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The administrative divisions of mainland France as 2D random cellular structures

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Résumé. — Les propriétés topologiques et quelques propriétés métriques des découpages administratifs de la France continentale en départements et en arrondissements ont été étudiées. Elles ne diffèrent pas des propriétés qui sont mesurées habituellement dans les tissus biologiques. Ces résultats confirment la variabilité limitée des corrélations topologiques des structures cellulaires 2D désordonnées qui ont des valeurs comparables de μ_2, variance de la distribution du nombre de côtés des cellules. Les processus ponctuels planaires P1 et P2, constitués respectivement par les chefs-lieux de départements et d'arrondissements ont été comparés à un processus de référence formé par les valeurs propres de matrices aléatoires complexes et dissymétriques. Le processus P2 est correctement représenté par le modèle tandis que l'interaction entre les points du processus P1 est plus répulsive.

Abstract. — The topological properties and some metric properties of the administrative divisions of mainland France in departments and in districts have been investigated. They do not differ from the properties which are usually measured in biological tissues. This confirms the restricted variability of the topological correlations amongst disordered 2D cellular structures with comparable values of μ_2, the variance of the distribution of the number of cell sides. The planar point processes P1 and P2 constituted by the chief towns of departments and of districts respectively have been compared to a reference process formed by the eigenvalues of asymmetric complex random matrices. Process P2 is fairly accounted for by the latter model while the interaction between the points of P1 is more repulsive.

1. Introduction.

Random 2D and 3D cellular structures occur frequently in Nature from the microscopic scale to the astronomic scale [1, 2]. They involve all kinds of scientific fields, such as biology

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(epidermal tissues, roots, stems, etc., [3]), materials science (polycrystals (3D) and their cuts (2D) [4-5], cellular solidification of alloys [6], foams, [7]), geology (crack networks in basalt flows [8]), geography [9-10], astrophysics (large-scale structure of the universe [11]).

The characterization of 2D random space-filling cellular structures includes in general the distributions of metric and of topological properties of single cells. The properties investigated are typically : cell area, edge lengths, cell perimeter, angles, number n of edges of cells, $n_1 \leq n \leq n_2$, where $n_1 = 3$ in general. The two-cell correlations of metric or of topological properties have been less investigated except for $m(n)$, the mean number of edges of the first neighbour cells of n-sided cells (called here n-cells). A semi-empirical law, the Aboav-Weaire law [12-13], states that $nm(n)$ is linearly related to n by :

$$nm(n) = (6 - a)\, n + 6\, a + \mu_2 \tag{1}$$

where μ_2 is the variance of the distribution $P(n)$ of the number n of edges of cells : $\mu_2 = \langle n^2 \rangle - \langle n \rangle^2$, with $\langle n \rangle = 6$ for infinite networks because of Euler's relation in 2D [1] and $\langle nm(n) \rangle = \mu_2 + 36$ [13]. In many natural random cellular structures, the parameter « a » is of the order of 1 [12].

In a recent study, Peshkin *et al.* [14] have drawn again attention to an important two-cell correlation : $M_k(n)$ which is the average number of k-sided neighbours of an n-cell. The topological correlation functions A_{kn} ($A_{kn} \geq 0$) defined as [15] :

$$A_{kn} = M_k(n)/P(k) = M_n(k)/P(n) = A_{nk} \tag{2}$$

allow topological properties of tissues with different distributions $P(n)$ to be compared. In recent papers [16-17], we have compared the correlations A_{kn} in topological models of 2D random cellular structures with the correlations found in various natural or simulated cellular structures. The latter structures include biological tissues with $\mu_2 \leq 1.5$, polycrystal cuts with $\mu_2 \geq 2$, Voronoi froths and various simulated models. We have observed a restricted variability of the topological correlations in different structures that have similar values of the variance μ_2 whatever the scale and the field in which the structures occur. The correlations further exhibit a regular and smooth change when μ_2 increases. The shapes of cells do not in general depart too much from some kind of regularity. Additional conditions imposed by space-filling [18, 19] strongly constrain 2D random cellular structures and restrict the accessible ranges of these topological correlations.

As the latter point of view is not yet fully accepted, it is important to seek cellular structures that may *a priori* show topological properties that deviate significantly from the previous behaviour. Human activities have produced numerous « artificial » but disordered cellular structures. Such patterns, called cellular networks, have already been studied by geographers [9, 10 and references therein] with scopes which differ from ours. The aims of the present paper are to characterize the detailed topological properties of some 2D « artificial » cellular structures, namely the administrative division of mainland France in departments (Fig. 1) and its first division in districts (subunits called « arrondissements » in French) in order to compare them with the properties of various natural structures. Metric properties of single cells have also been investigated. Finally, the point processes which are associated with the chief towns of departments and of districts are compared with the point process generated from the eigenvalues of complex asymmetric random matrices as proposed previously by Le Caër [20-21] for Spanish towns [22-23].

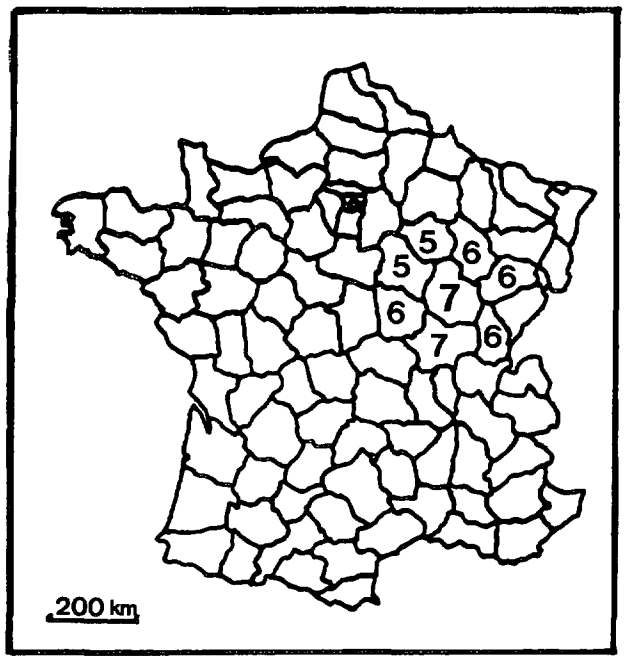

Fig. 1. — A schematic representation of the main administrative division of mainland France in departments. The number of sides of the department of Côte d'Or and of its neighbours has also been indicated. Every department is in turn divided into ~ 3 districts which are not shown.

2. Data and methods.

2.1 DATA. — The administrative divisions of France (French administrative cellular structures, called here FACES) have mainly been inherited from the French Revolution. The number of departments reached its maximum (130) in 1811 at the time of the Napoleonic conquests and included departments in the present areas of various European countries [24]. The number of departments in mainland France is now 94 while the number of districts is 322 (I.N.S.E.E.). Some of the main French cities (Paris, Lyon, Marseille) are divided in administrative units also called « arrondissements ». The latter divisions, which differ from districts, are not taken into account in the present study. In the following, departments or their districts will be called cells. In order to perform a meaningful comparison with natural or model structures, we will distinguish the cells called frontier cells (F) from the others (called NF). The NF cells are surrounded by cells (NF and F) which belong to the same family (Fig. 1). The boundaries of frontier cells are partly seaboards or partly frontiers with other European countries. A cell which shares its border with n different neighbouring cells will be called an n-cell. The part of a border which belongs to two cells will be called an edge (also called a contact by Boots [10]) and the intersection of z ($\geqslant 3$) edges will be called a z-valent vertex. An examination of a detailed administrative map [25] shows that the FACES are such that two departments (or districts) share at most one edge (except two enclaves, see below) and that all vertices are trivalent. The latter observation is usual in 2D random cellular structures as all vertices with $z > 3$ are structurally unstable, as their properties change by small deformations.

To determine the number of edges of every cell, the following simplifications have been performed :

— An enclave of the department of Vaucluse in the department of Drôme and an enclave of the department of Hautes-Pyrénées in the department of Pyrénées Atlantiques have been neglected ;

— The two districts Metz-ville and Metz-campagne and the two districts Strasbourg-ville and Strasbourg-campagne have been considered as forming two districts : Metz and Strasbourg ;

— The number of edges of some districts, close to the estuaries of the rivers Seine, Loire and Garonne, whose borders do not cross the corresponding river has been determined by assuming that the two river banks only form one edge ;

— The urban growth in the region that surrounds Paris has given rise in 1964 to a division of the departments of Seine and of Seine-et-Oise into seven new departments as well as to new districts. We have kept the previous department of Seine into account. We have neglected its recent mitosis to study FACES which do not differ too much from the initial structures.

The previous simplifications have only negligible or even no consequences as some of the previous departments or districts are frontier cells. The total number of departments and of districts is consequently reduced to 91 and 310, respectively. Among the latter cells, $N = 49$ departments and $N = 222$ districts share no frontier with the sea or with a European country. In the following the FACES are considered as particular samples that belong to an ensemble of networks which could have, under similar conditions, been inherited from the French Revolution. In other words, we assume that the various properties are subject to fluctuations. This point of view permits us to compare more fruitfully the FACES with « experimental » structures.

2.2 METHODS. — We have studied the point processes formed by the chief towns of departments and by the chief towns of districts that are located within a rectangle W (sidelengths : $a \times b \sim 540$ km $\times$ 640 km, Fig. 13) whose corners are close to the towns of Saint-Lô (northwest), Thionville (northeast), Apt (southeast) and Mont-de-Marsan (southwest). The previous processes are called P1 and P2 respectively. The chief towns of process P1 also belong to P2. These processes have been analyzed considering that the distance D between two towns is the distance as the crow flies. We have therefore not taken into account the geographical differences between the various regions which are for instance due to the presence of mountains, etc. If $Q(D)$ is the probability for having a point of the process inside each of the two circles of areas $dA1$ and $dA2$ whose centers are separated by D, the pair correlation function $g(D)$ is defined as [26, 27] :

$$Q(D) = \rho^2 g(D)\, dA1\, dA2 \tag{3}$$

where the density ρ is the number of chief towns per unit area (actually an estimator of ρ). For a Poisson point process, $g_P(D)$ is equal to 1 whatever D. The statistical determination of $g(D)$ is performed through an edge-corrected kernel estimator [27-28]. Besides $g(D)$, we have also calculated :

1) the cumulative distribution function $NN(D)$ of the distance D from a point of P_i $(i = 1, 2)$ to its nearest neighbour ;

2) the cumulative distribution function $NR(D)$ of the distance D from a point taken at random to the nearest point of P_i.

The function $NN(D)$ has been estimated with the help of two asymptotically unbiased estimators described by Stoyan *et al.* [26, Chapter 4]. The two estimators give close results which have been averaged. For estimating the distribution $NR(D)$, $\sim 5\,600$ sample points are chosen according to the stratified random sample scheme [29]. All the previous estimators have been satisfactorily checked with the help of computer simulations of Poisson ($NN(D) = NR(D) = 1 - \exp(-\pi\rho D^2)$ [9, 26, 29]) and of hard-disc processes.

3. Topological properties.

3.1 DEPARTMENTS. — The number of non-frontier departments is $N = 49$. Table I gives the associated distribution $P(n)$ of the number of edges and compares it with various distributions (see also Fig. 2). The average number of sides is $\langle n \rangle = 5.94$ and the second moment is $\mu_2 = 0.91$. The average correlation $nm(n)$ (introduction), which can only be calculated for $n = 5$ to 7, gives a slope of the Aboav-Weaire line (relation 1) of 1.2 ± 0.1. A trend for an asymmetry of $P(n)$ for $n = 5$ and 7 is observed in table I. Table I also shows a strong increase of $P(n)$ ($n \leqslant 5$) and a correlated decrease of $P(6)$ when all departments are taken into account (see also Tab. II). The number of sides of a border cell is indeed underestimated with respect to the number of sides of the same cell which would be surrounded by cells in an enlarged tissue (for instance France in 1811, section 2.1). The distribution $P(n)$ for departments is quite comparable to distributions $P(n)$ in biological tissues [3, 32]. The correlations A_{kn} (section 1) have not been determined as the number of NF departments is too small.

Table I. — *Distributions $P(n)$ of the number of sides of:* a) *all departments* $N = 91$ (*see* Sect. 2.2); b) *non-frontier* (NF) *departments* ($N = 49$, $\mu_2 = 0.91$); c) NF *departments + frontier departments which have their frontier much shorter than their other sides* (sF, $N = 56$, $\mu_2 = 0.93$); d) *an epidermal epithelium of a* 100 mm *cucumber* ($N = 1\,000$, $\mu_2 = 0.64$, [3]); e) *a cellular tissue in a human amnion* ($N = 1\,000$, $\mu_2 = 1.$, Lewis 1931, [3]); f) *a young soap froth (R. Delannay, unpublished)* ($N = 381$, $\mu_2 = 0.93$). *Errors have been calculated with the assumption that a sample of size N is drawn from an urn which contains an infinite number of cells with a proportion $P(n)$ of n-cells.*

n	$P(n)$ All Depts	$P(n)$ NF Depts	$P(n)$ sF Depts	$P(n)$ Cuc.	$P(n)$ Hum. Amnion	$P(n)$ young soap
3	0.022	0	0	0	0.004 ± 0.002	0.003 ± 0.003
4	0.165	0.041 ± 0.028	0.036 ± 0.025	0.020 ± 0.004	0.054 ± 0.007	0.050 ± 0.011
5	0.308	0.286 ± 0.064	0.250 ± 0.058	0.251 ± 0.014	0.248 ± 0.014	0.247 ± 0.022
6	0.275	0.429 ± 0.071	0.429 ± 0.066	0.474 ± 0.016	0.397 ± 0.015	0.444 ± 0.025
7	0.176	0.204 ± 0.058	0.232 ± 0.056	0.224 ± 0.013	0.241 ± 0.014	0.202 ± 0.021
8	0.022	0.020 ± 0.020	0.036 ± 0.025	0.030 ± 0.005	0.049 ± 0.007	0.047 ± 0.011
9	0.011	0.020 ± 0.020	0.018 ± 0.018	0.001 ± 0.001	0.007 ± 0.003	0.008 ± 0.005

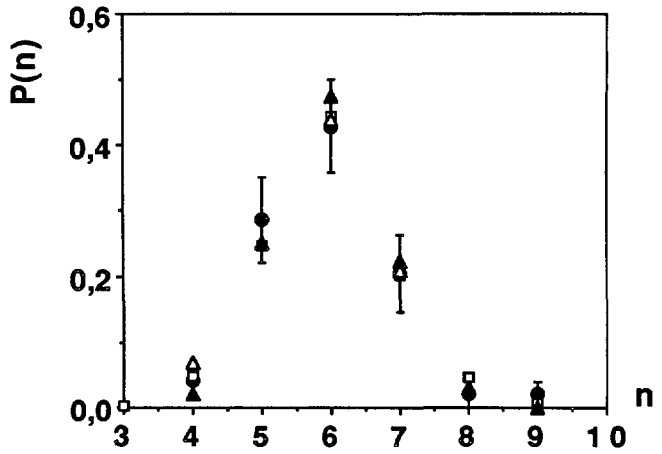

Fig. 2. — Distribution $P(n)$ of number of sides for NF departments (full circles with errors bars, $\mu_2 = 0.91$), for an epidermal epithelium of a 100 mm cucumber (Tab. I, full triangles, $\mu_2 = 0.64$, [3]), for 2D cellular arrays formed during the directional solidification of Pb-30 wt% Tl alloys with a growth velocity $V = 0.85$ cm/h (empty triangles, $\mu_2 = 0.85$, [6]) and for a young soap froth (empty squares, $\mu_2 = 0.93$, Delannay, unpublished).

3.2 DISTRICTS. — Table II and figure 3 give the distributions $P(n)$ of districts and of cellular structures taken from different scientific fields. Other comparisons, for instance with the computer simulation of a 2D froth [31], may be performed. The asymmetry $P(5) > P(7)$ is clearly observed for all structures. The average number of sides is $\langle n \rangle = 5.92$ while $\mu_2 = 1.41 \pm 0.12$. The error is estimated by the method explained in the caption of table I. Table III and figure 4 prove that $nm(n)$ is well described by assuming a linear variation with n (equation 1 with 6 replaced by $\langle n \rangle$). The parameter a is calculated from a weighted least-squares fit of $nm(n)$ with weights $w_n = P(n)$ [15] :

$$a_w = (\langle n \rangle \{\mu_2 + \langle nm(n) \rangle\} - \langle n^2 m(n) \rangle)/\mu_2 \tag{4}$$

which yields $a_w = 1.07$ which is typical of many natural cellular structures ($a \sim 1.$ -1.2). The $P(n)$ distribution for districts differs from the corresponding distribution in 2D soap froths in their scaling state [7, 30, 33]. The latter froths are characterized by $P(5) \sim P(6)$. However some disagreements remain about the detailed $P(n)$ of soap froths and the values of μ_2 (~ 1.4 or 1.2) [30, 33] which are nevertheless close to μ_2 of districts. The parameter a is 1 for the experimental study of 2D soap froths in the scaling state [34], while it is 1.2 ± 0.1 in the simulations of a 2D foam by Herdtle and Aref [33]. No detailed correlations (A_{kn} or related quantities) have been reported since.

Figure 5 shows a comparison between the correlations A_{kn} ($4 \leq n$, $k \leq 8$) for NF districts and for a vegetable tissue (Anthurium : $\mu_2 = 1.10$, $a = 1$ [32, 35]). Moreover, the A_{kn} are compared with the correlations expected for a linear dependence of A_{kn} in k and in n [15] :

$$A_{kn} = n + k - 6 - (a/\mu_2)(n + k - 6)\,. \tag{5}$$

An overall reasonable agreement holds between the experimental A_{kn} and relation 5 but the number of cells is too small for asserting the significance of the observed deviations. Moreover, we notice that relation 5 predicts here (Fig. 5) negative values for A_{k4} ($k \leq 4$) [15]. Relation 5 is expected from the application of the maximum entropy principle with constraints

Table II. — *Distributions* $P(n)$ $(3 \leq n \leq 10)$ *of the number of sides of:* a) *all districts* ($N = 310$, *see* Sect. 2.2); b) *non-frontier* (NF) *districts* ($N = 222$, $\mu_2 = 1.41$); c) *a biological tissue in the stationary state from a computer simulation which includes mitosis and growth* [35] *(average* $P(n)$, $P(2) = 0.014$, $\langle \mu_2 \rangle = 1.71$); d) *a planar section of a soap foam after* 30 h *of growth* ($N = 3\,623$, $\mu_2 = 1.30$, *Aboav* 1980, [12]); e) *the Voronoi froth generated from eigenvalues of asymmetric complex random matrices* ($\mu_2 = 1.23$, [20]). *Bracketed errors in* b) *have been calculated as explained in the legend of table* I.

n	$P(n)$ All Districts	$P(n)$ NF Districts	$P(n)$ Sim. Biol. Tiss. ($\pm \sim 0.02$)	$P(n)$ Soap Froth 30 h	$P(n)$ RMVF
3	0.023	0.009 (± 0.005)	0.014	0.011	0.0022 (± 0.0002)
4	0.148	0.104 (± 0.017)	0.077	0.059	0.069 (± 0.001)
5	0.284	0.252 (± 0.025)	0.245	0.265	0.2676 (± 0.0016)
6	0.287	0.338 (± 0.027)	0.339	0.373	0.356 (± 0.003)
7	0.174	0.203 (± 0.023)	0.214	0.198	0.217 (± 0.001)
8	0.064	0.077 (± 0.015)	0.077	0.074	0.0715 (± 0.0009)
9	0.019	0.018 (± 0.018)	0.020	0.018	0.0147 (± 0.0007)
10	0.019	0	0	0.001	0.0019 (± 0.0003)

imposed on the $P(n)$. The $M_k(n)$ (Eq. (2)) are predicted to be linear in n in order to reduce the number of independent constraints to 2 ($\langle 1 \rangle = 1$ and $\langle n \rangle = 6$) [14]. However, the observation of linear A_{kn} does not necessarily imply that $P(n)$ maximizes the entropy $S = -\Sigma P(n) \operatorname{Log} (P(n)/\pi(n))$ with a prior distribution $\pi(n)$. This is not the case for $\pi(n) = 1$ if $P(n)$ differs from $0.75^{(n-3)}/4$ $(n \geq 3)$.

3.3 DISCUSSION. — Lemaître *et al.* [36, 37] and Gervois *et al.* [38] have shown that the second moment μ_2 of the distribution $P(n)$ of the number of cell edges in random mosaics varies quasi-universally with the fraction $P(6)$ of six-sided cells. According to Lemaître *et al.* [37], the relation between μ_2 and $P(6)$ is the equivalent of the virial equation of state in statistical mechanics. An approximate relation, $\mu_2^{\alpha} P(6) = \partial$, has been recently discussed [17]. It yields $P(6)$ with a precision in general better than 5 %:

$$\mu_2^{0.513} P(6) = 0.389 \quad \text{if} \quad 0.1 \leq P(6) \leq 0.7 . \tag{6}$$

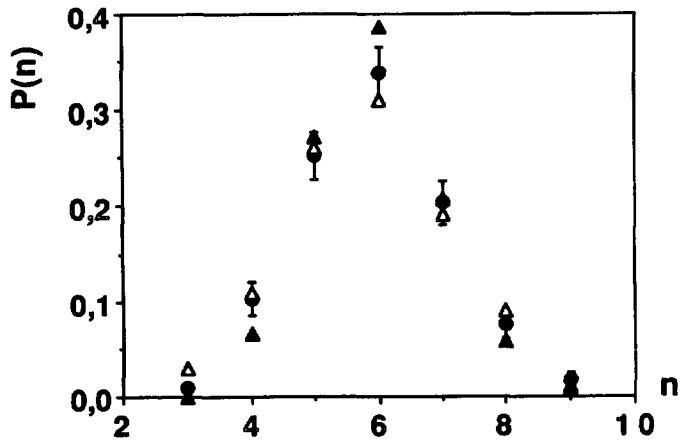

Fig. 3. — Distribution $P(n)$ of the number of edges for NF districts (full circles with error bars, $\mu_2 = 1.41$), for 2D cellular arrays formed during the directional solidification of Pb-30 wt% Tl alloys with a growth velocity $V = 0.75$ cm/h (empty triangles, $\mu_2 = 1.58$, [6]) and for an undifferentiated vegetable tissue of Anthurium ($N = 1\,295$, full triangles, $\mu_2 = 1.10$, [32]).

Table III. — *Average total number of sides* $nm(n)$ *of the cells which are first neighbours of n-cells for* NF *districts,* $nm(n)$ *calculated from an Aboav-Weaire relation with* $a_w = 1.067$ (Rel. (1) *with* $a = a_w$, $\mu_2 = 1.41$, $\langle n \rangle = 5.92$ *instead of* 6). *Average normalized area* $a(n)$ *for* NF *districts.*

n	$nm(n)$ for NF districts	$nm(n)$ calculated	average normal. area $a(n)$
3	22	22.38	0.52
4	26.39	27.24	0.70
5	31.63	32.10	0.90
6	37.04	36.95	1.
7	41.89	41.81	1.19
8	46.65	46.67	1.47
9	52.75	51.52	1.67

Allowing a variation of ~ 5 % in ∂, relation 6 predicts $P(6) = 0.41 \pm 0.02$ for departments as compared to $P(6) = 0.45 \pm 0.07$ (Tab. I) and $P(6) = 0.328 \pm 0.015$ for districts while $P(6) = 0.338 \pm 0.027$ is observed. As already discussed in [17], a large part of the quasi-universal behaviour may be ascribed to the strong constraints acting on such structures.

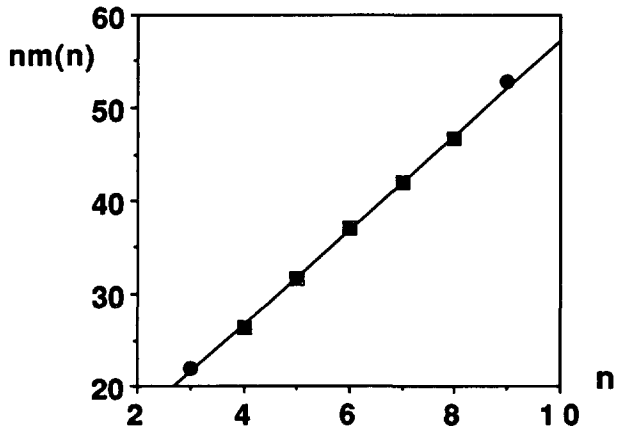

Fig. 4. — Average correlation $nm(n)$ as a function of n for NF districts (Aboav-Weaire line, see Tab. III). The statistical significance of 3 $m(3)$ and 9 $m(9)$ (full circles) is weak as there are only 2 and 4 districts with 3 and 9 edges respectively.

The similarities between the topological properties of districts and recently reported experimental results and computer simulations (Tab. III and Fig. 5, [32, 35]) of biological tissues are particularly impressive. All the previous comparisons prove that the topological properties of NF departments and NF districts do not differ from those of natural structures. These results agree with the observation of a restricted variability of the topological correlations for a given μ_2 or a limited range of μ_2 [17]. This does not explain why the topological properties of NF departments and of NF districts are typical of the class of biological tissues (see however section 6) although those of districts are close to the boundary region between biological tissues and soap froths. All geographical cellular networks do not belong to the same class of natural structures.

Boots (1980, [10]) has investigated the topological properties ($P(n)$, $m(n)$) of three cellular networks which represent the civil parishes for the English counties of Wiltshire ($\mu_2 = 2.91$), Somerset ($\mu_2 = 3.62$) and Devon ($\mu_2 = 4.62$). The quasi-universal relation $\mu_2 = f(P(6))$ predicts $P(6) = 0.225 \pm 0.010$, 0.20 ± 0.01 and 0.18 ± 0.01 for the latter counties respectively. The predicted values are in good agreement with the observed values (0.225, 0.219, 0.192), in spite of the existence of some parishes with 1 and 2 sides [10]. The distributions $P(n)$ are characteristic of polycrystal cuts. The topological and the metric properties of a planar cut of an alumina polycrystal with ~ 7 000 grains ($\mu_2 = 2.585$), have been recently characterized by Righetti *et al.* [39]. Figure 6 compares $P(n)$ and $m(n)$ of the alumina cut with the corresponding quantities for Wiltshire ($\mu_2 = 2.6$ if we omit the parishes with 2 sides). Even the very revealing representation $m(n) = f(n)$, instead of the usual $nm(n)$ plot, is convincing. Figure 7 further shows the distributions $P(n)$ and the correlations $m(n)$ for Devon, Somerset, for zone-refined iron isothermally annealed at 650 °C during 125 min [40] and for a topological model associated with a tiling by triangles (12-6-4, $\mu_2 = 3.12637$ [16]). Hu [40] gives the result of a fit $m(n) = 5 + 7.6/n$ but not the actual values (deviations take place for $n = 3$ and for large values of n). The average $m(n)$ for Somerset and Devon is $m(n) = 5 + 8.73/n$ for $3 \leqslant n \leqslant 11$. In France, the typical size of parishes (« communes » in French) is an order of magnitude less than the typical size of districts. The magnitude of μ_2, which reflects the topological disorder, seems to increase when the scale of the investigated cellular network decreases (sections 3.1 and 3.2). The characterization of the division of France in cantons and its subdivision in parishes would be worthwhile in order to

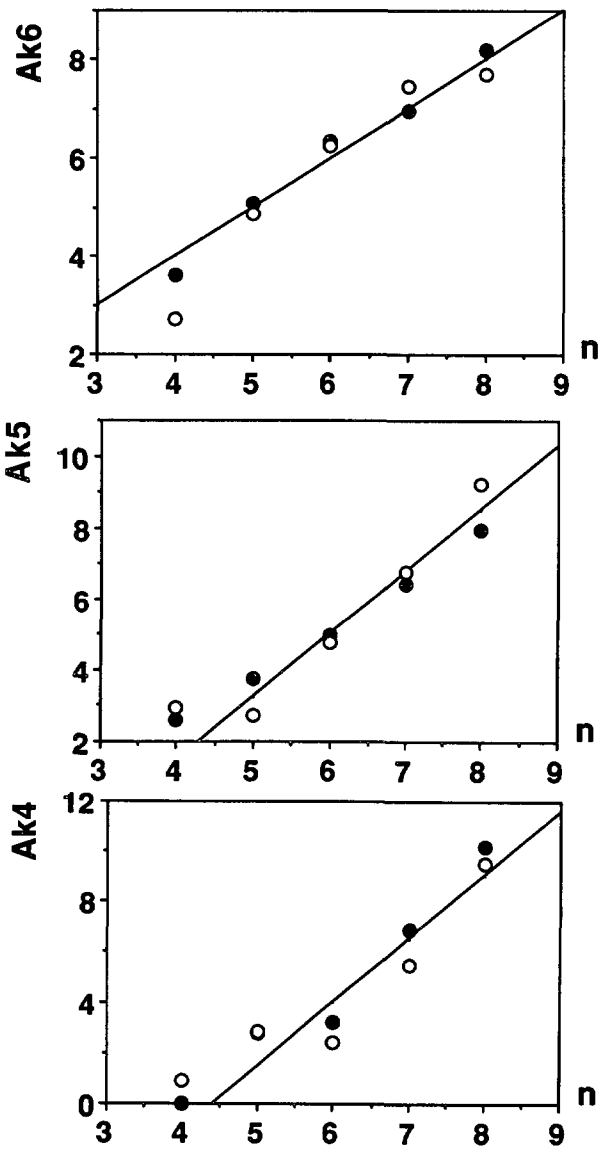

Fig. 5. — Correlations A_{kn} ($4 \leq n, k \leq 8$) for NF districts (full circles, $\mu_2 = 1.41$) and for a tissue of Anthurium (open circles, $\mu_2 = 1.10$. Mombach *et al.*, [32, 35]) (straight lines : Eq. (5) with $a/\mu_2 = 0.755$).

know if the successive networks shift progressively towards the class of polycrystal cuts. Preliminary investigations of parishes in the region of Lorraine agree completely with the latter assumption.

In conclusion, the topological properties of the various geographical cellular networks discussed here are similar to the properties of natural structures which have comparable values of μ_2.

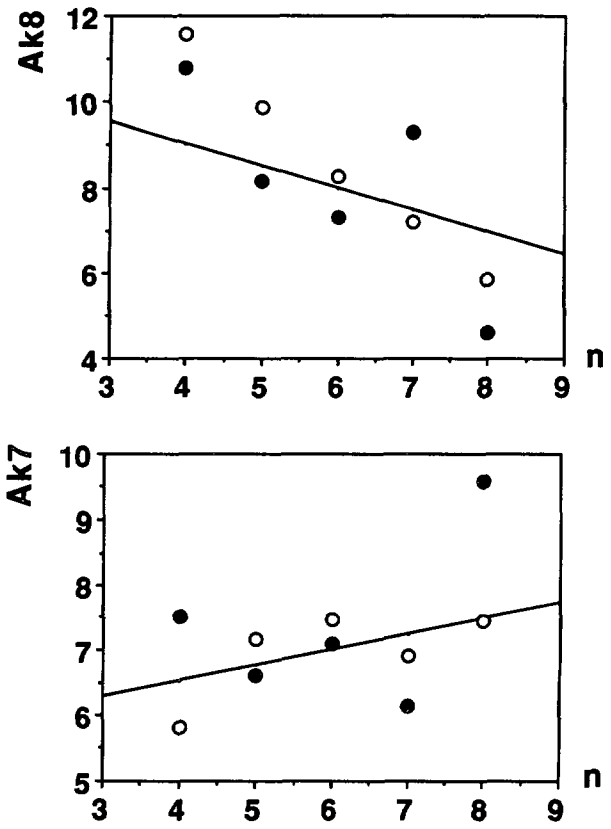

Fig. 5 *(continued).*

4. Metric Properties : cell areas.

The distribution of some metric properties (for instance department or district perimeters) will likely differ from the corresponding distributions in natural structures. As there is however a certain regularity in the cell shape, we have investigated the properties of the cell areas (data from I.N.S.E.E.). The empirical Lewis's law expresses the linear dependence of the average area $a(n)$ of n-cells with n :

$$a(n) = 1 + \lambda\,(n - 6) = (n - n_0)/(6 - n_0) \tag{7}$$

where $a(n)$ is normalized : $a(6) = 1$. For planar cuts of polycrystals, it is the average cell radius $r(n)\,\alpha\,\{a(n)\}^{1/2}$ and not $a(n)$ which depends linearly on n (Desch's law [2]). The average area $a(n)$ for NF districts increases regularly with n (Tab. III) and is compared in figure 8 with $a(n)$ for amnion [3]. A weighted fit of $a(n)$ yields $\lambda = 0.17$ and 0.19, that is $n_0 = 0.27$ and 0.76 for NF districts and amnion respectively. Due to « experimental errors », it is difficult to choose between Lewis's law and Desch's law although $r(n)$ seems to be more linear. Only three areas ($5 \leqslant n \leqslant 7$) can be safely calculated for NF departments. If we nevertheless assume that Lewis's law holds, we calculate $n_0 \sim 0.25$ which is quite similar to the previous values.

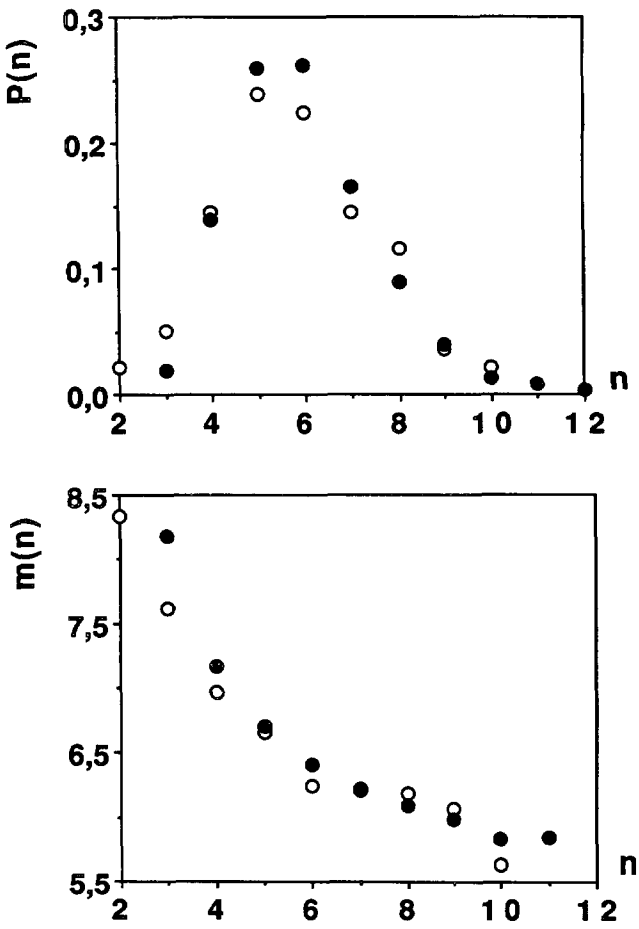

Fig. 6. — Distributions $P(n)$ and correlations $m(n)$ for Wiltshire (open circles, $\mu_2 = 2.91$, Boots, 1980, [10]) and for an alumina cut (full circles, $\mu_2 = 2.59$, Righetti *et al.*, [39]).

For biological tissues, the intercept n_0 is less than ~ 3 [3, 32, 35]. For 2D soap froths, Desch's law works significantly better than Lewis's law [7]. The curvature of $a(n)$ remains however sufficiently small for performing a reasonable linear fit. The extrapolation to $a(n) = 0$ gives n_0 larger than ~ 4 [7, 33, 42]. Comparable values are obtained in grain growth simulations with a Potts model [7] and in the alumina cut although it unambiguously follows Desch's law [39]. The cellular arrays formed during directional solidification of a binary Pb-Tl alloy, follow however Lewis's law with $n_0 \sim 2$ and have side distributions which are analogous with $P(n)$ of biological tissues ([6], Figs. 2 and 3).

Lewis's law has been derived by Rivier [2, 41] in the frame of the maximum entropy method. Rivier uses an argument about the reduction to two of the dimension of the space of constraints in order to derive equation (7). Rivier interprets the Lagrange multiplier λ as the ageing of the structure for soap froths and polycrystals. The intercept of Lewis's law $n_0 = 6 - 1/\lambda$ is larger for older froths which are also coarser. Biological tissues do not coarsen but evolve in a steady state by a combination of growth and cellular division. They have constant intercepts n_0 and λ does not measure time as before [2]. The department and district networks could be considered as young mosaics with a small λ, that is a small value of

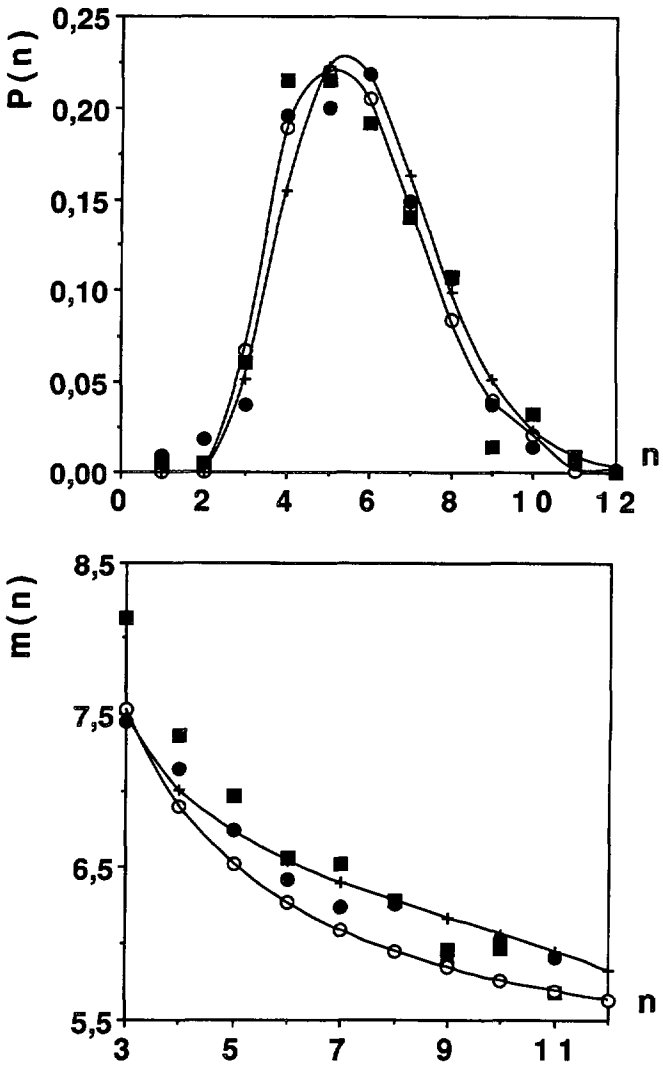

Fig. 7. — Distributions $P(n)$ and correlations $m(n)$ for Devon and Somerset (full squares, $\mu_2 = 4.62$, and full circles, $\mu_2 = 3.62$, respectively, Boots, 1980, [10]), for an iron polycrystal isothermally annealed at 650 °C during 125 min (open circles and bold line, $\mu_2 = 3.59$, $m(n)$ is given by $5 + 7.6/n$, Hu [40] and for a 12-6-4 topological model (crosses and bold line, $\mu_2 = 3.13$, [16]).

n_0 in agreement with the results. The latter interpretation is ruled out because the previous networks are not in statistical equilibrium as defined by Rivier [19]. They will not remain statistically unchanged under structural transformations (neighbour switching and cell disappearance) occurring independently in space or time.

To our surprise even the cell areas obey the usual semi-empirical laws known for natural cellular structures. From the point of view of these metric properties, the FACES would once more be classified in the same class as biological tissues although n_0 may be considered as a little too small.

The distributions of cell areas of various Voronoi tessellations or of planar cuts of

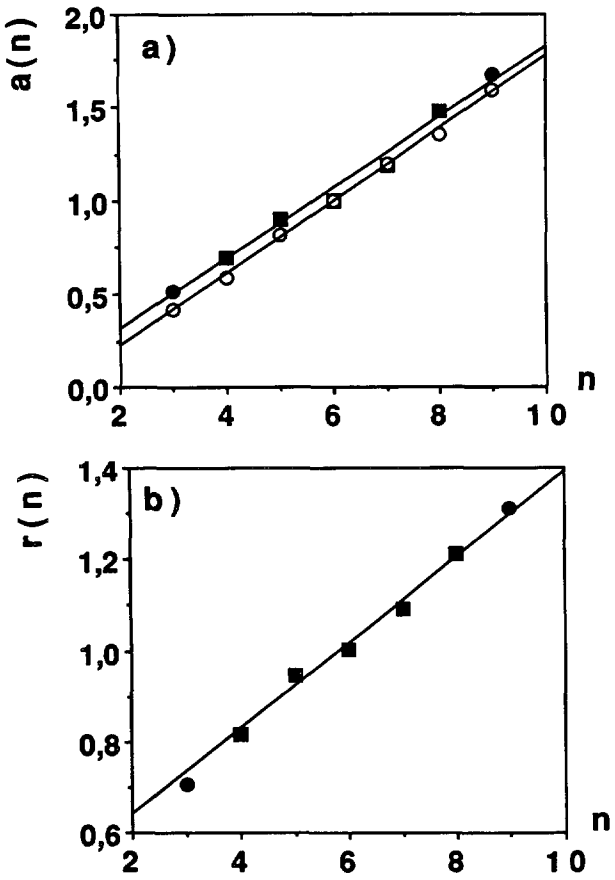

Fig. 8. — a) Average normalized area of n-cells $a(n)$ for NF districts (full squares and full circles) and for human amnion (open circles, Lewis 1928, [3] ; b) associated radius $r(n) = \{a(n)/\pi\}^{1/2}$ for NF districts as a function of n.

polycrystals are well fitted by a gamma distribution [37, 43, 44] with a density probability :

$$f(A) = (\alpha/\langle A\rangle)^{\alpha} A^{\alpha-1} \exp\{-\alpha A/\langle A\rangle\}/\Gamma(\alpha) \quad (8)$$

where α is ≈ 3.6 for the random Voronoi froth. Some geographical networks do not differ so much from Voronoi tessellations [45, 46]. Figure 9 shows a plot of $f(A)$ for all districts and the calculated gamma distribution with $\alpha = 4.38$ calculated from the average area $\langle A\rangle = 1\,722.7\ \text{km}^2$ and the standard deviation $\sigma_A = 823\ \text{km}^2$ ($\alpha = (\langle A\rangle/\sigma_A)^2$). The agreement is statistically significant as shown by a χ^2 test ($\chi^2 = 11$) with 16 degrees of freedom. The area distribution of departments is more concentrated and symmetric than the previous distribution. It is not satisfactorily accounted for by equation (8) ($\alpha = 12.57$). A normal distribution would better account for the observed area distribution. The radius $R = (\langle A\rangle/\pi)^{1/2} = 43.2$ km for the departments is favorably compared with the late rule prescribed by the deputies : 30-40 km [24]. Although the political discussion was mainly focussed on the problems of boundaries and of division modes [47], the problem of chief towns is also important. We have therefore investigated the point processes formed by these towns. We have compared them to a point process derived from statistical physics [20, 21]. The fact

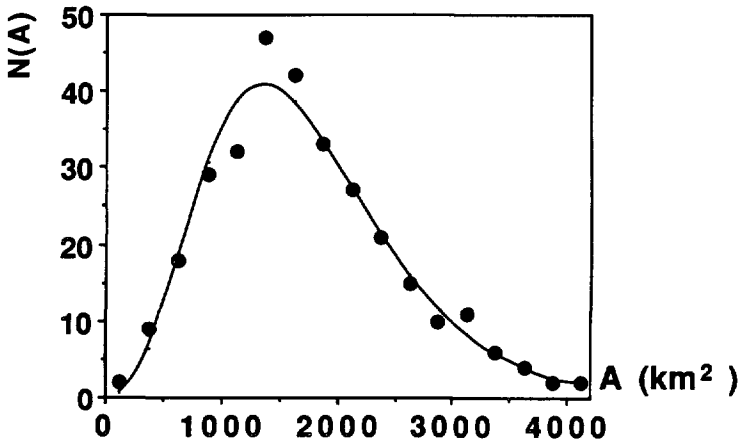

Fig. 9. — Distribution of the areas of all districts (full circles). The full curve (Eq. (8), α = 4.38) passes through the midpoints of the calculated bins.

that a central power played a major role in the final decision (1 790 for departments, 1 800 for districts) may justify that we consider distances in straight line and that we neglect the role of geography. The first project proposed in September 1789 by Thouret (Siéyès-Thouret project [47]) consisted of a chessboard division of France. This proves indeed that it was possible at that time to ignore geography for political reasons.

5. 2D Point Processes.

5.1 RANDOM MATRICES. — Many mathematical models have been developed to analyze random point patterns [9, 23, 26, 29, 45]. The interaction between N points of a spatial pattern in a finite planar region may be evaluated with the assumption that the points are distributed according to a Gibbs canonical distribution [23, 29, 45]. Ogata and Tanemura [23] describe for instance a statistical method for estimating a pair interaction potential $\Phi(D)$ which is a function of the distance D between points of a given process. To investigate the softness of the repulsive interaction between points, they have chosen models of very-soft-core, soft-core and hard-core pair potentials. The point process associated with the eigenvalues of asymmetric complex random matrices (PPRM) is, among other things, characterized by a very soft repulsion effect between eigenvalues. The distribution of nearest-neighbor spacings goes like D^3 (like D for a Poisson point process) when the distance D between two eigenvalues goes to zero [48-50].

The distribution of eigenvalues of asymmetric $N \times N$ complex random matrices M_N has been theoretically investigated by Ginibre [48] and Mehta [49]. The real and imaginary parts of the matrix elements are independently and identically distributed (I.I.D.) according to a Gaussian distribution with mean 0 and variance $\sigma^2/2$. When $N \to \infty$, the eigenvalues of $M_N/N^{1/2}$ are distributed uniformly over a disc of radius σ. For finite N, the density ρ is isotropic, nearly constant and equal to the asymptotic value :

$$\rho = 1/(\pi\sigma^2) \tag{9}$$

for $D \leqslant \sigma N^{1/2}$. The density goes to zero in an interval of order σ around $\sigma N^{1/2}$ [48]. In the tail of the distribution, there are of the order of $(N/2\pi)^{1/2}$ ($N \gg 1$) eigenvalues [48]. These properties are particularly convenient for numerical simulations. They give, for a finite N, a

very good approximation of the point pattern for an infinite N. The process is isotropic whatever N and homogeneous when $N \to \infty$ with uncorrelated eigenvalues for large distances [48, 49]. Due to the repulsion effect, this process is more regular than a 2D Poisson point process [20]. The mean distance $\langle D \rangle$ between a point and its nearest neighbour is given by $0.5\,\rho^{-1/2}$ and $0.644829\,\rho^{-1/2}$ for the 2D Poisson point process and the PPRM [50] respectively. It is $1.07457\,\rho^{-1/2}$ for the vertices of a triangular lattice. The eigenvalue distribution is identical with the distribution of the equilibrium positions of charges of a 2D Coulomb gas in a harmonic oscillator potential at a temperature $kT = 0.5$ [48].

The latter 2D eigenvalue distribution is universal. It is obtained for a very broad class of distributions of the matrix elements as shown numerically [20] and theoretically in the frame of quantum chaos [50]. It has been conjectured [20, 49] that only the mere existence of the mean and of the variance of the distribution for I.I.D. matrix elements yields the universal distribution in the limit $N \to \infty$. A unique point process is therefore associated with the eigenvalues of fully asymmetric complex matrices with I.I.D. matrix elements. The cumulative distribution function of the distance D from a point chosen at random to the nearest eigenvalue $NR(D)$ is [49, 51] :

$$NR(D) = 1 - \lim_{N \to \infty} \prod_{k=0}^{N-1} \left\{ \exp(-D^2/\sigma^2)\, e_k(D) \right\} \tag{10}$$

with :

$$e_k(D) = \sum_{i=0}^{k} (D^2/\sigma^2)^i / i! \tag{11}$$

The existence of a limit in equation (10) has been proved by Mehta [49]. The cumulative distribution function of the distance D from an eigenvalue to the nearest eigenvalue $NN(D)$ is [50, 51] :

$$NN(D) = 1 - \lim_{N \to \infty} \prod_{k=1}^{N-1} \left\{ \exp(-D^2/\sigma^2)\, e_k(D) \right\} . \tag{12}$$

The products in equations (10) and (12) converge very rapidly [20, 50]. Finally, the pair correlation function $g(D)$ is [48, 49] :

$$g(D) = 1 - \exp(-D^2/\sigma^2) . \tag{13}$$

Besides its application to various fields, the PPRM represents our psychological intuition about a homogeneous « disordered » distribution of points in a plane better than a Poisson point process does. If somebody is asked to put N points in a planar domain « at random », without any further specification, he will avoid the clumps which often occur in a Poisson point process. As shown by preliminary experiments, the resulting processes show a very soft to soft repulsion between points. This discussion may not be irrelevant in the context of the present paper (end of section 4, section 6).

New processes (Le Caër, unpublished) may be obtained by applying to the 2D-PPRM the basic thinning, clustering or superposition operations [26, 29, 45]. Slightly stratified and anisotropic point patterns are obtained from quaternion matrices [48, 49]. Universality of the resulting point pattern also holds for the latter matrices (Le Caër, unpublished). In conclusion, the use of the 2D-PPRM as a reference point process for very soft interaction between points is justified by :

1) the universality of the eigenvalue pattern (with a D^3 repulsion for $D \to 0$) ;

2) the ability to perform exact calculations of $NN(D)$, $NR(D)$ and $g(D)$ without the need of computer simulations.

5.2 SPANISH TOWNS. — The very soft-core pair potential of Ogata and Tanemura [23] $\Phi_{VSC} = -\text{Ln}\,[1 - \exp(-D^2/\sigma^2)]$ accounts satisfactorily for the distribution of towns of a Spanish plateau (69 towns in a homogeneous 40×40-miles^2 area, $\sim 4\,144$ km^2 [22]). The parameter $\sigma = 2.653$ miles (~ 4.27 km) and the pair correlation $g(D)$ are obtained from Monte Carlo simulations [23]. From the actual density, $\rho = 1.665\ 10^{-2}$ km^{-2}, we derive (Eq. (9)) the sole parameter of the PPRM model $\sigma = (\pi\rho)^{-1/2} = 2.717$ miles (~ 4.37 km). The calculated and « experimental » pair correlation functions, distributions NN and NR are compared in figure 10. The average distance $\langle D \rangle$ between nearest neighbours is 4.96 km while 5 km is calculated for the PPRM with density ρ. As shown in figure 11, the fluctuations of $g(D)$ are accounted for by the sample size. The agreement with the PPRM is statistically significant. The PPRM model is simpler than the models proposed until now for the spatial repartition of these Spanish towns.

5.3 CHIEF TOWNS OF DEPARTMENTS : PROCESS P1. — Sixty-four chief towns are located in the rectangle W (section 2.2 and Fig. 13). The average distance between nearest neighbours of P1 is $\langle D \rangle = 55$ km while 48.2 km would be expected for the PPRM from the estimated density. Figure 11 compares the pair correlation function of P1 to $g(D)$ of a sample of a PPRM process with 66 points. The latter points are included in a rectangle W′ (W′ and W are homothetic) which is thrown in the central part of a pattern which consists of the eigenvalues of a 600×600 asymmetric complex random matrix. The real and imaginary parts of the matrix elements are independently distributed according to a Gaussian distribution with mean 0. The fluctuations of the correlation function in the case of the random matrix sample give an idea of the variations which are due to the sample size and which are still consistent with the PPRM model. The distance $\langle D \rangle$ measured for the PPRM sample is about 7 % larger than the value expected from the exact density. Before being able to conclude in favor of the PPRM model, we must also calculate $NN(D)$ and $NR(D)$. From the significant deviations observed in figure 12, we deduce that the mutual interaction between the points of P1 is more repulsive than predicted by the PPRM model. This result is consistent with the historical analysis [47] which indicates that the chief towns are geometric centers, whose choice has mainly been subordinated to the division of the territory in departments.

5.4 CHIEF TOWNS OF DISTRICTS : PROCESS P2. — The rectangle W (East-West side ≈ 544 km, North-South side ≈ 636 km, Fig. 13) includes 188 chief towns of districts ($\rho = 5.43\ 10^{-4}$ km^{-2}). Figure 13 also shows 184 and 187 points distributed according to the PPRM model (section 5.3) and to a Poisson point process respectively. Except for the recent cluster which includes Paris and the nearby chief towns, the chief-town and the PPRM processes look quite similar. Process P2 shows apparently a slightly smaller number of pairs with very close points than the PPRM does. The pair correlation functions of P2 and of the random matrix sample (Fig. 14) are well accounted for by relation 13, using σ values derived from the estimated density ρ. Clearly, $g(D) = 1$ cannot be obtained from a finite size sample of a Poisson point process. The pair correlation function of the Poisson sample also looks like the $g(D)$ of the P2 and PPRM processes. However, the parameter σ which is fitted from relation 13 is four times smaller than the parameter $(\pi\rho)^{-1/2}$ which is expected for a PPRM model. The average distance between nearest neighbours of P2 is $\langle D \rangle = 29.6$ km while the theoretical value $\langle D \rangle_{PPRM}$ is 27.7 km. For the random matrix sample, $\langle D \rangle = 28.4$ km agrees very well with $\langle D \rangle_{PPRM} = 28$ km. Finally $NR(D)$ (Fig. 15) is quite consistent with the PPRM model. A more detailed study would be necessary to know whether the deviations between the

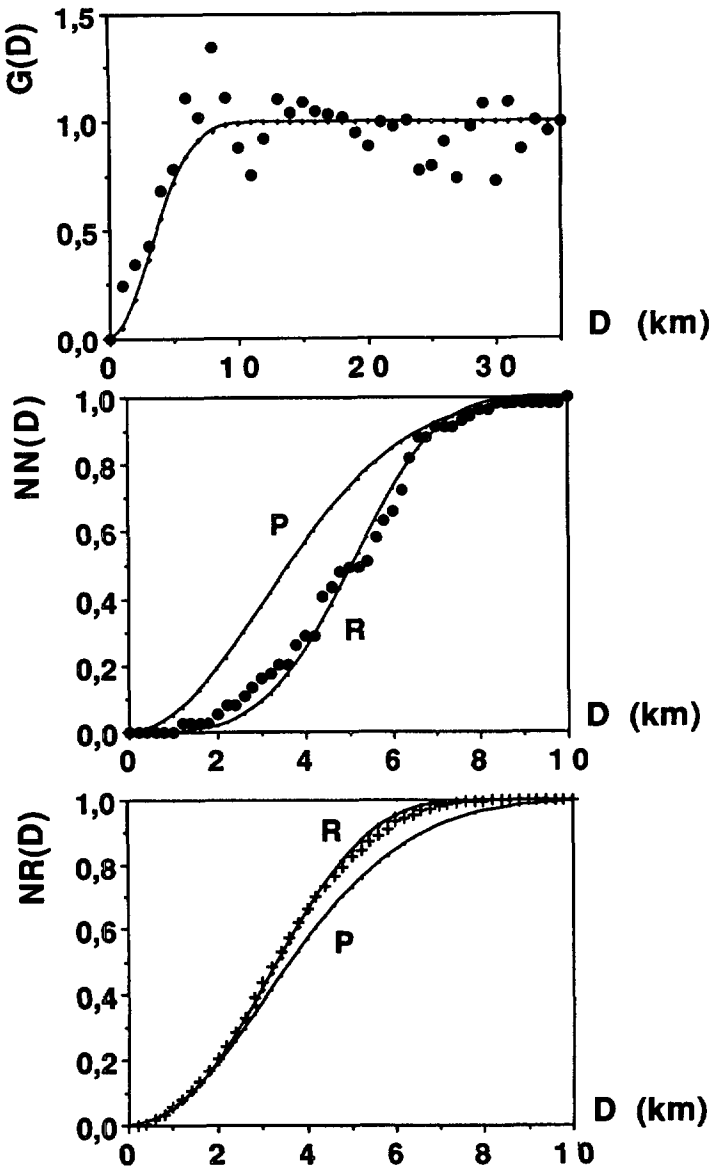

Fig. 10. — The pair correlation function $g(D)$, the distributions $NN(D)$ and $NR(D)$ measured for spanish towns (full circles and crosses, Sect. 2.2, [22, 23]). Bold lines are calculated from equations (13) ($g(D)$), (12) (R), (10) (R) respectively for $\sigma = 4.372$ km (Rel. (9)). The bold lines marked P are calculated for a Poisson point process ($NN(D) = NR(D) = 1 - \exp(-D^2/\sigma^2)$).

measured and the theoretical $NN(D)$ are statistically significant or not. In any case, P2 is a soft repulsive point process which is fairly accounted for by the PPRM although P2 seems slightly more repulsive than the later process (Fig. 13).

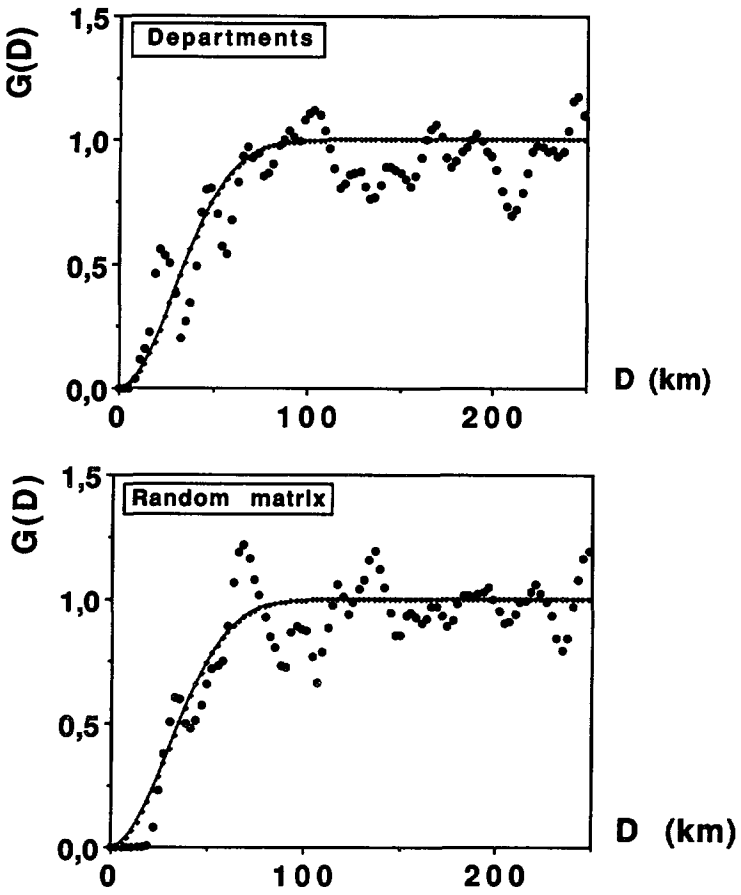

Fig. 11. — The pair correlation function for chief towns of departments ($N = 64$) in W (Fig. 13) and for $N = 66$ eigenvalues of an asymmetric complex random matrix taken in a rectangle W′ (W and W′ are homothetic). Bold lines are calculated using relations (9) and (13).

6. Conclusion.

The topological properties and even some metric properties of the administrative divisions of mainland France do not differ from the properties which are measured in natural disordered cellular structures with similar values of μ_2. The disorder, quantified by μ_2, in the geographical networks investigated seems to increase when the scale decreases. The results of the present paper, as well as those mentioned for English parishes [10], strengthen previous observations of a reduced variability of the topological correlations of disordered structures in a limited range of μ_2 and of a rather regular evolution of these correlations when μ_2 increases [16, 17]. More generally, we have been unable up to now to find disordered 2D cellular structures whose topological correlations differ markedly from those of structures with comparable μ_2.

The initial project for departments (Sieyès-Thouret project, September 29th 1789 [47]) was to perform a chessboard-type division of France according to a map [52] published in 1780 by a geographer, Robert de Hesseln. A first map, which differed strongly from the later project,

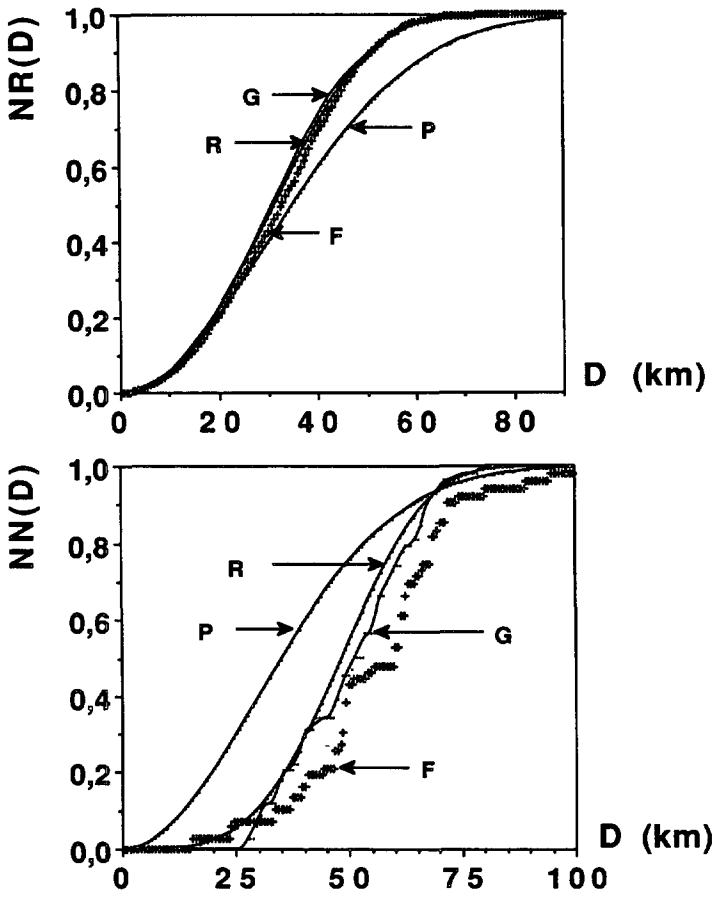

Fig. 12. — The distributions $NN(D)$ and $NR(D)$ measured for 64 chief towns of departments (crosses and F) and for a sample of 66 eigenvalues of a complex random matrix (bold lines indicated by G). Bold lines with a mark R are calculated from equations (12) and (10) respectively while the bold lines with a mark P are calculated for a Poisson point process.

was displayed on October 3rd 1789 [47]. The final decision resulted from dialogues and compromises between the central and the local forces [47]. Although the final division was far from the very ordered initial project, the later geometric proposal had an influence and may explain that the FACES are not too disordered with moderate values of μ_2. It is finally amazing to draw a parallel between the fact that the deputies often referred to biology and mechanics [47] and the fact that the main administrative divisions of France belong to the same topological class as biological tissues.

The usefulness of the point process constituted by the eigenvalues of asymmetric complex random matrices (PPRM) as a reference for very soft core processes has been exemplified. The later point process constitutes a fairly satisfactory model for the spatial pattern of chief towns of districts with distances in straight line. We have not been able to find elements about the criteria which controlled the choice of district chief towns and about the role of the central power. The interaction between the chief towns of departments is more repulsive than it is for districts.

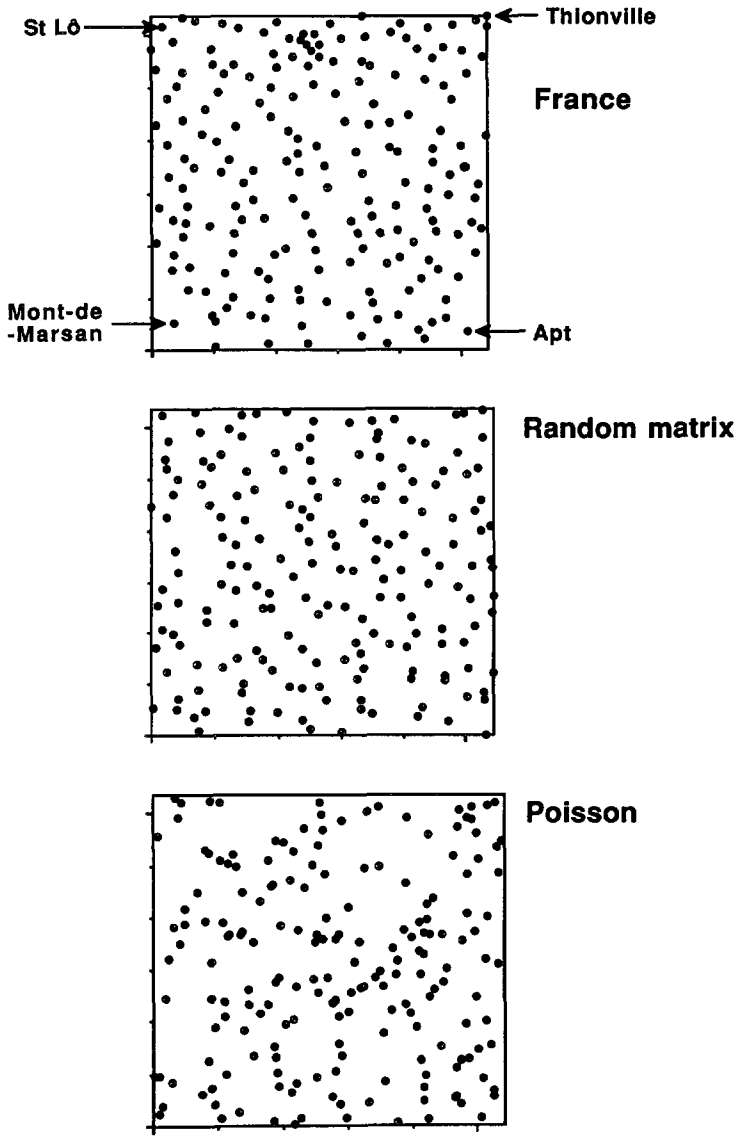

Fig. 13. — 188 chief towns of districts in a rectangle W (544 × 636 km), 184 eigenvalues of a complex 600 × 600 random matrix in a rectangle W', 187 points of a Poisson point process in W' (W and W' are homothetic).

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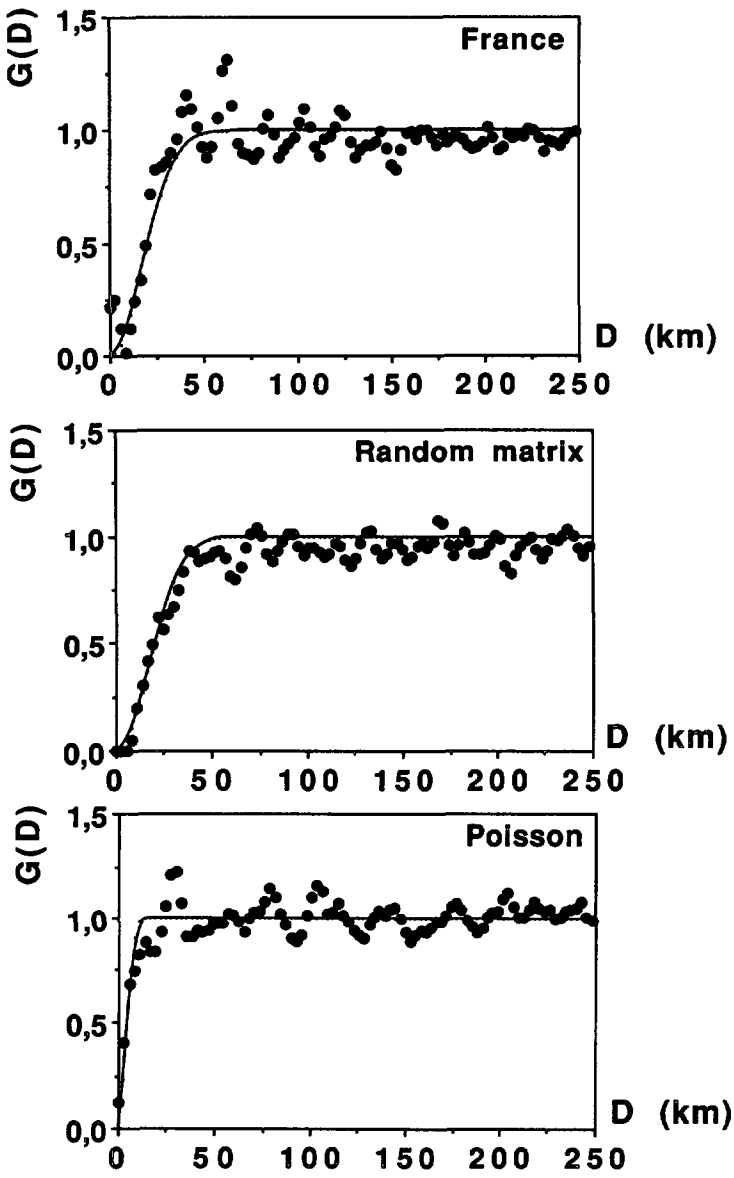

Fig. 14. — Pair correlation functions $g(D)$ for the point processes of figure 13 (bold lines : Rel. (13), σ is calculated from Rel. (9) except for the Poisson point process).

Dr. N. Rivier (Imperial College) for useful preprints, Prof. D. Bideau (Université de Rennes) and Dr. A. Gervois (Saclay) for useful discussions. We also thank Prof. A. Mocellin (Ecole des Mines de Nancy), Prof. T. Liebling, Dr. F. Righetti (E.P.F.L., Lausanne) for the communication of experimental data about the alumina cuts.

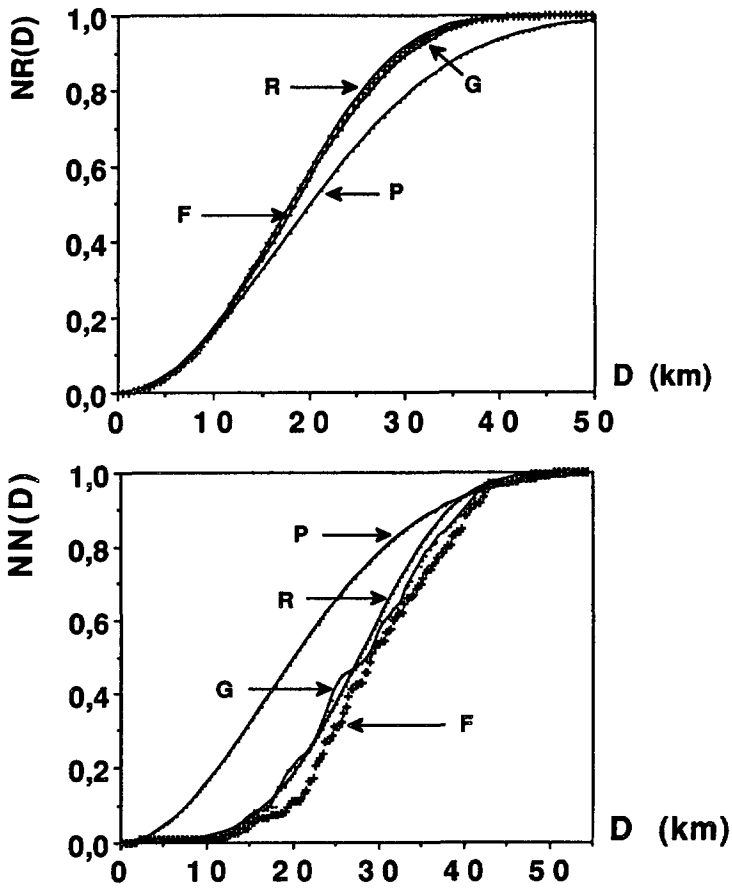

Fig. 15. — The distributions $NN(D)$ and $NR(D)$ measured for 188 chief towns of districts (crosses and F) and for a sample of 184 eigenvalues of a complex random matrix (bold lines indicated by G). Bold lines with a mark R are calculated from equations (12) and (10) respectively while the bold lines with a mark P are calculated for a Poisson point process.

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