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Supersymmetry in spin glass dynamics

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Abstract. — The spin glass dynamic equations can be written in a way that makes the supersymmetry associated with such a Langevin process explicit. In such a framework the fluctuation-dissipation relations and time homogeneity properties are implicit in the symmetries of the action functional. The spin glass phase transition can be discussed in terms of supersymmetry breaking. The superspace notation appears naturally. In this notation the dynamics is expressed in terms of a single superspace function Q , and the dynamic problem bears a striking formal similarity with its static replica-treatment counterpart. In particular, this similarity can be used to show explicitly that the supersymmetric dynamical solution and the replica symmetric solutions yield the same static results for a wide range of models.

1. Introduction.

The study of long range spin glasses has been tackled analytically using several methods: the replica trick, the TAP equations, the cavity method and the dynamic approach [1]. Even though each of these methods gives a particular insight into some aspect of the spin glass physics, the replica trick has been by far the most successful in uncovering the novel features of this problem, as well as in practical calculations.

In the dynamic approach one considers a Langevin process (gradient descent + noise) associated with the spin glass Hamiltonian. The physical information is retrieved from the correlation functions of magnitudes (e.g. spins) at different times. The probability distribution of the system, averaged over the noise, can be shown to converge at long times (at least for finite number of spins N) to the canonical distribution.

In practice one constructs a Martin-Siggia-Rose [2] functional integral over the trajectories with different noise realizations. It was observed by De Dominicis [3] that since the sourceless path integral is independent of the Hamiltonian, one can calculate directly the averages over the couplings for the correlation functions, without introducing replicas. This program was firstly implemented in this context by Sompolinsky [4].

A Langevin dynamic system such as this one leads to an action that has associated with it a Parisi-Sourlas supersymmetry (SUSY) which, as these authors point out, encompasses the information that the system came from a stochastic equation. It also contains the fluctuation-dissipation theorems (FDT) (valid at all temperatures for finite number of spins N) which appear as Ward identities, and it implies the time homogeneity of equilibrium correlation functions [5, 6].

In the dynamic treatments of spin glasses so far, the SUSY has been rather hidden. The purpose of this paper is to cast the spin glass dynamic problem in a manifestly SUSY way, and to make the first few steps in its exploration from this point of view. The main encouraging feature is that in so doing the dynamic approach comes formally closer to the static replica treatment, where much more is known. In order to emphasize the generality of the arguments, we treat simultaneously several long range models.

As is well known, the low temperature spin glass phase is characterized by the appearance of many metastable states. Dynamically, because of the ruggedness of the energy landscape and the diverging height of the barriers (at least for fully connected systems), the ergodicity is broken.

In the static replica treatment this manifests itself in the breaking of the replica symmetry by the saddle point solutions for the thermodynamic limit ($N \rightarrow \infty$) in the low temperature phase. In the dynamic approach, Sompolinsky and Zippelius [4] have considered for this phase a FDT-violating saddle point solution; *which in the SUSY language reads as a SUSY-breaking solution.*

In order to clarify the relationship between SUSY-breaking and the spin glass phase transition it is illuminating to have in mind the simple example of a Langevin dynamics of a particle moving on a one dimensional symmetric double well, for which the SUSY can also be made manifest [6]. In the "semiclassical" limit of low noise ergodicity is broken (the barrier is not surmountable). One finds the time-independent solutions corresponding to the particle sitting at the stationary values of the energy. To obtain the correct equilibrium distribution, independently of the initial conditions, one has to allow for barrier penetration effects. These are represented by instantons, which can be described as SUSY-breaking saddle point solutions (time-translation, which instantons obviously break, is a part of the SUSY group).

Bearing this in mind, one can expect that in the low temperature spin glass phase taking into consideration SUSY-breaking (FDT-violating) saddle point solutions one can get the correct equilibrium results, independently of the initial conditions.

An alternative strategy is to stay at the level of SUSY solutions, but then the initial conditions must be chosen with the correct statistical weight. This is just a way of restating what has been done in [8], where the price is paid of the reintroduction of replicas.

The breaking of SUSY by a solution is, in any case, an artifact of the saddle point evaluation, physical values have to be summed over all the different saddle points, and hence the SUSY is restored. Again this is analogous to the breaking (and further restoration) of time-invariance by instantons.

It is interesting to note here that the spin glass phase transition has already been displayed as a SUSY breaking transition in the different context of the mean field equations [7].

In section 2 we make a Hilbert space construction of the Fokker-Planck problem associated with the dynamics. This allows us to discuss boundary conditions in the resulting path integral, we consider a choice that is suitable for the calculation of equilibrium quantities. It also seems a good starting point for generalizations to Glauber processes. This section is rather independent from the rest of the paper, it can be skipped without much loss of understanding of the rest.

In writing the dynamic equations in a manifestly SUSY way, one finds that a great simplification is obtained by writing the time dependent order parameters as functions of *superspace*

(Sect. 3), consisting of the two times t_1 and t_2 and their associated *Grassmann* coordinates $\theta_1, \bar{\theta}_1$ and $\theta_2, \bar{\theta}_2$ respectively.

In this notation, the object which appears naturally is the *single* superspace-dependent field $Q(1, 2)$ (where “1” and “2” stand for the two sets of superspace coordinates). It encodes all the relevant correlation and response functions and plays much the same role as the matrix $Q_{\alpha\beta}$ does in the replica treatment. In section 4 we construct the averaged dynamics for a range of models. This leads to (non-local) field equations for $Q(1, 2)$.

The action functional and the saddle point equations are seen to have a striking similarity with corresponding static replica expressions. In order to show that this is a very general fact we show that diagrams involved (if any) in the computation of the action functional when written in superspace (“superdiagrams”) have the same form as the corresponding ones in the replica treatment (this may also be of use in the perturbative corrections, but we do not discuss that here).

In section 5 we discuss the symmetries of the action functional. In particular one finds the SUSY appears as a symmetry in the action with respect to transformations in superspace. Above T_c the solution in [4] does not break any of these symmetries.

Quite generally the dynamic action is the sum of a “potential” term, which is formally the same as the replica expression for the averaged free energy; plus a purely “kinetic” term, which depends on the particular dynamics one has given to the spin glass problem (and contains the causal information). Such simplicity, plus the requirement of SUSY and causality, can possibly help as a guideline to the construction of simplified dynamical models (e.g. truncated models near T_c).

Finally, in section 6 we take advantage of the form of the dynamic equations in superspace to show that they can be solved in a form that makes close contact with their static replica counterpart. This is done however only above T_c , the more interesting case of low temperature phase is being studied currently.

2. Hilbert space.

In this section we make a Hilbert space construction of the Fokker-Planck dynamics. The purpose is twofold. Firstly we wish to show clearly what it is that we are calculating in the functional approach. In particular, the boundary conditions in the path integral are important since they can explicitly break the SUSY. The boundary conditions proposed in [5], which are periodic in both the boson and the fermion variables, are seen to have an interesting algebraic meaning. For example the sourceless path integral corresponds to the index, a topological invariant studied extensively in [9] which gives useful information: SUSY is shown to be unbroken for N finite. If as a result of the ($N \rightarrow \infty$) saddle point evaluation it is broken it must be restored as soon as one goes beyond that approximation. These boundary conditions will be shown to yield the equilibrium correlations.

Secondly, this Hilbert space construction lends itself easily to generalizations to processes such as Glauber dynamics; for which SUSY can also be introduced.

2.1 SUPERSYMMETRIC HAMILTONIAN. — Consider a family of soft p -spin models with energies:

$$E = - \sum_{1 \leq i_1 \leq i_2 \dots < i_p \leq N} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} - h \sum_i s_i + h(s) \tag{1}$$

where the $J_{i_1 \dots i_p}$ are independent random variables with zero mean and variance $J^2 p! / 2N^{p-1}$. The factor $h(s)$ is a spin length probability that dominates over the interaction term for large

s_i , for example (for $p = 2$) [4] :

$$h(\mathbf{s}) = \sum_i \left[\frac{1}{2} r_0 s_i^2 + u s_i^4 \right] \quad (2)$$

We shall also consider a spherical constraint:

$$h_{sc}(\mathbf{s}) = \lim_{a \rightarrow \infty} a \left(\sum_i s_i^2 - N \right)^2 \quad (3)$$

The Langevin dynamics for this system at inverse temperature β is given by

$$\Gamma_0^{-1} \frac{ds_i}{dt} = -\frac{\partial}{\partial s_i} (\beta E) + \xi_i(t) \quad (4)$$

where $\xi_i(t)$ are Gaussian random variables with zero mean and variance

$$\langle \xi_i(t) \xi_j(t') \rangle = \frac{2}{\Gamma_0} \delta_{ij} \delta(t - t') \quad (5)$$

Starting from an initial condition the system evolves into some point in phase space which depends on the particular noise realization. If we consider the dynamics averaged over the noise realizations we obtain a probability $P(\mathbf{s}, t)$ that the system is in a point \mathbf{s} of phase space. The time evolution of $P(\mathbf{s}, t)$ is described by the Fokker-Planck equation associated with (4) which reads (see [6], chapt. 3):

$$\begin{aligned} \frac{\partial P}{\partial t} &= \Gamma_0 \sum_i \frac{\partial}{\partial s_i} \left[\frac{\partial}{\partial s_i} + \beta \frac{\partial E}{\partial s_i} \right] P = -\Gamma_0 \sum_i p_i (p_i - i\beta \frac{\partial E}{\partial s_i}) P \\ &= -H_{FP} P \end{aligned} \quad (6)$$

where

$$p_i = -i \frac{\partial}{\partial s_i} \quad (7)$$

The Fokker-Planck operator H_{FP} is non-hermitian:

$$H_{FP} = \sum_i A_{-i} A_{+i} \quad (a)$$

$$A_{-i} = p_i ; A_{+i} = p_i - i\beta \frac{\partial E}{\partial s_i} \quad (b) \quad (8)$$

but can be taken to an Hermitian form by the non-unitary transformation \hat{T} :

$$A'_{\pm i} = \hat{T}^{-1} A_{\pm i} \hat{T} = p_i \mp \frac{i\beta}{2} \frac{\partial E}{\partial s_i} \quad (9)$$

$$H'_{FP} = \hat{T}^{-1} H_{FP} \hat{T} = \sum_i A'_{-i} A'_{+i} = H_{FP}^\dagger \quad (10)$$

where:

$$\hat{T} = \exp \left[-\frac{1}{2} \beta E(\mathbf{s}) \right] \quad (11)$$

From the form of (10) H'_{FP} (and hence H_{FP}) have eigenvalues ≥ 0 . The right eigenvector of H_{FP} associated with the eigenvalues zero is proportional to the canonical weight

$$P_{eq} \sim e^{-\beta E(s)} \tag{12}$$

while the left eigenvector is proportional to a constant, thus insuring probability conservation.

At this point we introduce $2N$ fermion operators a_i^\dagger, a_j :

$$[a_i^\dagger, a_j]_+ = \delta_{ij} \tag{13}$$

and the SUSY operators

$$Q_+ = \sum_i A_{+i} a_i^\dagger, \quad Q_- = \sum_i A_{-i} a_i \tag{14}$$

their combinations:

$$Q_1 = Q_+ + Q_- \quad ; \quad Q_2 = \frac{1}{i} (Q_+ - Q_-) \tag{15}$$

and the fermion number operator:

$$N_F = \sum_i a_i^\dagger a_i \tag{16}$$

From (8b) they are seen to verify:

$$\begin{aligned} Q_\pm^2 &= 0 \\ [Q_\pm, N_F]_- &= \mp Q_\pm \end{aligned} \tag{17}$$

Consider now the Hamiltonian:

$$\begin{aligned} H &= \Gamma_0 [Q_+, Q_-]_+ && \text{(a)} \\ &= \Gamma_0 Q_1^2 = \Gamma_0 Q_2^2 && \text{(b)} \\ &= \frac{\Gamma_0}{2} \left\{ \sum_i [A_{-i}, A_{+i}]_+ + \sum_{i,j} [A_{-i}, A_{+j}]_- [a_i, a_j^\dagger]_+ \right\} \\ &= \Gamma_0 \left\{ \sum_i p_i \left(p_i - i \frac{\partial E}{\partial s_i} \right) + a_i^\dagger a_j \frac{\partial^2 E}{\partial s_i \partial s_j} \right\} && \text{(c)} \end{aligned} \tag{18}$$

It commutes with the SUSY charges and the fermion number:

$$[H, Q_\pm] = 0 \tag{a} \tag{19}$$

$$[H, N_F] = 0 \tag{b} \tag{20}$$

Transforming Q_\pm through (11) they become hermitian conjugates; and Q_1, Q_2, H transform into hermitian operators:

$$\begin{aligned} Q'_\pm &= \hat{T}^{-1} Q_\pm \hat{T} = Q'_\mp \\ Q'_1 &= \hat{T}^{-1} Q_1 \hat{T} = Q_1^\dagger \quad ; \quad Q'_2 = \hat{T}^{-1} Q_2 \hat{T} = Q_2^\dagger \\ H' &= \hat{T}^{-1} H \hat{T} = H^\dagger \end{aligned} \tag{21}$$

and

$$H' = \Gamma_0 Q_1'^2 = \Gamma_0 Q_2'^2 \tag{22}$$

These hermiticity properties are important, though the primed representation is not convenient for actual calculations because H' is quadratic in the couplings $J_{i_1 \dots i_p}$. In the next sections we shall only use unprimed quantities.

Again, since H' can be seen to be the square of an hermitian operator Q_1' or Q_2' its spectrum (and hence that of H) consists of eigenvalues that are greater or equal than zero.

Denoting $|\tilde{\varphi}_i\rangle$ the (orthonormal) eigenvectors of H' we have:

$$H'|\tilde{\varphi}_i\rangle = \hat{T}^{-1}H\hat{T}|\tilde{\varphi}_i\rangle = \varepsilon_i|\tilde{\varphi}_i\rangle \tag{23}$$

so that the right and left eigenvectors of H are:

$$|\varphi_{Ri}\rangle = \hat{T}|\tilde{\varphi}_i\rangle; \quad \langle\varphi_{Li}| = \langle\tilde{\varphi}_i|\hat{T}^{-1} \tag{24}$$

Each pair $|\varphi_{Ri}\rangle, |\varphi_{Li}\rangle$ has well defined fermion number.

Looking at (18) we note that within the zero ghost subspace $H = H_{FP}$. Hence the states:

$$\begin{aligned} |\varphi_{R0}\rangle &= \frac{e^{-\beta E}}{[\int e^{-\beta E} \pi_i ds_i]^{1/2}} \otimes |0 - \text{ghost}\rangle \\ \langle\varphi_{L0}| &= \frac{1}{[\int e^{-\beta E} \pi_i ds_i]^{1/2}} \otimes \langle 0 - \text{ghost}| \end{aligned} \tag{25}$$

are stationary right and left eigenvectors with zero eigenvalue. We show below that they are the only ones.

2.2 INVARIANTS. — To show the uniqueness of the equilibrium distribution, and hence that the fermions have not added anything spurious, we shall briefly review the discussion in [9] as applied to our case. We refer the reader to that reference for the derivations.

Let $n_B^{E=0}$ and $n_F^{E=0}$ be the numbers of zero energy states having even and odd fermion numbers respectively. Consider first the total number ($n_B^{E=0} + n_F^{E=0}$). It is shown in [9] that this quantity is invariant with respect to a group of (“conjugation”) transformations in the Hamiltonian. One can easily construct a subset of such transformations that effects an arbitrary change in the couplings $J_{i_1 \dots i_p}$ while leaving the spin length probability (which dominates for large s_i) invariant.

Secondly, and more important for us, the quantity $n_B^{E=0} - n_F^{E=0}$ (the “index” of H) is also left invariant by such changes.

Hence, both $n_B^{E=0}$ and $n_F^{E=0}$ are the same for any realization of the $J_{i_1 \dots i_p}$; in particular for $J_{i_1 \dots i_p} = 0$. But it is easy to see that in the uncoupled case $n_B = 1, n_F = 0$ if the spin length probability is reasonably chosen.

This proves that for any (finite) number of spins the only equilibrium (zero eigenvalue) state is indeed the canonical distribution which is annihilated by the charges Q_{\pm} . Hence for N finite the SUSY is unbroken.

2.3 CORRELATION FUNCTIONS. — As is well known [4], the dynamical approach can be implemented without replicas because the path integral without sources does not depend on the realization of the couplings.

We construct now the correlation functions in a manner that is related to the topological index and that yields the fluctuation-dissipation theorems (FDT) in a particularly convenient

way. They are based on a trace, and are useful for the equilibrium properties, but do not refer to a particular non-equilibrium initial condition.

Consider first the quantity

$$\text{tr}[e^{+i\pi N_F} e^{-TH}] \quad (26)$$

where T is *not* the physical inverse temperature β but a mathematical artifact that on going to infinity selects the equilibrium state. It is convenient to pass to the basis in which the Hamiltonian is hermitian:

$$\begin{aligned} \text{tr}[e^{+i\pi N_F} e^{-TH}] &= \text{tr}[\hat{T}^{-1} e^{+i\pi N_F} e^{-TH} \hat{T}] = \\ &= \text{tr}[e^{+i\pi N_F} e^{-TH'}] = \sum_i e^{-T\epsilon_i} \langle \tilde{\varphi}_i | e^{i\pi N_F} | \tilde{\varphi}_i \rangle \end{aligned} \quad (27)$$

Now, the positive energy eigenvectors of a supersymmetric Hamiltonian come in pairs of even and odd fermion numbers [9]. Hence, only zero energy states contribute to (27) and we get:

$$\text{tr}[e^{+i\pi N_F} e^{-TH}] = n_B^{E=0} - n_F^{E=0} = 1 \quad (28)$$

i.e. the index *for all* T . Note that where it not for the factor $e^{i\pi N_F}$ this would only hold for $T \rightarrow \infty$.

This suggests that we define (note the first factor in the trace):

$$\langle O_a(t+\tau) O_b(t) \rangle_T = \text{tr}[e^{i\pi N_F} e^{-TH} O_a e^{-\tau H} O_b e^{\tau H}] \quad (29)$$

Again, making the similarity transformation as in (27) we get:

$$\begin{aligned} \langle O_a(t+\tau) O_b(t) \rangle_T &= \sum_{i,j} (-1)^{N_{F_i}} e^{-T\epsilon_i - \tau(\epsilon_j - \epsilon_i)} \\ &\langle \tilde{\varphi}_i | T^{-1} O_b \hat{T} | \tilde{\varphi}_j \rangle \langle \tilde{\varphi}_j | \hat{T}^{-1} O_a \hat{T} | \varphi_i \rangle \\ &= \sum_{i,j} (-1)^{N_{F_i}} e^{-T\epsilon_i - \tau(\epsilon_j - \epsilon_i)} \langle \varphi_{L_i} | O_a | \varphi_{R_j} \rangle \langle \varphi_{L_j} | O_b | \varphi_{R_i} \rangle \end{aligned} \quad (30)$$

Now, if $T \rightarrow \infty$ (τ finite);

$$\begin{aligned} \langle O_a(t+\tau) O_b(t) \rangle &= \lim_{T \rightarrow \infty} \langle O_a(t+\tau) O_b(t) \rangle_T \\ &= \sum_j e^{-\tau\epsilon_j} \langle \varphi_{L_0} | O_a | \varphi_{R_j} \rangle \cdot \langle \varphi_{L_j} | O_b | \varphi_{R_0} \rangle \end{aligned} \quad (31)$$

and if further $\tau \rightarrow \infty$ (but still $T - |\tau| \rightarrow \infty$)

$$\lim_{\tau \rightarrow \infty} \langle O_a(t+\tau) O_b(t) \rangle = \langle \varphi_{L_0} | O_a | \varphi_{R_0} \rangle \langle \varphi_{L_0} | O_b | \varphi_{R_0} \rangle \quad (32)$$

which is the product of the canonical expectations: the static quantity of interest.

2.4 FLUCTUATION DISSIPATION RELATIONS. — Thanks to the introduction of the factor $e^{i\pi N_F}$ in the trace we have fluctuation dissipation relations valid for all times T . Let us derive one of them:

$$\frac{\partial}{\partial \tau} \langle O_a(t+\tau) O_b(t) \rangle_T = \text{tr} \left[e^{i\pi N_F} e^{-TH} O_a e^{-\tau H} [O_b H]_- e^{+\tau H} \right] \quad (33)$$

Using (18a), the Jacobi identity and the fact that:

$$e^{i\pi N_F} Q_{\pm} = -Q_{\pm} e^{i\pi N_F} \quad (34)$$

we get after a short computation:

$$\begin{aligned} \frac{\partial}{\partial \tau} \langle O_a(t+\tau) O_b(t) \rangle_T &= \Gamma_0 \text{tr} \left[e^{i\pi N_F} e^{-T\mathcal{H}} O_a e^{-\tau\mathcal{H}} \left[[O_b, Q_+]_-, Q_- \right]_+ e^{+\tau\mathcal{H}} \right] \\ &\quad - \Gamma_0 \text{tr} \left[e^{i\pi N_F} e^{-T\mathcal{H}} \left[[O_a, Q_+]_-, Q_- \right]_+ e^{-\tau\mathcal{H}} O_b e^{\tau\mathcal{H}} \right] \end{aligned} \quad (35)$$

For example, putting $O_a = O_b = s_k$ in (35) we get:

$$\begin{aligned} \frac{\partial}{\partial \tau} \langle s_k(t+\tau) s_k(t) \rangle_T &= \\ &= i\Gamma_0 \langle s_k(t+\tau) p_k(t) \rangle_T - i\Gamma_0 \langle p_k(t+\tau) s_k(t) \rangle_T \end{aligned} \quad (36)$$

2.5 CAUSALITY. — The following “causality” relations use explicitly the structure of the ground state and are valid only for $T - |\tau| \rightarrow \infty$ (unlike the FDT). They are easily read from (31):

$$\langle p_i(t+\tau) O(t) \rangle = \langle a_i^\dagger(t+\tau) O(t) \rangle = 0 \quad (37)$$

for any operator O , since p_i and a_i^\dagger annihilate the left eigenvector $\langle \varphi_{L0} |$.

Finally, we shall use the relations:

$$\begin{aligned} \lim_{\tau \rightarrow 0_+} \langle s_i(t+\tau) p_i(t) \rangle &= i \\ \lim_{\tau \rightarrow 0_+} \langle a_i(t+\tau) a_i^\dagger(t) \rangle &= 1 \end{aligned} \quad (38)$$

which are easily derived from the commutation relations.

2.6 SUPERSYMMETRY BREAKING. — The previous arguments about unicity of the zero energy ground state hold for *finite number N of spins*. In particular they show that SUSY is unbroken.

However, in the saddle point approximation ($N \rightarrow \infty$) the SUSY can be broken, and hence the FDT violated, by some saddle point solutions. The SUSY is recovered for large but finite N by summing over the inequivalent SUSY breaking saddle point solutions.

This concludes the discussion in the hilbert space. In the following sections we work only with functional methods.

3. Superspace.

Starting from the trace formula (29) we can now construct the path integral expressions for the correlation functions. This can be done in any standard way; the only special feature is that the presence of the factor $e^{i\pi N_F}$ leads to ([6] chapt. 2) paths that are periodic both for fermions and bosons, as opposed to the usual antiperiodic conditions in fermions.

We obtain the Martin-Siggia-Rose [2] path integral with its jacobian exponentiated through ghosts:

$$Z = \int \exp(-S) D[s_i] D[p_i] D[\eta_i] D[\bar{\eta}_i]$$

$$S = \int_0^T \left\{ \Gamma_0^{-1} \sum_i (\dot{\eta}_i \bar{\eta}_i + p_i s_i - p_i^2) + \sum_i \frac{\partial E}{\partial s_i} p_i + \sum_{i,j} \frac{\partial^2 E}{\partial s_i \partial s_j} \eta_i \bar{\eta}_j \right\} \quad (39)$$

where we have rescaled the ghosts and momenta for simplicity of notation.

If at this point one integrates away the fermionic variables $\eta_i, \bar{\eta}_i$ one obtains [5] the expression used in [4] where the fermionic term yields an extra term in the action S. We do not do so here but keep these variables.

The expression for S can be written in a compact form in superspace ([6] chapt. 15), we introduce two anticommuting Grassmann variables $\theta, \bar{\theta}$:

$$[\theta \bar{\theta}]_+ = \theta^2 = \bar{\theta}^2 = 0 \quad (40)$$

The integrals over these variables are defined as:

$$\begin{aligned} \int 1 d\theta &= \int 1 d\bar{\theta} = 0 \\ \int \theta d\theta &= \int \bar{\theta} d\bar{\theta} = 1 \end{aligned} \quad (41)$$

We also introduce the superfields:

$$\phi_i = s_i + \bar{\theta} \eta_i + \bar{\eta}_i \theta + p_i \bar{\theta} \theta \quad (42)$$

In terms of these fields it is easy to check using (40), (41) and (42) that:

$$\begin{aligned} Z &= \int \Pi_i D[\phi_i] \exp[-S_{\text{KIN}} - S_{\text{POT}}] \\ S_{\text{KIN}} &= \Gamma_0^{-1} \int_0^T d\theta d\bar{\theta} dt \sum_i \frac{\partial \phi_i}{\partial \theta} \left(\frac{\partial \phi_i}{\partial \bar{\theta}} - \theta \frac{\partial \phi_i}{\partial t} \right) \\ S_{\text{POT}} &= \int_0^T d\theta d\bar{\theta} dt E(\phi) \end{aligned} \quad (43)$$

The third equality is readily obtained by expanding $E(\phi)$ around s_i , a Taylor series which terminates at the second derivatives because of (40).

At this point it is useful to introduce the two sets of SUSY operators:

$$\begin{aligned} \bar{D} &= \frac{\partial}{\partial \theta} \\ D &= \frac{\partial}{\partial \bar{\theta}} - \theta \frac{\partial}{\partial t} \\ [D, \bar{D}]_+ &= -\frac{\partial}{\partial t} \end{aligned} \quad (44)$$

and:

$$\begin{aligned} \bar{D}' &= \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial t} \\ D' &= \frac{\partial}{\partial \bar{\theta}} \\ [D', \bar{D}']_+ &= \frac{\partial}{\partial t} \end{aligned} \quad (45)$$

The primed operators commute with the unprimed, and all D's are nilpotent:

$$D^2 = \bar{D}^2 = D'^2 = \bar{D}'^2 = 0 \quad (46)$$

An operator which we will make frequent use of is:

$$D^{(2)} = [\bar{D}, D]_- = 2 \frac{\partial^2}{\partial \theta \partial \bar{\theta}} + 2\theta \frac{\partial^2}{\partial \theta \partial t} - \frac{\partial}{\partial t} \quad (47)$$

$$[D^{(2)}]^2 = \frac{\partial^2}{\partial t^2} \quad (48)$$

We also write the grassmann delta function:

$$\delta^2(\theta_1 - \theta_2) = (\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2) \quad (49)$$

and the superspace delta function:

$$\delta(1 - 2) = \delta(t_1 - t_2) \delta^2(\theta_1 - \theta_2) \quad (50)$$

Here and in what follows we will denote:

$$1 \equiv (t_1, \bar{\theta}_1, \theta_1), \quad 2 \equiv (t_2, \bar{\theta}_2, \theta_2) \quad \text{etc.} \quad (51)$$

Integrating by parts the kinetic term becomes:

$$S_{\text{KIN}} = -\frac{\Gamma_0^{-1}}{2} \int d\theta d\bar{\theta} dt \sum_i (\phi_i D^{(2)} \phi_i) \quad (52)$$

In the superspace notation all correlation and response functions are encoded in a single superspace function:

$$\langle Q(1, 2) \rangle = \frac{1}{N} \sum_i \langle \phi_i(1) \phi_i(2) \rangle \quad (53)$$

this contains 16 components which can be read from (42) (the first magnitude in a bracket in (54) is evaluated at t_1 and the second at t_2):

$$\begin{aligned} \langle Q(1, 2) \rangle = \frac{1}{N} \sum_i \left\{ \right. & \langle s_i s_i \rangle + \bar{\theta}_2 \theta_2 \langle s_i p_i \rangle + \bar{\theta}_1 \theta_1 \langle p_i s_i \rangle + \\ & -\bar{\theta}_1 \theta_2 \langle \eta_i \bar{\eta}_i \rangle + \bar{\theta}_1 \bar{\theta}_2 \langle \bar{\eta}_i \eta_i \rangle + \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2 \langle p_i p_i \rangle + \\ & -\theta_1 \theta_2 \langle \bar{\eta}_i \bar{\eta}_i \rangle - \bar{\theta}_1 \bar{\theta}_2 \langle \eta_i \eta_i \rangle + \bar{\theta}_2 \langle s_i \eta_i \rangle + \bar{\theta}_1 \langle \eta_i s_i \rangle + \\ & -\theta_1 \langle \bar{\eta}_i s_i \rangle - \theta_2 \langle s_i \bar{\eta}_i \rangle - \bar{\theta}_1 \theta_1 \theta_2 \langle p_i \eta_i \rangle + \bar{\theta}_1 \theta_1 \bar{\theta}_2 \langle p_i \bar{\eta}_i \rangle + \\ & \left. + \theta_1 \bar{\theta}_2 \theta_2 \langle \bar{\eta}_i p_i \rangle + \bar{\theta}_1 \bar{\theta}_2 \theta_2 \langle \eta_i p_i \rangle \right\} \quad (54) \end{aligned}$$

Most of these correlations, as we shall see, vanish.

4. Action functional for the dynamics averaged over the couplings.

4.1 INTERACTION TERMS. — We turn now to the calculation of the path integral averaged over the couplings. Let us first compute the average of the interaction term:

$$\exp \left[- \int d1 \sum_{1 \leq i_1 \leq \dots \leq i_p \leq N} J_{i_1 \dots i_p} \phi_{i_1}(1) \cdot \phi_{i_p}(1) \right] \tag{55}$$

where $d1 \equiv (d\theta_1 d\bar{\theta}_1 dt_1)$. In an exactly analogous manner as in the static replica case (see [10]) once the Gaussian integral over the couplings is performed, the average can be expressed in terms of the magnitude:

$$Q(1,2) = \frac{1}{N} \sum_i \phi_i(1)\phi_i(2) \tag{56}$$

This suggests we multiply the functional integral by:

$$\int D[Q] \delta(NQ(1,2) - \sum_i \phi_i(1)\phi_i(2)) = \int D[Q] D[\lambda] \exp \left[-\frac{1}{2} \int d1d2 (NQ(1,2)\lambda(1,2) - \lambda(1,2) \sum_i \phi_i(1)\phi_i(2)) \right] \tag{57}$$

The functional integration is over $Q(1,2)$, $\lambda(1,2)$ such that $Q(1,2) = Q(2,1)$ and $\lambda(1,2) = \lambda(2,1)$; both 16-component superspace functions, periodic in the two times t_1, t_2 .

Following the same steps as in the static replica calculation we have for the action functional, to leading order in N :

$$\begin{aligned} S = S_{\text{KIN}} - N \frac{(\beta J)^2}{4} \int d1d2 Q^p(1,2) + \frac{N}{2} \int d1d2 \lambda(1,2) Q(1,2) \\ - \frac{1}{2} \int d1d2 [\lambda(1,2) \sum_i \phi_i(1)\phi_i(2)] \\ - \beta h \int d1 (\sum_i \phi_i(1)) - \int d1 h(\phi) \end{aligned} \tag{58}$$

4.2 SUPERDIAGRAMS. — In the computation of the preceding subsection we have begun to encounter a formal similarity between the static replica calculation and the dynamics when written in superspace. This similarity becomes more complete by carrying further the functional integration over the superfields ϕ_i . For this we need to consider (but only very superficially) Feynman diagrams in superspace (“superdiagrams”).

We will consider two cases: the case in which the spin probability weight is of the form:

$$h(s) = \frac{1}{2} r_0 \sum_i s_i^2 + \sum_i f(s_i) \tag{59}$$

where $f(s_i)$ is of higher order in the s_i (for example quartic in [4]), and the case in which in addition the spins are spherically constrained:

$$\sum_i s_i^2 = N \tag{60}$$

For simplicity we shall assume zero magnetic field in both cases, and indicate at the end how the derivation has to be modified for finite h .

Consider first a spin weight (59). The superfields uncouple and we get:

$$\frac{S}{N} = - \int d1d2 Q^p(1,2) + \frac{1}{2} \int d1d2 \lambda(1,2) Q(1,2) + W(\lambda) \quad (61)$$

where:

$$\exp[W(\lambda)] = \int D[\phi] \exp[-\frac{1}{2} \int d1d2 \phi(1) K(1,2) \phi(2) + \int d1 f(\phi)] \quad (62)$$

and:

$$K(1,2) = [\Gamma_0^{-1} D^{(2)}(2) + r_0] \delta(1-2) + \lambda(1,2) \quad (63)$$

where $D^{(2)}(2)$ is the operator (47) written in terms of the variables $t_2, \bar{\theta}_2, \theta_2$. $W(\lambda)$ can in principle be expressed as a sum over connected (super) diagrams with propagator $[K]^{-1}$

If we now demand that S be stationary with respect to $\lambda(1,2)$ we get:

$$Q(1,2) = -\frac{\delta W}{\delta \lambda(1,2)} = \langle \phi(1)\phi(2) \rangle \quad (64)$$

where the bracket " $\langle \quad \rangle$ " denotes average taken with (62). We observe that $Q(1,2)$ is the "dressed" propagator; reexpressing $W(\lambda)$ in terms of Q amounts to performing mass renormalization. It is well known ([11], see the presentation in [12]) that W expressed in terms of Q is given by (the extension of the proof for superdiagrams is trivial):

$$W(Q) = -\frac{1}{2} \text{tr}[\ln \hat{Q}] - \frac{1}{2} \int d1d2 K(1,2) Q(1,2) + \Phi(Q) \quad (65)$$

where $\Phi(Q)$ is the sum of two-line irreducible diagrams of (62) calculated replacing the bare propagator \hat{K}^{-1} by the dressed propagator Q .

We have also introduced the supertrace:

$$\text{Tr}[A(1,2)] = \int d1 A(1,1) \quad (66)$$

and $[\ln \hat{Q}]$ stands for the logarithm of Q considered as an operator i.e.:

$$\int Q(1,2) \psi(2) d2 = [\hat{Q}\psi](1) \quad (67)$$

Replacing this expression in the action we finally get:

$$\begin{aligned} \frac{S(Q)}{N} = & -\frac{(BJ)^2}{4} \int d1 d2 Q^p(1,2) - \frac{1}{2} \text{tr}[\ln \hat{Q}] \\ & - \frac{1}{2} \int d1d2 \{[(\Gamma_0^{-1} D^{(2)}(2) + r_0)\delta(1-2)]Q(1,2)\} + \Phi(Q) \end{aligned} \quad (68)$$

Consider this expression. The important point is that if we had been dealing with a static replica calculation with the order parameter $Q_{\alpha\beta}$ we would have gone (at least in principle) through exactly *the same steps*. The superdiagrams with dressed propagator $Q(1,2)$ would have been replaced by ordinary diagrams of *the same form* with dressed "propagator" $Q_{\alpha\beta}$, the

supertraces would have been substituted by ordinary traces, etc. Hence except for the kinetic term in (68), proportional to Γ_0^{-1} , we would have obtained an expression that is formally identical. We shall return to this.

Before finishing this subsection let us see how expression (68) is modified by a spherical constraint. The addition of a constraint term (3) in the path integral can be seen ([6] Sect. 16.7) to be equivalent to having a factor: $\delta(\sum_i \phi_i^2 - N)$ in the path integral. Notice that this encodes *four* relations (*two* relations if the fermions are integrated away). This leads to be constraint:

$$Q(1, 1) = 1 \tag{69}$$

again involving four relations. Exponentiating (69) with a Lagrange multiplier *superfield* $Z(1)$ yields an extra term in the action

$$\frac{1}{2} \int d1 Z(1)Q(1, 1) - \frac{1}{2} \int d1 Z(1) \tag{70}$$

The Lagrange multiplier has four components:

$$Z(1) = Z_0(t_1) + \bar{Z}_1(t_1)\theta + \bar{\theta}Z_1(t_1) + Z_2(t_1)\bar{\theta}\theta \tag{71}$$

Let us now briefly mention how this has to be modified in the presence of a magnetic field (or any term linear in the spins in $h(s)$). In such a case [11, 12]. One first has to make the Legendre transform to reexpress the functional in terms of the averages $\langle \phi_i \rangle$, and only then perform mass renormalization on the one-line irreducible diagrams that result. Again, the important thing here is that such a procedure is paralleled by a similar procedure in the replica treatment.

4.3 CORRELATION FUNCTIONS. — The correlation functions are given by:

$$\langle Q(1, 2) \rangle = \int D[Q] Q(1, 2) \exp[-S(Q)] \tag{72}$$

or, in a spherical model

$$\langle Q(1, 2) \rangle = \int D[Q] D[Z] Q(1, 2) \exp[-S(Q, Z)] \tag{73}$$

The normalization is absent because it is unity.

Since $S(Q)$ is multiplied by N in a long range system, one can evaluate this by saddle point. If there is a single saddle point we have:

$$\langle Q(1, 2) \rangle_{N \rightarrow \infty} = Q(1, 2)|_{\text{saddle point}} \tag{74}$$

However, if there is more than one relevant saddle point, their contributions should be added. This will certainly happen when the saddle point $Q(1, 2)$ breaks a symmetry of the action, as is the case below the critical temperature.

4.4 TWO EXAMPLES. — Let us end this section by giving two examples. The first has been considered (statically) by Jagannathan and Rudnik [14], it is a spherical spin system with ordinary two spin interaction but with a spin length probability perturbed by a small quartic term $\sim (\sum_i s_i^2 - 1)^2$.

The replica expression for the free energy and the dynamic action functional read, respectively:

$$\frac{G}{N} = -\left(\frac{\beta^2 J^2}{4}\right) \sum_{\alpha\beta} (Q_{\alpha\beta})^2 - \frac{1}{2} \text{Tr}[\ln \hat{Q}] - u \sum_{\alpha,\beta} (Q_{\alpha\beta})^4 + \frac{1}{2} \sum_{\alpha} Z_{\alpha} (1 - Q_{\alpha\alpha}) \quad (75)$$

$$\begin{aligned} \frac{S}{N} = & -\left(\frac{\beta^2 J^2}{4}\right) \int d1 d2 (Q(1,2))^2 - \frac{1}{2} \text{Tr}[\ln \hat{Q}] - u \int d1 d2 [Q(1,2)]^4 \\ & + \frac{1}{2} \int d1 Z(1) (1 - Q(1,1)) - \frac{\Gamma_0^{-1}}{2} \int d1 d2 [D^{(2)}(1) \delta(1-2)] Q(1,2) \end{aligned} \quad (76)$$

The other example is a p-spin model with a spherical constraint (and no other spin length probability weights) that has been considered both statically [10] and dynamically [13] by Crisanti and Sommers

$$\frac{G}{N} = -\frac{\beta^2 J^2}{4} \sum_{\alpha\beta} (Q_{\alpha\beta})^p - \frac{1}{2} \text{Tr}[\ln \hat{Q}] - \sum_{\alpha} Z_{\alpha} (1 - Q_{\alpha\alpha}) \quad (77)$$

$$\begin{aligned} \frac{S}{N} = & -\frac{\beta^2 J^2}{4} \int [Q(1,2)]^p d1 d2 - \frac{1}{2} \text{Tr}[\ln \hat{Q}] - \int d1 Z(1) (1 - Q(1,1)) \\ & - \frac{\Gamma_0^{-1}}{2} \int d1 d2 [D^{(2)}(1) \delta(1-2)] Q(1,2) \end{aligned} \quad (78)$$

We note that, roughly speaking, the dynamical action is obtained from the replica free energy by substituting replica indices by superspace variables, Kroeneker deltas by superspace deltas, etc; and then adding a purely kinetic term proportional to Γ_0^{-1} .

This formal analogy can be handled better if we denote simultaneously for the dynamic and the static case the two products:

$$(AB)(1,2) = \int d3 A(1,3) B(3,2); \quad (AB)_{\alpha\beta} = \sum_{\gamma} A_{\alpha\gamma} B_{\gamma\beta} \quad (79)$$

(operator product) and:

$$(A \bullet B)(1,2) = A(1,2) B(1,2); \quad (A \bullet B)_{\alpha\beta} = A_{\alpha\beta} B_{\alpha\beta} \quad (80)$$

(“pointwise” or “element” product); and the corresponding powers:

$$A^r(1,2) = [AA \dots A](1,2); \quad (A^r)_{\alpha\beta} = [AA \dots A]_{\alpha\beta} \quad (81)$$

$$A^{(r)}(1,2) = [A(1,2)]^r; \quad (A^{(r)})_{\alpha\beta} = (A_{\alpha\beta})^r \quad (82)$$

With these notations the dynamical and static replica exponents become (typographically) identical except for the kinetic term proportional to Γ_0^{-1} . Note that this is quite general.

5. Invariances.

The action functional $S[Q]$ for $(S(Z, Q)$ for spherical constraint) has several invariance properties related to those we have already described in section 2.

5.1 SUPERSYMMETRY. — Consider the SUSY algebra:

$$\begin{aligned} \mathbf{D}' &= D'(1) + D'(2) \\ \bar{\mathbf{D}}' &= \bar{D}'(1) + \bar{D}'(2) \\ [D', \bar{D}']_+ &= \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \end{aligned} \tag{83}$$

Using the periodicity in the boundary conditions, one can check directly that $S(Q)$ is invariant with respect to a change $Q \rightarrow Q + \delta Q$, where δQ is generated by any of the three operators in (83). If there is a spherical constraint, then $S(Z, Q)$ is invariant when Q is varied as before but simultaneously $Z \rightarrow Z + \delta Z$ where $\delta Z(1)$ is $D'(1)Z(1)$ or $\bar{D}'(1)Z(1)$ or $\frac{\partial Z(1)}{\partial t_1}$ respectively.

This SUSY of the action is a general fact and is valid for any Langevin process whose drift force derives from a potential.

5.2 GHOST NUMBER. — The invariance with respect to the ghost number N_F described in section 2 reflects here in the fact that $S(Q)$ (or $S(Q, Z)$) is invariant with respect to changes generated by:

$$N(1) + N(2) = \theta_1 \frac{\partial}{\partial \theta_1} - \bar{\theta}_1 \frac{\partial}{\partial \bar{\theta}_1} + \theta_2 \frac{\partial}{\partial \theta_2} - \bar{\theta}_2 \frac{\partial}{\partial \bar{\theta}_2} \tag{84}$$

in $Q(1, 2)$ and by $N(1)$ in $Z(1)$ for spherical constraints.

5.3 CAUSALITY. — The causality relations discussed in section 2 imply a number of restrictions on the components of $\langle Q(1, 2) \rangle$ for $T - |\tau| \rightarrow \infty$ and N finite (c.f. (37) and (54)).

- The coefficients proportional to $\bar{\theta}_2 \theta_2$ and $\bar{\theta}_1 \theta_2$ ($\bar{\theta}_1 \theta_1$ and $\theta_1 \bar{\theta}_2$) are zero for $t_1 < t_2$ ($t_2 < t_1$).
- The coefficient of $\bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2$ is zero .

The action is also invariant with respect to a discrete transformation involving time reversal plus ghost exchange.

5.4 UNBROKEN SYMMETRIES. — In the saddle point approximation, the function $Q(1, 2)$ (and $Z(1)$) may or may not break the symmetries we have discussed.

If the ghost number symmetry is unbroken then:

$$[N(1) + N(2)]Q(1, 2) = N(1)Z(1) = 0 \tag{85}$$

This implies that all but the six components of $Q(1, 2)$ having an equal number of (“unbarred”) θ_i ’s and (“barred”) $\bar{\theta}_j$ ’s cancel. In particular the only correlation functions that do not vanish are boson-boson or fermion-fermion . For $Z(1)$ it implies that

$$Z(1) = Z_0(t_1) + \bar{\theta}_1 \theta_1 Z_2(t_1) \tag{86}$$

Consider SUSY: it can be partially or totally broken. If it is totally unbroken then:

$$D'Q(1, 2) = \bar{D}'Q(1, 2) = \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) Q(1, 2) = 0 \tag{87}$$

(the last relation implying time homogeneity), and:

$$D'(1)Z(1) = \bar{D}'(1)Z(1) = \frac{\partial}{\partial t_1} Z(1) = 0 \quad (88)$$

For $Z(1)$ one can easily check that (88) imply that $Z = Z_0$, a constant independent of $t_1, \bar{\theta}_1, \theta_1$. It is easy to construct the most general supersymmetric $Q(1, 2)$; however if we in addition require zero ghost number and causality, then it is [6] (chapt.16):

$$\begin{aligned} Q(1, 2) &= \left\{ 1 + \frac{1}{2}(\bar{\theta}_1 - \bar{\theta}_2) [\theta_1 + \theta_2 - (\theta_1 - \theta_2) \epsilon(t_1 - t_2)] \frac{\partial}{\partial t_1} \right\} C(t_1 - t_2) \\ &= \frac{1}{2} [D^{(2)}(1) - \epsilon(t_1 - t_2) \frac{\partial}{\partial t_1}] \delta^2(\theta_1 - \theta_2) C(t_1 - t_2) \end{aligned} \quad (89)$$

where $\epsilon(\tau)$ is the sign of τ . This is the most general ansatz for Q satisfying fluctuation-dissipation relations and causality. It is, as we shall see, the general form of Q (for $T \rightarrow \infty$ (see Sect. 2.5)) for spin glasses *above* the critical temperature.

5.5 SOME USEFUL ALGEBRAIC RELATIONS. — Consider two superspace functions $A(1, 2), B(1, 2)$ and their two products:

$$\begin{aligned} C_1 &= AB = \int d3 A(1, 3) B(3, 2) \\ C_2 &= A \bullet B = A(1, 2) B(1, 2) \end{aligned} \quad (90)$$

- i) If both A and B possess any of the symmetries discussed above, the ir products C_1 and C_2 also do.
- ii) If A and B are both causal, the C_1 and C_2 are also causal.
- iii) The functions $\delta(1 - 2)$ and $D^{(2)}(1)\delta(1 - 2)$ are supersymmetric.
- iv) The supertrace of a supersymmetric operator is zero.

These relations are easy to verify.

As an application this immediately implies that in general, since the action S is constructed using the products and the operators (iii) then:

- v) $S(Q, Z) = 0$ (as it should) if Z, Q are supersymmetric.

Note that the converse is not true, the action could be zero even for non supersymmetric solutions.

6. Replica symmetric statics and supersymmetric dynamics.

Above the critical temperature, in the replica approach, replica symmetry is unbroken. From the dynamic point of view, Sompolinsky and Zippelius [4] have proposed that the solution satisfies the FDT, which we recognized in the previous section (in the superspace language) as the fact that also SUSY is unbroken.

In this section, dirgarding the boundary conditions, we show that there is a general direct connection between (unbroken) replica symmetry and (unbroken) SUSY plus causality. More

precisely, we show that if one lets the time scale $\Gamma_0 \rightarrow \infty$ (“fast motion”) the superspace becomes in a certain sense equivalent to $(n \rightarrow 0)$ replica space.

This connection holds for any model, but let us consider for definiteness the p -spin spherical model (77) and (78). Differentiating (77) with respect to $Q(1, 2)$ and $Z(1)$ we have the equations of motion:

$$Q(1, 1) = 1$$

$$\Gamma_0^{-1} D_1^{(2)} \delta(1 - 2) + Z(1) \delta(1 - 2) + \frac{(\beta J)^2}{2} p Q^{(p-1)}(1, 2) + Q^{-1}(1, 2) = 0 \quad (91)$$

The static replica saddle point equations are:

$$Q_{\alpha\alpha} = 1$$

$$Z_\alpha \delta_{\alpha\beta} + \frac{(\beta J)^2}{2} p [Q^{(p-1)}]_{\alpha\beta} + (Q^{-1})_{\alpha\beta} = 0 \quad (92)$$

If we now make the supersymmetric-causal ansatz (89) in (91) (which also implies $Z = Z_0$) we obtain the “one time scale” dynamic equation proposed and solved in [13] for $C(t_1 - t_2)$. The spin-spin correlation function decays from $C(0) = 1$ to $C(t_1 - t_2) \rightarrow q$ as $|t_1 - t_2| \rightarrow \infty$.

Let us consider instead the “fast motion” dynamics for Γ_0 large, we intend to compare it with the static replica solution. Let us first make a brief digression concerning the static replica symmetric solution, solving (92) in a fancy way. Put:

$$Q = (1 - q) U_1 + q U_0$$

$$Z_\alpha = Z_0 \quad (93)$$

where we have denoted:

$$[U_1]_{\alpha\beta} = \delta_{\alpha\beta}$$

$$[U_0]_{\alpha\beta} = 1 \quad \forall \alpha\beta \quad (94)$$

Consider the two products (in the notation of Eqs. (79) and (80)), the “element to element” product:

$$U_1 \bullet U_0 = U_0$$

$$U_0 \bullet U_0 = U_0$$

$$U_1 \bullet U_1 = U_1 \quad (95)$$

and the operator product:

$$U_1 U_0 = U_1$$

$$U_0 U_0 = U_0$$

$$U_1 U_1 = 0 \quad (96)$$

In the last relation we have made (abstractly) for the operators the limit $n \rightarrow 0$ ($U_1^2 = n U_1 = 0$). In particular:

$$[(1 - q)U_1 + q U_0]^{-1} = (1 - q)^{-1} U_1 - q(1 - q)^{-2} U_0 \quad (97)$$

It is easy to see that if one inserts the ansatz (93) in (92) using the algebraic relations (95) and (96) one gets the equations for q and Z_0 (with the limit $n \rightarrow 0$ already taken).

Now let us turn back to the ("fast") dynamic approach. In order to make manifest the relation between static and dynamic approach we will define, a "causal-supersymmetric" regularization of the superspace delta function. Consider *any* function $A(|\tau|)$ such that

$$A(0) = 1 \quad (a) \quad (98)$$

$$\lim_{\tau \rightarrow \infty} A(|\tau|) = 0 \quad (b) \quad (99)$$

and let us construct a rapidly decaying supersymmetric-causal function:

$$\delta_{SC}(1-2) = \lim_{\Delta \rightarrow \infty} \left\{ 1 + \frac{1}{2}(\bar{\theta}_1 - \bar{\theta}_2)[\theta_1 + \theta_2 - (\theta_1 - \theta_2)\epsilon(t_1 - t_2)\frac{\partial}{\partial t_1}] \right\} A(\Delta|\tau|) \quad (100)$$

(c.f. Eq. (89)). It is easy to see that $\delta_{SC}(1-2)$ indeed tends to the ordinary superspace delta (50) as $\Delta \rightarrow \infty$. Moreover, by construction we have that evaluated at $t_1 = t_2, \theta_1 = \theta_2, \bar{\theta}_1 = \bar{\theta}_2$:

$$\delta_{SC} = 1 \quad (101)$$

A more suprising property is:

$$\delta_{SC}(1-2)\delta_{SC}(1-2) = \delta_{SC}(1-2) \quad (102)$$

by which we mean that the product is made of two expressions like (100) (with different functions $A_1(|\tau|)$ and $A_2(|\tau|)$ both satisfying (99)) and then the limit $\Delta \rightarrow \infty$ is taken.

If we now define

$$U_1 = \delta_{SC}(1-2) \quad (a)$$

$$U_0(t_1, \bar{\theta}_1, \theta_1, t_2, \bar{\theta}_2, \theta_2) = 1 \quad (b) \quad (103)$$

(where in (b) "1" stands for the ordinary number 1!) we immediately check that they satisfy the same rules (95), (96) for their corresponding two products. The ansatz (93) (with, now, U_0 and U_1 of (103)) now describes a rapid decay of the correlation functions, satisfying FDT and with $C(t_1 - t_2) \rightarrow q$ as $|t_1 - t_2| \rightarrow \infty$.

But we also have that (103) provides a concrete realization of the algebra (95), (96) (equivalent to that of "dimension = 0 matrices"). Furthermore, if we neglect the kinetic term proportional to Γ_0^{-1} in (91) it is easy to see that the formal identity of the equations plus the same algebra in the solutions leads to the same values of q, Z_0 . (Note however that the kinetic term has not been completely neglected, in that it has determined causality).

One can easily convince oneself of the generality of this argument.

We now compute the eigenvalues of the quadratic fluctuations around this solution. We find, perhaps at this point not surprisingly, the same eigenvalues as in the static replica treatment. Crisanti and Sommers [10] have calculated the eigenvalues of these quadratic fluctuations $\delta q_{\alpha\beta}$ around the replica symmetric saddle point. They found three eigenvalues:

$$\begin{aligned} \Lambda_1 &= -\frac{(\beta J)^2}{2}(p-1)q^{p-1} + A^2 & ; & \quad \text{for } \sum_{\beta} \delta q_{\alpha\beta} = 0 \quad \forall \alpha \\ \Lambda_2 &= \Lambda_1 + (n-2)AB & ; & \quad \text{for } \sum_{\alpha\beta} \delta q_{\alpha\beta} = 0, \quad \sum_{\beta} \delta q_{\alpha\beta} \neq 0 \\ \Lambda_3 &= \Lambda_2 + nAB + n(n-1)B^2 & ; & \quad \text{for } \sum_{\alpha,\beta} \delta q_{\alpha\beta} = 0 \end{aligned} \quad (104)$$

where:

$$\begin{aligned} A &= (1 - q)^{-1} \\ B &= -q(1 - q)^{-1} [1 + (n - 1)q]^{-1} \end{aligned} \tag{105}$$

note that as $n \rightarrow 0$, $\Lambda_2 = \Lambda_3$. Consider now the dynamic equation (91). If we solve for $Z(1)$ (neglecting the kinetic term) by multiplying both sides of eq. (91) by δ_{SC} :

$$-\delta(1 - 2)Z(1) = \delta_{SC}(1 - 2) \left[\frac{\beta^2 J^2}{4} p Q^{(p-1)}(1, 2) + Q^{-1}(1, 2) \right]. \tag{106}$$

reintroduce this into (91), differentiate with respect to $Q(3, 4)$ and substitute the ansatz (93) we obtain the equation for the eigenvalues Λ :

$$\begin{aligned} (\Lambda_1 - \Lambda)\delta q(1, 2) &+ AB_0[\int d3 \delta q(1, 3) + \int d3 \delta q(3, 2)] + B_0^2 \int \delta q(3, 4)d3 d4 \\ &- \delta(1 - 2)[2AB_0(\int d3 \delta q(1, 3)) + B_0^2 \int \delta q(3, 4)d3 d4] = 0 \end{aligned} \tag{107}$$

where B_0 is the value of B at $n = 0$ and we have used the fact that $\delta q(1, 1) = 0$. By integrating first over the variables "1" and "2", we easily conclude that unless:

$$\int \delta q(1, 2) d1 d2 = 0 \tag{108}$$

then:

$$\Lambda = \Lambda_1 - 2AB_0 \tag{109}$$

If instead (108) holds we integrate (107) only over "1" to find the same eigenvalue unless

$$\int \delta q(1, 2) d1 = \int dq(2, 1) d1 = 0 \tag{110}$$

in which case the eigenvalue is:

$$\Lambda = \Lambda_1 \tag{111}$$

The steps parallel closely the ones taken in the replica case; and, furthermore we obtain the same results (104) *but with the limit $n \rightarrow 0$ already taken.*

7. Conclusions.

We have presented the spin glass dynamic problem in a manifestly supersymmetric way. In order to make the paper roughly self-contained, a considerable portion of this paper is spent in notation, but possibly a very suggestive one.

The superspace presentation puts the problem of spin glass dynamics in a rather more standard field theoretic language, i.e. SUSY breaking and restoration, Ward identities, etc. It also brings the dynamical problem into closer formal contact with the replica treatment.

Above the critical temperature, the connection between superspace and $n \rightarrow 0$ replica space has been explicitated; and one can see how the equivalence of static results obtained in both cases comes about formally.

However, the real challenge is to complete this connection to the case of replica symmetry breaking below T_c . This would have several advantages, the first obvious (formal) one as verification of the replica trick. At a more physical level, finding the solution in all its details would yield new information such as the time scales.

A possible way in which the connection could come about (at the level of "fast dynamics") could be to generalize the procedure of the last section by finding a set of superspace functions representing the algebra of " $n = 0$ Parisi matrices", and then to write an ansatz as a combination of them. Work is under way on these lines.

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