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► **To cite this version:**

Christian Maes, Frank Redig. Anisotropic perturbations of the simple symmetric exclusion process : long range correlations. *Journal de Physique I*, 1991, 1 (5), pp.669-684. 10.1051/jp1:1991161 . jpa-00246361

HAL Id: jpa-00246361

<https://hal.science/jpa-00246361>

Submitted on 4 Feb 2008

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Classification

Physics Abstracts

05.40 — 02.90 — 46.10

Anisotropic perturbations of the simple symmetric exclusion process : long range correlations

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(Received 25 October 1990, accepted in final form 22 January 1991)

Abstract. — Consider a lattice spin system in which nearest neighbor spins are exchanged at rates which weakly depend on the neighboring configuration. The system has the same symmetry of the lattice except possibly for lattice rotations. We set up a perturbation expansion for the correlation functions at time t around the simple symmetric exclusion process. Convergence is proven for small t and the formal $t \rightarrow \infty$ limit reproduces the usual high temperature expansion in the case of detailed balance. If the system is isotropic, then each term in the expansion is strictly local. If not, then, generically, the two points function has the direction dependence of a quadrupole field and decays only like a power r^{-d} , where r is the spatial separation and $d \geq 2$ is the dimension.

1. Introduction.

Gibbs states at high temperatures have correlations decaying in the same sense as the interaction potential. This well known fact from equilibrium statistical mechanics has been established by various methods such as the cluster expansion [1-3], or a Dobrushin type analysis [4-6], and is in last instance a consequence of the so called local Markov property of Gibbs states : for a finite range interaction, given the configuration inside the appropriate boundary layer surrounding a volume, the inside and outside are conditionally independent. It allows to follow the interdependence of certain events in different regions in space *via* intermediate points (or, « polymers ») along which information is transported (and lost). These « connections » are explicitly constructed in the cluster expansion, starting from the Boltzmann factor $Z^{-1} \exp(-\beta H)$, β small, for an interaction H . For example, to first order in β , for a lattice spin system, only the neighborhood of a site, as determined by H , can influence the spin there. More probabilistic methods use the (quasi-) locality of the conditional distributions to estimate the dependence of a spin on its neighborhood. This explains in fact why we expect that local interactions generically (that is, away from critical points) give rise to a finite correlation length.

Turning to real non-equilibrium situations, we cannot expect such methods to be applicable by only considering the spatial structure. For a general lattice spin dynamics, we will not find a simple local mechanism by which to describe the mutual influence of spins in different

regions. The spatial structure of non-Gibbsian stationary states can best be studied by embedding it in the spatio-temporal process. Simple examples for which one readily recognizes the advantages of connecting spatially separated spins *via* their common history are interacting particle systems or probabilistic cellular automata [7-8]. In fact, such an analysis naturally arises for general non-equilibrium phenomena. This may seem obvious but it has important consequences. If the stationary state is not Gibbsian, and the spatial correlations must be thought off as intermediated *via* events in the past, then the spatial decay will be related to the temporal correlations. In particular if the latter have a weak decay such as for diffusive systems, then a similar behavior can be expected for the stationary correlations.

This, in short, was the starting point of the analysis in [9] to study the phenomenon of long range spatial correlations in conservative non-equilibrium dynamics. Here we wish to investigate in more detail the associated perturbation expansion around an exactly solvable lattice gas dynamics. The system consists of spins $\sigma(x) = \pm 1$, $x \in Z^d$, on the d -dimensional lattice. A dynamics is introduced in which nearest neighbor spins are exchanged with rates which depend weakly on the configuration in a finite neighborhood. We are interested in the behavior of the correlation functions $\left\langle \prod_{x \in A} \sigma(x) \right\rangle$, for some finite $A \subset Z^d$, in the stationary

states. To study them, we make an expansion around the product state which is invariant for the so-called symmetric simple exclusion [7]. It can be viewed as a system of random walkers which interact *via* a hard core potential only. Although the stationary states are trivial for this system, the temporal correlations decay weakly. Typically, fluctuations in the density of the walkers decay diffusively in time like $t^{-d/2}$. This decay enters in the perturbation expansion as the zero mass of the free propagator. While this makes our analysis considerably more complicated, it has the interesting feature to produce, for generic perturbations, also a weak decay in space. Putting $r^2 \sim t$ as the usual space-time scaling for diffusive systems and assuming that the spatial decay is intermediated *via* the temporal diffusive decay, we end up with the prediction that there is some constant Q such that $\langle \sigma(0); \sigma(\mathbf{r}) \rangle = Q |\mathbf{r}|^{-d}$, $|\mathbf{r}| \rightarrow \infty$ for the stationary two points function. While, *a priori*, this scenario is completely general, the details of the dynamics enter in the prefactor Q . We will argue that generically $Q \neq 0$ but there are special (non-generic) situations in which by symmetry $Q \approx 0$. Those situations can be divided into two (overlapping) classes.

The first important special case occurs when this « self-organized criticality » disappears by arranging the dynamics in such a way that it satisfies the condition of detailed balance, and the appropriate Gibbs states become stationary. In that case we recover (in an unusual way) the usual high-temperature expansion. In the light of the discussion that follows we emphasize that this is as expected and is independent of the (an-)isotropy of the corresponding Hamiltonian — the symmetry that we need is completely contained in the detailed balance condition.

Secondly, as will become clear, an essential (necessary) ingredient in our analysis is the anisotropy of the dynamics. All our arguments leading to long range correlations are based on the fact that the systems we consider do not possess the full symmetry of the lattice. Moreover, we will show that starting from the formal expansion for the stationary correlation functions, any fully isotropic perturbation *must* have short range correlations. This is consistent with the results of [10] for similar dynamics, but where it is not excluded to have more than one particle per site.

Finally, it may still happen that by some other symmetry $Q = 0$ even though the model does not fit in one of the two classes described above. An example of this is presented at the end of section 4 and it is found there that the correlations still have a power law decay with the

direction dependence of an octopole field (instead of the quadrupole decay for the case $Q \neq 0$).

The difference of our work with [9] lies in the details of the perturbation expansion, the rigorous (but, admittedly, weak) results concerning the convergence and the thorough analysis of the role of the anisotropy. We also add complete and explicit results for a specific model and we find the possibility of octopole-like decay.

In the next section we define the model. Section 3 is devoted to the derivation of the perturbation expansion. We can only show that it converges for small times, or, when the perturbation is strictly finite, for all finite times. Still taking the formal $t \rightarrow \infty$ limit, we discuss in section 4 the presence of long range correlations in the stationary measure. The two points function is studied term per term in the expansion, and we find that generically its decay is like r^{-d} , where r is the spatial separation, and $d \geq 2$. We emphasize the role of the anisotropy. This behavior is explicitly calculated up to first order in the expansion for a specific perturbation. In the Appendix we apply our formal expansion to the Kawasaki dynamics and give this expansion more significance by recovering — without using the standard Gibbs formalism — the well known high temperature expansion for the equilibrium correlation functions.

2. The model.

We consider a stochastic time evolution for spins $\sigma(x) = \pm 1$, $x \in \mathbb{Z}^d$ in which the rates $c(x, y, \sigma)$ at which the spin at site x and y , $|x - y| = 1$ are exchanged depend weakly on the configuration at other sites in a finite neighborhood of the bond $\langle xy \rangle$. For a finite volume $\Lambda \subset \mathbb{Z}^d$, the probability $P_t^\Lambda(\sigma)$ to find configuration $\sigma \in \{-1, +1\}^\Lambda$ in Λ , say with periodic boundary conditions, is governed by the master equation

$$\frac{d}{dt} P_t^\Lambda(\sigma) = \sum_{\langle xy \rangle \in \Lambda} (c(x, y, \sigma^{xy}) P_t^\Lambda(\sigma^{xy}) - c(x, y, \sigma) P_t^\Lambda(\sigma)), \quad (2.1)$$

where σ^{xy} is the configuration obtained from σ after exchanging the nearest neighbor sites $\langle xy \rangle$:

$$\begin{aligned} \sigma^{xy}(z) &= \sigma(z) && \text{if } z \neq x, \quad z \neq y, \\ &= \sigma(y) && \text{if } z = x, \\ &= \sigma(x) && \text{if } z = y. \end{aligned} \quad (2.2)$$

The process σ_t , $t \geq 0$, in the infinite volume limit is conveniently defined *via* its generator L , which on local functions $f(\sigma)$, $\sigma \in \{-1, +1\}^{\mathbb{Z}^d}$, is defined by

$$Lf(\sigma) = \sum_{\langle xy \rangle} c(x, y, \sigma) (f(\sigma^{xy}) - f(\sigma)). \quad (2.3)$$

The sum in (2.3) is over nearest neighbor sites $x, y \in \mathbb{Z}^d$, $|x - y| = 1$, and of course $c(x, y, \sigma) = c(y, x, \sigma) \geq 0$. Further assumptions on the rates must be imposed but their formulation is postponed until after we have introduced some well known type of such dynamics.

The simplest example corresponds to an infinite temperature Kawasaki dynamics, where the rates satisfy the condition

$$c_0(x, y, \sigma) = c_0(x, y, \sigma^{xy}). \quad (2.4)$$

Clearly, all Bernoulli measures with average spin $\langle \sigma_x \rangle_0 = m \in [-1, +1]$, are invariant for this time evolution. Finite temperature Kawasaki dynamics have rates $c_\beta(x, y, \sigma)$, $\beta \geq 0$, satisfying the condition of detailed balance

$$c_\beta(x, y, \sigma) = c_\beta(x, y, \sigma^{xy}) \exp(-\beta (H(\sigma^{xy}) - H(\sigma))), \tag{2.5}$$

where

$$H(\sigma) \equiv - \sum_A J_A \prod_{x \in A} \sigma(x) \tag{2.6}$$

is a local translation invariant Hamiltonian, i.e. $\langle J_A, A \subset Z^d \text{ finite} \rangle$ are real numbers such that $J_{A+x} = J_A$, and $J_A = 0$ whenever the diameter of A is too large, say if $\text{diam}(A) \geq R$. The Gibbs-measures μ_β for (2.6) are then reversible measures for the dynamics, and we know from equilibrium statistical mechanics how to express the correlation functions

$$\langle \sigma_A \rangle_\beta \equiv \int \prod_{x \in A} \sigma(x) d\mu_\beta(\sigma) \tag{2.7}$$

as a convergent high temperature expansion around $\beta = 0$. Of course, we may always add a magnetic field term $h \sum_x \sigma(x)$ to (2.6) and (2.5) remains verified. As in the case $\beta = 0$, there is at least a one parameter family of Gibbs measures corresponding to different magnetizations, which are invariant for the dynamics. They are the so called canonical Gibbs measures, see [11] for a further discussion.

We are interested in dynamics with rates not satisfying (2.5) for any local Hamiltonian. We therefore look to generic perturbations of the infinite temperature dynamics, having constant rates $c_0(x, y, \sigma) = \frac{1}{2}$. The perturbed process is then assumed to have rates

$$c(x, y, \sigma) = \frac{1}{2} \left(1 + \sum_A q_A(x, y) \sigma_A \right), \tag{2.8}$$

where $\sigma_A \equiv \prod_{x \in A} \sigma(x)$. The coefficients $\{q_A(x, y)\}$ must be taken so that the rates (2.8) are non-negative, local and bounded functions of the configuration σ (see (2.12) and (2.13)). We also assume that the rates are even : $q_A(x, y) = 0$ whenever the number of elements in A , $|A|$, is odd. Translation invariance is imposed by the condition that

$$q_A(x, y) = q_{A+a}(x+a, y+a) \tag{2.9}$$

for any $a \in Z^d$. Finally, we assume that the model is reflection symmetric over all d coordinate axes, i.e. if $\{e_\alpha\}_{\alpha=1}^d$ are the unit vectors in Z^d , then

$$q_{\theta_\alpha A}(0, e_{\alpha'}) = q_A(0, e_{\alpha'}) \quad \text{if} \quad \alpha \neq \alpha',$$

and

$$q_{\theta_\alpha A}(0, e_\alpha) = q_A(0, -e_\alpha), \quad \alpha = 1, \dots, d, \tag{2.10}$$

where

$$\theta_\alpha A \equiv \{(x_1, \dots, -x_\alpha, \dots, x_d) : x = (x_1, \dots, x_\alpha, \dots, x_d) \in A\}.$$

Let

$$q_{AB} \equiv \frac{1}{2} \sum_{\langle xy \rangle} (q_{A^{xy} \Delta B}(x, y) - q_A \Delta B(x, y)) \tag{2.11}$$

with « Δ » the symmetric difference between sets, and A^{xy} is obtained from A by exchanging the labels of the sites $\langle xy \rangle$. We further require that

$$q_{AB} = 0 \text{ if } \text{diam} (A \Delta B) > R, \tag{2.12}$$

and

$$|q_{AB}| \leq \gamma \text{ for all } A, B \tag{2.13}$$

for certain constants $R < \infty$ and $\gamma \ll 1$. There is no *a priori* information on the stationary measures. We try therefore to use γ as an expansion parameter and set up a formal perturbation theory.

In order not to loose oneself in the notations for the most general model it is good to keep in mind the following simple example which we will take up again at the end of section 4. The model is two dimensional and the exchange rates are for $\alpha = 1, 2$ given by

$$c(x, x + e_\alpha, \sigma) = \frac{1}{2} \exp \left(-\frac{1}{2} \beta_\alpha [H(\sigma^{x, x+e_\alpha}) - H(\sigma)] \right) \tag{2.14}$$

for « Hamiltonian »

$$H(\sigma) = -2 \sum_x [K_1 \sigma(x) \sigma(x + e_1) + K_2 \sigma(x) \sigma(x + e_2)]. \tag{2.15}$$

Here β_1, β_2 are small and if they are both equal to some β , then (2.14) satisfies the condition of detailed balance (2.5). The main problem this paper investigates might be summarized by the question, what happens to the stationary correlations if $\beta_1 \neq \beta_2$.

3. The expansion.

The unperturbed process corresponds to the simple symmetric exclusion process. The main tool in its analysis is the self-duality [7]. In this case, the generator (2.3) is :

$$L_0 f(\sigma) = \frac{1}{2} \sum_{\langle \lambda y \rangle} [f(\sigma^{xy}) - f(\sigma)]. \tag{3.1}$$

If we define for any finite $A \subset Z^d$, the function $D(A, \sigma) \equiv \prod_{x \in A} \sigma(x) \equiv \sigma_A$, then it is easy to check that $L_0 D$ is the same both when it acts on A and on σ , i.e.

$$L_0 D(A, \sigma) = \frac{1}{2} \sum_{\langle xy \rangle} (D(A^{xy}, \sigma) - D(A, \sigma)). \tag{3.2}$$

We therefore consider the so called dual process $A(t)$, $t \geq 0$, on the finite subsets of Z^d , with generator

$$\hat{L}_0 f(A) = \frac{1}{2} \sum_{\langle xy \rangle} [f(A^{xy}) - f(A)], \tag{3.3}$$

and let $p_t(A, B)$ be the corresponding transition probability, i.e. the probability that a collection of random walkers starting from the sites in A and subject only to a hard core interaction, end up in the set B at time t . Obviously, $p_t(A, B)$ is zero unless $|A| = |B|$, and formally,

$$\frac{1}{2} \sum_{xy} \int_0^\infty dt (p_t(A, B) - p_t(A^{xy}, B)) = \delta_{A, B} \tag{3.4}$$

is one if $A = B$ and zero otherwise. From (3.2) one derives the duality relation :

$$E_0^\mu(D(A, \sigma_t)) = \int d\mu(\sigma) \sum_B p_t(A, B) D(B, \sigma), \quad (3.5)$$

where E_0^μ denotes the expectation in the simple symmetric exclusion process started from the measure μ .

Consider now the perturbed process with generator (2.3) $L = L_0 + L_1$ and let ξ denote the expectation with respect to the product of the A_t -process (with generator \hat{L}_0) and the σ_t -process (with generator L). \hat{L}_0 commutes with L and since $\hat{L}_0 D(A, \sigma) = L_0 D(A, \sigma)$

$$\frac{d}{dt} \xi[D(A_{t-s}, \sigma_s)] = \xi[L_1 D(A_{t-s}, \sigma_s)]. \quad (3.6)$$

By integrating (3.6) over s from 0 to t we get

$$E^\mu[D(A, \sigma_t)] = \sum_B p_t(A, B) \mu[D(B, \sigma)] + \int_0^t ds \sum_B p_{t-s}(A, B) E^\mu[L_1 D(B, \sigma_s)], \quad (3.7)$$

with E^μ the expectation in the perturbed σ_t -process with initial measure μ .

Suppose the perturbed process with rates (2.8) has as initial state μ the Bernoulli measure with magnetization $\langle \sigma_x \rangle_0 = 0$. Then, from (3.7), the time evolved spin-spin correlations satisfy the equation

$$\langle \sigma_A \rangle_t = \delta_{A, \emptyset} + \int_0^t ds \sum_B p_{t-s}(A, B) v_s(B) \quad (3.8)$$

with

$$v_s(B) \equiv \langle L_1 D(B, \sigma) \rangle_s$$

and from (2.8), (2.11),

$$\begin{aligned} v_s(B) &= \sum_{\langle xy \rangle} \left\langle (\sigma_{B^{xy}} - \sigma_B) \left(c(x, y, \sigma) - \frac{1}{2} \right) \right\rangle_s \\ &= \sum_C q_{BC} \langle \sigma_C \rangle_s \end{aligned} \quad (3.9)$$

Substituting (3.9) into (3.8) we can iterate to obtain

$$\langle \sigma_A \rangle_t = \sum_{n=0}^{\infty} V_A^n(t), \quad (3.10)$$

where $V_A^0(t) \equiv \delta_{A, \emptyset}$, and

$$V_A^n(t) = \int_0^t ds \sum_{B, C} p_{t-s}(A, B) q_{BC} V_C^{n-1}, \quad n \geq 1. \quad (3.11)$$

The first question is to see when (3.10), (3.11) defines a convergent expansion for $\langle \sigma_A \rangle_t$.

PROPOSITION 1. — *If $t\gamma R < 1$, then (3.10)-(3.11) defines a convergent expansion for the time evolved correlations. For fixed time t , $|\langle \sigma_A \rangle|_t$ is of the order $\gamma^{|A|/R}$, γ small.*

Proof: By locality, see (2.12),

$$\sum_B p_t(A, B) q_{BC} = 0 \quad \text{if} \quad |A| > |C| + R. \tag{3.12}$$

Therefore, by induction from (3.11)

$$V_A^n(t) = 0 \quad \text{whenever} \quad |A| > nR. \tag{3.13}$$

On the other hand, from (2.13):

$$|V_A^1(t)| \leq t\gamma \quad \text{if} \quad |A| \leq R. \tag{3.14}$$

Assuming that for some $n > 1$

$$|V_A^{n-1}(s)| \leq \gamma^{n-1} R^{n-2} s^{n-1}, \tag{3.15}$$

we have from (3.13) that

$$|V_A^n(t)| \leq \int_0^t ds (nR\gamma) (\gamma^{n-1} R^{n-2} s^{n-1}) \leq \gamma^n R^{n-1} t^n \tag{3.16}$$

so that (3.10) converges for $\gamma R t < 1$.

By construction this series satisfies the evolution equations (3.8).

PROPOSITION 2. — *If*

$$\sup_A \sum_B |q_{AB}| \equiv \chi < \infty, \tag{3.17}$$

then (3.10)-(3.11) converges for all times $t < \infty$.

Proof: Since the perturbation is finite, it follows from similar estimates as above that

$$|V_A^n(t)| \leq \frac{t^n}{n!} \chi^n, \tag{3.18}$$

and we are done.

Note that condition (3.17) requires the perturbation to be strictly finite in the sense that $c(x, y, \sigma) \neq 1/2$ for only a finite number of bonds $\langle xy \rangle$. This breaks the translation invariance and we do not consider this case here any further.

Continuing with (3.10), (3.11) we take the formal $t \rightarrow \infty$ limit to obtain

$$\langle \sigma_A \rangle_\infty = \sum_{n=0}^{\infty} V_A^n \tag{3.19}$$

where $V_A^0 \equiv \delta_{A\emptyset}$,

$$V_A^n \equiv \sum_{B,C} G(A, B) q_{BC} V_C^{n-1}, \quad n \geq 1 \tag{3.20}$$

and

$$\sum_B G(A, B) f(B) \equiv \lim_{t \uparrow \infty} \int_0^t \sum_B p_s(A, B) f(B) ds. \tag{3.21}$$

There are of course many problems with (3.19)-(3.21). Since we cannot control the expansion (3.10) for every finite time, not only is the limit $t \uparrow \infty$ purely formal, but we have no convergence result for (3.19). On the other hand, we have two additional results that may indicate that (3.19) is meaningful at least in some respects.

First, we argue that the series is well defined termwise due to the special form of the coefficients q_{BC} . They reflect the symmetry and the conservation law present in the dynamics. More generally, the question is whether the limit (3.21) exists for functions f of the form

$$f(B) = \sum_C q_{BC} K_C . \tag{3.22}$$

If we assume that the coefficients K_C in (3.22) are such that for all numbers $N = 1, 2, \dots$,

$$Q_\alpha(N, f) \equiv \sum_{B: |B|=N} \sum_{x \in B} x_\alpha^2 f(B) < \infty , \quad \alpha = 1, \dots, d \tag{3.23}$$

then, $f(B)$ is a « quadrupole » :

$$\begin{aligned} \sum_{B: |B|=N} f(B) &= 0 \\ \sum_{B: |B|=N} \sum_{x \in B} x_\alpha f(B) &= 0 \\ \sum_{B: |B|=N} \sum_{x \in B} x_\alpha x_{\alpha'} f(B) &= Q_\alpha(N, f) \delta_{\alpha, \alpha'} . \end{aligned} \tag{3.24}$$

Properties (3.23), (3.24) are probably sufficient for Gf to be defined by $Gf(A) \equiv \sum_B G(A, B) f(B)$ but we have no proof for general sets A . We now show how it can be done in the case where $A = \{0, x\}$ consists of at most two sites. From (3.4) $Gf(\{0, x\}) \equiv V(x)$ can be found as the unique solution of the equation

$$\frac{1}{2} \sum_{(y:z)} (Gf(\{0, x\}^{yz}) - Gf(\{0, x\})) = -f(\{0, x\}) , \tag{3.25}$$

or,

$$\sum_{\alpha=1}^d \Delta_\alpha (V(x) + \delta_{x,0} V(e_\alpha)) = \rho(x) , \tag{3.26}$$

with boundary conditions $V(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In (3.26), $\Delta_\alpha g(x) \equiv g(x + e_\alpha) + g(x - e_\alpha) - 2g(x)$ is the discrete Laplacian in the direction α and $\rho(x) = -f(\{0, x\})$ is a quadrupole. Using the properties (3.23), (3.24), we have that

$$\rho(x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} dk e^{ikx} \sum_\alpha (1 - \cos k_\alpha) \chi_\alpha(k)$$

for some $\chi_\alpha(k)$ bounded at $k = 0$. This will be derived explicitly in section 4, equation (4.7). Here, it suffices to note that substituting the above expression in (3.26) allows to find a well defined solution $V(x)$.

A second argument yielding substance to the formal expansion (3.17)-(3.19) is that it is meaningful for certain dynamics on which we have detailed information about the stationary measures. We turn therefore to the Kawasaki dynamics defined in (2.5) for which the Gibbs measures are reversible. In the Appendix we derive the usual high temperature expansion for

the equilibrium correlations without using any properties of the corresponding Gibbs measures but fully relying on our formal expansion (3.19). We thus find how to use the dynamical « symmetry » (2.5) to simplify each term in the expansion to get a strictly local function.

4. Long range correlations.

In the Appendix we show that the formal perturbation expansion around the infinite temperature dynamics reproduces the strictly local high temperature expansion in the case of detailed balance dynamics. Generic perturbations are not expected to satisfy such a condition. Here we investigate what the behavior is of the two points function as predicted from each term in the expansion, for such a generic perturbation. Each term $V_{\{0, x\}}^n$, $n \geq 1$, $x \in \mathbb{Z}^d$, in the expansion (3.19) satisfies the equation (3.26) with

$$\rho(x) \equiv \frac{1}{2} \sum_{\langle yz \rangle} (Q_{\{0, x\}}^n(y, z) - Q_{\{0, x\}}^n(z, y)) \tag{4.1}$$

and

$$Q_A^n(y, z) \equiv \sum_B q_{A \Delta B}(y, z) V_B^{n-1}$$

The translation invariance of the model, (2.9), implies that

$$Q_A^n(y, z) = Q_{A+a}^n(y+a, z+a) \tag{4.2}$$

for any $a \in \mathbb{Z}^d$, and by reflection symmetry (2.10),

$$Q_A^n(0, -e_\alpha) = Q_{-A}^n(0, e_\alpha). \tag{4.3}$$

As a consequence, (4.1) can be rewritten as

$$\begin{aligned} \rho(x) = \sum_{\alpha=1}^d (Q_{\{0, -x\}}^n(0, e_\alpha) - Q_{\{0, x+e_\alpha\}}^n(0, e_\alpha) + Q_{\{0, x\}}^n(0, e_\alpha) - Q_{\{0, -x+e_\alpha\}}^n(0, e_\alpha)) - \\ - \sum_{\alpha=1}^d \Delta_\alpha \delta_{x, 0} Q_{\{0, e_\alpha\}}^n(0, e_\alpha). \end{aligned} \tag{4.4}$$

The behavior of $Q_{\{0, x\}}^n(0, e_\alpha)$ as $|x| \rightarrow \infty$ depends on the $(n-1)$ -th order correlation functions $V_{A \Delta \{x\}}^{n-1}$, for fixed sets A around the origin. Indeed, if $|x| > R$, then

$$Q_{\{0, x\}}^n(0, e_\alpha) = \sum_A q_{\{0\} \Delta A}(0, e_\alpha) V_{A \Delta \{x\}}^{n-1} \tag{4.5}$$

First consider the situation where the $V_{A \Delta \{x\}}^{n-1}$ are quasi-local in the sense that they decay at least exponentially fast to zero as $|x| \rightarrow \infty$. That is certainly verified for $n = 1$. Then, this also holds for $\rho(x)$. Taking Fourier transforms, $k = (k_1, \dots, k_\alpha, \dots, k_d) \in [-\pi, \pi]^d$,

$$\hat{q}_\alpha(k) \equiv \sum_x Q_{\{0, x\}}^n(0, e_\alpha) e^{ikx}, \tag{4.6}$$

we get

$$\hat{\rho}(k) = \sum_x \rho(x) e^{ikx} = \sum_{\alpha=1}^d (1 - \cos k_\alpha) \chi_\alpha(k) \tag{4.7}$$

where

$$\chi_\alpha \equiv \frac{(1 - e^{-ik_\alpha}) \hat{q}_\alpha(k) + (1 - e^{ik_\alpha}) \hat{q}_\alpha(-k)}{(1 - \cos k_\alpha)} + 2 Q_{\{0, e_\alpha\}}^n(0, e_\alpha) \quad (4.8)$$

is analytic at $k = 0$. One can first find the nearest neighbor correlation functions to equal

$$V_{\{0, e_\alpha\}}^n = \sum_{\gamma=1}^d (T^{-1})_{\alpha\gamma} M_\gamma, \quad (4.9)$$

Where T^{-1} is the inverse of the matrix T given by

$$T_{\alpha\gamma} \equiv \delta_{\alpha\gamma} + \frac{1}{(2\pi)^d} \int dk e^{-ik_\alpha} \frac{(1 - \cos k_\gamma)}{\sum_{\alpha'} (1 - \cos k_{\alpha'})}. \quad (4.10)$$

and

$$M_\gamma \equiv \frac{1}{(2\pi)^d} \int dk e^{-ik_\gamma} \frac{\sum_{\alpha} (1 - \cos k_\alpha) \chi_\alpha(k)}{2 \sum_{\alpha} (\cos k_\alpha - 1)} \quad (4.11)$$

Equation (3.26) is then solved by putting $\tilde{\chi}_\alpha(k) \equiv -\frac{1}{2} \chi_\alpha(k) - V_{\{0, e_\alpha\}}^n$ to obtain the n -th order structure function as

$$S^n(k) = \sum_x V_{\{0, x\}}^n e^{ikx} = \frac{\sum_{\alpha} (1 - \cos k_\alpha) \tilde{\chi}_\alpha(k)}{\sum_{\alpha} (1 - \cos k_\alpha)} \quad (4.12)$$

Hence, if the system is isotropic, $\tilde{\chi}_\alpha(k) = \tilde{\chi}(k)$ for all $\alpha = 1, \dots, d$, then $S^n(k) = \tilde{\chi}(k)$ is analytic at $k = 0$, and the two points function $V_{\{0, x\}}^n$ decays to zero as $|x| \rightarrow \infty$ at least exponentially fast. We thus see that the locality of the correlation functions at order $n - 1$ in the expansion, is reproduced, at least for the two points functions, at order n , if we assume that the system has the full symmetry of the lattice. In particular this is true in $d = 1$. This is consistent with [10], where it is also the case that an isotropic perturbation of an infinite temperature dynamics still gives rise to short range spatial correlations in the stationary state. If however the $\chi_\alpha(k)$ are *not* equal, then $S^n(k)$ is not analytic at $k = 0$. So, even if at order $n - 1$ the correlations are strictly local, e.g. $n = 1$, anisotropic perturbations may lead to long range stationary correlations at order n . From (4.12), the two points function then decays as

$$V_{\{0, x\}}^n \simeq \sum_{\alpha} b_{\alpha}^n \frac{x_{\alpha}^2}{|x|^{d+2}}, \quad |x| \rightarrow \infty, \quad (4.13)$$

for some constants b_{α}^n .

Note that anisotropy is certainly not sufficient to produce this weak decay. It may for example happen that $\chi_\alpha(k) = -2 S^n(k) - 2 V_{\{0, e_\alpha\}}^n$. As an illustration, suppose that

$$Q_{\{0, x\}}^n = J(x - e_\alpha) - J(x) + J(e_\alpha) [\delta_{x, e_\alpha} - \delta_{0, x}], \quad (4.14)$$

with $J(x)$ strictly local, reflection symmetric, $J(0) = 0$, and having Fourier transform $\hat{J}(k)$. Then, $\chi_\alpha(k) = -4 \hat{J}(k) - 4 J(e_\alpha)$ and $V_{\{0,x\}}^n = 2 J(x)$, no matter whether $J(x)$ is isotropic or not. This in fact is the scenario for a detailed balance dynamics, see (A24). However, generic anisotropy does not satisfy such relations. The simplest example is the multiple temperature model of [9] which we announced in (2.14). Choose β_α close to zero, $\alpha = 1, \dots, d$, and let the exchange rates be given by

$$c(x, x + e_\alpha, \sigma) \equiv \frac{1}{2} \left(1 - \frac{1}{2} \beta_\alpha [H(\sigma^{x, x+e_\alpha}) - H(\sigma)] \right). \tag{4.15}$$

The « energy »

$$H(\sigma) \equiv - \sum_{x,y} J(x-y) \sigma(x) \sigma(y) \tag{4.16}$$

is parametrized by the local functions $J(x)$ as in (4.14). If all $\beta_\alpha = \beta$, then (4.16) is the expansion of the detailed balance exchange rates (2.5), (2.14) or (A1), up to first order in β . Now, (4.14) has to be replaced for $n = 1$ by

$$Q_{\{0,x\}}^1 = q_{\{0,x\}}(0, e_\alpha) = \beta_\alpha [J(x - e_\alpha) - J(x)] + \beta_\alpha J(e_\alpha) (\delta_{x, e_\alpha} - \delta_{x, 0}). \tag{4.17}$$

As a consequence, $\chi_\alpha(k) = -4 \beta_\alpha [\hat{J}(k) + J(e_\alpha)]$ and correspondingly $\tilde{\chi}_\alpha(k)$ in (4.12) must depend on α whenever the β_α are not all equal. In $d = 2$ with nearest neighbor coupling

$$J(x) = K_1 (\delta_{x, e_1} + \delta_{x, -e_1}) + K_2 (\delta_{x, e_2} + \delta_{x, -e_2}), \tag{4.18}$$

we find, after doing the integrals (4.9)-(4.11),

$$\begin{aligned} V_{\{0, e_1\}} &= (\beta_1 + \beta_2) K_1 + \frac{\beta_1 - \beta_2}{\pi - 2} \left(\left(4\pi - 18 + \frac{16}{\pi} \right) K_1 + \right. \\ &\quad \left. + \left(-2\pi + 12 - \frac{16}{\pi} \right) K_2 \right) \equiv V_1 \\ V_{\{0, e_2\}}^1 &= (\beta_1 + \beta_2) K_2 + \frac{\beta_1 - \beta_2}{\pi - 2} \left(\left(2\pi - 12 + \frac{16}{\pi} \right) K_1 + \right. \\ &\quad \left. + \left(-4\pi + 18 - \frac{16}{\pi} \right) K_2 \right) \equiv V_2. \end{aligned} \tag{4.19}$$

Therefore, the

$$\tilde{\chi}_\alpha(k) = 4 \beta_\alpha [K_1 \cos k_1 + K_2 \cos k_2] + 2 \beta_\alpha K_\alpha - V_\alpha, \quad \alpha = 1, 2 \tag{4.20}$$

are explicitly determined. Inverse Fourier transforming (4.12) gives

$$\begin{aligned} V_{\{0,x\}}^1 &= 2 \beta_2 J(x) + \sum_{\alpha=1,2} \left(\frac{\beta_\alpha K_\alpha}{2} - \frac{V_\alpha}{4} \right) \Delta_\alpha I(x) + \\ &\quad + \frac{\beta_1 - \beta_2}{2} [K_1 \Delta_1 (I(x - e_1) + I(x + e_1)) + K_2 \Delta_1 (I(x - e_2) + I(x + e_2))] \end{aligned} \tag{4.21}$$

where

$$I(x) \equiv \frac{1}{(2\pi)^2} \int dk \frac{1 - \cos k \cdot x}{1 - (1/2)[\cos k_1 + \cos k_2]} \tag{4.22}$$

is the potential kernel for simple random walk on the plane. We immediately verify from (4.21) that if $\beta_1 = \beta_2 = \beta$, then $V_{\{0,x\}}^1 = 2\beta J(x)$. If $\beta_1 \neq \beta_2$, then the asymptotic behavior can be derived from

$$\Delta_1 I(x) = -\Delta_2 I(x) \approx -\frac{2}{\pi} \frac{[x_1^2 - x_2^2]}{(x_1^2 + x_2^2)^2}, \quad |x| \rightarrow \infty. \quad (4.23)$$

Hence, as $|x| \rightarrow \infty$,

$$V_{\{0,x\}}^1 \approx -\frac{1}{2\pi} \frac{(\tilde{\chi}_1(0) - \tilde{\chi}_2(0))(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} \quad (4.24)$$

and

$$\tilde{\chi}_1(0) - \tilde{\chi}_2(0) = \frac{3\pi - 4}{\pi - 2} (\beta_1 - \beta_2)(K_1 + K_2) \quad (4.25)$$

(4.24) is the typical quadrupole field-like decay predicted in (4.14). Note that if $K_1 = -K_2 \equiv K$, then (4.25) is zero but the two points function still decays as a power: in this case,

$$\begin{aligned} V_{\{0,x\}}^1 &= 2\beta_2 K (\delta_{x,e_1} + \delta_{x,-e_1} - \delta_{x,e_2} - \delta_{x,-e_2}) \\ &\quad + (\beta_1 - \beta_2) K \Delta_1^2 I(x) - 2(\beta_1 - \beta_2) K \Delta_1 \delta_{x,0} \\ &\approx -\frac{12}{\pi} (\beta_1 - \beta_2) K \frac{x_1^4 + x_2^4 - 6x_1^2 x_2^2}{(x_1^2 + x_2^2)^4}, \quad |x| \rightarrow \infty, \end{aligned} \quad (4.26)$$

has the long distance behavior of an octopole field. We thus get always long range correlations for this two temperature model provided that $\beta_1 \neq \beta_2$.

Summarizing, we find that the correct induction hypotheses on the $V_{\Delta x}^{n-1}$, $n \geq 1$ must be that their decay, as $|x| \rightarrow \infty$, is not slower than described in (4.14). $\tilde{\chi}_\alpha(k)$ is no longer necessarily analytic at $k=0$ but the derivation of (4.13) remains meaningful since the corresponding $\tilde{\chi}_\alpha(k)$ are then still bounded near $k=0$. Generic anisotropy then gives rise to the power law decay (4.14) also for the next order in the expansion.

Acknowledgement.

We are indebted to Jan Naudts for a careful reading of the manuscript, and for useful criticism.

Appendix.

Derivation of the high temperature expansion *via* Kawasaki dynamics.

We choose the rates $c(x, y, \sigma)$ of the form

$$c(x, y, \sigma) = \frac{1}{2} \Phi(\beta(H(\sigma^{xy}) - H(\sigma))) \quad (A1)$$

with

$$\Phi(z) = e^{-z/2} \Psi(z), \quad \Psi(z) = \Psi(-z). \quad (A2)$$

We can assume that $\Psi(0) = 1$ and that $\Psi(z)$ is analytic around $z = 0$;

$$\Psi(z) = \sum_{\ell=0, \text{even}} b_{\ell} z^{\ell} \tag{A3}$$

Other cases, as for the Metropolis rates where $\Psi(z) = e^{-|z|^{1/2}}$, can be treated *via* uniform approximation.

We then have that the derivatives of $\Phi(z)$ at $z = 0$ satisfy the following relations : if $\Phi^{(k)}(0) \equiv a_k$, then

$$(-1)^n a_n = \sum_{k=0}^n a_{n-k} \binom{n}{k} \tag{A4}$$

and

$$(-1)^n \sum_{k=0}^m a_{n-k} \binom{m}{k} = \sum_{k=0}^{n-m} a_{n-k} \binom{n-m}{k}, \quad 0 \leq m \leq n \tag{A5}$$

(A4) is an immediate consequence of $\Phi(z) = e^{-z} \Phi(-z)$. As for (A5), we note that from (A2) and (A3)

$$a_k = \sum_{\ell=0, \text{even}} b_{\ell} \frac{d^{\ell}}{dz^{\ell}} [(z^k)]_{z=-1/2} \tag{A6}$$

so that (A5) is equivalent to

$$(-1)^n \frac{d^{\ell}}{dz^{\ell}} \left[\sum_{k=0}^m z^{n-k} \binom{m}{k} \right]_{z=-1/2} = \frac{d^{\ell}}{dz^{\ell}} \left[\sum_{k=0}^{n-m} z^{n-k} \binom{n-m}{k} \right]_{z=-1/2} \tag{A7}$$

for all ℓ even and $0 \leq m \leq n$. But this is obvious because the function

$$\xi(z) \equiv (-1)^n \left(z - \frac{1}{2}\right)^{n-m} \left(z + \frac{1}{2}\right)^m - \left(z - \frac{1}{2}\right)^m \left(z + \frac{1}{2}\right)^{n-m} \tag{A8}$$

is odd.

Since we want to obtain an expansion in β , we have to separate the different orders in β as they appear in the rates and define

$$q_A^{\ell}(x, y) \equiv \frac{\beta^{\ell}}{\ell!} a_{\ell} \sum_{A_1 \Delta \dots \Delta A_{\ell} = A} \prod_{r=1}^{\ell} (J_{A_r} - J_{A_r^{xy}}) \tag{A9}$$

$$q_{BC}^{\ell} \equiv \frac{1}{2} \sum_{\langle xy \rangle} (q_{B^{xy} \Delta C}^{\ell}(x, y) - q_{B \Delta C}^{\ell}(x, y)), \quad \ell = 1, 2, \dots \tag{A10}$$

so that

$$c(x, y, \sigma) = \frac{1}{2} \left[1 + \sum_{\ell=1}^{\infty} \sum_A q_A^{\ell}(x, y) \sigma_A \right]. \tag{A11}$$

The expansion (3.19) now has the form

$$\langle \sigma_A \rangle_{\infty} = \sum_{n=0}^{\infty} V_A^n \tag{A12}$$

with

$$V_A^n \equiv \sum_{k=0}^{n-1} \sum_{B, C} G(A, B) q_{BC}^{n-k} V_C^k, \quad n \geq 1 \tag{A13}$$

and $V_A^0 = \delta_{A\emptyset}$.

We start by computing the first order term

$$\begin{aligned}
 V_A^1 &= \sum_B G(A, B) q_{B\emptyset}^1 \\
 &= \frac{\beta}{2} \sum_B G(A, B) \sum_{\langle \lambda, \nu \rangle} (J_B - J_{B^{\nu\lambda}})
 \end{aligned}
 \tag{A14}$$

so that by (3.4) $\hat{L}_0 V_A^1 = \hat{L}_0 J_A$. Using the boundary condition $V_A^1 \rightarrow 0$ as $\text{diam}(A) \rightarrow \infty$, we conclude that $V_A^1 = J_A$. We will repeatedly use this argument in the derivation of the following.

PROPOSITION 3

$$V_A^n = \frac{\beta^n}{n!} \sum_{A_1 \Delta \dots \Delta A_n = A} \prod_{r=1}^n J_{A_r}.
 \tag{A15}$$

It is clear that we thus get a converging high temperature expansion for β small, with exponential decay of spin-spin correlations.

Proof: By induction, assume that

$$V_A^k = \frac{\beta^k}{k!} \sum_{A_1 \Delta \dots \Delta A_k = A} \prod_{r=1}^k J_{A_r}, \quad 1 \leq k \leq n-1.
 \tag{A16}$$

$k = 1$ was computed above. A straightforward calculation gives

$$\begin{aligned}
 \sum_{B, C} G(A, B) q_{BC}^{n-k} V_C^k &= \frac{\beta^n}{2} a_{n-k} \sum_{A_1, \dots, A_n} \prod_{r=1}^n J_{A_r} \sum_{\langle \lambda, \nu \rangle} \sum_{\ell=k}^n \frac{(-1)^{n-\ell}}{\ell!(n-\ell)!} \binom{\ell}{k} \\
 &\quad \times (g_{A, \ell}^{xy}(A_1, \dots, A_n) - g_{A, n-\ell}^{xy}(A_1, \dots, A_n)),
 \end{aligned}
 \tag{A17}$$

where, for $m = 0, \dots, n$,

$$g_{A, m}^{xy}(A_1, \dots, A_n) \equiv G(A, A_1^{xy} \Delta \dots \Delta A_m^{xy} \Delta A_{m+1} \Delta \dots \Delta A_n).
 \tag{A18}$$

Hence,

$$\begin{aligned}
 V_A^n &= \frac{\beta^n}{2} a_n \frac{(-1)^n}{n!} \sum_{A_1, \dots, A_n} \prod_{r=1}^n J_{A_r} \sum_{\langle xy \rangle} (g_{A, 0}^{xy}(A_1, \dots, A_n) - g_{A, n}^{xy}(A_1, \dots, A_n)) \\
 &+ \frac{1}{2} \sum_{k=0}^{n-1} \frac{\beta^n}{n!} a_{n-k} \binom{n}{k} \sum_{A_1, \dots, A_n} \prod_{r=1}^n J_{A_r} \sum_{\langle \lambda, \nu \rangle} (g_{A, n}^{xy}(A_1, \dots, A_n) - g_{A, 0}^{xy}(A_1, \dots, A_n)) \\
 &\quad + \frac{\beta^n}{2 n!} \sum_{A_1, \dots, A_n} \prod_{r=1}^n J_{A_r} \sum_{m=1}^{n-1} \frac{(-1)^m}{m!(n-m)!} \sum_{\langle xy \rangle} g_{A, m}^{xy}(A_1, \dots, A_n) \\
 &\quad \times \left((-1)^n \sum_{k=0}^m \binom{m}{k} a_{n-k} - \sum_{k=0}^{n-m} \binom{n-m}{k} a_{n-k} \right)
 \end{aligned}
 \tag{A19}$$

and we use (A4) and (A5) to conclude that

$$\begin{aligned}
 V_A^n &= \frac{\beta^n}{n!} (-1)^n a_n \sum_{A_1 \Delta \dots \Delta A_n = A} \prod_{r=1}^n J_{A_r} + \frac{\beta^n}{n!} [1 - (-1)^n a_n] \sum_{A_1 \Delta \dots \Delta A_n = A} \prod_{r=1}^n J_{A_r} \\
 &= \frac{\beta^n}{n!} \sum_{A_1 \Delta \dots \Delta A_n = A} \prod_{r=1}^n J_{A_r}.
 \end{aligned}
 \tag{A20}$$

The local character of each of the terms V_A^n can be understood directly from the detailed balance condition. It implies that for each bond $\langle xy \rangle$ separately, we must have that

$$\langle c(x, y, \sigma)(\sigma_{A^{xy}} - \sigma_A) \rangle_\infty = 0 \tag{A21}$$

or, using (A11), for each order $n \geq 1$,

$$\sum_{k=0}^{n-1} \sum_B (q_{A^{xy} \Delta B}^{n-k}(x, y) - q_{A \Delta B}^{n-k}(x, y)) V_B^k = K_A - K_{A^{xy}}. \tag{A22}$$

for some $\{K_A\}$. Assume inductively that $V_B^k = 0$ whenever the diameter of the set B is larger than kR , $0 \leq n - 1$, as we know is certainly the case for $k = 0, 1$. Then (A.22) has the unique solution $K_A = V_A^n$ by imposing the boundary condition $K_A \rightarrow 0$ as $\text{diam}(A) \rightarrow \infty$. (A22) implies that $V_A^n = 0$ whenever $\text{diam}(A) > nR$. In fact, put

$$Q_A^n(x, y) \equiv \sum_{k=0}^{n-1} \sum_B q_{A \Delta B}^{n-k}(x, y) V_B^k, \tag{A23}$$

so that

$$Q_{A^{xy}}^n(x, y) - Q_A^n(x, y) = V_A^n - V_{A^{xy}}^n, \tag{3.48}$$

and let $(x(1), \dots, x(r))$ be a sequence of nearest neighbor sites $\langle x(s) x(s+1) \rangle$, $s = 1, \dots, r - 1$, such that $x(1) \in A$, $x(s) \notin A$ for $s = 2, \dots, r - 1$, and $\text{diam}(A^{x(1) x(r)}) > nR$. Then

$$\begin{aligned} V_A^n &= V_A^n - V_{A^{x(1) x(2)}}^n + V_{A^{x(1) x(2)}}^n - V_{A^{x(1) x(3)}}^n + \dots + V_{A^{x(1) x(r-1)}}^n - V_{A^{x(1) x(r)}}^n \\ &= Q_{A^{x(1) x(2)}}^n(x(1), x(2)) - Q_A^n(x(1), x(2)) + Q_{A^{x(1) x(3)}}^n(x(1), x(3)) - Q_{A^{x(1) x(2)}}^n \\ &\quad + \dots + Q_{A^{x(1) x(r)}}^n(x(r-1), x(r)) - Q_{A^{x(1) x(r-1)}}^n(x(r-1), x(r)) \end{aligned} \tag{A24}$$

gives the solution of (A22).

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