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## A non-linear differential equation and a Fredholm determinant

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**Abstract.** — In several branches of mathematical physics one comes across the Fredholm determinant of the kernel  $\sin(x-y)\pi/(x-y)\pi$  on the finite interval  $(-t, t)$ . Jimbo, Miwa, Mori and Sato derived a non-linear differential equation for it. We reinvestigate this problem, find five equations satisfied by three functions  $A(t)$ ,  $B(t)$  and  $S(t)$  related to this Fredholm determinant, and as a consequence deduce the differential equation of Jimbo, Miwa, Mori and Sato.

### 1. Introduction.

In a long and profound article on monodromy preserving deformations, Jimbo, Miwa, Mori and Sato [1] found, among other things, that the Fredholm determinant

$$F(z, t) = \prod_{i=0}^{\infty} \{1 - z\lambda_i(t)\}, \quad (1.1)$$

of the integral equation

$$\lambda f(x) = \int_{-t}^t K(x, y)f(y) dy. \quad (1.2)$$

with the kernel  $K$

$$K(x, y) = \frac{\sin(x-y)\pi}{(x-y)\pi} \quad (1.3)$$

satisfies a non-linear second order differential equation. This same Fredholm determinant occurs in the study of one dimensional models, for example, the correlation functions of the transverse Ising chain [2] or the theory of random matrices [3] among others.

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Ordering the eigenvalues  $\lambda_i$  as  $1 \geq \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq 0$ , we define

$$F_+(z, t) = \prod_{i=0}^{\infty} \{1 - z\lambda_{2i}(t)\}, \quad (1.4)$$

$$F_-(z, t) = \prod_{i=0}^{\infty} \{1 - z\lambda_{2i+1}(t)\}. \quad (1.5)$$

Thus  $F_+$  (resp.  $F_-$ ) is the Fredholm determinant of the integral equation (1.2) with the kernel  $K_+$  (resp.  $K_-$ ), where

$$K_{\pm}(x, y) = \frac{1}{2} [K(x, y) \pm K(-x, y)]. \quad (1.6)$$

The integral equation (1.2) is known for a long time [4], its solutions are known as spheroidal functions; they are either even or odd; the even (odd) solutions correspond to the  $\lambda$ 's with an even (odd) index. Suppressing the dependence on  $z$  for simplicity, let us set

$$A(t) = -\frac{1}{2} \frac{\partial}{\partial t} [\log F_+(z, t) + \log F_-(z, t)] \quad (1.7)$$

$$B(t) = -\frac{1}{2} \frac{\partial}{\partial t} [\log F_+(z, t) - \log F_-(z, t)] \quad (1.8)$$

We will be concerned with the following relations

$$\frac{d}{dt} A(t) = 2B^2(t), \quad (1.9)$$

$$\frac{d}{dt} (tA) = z|S(t)|^2, \quad (1.10)$$

$$\frac{d}{dt} (tB) = z \operatorname{Re} (S(t)^2), \quad (1.11)$$

and

$$2\pi(tB) = z \operatorname{Im} (S(t)^2), \quad (1.12)$$

with

$$\begin{aligned} S(t) &= e^{i\pi t} + \sum_{n=1}^{\infty} \left(\frac{zt}{2}\right)^n \int_{-1}^1 dx_1 \dots dx_n \int_{-1}^1 dy_1 \dots dy_n \\ &\quad \exp i\pi t (x_1 - x_1 y_1 + x_2 y_1 - x_2 y_2 + \dots + x_n y_{n-1} - x_n y_n + y_n) \\ &= e^{i\pi t} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_n \\ &\quad \exp i\pi (tx_1 - x_1 y_1 + x_2 y_1 - x_2 y_2 + \dots + x_n y_{n-1} - x_n y_n + y_n) \end{aligned} \quad (1.13)$$

and "Re (Im)" means the "real (imaginary) part of".

The function  $S(t)$ , with complex values, satisfies the first order non-linear differential equation

$$-i \frac{dS}{dt} = \pi S - \frac{z}{2\pi t} S^* (S^2 - S^{*2}), \quad (1.14)$$

(here  $i = \sqrt{-1}$  and  $S^*$  is the complex conjugate of  $S$ ). Actually equation (1.14) with the initial condition,  $S(0) = 1$ , fixes completely the function  $S(t)$ . For example, expanding  $S$  in powers of  $t$ , the successive coefficients can be determined iteratively from equation (1.14). This gives complete information about  $S$  near  $t = 0$  and using equations (1.7)-(1.12) we can get the same information about  $F_+$  and  $F_-$ . However, it is also of interest to know their behaviour when  $t \rightarrow \infty$ .

Equation (1.9) was proved by Gaudin [3]; we will reproduce his proof here in section 2 for completeness. It was quoted by Jimbo, Miwa, Mori and Sato [1] in the form (our notation)

$$\log F_{\pm}(1, t) = \frac{1}{2} \log F(1, t) \mp \frac{1}{2} \int_0^t dt' \left\{ -\frac{d^2}{dt'^2} \log F(1, t') \right\}^{1/2} \tag{1.15}$$

(their Eq.(7.118)).

If we take equations (1.12) and (1.9) as the definitions of  $B(t)$  and  $A(t)$  in terms of  $S(t)$ , then equations (1.10) and (1.11) are consequences of equation (1.14). Moreover, from equations (1.9)-(1.12) one can deduce the differential equation of Jimbo, Miwa, Mori and Sato [1], (their Eq. (7.104)), which in our notation reads

$$\left( \frac{t}{2} \frac{d^2 A}{dt^2} + \frac{dA}{dt} \right)^2 + 4\pi^2 t^2 \left( \frac{dA}{dt} \right)^2 - 2 \frac{dA}{dt} \left( t \frac{dA}{dt} + A \right)^2 = 0, \tag{1.16}$$

or the differential equation

$$\left( \frac{t}{2} \frac{d^2 B}{dt^2} + \frac{dB}{dt} + 2\pi^2 t B \right)^2 = 4B^2 \left\{ \left( t \frac{dB}{dt} + B \right)^2 + 4\pi^2 t^2 B^2 \right\}. \tag{1.17}$$

Despite several efforts with the help of prominent persons, including the four authors, we never understood the original proof of equation (1.16). Here we deduce it from equations (1.9)-(1.12) in section 2. This constitutes an elementary, and perhaps a different, proof of it. Moreover, the four relations, equations (1.10)-(1.12) and (1.14), are probably new.

The series expansions of  $S$ ,  $A$  or  $B$  in powers of  $t$ , or in powers of  $z$ , does not show that  $z = 1$  has any special significance. However, as  $1 - \lambda_0(t)$  decreases very fast to zero for large  $t$ , these series expansions are surely divergent when  $|z| > 1$ .

Equation (1.16) was used by McCoy and Tang [5] in their study of the correlation functions of the transverse Ising chain, and by Basor, Tracy and Widom [6] to study the level spacing functions in the theory of random matrices.

In another article we will use equations (1.7)-(1.14) to derive the power series and the asymptotic behaviours of the spacing probability functions in the theory of random matrices. In particular, it will be seen that equations (1.14) and (1.16) are related to the fifth Painlevé transcendental functions, while equation (1.17) is related to the third Painlevé transcendent.

**2. Fredholm determinants, resolvents and differential equations.**

From the Fredholm theory of integral equations [7], one can write symbolically,

$$\begin{aligned} \log F(z, t) &= \log \det [1 - z\mathbf{K}_t] \\ &= \text{Tr} \log [1 - z\mathbf{K}_t] \\ &= - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr} [\mathbf{K}_t^{(n)}] \\ &= - \sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-t}^t dx_1 \dots dx_n K(x_1, x_2) K(x_2, x_3) \dots K(x_n, x_1). \end{aligned} \tag{2.1}$$

Here “det” and “Tr” respectively mean “determinant” and “trace”, and  $\mathbf{K}_t$  is the integral operator with the kernel  $K$  on the finite interval  $(-t, t)$ . For  $F_{\pm}(z, t)$ , this same equation holds provided we replace  $K(x, y)$  by  $K_{\pm}(x, y)$ . Differentiating this equation with respect to  $t$ , we have

$$\begin{aligned} \frac{\partial}{\partial t} \log F(z, t) &= - \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n \{K(t, x_1)K(x_1, x_2) \dots K(x_n, t) \\ &\quad + K(-t, x_1)K(x_1, x_2) \dots K(x_n, -t)\} \\ &= -2 \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots K(x_n, t). \end{aligned} \tag{2.2}$$

To derive the last equality, we have used the fact that

$$K(x, y) = K(-x, -y) \tag{2.3}$$

The series in equations (2.1) and (2.2) are absolutely convergent for  $|z| < 1/\lambda_0$ , where  $\lambda_0$  is the largest eigenvalue of  $K$ . The heuristic derivation given here can be perfectly justified [3, 7]. Similarly we have

$$\begin{aligned} \frac{\partial}{\partial t} \log F_{\pm}(z, t) &= -2 \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K_{\pm}(t, x_1)K_{\pm}(x_1, x_2) \dots K_{\pm}(x_{n-1}, x_n)K_{\pm}(x_n, t) \\ &= -2 \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots K(x_{n-1}, x_n)K_{\pm}(x_n, t) \end{aligned} \tag{2.4}$$

Thus  $A(t)$  and  $B(t)$  defined in equations (1.7) and (1.8) can be written as follows

$$A(t) = \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots K(x_{n-1}, x_n)K(x_n, t) \tag{2.5}$$

and

$$B(t) = \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots K(x_{n-1}, x_n)K(x_n, -t). \tag{2.6}$$

Differentiating  $A(t)$  we get

$$\frac{dA(t)}{dt} = A_1 + A_2 + A_3 + A_4, \tag{2.7}$$

where

$$A_1 = \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n \frac{\partial K(t, x_1)}{\partial t} K(x_1, x_2) \dots K(x_{n-1}, x_n)K(x_n, t), \tag{2.8}$$

$$A_2 = \sum_{n=0}^{\infty} z^{n+1} \int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots K(x_{n-1}, x_n) \frac{\partial K(x_n, t)}{\partial t}, \tag{2.9}$$

$$A_3 = \sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^n \int_{-t}^t dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n K(t, x_1) \dots K(x_{j-1}, t)K(t, x_{j+1}) \dots K(x_n, t), \tag{2.10}$$

$$A_4 = \sum_{n=1}^{\infty} z^{n+1} \sum_{j=1}^n \int_{-t}^t dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n K(t, x_1) \dots K(x_{j-1}, -t)K(-t, x_{j+1}) \dots K(x_n, t). \tag{2.11}$$

Now as  $K(x, y)$  depends only on the difference  $x - y$ , we can replace  $\partial K(x_n, t)/\partial t$  in  $A_2$  by  $-\partial K(x_n, t)/\partial x_n$ . Integration by parts with respect to  $x_n$  pushes the derivative one step left. In the integral

$$\int_{-t}^t dx_1 \dots dx_n K(t, x_1)K(x_1, x_2) \dots \frac{\partial K(x_{n-1}, x_n)}{\partial x_n} K(x_n, t) \tag{2.12}$$

replace  $\partial K(x_{n-1}, x_n)/\partial x_n$  by  $-\partial K(x_{n-1}, x_n)/\partial x_{n-1}$  and integrate by parts over  $x_{n-1}$ . And so on, till the partial derivative is pushed to the extreme left. These step by step integrations give

$$A_2 = -A_1 - A_3 + A_4. \tag{2.13}$$

Also it is easy to convince oneself from equations (2.6) and (2.11) that

$$A_4 = B^2(t) \tag{2.14}$$

With equations (2.7), (2.13) and (2.14) Gaudin's proof of equation (1.9) is complete. Note that we have used only the property that  $K(x, y)$  is an even function of  $x - y$ .

To arrive at equations (1.10)-(1.12) we need to use the explicite form (1.3) of  $K(x, y)$ . With the notation,

$$\begin{aligned} k(x, y) &= \frac{\sin(x - y)\pi t}{(x - y)\pi} \\ &= \frac{1}{2} \int_{-t}^t d\xi \exp i\pi(x - y)\xi. \end{aligned} \tag{2.15}$$

we get from equation (2.5) by a change of variables  $x_j \rightarrow tx_j$ ,

$$\begin{aligned} tA(t) &= \sum_{n=0}^{\infty} z^{n+1} \int_{-1}^1 dx_1 \dots dx_n k(1, x_1)k(x_1, x_2) \dots k(x_{n-1}, x_n)k(x_n, 1) \\ &= \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{n+1} \\ &\quad \exp i\pi(y_1 - y_1x_1 + y_2x_1 - y_2x_2 + \dots + y_nx_{n-1} - y_nx_n + y_{n+1}x_n - y_{n+1}) \end{aligned} \tag{2.16}$$

Similarly

$$\begin{aligned} tB(t) &= \sum_{n=0}^{\infty} z^{n+1} \int_{-1}^1 dx_1 \dots dx_n k(1, x_1)k(x_1, x_2) \dots k(x_{n-1}, x_n)k(x_n, -1) \\ &= \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{n+1} \\ &\quad \exp i\pi(y_1 - y_1x_1 + y_2x_1 - y_2x_2 + \dots + y_nx_{n-1} - y_nx_n + y_{n+1}x_n + y_{n+1}) \end{aligned} \tag{2.17}$$

A differentiation with respect to  $t$  gives

$$\frac{d}{dt}(tA(t)) = X_1 + X_2, \tag{2.18}$$

$$X_1 = \frac{z}{2} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^{n+1} \sum_{j=1}^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_{n+1} \\ \exp i\pi (y_1 - y_1 x_1 + \dots - y_{j-1} x_{j-1} + t x_{j-1} - t x_j + y_{j+1} x_j - \dots + y_{n+1} x_n - y_{n+1}), \quad (2.19)$$

$$X_2 = \frac{z}{2} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^{n+1} \sum_{j=1}^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_{n+1} \\ \exp i\pi (y_1 - y_1 x_1 + \dots - y_{j-1} x_{j-1} - t x_{j-1} + t x_j + y_{j+1} x_j - \dots + y_{n+1} x_n - y_{n+1}), \quad (2.20)$$

Now it is easy to convince oneself that

$$X_1 = \frac{z}{2} S S^* = X_2, \quad (2.21)$$

with  $S = S(t)$  given by equation (1.13), and  $S^*$  is its complex conjugate.

Similar manipulations will give

$$\frac{d}{dt}(tB(t)) = B_1 + B_2, \quad (2.22)$$

$$B_1 = \frac{z}{2} e^{2i\pi t} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^{n+1} \sum_{j=1}^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_{n+1} \\ \exp i\pi (y_1 - y_1 x_1 + \dots - y_{j-1} x_{j-1} + t x_{j-1} - t x_j + y_{j+1} x_j - \dots + y_{n+1} x_n + y_{n+1}), \\ = \frac{z}{2} S^2 \quad (2.23)$$

and

$$B_2 = \frac{z}{2} e^{-2i\pi t} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^{n+1} \sum_{j=1}^{n+1} \int_{-1}^1 dx_1 \dots dx_n \int_{-t}^t dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_{n+1} \\ \exp i\pi (y_1 - y_1 x_1 + \dots - y_{j-1} x_{j-1} - t x_{j-1} + t x_j + y_{j+1} x_j - \dots + y_{n+1} x_n + y_{n+1}), \\ = \frac{z}{2} S^{*2} \quad (2.24)$$

Equations (2.18) and (2.21) give equation (1.10), while equations (2.22)-(2.24) give (1.11).

To get equation (1.12) we will calculate the imaginary part of  $S^2(t)$ . For this purpose let us first integrate over all the variables  $y_j$  in  $S(t)$  to write equation (1.13) in the form

$$S(t) = e^{i\pi t} + \sum_{n=1}^{\infty} z^n \int_{-1}^1 dx_1 \dots dx_n e^{i\pi t x_1} k(x_1, x_2) k(x_2, x_3) \dots k(x_{n-1}, x_n) k(x_n, 1). \quad (2.25)$$

where  $k(x, y)$  is given by equation (2.15). Changing the sign of each of the integration variables, we can write this equation also as

$$S(t) = e^{i\pi t} + \sum_{n=1}^{\infty} z^n \int_{-1}^1 d\xi_1 \dots d\xi_n e^{-i\pi t \xi_1} k(\xi_1, \xi_2) k(\xi_2, \xi_3) \dots k(\xi_{n-1}, \xi_n) k(\xi_n, -1). \quad (2.26)$$

Multiplying the two expressions (2.25) and (2.26) we see that the coefficient of  $z^n$  in  $S^2(t)$  is

$$\begin{aligned} & \int_{-1}^1 d\xi_1 \dots d\xi_n e^{i\pi t(1-\xi_1)} k(\xi_1, \xi_2) \dots k(\xi_{n-1}, \xi_n) k(\xi_n, -1) \\ & + \int_{-1}^1 dx_1 \dots dx_n e^{i\pi t(1+x_1)} k(x_1, x_2) \dots k(x_{n-1}, x_n) k(x_n, 1) \\ & + \sum_{j=2}^{n-1} \int_{-1}^1 dx_1 \dots dx_j d\xi_1 \dots d\xi_{n-j} e^{i\pi(x_1-\xi_1)} k(x_1, x_2) \dots k(x_{j-1}, x_j) k(x_j, 1) \\ & \qquad \qquad \qquad k(\xi_1, \xi_2) \dots k(\xi_{n-j-1}, \xi_{n-j}) k(\xi_{n-j}, -1) \end{aligned} \tag{2.27}$$

Its imaginary part is therefore

$$\begin{aligned} & \int_{-1}^1 dx_1 \dots dx_n k(1, x_1) k(x_1, x_2) \dots k(x_{n-1}, x_n) k(x_n, -1) \\ & \qquad \qquad \qquad \pi \left( 1 - x_1 + x_n + 1 + \sum_{j=2}^{n-1} (x_{j-1} - x_j) \right) \\ & = 2\pi \int_{-1}^1 dx_1 \dots dx_n k(1, x_1) k(x_1, x_2) k(x_1, x_2) \dots k(x_{n-1}, x_n) k(x_n, -1). \end{aligned} \tag{2.28}$$

But on integrating over all the variables  $y_j$  in equation (2.17) one sees that the coefficient of  $z^{n+1}$  in  $tB(t)$  is

$$\int_{-1}^1 dx_1 \dots dx_n k(1, x_1) k(x_1, x_2) k(x_1, x_2) \dots k(x_{n-1}, x_n) k(x_n, -1). \tag{2.29}$$

Thus

$$2\pi t B(t) = z \text{Im } S^2(t). \tag{2.30}$$

This is equation (1.12). Note that unlike equation (1.9), this one is valid only for the particular kernel (1.3).

From equations (1.10)-(1.12) we have

$$\left( t \frac{dA}{dt} + A \right)^2 = \left( t \frac{dB}{dt} + B \right)^2 + (2\pi t B)^2 \tag{2.31}$$

Also from equation (1.9) with one differentiation we have

$$\frac{t}{2} \frac{d^2 A}{dt^2} + \frac{dA}{dt} = 2B \left( t \frac{dB}{dt} + B \right). \tag{2.32}$$

Eliminating  $t \frac{dB}{dt} + B$  between equations (2.31) and (2.32) and replacing  $2B^2$  with  $dA/dt$ , equation (1.9), we get equation (1.16).

On the other hand, if we differentiate equation (2.31), use equation (2.32), and cancel out the non-zero factor  $t \frac{dB}{dt} + B$ , we get

$$4B \left( t \frac{dA}{dt} + A \right) = t \frac{d^2 B}{dt^2} + 2 \frac{dB}{dt} + 4\pi^2 t B \tag{2.33}$$



Squaring this and using equation (2.31) again to eliminate, this time,  $t \, dA/dt + A$ , we get equation (1.17).

Equation (1.17) could have been obtained also as follows. Multiply equation (1.14) throughout by  $S$  and write it as a differential equation for  $S^2$ . Separating the real and imaginary parts of this equation and eliminating the real part one gets a second order differential equation for the imaginary part of  $S^2$ , i.e. equation (1.17) for  $B(t)$ .

To differentiate  $S(t)$ , it is convenient first to write it in the form, (Eqs. (1.13), (2.15) and (1.3)),

$$\begin{aligned}
 S(t) &= e^{i\pi t} + \sum_{n=1}^{\infty} z^n \int_{-1}^1 dx_1 \dots dx_n e^{i\pi t x_1} k(x_1, x_2) \dots k(x_{n-1}, x_n) k(x_n, 1) \\
 &= e^{i\pi t} + \sum_{n=1}^{\infty} z^n \int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots K(x_{n-1}, x_n) K(x_n, t) \quad (2.34)
 \end{aligned}$$

so that

$$\frac{dS}{dt} = i\pi e^{i\pi t} + S_1 + S_2 + S_3 \quad (2.35)$$

$$S_1 = \sum_{n=1}^{\infty} z^n \int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots K(x_{n-1}, x_n) \frac{\partial K(x_n, t)}{\partial t} \quad (2.36)$$

$$S_2 = \sum_{n=1}^{\infty} z^n \sum_{j=1}^n \int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots K(x_{j-1}, t) K(t, x_{j+1}) \dots K(x_{n-1}, x_n) K(x_n, t) \quad (2.37)$$

$$S_3 = \sum_{n=1}^{\infty} z^n \sum_{j=1}^n \int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots K(x_{j-1}, -t) K(-t, x_{j+1}) \dots K(x_{n-1}, x_n) K(x_n, t) \quad (2.38)$$

For  $S_1$  we follow the method of Gaudin; replace  $\partial K(x_n, t)/\partial t$  by  $-\partial K(x_n, t)/\partial x_n$  and integrate by parts on  $x_n$ , then in the integral

$$\int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots \frac{\partial K(x_{n-1}, x_n)}{\partial x_n} K(x_n, t) \quad (2.39)$$

replace  $\partial K(x_{n-1}, x_n)/\partial x_n$  by  $-\partial K(x_{n-1}, x_n)/\partial x_{n-1}$  and integrate by parts over  $x_{n-1}$ , and so on, till the differentiation sign is pushed to the extreme left, where it finally disappears. Thus

$$S_1 = -S_2 + S_3 + i\pi \sum_{n=1}^{\infty} z^n \int_{-t}^t dx_1 \dots dx_n e^{i\pi x_1} K(x_1, x_2) \dots K(x_n, t), \quad (2.40)$$

and from equations (2.34), (2.35) and (2.40),

$$\frac{dS}{dt} = i\pi S + 2S_3 \quad (2.41)$$

It is easy to convince oneself from equations (2.6) and (2.38) that

$$S_3 = S^* \cdot B. \quad (2.42)$$

Equations (2.41), (2.42) and (1.12) imply equation (1.14).

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