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The dynamical thermal fuse model

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Abstract. — A dynamical version of the random fuse model is presented in which the temperature T of a fuse of resistance r carrying the current i obeys the equation $dT/dt = ri^b - aT$, where ri^b accounts for a generalized Joule heating and $-aT$ describes the coupling to a thermal bath. For random resistances with the same temperature threshold T_r for rupture, the *quasi-static* random fuse model is recovered in the limit $b \rightarrow +\infty$, whereas the other extreme $b \rightarrow 0$ corresponds to the percolation model. In the intermediate regime, the competition between the two time scales T_r/ri^b and a^{-1} of the temperature field produces a rich phenomenology of rupture patterns which present sensitive dependence upon the input current.

1. Introduction.

Rupture of random materials within a statistical physics framework has been studied intensively these last few years and a partial classification of different possible regimes has emerged [1, 2]. The statistical physics approach seems unavoidable in order to tackle the many important difficulties underlying this field, for instance the presence of many interacting defects, the long-range nature of electric or elastic Green functions and the irreversible evolution of the rupture, to cite a few.

As a consequence of the extraordinary richness of the phenomenology of fracture mechanics, a large variety of lattice models has been defined, each model capturing to some degree a part of a realistic situation. These models can be classified [3] according to 1) the nature of the physical problem and governing equation (scalar, central force, beam model...), 2) the boundary condition (circular, uniaxial, shear, surface cracking), 3) the connectivity constraint, 4) the nature of the disorder (strength, conductivity or elastic moduli, dilution, thermally activated, probabilistic or DLA type). However, all these models have one fundamental property in common, namely the drastic hypothesis that the evolution of the rupture is *quasi-static*. Indeed, the evolution of the system is obtained by solving the equations and the first bond which fulfils the rupture criterion is singled out and broken. For instance,

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taking a rupture criterion on the strain, the bond for which the ratio strain over threshold is largest is chosen to be broken. Then, the same process is iterated on the new system and so on until a macroscopic fracture appears. In these models, there is no real dynamics but only a quasi-static irreversible process. Of course, a time dependence can be artificially introduced from the probability to rupture a given bond [4, 5]. This is exactly the spirit of the growth models such as DLA (diffusion limited aggregation), which describe the *quasi-static* irreversible evolution of complex interfaces [6]. It is thus not a coincidence that statistical models of rupture in random systems have been and are still considered as belonging to the realm of fractal and non-fractal (quasi-static) growth processes.

In this paper, a new point of view is proposed within the framework of statistical physics, namely the relevance of the dynamical nature of rupture. Indeed, natural materials very rarely break down without presenting time dependent effects. The rupture of a window screen by a stone, more generally any breaking of colliding objects, cracking under explosions, rupture under fatigue, etc. are a few examples showing that the time evolution of a system is most important in order to describe its rupture adequately. Here, our purpose is not to dwell on specific time-dynamical effects in relation to rupture but rather to present a simple but already rich model which encaptures the qualitatively novel aspect of relaxation processes, a feature which is absent in *quasi-static* models. We will show below that, as a consequence of the existence of these relaxation processes, new fascinating behaviors appear which bear resemblance to some real-life time-dependent rupture processes.

The paper is organized as follows. In section 2, we introduce the thermal fuse model in the electric context and outline its relation with its mechanical formulation. In section 3, the connections and differences between the thermal fuse model and other previously studied models of rupture are discussed. Section 4 constitutes the main part which describes in some detail the properties which have been found in the general case. We study more specifically the dependence of the fractal dimension of the main crack as a function of the initial quenched disorder of the conductances and on the exponent b describing the degree of nonlinearity of the response of the fuse heating rates as a function of the electrical current. Note that the rôle of this exponent b is similar to the η -exponent introduced by Niemeyer *et al.* [7] in the Dielectric Breakdown model, a generalization of the DLA model, and to recent calculations on rupture in networks presenting an annealed disorder [4, 5]. The time dependence of the time t_f needed for total rupture is studied both analytically in some cases and numerically, notably by monitoring the time evolution of the total conductance of the network. A short account of this work has been presented elsewhere [8]. Other results on dendritic patterns and front propagations have been obtained when cracks grow from a well-defined nucleation center in absence of disorder. They show that the thermal fuse model of dynamical rupture possesses properties in common with other general surface growth phenomena. These results will be published in a future paper [9]. The mechanical version of this model can also be used to rationalize the remarkable observation, quantified by the so-called Omori's law, that seismicity prior to a great earthquake often shows a marked increase of activity [10]. In the future, we hope to extend further the present analysis of this model and compare it with well-defined experiments [11].

2. The time dependent thermal fuse model.

As scalar fields (voltages and currents) are simpler to deal with, we shall restrict our attention to a 2D electrical fuse model. Let us consider a square lattice of unit mesh oriented at 45° with respect to the two borders a distance $L/\sqrt{2}$ apart. Periodic boundary conditions are assumed in the direction parallel to two opposite borders acting as bus bars. The bonds are electric

resistances. We assume that the resistances are distributed according to a probability distribution $P(r)$. To each bond n is thus associated a resistance r_n chosen with the probability distribution $P(r_n)$. This choice is made once and for all at the beginning, corresponding to a quenched disorder on the electrical resistances.

In the *quasi-static* fuse model with quenched disorder [12], it is usually assumed that the bonds turn irreversibly into insulators if their current exceeds a chosen threshold value. In order to introduce a dynamics in the rupture process, many possibilities exist. Within the electrical formulation, a very natural way would be to consider that the bonds of the network carry generalized impedances, which possess both a real part (resistance) and an imaginary part (capacity and inductance). This leads to rather intricate numerical problems. Maybe the simplest way is to come back to the formulation of the quasi-static random fuse model, however with a rupture criterion modified according to the rule that a bond will break down if its intensity *integrated over time* reaches a given threshold. In this case, the rupture scenario indeed depends on the value of the intensity, not only at time t , but also at all previous stages of the rupture process. This is not yet the model that will be studied in the present paper. However, it will be shown that this model corresponds to the special case obtained by fixing the two parameters ($b = 1$, $a = 0$) of the thermal fuse model that we now introduce.

Inspired from the physics of a real fuse which burns out by melting, we will introduce the additional temperature field in terms of which the rupture criterion will be defined. Therefore, in addition to being characterized by its electrical current i_n , we will postulate that a bond n is also characterized by its temperature T_n . The coupling between the two fields is chosen to be given by the following heat equation :

$$C \, dT_n/dt = r_n i_n^b(t) - aT_n(t) \quad (1)$$

where the specific heat C is chosen equal to 1 in all fuses. Equation (1) describes the heating of resistance « n » due to a generalized Joule effect ri^b characterized by the exponent « b ». The case $b = 2$ yields the standard Joule effect. The second term of the r.h.s. of equation (1) describes the coupling of the bond to a thermal bath at zero temperature. Note that any other choice for the bath temperature just shifts the temperature scale. The coupling with the thermal bath is characterized by a time scale a^{-1} . The rupture of a bond is now prescribed to occur when its temperature attains a threshold T_r , chosen equal to unity for all bonds. This is the way the temperature field reacts back on the current field, i.e. the thermal rupture of a fuse changes the topology of the electric network and therefore the current distribution. It is possible to consider different types of coupling between the electric and thermal fields, by considering for instance a dependence of the resistances as a function of the temperature (thermo-electric effects..). This will be neglected in our study but may lead to interesting cooperative behaviors.

Note that three levels of complexity can be distinguished in the thermal fuse model as specified by equation (1). 1) For $a = 0$ and $b \rightarrow +\infty$, we get back the irreversible quasi-static random fuse model studied previously since only the bond carrying the largest current will be heated and will eventually break down. In this limit, the only relevant « memory » effect comes from the irreversible evolution of the bond rupture pattern which in turn controls the electrical current field. 2) For $a = 0$ and $0 < b < +\infty$, in addition to the previous « memory » effect which is of course present, the temperature of a given fuse at time t is an additive function of all the previous heating rates. Note that in particular for $a = 0$ and $b = 1$, one obtains a random fuse model with a rupture criterion such that a bond will break down if its intensity integrated over time reaches a given threshold, a case which has been mentioned above. 3) In the general case ($a \neq 0$ and $0 < b < +\infty$), in addition to the two previous « memory » effects, the model possesses a kind of « healing » or « hardening »

mechanism, since heat can be exchanged with a bath and the bond temperature can decay away from the temperature threshold for rupture.

The simulations have been carried out with the following procedure. At time $t = 0$, a constant current I is suddenly applied and flows from one bus bar to the other. The current per bond would thus be $i = I/L$ in absence of resistance disorder. We postulate that the electrical voltages and currents have an infinitely short response. We thus calculate the current distribution in the network, i.e. the current i_n in each bond, as for a static input current. We have used both a relaxation method and the conjugate gradient technique, using an error criterion $\varepsilon \leq 10^{-20}$. No significant difference was found between the two numerical schemes. Once the currents in each bond is known, it is reported in equation (1) which gives the time evolution of the n -th fuse temperature $T_n(t)$ (see Eqs. (2), (3) below). Starting from a reference temperature equal to that of the thermal bath, the fuses are progressively heated by the generalized Ohm effect (1). As long as no fuse burns out, the current in each bond stays constant. Knowing the current $i_n(0)$ flowing in bond n , equation (1) yields

$$T_n(t) = (r_n i_n^b(0)/a)[1 - \exp\{-at\}] \quad (2)$$

This time evolution (2) is valid up to the time t_1 at which a bond first reaches the threshold $T_r = 1$ for rupture. At t_1 , this bond is broken, i.e. its resistance is put equal to infinity. The current distribution in all remaining bonds is computed again. We note $i_n(t_1)$ the value of the new current in bond n after the breaking of the first bond. This set of currents $\{i_n(t_1), \text{ for all bonds}\}$ is injected back into the heat equation (1), with the additional information that the temperature attained by bond n at t_1 is $T_n(t_1)$. Iterating the procedure again and again after each bond rupture and noting t_i the time at which the i -th bond has been blown out, we obtain the following general time dependent temperature expression for the n -th bond, provided that it has not reached the threshold temperature $T_r = 1$:

$$T_n(t) = T_n(t_i) e^{-a(t-t_i)} + (r_n i_n^b(t_i)/a)[1 - \exp\{-a(t-t_i)\}] \quad \text{for } t_i \leq t \leq t_{i+1} \text{ if } T_n(t) \leq 1. \quad (3)$$

Note that the ordering of the successive bonds rupture is well-defined, thus enabling the introduction of the ordered breaking times $0 < t_1 < t_2 < \dots < t_i < t_{i+1} < \dots \leq t_f$, where t_f is the time of the last bond rupture, which leads to a complete disconnection of the network in at least two pieces. Equation (3) allows to obtain the formal solution of the temperature of the n -th bond under the form

$$T_n(t) = (r_n/a) \sum_{j=0}^{i-1} [i_n(t_j)]^b e^{-a(t_{j+1}-t_j)} [1 - \exp\{-a(t_{j+1}-t_j)\}] + (r_n [i_n(t_i)]^b/a) [1 - \exp\{-a(t-t_i)\}] \quad (4)$$

for $t_i \leq t \leq t_{i+1}$ if $T_n(t) \leq 1$.

Let us make a few comments concerning this model. First note that the time dependence enters *via* the existence of two time scales $\tau_1 = T_r/r_i^b$ and $\tau_2 = a^{-1}$. τ_1^{-1} represents the rate of heating due to the generalized Joule effect while τ_2 is a relaxation time with a bath. τ_2 allows for the existence of an asymptotic temperature $T_n(+\infty) = r_n i_n^b/a$ under a constant current. It also determines the way this asymptotic temperature is reached. We will show below that it is the competition between these two times scales which creates a remarkable sensitivity of the rupture dynamics and of the final crack pattern as a function of the input current.

Secondly, note that the coupling between the electrical and the thermal fields are not symmetrical: indeed, only the electrical current field couples bonds at large distances whereas the temperature field is a local quantity, coupled only locally on each bond to the electrical current in that bond. It is only when thermal rupture occurs on a bond that a long range reorganisation of the electric field appears due to the change of the electrical network topology and to the long-range nature of the electrical Green function. The simplicity of the model stems from the separation of the time evolution of the electrical and thermal fields: the electric current distribution evolves instantaneously under rupture of a new bond and the thermal field changes continuously under the fixed current distribution until the next rupture occurs. This feature simplifies a lot the analysis and the numerical computations. Furthermore, this may be of some relevance to real material systems: considering the rupture of metals for instance, one could imagine to relate the thermal field to the plastic component of the deformation. The plastic deformation is indeed known to be preferentially localized in regions of large elastic deformation, for instance in the vicinity of crack tips. In particular, the theoretical $r^{-1/2}$ divergence of the stress at the tip of a crack is smoothed out by the appearance of a plastic deformation localized near the crack tip. Thus, similarly to our model, the plastic component of the deformation is controlled by the value of the elastic component. Furthermore, the growth of a crack involves first the restructuration of the plastic zone near the crack tip. Thus, the rupture is controlled by the second (plastic) field, in a fashion which is similar to what occurs in our dynamical thermal model.

In fact, the random fuse model introduced above possesses an exact mechanical analog. In this analogy, each bond carries, not a conductance, but an elasto-plastic element. It is also assumed that each element is allowed to deform only along its *anti-plane* component. The anti-planar simplification provides simple scalar fields (stress s and strain ε). It is assumed that the total strain ε can be written as the sum of elastic ε_e and plastic ε_p components. The elastic displacement w_e in the direction normal to the lattice plane is given by the solution of

$$\nabla \cdot [g(x, y) \nabla w_e(x, y)] = 0,$$

where $g(x, y)$ is the elastic shear constant on the element at position (x, y) . This equation is the exact analog of Kirchoff's equation of the electric problem. Hooke's law $s = g \varepsilon_e$ replaces $i = gv$ for the electrical case. Then, $\varepsilon_{ex} = \partial w_e / \partial x$ and $\varepsilon_{ey} = \partial w_e / \partial y$ for lattice elements that point in the x - or y -directions. To pursue the analogy, we furthermore assume that each element of the lattice, which is submitted to a given stress s , undergoes a creep deformation ε_p which obeys the equation

$$d\varepsilon_p/dt = G^{-1} s^b - a\varepsilon_p. \quad (5)$$

$d\varepsilon_p/dt$ is the rate of the creep deformation process which is increasing with the applied stress as a powerlaw with an exponent b . The second term accounts for a possible hardening process (so-called « work hardening »). The plastic or ductile deformation ε_p is thus supposed to be controlled by the elastic stress and obeys equation (5). The important idea is thus to consider two degrees of freedom ε_e and ε_p controlling the total deformation, coupled together *via* the stress field. Similarly to the electrical version, the definition of the mechanical model is completed by the rule that an element breaks down irreversibly when its plastic deformation ε_p under creep reaches a given threshold. This rupture criterion can be justified by thinking of the creep deformation as a plastic component of the deformation which controls the rupture process. We thus assume a plastic-controlled rupture or in other words only when the plastic deformation is sufficiently large, does the system suffer break down.

3. Relation with other models.

It is worth noting that the thermal fuse model contains as natural limits two well-studied statistical models of rupture, as we now discuss.

3.1 THE RANDOM FUSE MODEL AS THE LIMIT $b \rightarrow +\infty$. — Consider first the limit of large values of the exponent « b » entering the generalized Joule heating effect introduced in equation (1). Suppose that the time is normalized by the time scale $\tau(i_{\max}) = T_r/r_n i_{\max}^b$, where i_{\max} is the largest current in the network. Then, equation (1) becomes

$$\begin{aligned} dT_n/d(t/\tau) &= [i_n(t)/i_{\max}]^b - \{aT_r/r_n i_{\max}^b\} T(t) \\ &\approx [i_n(t)/i_{\max}]^b \text{ in the limit of large } b. \end{aligned} \quad (6)$$

In the limit $b \rightarrow +\infty$, all terms $[i_n(t)/i_{\max}]^b$ tend to zero except the one for which $i_n(t) = i_{\max}$, which is then equal to one. Thus, in this limit, only the bond which carries the largest current is heated. As a consequence, all bonds remain intact except the bond which carries the largest current, which eventually reaches the rupture threshold $T_r = 1$ and breaks down irreversibly. After this breaking, the new current distribution is calculated which allows to define again the bond which carries the largest current. The specific value of this largest current is not constant and evolves with the rupture process. However, the important observation is that the thermal heating amounts to select and always break down the bond which carries the largest electrical current. Forgetting for the moment the time dependence of this process, we thus recover exactly in the limit $b \rightarrow +\infty$ the *quasi-static* random fuse model [12]. A large amount of work has been done on this model, especially concerning the scaling properties of voltage and current fields at the end of the breakdown process [1, 2].

In addition to the correspondence between the ordering and location of the bond breakdown in the two models, the thermal fuse model is characterized by a natural time evolution. Using the notations introduced to formulate equations (3) and (4), the total time t_f needed for global rupture of the network can be written as

$$t_f = \sum_{j=0}^{i-1} \{T_r/r_{\max}(t_j) i_{\max}^b(t_j)\} \quad (7)$$

which is the sum of the time intervals between two successive bond breakdowns. T_r is the temperature threshold for rupture equal to one in the computations. $i_{\max}(t_j)$ is the largest current among all bonds in the network just after the breakdown of the j -th bond at time t_j and $r_{\max}(t_j)$ is the value of the resistance of this bond which carries the largest current just after the breaking of the j -th bond. Equation (7) is valid if each interval between two breaking events $\tau_1 = T_r/r_{\max}(t_j) i_{\max}^b(t_j)$ is much shorter than the relaxation time $\tau_2 = a^{-1}$. The choice of the thermal coupling constant a and all the resistances r_n being made at the beginning, this condition can always be fulfilled in a finite system for a large enough value of b . Note that from equation (7), the total time t_f for global rupture of the network is analogous to an integrated conductance jump distribution [1]. In this limit of very large b 's, changing the input current I by a factor λ only rescales the time by the factor λ^{-b} , without changing the ordering and the location of the bond breakdowns which are characteristic of the *quasi-static* random fuse model.

3.2 THE BOND-PERCOLATION MODEL AS THE LIMIT $b \rightarrow 0$. — Let us now consider the other limit $b \rightarrow 0$. In this case, equation (1) becomes

$$dT_n/dt = r_n - aT_n(t). \quad (8)$$

If we apply equation (8) for all bonds, whatever their current, finite or vanishing, the geometrical characteristics of the rupture process can be mapped exactly onto a bond percolation model since the thermal field and thus the rupture process decouple completely from the electrical current field. The thermal heating is function only of the quenched value of the resistance r_n . Note that this definition of the model in the limit $b \rightarrow 0$ amounts to take the following ordering of the two limits $b \rightarrow 0$ and $i \rightarrow 0$ in the expression i^b , where i is the current in a bond : $\lim_{i \rightarrow 0} \lim_{b \rightarrow 0} i^b = 1$. Indeed, this ordering of limits does not distinguish between bonds

which do or do not carry a current.

This model is rather peculiar since it amounts to heat bonds which carry no current at all. For any given non-zero b value, for a bond to be heated it must carry a non-vanishing current. For small values of the exponent b , the heating power becomes weakly dependent upon the current. However, one still has $0^b = 0$ for $b \neq 0$, i.e. bonds which do not carry any current are not heated. It is thus more natural to consider an alternative way to take the limit $b \rightarrow 0$ which is the extension of ($b \neq 0$)-thermal fuse models, in which only bonds which carry a non-zero current are heated and can eventually break down. We thus consider the $b \rightarrow 0$ limit of the thermal fuse model in which, only those bonds which carry a current, whatever its value, are described by equation (8). For intact bonds which are perfectly screened and carry no current, no thermal heating occurs at all. This means that we cannot simply consider each bond independently of each other. The electrical current field still retains some influence on thermal heating in the sense that only the subset of all intact bonds which carry a non-vanishing current can be heated according to equation (8). This implies that the geometrical structure of the system of ruptured bonds at complete rupture is slightly modified in comparison with the incipient infinite percolation cluster. In particular, loops are no more present (they cannot close) since the bonds in the interior of a loop are perfectly screened electrically and carry no current at all.

Mathematically, the present definition of the « correlated » percolation model amounts to take the following ordering of the two limits $b \rightarrow 0$ and $i \rightarrow 0$ in the expression i^b , where i is the current in a bond : $\lim_{b \rightarrow 0} \lim_{i \rightarrow 0} i^b = 0$. Numerically, it is not straightforward to

implement this limit due to inherent truncations. The concrete numerical problem is to decide when a bond carries a zero current, when considering the finite precision ($\geq 10^{-12}$) with which the currents are calculated. To solve in a satisfying manner this problem, we have introduced a threshold i_0 such that all currents found, below i_0 , are put exactly equal to zero. Numerical simulations have then been performed for a wide range of i_0 values, from 10^{-20} to 1. We have found a range $10^{-11} \leq i_0 \leq 10^{-4}$ such that the set of rupture bonds at the end of the process is found unchanged. This rupture pattern is the one which qualifies as the real thermal fuse model with the correct limit $\lim_{b \rightarrow 0} \lim_{i \rightarrow 0} i^b = 0$. Figure 2a shows the incipient

cluster obtained with this procedure with $i_0 \approx 10^{-10}$. A total of 2822 bonds have burned out. In comparison, the exact percolation model, obtained for instance with a threshold $i_0 \approx 10^{-20}$ which is not able to distinguish the bonds which carry or not a current, corresponds to 3212 broken bonds in the same disorder realization. The difference 3212-2822 is significant. However, the fractal dimension of the incipient cluster of the correlated percolation model shown in figure 3a is almost unchanged $D \approx 1.85$ from the value of uncorrelated percolation (≈ 1.89). With the available numerical accuracy, it is not possible to access whether this correlated percolation model is in a different universality class.

If we neglect the influence of the subtle correlation introduced by the electrical current, the mathematical mapping of the thermal fuse model with equation (8) to the percolation model can be done as follows. We consider the normalized distribution $P(r)$ of the resistance of the

bonds in the electrical network. To each bond resistance r corresponds a time-dependent temperature given by integration of equation (8) :

$$T(t) = (r/a)[1 - \exp(-at)] . \quad (9)$$

The distribution $Q_t(T)$ of the temperatures at time t in the network is thus obtained by performing the change of variable $r \rightarrow T(r, t)$ in the initial resistance distribution $P(r)$:

$$Q_t(T) = a[1 - \exp(-at)]^{-1} P(aT[1 - \exp(-at)]^{-1}) . \quad (10)$$

Equation (10) describes 1) an average translation of the temperature distribution to higher values as the time increases, 2) a dispersion of the temperatures around their average due to the fact that high resistances are heated more rapidly than smaller ones and 3) a corresponding normalization of the distribution due to the dispersion. The quantity p defined by

$$p = \int_1^{+\infty} Q_t(T) dT \quad (11)$$

thus gives the fraction of the bonds which have attained the rupture threshold $T_r = 1$. Since the resistances are independent random variables, the successive bond breakdowns are independent random events solely controlled by the distribution and spatial position of the electrical resistances. The global rupture will be reached when p has attained the percolation threshold p_c ($= 1/2$ for a square lattice in two dimensions). In the limit $b \rightarrow 0$, the ordering and location of the bond breakdowns and thus the entire geometrical properties of the rupture process is thus equivalent to the bond-percolation model. Note that the procedure implied in expression (11), which consists in reaching the percolation threshold by progressively filling a distribution of random numbers, constitutes an elegant way to define the percolation model and its threshold [13].

In contrast with other values of b , there is no current threshold for rupture in the limit $b = 0$. However, there is a critical value of the thermal coupling constant a . Since, according to equation (9), $T(t)$ does not grow indefinitely but tends asymptotically to r/a , the relaxation rate « a » must not be too large. Indeed, at large times, the temperature distribution tends to

$$Q_{t \rightarrow +\infty}(T) = aP(aT) . \quad (12)$$

If a is large, the bonds at threshold (i.e. $T = T_r = 1$) correspond to the tail of P for large r 's and may not attain the percolation threshold p_c . Therefore, for a given resistance distribution $P(r)$, there exists a critical value a_c above which global rupture is never attained. The threshold a_c is determined by the condition

$$\int_1^{+\infty} a_c P(a_c T) dT = \int_{a_c}^{+\infty} P(R) dR = p_c . \quad (13)$$

The discussion of the global rupture time in the case $b = 0$ is presented in appendix A.

4. Properties of the dynamical thermal fuse model.

Let us now study the generic case $0 < b < +\infty$. We have performed a large set of different computations from which we now present the salient features. We first discuss the limit of small a 's or equivalently of very large applied electrical currents (much larger than the threshold current I_c defined below) and then the more complex case of a finite « a ».

4.1 THE HIGH HEATING POWER OR CRACK-CRACK FUSION REGIME ($I \gg I_c$ OR $a \rightarrow 0$).

4.1.1 *Dilution versus crack growth and fusion.* — Figure 1 shows the damage patterns at three times of the rupture dynamics of the same system for $b = 2$ and a disorder parameter $\Delta g = 0.2$ (the bond conductances are uniformly sampled in the interval $[0.9 ; 1.1]$) in a square lattice of size 180×180 tilted at 45° . Figure 1a corresponds to a fraction of ruptured elements equal to 50 % of the total number (3423 in this specific case) needed to complete the global rupture. The reduced time of this snapshot is $t/t_f = 0.9912$. In this first picture, one observes essentially isolated independent breakdown events leading to a continuously increasing damage of the system. One may however notice the existence of a few relatively large clusters of broken elements which tend to dominate the rupture in the continuing rupture process. Note that the damage occurs rather late in the dynamics : for instance, the first element breaks down at $t/t_f = 0.886$, the second one is at $t/t_f = 0.895$, etc. Figure 1b shows the same system at a latter time $t/t_f = 0.9982$ at which 80 % of the bonds needed to complete the global rupture have ruptured. Many large cracks are competing and, from their observation, it is very difficult to predict where will be the chosen path of the macroscopic rupture. A small change in the initial disorder realization may drastically change the final rupture pattern. Figure 1c shows the system at complete rupture at $t/t_f = 1$. The first important information obtained from figure 1 is the existence of two regimes. At relatively small times, the progressive deterioration is similar to a random bond dilution process. In this time interval, the initial quenched disorder on the conductances dominates the dynamics. The bonds which present the largest Joule energy ri^2 break down first. Note that a similar observation has been made in the quasi-static random fuse model [1, 2]. As the approximately uncorrelated dilution process proceeds, clusters of various shapes and sizes beging to form. At larger times, the dilution process gives place to the regime where fusion between cracks and the growth of clusters dominate. This is the regime where current enhancement and screening effects are most important. Long-range spatial and time correlations appear and form a large defect which eventually breaks the network in at least two pieces.

4.1.2 *Existence of a well-defined limit of large I 's or small a 's.* — For a given network, i.e. for a given conductance disorder configuration, we have observed numerically that the ordered sequence of bond breaks down and therefore the final crack pattern at the end of rupture do not change in the limit of large input currents I . In other words, there is a well-defined limit of the rupture process and final crack pattern for large I 's. This is in contrast with the other regime, studied in the following section 4.2, where the applied current I is not large compared to the rupture threshold I_c , and a small change of the input current I may drastically modify the rupture pattern. The analytical proof of the existence of this well-defined $I \rightarrow +\infty$ limit is given in appendix B. Concerning the time scales of the rupture, it is shown in appendix B that all time intervals are reduced by the factor λ^{-b} when increasing the total applied current by the factor λ^b . In particular, this indicates that the total time needed for global rupture of the network scales as

$$t_f(I) \sim I^{-b} \quad (14)$$

for large I 's.

4.1.3 *The effect of the exponent b on damage patterns.* — Figure 2 shows a series of final rupture patterns obtained in this limit of large I 's for different values of b : $b = 0$ (a) ; $b = 0.1$ (b) ; $b = 0.5$ (c) ; $b = 1$ (d) ; $b = 2$ (e) ; $b = 4$ (f) and $b = 8$ (g). The set of ruptured bonds which participate to the « infinite » crack which disconnects the system in at least two pieces are outlined. For small b 's, the damage is dense and the « infinite » cluster of broken

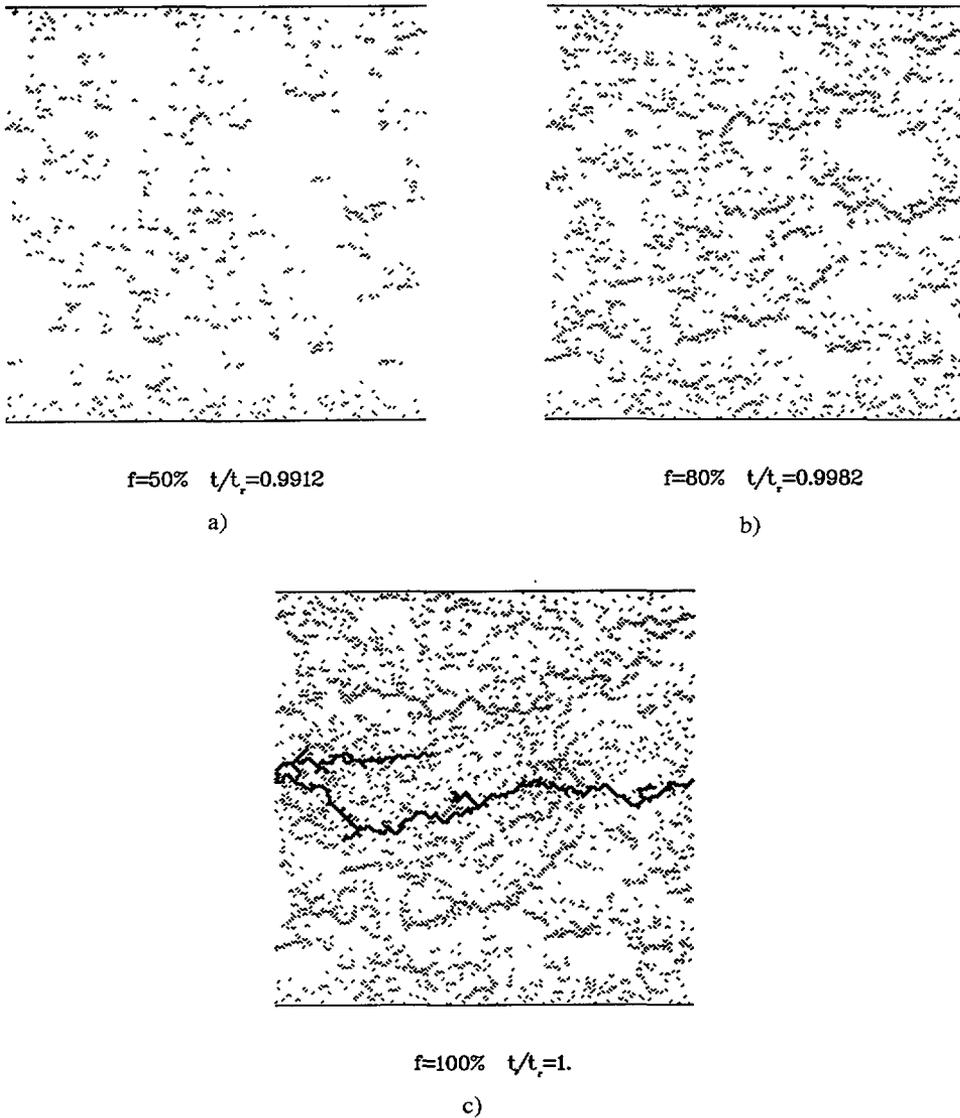


Fig. 1. — Typical crack patterns at three typical times of the rupture dynamics of the same system for $b = 2$ in a square lattice of size 180×180 tilted at 45° . For clarity of the figure, only the broken bonds are displayed. The total current flows between the top and bottom bus bars. The conductances are uniformly sampled in the interval $[0.9, 1.1]$ ($\Delta g = 0.2$). a) Fraction of rupture elements equal to 50 % of the total number (3423) needed to complete the global rupture ($t/t_r = 0.9912$). One observes essentially isolated independent breakdown events leading to a continuously increasing damage of the system. One may however notice the existence of a few relatively large clusters of broken elements which tend to dominate the rupture in the continuing rupture process. Note that the damage occurs rather late in the dynamics : for instance, the first element breaks down at $t/t_r = 0.886$, the second one is at $t/t_r = 0.895\dots$ b) Fraction of rupture elements equal to 80 % of the total number needed to complete the global rupture ($t/t_r = 0.9982$). Many large cracks are competing and, from their observation, it is very difficult to predict where will be the chosen path of the macroscopic rupture. A small change in the initial disorder realization may drastically change the final rupture pattern. c) System at complete rupture (100 % broken elements and $t/t_r = 1$). The main crack which disconnects the network in two pieces is represented by boldface bonds.

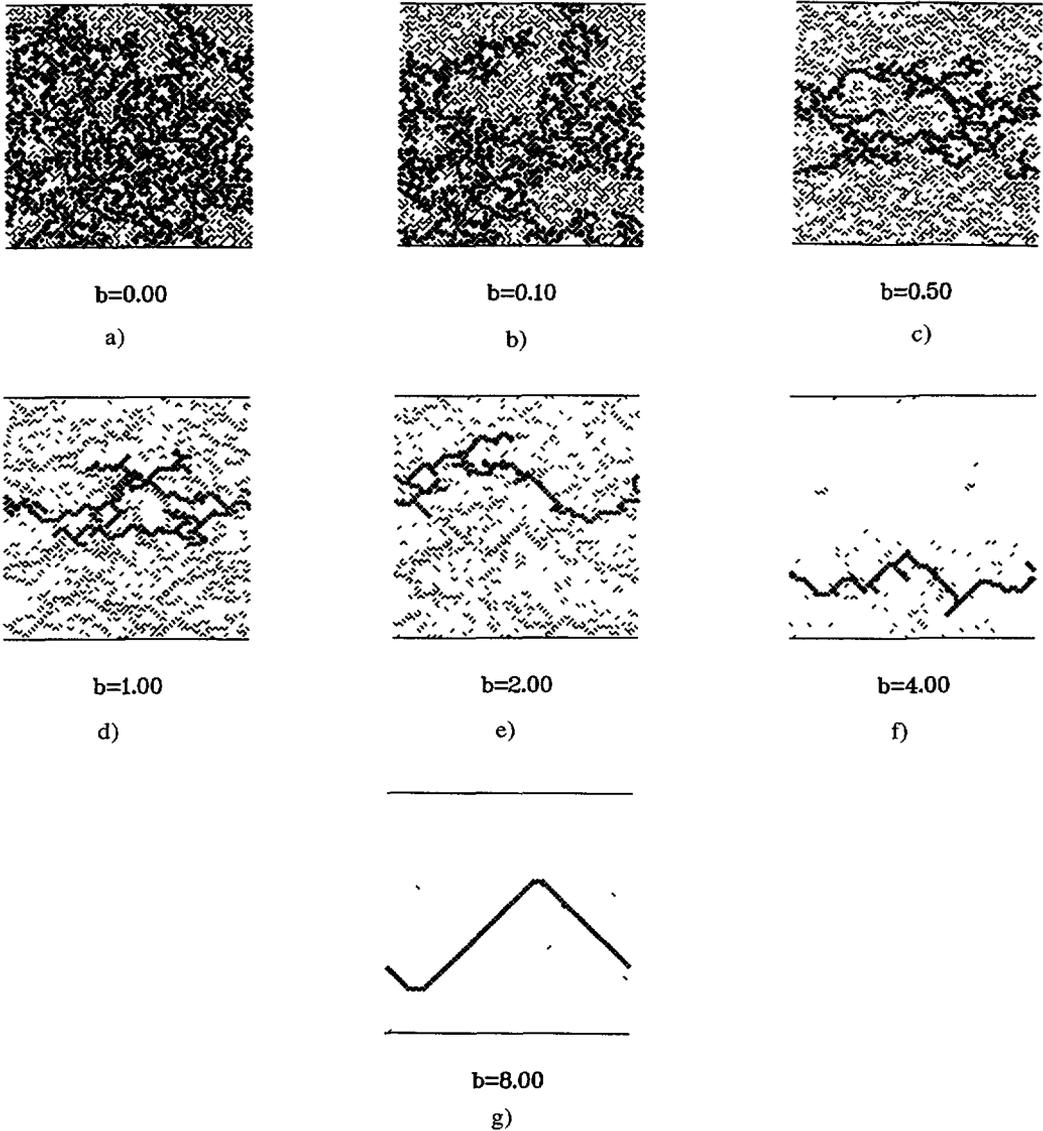


Fig. 2. — Final rupture patterns in the limiting case $a = 0$ (i.e. for very large currents if $a \neq 0$), for $b = 0$ (a); $b = 0.1$ (b); $b = 0.5$ (c); $b = 1$ (d); $b = 2$ (e); $b = 4$ (f); $b = 8$ (g). The conductances are uniformly sampled in the interval $[0.9, 1.1]$ ($\Delta g = 0.2$). As in figure 1, only the broken bonds are displayed. The main crack which disconnects the network is outlined.

bonds possesses a large fractal dimension whereas for large b 's the rupture occurs essentially by the growth of a single one-dimensional crack. This corresponds to a cross-over from the uncorrelated percolation limit to a regime dominated by a current enhancement effect at the crack tips.

These qualitative observations can be quantified by calculating the fractal dimensions of the « infinite » crack. Two different methods have been used for measuring the self-similar structure of these cracks. The first method computes a « growth » fractal dimension D_1 of the largest crack, at different times during the rupture, by calculating the total number

of bonds it contains as a function of its projected lengths ℓ_x (resp. ℓ_y and $(\ell_x \ell_y)^{1/2}$) along its principal growth direction (resp. along the normal to this direction and their geometrical mean). The second method analyzes the macroscopic crack which is finally obtained at the end of the rupture, by measuring the number of broken bonds belonging to this crack in a box of size ℓ^2 as a function of ℓ (« capacity » fractal dimension D_2). For systems of size $L = 80$, the « capacity » and « growth » fractal dimensions are estimated by a linear fit in a log-log plot over one and almost two decades respectively. Figure 3 shows in log-log plots the number of broken bonds belonging to the crack as a function of its linear extension for the different b -values shown in figure 2. The two methods give similar results with deviations which are however systematic, indicating, in a manner similar to DLA growth processes, that perimeter bonds on which the action is taking place are not characterized by exactly the same scaling as the « dead » bonds in the interior. We find for instance $D_1 = 1.11 \pm 0.02$ and $D_2 = 1.18 \pm 0.02$ for $b = 2$. It is interesting to note that these dimensions do not depend on the disorder in the range we have considered from $\Delta g = 0.1$ to 1.6. Figure 4 summarizes the evolution of D_1 and D_2 as a function of b . Note that the value of the percolation model $D_1 = D_2 = 2 - \beta/\nu = 129/144 \approx 1.9$ is recovered for very small b 's. For $b > 2$, the fractal dimension decreases down to $D_1 = D_2 \approx 1$ as found for $b = 8$. This result is to be compared with the value for the quasi-static random fuse model [14]: $D = 1.1 \pm 0.1$. Eventually, additional simulations for systems of size $L = 40$ and $L = 160$ gave identical values of D_1 and D_2 as a function of b .

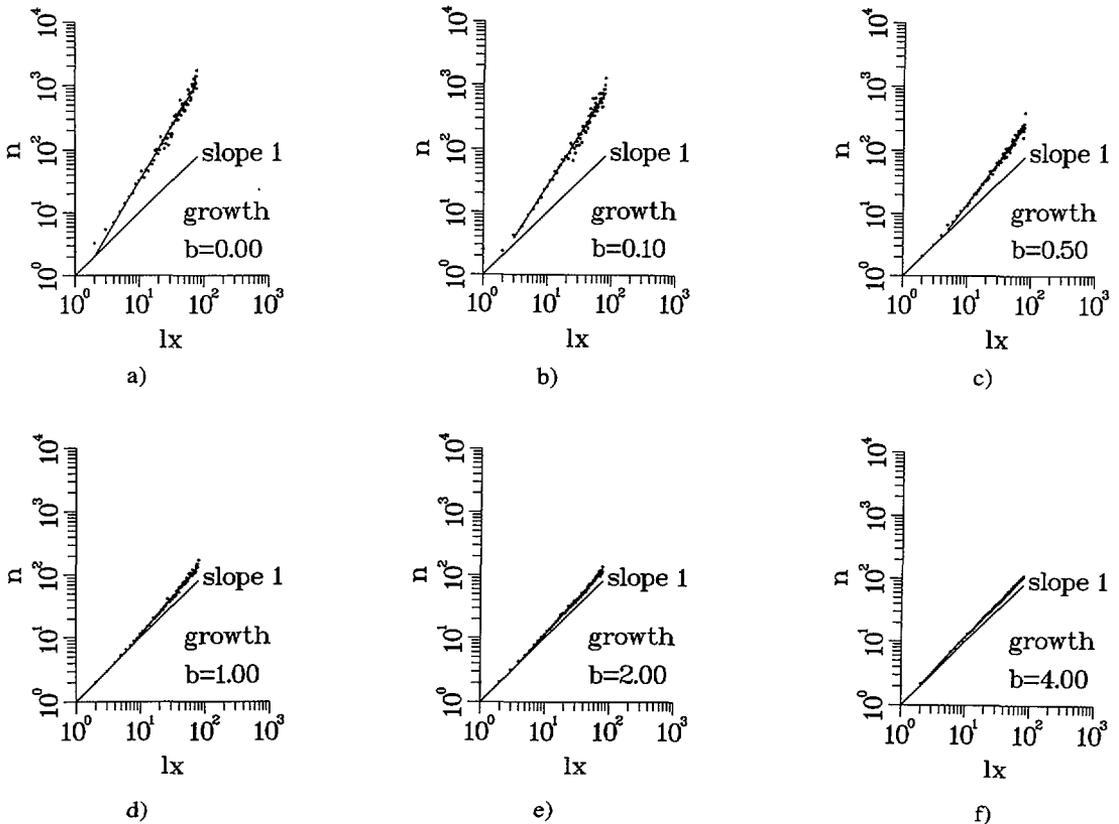


Fig. 3. — Log-log plot showing the fractal dimensions D_1 (growth) and D_2 (capacity) of the « infinite » cracks. Each curve has been obtained by averaging over typically 30 different disorder realisations ($\Delta g = 0.2$) giving rise to rupture patterns similar to those displayed in figure 2.

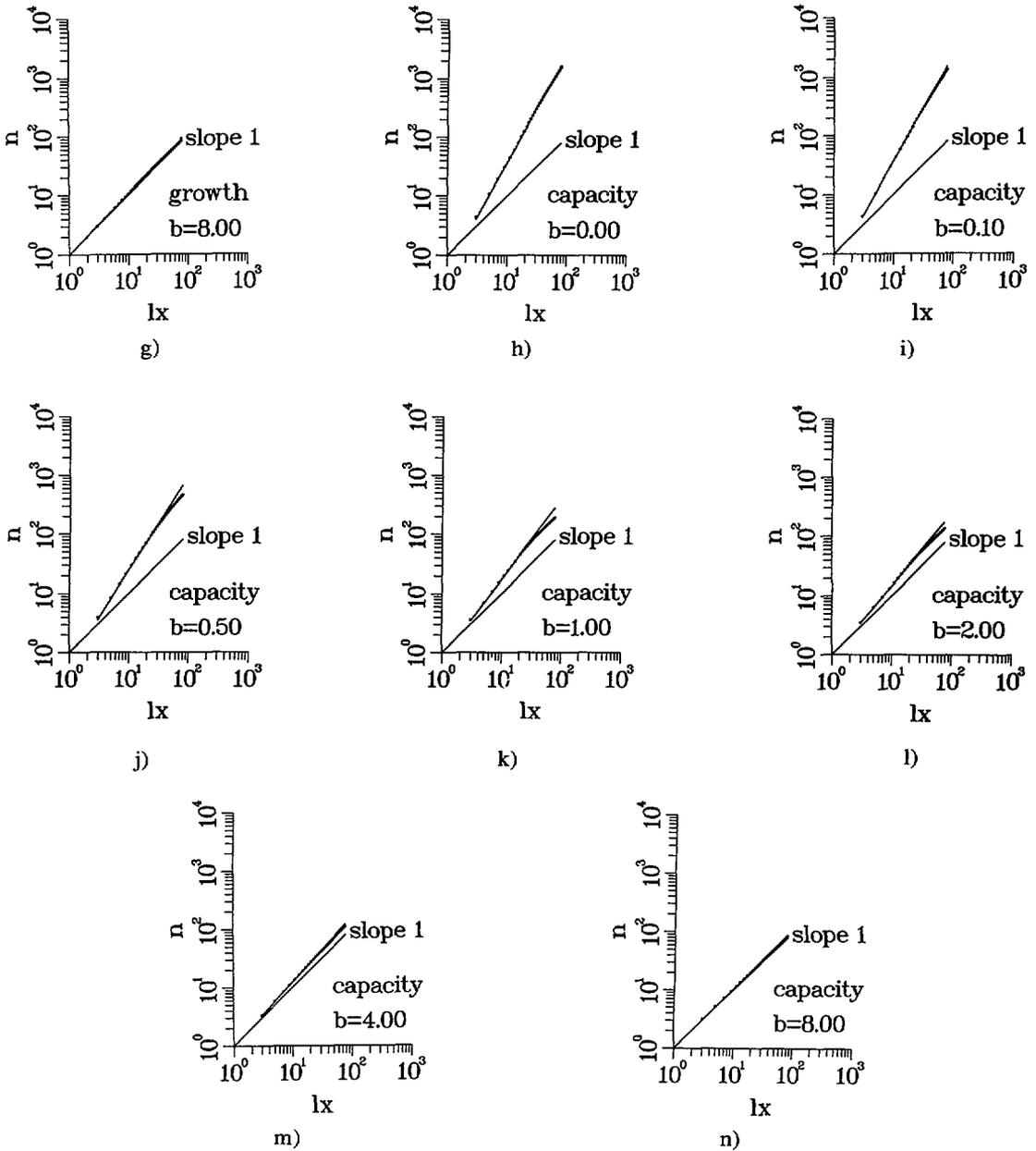


Fig. 3 (continued).

4.1.4 The effect of the exponent b on rupture dynamics. — It is convenient to monitor the dynamics of the damage and rupture process through the time evolution of the network total conductance G or its inverse G^{-1} , which is proportional to the dissipated electric power since we are working at constant applied current I . Figure 5 shows, for $b = 2$, in log-log units the time dependence of G^{-1} as a function of $(t_j - t)/t_j$ where t_j is the time of occurrence of a macroscopic rupture in the network. Each curve corresponds to a different value of the disorder Δg , the lower curve corresponding to the smallest disorder $\Delta g = 0.1$ and the upper curve corresponding to the largest disorder $\Delta g = 1.6$ which have been considered. Each curve

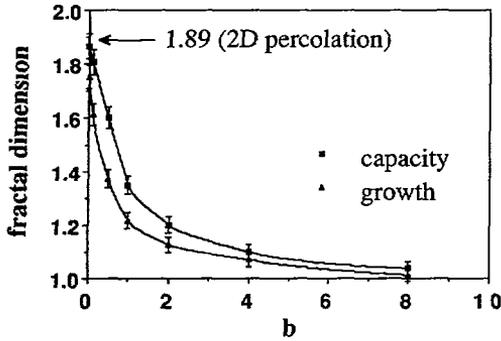


Fig. 4.

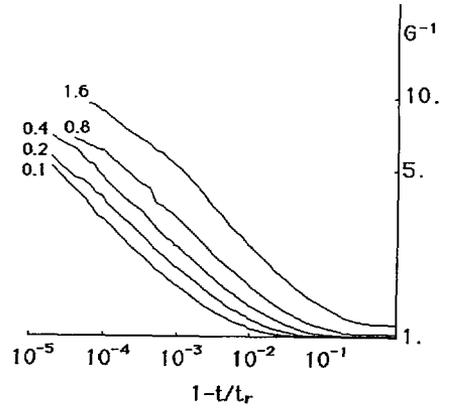


Fig. 5.

Fig. 4. — Dependence of the « growth » D_1 (triangles) and « capacity » D_2 (squares) dimensions as a function of the heating power b -exponent. The lines serve as guides to the eye.

Fig. 5. — In log-log units, the time dependence of G^{-1} is shown as a function of $(t_f - t)/t_f$ for $b = 2$ where t_f is the time of occurrence of a macroscopic rupture in the network. Each curve corresponds to a different value of the disorder Δg , the lower one corresponding to the smallest disorder $\Delta g = 0.1$ which has been studied and the upper curve corresponding to the largest disorder $\Delta g = 1.6$ which has been considered. Each curve has been obtained by averaging over typically 25 different network configurations (« ensemble » average) with the same average disorder Δg . The average slope of the curves in their linear portion is approximately $\alpha \approx 0.3$ for $b = 2$.

has been obtained by averaging over typically 25 different network configurations (« ensemble » average) with the same average disorder Δg . We observe that G^{-1} seems to follow a powerlaw as a function of $(t_f - t)/t_f$. $G^{-1} \sim [(t_f - t)/t_f]^{-\alpha}$ with an exponent $\alpha \approx 0.3$ for $b = 2$. One can observe that the exponent α does not seem to depend on the strength of the disorder Δg since all curves are approximately parallel. However, the exponent α is sensitive to the value of b as summarized in figure 6, which shows the evolution of α as a function of b : for $b = 0$ we find $\alpha \approx 1.2 \pm 0.1$; for $b = 0.1$, $\alpha \approx 1.15 \pm 0.1$; for $b = 1$, $\alpha \approx 0.7 \pm 0.1$; for $b = 4$, $\alpha \approx 0.15 \pm 0.02$; for $b = 8$, $\alpha \approx 0.05 \pm 0.02$. Note that predictions can be made in the two limiting cases $b = 0$ and $b \rightarrow +\infty$.

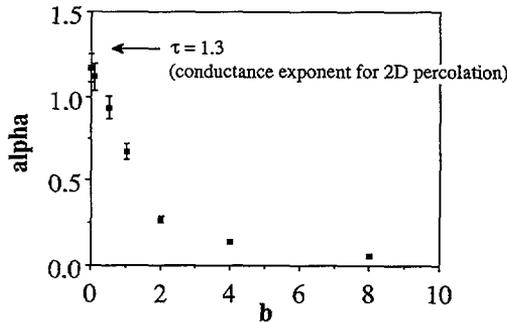


Fig. 6. — Dependence of the exponent α as a function of b .

Large b limit. — In this case, the rupture proceeds *via* the formation of essentially linear cracks of fractal dimension very close to 1 in two dimensions as shown in figures 2g and 3g. Let us now consider the growth of a linear macroscopic crack obeying the heat equation (1) (or creep growth law in its mechanical version) with $a = 0$. Neglecting some transient behaviors concerning the initial stage of the crack growth, one can forget about the initial disorder when the macroscopic crack is sufficiently large, since the rupture dynamics will be controlled by the very large crack tip stress enhancement effect. According to equation (3), the time needed for breaking an element submitted to a current i is $t = (ri^b)^{-1}$. When the macroscopic crack length is ℓ , the current applied on the element at its tip is proportional to $\ell^{1/2}$ due to current enhancement effect [15]. We can obtain an upper bound on the exponent α by assuming that the time for breaking the element, just placed at the crack tip, is essentially controlled by the large applied current, and only secondarily sensitive to previous thermal heating (plastic deformations) which has occurred when the crack tip was at a distance larger than ℓ from the crack tip. This simplifying assumption becomes all the more valid in the terminal stage of rupture when the macroscopic crack becomes very large such that the dynamical rupture process accelerates drastically. In this limit, an element far away from the tip is submitted to a relatively small current and its temperature has not the time to increase markedly. Within this assumption, we obtain the time t^* needed for breaking the element at the tip of the macroscopic crack of length ℓ which is proportional to $t^* \sim \ell^{-b/2}$. During this time, the crack grows by one lattice mesh. Therefore, we can write down the following differential equation for this crack growth: $d\ell/dt = 1/t^* \sim \ell^{b/2}$. Integrating this simple equation for $b > 2$, we find $\ell(t) \sim (t_f - t)^{-2/(b-2)}$ with the time for complete rupture of the network given by $t_f = [2/(b-2)] \ell_{\min}^{-(b-2)/2}$ where ℓ_{\min} is the initial size of the crack at our origin of time. In an infinite system, the incremental dissipated electric power due to a crack of length ℓ is proportional to ℓ^2 in two dimensions. We thus estimate G^{-1} which diverges as $G^{-1} \sim (t_f - t)^{-4/(b-2)}$. For large b 's, the obtained upper bound $4/(b-2)$ of the exponent α tends to zero. This implies that α goes to zero at large b 's. We do not expect this estimation to remain good for finite values of b due to the fact that the rupture process becomes more complex (see below) and our «tip» approximation becomes unwarranted. Indeed, this analysis would predict an exponent 0.66 for $b = 8$ and we find numerically (see Fig. 6) that G^{-1} does diverges at t_f with an exponent close to 0.05. The above simplified model thus becomes less and less reliable when b decreases. It is possible to improve on this upper bound by taking into account the effect of the finite size L of the lattice compared to the length of the crack, while still neglected the pre-heating of the bond at the tip. In reference [15], it is shown that finite size effects amount to replace the $\ell^{1/2}$ dependence of the current at the crack tip by $\ell^{1/2} [1 - (\ell/L)^2]^{-1/2}$. Solving the equation $d\ell/dt = 1/t^* \sim \ell^{b/2} [1 - (\ell/L)^2]^{-b/2}$ yields $1 - (\ell/L) \sim (t_f - t)^{1/(1+b/2)}$. In a finite lattice, the conductance is simply proportionnal to $1 - (\ell/L)$ when $\ell \rightarrow L$, up to logarithmic corrections. We thus obtain the improved upper bound $1/(1 + b/2)$ for the exponent α . This upper bound goes to zero for large b faster ($2/b$) than the previous estimate ($4/b$) and yields a better value = 0.2 for $b = 8$. However, this estimate is still not good for finite b 's due to the neglect of pre-heating.

Small b limit. — For $b = 0$, equation (3) simplifies into $T_n = r_n t$. The time needed for element n to reach the rupture threshold $T_r = 1$ is exactly given by $t_n = r_n^{-1} = g_n$. It is easier for our argument to introduce the distribution $P_g(g)$ of the conductances rather than the distribution $P(r)$ of the resistances considered previously. The number of rupture events $n(t) dt$ during an infinitesimal time interval dt is thus simply given by $n(t) = P_g(g) |dg/dt|$. As $t = g$, one finds $n(t) = P_g(t)$. The fraction $p(t)$ of intact bonds at time t is simply given by $p(t) = 1 - \int_0^t n(t) dt$. From percolation theory, the global conductance G of the network

goes to zero as p goes down to p_c according to the powerlaw $G \sim (p - p_c)^\tau$ with a critical exponent $\tau \approx 1.300$ in 2D and 2.0 in 3D. Therefore, the dissipated electrical power G^{-1} scales as $(p(t) - p_c)^{-\tau}$ with $p(t) = 1 - \int_0^t P_g(t) dt$. For the sake of concreteness, consider the case of the uniform distribution of conductances $P_g(g) = 1/\Delta g$ in the interval $[1 - \Delta g/2, 1 + \Delta g/2]$ used in our numerical simulations. Then, $p(t) = 1 - t/\Delta g$ and $G^{-1} \sim (t_f - t)^{-\tau}$ with $t_f = \Delta g/2$. Thus, we predict $\alpha = \tau \approx 1.3$ in 2D in reasonable agreement with the value $\alpha = 1.2 \pm 0.1$ found in our numerical simulations. One can show that this scaling law with the exponent $\alpha = \tau$ keeps holding for a very broad class of conductance distributions.

4.2 THE NUCLEATION AND BRANCHING REGIME ($I \geq I_c$). — We now turn our attention to the most general case where $a \neq 0$ and I is not too large so that the relaxation term $-aT$ is not completely dominated by the heating term ri^b . In this case, in contrast to the previous regime studied in 4.1, the physics of the rupture depends on the value of the applied current. This is illustrated in figure 7 which shows three final rupture patterns for three different applied currents I , for the same given realisation of the initial conductance disorder and same b . As before, the numerical procedure consists in applying at $t = 0$ a constant current I which flows from one bus bar to the other. The current per bond would thus be I/L in absence of conductance disorder. For a given bond resistance distribution, there exists a minimum value I_c of the input current necessary for global rupture to occur. Figure 7 has been obtained with the following choice of the parameters : $a = 1$ (this will remain our choice in the sequel), g is randomly distributed within the interval $[1 - \Delta g/2, 1 + \Delta g/2]$ with $\Delta g = 0.2$. Figure 7a corresponds to I very close to I_c , figure 7b to a larger value $I \approx 1.1 I_c$ and figure 7c to a very large value $I = 35 I_c$. Comparing the three figures, it is clear that the physics of the rupture for I close or slightly larger than I_c is dramatically different from that for large I 's discussed in the previous section 4.1.

For I very close to I_c , the rupture process can be decomposed into two steps. The first step is the breaking of the first bond, which will then act as a nucleation center. The second step is the growth of a large crack developing from this nucleation center. In the first regime, the Joule heating breaks down the bond which presents the strongest heating power. Once this « hottest » bond is broken, the defect strongly distorts the electric current field around it. Namely, the intensity in the neighboring bonds is typically $(4/\pi)I$ for the square lattice tilted at 45° with respect to the bus bars. These bonds are thus heated much more efficiently than all the others in the network. They reach the rupture threshold T_r first. The process goes on with the *connected* growth of a large crack until the final blow up of the total network. We call this regime the « nucleation » regime, since once a bond breaking has been initiated, the evolving rupture grows from this « nucleation » center. Of course, this « nucleation » regime occurs only for sufficiently small disorder, so that the Joule heating power $\approx [(4/\pi) i]^b$ in the bonds at the tip of the first broken bond is significantly larger than all the other heating powers in the other bonds. For larger disorders (larger Δg 's), one still observes the nucleation of a few cracks which then grow and fuse until total breakdown. As the disorder increases, more nucleation centers appear at the beginning of the dynamics, and the final crack is more tortuous, reflecting the fusion of a larger number of initial cracks.

The total time needed to break the entire network diverges when $I \rightarrow I_c$. We can determine the nature of this divergence by noting that the main fraction of the time needed for the complete breakdown of the network is in fact spent in nucleating the first crack. After this nucleation, the heating of the bonds at the tip of the crack is much stronger and the time needed for the subsequent ruptures remains relatively smaller as $I \rightarrow I_c$, since it is a

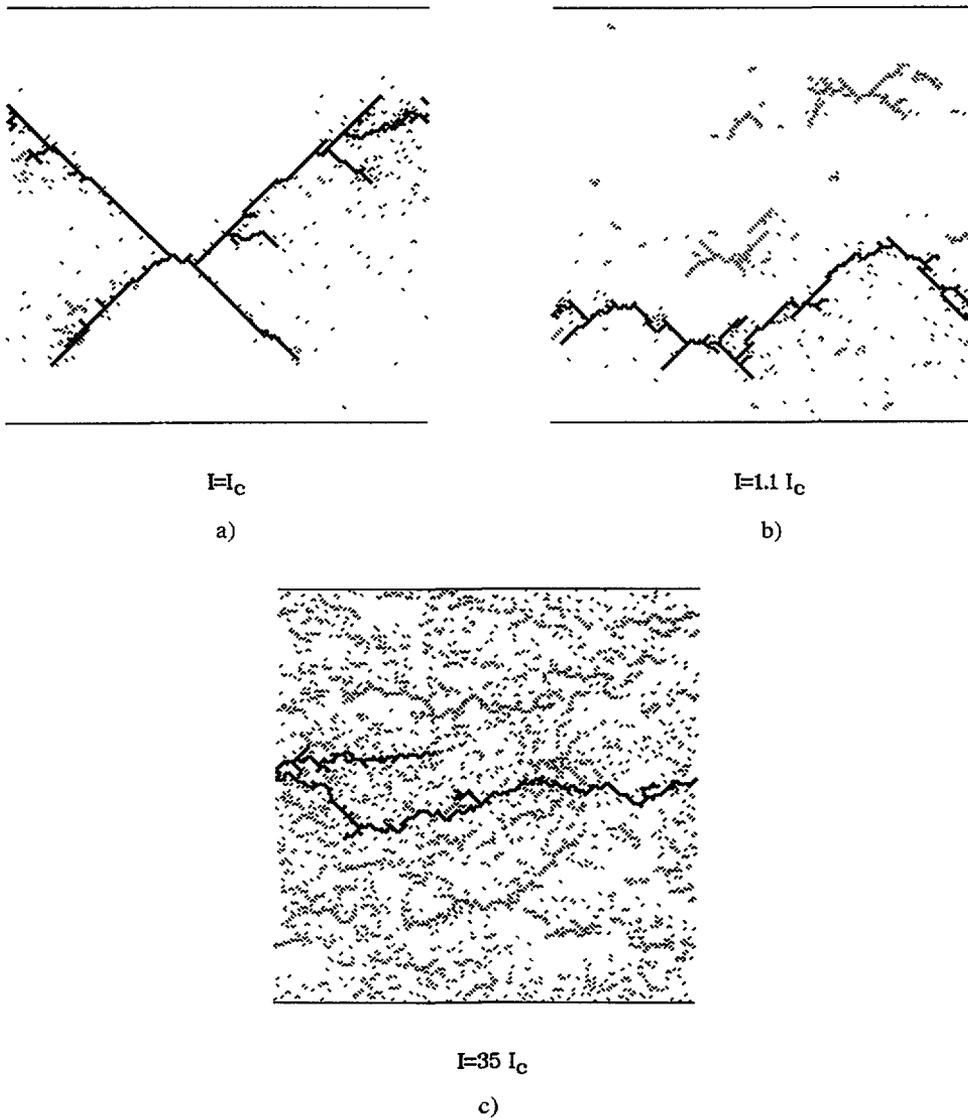


Fig. 7. — For a given realisation of the resistance disorder and for three values of the applied current I per bonds, this figure shows the final stage of rupture. The resistances are sampled in the interval $[1 - \Delta g/2, 1 + \Delta g/2]$ with $\Delta g = 0.2$. a) $I \approx I_c$. the applied current per bond is close to the threshold. b) $I \approx 1.1 I_c$. c) $I = 35 I_c$.

characteristic of the propagation on a single linear crack. In fact, the time t_f^* needed to break the first bond is a lower bound of the total breaking time t_f . Since it diverges as $I \rightarrow I_c$, so does t_f .

Note that for a given realization, the first bond to break down is always the same for any value of the total current I above the critical value I_c for rupture. This is the bond where the heating power ri^b is the largest in the network before any breaking event. The current i carried by this bond and the time t_f^* needed to break it down obey equation (2) where $T = T_r = 1$, i.e. $ri^b[1 - \exp(-at_f^*)]/a = 1$. At the onset of rupture t_f^* diverges and the

critical value i_c is given by $ri_c^b/a = 1$. Using $i/i_c = I/I_c$, the lower bound t_f^* of the total time t_f for rupture is solution of $(I/I_c)^b [1 - \exp(-at_f^*)] = 1$, which yields $t_f^* \sim a^{-1} \text{Log} [I_c/(I - I_c)]$. If instead of the relaxation term $-aT$ of equation (1), we were to choose a general powerlaw relaxation of the form $-aT^\eta$, we would again obtain a logarithmic dependence of t_f^* as a function of $I - I_c$. For instance, the analytically non-trivial soluble case $\eta = 2$ yields $T(t) = (ri^b/a)^{1/2} \{1 - 2[\exp\{2(ari^b)^{1/2}t\} + 1]^{-1}\}$ and thus again $t_f^* \sim a^{-1} \text{Log} \{I_c/(I - I_c)\}$. This case is particularly intriguing since it shows that there is a whole set of relaxation times $(ari^b)^{-1/2}$, one for each bond resistance r of the network, which are function of the current flowing in each particular bond. As the currents evolve with the topology of the network during the rupture process, the distribution of relaxation times also evolves. Since larger and larger currents appear at the end of the breaking process, the relaxation times becomes shorter as $i^{-b/2}$, where i is the current flowing in a bond. This increases the acceleration of the dynamics already observed for the simple case $\eta = 1$.

For a total applied current I slightly above I_c , we observe again that the crack patterns as a function of time are characterized by the growth and fusion of cracks nucleating at a few centers. In this regime, increasing slightly the applied current may bring a local or even a macroscopic change in the breaking pattern, due to the nucleation of small cracks at different positions. This regime is thus characterized by a remarkable sensitivity with respect to the applied current. Indeed, in some instances, changing I by a very small amount leads to a catastrophic change in the breaking pattern. This is particularly clear in figure 8, where one observes a main crack at complete rupture which lies in the upper (U) half of the network, then for a slightly larger applied current, it changes into another crack in the lower (L) half of the network, then goes up again, then goes down and up finally, defining the symbolic sequence ULULU ! In addition to this global evolution, there are many modifications of the crack patterns at smaller scales. The observed remarkable sensitivity with respect to the applied current can also be guessed in figure 7b where, in addition to the main disconnecting crack, two large cracks bear witness of the existence of two other competing rupture paths. This property is analogous to the drastic alteration found on the best path in a random medium which undergoes slow drifts [16]. However, the drift concerns, here, the input current in the *same* random network. In addition to macroscopic rupture pattern alterations, we have also observed a hierarchy of evolutions of the crack patterns at smaller scales, as the applied current I is varied. This observation suggests the existence of a hierarchical structure of the main rupture path, with global attractives basins of relatively wide extend in the I variable, which break into smaller basins of attraction, corresponding to local reajustments of the path of rupture. This qualitative idea could be quantified by studying the overlap between two crack rupture patterns corresponding to two different applied currents.

The existence of this wealth of behaviors and patterns relies on the competition between the two characteristic time scales of the problem: $\tau_1 = T_r/ri^b$ and $\tau_2 = a^{-1}$. Recall that τ_1^{-1} represents the rate of heating due to the generalized Joule effect while τ_2 controls the long time regime, i.e. it determines both the existence of an asymptotic temperature $T_n(+\infty) = r_n i_n^b/a$ under constant current and also the relaxation to it. If only the heating time scale τ_1 was present, i.e. $\tau_2 = +\infty$, then we would be in the regime $I \gg I_c$ discussed in section 4.1 characterized by a complete lack of dependence of the rupture process with respect to the applied current. Therefore, the rich dependence found in figure 8 as a function of I would be completely absent. When the two time scales τ_1 and τ_2 are finite, a subtle *memory* effect appears whereby a bond with an intermediate heating power may break earlier than a bond which first has a lower heating rate and later on a higher heating rate. Indeed, from equation (1), the slope dT_n/dt_i at the beginning t_i of the $(i + 1)$ -th heating sequence of the n -th resistance is $r_n i_n^b(t_i) - aT_n(t_i)$. Thus, the largest current does not lead necessarily to

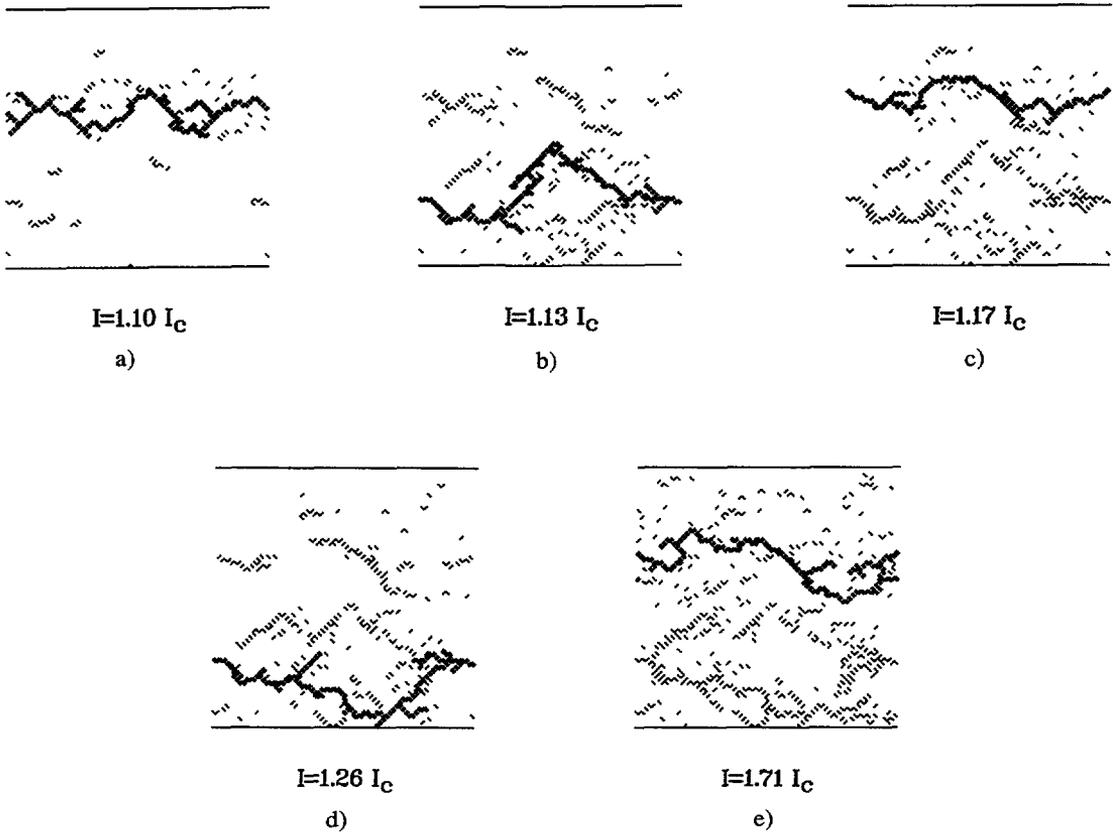


Fig. 8. — Final patterns of rupture obtained for the same system realization with $\Delta g = 0.8$ and $a \neq 0$ as the total input current I is slightly increased. a) $I/I_c \approx 1.10$; b) $I/I_c \approx 1.13$; c) $I/I_c \approx 1.17$; d) $I/I_c \approx 1.26$; e) $I/I_c \approx 1.71$.

the largest heating efficiency if the temperature $T_n(t_i)$ is already high, since then a large heat flux is directed to the thermal bath. As a consequence, the bond which carries the instantaneous largest current is not generally the one chosen for rupture at a given time, in contrast to the rule of the quasi-static random fuse model [1, 2, 12]. For a complete analysis of the alterations of the time ordering in two successive bond ruptures when changing the input current I , one needs to consider three sets of inequalities on the currents i_1 and i_2 , temperatures T_1 and T_2 , and heating rates dT_1/dt and dT_2/dt of the two bonds.

This reasoning also gives a clue to understand qualitatively the existence of the dilution regime at short times in a network fed with an applied current significantly larger than I_c . Indeed, the above memory effect implies that, at early times, the bonds which possess the largest initial heating power will be those breaking first. The initial values of the heating powers thus control the rupture process at small times. Since they can be considered to be conditioned by the initial quenched disorder on the resistances, this explains the occurrence of a percolation dilution regime at short times. As the exponent b decreases, the memory effect increases in strength over a larger time interval until it is all dominant at $b = 0$. In other words, the small time regime for large I 's and arbitrary $b \neq 0$ is similar to the rupture process at $b = 0$. Note that even in the limiting case $b \rightarrow +\infty$, one still observes a percolation-like initial deterioration regime [1, 2].

These different considerations can be alternatively synthesized by examining figures 9 and 10 respectively. Figure 9a (resp. 9b) shows a graph of the number of cracks N_s (clusters of broken bonds) as a function of the number of broken bonds N_r at each breaking event t_i , for different values of the total applied current I , in the case $b = 2$ (resp. for different values of b , for $a = 0$ or I large). Both quantities N_s and N_r are normalized to the total number of cracks (N_s) and to the total number of broken bonds (N_r) at the final stage of rupture respectively (thus, $0 \leq N_s, N_r \leq 1$). For $I \approx I_c$ and small N_r , the number of cracks increases slowly (Fig. 9a). This expresses the fact that once a first nucleation center has been broken, a single crack begins to grow from it. At a latter stage of the rupture process, a diffuse damage may occur then ending into the final connection of the large crack with the network borders. For large I 's and small N_r , N_s increases linearly with N_r , revealing the uncorrelated damage regime. For larger N_r , N_s tends to saturate meaning that no further crack nucleation occurs and that one enters a crack growth and fusion regime. Figure 9b shows that the number of cracks first increases drastically for small b 's up to a maximum and then decrease again. The first regime corresponds to the uncorrelated damage process akin to a dilution process in the percolation model. After the maximum, the system enters a regime of crack fusion. The separation between the two regimes is less and less well-defined as b increases. For very large b 's (here $b = 8$), the asymptotic value for N_s is very rapidly reached since only one large crack is present and grows in the system.

Figure 10 shows the dependence of the mass \mathcal{M}_{\max} of the largest crack as a function of N_r , for different values of the total applied current I (Fig. 10a) and different values of b for

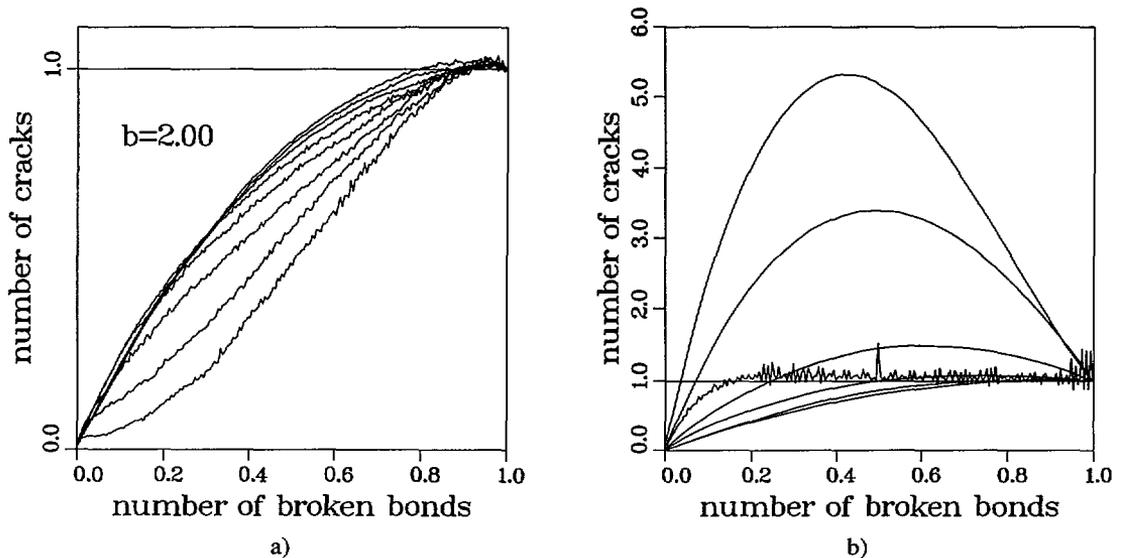


Fig. 9. — a) Number of cracks N_s (clusters of broken bonds) as a function of the number of broken bonds N_r at each breaking event t_i , for different values of the total applied current I increasing from the bottom to the top curve ($1 \leq I/I_c \leq 11$, $b = 2.00$, $\Delta g = 0.2$). Both quantities are normalized to the total number of cracks and to the total number of broken bonds at the final stage of rupture respectively ($0 \leq N_s, N_r \leq 1$). b) Same as a) for different b 's at large current I (or $a = 0$). The different curves correspond, from the top to the bottom, to $b = 0, 0.1, 0.5, 1, 2$ and 4 . The curve obtained for $b = 8$ has a special status since N_s is already very close to 1 for $N_r \approx 0.1$. This is due to the fact that essentially one large crack appears in the system. The wiggles decorating the curve for $b = 8$ come from the fact that the statistics are poor in this case, since only one main crack appears for each system realization.

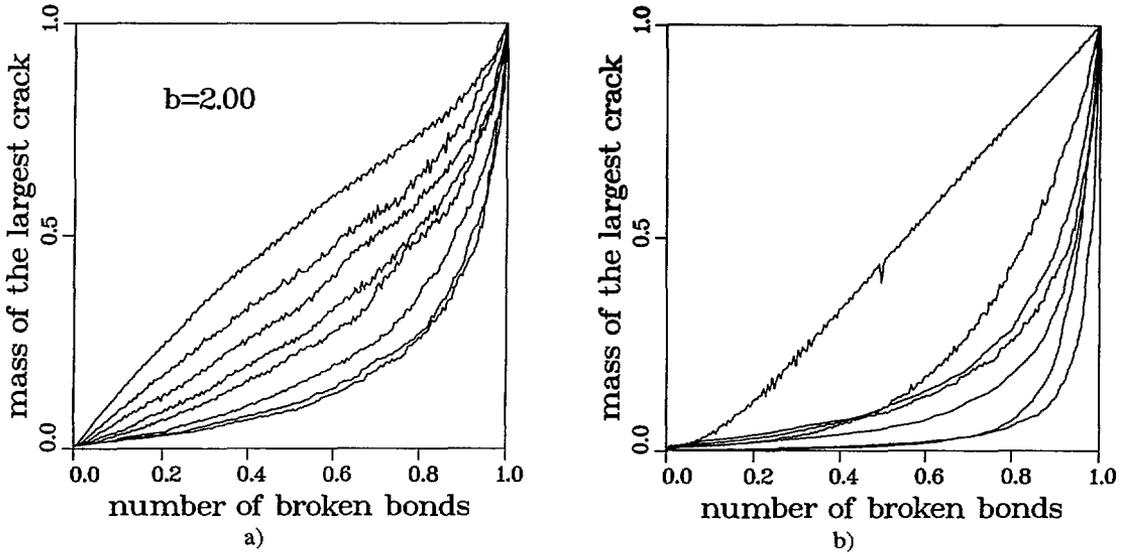


Fig. 10. — a) Dependence of the mass \mathcal{M}_{\max} of the largest crack as a function of the number of broken bonds N_r for different values of the total applied current I increasing from the top to the bottom curve ($1 \leq I/I_c \leq 11$, $b = 2.00$, $\Delta g = 0.2$). Both quantities are normalized to the total number of cracks and to the mass of the largest crack at the final stage of rupture respectively ($0 \leq \zeta_{\max}, N_r \leq 1$). b) Same as a) for different b 's at large current I (or $a = 0$). The different curves correspond, from the top to the bottom, to $b = 8, 4, 2, 1, 0.5, 0.1$ and 0 .

$a = 0$ or I large (Fig. 10b). The regimes discussed in figure 9 are also clearly exhibited. For $I \approx I_c$ (near the current rupture threshold), \mathcal{M}_{\max} increases steadily during the initial regime of single crack growth, then its increase levels off somewhat until it diverges at the final stage of rupture. For large I 's, one observes a completely different behavior: \mathcal{M}_{\max} increases very slowly at first, exemplifying the existence of the uncorrelated damage regime. Then it crosses over to a second regime where \mathcal{M}_{\max} grows very rapidly as a consequence of growth and fusion processes. Figure 10b shows that for large b 's (here $b = 8$), \mathcal{M}_{\max} varies essentially linearly with the number of broken bonds since only one main crack is present. As b decreases, \mathcal{M}_{\max} decreases in general for a given N_r , reflecting the existence of a less strongly correlated growth. For small N_r , \mathcal{M}_{\max} increases very slowly, reflecting the diffuse damage regime. It is only in the last stage of the process that \mathcal{M}_{\max} diverges with an exponent with increases as $b \rightarrow 0$. In particular for $b = 0$, it is easy to show that $\mathcal{M}_{\max} \sim (1 - N_r)^{-\nu d_f}$, where ν is the percolation correlation exponent ($\nu = 4/3$ in 2D) and $d_f = 2 - \beta/\nu \approx 1.89$ is the fractal dimension of the percolation clusters, β being the percolation order parameter exponent.

5. Conclusion.

A new model of statistical rupture in heterogeneous media has been introduced which contains a genuine dynamics with two levels of memory effect. The first level comes through the irreversible evolution of the electric network, similarly to previously studied random fuse models. The second level stems from the memory contained in the additional temperature field which is an additive function of all the previous thermal history of each bond in the network. Moreover, the thermal coupling between each fuse and a bath allows the temperature of each fuse to evolve towards a steady value with a characteristic relaxation

time. A major feature of this model is its simplicity which stems from the separation of the time evolution of the electrical and thermal fields : the electric current distribution evolves instantaneously under rupture of a new bond and the thermal field changes continuously under the fixed current distribution until the next rupture occurs. This feature simplifies a lot the analysis and the numerical computations. Therefore, this model is maybe the simplest one can think of which incorporates naturally a dynamics in a statistical model of rupture.

We have been able to show that several interesting scaling behaviors can be computed in this model as a function of the two main control parameters, the total input current I and the heating power exponent b . The crack structures have been shown to be fractals with fractal dimensions changing *continuously* from 1.9 to 1 as b goes from 0 to $+\infty$. The dynamics of the rupture has also been characterized by the scaling law governing the divergence of the electrical dissipated power G^{-1} as time approaches the final rupture time t_f . Here also the exponent α of this divergence is a continuous function of b , evolving from 1.3 to 0 as b goes from 0 to $+\infty$. Another striking feature which we have observed and rationalized is the extreme sensitivity of the rupture pattern when modifying slightly the total input current. Much remains to be done that will reveal many other interesting features of rupture encaptured in this simple model, which we intend to cover in future works.

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Appendix A.

For $b = 0$ (Sect. 3.2), it is possible to derive some analytical time-dependent properties when neglecting the correlation induced by the electrical current field. In other word, we do not distinguish here between the « correlated percolation » model and the standard uncorrelated percolation model. In this case, there exists a threshold value a_c of the thermal coupling constant instead of a threshold value of the input current. Analytical expressions of the global rupture time in the vicinity of a_c can be obtained. At $a = a_c$, the rupture takes an infinite time to occur due to the asymptotic convergence of the temperature given by equation (9). For « a » close but smaller than a_c , the time t_f needed to reach global rupture (i.e. the percolation threshold) is equal to the time taken by the smallest resistance, needed to barely reach the percolation threshold, to attain the temperature breakdown threshold $T_r = 1$. From the definition of a_c , this smaller resistance is precisely equal to a_c with $T_r = 1$. The time t_f for global rupture is thus given by $(a_c/a)[1 - \exp(-at_f)] = 1$, which yields

$$t_f = a_c^{-1} \text{Log} \{ a_c/a (a_c - a) \} . \quad (\text{A1})$$

The logarithmic divergence of the global rupture time t_f as $a \rightarrow a_c$ reflects the exponentially rapid thermal relaxation process due to the second term $-aT$ of the r.h.s. of equation (1). If instead of the expression $-aT$, we were to choose a general powerlaw relaxation of the form $-aT^\eta$, we would again obtain a logarithmic dependence of t_f as a function of $a_c - a$. For instance, the analytically non-trivial soluble case $\eta = 2$ yields $T(t) = (r/a)^{1/2} \{1 - 2[\exp\{2(ar)^{1/2}t\} + 1]^{-1}\}$ instead of (9) and thus $t_f \sim (2a_c)^{-1} \text{Log} \{4 a_c/a (a_c - a)\}$. This possibility has not been persued in our investigations. In the future, it may be of interest to investigate it since it introduces a distribution of relaxation times $(ar)^{-1/2}$, instead of the single relaxation time a^{-1} , which are now function of the value of the resistance r . Note also that a value of η

different from 1 must be considered for describing empirical laws of hardening processes in the mechanical version of the model.

Appendix B.

Existence of a well-defined limit for large P s or small a 's.

The limit of large input currents essentially amounts to the fact that the first term $r_n i_n^b(t)$ of the r.h.s. of equation (1) is much larger than the second term $-aT_n(t)$. In this limit, it is thus possible to neglect the second term $-aT_n(t)$ of the r.h.s. of equation (1). Of course, this limit is also reached trivially by taking $a = 0$ from the beginning. In this case, the rupture process and crack patterns become completely independent from the applied current I , as we now demonstrate.

For this purpose, it is sufficient to show that the delicate order of the rupture sequence, by which one bond after another is broken, is not changed when changing the value of the applied current I . Associated to the ordering of the successive bonds rupture obtained for a given network and for an applied current I , we have introduced in section 2 the ordered breaking times $0 < t_1 < t_2 < \dots < t_i < t_{i+1} < \dots < t_f$. Let us also note $0 < T_n(t_1) < T_n(t_2) < \dots < T_n(t_i) < T_n(t_{i+1}) < \dots < 1$ the successive temperatures of the bond n at the successive breaking times. Let us now call $0 < t_1^\lambda < t_2^\lambda < \dots < t_i^\lambda < t_{i+1}^\lambda < \dots < t_f^\lambda$ the ordered breaking times and $0 < T_n^\lambda(t_1^\lambda) < T_n^\lambda(t_2^\lambda) < \dots < T_n^\lambda(t_i^\lambda) < T_n^\lambda(t_{i+1}^\lambda) < \dots < 1$ the successive temperatures of the bond n at these successive breaking times for the same network but submitted to the applied current λI . Let us now reason in a recurrent way and assume that up to the i -th bond breaking, all the ruptures occurred exactly on the same bonds in the same order with the same values of the temperatures ($T_n^\lambda(t_j^\lambda) = T_n(t_j)$ for $j = 1$ to i and all n 's) for each bond in the two cases I and λI , however with possibly different time scales (but not different time orders). If we show that the $(i + 1)$ -th breaking occurs on the same bond at the same temperatures ($T_n^\lambda(t_{i+1}^\lambda) = T_n(t_{i+1})$) for all bonds n for the two applied currents I and λI , then by recurrence this will be true for all times.

After the blow up of the i -th bond, all temperatures of the intact remaining bonds evolve according to the following laws, derived from equation (3) by posing $a = 0$:

$$T_n(t) = T_n(t_i) + r_n i_n^b(t_i)(t - t_i) \quad \text{for the applied current } I \quad \text{(B1a)}$$

$$T_n^\lambda(t) = T_n^\lambda(t_i^\lambda) + r_n [i_n^\lambda(t_i^\lambda)]^b (t - t_i^\lambda) \quad \text{for the applied current } \lambda I . \quad \text{(B1b)}$$

Let us denote r_{n^*} the conductance of the bond n^* which first reaches the rupture threshold temperature T_r at time t_{i+1} for the applied current I . From equation (B1a), this means that $T_{n^*}(t_i) + r_{n^*} i_{n^*}^b(t_i)(t_{i+1} - t_i)$ reaches 1 the first one. We note first that $i_n(t_i^\lambda) = \lambda i_n(t_i)$ since the two networks fed by I and λI have exactly the same geometry up to the i -th bond breaking event. As a consequence, the current in each bond of the network is simply proportional to the total applied current, therefore the factor λ . Furthermore, we have assumed that $T_n^\lambda(t_i^\lambda) = T_n(t_i)$. As a consequence, equation (B1b) transforms into

$$T_n^\lambda(t) = T_n(t_i) + \lambda^b r_n i_n^b(t_i)(t - t_i^\lambda) \quad \text{for the applied current } \lambda I . \quad \text{(B2)}$$

Rupture occurs at time t_{i+1}^λ which is the minimum of $\lambda^{-b}[1 - T_n(t_i)]/r_n i_n^b(t_i)$ over all bonds in the network. This rupture occurs on the bond which realizes this minimum : this is precisely the bond n^* which minimizes the expression $[1 - T_n(t_i)]/r_n i_n^b(t_i)$ for the applied current I (note that the factor λ^{-b} does not modify the realization of the minimum). This proves that the $(i + 1)$ -th rupture occurs on the same bond n^* in the two cases I and λI . We have

$t_{i+1} - t_i = [1 - T_n^*(t_i)]/r_n^* i_n^{b_n^*}(t_i)$ and $t_{i+1}^\lambda - t_i^\lambda = \lambda^{-b}(t_{i+1} - t_i)$. From equations (B1a) and (B2), this shows that $T_n^\lambda(t_{i+1}^\lambda) = T_n(t_{i+1})$ for all bonds n for the two applied currents I and λI . We have thus proved that the assumed recurrence is true for all times and that the limit of large I 's (or $a = 0$) is well-defined. The rupture process is in fact completely defined by the set $\{0 < T_n^1 < T_n^2 < \dots < T_n^i < T_n^{i+1} < \dots < 1, \text{ for all bonds } n\}$ of the successive temperatures of all bonds at the successive breaking events, where we have noted $T_n(t_i) = T_n^i$ since it is independent from the value of i -th breaking time.

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