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► **To cite this version:**

Quoc Son Nguyen. Mechanical modelling of anelasticity. *Revue de Physique Appliquée*, 1988, 23 (4), pp.325-330. 10.1051/rphysap:01988002304032500 . jpa-00245777

HAL Id: jpa-00245777

<https://hal.science/jpa-00245777>

Submitted on 4 Feb 2008

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Mechanical modelling of anelasticity

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(Reçu le 26 mai 1987, accepté le 15 janvier 1988)

RESUME - On présente une étude de synthèse de l'approche du mécanicien dans la modélisation mécanique de la plasticité afin d'illustrer les concepts et les méthodes fondamentaux de la description macroscopique des milieux continus. Cette approche possède des avantages incontestables concernant ses caractères systématiques et opérationnels. En plasticité classique, la donnée des deux potentiels de l'énergie libre et du pseudo-potentiel de dissipation conduit aux modèles des matériaux standards généralisés. Les modèles usuels de plasticité parfaite ou d'écroutissage isotrope et cinématique entrent dans cette description. Cette étude est illustrée par une description de monocristal et par une analyse de bifurcation et de stabilité. La technique de macro-homogénéisation est décrite en détail.

ABSTRACT - A review of the mechanical modelling of plasticity is given in order to illustrate the preceding concepts and preceding methods of the mechanician in the macroscopic approach of continuous continua. This approach presents uncontestable advantages concerning its systematical and operational characteristics. In classical plasticity, the expressions of the free energy density and of the pseudo-potential of dissipation lead to generalized standard models of plasticity. Usual models of perfect plasticity or of isotropic and kinematic hardening can be described in this unified presentation and are involved with internal parameters which are plastic strains, plastic path length or plastic works. The analysis is illustrated by a description of single crystals and by an analyse of bifurcation and stability in quasi-static evolution. The technique of macro-homogenization is underlined.

1. INTRODUCTION

The objective of this communication is to give a review of the mechanical modelling of plasticity. This modelling illustrates the macroscopic phenomenological approach of anelasticity in relation with thermodynamical considerations as it has been sketched out in the previous paper by P. Germain.

2. MODELLING OF METAL PLASTICITY

The mechanical modelling of plasticity is an old problem in Solid Mechanics. Basic ideas of plasticity as a feasible description of the behaviour of common metals were introduced very early on, almost at the same time as linear elasticity. But their development as a satisfactory mathematical theory only began with the fundamental works of Melan (1936), Prager (1937), Mandel (1942), Hill (1950), Drucker (1964), Koiter (1960), etc... Nowadays, this description is widely accepted and successfully applied in the resolution of practical engineering structural problems, in particular in relation with numerical analysis by finite element discretization.

In the context of small strain, let us recall first some of its basic elements. The history dependence $\sigma = H \left\{ \frac{\sigma}{\sigma_0} \right\}$ of stress vs strain is condensed

via the present value of strain and of a set of internal parameters α which represents the plastic strain and eventually other material parameters β , $\alpha = (\epsilon^p, \beta)$. The variation of α corresponds to irreversible evolution of the material. Principal governing equations are :

- Stress-elastic strain relation :

$$(1) \begin{cases} \epsilon = \epsilon^e + \epsilon^p \\ \sigma = L \cdot \epsilon^e = L (\epsilon - \epsilon^p) \end{cases}$$

- Plastic criterion :

$$(2) \begin{cases} f(\sigma, \alpha) \leq 0 \text{ defines admissible states of stress} \\ \text{If } f < 0, \text{ the behaviour is purely elastic.} \end{cases}$$

- Evolution law :

Internal parameters $\alpha = (\epsilon^p, \beta)$ follow a time-independent incremental law :

$$(3) \begin{cases} \dot{\epsilon}^p = \lambda n(\sigma, \alpha) \\ \dot{\beta} = \lambda \ell(\sigma, \alpha) \end{cases}$$

where λ denotes the plastic multiplication which is such that $\lambda \geq 0$ and $\lambda f = 0$.

Equations (1), (2), (3) give completely the stress-strain behaviour as an incremental law $\dot{\sigma} = \dot{\sigma}(\dot{\epsilon})$, i.e. a hypoelastic behaviour.

Standard models of plasticity is obtained if $n(\sigma, \alpha) = \frac{\partial f}{\partial \sigma}$, i.e. if the evolution of the plastic strain is the normality law. In this case, the criterion function f is also called the plastic potential and the preceding incremental relations $\dot{\sigma}(\dot{\epsilon})$ can be explicitly written as :

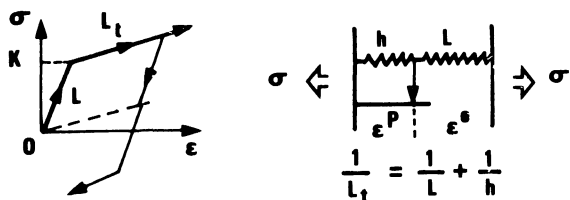
$$(4) \left\{ \begin{aligned} \dot{\epsilon}^p &= \lambda \frac{\partial f}{\partial \sigma} \\ \lambda &= \frac{1}{h + \frac{\partial f}{\partial \sigma} \cdot L \cdot \frac{\partial f}{\partial \sigma}} \left\langle \frac{\partial f}{\partial \sigma} \cdot L \cdot \dot{\epsilon} \right\rangle, h = - \frac{\partial f}{\partial \alpha} \cdot \ell \\ \dot{\sigma} &= \frac{\partial U}{\partial \dot{\epsilon}}(\dot{\epsilon}) \\ \text{with } U(\dot{\epsilon}) &= \frac{1}{2} \dot{\epsilon} \cdot L \cdot \dot{\epsilon} - \frac{1}{2} \frac{\left\langle \frac{\partial f}{\partial \sigma} \cdot L \cdot \dot{\epsilon} \right\rangle^2}{h + \frac{\partial f}{\partial \sigma} \cdot L \cdot \frac{\partial f}{\partial \sigma}} \end{aligned} \right.$$

in the plastic region $f(\sigma, \alpha) = 0$.

The extension to the case of multiple plastic potential has been introduced by Mandel (1965). If the plastic criterion is given by n inequality $f^i(\sigma, \alpha) \leq 0$, $i=1, n$ then the associated evolution law must be written as

$$(5) \left\{ \begin{aligned} \dot{\epsilon}^p &= \lambda_i \frac{\partial f^i}{\partial \sigma} \quad \text{where } \lambda_i \geq 0, f^i \leq 0 \\ \dot{\beta} &= \lambda_i \ell^i \quad \lambda_i f^i = 0 \end{aligned} \right.$$

Elastic plastic equations can be illustrated by simple examples. The simplest one corresponds to classical rheological models of springs and slides. The following model :



shows clearly the significance of hardening modulus h and represents an unidimensional representation of the well-known Ziegler-Prager's model of kinematic hardening. Here, the internal parameter reduces to the plastic strain ϵ^p and the plastic criterion is written as :

$$f(\sigma, \epsilon) \equiv | \sigma - h \cdot \epsilon^p | - k^2 \leq 0$$

3. STANDARD MODELS AND THERMODYNAMIC CONSIDERATIONS

However, the study of rheological models and of usual models of plasticity shows that, in fact, the incremental relations (4) are intimately related to an energetic description since in these models the notion of energy and dissipation are extremely clear. For example, Ziegler-Prager's model is related to an reversible energy :

$$W = \frac{1}{2} (\epsilon - \epsilon^p) \cdot L \cdot (\epsilon - \epsilon^p) + \frac{1}{2} \epsilon^p \cdot h \cdot \epsilon^p$$

and the associated dissipation is $D = (\sigma - h \cdot \epsilon^p) \cdot \dot{\epsilon}^p$.

Energetic considerations can be best studied in a classical thermodynamic framework as shown by Germain [1] and give rise to a general description of anelastic behaviours of materials. The preceding elastic plastic relations correspond to a particular case of the following thermodynamic description based upon the two potentials : thermodynamic potential and pseudo-potential of dissipation.

More precisely, in this framework, the material behaviour can be described by state variables (ϵ, α) with an associated free energy density $W(\epsilon, \alpha)$. If irreversible stress is assumed to be excluded, the associated forces are :

$$(6) \quad \sigma = \frac{\partial W}{\partial \epsilon} \quad ; \quad A = - \frac{\partial W}{\partial \alpha}$$

in isothermal process and the dissipation is

$$(7) \quad D = \sigma \dot{\epsilon} - \dot{W} = A \dot{\alpha}$$

If a criterion is assumed concerning physically admissible forces $f(A) \leq 0$, the normality law is again introduced : concerning the evolution of state variables α :

$$(8) \quad \dot{\alpha} = \lambda \frac{\partial f}{\partial A}, \quad \lambda \geq 0, f \leq 0, \lambda f = 0$$

In general, the set of admissible forces depends on the present state (ϵ, α) and one should write correctly $f(A) = f[A; \epsilon, \alpha]$.

This modelization furnishes a general description of a class of time independent anelastic behaviour of materials such as plasticity, brittle damage and brittle fracture. The reader may refer to [2] for a more detailed presentation of the covered subjects.

Most often, when there is no mechanical or physical configuration change, the working assumption of state independence of the criterion can be introduced. The obtained description corresponds then to the generalized standard models (G.S.M.) [3] which are characterized by the dependence of f on generalized force A alone.

It is important to note that state variable (ϵ, α) can be of physical or mechanical nature. For example, ϵ and ϵ^p are mechanical variables since they are not directly related to the physical state of the material, while $\epsilon^e = \epsilon - \epsilon^p$ can be considered as a physical variable.

The G.S.M. models of plasticity [3] correspond to the particular cases with $\alpha = (\epsilon^p, \beta)$, $W(\epsilon, \alpha) = \frac{1}{2} (\epsilon - \epsilon^p) \cdot L \cdot (\epsilon - \epsilon^p) + W^a(\epsilon^p, \beta)$. In the expression of energy one can separate the elastic part W^e due to elastic strain from the anelastic part W^a due to different microscopic contributions by residual stresses or internal structural changes, etc. Force relations are :

$$(9) \quad \sigma = \frac{\partial W}{\partial \epsilon}, \quad A^p = - \frac{\partial W}{\partial \epsilon^p}, \quad A^\beta = - \frac{\partial W}{\partial \beta}$$

The plastic criterion may be written as :

$$(10) \quad f(A^p, A^\beta) \leq 0$$

and the normality law as :

$$(11) \left\{ \begin{aligned} \dot{\epsilon}^p &= \lambda \frac{\partial f}{\partial A^p} & \lambda \geq 0, f \leq 0 \\ \dot{\beta} &= \lambda \frac{\partial f}{\partial A^\beta} & \lambda f = 0 \end{aligned} \right.$$

It is not difficult to verify that all rheological models composed of springs and slides are G.S.M.

The Ziegler-Prager's model of kinematic hardening is G.S.M. as well as all models of combined kinematic and isotropic hardening. A more interesting example is given by Mandel's description of single crystal [4] :

If N slip systems defined by the slip planes and slip directions is assumed and Γ_i denotes the amplitude of slip of the i-th mechanism, the Kinematic implies :

$$\epsilon_{ij}^p = C_{ij}^K \Gamma_K$$

while Schmid's law must be expressed as :

$$f^K \equiv C_{ij}^K \sigma_{ij} - g^K(\Gamma) - \tau_0 \leq 0$$

The evolution equations are :

$$\begin{cases} \dot{\Gamma}_K = \lambda_K \\ \dot{\epsilon}_{ij}^p = \lambda_K \frac{\partial f^K}{\partial \sigma_{ij}} \end{cases}$$

Mandel's model of single crystal is G.S.M. Indeed, state variables are $\alpha = (\epsilon^p, \Gamma)$ with :

$$W = \frac{1}{2}(\epsilon - \epsilon^p) \cdot L \cdot (\epsilon - \epsilon^p) + W^a(\Gamma)$$

where the anelastic energy $W^a(\Gamma)$ is obtained from α^K by the relation $g^i = - \partial W^a / \partial \Gamma_i$ when Mandel's assumption of symmetry of the interaction matrix $H_{ij} = \partial g^i / \partial \Gamma_j$ is satisfied, $H_{ij} = - \partial^2 W^a / \partial \Gamma_i \partial \Gamma_j$.

Generalized force is $A = (\sigma, g)$ and one obtains effectively :

$$\begin{cases} \dot{\epsilon}_{ij}^p = \lambda_K \frac{\partial f^K}{\partial \sigma_{ij}} = C_{ij}^K \dot{\Gamma}_K \\ \dot{\Gamma}_m = \lambda_K \frac{\partial f^K}{\partial g^m} = \lambda_m \end{cases}$$

4. MACRO HOMOGENIZATION

The macroscopic behaviour of a material must result from the underlying micromechanisms. In this section, it will be assumed that continuum approach is still applicable at the local scale and our purpose is to give a rigorous discussion on the resulting global behaviour when the local one and all the micromechanisms are assumed to be known. Such a discussion is useful in the study of polycrystalline aggregates as well as in the study of composites.

The first part is devoted to homogenization process of G.S.M. The principal obtained results correspond to the fact that the overall behaviour is also G.S.M. but involves an infinite number of internal parameters.

Let us assume that the local behaviour corresponds to G.S.M. If V denotes a representative volume element, at each material point y of V, the material is defined by constitutive equations (6), (7), (8).

It is useful to recall first that if $\langle a \rangle$ denotes the mean value of a physical quantity a,

$$(12) \quad \langle a \rangle = \frac{1}{V} \int_V a \, dV$$

then Hill's lemma is satisfied for any local stress and strain fields σ, ϵ such that :

$$(13) \quad \left\{ \begin{array}{l} \cdot \epsilon \text{ is kinematically admissible C.A.} \\ \text{i.e. : } \epsilon = (\nabla u) \text{ in } V \\ \cdot \sigma \text{ is statically admissible S.A.} \\ \text{i.e. : } \text{div } \sigma = 0 \text{ in } V \\ \cdot \sigma, \epsilon \text{ satisfy on } \partial V \text{ the localization condition (for example } \sigma, \epsilon \text{ periodic on } \partial V \text{ with respect to the cell geometry or } \sigma, \epsilon \text{ satisfy Hill's macro-homogeneity assumption).} \end{array} \right.$$

Hill's lemma is expressed by the condition :

$$(14) \quad \langle \epsilon \rangle \langle \sigma \rangle = \langle \sigma \cdot \epsilon \rangle$$

If Σ and E denote the global stress and strain, relations between Σ and E can be obtained via the resolution of the localization problem which can be written for periodic composites for example under the following form :

$$(15) \quad \left\{ \begin{array}{l} \underline{\alpha} \text{ is given in } V \\ \sigma \text{ and } \epsilon \text{ satisfy :} \\ \cdot \epsilon \text{ C.A. , } \langle \epsilon \rangle = E \\ \cdot \sigma \text{ S.A. , } \langle \sigma \rangle = \Sigma \\ \cdot \sigma = \frac{\partial W}{\partial \epsilon}(\epsilon, \alpha) \\ \cdot \sigma, \epsilon \text{ periodic on } \partial V \end{array} \right.$$

which is a purely elastic problem. If E is given then the resolution of (15) gives $\sigma = \sigma(E, \underline{\alpha})$, $\epsilon = \epsilon(E, \underline{\alpha})$ and thus $\Sigma = \langle \sigma(E, \underline{\alpha}) \rangle$.

Let us verify that the overall behaviour is effectively G.S.M. The global energy density is clearly

$$(16) \quad W = \langle w \rangle = W(E, \underline{\alpha})$$

One obtains :

$$(17) \quad \left\{ \begin{array}{l} W_{,E} = \langle w_{, \epsilon} \cdot \epsilon_{, E} \rangle = \langle \sigma \cdot \epsilon_{, E} \rangle \\ \quad = \langle \sigma \rangle \langle \epsilon_{, E} \rangle = \langle \sigma \rangle = \Sigma \end{array} \right.$$

since $\langle \epsilon_{, E} \rangle = 1$ by definition from (15).

Generalized force field A associated to the internal parameter field $\underline{\alpha}$ is by definition :

$$\left\{ \begin{array}{l} \underline{A} \cdot \delta \underline{\alpha} = - W_{, \alpha} \cdot \delta \alpha = - \langle w_{, \epsilon} \cdot \epsilon_{, \alpha} \cdot \delta \alpha \rangle + \\ \quad \langle - W_{, \alpha} \cdot \delta \alpha \rangle = \langle A \cdot \delta \alpha \rangle \end{array} \right.$$

since $\langle \sigma \cdot \epsilon_{, \alpha} \delta \alpha \rangle = 0$ from (15).

The overall dissipation D is :

$$D = \langle \sigma \dot{\epsilon} - \dot{w} \rangle = \langle A \dot{\alpha} \rangle = \underline{A} \cdot \dot{\underline{\alpha}}$$

Normality law is globally conserved in the sense that :

$$(18) \quad \left\{ \begin{array}{l} (\underline{A} - \underline{A}^*) \dot{\underline{\alpha}} = \langle (A - A^*) \dot{\alpha} \rangle \geq 0 \\ \forall A^* \quad f(A^*) \leq 0 \text{ in } V \end{array} \right.$$

The overall behaviour is thus given by state variables $(E, \underline{\alpha})$ where $\underline{\alpha}$ is the local field of internal parameters. Such a model is particularly complete because of the nature of $\underline{\alpha}$.

The second part of this section is devoted to the

special case of elastoplasticity. The assumption of linear elasticity enables us to perform a proper analysis as it has been done by Suquet [5].

To simplify, the local behaviour is assumed to be elastic-perfectly plastic (this is not a restriction, one can also, for example, assume Mandel's model of single crystal). Governing equations are then :

$$(19) \left\{ \begin{array}{l} \text{State variable } (\epsilon, \epsilon^P) \\ W = \frac{1}{2} (\epsilon - \epsilon^P) L (\epsilon - \epsilon^P) \quad \text{Energy} \\ \sigma = \frac{\partial W}{\partial \epsilon} = L (\epsilon - \epsilon^P) \quad \text{Forces} \\ A = - \frac{\partial W}{\partial \epsilon^P} = \sigma, \quad f(\sigma) \leq 0 \\ \dot{\epsilon}^P = \lambda \frac{\partial f}{\partial \sigma}, \quad \lambda \geq 0, \quad \lambda f = 0 \quad \text{Normality.} \end{array} \right.$$

In this case, the localization problem (17) can be explicitly written as :

$$(20) \left\{ \begin{array}{l} E, \epsilon^P \text{ are given} \\ \text{Find } \sigma, \epsilon \text{ such that} \\ \cdot \epsilon = (\nabla u) \\ \cdot \text{Div } \sigma = 0 \quad \text{in } V \\ \cdot \sigma = L (\epsilon - \epsilon^P) \\ \cdot \sigma, \epsilon \text{ periodic on } \partial V \\ \cdot \langle \sigma \rangle = \Sigma, \quad \langle \epsilon \rangle = E \end{array} \right.$$

This is a linear elastic problem with residual strain and appropriated boundary conditions. It is then interesting to introduce the following decompositions of stress and strain

- Strain deformation : $\epsilon = e + z$ with :

$$(21) \left\{ \begin{array}{l} \cdot e \quad \text{C.A.} \\ \cdot Le \quad \text{S.A.} \\ \cdot \langle e \rangle = E \end{array} \right.$$

$$(22) \left\{ \begin{array}{l} \cdot z \quad \text{C.A.} \\ \cdot L(z - \epsilon^P) \quad \text{S.A.} \\ \cdot \langle z \rangle = 0 \end{array} \right.$$

- Stress decomposition $\sigma = s + r$ with :

$$(23) \left\{ \begin{array}{l} \cdot L^{-1} s \quad \text{C.A.} \\ \cdot s \quad \text{S.A.} \\ \cdot \langle s \rangle = \Sigma \end{array} \right.$$

$$(24) \left\{ \begin{array}{l} \cdot L^{-1} r + \epsilon^P \quad \text{C.A.} \\ \cdot r \quad \text{S.A.} \\ \cdot \langle r \rangle = 0 \end{array} \right.$$

The significance of e and z follows from (21), (22). It is clear that $e = D \cdot E$ where D denotes the linear operation of strain concentration and $z = Z \cdot \epsilon^P$ where Z denotes the linear operator of strain incompatibility. From (23), (24), it is also clear that $s = C \cdot \Sigma$, where C denotes the linear operator of stress concentration and $r = R \cdot \epsilon^P$ where R denotes the linear operator of stress incompatibility. Note that C and D are classical operators in the study of inhomogeneous elastic aggregates and Z and R are also wellknown in the mathematical theory of dislocation. Operator Z

generates the strain response from a given incompatibility and operator R generates the stress response resulting from a given incompatibility.

As before, it is clear that the overall behaviour is G.S.M. with the state variables (E, ϵ^P) . Our objective here is to introduce the overall plastic strain E^P as a mechanical variable and to prove that the set (E, E^P, ϵ^P) can be also chosen as state variables to derive again a G.S.M. as overall behaviour.

The macro plastic strain E^P can be introduced in a natural way by elastic unloading, thus by definition :

$$(25) \quad \Sigma = \bar{L} (E - E^P),$$

where \bar{L} denotes the overall modulus. From the decomposition $\epsilon = e + z$, one obtains :

$$\epsilon = D E + Z \epsilon^P$$

$$\sigma = L D E + L (Z - I) \epsilon^P$$

$$\bar{\sigma} = \langle \sigma \rangle = \langle L D \rangle E + \langle L (Z - I) \rangle \epsilon^P, \text{ thus}$$

$$(26) \quad \bar{L} = \langle L D \rangle$$

$$\text{and } \langle L D \rangle E^P = - \langle L (Z - I) \rangle \epsilon^P.$$

From Hill's lemma $\langle \rho \rangle = \langle D^T \rho \rangle$ if ρ is S.A., thus the second member can also be written as $- \langle D^T L (Z - I) \epsilon^P \rangle = - \langle \bar{L} C^T (Z - I) \epsilon^P \rangle$ since $C \bar{L} = L D$. Finally :

$$E^P = \langle C^T \cdot \epsilon^P \rangle - \langle C^T Z \epsilon^P \rangle.$$

$\langle C^T Z \epsilon^P \rangle = 0$ because C^T is S.A. and $Z \epsilon^P$ C.A. and $\langle Z \epsilon^P \rangle = 0$. It follows that :

$$(27) \quad E^P = \langle C^T \cdot \epsilon^P \rangle,$$

this relation has been obtained by Mandel [4] since 1964.

Now let us start with state variables (E, E^P, ϵ^P) with energy density :

$$(28) \left\{ \begin{array}{l} W = \langle W \rangle = \frac{1}{2} (E - E^P) \cdot \bar{L} \cdot (E - E^P) \\ + \frac{1}{2} \langle \epsilon^P (Z - I) \cdot L \cdot (Z - I) \cdot \epsilon^P \rangle \\ - \frac{1}{2} \langle C^T \epsilon^P \rangle \cdot \bar{L} \cdot \langle C^T \epsilon^P \rangle \end{array} \right.$$

If $\delta E, \delta E^P, \delta \epsilon^P$ are arbitrary variations of the state variables, compatible with the constraint (27), then one obtains :

$$(29) \left\{ \begin{array}{l} \delta W = \Sigma \delta E - \Sigma \delta E^P + \langle \delta \epsilon^P (Z - I) \cdot L \cdot (Z - I) \cdot \epsilon^P \rangle \\ - \langle C^T \cdot \delta \epsilon^P \rangle \cdot \bar{L} \cdot \langle C^T \epsilon^P \rangle \\ = \Sigma \delta E - \Sigma \delta E^P - \langle \delta \epsilon^P \cdot L (Z - I) \epsilon^P \rangle \\ - \langle \delta \epsilon^P \cdot C \bar{L} E^P \rangle \\ \delta W = \Sigma \delta E - \Sigma \delta E^P - \langle r \delta \epsilon^P \rangle \end{array} \right.$$

$$\text{since } r = \sigma - s = L(D E + (Z - I) \epsilon^P) - C \Sigma \\ = C \bar{L} E^P + L (Z - I) \epsilon^P.$$

Relation (29) shows clearly that $\Sigma = - W_E$ and Σ and r are respectively the associated forces of E^P and ϵ^P , thus generalized force associated to ϵ^P is the residual stress field r .

The plastic condition is expressed by multiple

plastic potential $f(\sigma(y)) \leq 0$, $y \in V$ or, in function of generalized forces, $f(C, \Sigma + r)|_{y \in V} \leq 0$, $\forall y \in V$. Normality law $\dot{\epsilon}^P = \lambda \frac{\partial f}{\partial \sigma}$ leads to :

$$(30) \left\{ \begin{aligned} \dot{\epsilon}^P(y) &= \lambda(y) \frac{\partial f}{\partial r(y)} \\ \dot{E}^P &= \langle \lambda(y) \frac{\partial f}{\partial \Sigma} \Big|_y \rangle = \langle C^T \cdot \dot{\epsilon}^P(y) \rangle \end{aligned} \right.$$

which proves that the overall behaviour is effectively G.S.M.

Remark : The preceding result is well known in other contexts of Solid Mechanics and actually adopted for practical applications. For example, the constitutive equation of elastic plastic shells is described by a G.S.M. which can be derived from a reduction of the three-dimensional problem to a bidimensional one [7], [8].

If γ, χ denotes respectively the plane extension and curvature tensor, state variables for a shell element are $(\gamma, \chi, \gamma^P, \chi^P, \epsilon^P(z))$, $z \in [-h/2, h/2]$ with energy $W = W^e(\gamma - \gamma^P, \chi - \chi^P) + W^a(\epsilon^P)$. Generalized forces associated to $\gamma^P, \chi^P, \epsilon^P$ are respectively N, M, r the in-plane force, moment tensor and residual stress distribution along the thickness of the shell element.

5. SOME GENERAL RESULTS ON SYSTEM BEHAVIOUR

In the preceding analysis, a cell element is in fact a structure in the sense of engineering structures and it may be then interesting to recall here some general results concerning the behaviour of a solid undergoing quasistatic transformation in response to a given loading path. The constitutive equations are assumed to be elastic plastic with energy $W(\epsilon, \alpha)$, forces $\sigma = \partial W / \partial \epsilon$, $A = - \partial W / \partial \alpha$, plastic criterion $f^i(A, \epsilon, \alpha) \leq 0$, $i = 1, N$ and normality law. The quasi-static evolution under a prescribed loading path of this solid has been discussed in the early works of Melan (1935), Prager (1937), Greenberg (1949), Hill (1950), Kofter (1960)... at least in small transformation. Its extension to finite strain has been introduced by Hill, [8] Halphen, [9], etc...

The analysis of the quasi-static response is based essentially on the formulation of the rate problem which gives the incremental response with respect to a load increment when the present state is assumed to be known.

To simplify the presentation, only surface forces F are prescribed on the boundary S of the solid Ω . Equilibrium equations and plastic equations

$$\left\{ \begin{aligned} \int_{\Omega} \sigma \delta \epsilon \, d\Omega &= \int_S F \cdot \delta u \, dS \\ \lambda_i f^i &= 0, \lambda_i \geq 0, f^i \leq 0 \end{aligned} \right.$$

after time differentiation, lead to :

$$(31) \left\{ \begin{aligned} \int_{\Omega} \delta \epsilon (W_{,\epsilon\epsilon} \dot{\epsilon} + W_{,\epsilon\alpha} f^i_{,A} \lambda_i) &= \int_S \dot{F} \delta u \, dS \\ \forall x \in \Omega \text{ if } f^i_x &= 0 \text{ and } \lambda_i(x) > 0 \text{ then} \\ \dot{f}^i &= -f^i_{,A} W_{,\alpha\epsilon} \dot{\epsilon} - f^i_{,A} W_{,\alpha\alpha} f^j_{,A} \lambda_j \\ &+ f^i_{,\epsilon} \dot{\epsilon} + f^i_{,\alpha} f^j_{,A} \lambda_j = 0 \end{aligned} \right.$$

Equations (31) can also be written under the form of a variational inequality which is :

- unsymmetric if $f^i_{,\epsilon} \neq 0$ or if $f^i_{,\alpha} f^j_{,A} \neq f^j_{,A} f^i_{,\alpha}$;
- symmetric if $f^i_{,\epsilon} = 0$ and if $f^i_{,\alpha} f^j_{,A} = f^j_{,A} f^i_{,\alpha}$.

For example, these conditions are fulfilled in the G.S.M. description.

The fact that equations (31) can be associated with a symmetric variational inequality enables us to derive an equivalent formulation of the rate problem as the stationnarity of a rate functional. For G.S.M., the associated variational inequality can be expliciteley written as :

$$(32) \left\{ \begin{aligned} \delta u \cdot (E_{,uu} \dot{u} + E_{,u\alpha} \dot{\alpha} + E_{,uF} \dot{F}) &= 0 \\ (\delta \alpha - \dot{\alpha}) \cdot (E_{,\alpha u} \dot{u} + E_{,\alpha\alpha} \dot{\alpha} + E_{,\alpha F} \dot{F}) &\geq 0 \end{aligned} \right.$$

$$(33) \left\{ \begin{aligned} \forall \delta \alpha \in N &= \left\{ \delta \alpha \mid \delta \alpha = \delta \lambda_i \frac{\partial f^i}{\partial A} \right. \\ \forall \delta u \in V &\quad \left. \delta \lambda_i \geq 0, f^i \leq 0, \delta \lambda_i f^i = 0 \right\}, \end{aligned} \right.$$

where E denotes the total potential energy of the system :

$$(34) E(u, \alpha, F) = \int_{\Omega} W(\epsilon, \alpha) \, d\Omega - \int_S F \cdot u \, dS,$$

and N the admissible rate of α , V the admissible rate of u .

The associated rate functional $U(\dot{u}, \dot{\alpha})$ is :

$$(35) \left\{ \begin{aligned} U(\dot{u}, \dot{\alpha}) &= \frac{1}{2} (\dot{u}, \dot{\alpha}) \begin{bmatrix} E_{,uu} & E_{,u\alpha} \\ E_{,\alpha u} & E_{,\alpha\alpha} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\alpha} \end{bmatrix} \\ &+ (\dot{u}, \dot{\alpha}) \begin{bmatrix} E_{,uF} \\ E_{,\alpha F} \end{bmatrix} \dot{F} \end{aligned} \right.$$

As it has been shown out by Hill [8], the description of the rate problem furnishes interesting results concerning global behaviour such as the stability of the present state and the possibility of bifurcation of the response from a trivial one. In the G.S.M. formalism, these statements depend essentially on the positivity of the second derivative of energy :

$$\begin{bmatrix} E_{,uu} & E_{,u\alpha} \\ E_{,\alpha u} & E_{,\alpha\alpha} \end{bmatrix} = E''.$$

Namely, the stability of the present state can be characterized by the positivity of E'' on the set $V \times N$.

On the other hand, the positivity of E'' on the set $V \times \bar{N}$ where \bar{N} denotes the vectorial space generated by N characterizes the uniqueness of the rate response and ensures no possible bifurcation of the response from a trivial one. The reader may also refer to [10] for a more detailed discussion on stability and bifurcation.

6. PHYSICAL INTERPRETATION-PRINCIPAL DIFFICULTY

Research on the physical basis of the introduced models has been considered since the early days of Plasticity. If the underlying mechanisms are now well

understood, a quantified description to obtain from microscopic physical mechanisms the nature of internal parameters α and the foundation of the macroscopic plastic criterion still remains an open problem. Knowledge obtained in Physics of Solid in the domain of plastic deformation of single crystal cannot, at the present time, be simply transcribed to obtain a simple and operational modelling of polycrystal.

In fact, macro-homogenization technique as shown in paragraph 4, gives theoretically the answer to obtain the overall behaviour. Its complexity is the major difficulty to be effectively adopted in the resolution of engineering problem. It is necessary to introduced some approximations, for example the self consistent models [11] may be used in certain situations.

However, it is clear that the progress obtained in the description of single crystal at finite strain, cf. Asaro [12] for example, furnishes principal results in the mechanical description of finite strain (Mandel, [13] ; Stolz, [14]) and suggests some macroscopic models to be developed for polycrystal aggregates.

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