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GAUGE GROUP AND PHASES OF SUPERFLUID <sup>3</sup>He

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Résumé.- On propose une classification des phases A et B qui ne fait appel ni à la contrainte d'unitarité, ni à des restrictions sur l'aimantation nucléaire et la susceptibilité.

Abstract.- The p-wave phases of superfluid <sup>3</sup>He are classified without the unitary constraint or the constraints on the net nuclear magnetization and the magnetic susceptibility.

In paper/1/ for different phases of superfluid <sup>3</sup>He in the state of p-wave pairing we studied the topology of spaces of the order parameter (i.e. vacuum manifolds) by means of the gauge group SO(3)<sub>1</sub> x SO(3)<sub>2</sub> x U(1) of the transformations

$$A \rightarrow e^{i\phi} R_1 A R_2^{-1} \tag{1}$$

Here A is a complex 3x3-matrix of the order parameter, R<sub>1</sub> is a rotation matrix in the spin part and R<sub>2</sub> is rotation matrix in the orbit part. In the present paper we suggest an algebraic method which enables to get explicit formulae for the order parameter and to amend some statements of paper/1/.

Under the constraints that the net nuclear magnetization vanished in the absence of an applied magnetic field and the magnetic susceptibility was unchanged from its normal state value N.D. Mermin and G. Stare found six distinct p-wave phases by minimizing the free energy functional/4/,

$$F = \alpha \text{tr}(AA^+) + \beta_1 |\text{tr}AA^t|^2 + \beta_2 |\text{tr}AA^+|^2 + \beta_3 [\text{tr}(AA^t)(AA^t)^*] + \beta_4 \text{tr}((AA^+)^2) + \beta_5 \text{tr}(({}^tAA)({}^tAA)^*) \tag{2}$$

They proved that two of these phases were unstable against variations of the order parameter, when the constraints were relaxed.

Another condition which enables to find the p-wave phases is the unitary restriction/4-5-6/,

$$\epsilon_{ijk} A_{pi}^* A_{qj} n_p n_q = 0 \tag{3}$$

where A<sub>pi</sub>, A<sub>qj</sub> are the entries of the order parameter matrix and n<sub>p</sub> is a real unit vector, n<sub>p</sub><sup>2</sup> = 1. There exist four p-wave phases with the unitary restriction, but only two of them are stable/5-6/. These stable phases are generally accepted as the A- and B-phases of <sup>3</sup>He.

We do not use the unitary assumption or cons-

traints on the net nuclear magnetization and the magnetic susceptibility. We study the order parameter by means of the gauge group and its subgroups. Indeed, in our opinion this problem mainly concerns the symmetries of the system. We do not study the problem of the minimization ; the stability of the phases is to be discussed elsewhere.

Following paper/1/ we assume that all the values A of the order parameter for a p-wave phase are generated by transformations (1) with A = A<sub>0</sub>. If there is no superdegeneracy ordering to the singularities of the space of the order parameter, this condition means that the phase is fixed by the value of A<sub>0</sub>. Then the problem of classification of the phases is reduced to the proper choice of A<sub>0</sub>; which can be described by the subgroup H = H(A<sub>0</sub>) of the gauge transformations leaving A<sub>0</sub> invariant. Thus we may say that the SO(3)<sub>1</sub> x SO(3)<sub>2</sub> x U(1) - gauge symmetry is broken down to the subgroup H = H(A<sub>0</sub>).

It is easy to indicate the spaces of the order parameter where the gauge symmetry is completely broken, e.g. if A<sub>0</sub> is a Jordan matrix of the rank equal to 3 then the subgroup H is trivial<sup>+</sup>. Another example to the effect is a diagonal matrix with unequal diagonal entries ; it is not hard to prove that for it the subgroup H = H(A<sub>0</sub>) is trivial. We do not know under what constraints on the coefficients

<sup>+</sup> To prove the statement we note that for any non-unit rotation matrix R we have RA<sub>0</sub> ≠ A<sub>0</sub>, A<sub>0</sub>R ≠ A<sub>0</sub>. This means that in SO(3)<sub>1,2</sub> we have no subgroups leaving A<sub>0</sub> invariant. If there were such subgroups in SO(3)<sub>1</sub> x SO(3)<sub>2</sub>, we should have RA<sub>0</sub>R<sup>-1</sup> = A<sub>0</sub> for some R. Since the Jordanian matrix A generates an irreducible matrix algebra  $\bar{A}$  and the matrix R must belong to the commutator algebra of  $\bar{A}$ . Schur's lemma requires that R be a unit matrix, (c.f. H. Weyl "The Classical Groups" for the specific information). The similar arguments are applied for the Jordanian matrix of rank 2, when the subgroup H is also trivial.

$\alpha, \beta_1, \dots, \beta_5$  these phases can be minima of free energy (2) and how the dipole interaction energy changes the form of the order parameter ; it presents a rather complicated analytical problem. The magnetic field contribution to the free energy is more tractable. If the coherence length  $\xi$ , the magnetic length  $\xi_H$  and the dipole interaction length  $\xi_D$  satisfy the constraint  $\xi \ll \xi_H \ll \xi_D$  we may cancel out the dipole interaction terms for the systems having the scale  $R$ ,  $\xi_H \ll R \ll \xi_D$ . Then it is particularly interesting to know what degeneracies of the order parameter still remain after the magnetic field  $\vec{H}$  is taken into account. The answer to this question shows that the phases with the trivial subgroup  $H(A_0)$  are highly anisotropic. Let us write down the magnetic energy contribution in the form/5/,

$$F_H = g_H^* \frac{A}{p_i} \frac{A}{q_i} \frac{H}{p} \frac{H}{q} = g_H^* A \frac{t}{H} \frac{t}{AH} \tag{4}$$

since  $A = R_1 A_0 R_2 e^{i\phi}$ , we have

$$F_H = g_H^* A_0 \frac{t}{H'} \cdot A_0 \frac{t}{H'} , \quad H' = R_1^{-1} \vec{H} \tag{5}$$

We shall minimize  $F_H$  with respect to  $\vec{H}'$  and then we shall make some conclusions concerning the order parameter. To simplify the calculations we suppose that  $A_0$  is a diagonal matrix,  $(A_0)_{ij} = \lambda_i \delta_{ij}$ ,  $\lambda_1 \neq \lambda_2 \neq \lambda_3$ . The case of  $A_0$  being a Jordan matrix of rank 2 or 3 is similar. Now we may write down  $F_H$

$$F_H = g_H \sum_{i=1}^3 |\lambda_i|^{-2} H_i'^2 \tag{6}$$

Since we are interested in the symmetry of the order parameter we may minimize (6) with respect to the direction of  $\vec{H}'$  or just the same under the constraint  $H^2 = \text{const}$ . The answer is straightforward, there are three directions of the field  $\vec{H}'$  which minimize (6) and which correspond to the axes of  $A_0$ . Now we return to equation (4) and want to find what constraints are imposed on the order parameter  $b$  by the original magnetic field  $H$ . Equation (5) tells, that we may put the answer for the minimization problem for (6) in the form  $A = R_2 A_0 R_1^{-1}$  where  $R_1^{-1}$  is a rotation matrix such that  $\vec{H}' = R_1^{-1} \vec{H}$  minimizes (6) and  $R_2$  is any. The rotation matrix  $R_1$  is defined by the value of  $\vec{H}'$  up to a  $SO(3)$ -matrix multiple  $R_H$  which leaves  $\vec{H}'$  invariant. Therefore we may conclude that in the magnetic field the degeneracy of the phase is not defined in the unique way like in the case of the A- and B-phase, but that in the spin part the space of the order parameter is three linked circles  $(S^1, S^1, S^1)_{\text{linked}} = L(S^1)$ . The whole space  $J_H$  of the order parameter is a product of  $L(S^1)$  and the group of the orbital rotations  $SO(3)_2$  and  $U(1)$ ,  $J_H = L(S^1)$

$\times SO(3) \times U(1)$ .

Now we turn to the less degenerate phases with the subgroup  $H(A_0)$  being non-trivial. The previous analysis indicates that we must exclude all  $A_0$  having Jordan blocks of rank more than 1. Our main idea is to study the possible forms of  $A_0$  by their rank and eigenvalues. To get around the difficulties generated by the complex eigenvalues we apply the following trick. We change the matrix  $A_0$  into a matrix  $D$  by means of the transformation

$$D = X^{-1} A_0 X \tag{7}$$

with a non-singular matrix  $X$ . Let us consider the group  $Gl(3)_1 \times Gl(3)_2$  of pairs of complex non-singular  $3 \times 3$  matrices and its action on complex  $3 \times 3$  matrices

$$(X_1, X_2)M = X_1 M X_2^{-1}$$

In the group  $Gl(3)_1 \times Gl(3)_2$  we consider the subgroup  $H_C$  which leaves the matrix  $D$  invariant, i.e.  $D = S_1 D S_2^{-1}$ . The original subgroup  $H = H(A_0)$  of  $SO(3)_1 \times SO(3)_2 \times U(1)$  is transformed by (7) into a subgroup of  $H_C$  since we have

$$S_1 X^{-1} (X D X^{-1}) X S_2^{-1} = X^{-1} (X S_1 X^{-1}) A_0 (X S_2^{-1} X^{-1}) X$$

It is important that  $H$  and therefore its conjugate in  $H_C$  are compact groups. We may take the matrix  $X$  such that the matrix  $D$  should be the most simple.

As we have demonstrated, if  $A_0$  has a Jordan block of rank  $\geq 2$  or all its eigenvalues are distinct, then the subgroup  $H = H(A_0)$  is trivial. Therefore we may consider now the opposite case, when (i) at least two of the eigenvalues are equal or (ii) the matrix  $A_0$  is of rank 1. Under these constraints we shall write down the order parameter.

(1) All eigenvalues are equal ;  $(A_0)_{ij} = \Delta \delta_{ij}$   
 $H(A_0) = SO(3)$  ; the order parameter space is  $SO(3) \times U(1)$ . It is the B-phase.

(2) Two eigenvalues of  $A_0$  are equal,  $\lambda_1 = \lambda_2 \neq 0$   
 We may take the matrix  $D$  in the form

$$D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

The subgroup  $H_C$  is pairs of non-singular matrices  $(S_1, S_2) : (S_1)_{ij} = (S_2)_{ij} ; (S_1)_{i3} = (S_1)_{3i} = \delta_{i3}$   
 The compact subgroups are the images of the diagonal subgroup in the group  $SO(3)_1 \times SO(3)_2 \times U(1)$ . The space of the order parameter is  $SO(3)_1 \times SO(3)_2 \times U(1) / SO(2)$   
 The case  $\mu = 0$  is often singled out (c.f. /5/) ; it

is called the 2D-phase.

(3) Two eigenvalues of  $A_0$  are equal to zero. The arguments similar to the previous  $n^\circ$  show that the subgroup  $H(A_0) = SO(2)_1 \times SO(2)_2$  and the space of the order parameter is  $SO(3)_1 \times SO(3)_2 \times U(1) / SO(2)_1 \times SO(2)_2$ . This is the polar phase/5/.

(4) The matrix  $A_0$  is of rank 1 and is not symmetric, then we may choose  $D$  in one of the two forms

$$D_1 = \left( \begin{array}{c|c} \mathbf{O} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \Delta_1 & \Delta_2 \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right) \quad D_2 = \left( \begin{array}{c|c} \mathbf{O} & \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \\ \hline 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right)$$

The matrices  $D_{1,2}$  are invariant under the transformations

$$D_1 \rightarrow D_1' = \left( \begin{array}{c|c} * & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline x_{31} & x_{32} \end{array} \middle| \begin{array}{c} 1 \\ 1 \end{array} \right) \mathcal{D}_1, \quad D_2 \rightarrow D_2' = \mathcal{D}_2 \left( \begin{array}{c|c} * & \begin{array}{c} y_{13} \\ y_{23} \end{array} \\ \hline 0 & 0 \end{array} \middle| \begin{array}{c} 1 \\ 1 \end{array} \right)$$

The corresponding compact subgroups are isomorph to  $SO(2)$ . Hence we obtain the subgroups  $SO(2)_1 \times \{1\}$ ,  $SO(2)_2 \times \{1\}$  of the gauge group  $SO(3)_1 \times SO(3)_2 \times U(1)$ . For the A-phase we need the subgroup  $SO(2)_1 \times \{1\}$ ; the order parameter  $A$  is of the form

$$A = \left( \begin{array}{c|c} \mathbf{O} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \Delta_1 & \Delta_2 \end{array} \middle| \begin{array}{c} 0 \\ 0 \end{array} \right)$$

where  $\Delta_1, \Delta_2$  are coordinates of the complex vector  $\vec{\Delta}$ , the real and imaginary part of  $\vec{\Delta}$  being unit vectors  $\vec{\Delta}^r, \vec{\Delta}^i$ . This constraint is very important since it influences the topological type of the space of the order parameter. Indeed, we see that the subsidiary condition on the real and imaginary part of  $\vec{\Delta}$  reduces the space of the order parameter

$SO(3)_1 \times SO(3)_2 \times U(1) / SO(2)_1$   
to the product  $S^2 \times SO(3)$ .

CONCLUSIONS.- We want to emphasize that the method we used to classify the p-wave phases can be successfully applied to higher pairing states. The necessity to study high dimensional representations of  $SO(3) \times SO(3) = SO(4)$  introduces some cumbersome details which can be successfully overcome.

P-wave phases with completely broken gauge symmetry, if they do exist, should have some interesting properties. They should be highly anisotropic. In the magnetic field the space of the order parameter is the product of  $SO(3)$  and three linked circles in another copy of  $SO(3)$ . This property sug-

gests the existence of complicated domain structures (c.f. /1-2/)

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#### Appendix

These are the following subgroups  $H(A_0)$  which we differentiate with respect to the imbedding into the gauge group.

1.  $H(A_0) = \{1\}$ ; the trivial subgroup.
2.  $H(A_0) = SO(3)$ ; the diagonal subgroup in  $SO(3) \times SO(3)$ . The B-phase is contained in this class.
3.  $H(A_0) = SO(2) = \{(R;R), R \text{ belongs to } SO(2)\}$ . The unstable 2D-phase is contained in this class.
4.  $H(A_0) = SO(2) \times SO(2) = \{(R_1;R_2), SO(2)_{1,2} \subset SO(3)_{1,2}\}$ . The polar phase is contained in this class.
5.  $H(A_0) = \{SO(2)_i, SO(2)_i \subset SO(3)_i, i = 1, 2\}$ . The A-phase is contained in this class for  $i = 1$ .