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ELASTIC WAVE PROPAGATION IN THREE-DIMENSIONAL PERIODIC COMPOSITE MATERIALS

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Résumé – Nous avons effectué l'analyse d'ondes élastiques dans un solide à structure périodique.

Abstract – An elastodynamic theory, based on the concepts of Brillouin theory has been developed for elastic wave propagation in periodic composite materials.

An important new class of periodic piezoelectric composite materials has recently been developed from considerations of strictly static elasticity theory. This paper gives a dynamic theory of such elastic lattices, neglecting piezoelectricity.

I – PROPAGATION IN AN INFINITE COMPOSITE

Elastic plane wave propagation in a homogeneous solid is governed by the Christoffel equation, obtained from the acoustic wave equation by replacing the spatial derivatives with i times the wave vector components /1,2/. The acoustic wave equation contains terms in which spatial derivatives act upon products of the elastic stiffness and the elastic strains. In a homogeneous medium the stiffnesses are constant and can be moved outside the derivative. This is not true for the cases considered here and these product terms must be treated by the rule of product differentiation, leading to extra, spatially-varying, terms in the wave equation.

The case of two-dimension composite lattices will be considered first. Figure 1 shows three types of unit cells that have been studied in detail. The restriction to square cells is not essential, but in all two-dimensional cases the lattice is uniform in the z -direction. In these lattices the mass density and stiffnesses are doubly periodic functions of x and y . For convenience, the density function is

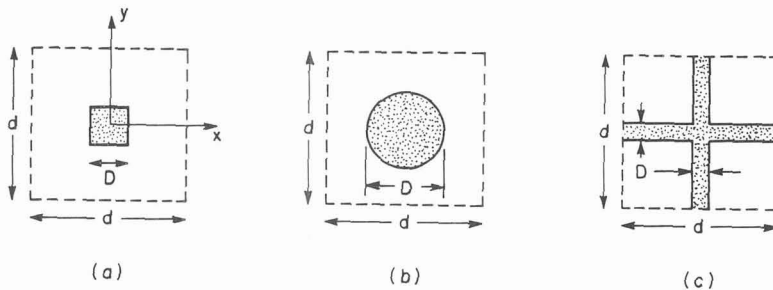


Fig. 1 – Examples of 2-D periodic composites (uniform in z).

separated into an average density $\bar{\rho}$ and a spatially-varying part $\Delta\rho(x,y)$, similarly for the stiffnesses. The elastic wave equation then takes the form

$$(\underline{\underline{\Gamma}} + \bar{\rho}\omega^2\underline{\underline{I}}) \cdot \underline{u} = \underline{p}(x,y) \cdot \underline{u} \tag{1}$$

$$\underline{p}(x,y) = -(\Delta\underline{\underline{\Gamma}} + \Delta\rho\omega^2\underline{\underline{I}}) - \underline{\Gamma}' \tag{2}$$

in matrix format, where the double underlined quantities are 3×3 matrices operating on the elastic displacement column vector. On the left side of Eq. (1) the first term is the wave equation matrix for the average material properties, obtained from the Christoffel matrix by the procedure described above, and the second term contains the unit matrix. The first term on the right contains the same matrices, but evaluated for the spatially-varying parts of the material parameters, and the second term is the extra term due to the spatial derivatives acting on the spatially-varying elastic constants.

A general solution to Eq. (1) is the elastic Bloch (or Floquet) wave /3/

$$\underline{u} = a_{\ell m \eta} \hat{u}_{\ell m \eta} \exp(-i\beta_{\ell m} \cdot R) \tag{3}$$

$$\beta_{\ell m} = \hat{x}(\beta_x + 2\pi\ell/d) + \hat{y}(\beta_y + 2\pi m/d) + \hat{z}\beta_z \tag{4}$$

where summation over repeated subscripts is assumed and R is the spatial position vector. The terms in this series, called space harmonics, are each described by three subscripts. The first two, summed from $-\infty$ to $+\infty$, define the order of the space harmonic and the third defines an elastic polarization vector (Fig. 2).

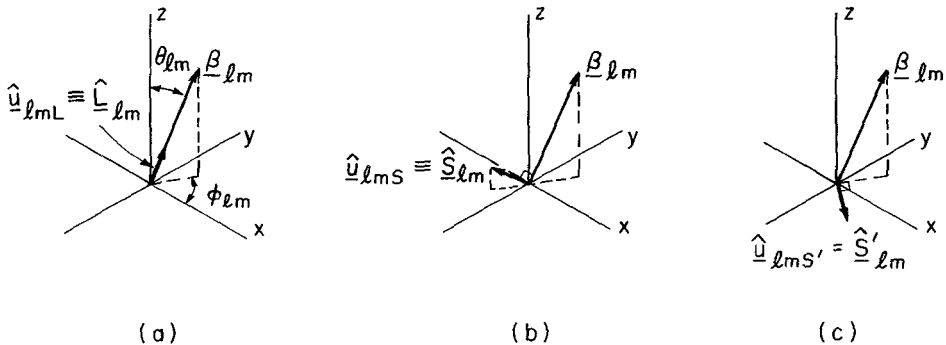


Fig. 2 - Three polarizations of the ℓm space harmonic. (a) Longitudinal. (b) Vertical shear. (c) Horizontal shear.

In the case of electronic Bloch waves each space harmonic is a scalar. For elastic Bloch waves they are vectors, which are conveniently decomposed into a basis consisting of the elastic wave polarizations for a homogeneous medium. Only isotropic media will be considered here and, in this case, the decomposition is as shown in Fig. 2. This means that the polarization vectors in Eq. (3) are eigenvectors of the left side of Eq. (1) and the equation becomes

$$(\bar{\rho}\omega^2 - \bar{C}_{\eta}\beta_{\ell m}^2) a_{\ell m \eta} \hat{u}_{\ell m \eta} \exp(-i\beta_{\ell m} \cdot R) = \underline{p}(x,y) \cdot a_{pq\zeta} \hat{u}_{pq\zeta} \exp(-i\beta_{pq} \cdot R) \tag{5}$$

where the average stiffnesses appearing on the left side are 11 for the longitudinal polarization and 44 for the shear. Since the two-dimensional Bloch function

exponentials are orthogonal over the unit cell, multiplication of Eq. (5) on the left by the complex conjugate of the $\ell m \eta$ space harmonic and integration over the unit cell yields

$$(\rho\omega^2 - \bar{c}_\eta \beta_{\ell m}^2) a_{\ell m \eta} = K_{\ell m \eta, p q \zeta} a_{p q \zeta} \quad (6)$$

$$K_{\ell m \eta, p q \zeta} = d^{-2} \int_{\text{Cell}} \hat{u}_{\ell m \eta} \cdot \underline{P} \cdot \underline{u}_{p q \zeta} \exp[-i(\beta_{\ell m} - \beta_{p q}) \cdot \underline{R}] dx dy \quad (7)$$

where it should be recalled that P is a function of x and y . Equations (6) and (7) define an infinite system of linear algebraic equations determining the space harmonic amplitudes in Eq. (3).

The coupling constants have been evaluated explicitly for all of the geometries shown in Fig. 1 but only case (a) will be presented here. In evaluating the integrals it is helpful to note, first, that the polarization vectors of Fig. 2 are also eigenvectors of the first term on the right side of Eq. (2). It will be recalled that the second term on the right of Eq. (2) contains spatial derivatives of the elastic constants. In the structures of Fig. 1 each of the two material regions shown is uniform, so that the spatial derivatives generate delta functions at the material boundaries. A general rule is that the coupling constants are zero for $\ell = p$ and $m = q$. For case (a)

$$\begin{aligned} K_{\ell m \eta, p q \zeta} &= (D/d)^2 \operatorname{sinc} \left(\frac{\ell - p}{d} \pi D \right) \operatorname{sinc} \left(\frac{m - q}{d} \pi D \right) \\ &\times \left\{ \Delta \rho \omega^2 \hat{u}_{\ell m \eta} \cdot \hat{u}_{p q \zeta} - \beta_{\ell m} \beta_{p q} \left[\Delta C_{12} \delta_\eta L \delta_\zeta L \right. \right. \\ &\left. \left. + \Delta C_{44} \left(\overline{\hat{u}_{\ell m \eta} \cdot \hat{L}_{p q}} \overline{\hat{L}_{\ell m} \cdot \hat{u}_{p q \zeta}} + \overline{\hat{u}_{\ell m \eta} \cdot \hat{u}_{p q \zeta}} \overline{\hat{L}_{\ell m} \cdot \hat{L}_{p q}} \right) \right] \right\} \quad (8) \end{aligned}$$

Here, $\Delta \rho$ etc. are differences between the two homogeneous regions. The general form of the result is the same for all three cases and consists of a geometric factor (containing sinc functions for the rectangular geometry and jinc functions for the circular geometry) multiplied by a material and polarization factor in curly brackets. The second factor has separate contributions from density and stiffness inhomogeneities and the 12 contribution to the latter occurs only for pairs of longitudinally polarized space harmonics.

Three-dimensional isotropic lattices are treated similarly, with the right side of Eq. (1) a function of three space coordinates. In Eqs. (3) and (4) there are now three space harmonic indices, with an appropriate modification of the third term in Eq. (4). These changes carry through to Eqs. (6) and (7), with the coupling coefficients now proportional to the inverse cube of the cell dimension (for a cubic lattice).

II - SOLUTION METHODS

Dispersion relations for the elastic Bloch waves [i.e., β in Eq. (4) as a function of ω] are obtained by truncating the infinite system of equations [Eq. (6)] and extending the number of equations included until satisfactory accuracy has been achieved [4]. Perturbation (or iteration) methods provide a rational method for guiding the choice of truncation, as well as providing physical insight into the wave behavior, if the coupling constants are weak enough to permit this approximation. A perturbation order parameter ϵ is introduced on the right side of Eq. (6) and the Bloch solution is expanded as a power series in this parameter

$$\underline{u}_\eta = \underline{u}_\eta^{(0)} + \epsilon \underline{u}_\eta^{(1)} + \dots \quad (9)$$

For a homogeneous medium ($e = 0$) the solutions, which are zero order in Eq. (9), are plane waves of longitudinal, vertical shear or horizontal shear types (Fig. 2), so that each Bloch solution belongs to one of these three types. Suppose, for example, a longitudinal type of Bloch wave. This has only the (00L) space harmonic in the limit of $e = 0$. The first order space harmonic amplitudes obtained by substituting into Eq. (6) and retaining first-order terms

$$a_{\ell m \eta}^{(1)} = (\bar{\rho}\omega^2 - \bar{c}_{\eta} \beta_{\ell m}^2)^{-1} K_{\ell m \eta, 00L} a_{00L}^{(0)} \quad (10)$$

contain all three polarizations because of cross coupling through the periodicity of the medium. However, due to the presence of the resonance denominator in Eq. (10), the space harmonics close to resonance have the largest amplitudes.

In the weak coupling approximation Eq. (6) is truncated by retaining only those space harmonics that are simultaneously near resonance. For two-dimensional propagation, i.e., in the xy plane, symmetry requires that the S space harmonics be decoupled from the L and S' space harmonics (Fig. 2). Consider, for example, longitudinal propagation along the x axis at frequencies close to the first stopband. In this case the two near resonance space harmonics have wave numbers

$$\beta_{00} \approx \pi/d \quad ; \quad \beta_{-10} \equiv \beta_{00} - 2\pi/d \approx -\pi/d \quad (11)$$

The truncated system of equations then takes the form

$$\begin{aligned} (\bar{\rho}\omega^2 - \bar{c}_{11}\beta_{00}^2) a_{00L} &= K_{00L, -10L} a_{-10L} \\ (\bar{\rho}\omega^2 - \bar{c}_{11}\beta_{-10}^2) a_{-10L} &= K_{-10L, 00L} a_{00L} \end{aligned} \quad (12)$$

and defines the structure of the dispersion relation near the first stopband. Horizontal shear waves (S') are not included because they are not near resonance except at particular combinations of cell dimensions and elastic constants. The band-gap is obtained by setting an exact equality on the left on Eq. (11). Similar considerations apply to the higher stopbands except that more than two space harmonics are strongly coupled, just as in the case of electronic Bloch waves.

For general three-dimensional propagation in a two-dimensional lattice the space harmonics are as shown in Fig. 2 and all three polarizations are coupled. Dispersion relations for these three-dimensional Bloch waves are obtained by taking θ and ϕ from the 00 space harmonic and solving a truncated version of Eq. (6) for β as a function of ω . (For both two- and three-dimensional propagation, the dispersion relation is anisotropic because of the lattice.) To study the structure of the first stopband in this case the space harmonic wave vectors in Eq. (12) must include the z component, which has a common value for all harmonics.

III — BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL LATTICES

The simplest boundary value problem that can be considered is Bloch wave scattering at the surface of a half space normal to the z axis of a two-dimensional lattice, or surface wave propagation on such a lattice. Suppose, for example, a half space $z < 0$ containing a lattice of type (a) in Fig. 1. Stress-free boundary conditions are defined by setting to zero the xz , yz and zz components of stress from the elastic constitutive relations, allowing for a difference in elastic constants between the shaded and unshaded parts of the lattice. Since the xz and yz stresses in an isotropic medium depend only on the 44 stiffness constant the boundary conditions reduce to

$$c_{44} \epsilon_{xz} = 0 \quad (13a)$$

$$C_{44}^{\nearrow} \epsilon_{yz} = 0 \quad (13b)$$

$$\bar{C}_{12}(\epsilon_{xx} + \epsilon_{yy}) + \bar{C}_{11}\epsilon_{zz} = -\Delta C_{12}(x,y)(\epsilon_{xx} + \epsilon_{yy}) - \Delta C_{11}(x,y)\epsilon_{zz} \quad (13c)$$

at $z = 0$. Consider, now, a longitudinal-type Bloch wave incident on this boundary. The strains ϵ_{xz} , etc. are now combinations of space harmonics. By noting the orthogonality of space harmonics over the unit cell one can reduce Eq. (13) to boundary coupling equations among the space harmonics, similar to Eq. (6). This interharmonic coupling occurs only through the right side of Eq. (13). In a homogeneous half-space all space harmonic couplings [in Eq. (6) and in Eq. (13)] disappear and the incident longitudinal Bloch wave becomes a simple longitudinal wave. This scatters into a reflected longitudinal and a reflected shear wave. If we consider only the spatially uniform part of the boundary conditions in Eq. (13), each space harmonic of the incident Bloch wave will experience similar polarization coupling at the boundary. The problem in making the weak coupling approximation is then to identify the significant space harmonics of the scattered field [i.e., the near resonance harmonics in Eqs. (6) and (13)]. It must be remembered that the z -component of the harmonic wave numbers is conserved in Eq. (6), while x - and y -components are conserved in Eq. (13). The latter condition leads to coupling between the incident Bloch wave and surface states. If none of the latter are resonant they may be neglected and the scattering problem treated in the weak coupling approximation by including only near resonant space harmonics in the incident and reflected Bloch waves.

IV - CONCLUSION

An extension of scalar Brillouin theory to the vector problem of elastic wave propagation in a periodic composite has been made. This shows that significant polarization effects occur because of the vector nature of the problem. The problem of bounded composites is not yet completely solved, but predictions from the theory have been confirmed by laser probe vibration pattern measurements on resonant composite plates /5/.

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