



**HAL**  
open science

# FINITE AUTOMATA AND ZERO TEMPERATURE QUASICRYSTAL ISING CHAIN

Joachim Allouche, M. Mendes-France

► **To cite this version:**

Joachim Allouche, M. Mendes-France. FINITE AUTOMATA AND ZERO TEMPERATURE QUASICRYSTAL ISING CHAIN. Journal de Physique Colloques, 1986, 47 (C3), pp.C3-63-C3-73. 10.1051/jphyscol:1986306 . jpa-00225716

**HAL Id: jpa-00225716**

**<https://hal.science/jpa-00225716>**

Submitted on 4 Feb 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FINITE AUTOMATA AND ZERO TEMPERATURE QUASICRYSTAL ISING CHAIN

J.P. ALLOUCHE and M. MENDES-FRANCE

*U.A. 226, U.E.R. de Mathématique et d'Informatique, 351, Cours de la Libération, F-33405 Talence Cedex, France*

Résumé

Un automate fini est une machine qui engendre des suites déterministes (périodiques ou non périodiques). Nous étudions de telles suites et nous considérons des chaînes d'Ising dont la suite des coefficients de couplage est "quasicristalline", c'est-à-dire engendrée par automate fini.

Abstract

A finite automaton is a machine which generates deterministic sequences (periodic or nonperiodic). We discuss such sequences and study Ising chains where the coupling coefficients form a "quasicrystalline" sequence (i.e. generated by automaton).

This report is a somewhat extended version of our article [1] even though we shall skip all proofs.

I. Two questions :

We start out with two seemingly unrelated problems. We shall see that their solution involves automata theory.

Problem 1 Are the binary digits of  $\sqrt{2}$  randomly distributed ?  
( $\sqrt{2} = 1.01101010 \dots$ )

Our second problem is longer to state. Let us first describe paperfolding [6]. Fold a sheet of paper in two and iterate the procedure to infinity (assuming the sheet of paper is infinitely wide).



Now unfold the sheet and observe the infinite sequence of creases V and A left by the folds. The sequence starts as follows :

V V Δ V V Δ Δ V V V Δ Δ V Δ Δ ...

Now replace V by 1 and Δ by 0 to obtain the binary expansion of the paperfolding number

$$x = .110110011100100 \dots$$

Problem 2 Is the paperfolding number algebraic ?

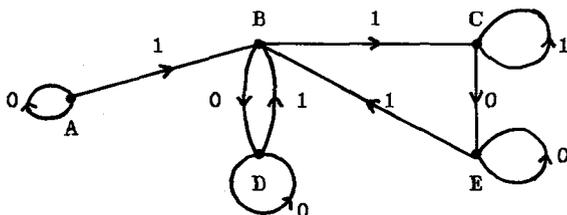
(An algebraic number  $x$  is a root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

where  $a_n \neq 0$  and where all coefficients are integers. A number which is not algebraic is called transcendental. Such are  $\pi$ ,  $e$ ,  $\log 2, \dots$  It is because  $\pi$  is transcendental that one cannot square the circle.)

II. Automatic sequences :

We first define what an automaton is. An automaton is composed of a finite number of states A,B,C, ... . In our example below we have five states. One of these states, say A, is singled out and is called the initial state. From every state two arrows leave which we name 0 and 1. These arrows link each state to some other state. Loops are permitted.



Each state is now coloured by a finite number of symbols. Here we choose to colour the states with a and b :

$$A \rightarrow a, B \rightarrow a, C \rightarrow b, D \rightarrow a, E \rightarrow b.$$

The automaton works as follows. The inputs are nonnegative integers. For example consider "nineteen" which in basis 2 is represented by 10011. Reading from left to right we follow the instructions 1,0,0,1,1 which take us from the initial state A through B,D,D,B to the final state C. The output function (colour of C) gives us b. Thus the specific automaton we have drawn maps "nineteen" on b. In the same way every integer  $n$  is mapped onto one of the symbols a or b, say  $a_n$ . The

infinite sequence  $a_0, a_1, a_2, \dots$  is said to be generated by the automaton. Here we obtain :

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
a <sub>n</sub>	a	a	a	b	a	a	b	b	a	a	a	b	b	a	b	b	a	...

Exercise : Show that the above sequence is the paperfolding sequence provided the first term  $a_0 = a$  is deleted ( $a=V, b=\Delta$ ).

Given an infinite sequence on a finite alphabet  $a, b, c, \dots$  one may ask whether there exists an automaton which generates it. If so, the sequence is said to be "automatic". Needless to say most sequences are not automatic (the set of automatic sequences is countable, whereas the set of all sequences on two symbols has power of the continuum).

It can be shown that finite deletion does not destroy automaticity. Ultimately periodic sequences are automatic. Aperiodic automatic sequences do exist (for instance the paperfolding sequence).

The complexity of a sequence may be measured through Fourier techniques. Given a complex-valued sequence  $(a_n)$ , define the correlation :

$$\gamma(k) = \lim_{N \rightarrow \infty} \left( \frac{\sum_{n=1}^N a_n a_{n+k}}{N} \right)$$

It is well known that  $\gamma$  is the Fourier transform of a measure  $M$  called the spectral power measure (or power spectrum) associated to the sequence  $(a_n)$  :

$$\gamma(k) = \int_0^1 \exp(2i\pi kx) M(dx).$$

If the sequence  $(a_n)$  is ultimately periodic,  $M$  consists of a finite sum of Dirac measures (with rational support). In the case of the paperfolding sequence,  $M$  is an infinite sum of Dirac measures (concentrated on dyadic rationals). The paperfolding sequence is thus almost-periodic (in the sense of J.P. Bertrandias). Actually, if one computes the Fourier-Bohr coefficients of the paperfolding sequence defined on the alphabet  $\pm 1$ ,

$$\hat{a}(x) = \lim_{N \rightarrow \infty} \left( \frac{\sum_{n=1}^N a_n \exp(-2i\pi nx)}{N} \right),$$

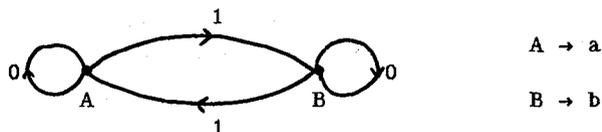
then one will notice that, in this specific case,

$$\sum |\hat{a}(x)|^2 = 1,$$

which proves that the sequence  $(a_n)$  is almost-periodic in the more restrictive sense of Besicovitch.

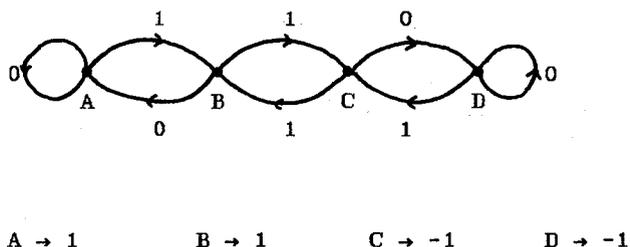
Two other examples of automatic sequences are the Thue-Morse sequence and the Rudin-Shapiro sequence.

The Thue-Morse sequence is generated by the automaton shown below :



the spectral measure  $M$  of this sequence is known to be continuous and singular ( $M(dx)$  cannot be written as  $m(x)dx$ ).

The Rudin-Shapiro sequence is generated by the automaton :



The spectral measure is the Lebesgue measure  $M(dx)=dx$ . A  $\pm 1$  random sequence would have the same spectral measure. So, in some sense, the Rudin-Shapiro sequence is more complex than the Thue-Morse sequence which itself is more complicated than the paperfolding sequence ...

Even though automatic sequences can behave in a rather apparently complicated fashion, their entropy is 0, (the entropy  $H$  of a sequence  $(a_n)$  is linked to the number  $p(k)$  of words of length  $k$  which occur in  $(a_n)$ ). Specifically :  $H \leq \overline{\lim}_{k \rightarrow \infty} (\log p(k))/k$ ).

Hence an automatic sequence stands in between the order of a crystal (periodic sequences) and random sequences, close however to the crystal order. We propose to rename automatic sequences quasicrystalline.

### III. Relationship with number theory :

Loxton and van der Poorten have proved that if the digits of a real irrational number form an automatic sequence, then the number is transcendental. This has two striking consequences :

Consequence 1 The digits of  $\sqrt{2}$  are "random" in that they are not generated by a finite automaton.

Consequence 2 The paperfolding number is transcendental.

### IV. Substitutions :

A substitution is a rule which assigns a word to each letter of a given alphabet. For example :

A → AB  
 B → DC  
 C → EC  
 D → DB  
 E → EB

Such a substitution generates an infinite sequence. Starting with A and iterating the substitution we obtain :

A  
 AB  
 ABDC  
 ABDCEBEC  
 ABDCEBECDBDCEBEC  
 ...

We now colour the letters A,B,C,D,E with a and b :

A → a    B → a    C → b    D → a    E → b .

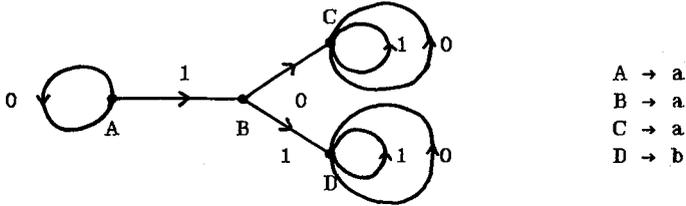
The infinite sequence then becomes

aaabaabbaaabbabb...

Ignoring the first term, we recognize once again the paperfolding sequence. This should not be a surprise since a general theorem states that automatic sequences can be generated by substitution and projection, and conversely [4], [5].

Remark 1 Several authors reminded us that the Penrose tiling can be constructed through a certain substitution procedure where each letter is to be replaced by words which do not have the same length. See for example [3], [8], [10]. Such sequences are not automatic in our sense.

Remark 2 In describing automatic sequences, we considered automata where the inputs were read from left to right. Actually every automatic sequence can be generated by an automaton where the input is read from right to left [4],[5]. Prove that the paperfolding sequence is generated by the right-left automaton :



Remark 3 The notion of automata can be generalized to any basis  $q$  greater than or equal to 2 :  $q$  arrows leave from each state and the input is expressed in the basis  $q$ .

V. The quasicrystal Ising chain :

The cyclic Ising chain is defined through the Hamiltonian :

$$\mathcal{H}(\sigma) = -J \sum_{q=0}^{N-1} \epsilon_q \sigma_q \sigma_{q+1} - H \sum_{q=0}^{N-1} \sigma_q ,$$

where  $\sigma_q = \pm 1$  represents the spin at site  $q$ ,  $\sigma_N = \sigma_0$ , where  $J > 0$  is the coupling constant,  $H$  the external field and where  $\epsilon_q = \pm 1$  is a given sequence. If the sequence  $(\epsilon_n)$  is automatic, we shall say that the Ising chain is quasicrystalline.

Solving the model at temperature  $T$  means computing the partition function

$$Z_N(T) = \sum_{\sigma} \exp(-\beta \mathcal{H}(\sigma)) , \quad \beta = 1/kT ,$$

(the summation is over all choices for  $\sigma = (\sigma_q)$  and  $\beta$  is the Boltzmann constant) and the thermodynamic limit

$$\lim_{N \rightarrow \infty} (\text{Log } Z_N(T))/N$$

In this paper we shall limit ourselves to calculating the induced magnetic field at  $T = 0$ .

It is well known that  $Z_N(T)$  is the trace of the matrix product :

$$P_N = \prod_{q=0}^{N-1} M_q$$

where

$$M_q = \begin{pmatrix} \epsilon_q + \alpha & -\epsilon_q + \alpha \\ z & z \\ -\epsilon_q - \alpha & \epsilon_q - \alpha \\ z & z \end{pmatrix},$$

with  $z = \exp(\beta J)$ ,  $\alpha = H/J$ .

Let  $Z_n^+$  denote the partition function of the Ising chain where the spin at site  $n$  is fixed to be  $+1$ ,  $Z_n^-$  is defined in a similar way. Then :

$$\begin{pmatrix} Z_n^+ \\ Z_n^- \end{pmatrix} = \left( \prod_{q=0}^{n-1} M_q \right) \begin{pmatrix} Z_0^+ \\ Z_0^- \end{pmatrix}.$$

$Z_n^+$  and  $Z_n^-$  are both generalized polynomials in  $z$ . As  $T$  vanishes,  $z$  increases to infinity, so that

$$Z_n^+ \sim p_n z^{a_n}, \quad Z_n^- \sim q_n z^{b_n},$$

where  $a_n, b_n, p_n, q_n$  depend only on  $n$ .

The difference  $d_n = a_n - b_n$  plays the rôle of the induced magnetic field of the chain at site  $n$ , see [7] for instance. It is easily seen that  $d_n$  is defined recursively :

$$\begin{cases} d_0 = a_0 - b_0 \\ d_{n+1} = 2\alpha + \epsilon_n \operatorname{sgn}(d_n) \min(2, |d_n|) \end{cases}$$

where  $\operatorname{sgn}(\ )$  is the sign function.

In a recent article [1] we prove that if we are given two maps, say  $f_+$  and  $f_-$  which both map a finite set  $S$  into itself, then the sequence

$$n \rightarrow f_{\epsilon(n)} f_{\epsilon(n-1)} \dots f_{\epsilon(1)} f_{\epsilon(0)} (a) , \quad a \in S$$

is automatic provided the sequence  $(\epsilon_n)$  is itself automatic.  
 Choosing

$$f_{\epsilon}(a) = 2\alpha + \epsilon \operatorname{sgn}(a) \min(2, |a|) ,$$

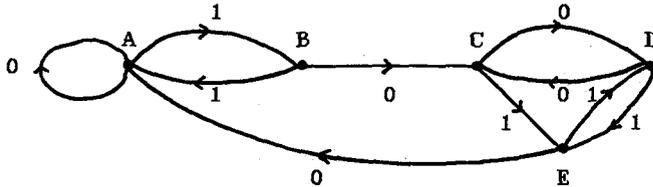
we then obtain the following theorem :

**Theorem** Consider an infinite quasicrystal Ising chain in a uniform external field. The magnetic field  $d_n$  induced on the  $n^{\text{th}}$  site at  $T = 0$  is quasicrystalline.

We conclude the paragraph with an explicit example. Suppose  $H = J$ , and suppose that  $(\epsilon_n)$  is the Thue-Morse sequence on the symbols  $+$  and  $-$  :

$+ - - + - + + - - + + - + \dots$

generated by the automaton described at the end of §2, where  $a$  is replaced by  $+$  and  $b$  by  $-$ .  
 Then the sequence  $(d_n^+) = (d_{n+2})$  is generated by the automaton



$A \rightarrow 0, \quad B \rightarrow 2, \quad C \rightarrow 4, \quad D \rightarrow 2, \quad E \rightarrow 4,$

and begins as follows

02402402440202402402024402 ...

We have chosen to describe the sequence  $(d_{n+2})$  because, as it happens, this sequence is independent of the initial values of  $d_0$  ( $d_0 = 0, 2, 4$ ). It goes without saying that a sequence is automatic if and only if the shifted sequence is automatic.

Remark 1 It is easy to compute the free energy  $\Psi$  per spin. Indeed :

$$\Psi = \lim_{\substack{N \rightarrow \infty \\ \beta \rightarrow \infty}} (\log Z_N(T)) / \beta N .$$

Now  $Z_N(T)$  is a polynomial in  $z = \exp(\beta J)$

$$Z_N(T) = Z_N^+ + Z_N^- = (p_N z^{a_N} + \dots) + (q_N z^{b_N} + \dots).$$

Let  $z$  go to infinity, then

$$-\Psi = \lim_{\substack{N \rightarrow \infty \\ \beta \rightarrow \infty}} (\log z^{c_N}) / \beta N ,$$

where  $c_N = \max(a_N, b_N)$  ,

hence  $-\Psi/J = \lim_{N \rightarrow \infty} c_N/N = \lim_{N \rightarrow \infty} a_N/N$  .

Now  $a_{q+1} - a_q = \alpha + \max(\epsilon_q, -d_q - \epsilon_q)$ .

We assume as in our example  $\alpha = 1$  and  $d_0 = 2$ . Then

$$a_{q+1} - a_q = 1 + (d_{q+1} - \min(2, d_q)) / 2$$

hence

$$\begin{aligned} \lim_{N \rightarrow \infty} a_N/N &= \lim_{N \rightarrow \infty} [ 1 + (\sum_1^N d_q) / 2N - (\sum_1^N \min(2, d_q)) / 2N ] \\ &= 1 + \lim_{N \rightarrow \infty} [ \sum_1^N (d_q - \min(2, d_q)) ] / 2N \end{aligned}$$

$$= 1 + \lim_{N \rightarrow \infty} (\text{Card} \{ q \leq N ; d_q = 4 \}) / N$$

$$= 1 + \text{freq}(4) ,$$

where  $\text{freq}(4)$  is the frequency of 4's in the sequence  $(d_n)$ . Using Peyrière's techniques described in [10], we easily conclude that  $\text{freq}(4) = 1/3$ .

Thus

$$\Psi = -4J/3 .$$

**Remark 2** We have not yet made a systematic study of the Fourier properties of the general sequence  $(d_n)$  in terms of the sequence  $(\varepsilon_n)$ . We can however show that  $(d_n)$  may have a purely continuous power spectrum  $M(dx)$  even though  $(\varepsilon_n)$  is almost-periodic. This would be the case for example in the zero external field  $H=0$  when  $(\varepsilon_n)$  is defined by

$$\varepsilon_{2n} = +1, \quad \varepsilon_{2n+1} = -\varepsilon_n .$$

Then :

$$\xi(x) = \lim_{N \rightarrow \infty} \left[ \sum_{n=0}^{N-1} \varepsilon_n \exp(-2i\pi n x) \right] / N$$

$$= \begin{cases} 1/3 & \text{if } x=0, \\ (4/3) ((-1)^{\nu} / 2^{\nu}) \exp(2i\pi(2k+1)/2^{\nu}) & \text{if } x=(2k+1)/2^{\nu}, \\ 0 & \text{in all other cases.} \end{cases}$$

One then verifies that

$$\sum_x |\hat{\varepsilon}(x)|^2 = 1 ,$$

which proves that  $(\varepsilon_n)$  is indeed almost-periodic :

$$\varepsilon_n \sim \sum_x \hat{\varepsilon}(x) \exp(2i\pi n x) .$$

Now the induced magnetic field  $(d_n)$  is

$$d_n = \prod_{q=0}^{n-1} \varepsilon_q, \text{ provided } d_0 = 1.$$

It is then easily proved that  $d_n = (-1)^n m_n$ , where  $(m_n)$  is the Thue-Morse sequence, and that  $(d_n)$  has continuous (singular) power spectrum.

Remark 3 In an article in preparation [2] we show that solving the quasicrystal Ising chain at temperature  $T$  (not necessarily 0) leads to the study of a polynomial iteration problem.

-----

- [1] J.-P. ALLOUCHE, M. MENDES FRANCE, Quasicrystal Ising chain and automata theory, *J. Stat. Phys.*, 42, 1986.
- [2] J.-P. ALLOUCHE, F. AXEL, M. KLEMAN, M. MENDES FRANCE, J. PEYRIERE Chafne d'Ising quasicristalline et itération polynomiale, Preprint.
- [3] E. BOMBIERI, J.E. TAYLOR, Quasicrystals, tilings and algebraic number theory, some preliminary connections, This volume.
- [4] G. CHRISTOL, T. KAMAE, M. MENDES FRANCE, G. RAUZY, Suites algébriques, automates et substitutions, *Bull. Soc. Math. France*, 108, 1980, 401-419.
- [5] A. COBHAM, Uniform tag sequences, *Math. Syst. Theory*, 6, 1972, 164-192.
- [6] F.M. DEKKING, M. MENDES FRANCE, A. van der POORTEN, Folds!, *Math. Intelligencer*, 4,3,1982, 130-138, and 4,4, 1982, 173-181 and 190-195.
- [7] E. GARDNER, B. DERRIDA, Zero temperature magnetization of one dimensional spin glass, *J. Stat. Phys.*, 39, 1985, 367-377.
- [8] T. JANSSEN, This volume.
- [9] M. MENDES FRANCE, A. van der POORTEN, Arithmetic and analytic properties of paperfolding sequences, *Bull. Austr. Math. Soc.* 24, 1982, 123-131.
- [10] J. PEYRIERE, This volume.