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Emmanuel Gutter

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Anharmonic theory of a stack of tethered membranes

Emmanuel Guitter (*)

Materials Department, College of Engineering, University of California, Santa Barbara CA 93106, U.S.A.

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Résumé. — Nous analysons par des techniques de théorie des champs le comportement à grande échelle d'un empilement de membranes réticulées ou polymérisées. Nous montrons que, dans la phase découplée où il n'y a pas de résistance au cisaillement entre membranes successives, la symétrie de rotation dans l'espace ambiant oblige à tenir compte de certains termes anharmoniques. Ces termes additionnels renormalisent les coefficients élastiques et conduisent à une élasticité anormale à grandes distances, de la même manière que pour une membrane réticulée isolée.

Abstract. — We analyse by field theoretical methods the large scale behavior of a stack of tethered or polymerized membranes. We show that, in the decoupled phase where there is no resistance to the shear between successive membranes, anharmonic terms dictated by rotational symmetry in the bulk have to be taken into account. They renormalize the elastic coefficients and lead to a large distance anomalous elasticity, as in the case of a single tethered membrane.

Recent theoretical studies have shown that tethered, or polymerized membranes exhibit rich thermodynamical behavior [1-6]. In contrast to linear polymers, they can at finite temperature remain 2-dimensional and asymptotically flat. Indeed, as a consequence of the coupling of out of plane fluctuations and internal phonon modes, the effective rigidity modulus of the membrane, which measures its effective bending free energy, increases as a power law of the scale of observation. At the same time, the internal elastic properties of the membrane are modified, and governed at large distances by a universal *anomalous elasticity* theory [1, 4, 5]. According to this theory, the effective Lamé coefficients vanish as power laws of the scale of observation and strong phonon fluctuations occur within the plane of the membrane. As a consequence, Hooke's law is no longer valid for the stretching of the membrane. These properties have been the subject of both recent analytical [4, 5] and numerical [6] investigations. Experimental tests are however not yet available. In fact, lamellar phases where the tethered membranes build a periodic stack appear as possible candidates for experiments. One may then ask how properties of the single membrane problem extend to the lamellar case, and in particular, whether membrane stacks also obey some anomalous elasticity law. An important step in answering this question is found in reference [7]. There it is shown that all elastic coefficients of the stack can be obtained from

(*) Service de physique théorique, C.E.N. Saclay, F-91191 Gif-sur-Yvette Cedex, France.

the single membrane theory. These elastic coefficients depend essentially on the distance d between successive membranes, in a way which directly reflects the single membrane anomalous behavior. Moreover, a transition was found for a critical value d_c of d , which separates a coupled phase ($d < d_c$) where the stack behaves like a standard uniaxial elastic 3-D solid from a decoupled phase ($d > d_c$) where successive membranes can translate freely on each other at no shear cost. This decoupled phase exhibits some 2-D elastic properties. However, for fixed d and in both phases, elastic coefficients were found independent of the scale of observation (*i.e.* not renormalized). In other words, no anomalous elasticity was found for the stack itself. This result was actually obtained on the basis of an harmonic theory for the stack. In this paper, we show that, *in the decoupled phase*, anharmonic terms have to be included to renormalize the elastic coefficients of the stack. We find that the process which leads to anomalous properties of a single tethered membrane extends to the stack in its decoupled phase. Although very attenuated, some large scale anomalous elasticity thus persists in the stack.

In section 1, we introduce the model for a stack of tethered membranes and show the relevance of anharmonic terms in the decoupled phase. In section 2, we discuss the renormalization of this model in a particular case and obtain the corresponding Renormalization Group Equation. This equation is solved in section 3 where the resulting anomalous elasticity is discussed.

1. The model.

We first introduce the continuous description of a stack of polymerized membranes in the general case of $D + 1$ dimensions. In this description, each D -dimensional membrane ($D = 2$ for the physical case) is labelled by its rest position in the $(D + 1)$ -dimensional stack : $z = n \cdot d$, with n integer and where d is the interlamellar distance. Each molecule is then labelled by its cartesian coordinates at rest $\sigma = (\sigma^i)_{i=1, \dots, D}$ within the membrane. A deformed configuration is then characterized by the position $\mathbf{X}(\sigma, z)$ of each molecule σ of each membrane z :

$$\mathbf{X}(\sigma, z) = \begin{pmatrix} \sigma^1 + u_1(\sigma, z) \\ \vdots \\ \sigma^D + u_D(\sigma, z) \\ z + u_z(\sigma, z) \end{pmatrix} \quad (1.1)$$

where we introduce here the displacement fields u_i and u_z corresponding respectively to internal phonon modes within the membranes and out of plane fluctuations of the membranes (*i.e.* phonons in the z direction). From these fields, we define the strain tensor invariant in $(D + 1)$ -dimensional space :

$$u_{\alpha\beta} = \frac{1}{2} (\partial_\alpha \mathbf{X} \cdot \partial_\beta \mathbf{X} - \delta_{\alpha\beta}) \quad (1.2)$$

with $\alpha, \beta = i$ or z , $\partial_i = \frac{\partial}{\partial \sigma^i}$, $\partial_z = \frac{\partial}{\partial z}$, namely :

$$\begin{cases} u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k + \partial_i u_z \partial_j u_z) \\ u_{iz} = u_{zi} = \frac{1}{2} (\partial_i u_z + \partial_z u_i + \partial_i u_k \partial_z u_k + \partial_i u_z \partial_z u_z) \\ u_{zz} = \frac{1}{2} (2 \partial_z u_z + \partial_z u_k \partial_z u_k + \partial_z u_z \partial_z u_z) . \end{cases} \quad (1.3)$$

One can then build in the standard way the corresponding elastic energy. Assuming isotropy within each membrane, one obtains the elastic Hamiltonian of an uniaxial $(D + 1)$ -dimensional solid [7]:

$$\mathcal{H}_0 = \int d^D \sigma dz \left[\mu_0^{\perp\perp} (u_{ij})^2 + \frac{1}{2} \lambda_0^{\perp\perp} (u_{ii})^2 + \mu_0^{\perp z} (u_{iz})^2 + \lambda_0^{\perp z} (u_{ii}) (u_{zz}) + \frac{1}{2} B_0 (u_{zz})^2 + \frac{1}{2} K_0 (\partial_{\perp}^2 u_z)^2 \right] \quad (1.4)$$

where $\partial_{\perp}^2 = \frac{\partial}{\partial \sigma^i} \frac{\partial}{\partial \sigma^i}$ is the Laplacian operator orthogonal to the z axis. The last term in equation (1.4) is the curvature (or bending) energy for the membranes. We have replaced the discrete sum over successive membranes by a continuous integral over z . The interlamellar distance d serves then as a short distance cut-off in the z direction.

The outstanding features that distinguish these stack from the general case of uniaxial solids are in the structure of their elastic constants, which have their origin in the fluctuations of a single membrane. K_0 , B_0 , $\mu_0^{\perp\perp}$, $\lambda_0^{\perp\perp}$ and $\lambda_0^{\perp z}$ behave as power laws of the interlamellar distance d . This dependence is discussed for $D = 2$ in reference [7] and is based on the anomalous elasticity theory for polymerized membranes [4, 5]. The main idea is that the interlamellar distance d fixes a maximal in plane wave length for the fluctuations of each membrane by controlling the maximal amplitude for out of plane deformations. The elastic coefficients of the stack are obtained from the renormalized elastic coefficients of the single membrane problem at the corresponding scale. The results of reference [7] extend to any dimension D between the lower critical dimension $D_{lc}^{(1)}$ for this single membrane anomalous theory (with $D_{lc}^{(1)}$ strictly smaller than 2) and the upper critical dimensions $D_{uc}^{(1)} = 4$. For $\mu_0^{\perp z}$, reference [7] predicts the existence of a critical interlamellar distance d_c such that:

$$\begin{aligned} \mu_0^{\perp z}(d) &> 0 \quad \text{for } d < d_c \\ \mu_0^{\perp z}(d) &= 0 \quad \text{for } d \geq d_c. \end{aligned} \quad (1.5)$$

For $d < d_c$, the stack behaves as a $(D + 1)$ -dimensional solid while for $d \geq d_c$, membranes can freely translate with no shear cost. This phase transition is actually peculiar to the $D = 2$ case where in plane fluctuations can be made arbitrarily small or large by varying d and thus strengthen or weaken shear cost. One can show that $\mu_0^{\perp z}$ is strictly positive for any d (i.e. $d_c \rightarrow \infty$) for $D > 2$ where in plane fluctuations are always small and $\mu_0^{\perp z} = 0$ for any d (i.e. $d_c = 0$) for $D < 2$ where in plane fluctuations are always large. In the following, we will be mostly interested in the shearless phase, also called the *decoupled phase* in reference [7], at $D = 2$ and will consider this phase at $D < 2$ for mathematical convenience only.

We now turn to the relevance of anharmonic terms in the Hamiltonian \mathcal{H}_0 , i.e. non-linear terms in u_{ij} , u_{iz} and u_{zz} as defined by equation (1.3):

— For $\mu_0^{\perp z} > 0$ ($d < d_c$), $[\sigma_i] = [z] = 1$ (where brackets indicate dimension in units of length, as defined by power counting), leading to $[u_i] = [u_z] = \frac{1-D}{2} < 0$ for any $D > D_{uc}^{(1)}$ ($D_{uc}^{(1)}$ is strictly larger than 1). As a consequence, all anharmonic terms in \mathcal{H}_0 are easily seen to be irrelevant and the strain tensor can be replaced by its linear form $u_{\alpha\beta} = \frac{1}{2} (\partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha})$. The curvature energy term is also irrelevant in this case and one ends up with the standard *harmonic* theory for $(D + 1)$ -dimensional uniaxial solid. We will not discuss this case further.

— For $\mu_0^{\perp z} = 0$ ($d \geq d_c$) however, the absence of u_{iz} term in the Hamiltonian allows σ^i and z to scale independently, making the result of power counting totally different. Their dimension is then fixed by comparing the compressibility term $(\partial_z u_z)^2$ and the curvature energy term $(\partial_z^2 u_z)^2$. This leads to $[z] = 2[\sigma^i] = 2$. The dimension of displacement fields is now $[u_i] = -\frac{D}{2}$ and $[u_z] = \frac{2-D}{2}$. Anharmonic terms involving $\partial_i u_z$ become relevant for $D \leq 2$. One is then led to the following *anharmonic* theory for the decoupled phase :

$$\mathcal{H}_0^{\text{dec}} = \int d^D \sigma dz \left[\frac{1}{2} K_0 (\partial_z^2 u_z)^2 + \frac{1}{2} B_0 (\partial_z u_z)^2 + \frac{1}{4} \mu_0^{\perp \perp} (\partial_i u_j + \partial_j u_i + \partial_i u_z \partial_j u_z)^2 + \right. \\ \left. + \frac{1}{8} \lambda_0^{\perp \perp} (2 \partial_i u_i + \partial_i u_z \partial_i u_z)^2 + \frac{1}{2} \lambda_0^{\perp z} (\partial_z u_z) (2 \partial_i u_i + \partial_i u_z \partial_i u_z) \right]. \quad (1.6)$$

The anharmonicity in equation (1.6) is similar to that of the anomalous theory for a single polymerized membrane. For a single membrane, the upper critical dimension is $D_{\text{uc}}^{(1)} = 4$ and for D strictly smaller than this critical dimension, including $D = 2$, the renormalized Lamé coefficients (respectively the rigidity modulus) vanish (respectively diverges) at large scales as power laws. This dependence can be obtained for instance in the framework of an $\varepsilon = 4 - D$ expansion. For the Hamiltonian of equation (1.6), the upper critical dimension is shifted to $D_{\text{uc}} = 2$ and the same results hold below this dimension. This will be shown in details in the following chapters in an $\varepsilon = 2 - D$ expansion formalism. The change of the upper critical dimension from 4 to 2 is just due to the added dimension of the extra z component. For $D < 2$, $\mu^{\perp \perp}$ and $\lambda^{\perp \perp}$ (respectively K) are renormalized to zero (respectively to infinity) as power laws of the scale of observation. For $D = 2$, the theory is asymptotically free at large distances and power laws are replaced by a logarithmic dependence. In reference [7] it was emphasized that elastic coefficients renormalize to finite values because of the suppression of large wave length fluctuations of one membrane by the presence of other membranes in the stack. However, *collective* long wave length fluctuations of several successive membranes (undulation modes) are still possible in the stack, with no upper bound on their amplitude. Such fluctuations which are suppressed in the coupled phase because they also imply a shear between the membranes, persist in the decoupled phase. In other words, one goes from a small wave length regime, where the amplitude of out of plane fluctuations is small, governed by the single membrane behavior, to a large wave length regime, where large out of plane amplitudes require fluctuations of several membranes. In the decoupled phase, these fluctuations then continue to renormalize the elastic Lamé coefficients (respectively K) to zero (respectively to infinity) at large distances. This effect in the stack is of course very attenuated in comparison with the single membrane problem, as indicated by the shift in the upper critical dimension. The study of these renormalizations is the purpose of the following chapters.

In conclusion, it should be noted that the situation described above is very similar to the non-linear elastic theory of smectic liquid crystals [8], which can be obtained from $\mathcal{H}_0^{\text{dec}}$ in equation (1.6) by ignoring u_i and by setting $2 \mu_0^{\perp \perp} + \lambda_0^{\perp \perp} = \lambda_0^{\perp z} = B_0$; non-linear terms dictated by rotational symmetry in bulk space lead to logarithmic renormalizations of elastic constants in the model for $D = 2$. As noticed in reference [7], the *harmonic* theory of a shearless stack of polymerized membranes is equivalent to the harmonic theory for smectics by simple integration over u_i . This equivalence breaks down at the *anharmonic* level which we consider here. We are thus in the presence of a new critical theory with new critical exponents which we will now investigate.

2. Renormalization group analysis.

2.1 RENORMALIZABILITY. — We now turn to the renormalization of Hamiltonian $\mathcal{H}_0^{\text{dec}}$ of the decoupled phase. $\mathcal{H}_0^{\text{dec}}$ is invariant under Euclidean transformation within the membranes, translation in the z direction and under the following symmetries :

$$\begin{cases} z \rightarrow -z \\ u_z \rightarrow -u_z \\ u_z \rightarrow u_z + A_i \sigma^i \\ u_i \rightarrow u_i - A_i(z + u_z) - \frac{1}{2} A_i A_j \sigma^j \\ u_i \rightarrow u_i + B_i z \end{cases} \quad (2.1)$$

for arbitrary A_i and B_i , corresponding respectively to reversing the z direction, rotating the stack in bulk space, and performing a shear transformation between successive membranes. Renormalizability requires the consideration of the most general local Hamiltonian satisfying these symmetries and involving relevant terms for $D \leq 2$, namely ⁽¹⁾ :

$$\mathcal{H}_0^{\text{dec}} = \mathcal{H}_0^{\text{dec}} + \int d^D \sigma dz \left[\frac{\tau_0^\perp}{2} (2 \partial_i u_i + \partial_i u_z \partial_i u_z) + \frac{\tau_0^z}{2} (\partial_z u_z) \right]. \quad (2.2)$$

As discussed in reference [5] for a single membrane, the first additional term corresponds to application of a lateral stress in the plane of the membranes. Such a term must be considered in general since the shrinking of the membrane due to thermal fluctuations generates an effective lateral tension when fixed boundary conditions are imposed on the stack. The second additional term has the same origin for compression in the z direction. A complete renormalization treatment thus involves *a priori* seven independent renormalizations ⁽²⁾ and is rather difficult. Here we decide for simplicity to consider only the special case of membrane stacks with $\lambda_0^{\perp z} = \tau_0^z = 0$. This choice strongly reduces the number of diagrams since the propagator is made diagonal and one vertex is suppressed. This property is moreover preserved by renormalization as a consequence of the new symmetry in this case $u_z \rightarrow -u_z$ (or, according to Eq. (2.1), the equivalent symmetry $z \rightarrow -z$). The $\tau_0^z = 0$ condition means that the membrane is free to relax to its equilibrium position in the z direction. The physical relevance of the $\lambda_0^{\perp z} = 0$ assumption is more difficult to motivate. However, one should keep in mind that $\mu_0^{\perp z} = 0$ (not $\lambda_0^{\perp z} = 0$) is the necessary condition for relevance of anharmonic terms in the Hamiltonian. In other words, the presence or the absence of $\lambda_0^{\perp z}$ does not modify *qualitatively* the large distance behavior of the stack, namely the vanishing or divergence of elastic coefficients as power laws or logarithmically. Whether it modifies *quantitatively* this behavior, i.e. changes the corresponding critical exponents, depends on the stability of the fixed manifold $\lambda^{\perp z} = 0$. We will return to this question at the end of this article.

⁽¹⁾ In principle, one should also add a term proportional to $\partial_\perp^2 \partial_z u_z$ but we will ignore it since $\int d^D \sigma dz (\partial_\perp^2 \partial_z u_z) = 0$.

⁽²⁾ One can actually get rid of the τ 's renormalizations in a dimensional regularization formalism by considering only stacks without external applied stress.

2.2 RENORMALIZATION GROUP EQUATION FOR $\lambda^{\perp z} = 0$. — It is convenient to redefine the variables according to :

$$\begin{aligned}\tilde{\sigma}^i &= \sigma^i, & \tilde{u}_i &= (K_0 B_0)^{\frac{1}{2}} u_i \\ \tilde{z} &= K_0^{\frac{1}{2}} B_0^{-\frac{1}{2}} z, & \tilde{u}_z &= (K_0 B_0)^{\frac{1}{4}} u_z\end{aligned}\quad (2.3)$$

and to omit the \sim in what follows. With this implicit rescaling, the Hamiltonian reduces to :

$$\begin{aligned}\mathcal{H} = \int d^D \sigma dz \left[\frac{1}{2} (\partial_{\perp}^2 u_z)^2 + \frac{1}{2} (\partial_z u_z)^2 + \frac{\hat{\tau}_0}{2} (2 \partial_i u_i + \partial_i u_z \partial_i u_z) + \right. \\ \left. + \frac{\hat{\mu}_0}{4} (\partial_i u_j + \partial_j u_i + \partial_i u_z \partial_j u_z)^2 + \frac{\left(\hat{H}_0 - \frac{2}{D} \hat{\mu}_0 \right)}{8} (2 \partial_i u_i + \partial_i u_z \partial_i u_z)^2 \right] \quad (2.4)\end{aligned}$$

and involves, as in the single membrane case, a mass parameter $\sqrt{\hat{\tau}_0} = \sqrt{\frac{\tau_0^{\perp}}{K_0}}$ with dimension -1 and two coupling constants :

$$\hat{\mu}_0 = \mu_0^{\perp \perp} K_0^{-\frac{3}{2}} B_0^{-\frac{1}{2}}, \quad \hat{H}_0 = \left(\lambda_0^{\perp \perp} + \frac{2}{D} \mu_0^{\perp \perp} \right) K_0^{-\frac{3}{2}} B_0^{-\frac{1}{2}} \quad (2.5)$$

with dimension $D - 2$. The combination $H_0^{\perp \perp} = \lambda_0^{\perp \perp} + \frac{2}{D} \mu_0^{\perp \perp}$ which appears in the definition of \hat{H}_0 is the compression modulus within a single membrane. Renormalization of Hamiltonian (2.5) is discussed in Appendix A in the framework of an $\varepsilon = 2 - D$ expansion. It requires *a priori* five independent renormalizations :

- one common multiplicative wave function renormalization Z for both u_i and u_z ,
- a multiplicative renormalization Z_z of the z coordinate,
- two multiplicative renormalizations Z_{μ} and Z_H for $\hat{\mu}$ and \hat{H} ,
- a « mass » renormalization Z_{τ} of $\hat{\tau}$, multiplicative in the ε expansion formalism.

It can be shown however (see Appendix B) that $Z_z = 1$ at all order in perturbation theory, as an additional consequence of the absence of a $u_{ij} u_{zz}$ term (i.e. of the $\lambda^{\perp z} = 0$ simplification) and one is left with only four renormalizations. The scaling behavior of the stack is then encoded in the Renormalization Group Equation for the renormalized 1-particle irreducible function $\Gamma_R^{(N, M)}$ with N external u_z legs and M external u_i legs. In momentum space, it reads (see Appendix A) :

$$\begin{aligned}\left[\sum_{\alpha} p_{\perp}^{\alpha} \frac{\partial}{\partial p_{\perp}^{\alpha}} + \left(2 - \frac{\gamma}{2} \right) \sum_{\alpha} p_{zR}^{\alpha} \frac{\partial}{\partial p_{zR}^{\alpha}} + (2 - \gamma_{\tau}) \hat{\tau}_R \frac{\partial}{\partial \hat{\tau}_R} - \beta_{\mu} \frac{\partial}{\partial \hat{\mu}_R} - \right. \\ \left. - \beta_H \frac{\partial}{\partial \hat{H}_R} - \left(D + 2 - \frac{\gamma}{2} \right) - \frac{N}{2} \left(2 - D - \frac{\gamma}{2} \right) - M \left(1 - D - \frac{\gamma}{2} \right) \right] \Gamma_R^{(N, M)} = 0 \quad (2.6)\end{aligned}$$

involving only four Wilson functions β_{μ} , β_H , γ and γ_{τ} , functions of the renormalized coupling constants $\hat{\mu}_R$ and \hat{H}_R only, corresponding to the remaining four independent

renormalizations. The Wilson functions are calculated at one loop order in Appendix B, with the following result :

$$\beta_\mu = -\varepsilon \hat{\mu}_R + \frac{\hat{\mu}_R^2}{32\pi} \left(\frac{\hat{\mu}_R + 19 \hat{H}_R}{\hat{\mu}_R + \hat{H}_R} \right), \quad \beta_H = -\varepsilon \hat{H}_R + \frac{\hat{H}_R^2}{16\pi} \left(\frac{10 \hat{\mu}_R + \hat{H}_R}{\hat{\mu}_R + \hat{H}_R} \right) \quad (2.7)$$

$$\gamma = \frac{3}{8\pi} \left(\frac{\hat{\mu}_R \hat{H}_R}{\hat{\mu}_R + \hat{H}_R} \right), \quad \gamma_\tau = \gamma + \frac{\hat{H}_R}{16\pi}.$$

3. Anomalous elasticity.

3.1 THE $D < 2$ CASE. — We will briefly discuss the anomalous elasticity in the case $D < 2$. However this case is academic since the lower critical dimension for anomalous elasticity is strictly larger than 1 and thus no integer dimension lies in the corresponding interval. Moreover, it is completely similar to the single membrane case for $D < 4$ which is extensively discussed in reference [5]. We will thus give its main properties, namely :

- (i) $\hat{\tau}_R = 0$ is a critical surface, corresponding to a stack with no applied lateral stress.
- (ii) In this surface, the $\hat{\mu}_R = 0$ and $\hat{H}_R = 0$ lines are renormalized onto themselves as a consequence of new symmetries in these cases (see Ref. [5]).
- (iii) The renormalization flow in the $\hat{\tau}_R = 0$ surface has four fixed points. One purely repulsive fixed point (in the infrared) at the origin, one partially attractive fixed point on each of the $\hat{\mu}_R = 0$ and $\hat{H}_R = 0$ lines, and one purely attractive fixed point with both $\hat{\mu}_R$ and \hat{H}_R non zero.

At first order, the three non trivial fixed points, and the corresponding anomalous dimensions are listed in table I.

Table I.

Fixed Point	γ^*	γ_τ^*
$\hat{\mu}_R = 0, \hat{H}_R = 16\pi\varepsilon$	0	ε
$\hat{\mu}_R = 32\pi\varepsilon, \hat{H}_R = 0$	0	0
$\hat{\mu}_R = \frac{32\pi}{7}\varepsilon, \hat{H}_R = \frac{16\pi}{7}\varepsilon$	$\frac{4}{7}\varepsilon$	$\frac{5}{7}\varepsilon$

It can be shown (see Ref. [5] for the derivation of an analogous result) that as a consequence of an equation of motion, the relation $\gamma_\tau^* + \frac{1}{2}\gamma^* = \varepsilon$ holds at all order in the ε expansion at the fixed points with $\hat{H}_R \neq 0$, reducing to one the number of independent critical exponents. This exponent appears for instance in the scaling forms for $\Gamma_R^{(2,0)}(\mathbf{p}_\perp, p_{zR})$ or $\Gamma_R^{(0,2)}(\mathbf{p}_\perp, p_{zR})$ in the absence of lateral stress ($\hat{\tau}_R = 0$) :

$$\Gamma_R^{(2,0)}(\mathbf{p}_\perp, p_{zR}) = |\mathbf{p}_\perp|^{4-\gamma^*} X \left(|\mathbf{p}_\perp|^{\frac{\gamma^*}{2}-2} p_{zR} \right)$$

$$\Gamma_R^{(0,2)}(\mathbf{p}_\perp, p_{zR}) = |\mathbf{p}_\perp|^{2+\varepsilon-\frac{3}{2}\gamma^*} Y \left(|\mathbf{p}_\perp|^{\frac{\gamma^*}{2}-2} p_{zR} \right). \quad (3.1)$$

Notice that from the first equation, one obtains the property $\Gamma_R^{(2,0)}(\mathbf{0}, p_{zR}) \propto p_{zR}^2$ which means that the stack compression modulus B_0 is not renormalized. This property, which is actually equivalent to the $Z_z = 1$ property, is however a consequence of the $\lambda_0^{\perp z} = 0$ assumption, i.e. the absence of coupling between compression modes in the z direction and the orthogonal direction (within the membranes). Finally, we obtain the modification of Hooke's law for the relation between the lateral elongation $e = \frac{1}{D} \langle \partial_i u_i \rangle$ and the applied lateral stretching force f from the corresponding Renormalization Group Equation :

$$\left[(2 - \gamma) f \frac{\partial}{\partial f} - \beta_\mu \frac{\partial}{\partial \hat{\mu}_R} - \beta_H \frac{\partial}{\partial \hat{H}_R} \right] e = \left(D + \frac{\gamma}{2} \right) e \quad (3.2)$$

which at the fixed point gives :

$$e \sim f^{\frac{D + \frac{\gamma^*}{2}}{2 - \gamma^*}}. \quad (3.3)$$

3.2 THE $D = 2$ CASE. — The properties (i) and (ii) described in the preceding paragraph extend to the $D = 2$ case. The four fixed points described in (iii) now merge into one single attractive fixed point at the origin. The theory is thus asymptotically free. The general solution of the Renormalization Group equation involves the Renormalization group flow $(\hat{\mu}(\ell), \hat{H}(\ell))$ defined by :

$$\begin{aligned} \ell \frac{\partial \hat{\mu}}{\partial \ell} &= \beta_\mu(\hat{\mu}(\ell), \hat{H}(\ell)) \\ \ell \frac{\partial \hat{H}}{\partial \ell} &= \beta_H(\hat{\mu}(\ell), \hat{H}(\ell)) \end{aligned} \quad (3.4)$$

with the initial conditions $\hat{\mu}(1) = \hat{\mu}_R$, $\hat{H}(1) = \hat{H}_R$, and the corresponding scaling functions $g(\ell)$ and $g_\tau(\ell)$ being solutions of :

$$\begin{aligned} \ell \frac{\partial g}{\partial \ell} &= \frac{1}{2} \gamma(\hat{\mu}(\ell), \hat{H}(\ell)) g(\ell) \\ \ell \frac{\partial g_\tau}{\partial \ell} &= \gamma_\tau(\hat{\mu}(\ell), \hat{H}(\ell)) g_\tau(\ell) \end{aligned} \quad (3.5)$$

with initial conditions $g(1) = g_\tau(1) = 1$. The general solution for $\Gamma_R^{(N,M)}$ satisfies :

$$\Gamma_R^{(N,M)}(\mathbf{p}_\perp^\alpha, p_{zR}^\alpha, \hat{\tau}_R) = \frac{\ell^{4-M}}{(g(\ell))^{1 + \frac{N}{2} + M}} \Gamma_R^{(N,M)}\left(\frac{\mathbf{p}_\perp^\alpha}{\ell}, \frac{p_{zR}^\alpha}{\ell^2} g(\ell), \frac{\hat{\tau}_R}{\ell^2} g_\tau(\ell), \hat{\mu}(\ell), \hat{H}(\ell)\right). \quad (3.6)$$

The solution of equation (3.4) for the one-loop values of the Beta functions (2.7) is given in Appendix C. To obtain the large distance properties of the stack, one needs only its behavior at small ℓ . This behavior depends only on whether or not one of the initial values $\hat{\mu}_R$ and \hat{H}_R vanishes (see Appendix C). For the general case where both are not zero, one has :

$$\hat{\mu}(\ell) \sim \frac{\ell \rightarrow 0}{7 \ln\left(\frac{1}{\ell}\right)} \frac{32 \pi}{}, \quad \hat{H}(\ell) \sim \frac{\ell \rightarrow 0}{14 \ln\left(\frac{1}{\ell}\right)} \frac{32 \pi}{}. \quad (3.7)$$

The corresponding solutions of (3.5) behave as :

$$g(\ell) \sim g^0 \left(\ln \left(\frac{1}{\ell} \right) \right)^{-\frac{2}{7}}, \quad g_r(\ell) \sim g_r^0 \left(\ln \left(\frac{1}{\ell} \right) \right)^{-\frac{5}{7}}. \tag{3.8}$$

From these equations, we can get the small p_\perp (or small p_z) behavior of the $\Gamma^{(N, M)}$ s. For $\Gamma_R^{(2, 0)}(\mathbf{p}_\perp, p_{zR})$ at $\tau_R = 0$ for instance, by choosing $\ell = |\mathbf{p}_\perp| \rightarrow 0$, the coupling constants $\hat{\mu}(|\mathbf{p}_\perp|)$ and $\hat{H}(|\mathbf{p}_\perp|)$ go to 0 and one can replace the right hand side of equation (3.6) by its classical value, namely :

$$\begin{aligned} \Gamma_R^{(2, 0)}(\mathbf{p}_\perp, p_{zR}) &\stackrel{|\mathbf{p}_\perp| \rightarrow 0}{\sim} \frac{|\mathbf{p}_\perp|^4}{(g(|\mathbf{p}_\perp|))^2} \left(1 + \left(\frac{p_{zR}}{|\mathbf{p}_\perp|^2} g(|\mathbf{p}_\perp|) \right)^2 \right) \\ &\sim \frac{|\mathbf{p}_\perp|^4}{(g^0)^2} \left(\ln \left(\frac{1}{|\mathbf{p}_\perp|} \right) \right)^{\frac{4}{7}} + p_{zR}^2. \end{aligned} \tag{3.9}$$

From this expression, we see that the compression modulus B_0 is not renormalized. As mentioned previously, this is a consequence of the $\lambda^{\perp z} = 0$ assumption. The effective rigidity modulus $K(|\mathbf{p}_\perp|)$ diverges at small $|\mathbf{p}_\perp|$ like :

$$K(|\mathbf{p}_\perp|) \propto \left(\ln \left(\frac{1}{|\mathbf{p}_\perp|} \right) \right)^{\frac{4}{7}}. \tag{3.10}$$

An analogous argument for $\Gamma_R^{(0, 2)}$ leads to the vanishing of the Lamé coefficients at small $|\mathbf{p}_\perp|$:

$$\begin{aligned} \mu(|\mathbf{p}_\perp|) \propto H(|\mathbf{p}_\perp|) &\propto \left(\ln \left(\frac{1}{|\mathbf{p}_\perp|} \right) \right)^{-\frac{1}{7}} \\ \frac{\mu(|\mathbf{p}_\perp|)}{H(|\mathbf{p}_\perp|)} &\stackrel{|\mathbf{p}_\perp| \rightarrow 0}{\rightarrow} 2. \end{aligned} \tag{3.11}$$

The same relations hold at small p_{zR} with $|\mathbf{p}_\perp|$ replaced by p_{zR} . Finally from equation (3.2), we obtain :

$$e(f) = \ell^2 g(\ell) e \left(\frac{f}{\ell^2} g^2(\ell), \hat{\mu}(\ell), \hat{H}(\ell) \right). \tag{3.12}$$

From this and the classical relation $e \propto \frac{f}{H}$ we obtain the modified Hooke's law :

$$e(f) \propto f \left(\ln \left(\frac{1}{f} \right) \right)^{\frac{1}{7}}. \tag{3.13}$$

4. Conclusion and discussion.

We have shown that, in its decoupled phase ($\mu_0^{\perp z} = 0$), a stack of tethered membranes obeys an anomalous elasticity law. As for a single tethered membrane in its flat phase, the Lamé coefficients vanish at large distances, while the rigidity modulus diverges. Power laws are,

however, replaced by logarithmic dependences (Eqs. (3.10), (3.11)). This is illustrated for instance by a modification of Hooke's law (Eq. (3.13)) for the relation between lateral stretching force and the corresponding lateral elongation. We have made these corrections explicit in the special case where no coupling is introduced between stack compression (in the z direction) and in plane compression, namely when $\lambda_0^{\perp z} = 0$. These logarithmic corrections could also in principle be seen in the structure factors (see Ref. [8] for an analogous result).

Let us finally discuss the effect of a non vanishing $\lambda^{\perp z}$ in the theory. This first of all does not modify the existence of logarithmic corrections, which are a consequence of the vanishing of $\mu_0^{\perp z}$ only. Whether or not the critical exponents obtained above for K , μ and H are modified depends on the stability of the $\lambda^{\perp z} = 0$ manifold. The relevance of the corresponding operator $\int d^D \sigma dz u_{ii} u_{zz}$ for $\varepsilon > 0$ is given by its dimension *at the non trivial attractive fixed point*, which is found to be $\frac{1}{2} \left(\varepsilon - \frac{3}{2} \gamma^* \right)$ in units of length. Since we want the exponent in Hooke's law (3.3) to be smaller than 1 [5], this dimension is easily seen to be positive (this can be verified with the one loop value obtained for γ^*); we thus have a *relevant perturbation*.

As far as K and B are concerned, one can integrate over the phonons u_i and get an effective theory for u_z . The results of this integration are :

(i) a shift of B : $B_0 \rightarrow B_0 - \frac{(\lambda_0^{\perp z})^2}{2 \mu_0^{\perp \perp} + \lambda_0^{\perp \perp}}$;

(ii) a long distance interaction within each membrane between Gaussian curvatures at different points of the membrane. This intra-membrane long range interaction is exactly the same as the one found in the single membrane theory [5]. It is in particular independent of $\lambda_0^{\perp z}$;

(iii) a long distance interaction within each membrane between the Gaussian curvature at one point and the local compression at some other point, namely :

$$\frac{\lambda_0^{\perp z} \mu_0^{\perp \perp}}{2 \mu_0^{\perp \perp} + \lambda_0^{\perp \perp}} \int dz d^D \sigma d^D \sigma' R(\sigma, z) \left(\frac{1}{-\partial_{\perp}^2} \right)_{\sigma \sigma'} \partial_z u_z(\sigma', z) \tag{4.1}$$

with

$$R = \varepsilon_{ij} \varepsilon_{kl} \partial_i \partial_k u_z \partial_j \partial_l u_z .$$

Besides introducing the simple shift (i), a non vanishing $\lambda^{\perp z}$ has thus the non-trivial effect of introducing a new long distance coupling (iii) between compression and curvature. Since (4.1) involves compression in the z direction, it seems unlikely that B remains unrenormalized. In any case, these issues, which require a full renormalization treatment, go beyond the scope of this paper. Further investigations of the $\lambda_0^{\perp z} \neq 0$ case are in progress. To end this discussion, we would like to mention that the actual importance of $\lambda^{\perp z}$ in the bare theory is measured by

the dimensionless (for any D) quantity $\alpha_0 = \frac{(\lambda_0^{\perp z})^2}{((2/D) \mu_0^{\perp \perp} + \lambda_0^{\perp \perp}) B_0}$ which, to satisfy a stability requirement, lies in the range $0 \leq \alpha_0 \leq 1$. According to the results of reference [7], it behaves as $\alpha_0(d) \sim d^{-4}$ at $D = 2$ and can thus be made arbitrarily small by increasing the distance d between successive membranes.

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Appendix A.

Renormalization requires to rewrite the Hamiltonian (2.4) in its renormalized form :

$$\mathcal{H} = \int d^D \sigma dz_R \left[\frac{Z}{2} (\partial_{\perp}^2 u_z^R)^2 + \frac{Z_z}{2} (\partial_{z_R} u_z^R)^2 + \frac{Z_{\tau} \hat{\tau}_R}{2} (2 \partial_i u_i^R + \partial_i u_z^R \partial_i u_z^R) + \right. \\ \left. + \frac{Z_{\mu} m^{\varepsilon} \hat{\mu}_R}{4} (\partial_i u_j^R + \partial_j u_i^R + \partial_i u_z^R \partial_j u_z^R)^2 + \right. \\ \left. + \frac{\left(Z_H m^{\varepsilon} \hat{H}_R - \frac{2}{D} Z_{\mu} m^{\varepsilon} \hat{\mu}_R \right)}{8} (2 \partial_i u_i^R + \partial_i u_z^R \partial_i u_z^R)^2 \right] \quad (\text{A1})$$

m is the renormalization mass scale. As discussed in Appendix B, one can actually choose $Z_z = 1$ at all orders in the expansion. Equation (A1) is then equivalent to define the renormalized quantities as :

$$z_R = Z^{-\frac{1}{2}} z \quad \left(\text{and } p_{zR} = Z^{\frac{1}{2}} p_z \right) \\ u_z^R = Z^{-\frac{1}{4}} u_z, \quad u_i^R = Z^{-\frac{1}{2}} u_i \\ Z_{\mu} m^{\varepsilon} \hat{\mu}_R = Z^{\frac{3}{2}} \hat{\mu}_0, \quad Z_H m^{\varepsilon} \hat{H}_R = Z^{\frac{3}{2}} \hat{H}_0 \\ Z_{\tau} \hat{\tau}_R = Z \hat{\tau}_0. \quad (\text{A2})$$

Using the equality of :

$$\int \prod_{\alpha=1}^{N+M} (d^D \boldsymbol{\sigma}^{\alpha} dz_I^{\alpha} d^D \mathbf{p}_{\perp}^{\alpha} dp_{zI}^{\alpha} e^{i(\mathbf{p}_{\perp}^{\alpha} \cdot \boldsymbol{\sigma}^{\alpha} + p_{zI}^{\alpha} z_I^{\alpha})}) \prod_{\beta=1}^N u_z^I(\boldsymbol{\sigma}^{\beta}, z_I^{\beta}) \prod_{\gamma=N+1}^{N+M} u_i^I(\boldsymbol{\sigma}^{\gamma}, z_I^{\gamma}) \times \\ \times \delta^{(D)} \left(\prod_{\zeta=1}^{N+M} \mathbf{p}_{\perp}^{\zeta} \right) \delta \left(\prod_{\eta=1}^{N+M} p_{zI}^{\eta} \right) \Gamma_{I_i^{(N, M)} \dots I_{N+M}}(\{\mathbf{p}_{\perp}^{\theta}\}, \{p_{zI}^{\kappa}\}, \hat{\tau}_I, \hat{\mu}_I, \hat{H}_I) \quad (\text{A3})$$

for the renormalized theory with $I = R$ and the bare one with $I = 0$, we get the relation :

$$\Gamma_0^{(N, M)}(\{\mathbf{p}_{\perp}^{\alpha}\}, \{p_{z0}^{\beta}\}, \hat{\tau}_0, \hat{\mu}_0, \hat{H}_0) = Z^{-\frac{1}{4}(N+2M+2)} \times \\ \times \Gamma_R^{(N, M)}(\{\mathbf{p}_{\perp}^{\alpha}\}, \{p_{zR}^{\beta}\}, \hat{\tau}_R, \hat{\mu}_R, \hat{H}_R). \quad (\text{A4})$$

Expressing first the fact that the left-hand side of equation (A4) is independent of the renormalization mass scale m , and using then the homogeneity equation for $\Gamma_R^{(N, M)}$ (which has dimension $D + 2 + N(1 + D/2) + M(1 + D)$ in units of mass), we obtain in the standard way the Renormalization Group Equation (2.6) with the definitions :

$$\gamma = m \frac{\partial}{\partial m} \ln Z \Big|_0, \quad \gamma_{\tau} = m \frac{\partial}{\partial m} \ln (Z/Z_{\tau}) \Big|_0 \\ \beta_{\mu} = m \frac{\partial}{\partial m} \hat{\mu}_R \Big|_0, \quad \beta_H = m \frac{\partial}{\partial m} \hat{H}_R \Big|_0 \quad (\text{A5})$$

where the index 0 means that the derivatives are made at constant bare values.

Appendix B.

We compute here the renormalization factors Z , Z_z , Z_μ , Z_H and Z_τ at one loop in $\varepsilon = 2 - D$ in a minimal subtraction scheme. Decomposing the renormalized form (A1) of \mathcal{K} into its classical part and counterterms, one can read the following diagrammatic rules :

— The $\langle u_z^R u_z^R \rangle$ propagator is :

$$\begin{array}{c} \vec{p} \\ \longrightarrow \end{array} = (|\mathbf{p}_\perp|^4 + \hat{\tau}_R |\mathbf{p}_\perp|^2 + p_{zR}^2)^{-1}. \tag{B1}$$

— The $\langle u_i^R u_j^R \rangle$ propagator is :

$$i \begin{array}{c} \vec{p} \\ \dashrightarrow \end{array} j = \frac{m^{-\varepsilon}}{|\mathbf{p}_\perp|^2} \left[\frac{1}{\hat{\mu}_R} \left(\delta_{ij} - \frac{p_\perp^i p_\perp^j}{|\mathbf{p}_\perp|^2} \right) + \frac{1}{\left(2 - \frac{2}{D}\right) \hat{\mu}_R + \hat{H}_R} \frac{p_\perp^i p_\perp^j}{|\mathbf{p}_\perp|^2} \right]. \tag{B2}$$

— A first vertex involves four u_z^R external legs :

$$\begin{array}{c} 1 \\ \diagdown \\ \diagup \\ 2 \end{array} \begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 4 \end{array} = m^\varepsilon \frac{\left(2 - \frac{2}{D}\right) \hat{\mu}_R + \hat{H}_R}{8} (\mathbf{p}_\perp^{(1)} \cdot \mathbf{p}_\perp^{(2)}) (\mathbf{p}_\perp^{(3)} \cdot \mathbf{p}_\perp^{(4)}). \tag{B3}$$

— A second vertex involves two u_z^R and one u_i^R external legs :

$$\begin{array}{c} 1 \\ \diagdown \\ \diagup \\ 2 \end{array} \dashrightarrow 3i = -\frac{i}{2} m^\varepsilon \left[\hat{\mu}_R ((\mathbf{p}_\perp^{(1)} \cdot \mathbf{p}_\perp^{(3)}) p_\perp^{(2)i} + (\mathbf{p}_\perp^{(2)} \cdot \mathbf{p}_\perp^{(3)}) p_\perp^{(1)i}) + \left(\hat{H}_R - \frac{2}{D} \hat{\mu}_R \right) (\mathbf{p}_\perp^{(1)} \cdot \mathbf{p}_\perp^{(2)}) p_\perp^{(3)i} \right]. \tag{B4}$$

The one loop diagrams divergent at $D = 2$ are topologically identical to the one loop diagrams, divergent at $D = 4$, of the single membrane theory. They are listed in reference [5] and will not be reproduced here.

The one-loop renormalizations of $\hat{\mu}$ and \hat{H} are obtained from the pole at $\varepsilon = 0$ of the one-loop contribution to $\Gamma^{(0,2)}$, namely :

$$i \begin{array}{c} \longrightarrow \\ \circlearrowleft \\ \dashrightarrow \end{array} j = \frac{m^{2\varepsilon}}{32 \pi \varepsilon} \frac{|\mathbf{p}_\perp|^2}{\hat{\tau}_R^{\varepsilon/2}} \left[\hat{\mu}_R^2 \left(\delta_{ij} - \frac{p_\perp^i p_\perp^j}{|\mathbf{p}_\perp|^2} \right) + (\hat{\mu}_R^2 + 2 \hat{H}_R^2) \frac{p_\perp^i p_\perp^j}{|\mathbf{p}_\perp|^2} \right] + O(1). \tag{B5}$$

This pole is removed by the corresponding counterterms with :

$$\begin{aligned} Z_\mu &= 1 + \frac{\hat{\mu}_R}{32 \pi \varepsilon} \\ Z_H &= 1 + \frac{\hat{H}_R}{16 \pi \varepsilon}. \end{aligned} \tag{B6}$$

The renormalization factors Z , Z_z and Z_τ are obtained in the same way from the pole in ε of the one loop contribution to $\Gamma^{(2,0)}$:

$$\begin{aligned}
 & \text{Diagram 1} + \text{Diagram 2} = \frac{-m^\varepsilon}{16 \pi \varepsilon \hat{\tau}_R^{1/2}} \left[6 |\mathbf{p}_\perp|^4 \frac{\hat{\mu}_R \hat{H}_R}{\hat{H}_R + \hat{\mu}_R} - \hat{\tau}_R |\mathbf{p}_\perp|^2 \hat{H}_R \right] + O(1) : \\
 & \hspace{15em} \text{(B7)}
 \end{aligned}$$

This pole is removed by choosing :

$$\begin{aligned}
 Z &= 1 - \frac{3}{8 \pi \varepsilon} \frac{\hat{\mu}_R \hat{H}_R}{\hat{\mu}_R + \hat{H}_R} \\
 Z_z &= 1 \\
 Z_\tau &= 1 + \frac{\hat{H}_R}{16 \pi \varepsilon} .
 \end{aligned}
 \tag{B8}$$

Since to all external legs of the vertices is attached a p_\perp^i , it is easy to see from the diagrammatic rules that the « self energy » $\Sigma(\mathbf{p}_\perp, p_{zR}) = \Gamma_R^{(2,0)}(\mathbf{p}_\perp, p_{zR}) - (Z |\mathbf{p}_\perp|^4 + Z_\tau \hat{\tau}_R |\mathbf{p}_\perp|^2 + Z_z p_{zR}^2)$ is proportional to $|\mathbf{p}_\perp|^2$. It is thus impossible to build a diagram with a divergence proportional to p_{zR}^2 only and the result $Z_z = 1$ thus holds *at all orders* in perturbation theory.

From (B6) and (B8), and their definition (A5), we get the Wilson functions (2.7).

Appendix C.

We discuss here the Renormalization Group flow $(\hat{\mu}(\ell), \hat{H}(\ell))$ in the $\hat{\tau}_R = 0$ plane at $D = 2$, as defined by equation (3.4) and calculated from the one-loop Beta functions (2.7) with $\varepsilon = 0$. One must distinguish three cases :

(i) Starting from arbitrary initial non vanishing values $(\hat{\mu}_R, \hat{H}_R)$ at $\ell = 1$ we get the general solution for the flow with decreasing ℓ :

$$\begin{cases}
 \hat{\mu}(\ell) = \left(\frac{7}{32 \pi} \ln \left(\frac{1}{\ell} \right) + \frac{1}{3} (\hat{\mu}_R^{-1} + \hat{H}_R^{-1}) + \frac{1}{3} (2 \hat{\mu}_R^{-1} - \hat{H}_R^{-1}) \left(1 + \frac{\frac{21}{32 \pi} \ln \left(\frac{1}{\ell} \right)}{\hat{\mu}_R^{-1} + \hat{H}_R^{-1}} \right)^{\frac{6}{7}} \right)^{-1} \\
 \hat{H}(\ell) = \left(\frac{14}{32 \pi} \ln \left(\frac{1}{\ell} \right) + \frac{2}{3} (\hat{\mu}_R^{-1} + \hat{H}_R^{-1}) - \frac{1}{3} (2 \hat{\mu}_R^{-1} - \hat{H}_R^{-1}) \left(1 + \frac{\frac{21}{32 \pi} \ln \left(\frac{1}{\ell} \right)}{\hat{\mu}_R^{-1} + \hat{H}_R^{-1}} \right)^{\frac{6}{7}} \right)^{-1} .
 \end{cases}
 \tag{C1}$$

This flow is represented in figure 1. The line $\hat{\mu} = 2 \hat{H}$ is a fixed line *at first order* of perturbation, with the corresponding flow :

$$\hat{\mu}(\ell) = 2 \hat{H}(\ell) = \frac{\hat{\mu}_R}{1 + \frac{7 \hat{\mu}_R}{32 \pi} \ln \left(\frac{1}{\ell} \right)} = 2 \frac{\hat{H}_R}{1 + \frac{14 \hat{H}_R}{32 \pi} \ln \left(\frac{1}{\ell} \right)} .
 \tag{C2}$$

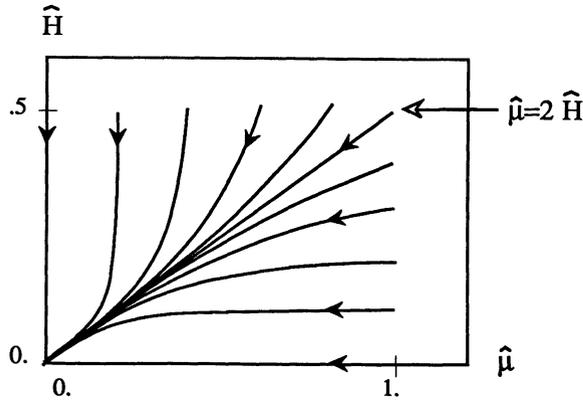


Fig. 1. — Renormalization Group flow of \hat{H} and $\hat{\mu}$ at $D = 2$ on the critical surface $\hat{\tau}_R = 0$. This flow is obtained from the one loop beta functions.

The small ℓ behavior is actually governed by this line for any initial value in regime (i) and is given by equation (3.7). It satisfies $\frac{\hat{\mu}(\ell)}{\hat{H}(\ell)} \xrightarrow{\ell \rightarrow 0} 2$.

(ii) Starting with $\hat{\mu}_R = 0$, one stays on the $\hat{\mu} = 0$ line, which is a fixed line at any order in perturbation (as explained in Ref. [5]). The corresponding flow at first order for \hat{H} is :

$$\hat{H}(\ell) = \frac{\hat{H}_R}{1 + \frac{\hat{H}_R}{16\pi} \ln\left(\frac{1}{\ell}\right)}. \quad (\text{C3})$$

(iii) Similarly, starting from $\hat{H}_R = 0$ leads to the following flow for $\hat{\mu}$ on the fixed (at all orders) line $\hat{H} = 0$:

$$\hat{\mu}(\ell) = \frac{\hat{\mu}_R}{1 + \frac{\hat{\mu}_R}{32\pi} \ln\left(\frac{1}{\ell}\right)}. \quad (\text{C4})$$

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