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Exact results for a two-dimensional one-component plasma on a sphere

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Résumé. — Pour étudier le modèle OCP du plasma classique à deux dimensions, nous proposons de placer les particules à la surface d'une sphère. Pour une valeur particulière de la température T_0 la fonction de partition ainsi que les fonctions de distribution peuvent être calculées explicitement. A cette température la limite thermodynamique de ce système coïncide avec celle du système plan.

Abstract. — In order to study the properties of the two-dimensional classical one-component plasma, we propose to confine the particles to the surface of a sphere. For a special value T_0 of the temperature, the partition and the distribution function can be calculated explicitly. The thermodynamic limit of this system coincides, for $T = T_0$, with the one obtained previously for a planar system.

In two recent articles Jancovici and Alastuey (JA) [1, 2] studied the equilibrium statistical mechanics of a classical two-dimensional one-component plasma (OCP) using methods of random matrices [3, 4]. For some special value T_0 of the temperature they calculated the free energy and the n -body distribution functions exactly.

In this letter we consider the case of a system of N particles confined to the surface of a sphere. We show that for the temperature T_0 this system has the same thermodynamic limit as the one considered by (JA).

The N particles of mass m and charge q are confined to the surface of a sphere centred in O and of radius R . The number density is

$$\rho = \frac{N}{4\pi R^2}. \quad (1)$$

The interaction potential between particles i and j located at M_i and M_j on the sphere is

$$v_{ij} = -q^2 \text{Log} \left(\frac{r_{ij}}{L} \right) \quad (2)$$

where L is a length scale and r_{ij} is the length of the chord joining M_i and M_j i.e.

$$r_{ij} = 2R \sin \left(\frac{\psi_{ij}}{2} \right) \quad (3)$$

with

$$\psi_{ij} = \text{Arc cos} \left(\frac{\mathbf{OM}_i \cdot \mathbf{OM}_j}{R^2} \right). \quad (4)$$

The particles are embedded in a uniform neutralizing background of opposite charge.

The total potential energy, taking into account the influence of the background, is

$$V_N(1, 2, \dots, N) = -\frac{q^2}{2} \sum_{i=1}^N \sum_{j>i}^N \text{Log} \frac{2R^2}{L^2} \times \\ \times (1 - \cos \psi_{ij}) - \frac{N^2 q^2}{4} \left(1 - \text{Log} \frac{4R^2}{L^2} \right). \quad (5)$$

1. **Excess free energy.** — For the temperature $T_0 = q^2/2k_B$ (k_B Boltzmann constant) the excess canonical partition function is

$$Z_N(T_0) = e^{N^2/2} \cdot \left(\frac{L}{2} \right)^N \cdot R^N \int \prod_{i=1}^N d\Omega_i \times \\ \times \prod_{k=1}^N \prod_{l>k}^N \left(\frac{1 - \cos \psi_{kl}}{2} \right) \quad (6)$$

where

$$d\Omega_i = \sin \theta_i d\theta_i d\varphi_i \quad (7)$$

(θ_i, φ_i spherical coordinates of M_i).

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Introducing the Cayley-Klein parameters defined by

$$\alpha_i = \cos \frac{\theta_i}{2} e^{i\varphi_i/2} \quad (8)$$

$$\beta_i = -i \sin \frac{\theta_i}{2} e^{-i\varphi_i/2} \quad (9)$$

we can write

$$1 - \cos \psi_{ij} = 2 |\alpha_i \beta_j - \alpha_j \beta_i|^2. \quad (10)$$

The integrand of (6) now takes the form

$$\prod_{i=1}^N \prod_{j>i} \left(\frac{1 - \cos \psi_{ij}}{2} \right) = \prod_{k=1}^N \beta_k^{N-1} \prod_{i=1}^N \prod_{j>i} \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right). \quad (11)$$

The second product in the r.h.s. of equation (11) is a Vandermonde [1] determinant. Expanding it and inserting into equation (6) we get

$$Z_N(T_0) = e^{N^2/2(2\pi L)^N} R^N N! \times \prod_{k=1}^N \frac{(k-1)!(N-k)!}{N!}. \quad (12)$$

This result has a structure similar to the expression derived by (JA) for a charged disk of radius R

$$Z_N(T_0) = K\pi^N N! \prod_{k=1}^N \gamma(k, N) \quad (13)$$

where $\gamma(k, N)$ is the incomplete gamma function [2], and K is a constant.

The difference is that our calculation involves complete gamma functions.

The excess free energy for $T = T_0$ is readily derived from (12)

$$F_N(T_0) = -k_B T_0 \text{Log} \frac{Z_N(T_0)}{(4\pi R^2)^N}. \quad (14)$$

In the thermodynamic limit ($N \rightarrow \infty$, $R \rightarrow \infty$, ρ constant) one obtains

$$\lim_{N \rightarrow \infty} \frac{F_N(T_0)}{N} = q^2 \left[\frac{1}{2} - \frac{1}{4} \text{Log} 2\pi \right] - \frac{q^2}{4} \text{Log}(\rho\pi L^2) + O\left(\frac{\text{Log} N}{N}\right) \quad (15)$$

which is identical to the result given in reference [1].

2. Distribution functions. — The n -particle distribution function for $T = T_0$ is given by

$$\rho_{N,T_0}^{(n)}(1, 2, \dots, n) = \frac{N!}{(N-n)!} \frac{1}{Z_N(T_0)} \times \int \prod_{k=n+1}^N d\Omega_k \prod_{i=1}^n \prod_{j>i} \left(\frac{1 - \cos \psi_{ij}}{2} \right). \quad (16)$$

Using arguments parallel to those given in reference [2] one finds

$$\rho_{N,T_0}^{(n)}(1, 2, \dots, n) = \rho^n \text{Det} \{ [\alpha_k \alpha_i^* + \beta_k \beta_i^*]^{N-1} \} \quad (17)$$

with

$$1 \leq k \leq n$$

$$1 \leq l \leq n.$$

In particular

$$\rho_{N,T_0}^{(1)}(1) = \rho \quad (18)$$

$$\rho_{N,T_0}^{(2)}(1, 2) = \rho^2 g_{N,T_0}^{(2)}(1, 2) \quad (19)$$

with

$$g_{N,T_0}^{(2)}(1, 2) = 1 - \left(\frac{1 + \cos \psi_{1,2}}{2} \right)^{N-1}. \quad (20)$$

One can note that for $\psi_{12} = \pi$, $\rho_{N,T_0}(12) = 1$.

The system appears to be homogeneous for all N and the distribution functions are invariant by rotation of the sphere. Indeed, under a rotation R_0 characterized by its Cayley-Klein parameters, the coordinates (α_i, β_i) of particle i transform in the following way

$$\begin{pmatrix} \alpha'_i \\ \beta'_i \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ -\beta_0^* & \alpha_0^* \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}. \quad (21)$$

Using the fact that

$$\alpha_0 \alpha_0^* + \beta_0 \beta_0^* = 1 \quad (22a)$$

and

$$\alpha_i \alpha_i^* + \beta_i \beta_i^* = 1 \quad \text{for all } i \quad (22b)$$

we conclude that

$$\alpha'_i \alpha_j^{*'} + \beta'_i \beta_j^{*'} = \alpha_i \alpha_j^* + \beta_i \beta_j^* \quad (23)$$

and consequently $\rho_{N,T_0}^{(n)}(1, \dots, n)$ is invariant under the rotation R_0 . In particular expression (20) shows that $g_{N,T_0}^{(2)}(12)$ depends only on the relative distance between particles 1 and 2.

The thermodynamic limit of the functions $\rho_{N,T_0}^{(n)}(1, \dots, n)$ is obtained in the following way: Define $\rho_k = R\theta_k$. The limit $N \rightarrow \infty$, $R \rightarrow \infty$, ρ constant, is taken by keeping ρ_k and φ_k constant for each particle k ($1 \leq k \leq n$). For an infinitely large sphere the particles will be situated in the tangent plane at the North Pole and there positions will be characterized by the polar coordinates (ρ_k, φ_k) .

An expansion of (17) to second order in ρ_k/R gives

$$\rho_{N,T_0}^{(n)}(1, 2, \dots, n) = \rho^n \exp \left[- \sum_{k=1}^n |Z_k|^2 \right] \times \text{Det} [\exp(Z_k^* Z_l)] \quad (24)$$

where Z_k is defined by

$$Z_k = \sqrt{\pi\rho} \rho_k e^{i\varphi_k}. \tag{25}$$

The expression (24) coincides with the thermodynamic limit of the n -particle distribution function of a planar system.

3. Conclusion. — The thermodynamic limit of the system we have considered is identical to that of the planar two-dimensional OCP. It seems likely that this result remains valid for any temperature.

It is thus possible to advantageously simulate, by molecular dynamics or Monte-Carlo, a two-dimensional OCP by placing N -particles interacting *via* the potential (2) on the surface of a sphere. The system will be homogeneous, whatever the value of N , and the pair distribution function $g^{(2)}(r)$ can thus be obtained over a larger range of r values than would be possible, for a given value of N , from a simulation in a plane.

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