actually, equation (3.28) yields already the first condition (3.24).

The second condition (3.25) means that the time delays of the different scattering modes are roughly identical (which looks natural for a chaotic cavity) and do not fluctuate (in comparison with their mean value). This second assumption is less natural, but can be justified if the number of channels is large (4).

In the following, we only consider a range for the parameter γ that is small enough for having Δ determined by the part H_0 of the Hamiltonian (3.15), i.e., $\Delta \simeq \Delta_0$ where Δ_0 is the level spacing of H_0 . This assumption is very reasonable since in the large \mathcal{N} limit the range $\gamma \sim 1/\sqrt{\mathcal{N}}$ yields already the wanted crossover (if H_0 is a GOE matrix). The number q_0 is now approximately independent of γ . Due to (3.28) and (3.27) it is given by $q_0 \simeq 2\pi/(M\Delta_0)$. Furthermore, since the variance v^2 is related to the level spacing Δ_{GUE} of H_{GUE} via [24] $v^2 = \mathcal{N} \Delta_{\text{GUE}}^2/\pi^2$, one gets for the Brownian motion time (3.26)

$$t = \frac{\mathcal{N}}{M^2} \frac{\Delta_{\text{GUE}}^2}{\Delta_0^2} \gamma^2 \quad (3.29)$$

where the diffusion constant $D = 4$. Assuming the validity of the conditions (3.24) and (3.25) with the approximation (3.28), the S -matrix of $H = H_0 + \gamma H_{\text{GUE}}(v^2)$ is described by the Brownian motion ensemble with an initial condition $S(t=0) = S(H_0)$ and a time parameter (3.29).

Now, we assume that the initial Hamiltonian is a Gaussian orthogonal random matrix $H_0 = H_{\text{GOE}}(v^2)$ and that $H(\gamma) = H_{\text{GOE}}(v^2) + \gamma H_{\text{GUE}}(v^2)$. The S -Brownian motion ensemble corresponds approximately to this choice with the S -matrix expression (3.11) and the time-translation $t = (\mathcal{N}\gamma^2)/M^2$. The Brownian motion ensemble can describe properly *averages and correlations between $\gamma = 0$ and $\gamma > 0$* (5).

(4) If H_0 is a GOE random matrix, we can consider the deviation $\delta Q = Q - 1/M \text{tr}(Q) 1_M$ from Q with the matrix $q_0 1_M$. A measure for the fluctuations δQ is just $\text{tr}(\delta Q^2) = \text{tr}(Q^2) - 1/M \text{tr}(Q)^2$. We have calculated the average of this quantity using supersymmetric techniques in leading order of a large channel number with the result $\langle \text{tr}(\delta Q^2) \rangle = (2\pi/\Delta_0)^2/M = \mathcal{O}(M^{-1})$ where Δ_0 is the level spacing of H_{GUE} . We do not present this calculation based on a perturbative treatment of the σ -model [28]. The relative variance $\langle |\delta Q_{ij}|^2 \rangle / \langle Q_{ii} \rangle^2$ of an arbitrary matrix element of δQ_{ij} is of the order $\mathcal{O}(M^{-1})$ thus verifying the second condition (3.25) in the limit

(5) We have still to ensure that the initial condition $\gamma = 0$ corresponds to the circular orthogonal ensemble. In general, the S -matrix described by equation (3.11) represents a wide class of random matrix ensembles depending on the independent matrix elements of the symmetric $M \times M$ -matrix $W^\dagger W$ [7]. In the case $H = H_{\text{GOE}}$, one expects [29] that there is one particular choice of $W^\dagger W$ in which $S(H)$ is exactly distributed as a unitary COE random matrix. This choice is characterized by the requirement that the average $\langle S(H) \rangle = 0$ vanishes corresponding to a maximal coupling between the scattering channels and the system [7]. At the energy range $|E| \ll \mathcal{N}\Delta_{\text{GOE}}$ one requires [7, 29] $W^\dagger W = \mathcal{N}\Delta_{\text{GOE}}/\pi^2 \cdot 1_M$. The typical value of the imaginary parts Σ_μ used in equation (3.28) can then be estimated as $\Sigma_\mu \simeq 1/\mathcal{N} \text{tr}(\pi WW^\dagger) = M\Delta_{\text{GOE}}/\pi$ and the condition for the validity of (3.28) reads as $M \gg 1$.

We can note that the random Hamiltonian $H(\gamma)$ is not identical to the Pandey-Mehta Hamiltonian (3.1). However, we can again use a decomposition of Gaussian variables which leads to the identification $H_\alpha \rightarrow \sqrt{1 - \alpha^2} H_{\text{GOE}}(v^2) + \sqrt{2} \alpha H_{\text{GUE}}(v^2)$. The factor $\sqrt{2}$ is due to the fact that the sum of the real and the imaginary parts of the (off diagonal) matrix elements has a doubled variance, i.e., $\langle (H_{\text{GOE},ij})^2 + (A_{ij})^2 \rangle = 2v^2 = 2\langle |H_{\text{GUE},ij}|^2 \rangle$. Since the typical scale of the parameter α is $\alpha \sim 1/\sqrt{\mathcal{N}}$, we can approximate $H_\alpha \simeq H_{\text{GOE}}(v^2) + \sqrt{2} \alpha H_{\text{GUE}}(v^2) = H(\sqrt{2}\alpha)$ and we obtain thus the relation $\gamma = \sqrt{2}\alpha$ leading to the final relation between t and α

$$t = t(\alpha) = \frac{2}{M^2} (\mathcal{N} \alpha^2) = \frac{2}{M^2} \pi^2 \lambda^2 \tag{3.30}$$

where λ is the crossover parameter used in reference [9].

We emphasize that the identification of the Pandey-Mehta Hamiltonian H_α with $H(\sqrt{2}\alpha)$ works only as far as averages are concerned. The correlations between $\alpha = 0$ and $\alpha > 0$ are then **not** properly described by the Brownian motion model and indeed the critical field scale for these correlations obtained from the Brownian motion ensemble is a factor $\sqrt{2}$ too small if compared with semiclassical results [5] and a supersymmetric treatment of the Pandey-Mehta Hamiltonian [30].

4. Validity and Limit of the S -Brownian Motion Ensemble

The S -Brownian motion time t is related to the crossover parameter $\lambda = \sqrt{\mathcal{N}}\alpha/\pi$ of the Pandey-Mehta Hamiltonian (3.1) via

$$t(\lambda) = \frac{2\pi^2 \lambda^2}{M^2} \tag{4.1}$$

This identification assumes the limit $M \gg 1$. One can express λ in terms of the magnetic flux: for a ballistic chaotic dot, we found in Subsection 3.1 the expression

$$\lambda = \sqrt{\frac{4\kappa \tilde{\kappa}}{\pi} \frac{\hbar v_F}{\Delta_0 2R} \left(\frac{\Phi}{\Phi_0} \right)} \tag{4.2}$$

where κ and $\tilde{\kappa}$ are numerical constants characteristic for the shape of a dot of typical size $2R$ and of level spacing Δ_0 without magnetic flux, v_F is the Fermi velocity.

We can now compare result (2.49) for the average magnetoconductance with that of reference [8]. From equation (4.1) we obtain for the S -Brownian motion ensemble

$$\langle g(\lambda) \rangle_{\text{BE}} = \frac{M}{4} - \frac{M}{4(M+1)} \exp\left(-\frac{8\pi^2 \lambda^2}{M}\right) \tag{4.3}$$

whereas the supersymmetric result of reference [8] reads

$$\langle g(\lambda) \rangle_{\text{SS}} \simeq \frac{M}{4} - \frac{M}{4(M+1)} \left(1 + \frac{8(M-1)\pi^2 \lambda^2}{M^2} \right)^{-1} \tag{4.4}$$

Equation (4.4) is not the exact analytical result, but an accurate fit obtained after the numerical evaluation of three integrals [8]. In leading order of $1/M$ and in the region $\lambda^2 \ll M/(8\pi^2)$, both results become identical, whereas for larger values of λ the difference between the Lorentzian and the Gaussian is more important. This can be understood from the argumentation developed in Subsection 3.3, which shows that the S -Brownian motion model cannot be exactly

mapped on the S -matrix expression (3.11) used in reference [8]. The expressions (3.19) and (3.20) for the averages of the increment δX show that in reality we have to deal with a more complicated stochastic process than the simplified one implied by equations (3.22) and (3.23). Both stochastic processes are only approximately in agreement for sufficiently small Brownian motion times or values of the symmetry breaking parameter, i.e., for $\lambda^2 \ll M/8$, where the results (4.3) and (4.4) are consistent.

The correlations between two different values λ and $\lambda + \Delta\lambda$ are only properly described if λ vanishes and $\Delta\lambda$ is sufficiently small. In addition, the comparison with semiclassical [5] and supersymmetric [19] results gives a factor $\sqrt{2}$ for the relevant $\Delta\lambda$ scale. This factor can be understood by the difference between a symmetry breaking due to an imaginary *antisymmetric* matrix or to a *fully hermitian* matrix (compare Subsection 3.3 and Ref. [30]).

In reference [19] it was shown that the Fokker-Planck equation for the transmission eigenvalues T_i can also be obtained for a more general matrix δX than described by equations (3.22) and (3.23). If we assume that this is true for the S -matrix (3.11) with the Pandey-Mehta Hamiltonian, we have only to determine correctly the relation between the S -Brownian motion time and the parameter λ . Then, the expression (3.30) will only be an approximation valid in the range $\lambda^2 \ll M/(8\pi^2)$ and one could use the supersymmetric result (4.4) to obtain a more appropriate expression

$$t = \frac{1}{4M} \ln \left(1 + \frac{8(M-1)\pi^2 \lambda^2}{M^2} \right) \quad (4.5)$$

This modified relation assumes that the problem resulting from the difference between the S -Brownian motion model and the microscopic S -matrix (3.11) can be solved by an appropriate rescaling of the Brownian motion time ⁽⁶⁾. This will work for the average conductance (suppression of the weak localization correction), but *not for the conductance-conductance correlations*.

It is worth underlining that the crossover from orthogonal to unitary symmetry happens at the critical M -dependent value $\lambda_C = \sqrt{M/8}/\pi$ for the parameter λ . The corresponding critical flux is given by

$$\Phi_C = \Phi_0 \sqrt{\frac{M}{32\pi\kappa\tilde{\kappa}} \frac{\Delta_0 2R}{\hbar v_F}} \quad (4.6)$$

This is due to the M -dependence of the typical time for an electron to stay in the dot. With increasing channel number $N = M/2$, this typical time is shorter. Therefore the critical flux needed to break the phase coherence between time reversed paths is larger. Semiclassical analysis [5] shows that the distribution of the effective area Θ surrounded by the electronic trajectories plays an important role. The comparison (of the quadratic term) of the Lorentzian [5] with the above expressions (4.3-4.4) yields just

$$\Phi_C = \frac{1}{2} \alpha_{cl} A \Phi_0 \quad (4.7)$$

where A is the area of the two-dimensional cavity and α_{cl} is the characteristic inverse area appearing in the exponential behavior of the probability density $p(\Theta) \sim \exp(-\alpha_{cl} |\Theta|)$ for Θ . Equations (4.6) and (4.7) yield directly $\alpha_{cl} \sim \sqrt{M}$ or $(|\Theta|) \sim 1/\sqrt{M}$.

The studied S -Brownian motion ensemble has the decisive advantage that many important properties (Section 2) can be calculated exactly, even for a finite dimension M of the S -matrix. It is possible to relate this model to the Pandey-Mehta Hamiltonian when there is a large

⁽⁶⁾ In fact, relation (4.5) gives the correct λ -dependence of the conductance fluctuations (in the limit $M \gg 1$) if we apply this relation to (2.54) (for $\tau = 0$) and compare it with a perturbative treatment of the σ -model [30].

number of channels up to a certain approximation and to express the Brownian motion time in terms of the parameter λ (directly related to the magnetic flux). This yields consistent results for the magnetoconductance for sufficiently small values of $\lambda \ll \lambda_C$ whereas in the other limit the deviation between a Lorentzian and a Gaussian behavior becomes relevant. This indicates that the S -matrix (3.11) with the Pandey-Mehta Hamiltonian (3.1) is not equivalent to the S -Brownian motion model. This latter model assumes a *uniform and irreversible* diffusion process in the S -matrix space being valid for short time scales. For longer times, a rescaling of the time scale, as proposed in equation (4.5), improves the validity of the model, as far as averages are concerned, e.g., average and variance of the conductance. It is shown however [30] that even this rescaling of the Brownian motion time is not sufficient for having the right autocorrelation function between different times.

To improve this exactly solvable model, one might reconsider its two main assumptions: first that the infinitesimal evolution $\exp(-i\delta X)$ of S is statistically independent of S , and second, that this evolution is isotropic with a time delay matrix $Q(H)$ fulfilling the conditions equations (3.25) and (3.24). In principle, these conditions are only valid in average. To remove the first hypothesis would imply a Brownian motion with memory effect or a diffusion constant becoming a random quantity on itself, i.e., a rather involved Brownian motion in a random medium.

It might be interesting for future works to use the S -Brownian motion ensemble for a probably simpler problem: the parametric correlations with respect to the energy. In this case, the time delay matrix $Q(H)$ appears in a very direct way as $\delta X/\delta E$ and just the fluctuations around its average (which we have neglected in the last subsection) are decisive to generate the Brownian motion dynamics. One could numerically investigate if the statistic of $Q(H)$ in a chaotic cavity is at least approximately independent of the statistic of S and obeys the isotropy condition necessary for our model. Furthermore, one could try to formulate a non-isotropic Brownian motion if imposed by a more realistic time delay matrix.

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Appendix A

Calculation of the Skew Orthogonal Polynomials

In this Appendix, we sketch the calculation of the skew orthogonal polynomials $q_n^{(t)}(x)$ for the antisymmetric scalar product (2.11). First we consider the case $t = 0$. Due to $g(x, y; 0) = \delta(x - y)$ and (2.7) the scalar product then becomes

$$\langle f, g \rangle_R := \langle f, g \rangle^{(0)} = \frac{1}{2} \int_{-1}^1 dy_1 \int_{-1}^1 dy_2 \frac{\varepsilon(y_2 - y_1)}{\sqrt{(1 - y_2)(1 - y_1)}} f(y_1) g(y_2) \quad (\text{A.1})$$

In addition, we need the conventional symmetric scalar product given by

$$\langle f, g \rangle_C = \int_{-1}^1 dy f(y) g(y) \quad (\text{A.2})$$

with the Legendre polynomials $P_n(x)$ as orthogonal polynomials, i.e., $\langle P_n, P_m \rangle_C = 2/(1+2n) \delta_{nm}$. In the following we omit the superscript $(\dots)^{(0)}$ and denote $q_n(x)$ the skew orthogonal polynomials (compare equation (2.12)). In usual random matrix applications [24] the two type of scalar products contain the same weight function and one can exploit a simple symmetry due to parity $x \rightarrow -x$. The skew orthogonal polynomials are then directly related with the conventional orthogonal polynomials and their derivatives [24]. The situation here is a bit more complicated and our aim is to expand the $q_n(x)$ in terms of the Legendre polynomials. We use therefore the ansatz

$$q_n(x) = \sum_{m=0}^n b_{nm} \frac{1}{2}(1+2m) P_m(x) \quad (\text{A.3})$$

where the b_{nm} are coefficients to be determined. We note that both scalar products are related via the following lemma

$$\langle f, g(x) - g(-1) \frac{1}{\sqrt{2(1-x)}} \rangle_C = \frac{1}{2} \langle f, g(x) - 2(1-x)g'(x) \rangle_R \quad (\text{A.4})$$

where x is the argument of the function on the right sides. Lemma (A.4) is easily verified by two-times partial integration of $\langle f, g \rangle_R$ (and using that $\varepsilon(u) = \text{sign}(u)$). Using the expansion

$$\frac{1}{\sqrt{2(1-x)}} = \sum_{k=0}^{\infty} P_k(x) \quad (\text{A.5})$$

and standard identities which relate the Legendre polynomials and their derivatives we find for an arbitrary function $f(x)$ and for $g(x) = P_m(x)$ from (A.4)

$$\begin{aligned} \langle f, P_m - (-1)^m \sum_{k=0}^{\infty} P_k \rangle_C &= \\ &= \frac{1}{2} \langle f, (1+2m)P_m - 2(2m-1)P_{m-1} + 2P'_{m-1} + 2P'_{m-2} \rangle_R \quad . \quad (\text{A.6}) \end{aligned}$$

The coefficients b_{nm} can be evaluated by $b_{nm} = \langle q_n, P_m \rangle_C$ which follows directly from (A.3). This identity holds also for $n < m$ if we put in this case $b_{nm} = 0$. The skew orthogonality of the q_n implies $\langle q_n, r_m \rangle_R = 0$ if r_m is an arbitrary polynomial of degree $m \leq n-2$ (or $m \leq n$ if n is an even number). We put $f = q_n$ if $n \geq m$ and obtain from (A.6) and (A.3)

$$b_{nm} - (-1)^m \sum_{k=0}^n b_{nk} = \frac{1}{2} \langle q_n, (1+2m)P_m - 2(2m-1)P_{m-1} \rangle_R \quad (\text{A.7})$$

where we have used that P'_{m-1} and P'_{m-2} are polynomials with a degree $\leq m-2 \leq n-2$. In general, this equation holds for $m \leq n$ but in the case of even n it is valid for $m = n+1$, too.

We consider first the case of even $n = 2l$. If $m \leq 2l$ the r.h.s. of (A.7) vanishes and we get therefore

$$b_{2l,m} = (-1)^m \sum_{k=0}^{2l} b_{2l,k} = (-1)^m b_{2l,2l} \quad (\text{A.8})$$

where the second equality holds because the sum does not depend on m . For $m = 2l+1$ equation (A.7) gives

$$b_{2l,2l} = \sum_{k=0}^{2l} b_{2l,k} = \frac{1}{2} \langle q_{2l}, (3+4l)P_{2l+1} \rangle_R = (b_{2l+1,2l+1})^{-1} \quad (\text{A.9})$$

The last equality is valid because the Legendre polynomials can be expanded in terms of the q_n by inversion of (A.3). The skew orthogonality implies that for (A.9) only the first coefficient of this expansion is needed.

We exploit now (A.7) for the case of odd $n = 2l + 1$ which is a bit more involved since three subcases $m < 2l$, $m = 2l$, $m = 2l + 1$ have to be considered yielding the following equations

$$b_{2l+1,m} = (-1)^m s_{2l+1} \quad , \quad m < 2l \quad , \quad (\text{A.10})$$

$$b_{2l+1,2l} = s_{2l+1} - \frac{1}{b_{2l,2l}} \quad (\text{A.11})$$

$$b_{2l+1,2l+1} = -s_{2l+1} + \frac{b_{2l+1,2l}}{b_{2l+1,2l+1} b_{2l,2l}} + \frac{2}{b_{2l,2l}} \quad (\text{A.12})$$

where we have used the abbreviation

$$s_{2l+1} = \sum_{k=0}^{2l+1} b_{2l+1,k} \quad . \quad (\text{A.13})$$

For (A.12) the first two coefficients in the expansion of P_{2l+1} in terms of the q_n were needed.

Equations (A.9), (A.11) and (A.12) yield two independent equations for the four quantities $b_{2l,2l}$, $b_{2l+1,2l}$, $b_{2l+1,2l+1}$ and s_{2l+1} . The other $b_{2l,m}$, $b_{2l+1,m}$ (with $m < 2l$) are determined by equations (A.8) and (A.10). We have therefore two free parameters that are related to the simple transformation $q_{2l} \rightarrow A \cdot q_{2l}$ and $q_{2l+1} \rightarrow A^{-1} q_{2l+1} + B \cdot q_{2l}$ (with arbitrary constants $A \neq 0$, B) that does not change the skew orthogonality of the q_n . Hence, we can choose $b_{2l,2l} = 1$ and $b_{2l+1,2l} = 0$ which result in $b_{2l+1,2l+1} = 1$, $s_{2l+1} = 1$ and $b_{2l,m} = b_{2l+1,m} = (-1)^m$ for $m < 2l$. With these values of the coefficients b_{nm} the ansatz (A.3) yields the skew orthogonal polynomials for $\langle \dots, \dots \rangle^{(t)}$ at the time $t = 0$.

It remains the task to find the skew orthogonal polynomials for an arbitrary time $t > 0$. We define now the polynomials

$$q_n^{(t)}(x) = \sum_{m=0}^n b_{nm} \frac{1}{2}(1+2m) P_m(x) e^{\varepsilon_n t} \quad (\text{A.14})$$

where ε_n is given by $\varepsilon_n = (1+2n)^2$ (compare equation (2.4)). The orthogonality of the Legendre polynomials and (2.4) imply directly that

$$\int_{-1}^1 dy g(x, y; t) q_n^{(t)}(y) = q_n^{(0)}(x) = q_n(x) \quad (\text{A.15})$$

with $q_n(x)$ given explicitly by (A.3) and the above stated values of b_{nm} . This relation shows immediately by virtue of equation (2.7) that the $q_n^{(t)}(x)$ are indeed a set of skew orthogonal polynomials with respect to (2.11). The result (A.14) is just equation (2.29) stated in Section 2.3.

Appendix B

Modifications in the Quaternion Formulation in the Case of Odd Dimension

The quaternion formulation as described in Section 2.2 is only valid for the case of an even number N of channels. Here, we briefly sketch the modifications for odd N . It turns out that

this case can be treated in a similar way as the even N case, but nevertheless some technical complications appear. The following treating is rather parallel to the one of reference [17], but adapted to the concrete situation given here. Since for an odd dimension the determinant of an antisymmetric matrix vanishes, equation (2.6) makes of course no sense if N is odd. The method of integration over alternate variables yields now for the integral (2.1) the expression

$$p(x, t) = \alpha \prod_{i>j} |x_i - x_j| e^{-C_N t} \left[\det \begin{pmatrix} H(x_i, x_j; t) & F(x_i; t) \\ -F(x_j; t) & 0 \end{pmatrix} \right]^{1/2} \quad (\text{B.1})$$

where

$$F(x; t) = \int_{-1}^1 dy \frac{1}{\sqrt{2(1-y)}} g(x, y; t) = \sum_{n=0}^{\infty} P_n(x) e^{-\varepsilon_n t} \quad (\text{B.2})$$

$g(x, y; t)$ is given by equation (2.4) and the second equality holds due to the expansion (A.5) of the square root in terms of Legendre polynomials. $H(x_i, x_j; t)$ stands for an $N \times N$ subblock of an $(N+1) \times (N+1)$ -matrix. $F(x_i; t)$ corresponds to the $(N+1)^{\text{th}}$ column and $-F(x_j; t)$ corresponds to the $(N+1)^{\text{th}}$ row. From equation (2.29) we get directly the identity

$$\int_{-1}^1 dx q_n^{(t)}(x) F(x; t) = 1 \quad . \quad (\text{B.3})$$

Therefore, it is useful to modify the skew orthogonal polynomials by

$$\tilde{q}_n^{(t)}(x) = q_n^{(t)}(x) - q_{N-1}^{(t)}(x) \quad , \quad n < N-1 \quad (\text{B.4})$$

$$\tilde{q}_{N-1}^{(t)}(x) = q_{N-1}^{(t)}(x) \quad . \quad (\text{B.5})$$

In addition, we denote with $\tilde{Q}_n^{(t)}(x)$ the dual functions that correspond to the $\tilde{q}_n^{(t)}(x)$, i.e., they are given by the integral (2.13) where the $q_n^{(t)}(x)$ are replaced by $\tilde{q}_n^{(t)}(x)$. equations (B.3-B.5) imply the "orthogonality" relation

$$\int_{-1}^1 dx \tilde{q}_n^{(t)}(x) F(x; t) = \delta_{n, N-1} \quad , \quad n \leq N-1 \quad (\text{B.6})$$

Using this notations, we redefine the quaternion function $\sigma(x, y; t)$ by

$$\sigma(x, y; t) = \begin{pmatrix} \sigma_{11}(x, y; t) & \sigma_{12}(x, y; t) \\ \sigma_{21}(x, y; t) & \sigma_{22}(x, y; t) \end{pmatrix} \quad , \quad (\text{B.7})$$

with

$$\begin{aligned} \sigma_{11}(x, y; t) &= -K_{\tilde{q}\tilde{q}}(x, y; t) + \tilde{q}_{N-1}^{(t)}(x) F(y; t) \quad , \\ \sigma_{12}(x, y; t) &= K_{\tilde{q}\tilde{q}}(x, y; t) \quad , \\ \sigma_{21}(x, y; t) &= H(x, y; t) - K_{\tilde{Q}\tilde{Q}}(x, y; t) + \\ &\quad + \tilde{Q}_{N-1}^{(t)}(x) F(y; t) - F(x; t) \tilde{Q}_{N-1}^{(t)}(y) \quad , \\ \sigma_{22}(x, y; t) &= K_{\tilde{Q}\tilde{q}}(x, y; t) + F(x; t) \tilde{q}_{N-1}^{(t)}(y) \end{aligned}$$

The function K is defined as in equation (2.15) but with the peculiarity that the sum runs over the range $n, m = 0, 1, \dots, N-2$. Equations (2.16-2.19) remain valid, but in (2.17) one has to replace N with $N-1$. Again the three main properties (2.21-2.23) are valid and ensure

the expression (2.28) for the m -point correlation function. The verification of (2.21-2.22) is a rather lengthy but straightforward calculation using (2.16-2.19) and (B.5). The non-trivial property is again (2.23) on which we will now concentrate. Instead of equation (2.26), we find the identity

$$A(\sigma(x_i, x_j; t)) = \lim_{\delta \rightarrow 0} M_1(\delta) M_2(\delta) \quad (\text{B.8})$$

where $M_{1,2}(\delta)$ stand for the two $2N \times 2N$ -matrices

$$M_1(\delta) = \begin{pmatrix} \tilde{q}(x_i)^T Z & \delta \cdot \tilde{q}_{N-1}^{(t)}(x_i) & 0 \\ \tilde{Q}(x_i)^T Z & \delta \cdot \tilde{Q}_{N-1}^{(t)}(x_i) + \delta^{-1} F(x_i; t) & 1 \end{pmatrix}, \quad (\text{B.9})$$

$$M_2(\delta) = \begin{pmatrix} -\tilde{Q}(x_j) & \tilde{q}(x_j) \\ -\delta \cdot \tilde{Q}_{N-1}^{(t)}(x_j) + \delta^{-1} \cdot F(x_j; t) & \delta \cdot \tilde{q}_{N-1}^{(t)}(x_j) \\ H(x_i, x_j; t) - \delta^{-2} \cdot F(x_i; t) F(x_j; t) & 0 \end{pmatrix} \quad (\text{B.10})$$

with a quasi block notation similar as in equation (2.26). The left column in M_1 contains two $N \times (N-1)$ -matrices and the first row in M_2 contains two $(N-1) \times N$ -matrices. The column (row) in the middle of M_1 (M_2) is of dimension $2N \times 1$ (or $1 \times 2N$). The blocks in the right column of M_1 (or in the lower row of M_2) are just two square matrices of dimension $N \times N$. The determinant of (B.8) is now evaluated as

$$\det(A(\sigma(x_i, x_j; t))) = \lim_{\delta \rightarrow 0} \left[\delta^2 \cdot \det(\tilde{q}_n^{(t)}(x_i))^2 \det(-H(x_i, x_j; t) + \delta^{-2} F(x_i; t) F(x_j; t)) \right]. \quad (\text{B.11})$$

Using the abbreviations $a_{ij} = -H(x_i, x_j; t)$, $v_i = \delta^{-1} F(x_i; t)$ and the identities

$$\begin{pmatrix} \mathbf{1} & -v \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} a + vv^T & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ v^T & \mathbf{1} \end{pmatrix} = \begin{pmatrix} a & -v \\ v^T & \mathbf{1} \end{pmatrix}, \quad (\text{B.12})$$

$$\det \begin{pmatrix} a & -v \\ v^T & \mathbf{1} \end{pmatrix} = \det \begin{pmatrix} a & 0 \\ v^T & \mathbf{1} \end{pmatrix} + \det \begin{pmatrix} a & -v \\ v^T & 0 \end{pmatrix} = \det \begin{pmatrix} a & -v \\ v^T & 0 \end{pmatrix} \quad (\text{B.13})$$

we find that the expression in equation (B.11) becomes equal (apart from the normalization constant) to the square of the joint probability distribution (B.1). The second equality in (B.13) holds because the determinant of an antisymmetric matrix in an odd-dimensional space vanishes. We have so far proved the last property (2.23), too. Hence, the m -point functions are again given by (2.28), but now with the quaternion function (B.7). It turns now out that the results (2.36-2.42) are valid for the odd N -case, too.

Appendix C

Semiclassical Evaluation of the Constants κ and $\tilde{\kappa}$

In Subsection 3.1, we introduced the numerical constants κ and $\tilde{\kappa}$ that depend on the dot geometry. In this Appendix, we give a semiclassical estimation of them for some simple geometries. The constant κ is defined through the relation $\langle L_z^2 \rangle_{Q, V} = \kappa m^2 v_F^2 R^2$ where the brackets denote a combined quantum mechanical and statistical average. In the disordered case the position \mathbf{r} and the momentum \mathbf{p} of the electron fluctuate on two different time scales $\tau_T \sim R^2/D$ (D is the diffusion constant) for \mathbf{r} and $\tau_{el} \ll \tau_T$ for \mathbf{p} . In the semiclassical limit we can therefore assume that \mathbf{r} and \mathbf{p} are statistically independent and the average consists of two independent

integrals of \mathbf{r} in the particle volume and of \mathbf{p} on the Fermi surface $|\bar{\mathbf{p}}| = m v_F$. For a circle of radius R in $d = 2$ dimensions, we obtain $\kappa = \frac{1}{4}$ and for a sphere of radius R in $d = 3$ dimensions the result is $\kappa = \frac{2}{15}$. One can also consider a d -dimensional ring ($d = 1, 2, 3$) on the xy -plane with a thickness much smaller than R but larger than the mean free path. The integrations then yield $\kappa = \frac{1}{d}$.

In the ballistic case we need the two constants κ and $\bar{\kappa}$ which appear only in the combination $\kappa \bar{\kappa}$. They are defined by

$$\kappa \bar{\kappa} \left(\frac{2R}{v_F} \right) m^2 R^2 v_F^2 = \int_0^\infty dt \langle L_z(0) L_z(t) \rangle_{Q,V} \quad (\text{C.1})$$

where the time dependence of the angular momentum is determined by the dynamics of the Hamiltonian without magnetic field. In equation (C.1) already the time reversal invariance was exploited. Let us now consider the classical ballistic motion of the electron in the quantum billiard. Let \mathbf{r}_i be the i^{th} reflection point on the surface (with $i = 0, 1, 2, \dots$ and $\mathbf{r}_0 = \mathbf{r}(t = 0)$) and t_i the corresponding time of the reflection. The classical path is then given by $\mathbf{r}(t) = \mathbf{r}_i + v_F t \mathbf{e}_i$, where $t_i \leq t < t_{i+1}$ and \mathbf{e}_i is the unit vector in the direction of $\mathbf{r}_{i+1} - \mathbf{r}_i$. We can now apply the semiclassical approximation to the average

$$\int_0^\infty dt \langle \mathbf{L}(0) \cdot \mathbf{L}(t) \rangle_{Q,V} \simeq \sum_{i=0}^\infty \langle (t_{i+1} - t_i) \mathbf{L}(t_0) \cdot \mathbf{L}(t_i) \rangle_{\text{average}} \quad (\text{C.2})$$

with

$$\mathbf{L}(t_i) = m v_F \frac{\mathbf{r}_i \times \mathbf{r}_{i+1}}{|\mathbf{r}_{i+1} - \mathbf{r}_i|}$$

Let us now consider a surface that is very rough on a scale much smaller than the diameter of the cavity. In addition, we consider again a circle in two dimensions or a sphere in three dimensions. We assume that the surface-roughness implies that all reflection points \mathbf{r}_i are statistically independent and that the average with respect to \mathbf{r}_i is just an integration over a circle (or sphere) with radius R . Then in equation (C.2) only the term with $i = 0$ yields a non vanishing contribution. Let ϑ be the angle between \mathbf{r}_0 and \mathbf{r}_1 . In two dimensions, equations (C.1) and (C.2) yield after some obvious vector algebra

$$\kappa \bar{\kappa} = \langle \sin(\frac{1}{2}\vartheta) \cos^2(\frac{1}{2}\vartheta) \rangle_{\text{average in } d=2} = \frac{2}{3\pi} \quad (\text{C.3})$$

In three dimensions the angular momentum has also an x - and a y -component and we obtain

$$\kappa \bar{\kappa} = \frac{1}{3} \langle \sin(\frac{1}{2}\vartheta) \cos^2(\frac{1}{2}\vartheta) \rangle_{\text{average in } d=3} = \frac{4}{45} \quad (\text{C.4})$$

The relation (4.2) has then the form

$$\lambda = C_d \sqrt{\frac{\hbar v_F}{\Delta_0 R}} \left(\frac{\Phi}{\Phi_0} \right) \quad (\text{C.5})$$

with

$$C_2 = \frac{1}{\pi} \sqrt{\frac{4}{3}} \simeq 0.3676 \quad , \quad (\text{C.6})$$

$$C_3 = \sqrt{\frac{8}{45\pi}} \simeq 0.2379 \quad (\text{C.7})$$

The square-root factor in equation (C.5) can be calculated by standard expressions that relate the Fermi energy and the number n of electrons in the cavity with the result:

$$\sqrt{\frac{\hbar v_F}{\Delta_0 R}} = n^{1/4} \quad \text{in } d = 2 \quad , \quad (\text{C.8})$$

$$\sqrt{\frac{\hbar v_F}{\Delta_0 R}} = (6/\pi)^{1/6} n^{1/3} \quad \text{in } d = 3 \quad . \quad (\text{C.9})$$

The numerical results for an asymmetric stadium on page 176 of reference [27] yield a relation $\lambda = 1.5 \cdot (\Phi/\Phi_0)$ for the first 500 energy levels. A comparison with (C.5) and (C.8) imply that the corresponding number of electrons is just $n \simeq 280$, which is just in the middle of the considered energy range. The agreement is very good if we consider the simplicity of the “rough-surface” model used above. The numerical values (C.6-C.7) should therefore be considered as (rather good) estimations for a typical quantum billiard.

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