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## On the replica approach to random directed polymers in two dimensions

Giorgio Parisi

Dipartimento di Fisica, INFN sezione Tor Vergata, II Università di Roma Tor Vergata, Prolungamento di Via del Fontanile di Carcaricola, Roma 00173, Italy

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**Abstract.** — In this note we study directed polymers in a two dimensional random medium with short range noise using the replica approach. We find the predictions of the replica symmetric theory and we compare them with exact results. We consider the possibility of spontaneous symmetry breaking and we suggest that replica symmetry is weakly broken also in this two dimensional model.

### 1. Introduction.

The study of directed polymers in a random medium is very interesting [1]. In a nutshell the problem of directed random polymers consists in finding the probability distribution of

$$G_\eta(x, t) = \int d\mu[\omega]_{x,t},$$

$$d\mu[\omega]_{x,t} \equiv dW[\omega]_{x,t} \exp\left(\int_0^t d\tau \eta(\omega(\tau), \tau)\right), \quad (1.1)$$

where  $dW[\omega]_{x,t}$  is the usual Wiener measure for going from 0 to  $x$  in time  $t$  and  $\eta$  is the noise. We notice that equation (1.1) can be also read as the probability distribution of an heteropolymer (without self excluding effects) on a substrate, where the interaction among the substrate and each component of the heteropolymer is random.

Typical quantities we would like to compute are

$$P(x, t) = \langle P_\eta(x, t) \rangle,$$

$$P(x, y, t) = \langle P_\eta(x, t) P_\eta(y, t) \rangle, \quad (1.2)$$

where the brackets denote the average over the noise  $\eta$  and  $P_\eta(x, t)$  is the normalized probability for finding a directed polymer at point  $x$  at time  $t$ , which is given by :

$$P_\eta(x, t) = G_\eta(x, t)/G_\eta(t),$$

$$G_\eta(t) \equiv \int dy G_\eta(y, t). \quad (1.3)$$

We are also interested in quantities like the probability distribution of the self overlap  $q$ , which is defined as :

$$q = \int dx dy d\mu[\omega_1]_{x,t} d\mu[\omega_2]_{y,t} \int d\tau \delta(\omega_1(\tau) - \omega_2(\tau)) / (G_\eta(t)^2). \quad (1.4)$$

Different models are obtained by choosing different probability distributions for the noise [2, 3]. In two dimensions (one space, one time) a very interesting case is for the white noise limit, i.e. the noise is Gaussian, with zero average and with variance

$$\langle \eta(x_1, t_1) \eta(x_2, t_2) \rangle = \lambda \delta(t_1 - t_2) \delta(x_1 - x_2). \quad (1.5)$$

The replica approach is based on the following observation : the expectation value of the products of  $G$ 's, e.g.

$$\langle G_\eta(x_1, t) G_\eta(x_2, t) \dots G_\eta(x_n, t) \rangle = G^{(n)}(x_1, x_2, \dots, x_n; t), \quad (1.6)$$

satisfies a Schroedinger type equation at imaginary time (i.e.  $\partial G / \partial t = -HG$ ) with an  $n$ -body Hamiltonian given by

$$H_n = \sum_{i=1,n} 1/2 p_i^2 - \sum_{i=1,n} \sum_{k=1,n} \lambda \delta(x_i - x_k). \quad (1.7)$$

The most likely behaviour of  $G_\eta(x_1, t)$  can be related to the properties of  $G^{(n)}$  near  $n = 0$ .

This model is particularly interesting as far as the Hamiltonian (1.7) is soluble. The ground state wave function and energy are given by

$$\begin{aligned} \psi(x_1, x_2, \dots, x_n) &= \exp\left(-\lambda \sum_{i,k=1,n} |x_i - x_k|\right) \\ E_n &= \lambda^2 n(1 - n^2), \end{aligned} \quad (1.8)$$

where we have added to the Hamiltonian a constant proportional to  $n\lambda \delta(0)$  in order to remove the divergent terms which appear when  $i = k$  in equation (1.7).

These observations were used by Kardar in his original study of the model [4]. In this note we discuss more carefully the properties of the model. After this introduction, we will analyze the results that we obtain neglecting the possibility of breaking of the replica symmetry. In the third section we compare these predictions with the known results coming from the mapping onto the random stirred Burger equation. In the fourth section we will show that replica symmetry is (in some sense) broken and finally, in the last section, we present our conclusions.

There are also two appendices in which some technical problems are discussed. In the first appendix we study some properties of random walks, while in the second appendix we present some exact computations of the Green function for the Hamiltonian  $H_n$  for  $n \neq 0$ .

## 2. The replica symmetric approach.

In the replica approach the behaviour of the energy as function of  $n$  is related to the cumulants of

$$F_\eta(t) \equiv -\ln(G_\eta(t)). \quad (2.1)$$

Indeed we have that

$$\langle (F_\eta(t)^k) \rangle_c \approx d^k E_n / dn^k . \tag{2.2}$$

The absence of a term proportional to  $n^2$  in the energy lead Kardar [4] to the conclusions that the fluctuations in  $F_\eta(t)$  are of order  $t^\omega$  with  $\omega = 1/3$ . Using scaling arguments this result implies that

$$\langle x^2(t) \rangle \equiv \int dx x^2 P(x, t) \approx t^{2\nu} , \tag{2.3}$$

with

$$2\nu = 1 + \omega = 4/3 . \tag{2.4}$$

These results agree with the analysis done using the mapping onto the random stirred Burger equation [5] (see next section) : one proves that the stationary probability distribution for

$$\varphi_\eta(x, t) = \log G_\eta(x, t) + \text{constant} \tag{2.5}$$

is proportional to

$$dP[\varphi] \equiv d[\varphi] \exp \left\{ -1/(2\lambda) \int dx (d\varphi/dx)^2 \right\} , \tag{2.6}$$

i.e. to the Wiener measure where now  $x$  plays the role of the time. It is crucial that  $\varphi(x)$  is defined apart from an overall constant ; indeed, if we use the measure (2.6), we get  $\langle \varphi^2(x) \rangle = \infty$ , but  $\langle (\varphi(x) - \varphi(y))^2 \rangle < \infty$ .

The consequences of this form of the ground state wave function have never been investigated ; they will be the subject of the rest of this section.

Let us assume that for large time and fixed  $x$

$$G^{(n)}(x_1, x_2, \dots, x_n ; t) \propto \psi^{(n)}(x_1, x_2, \dots, x_n) . \tag{2.7}$$

This assumption is obviously valid for integer  $n$ , but it is less clear if it is correct also in the limit  $n \rightarrow 0$ .

It is evident that

$$\langle G(x) G(y)^{n-1} \rangle \propto \psi^{(n)}(x, y, \dots, y) . \tag{2.8}$$

where  $G(x)$  is a short hand notation for  $G_\eta(x, t)$ .

Using previous equations we finally obtain for  $n = 0$  :

$$\langle \exp(\varphi(x) - \varphi(y)) \rangle = \exp \lambda |x - y| . \tag{2.9}$$

The apparent decrease of the wave function at infinity has transformed itself in an exponentially increase !

We can easily compute

$$\left\langle \prod_i G(x_i) a_i \right\rangle = \left\langle \exp \left( \sum_i a_i \varphi(x_i) \right) \right\rangle = \exp \left( \sum_{i,k} \lambda a_i a_k |x_i - x_k| \right) , \tag{2.10}$$

where we assume that

$$\sum_i a_i = 0. \quad (2.11)$$

Only if equation (2.11) is satisfied (or  $n = 0$  in Eq. (2.8)) the result is independent from the absolute normalization of  $G$ , otherwise we should always write *proportional to* instead of *equal*.

By now the educated reader should have recognized the equivalence of equation (2.6) and equation (2.8). The form of the ground state wave function equation (1.8) implies the stationarity of the probability distribution (2.6). In particular the exponential decay of the ground state wave function implies that

$$\langle (\varphi(x) - \varphi(y))^2 \rangle \propto |x - y|^\gamma, \quad \gamma = 1. \quad (2.12)$$

At this stage one may be inclined to argue that the exponential decay of the wave function is generic for bound states independently from the dimensions, and therefore the value  $\gamma = 1$  should be superuniversal, i.e. valid in any dimensions, of course in the region where a bound state is present in the limit  $n \rightarrow 0$ . This argument is strongly suggestive, but it is not clear to me if hidden loopholes are present.

If we want to compute the functions  $P(x, t)$  something must be said about the Green functions. We know that for large times the Green functions must be proportional to the ground state wave functions, while for short time they should be given by the free one.

The simplest hypothesis is that for large times

$$G^{(n)}(x_1, x_2, \dots, x_n; t) \propto \prod_i D(x_i, t) \psi^{(n)}(x_1, x_2, \dots, x_n), \quad (2.13)$$

where  $D(x, t)$  is the free propagator given by

$$1/(2\pi t)^{1/2} \exp(-x^2/2t). \quad (2.14)$$

Equation (2.13) is the simplest interpolation between the behaviour at  $t = 0$ , where the free term dominates, and the asymptotic behaviour for large times. An other simple approximation, i.e. the ground state function multiplied by the effect of diffusion of the center of mass, has the serious disadvantage of not reproducing the correct behaviour of the Green functions at small time.

Now we have

$$P(x, t) = \lim_{n \rightarrow 0} \int dx_2, \dots, dx_n G^{(n)}(x, x_2, \dots, x_n; t). \quad (2.15)$$

If we use equation (2.13) in equation (2.15) the result for  $P$  is reduced to quadratures; unfortunately I have been unable to find a nice representation for the results of the integral and to extract the result in the limit  $n \rightarrow 0$ . A way out is to use the following representation for the wave function

$$\psi^{(n)}(x_1, x_2, \dots, x_n) \propto \int dP[\varphi] \exp\left(\sum_i \varphi(x_i)\right), \quad (2.16)$$

where we have absorbed the infinities (arising from the divergence in  $\langle \varphi^2 \rangle$ ) in the constant of proportionality. It is evident that these infinities worry us only when  $n \neq 0$ .

Putting everything together we finally find

$$P(x, t) = \int dP[\varphi] \exp(-x^2/(2t) + \varphi(x)) / \int dy \exp(-y^2/(2t) + \varphi(y)). \quad (2.17)$$

It is not strange that equation (2.17) can also be derived starting from the stirred Burgers equation (see next section). The evaluation of this formula is non trivial. Some results may be obtained by dimensional analysis using the fact that  $|\varphi(x) - \varphi(y)| = O(|x - y|^{1/2})$ .

The integrand will be concentrated for large  $t$  near the maximum which is located for

$$x_M^2/t = O(x_M^{1/2}), \quad (2.18)$$

i.e. for  $x_M \propto t^{2/3}$  in agreement with equation (1.5).

It would be interesting to evaluate the width of the region where the difference of the argument of the exponential with its maximum remains of order 1. In principle we can use the formula

$$P(x, y, t) = \int dP[\varphi] \exp(-(x^2 + y^2)/(2t) + \varphi(x) + \varphi(y)) / \left\{ \int dz \exp(-(z^2)/(2t) + \varphi(z)) \right\}^2. \quad (2.19)$$

and compute

$$\Delta(t) \equiv \int dx dy P(x, y, t)(x - y)^2. \quad (2.20)$$

It has been shown [6, 7] that

$$\Delta(t) \propto t \quad (2.21)$$

for large  $t$ . A probabilistic argument which leads to (2.21) is presented in the appendix.

More precisely, if we define

$$\Delta_\eta(t) = \int dx dy P_\eta(x) P_\eta(y) (x - y)^2 \quad (2.22)$$

the probability distribution of  $\Delta_\eta$  has a tail proportional to

$$d\Delta/\Delta^{5/4}, \quad (2.23)$$

in the region  $\Delta \ll t^{4/3}$ . The linear increase of  $\Delta$  in equation (2.21) is due to rare events in the tail of the probability distribution, where  $\Delta = O(t^{4/3})$ ; these events small probability proportional to  $t^{-1/3}$ .

This replica symmetric analysis gives a precise picture of the model and it is amazing how much information is coded in the form of the wavefunction. We will discuss in the next section some of the difficulties which arise from the assumption that replica symmetry is exact and we will explore the possibility of its breaking.

### 3. The random stirred Burger equations.

It is evident that  $G_\eta(x, t)$  is also a solution of the stochastic differential equation :

$$\partial G(x, t)/\partial t = (1/2 \partial^2/\partial x^2 + \eta(x, t)) G(x, t), \quad (3.1)$$

where we have omitted the obvious dependence of  $G$  on  $\eta$ .

We can now use the mapping (2.5) for writing an equation of evolution for  $\varphi(x, t)$ . We readily get

$$\partial \varphi / \partial t = 1/2 \partial^2 \varphi / \partial x^2 + 1/2 (\partial \varphi / \partial x)^2 + \eta(x, t), \quad (3.2)$$

which is the random stirred Burger equation, which has been well studied in the past [5].

Equation (3.2) induces a functional differential equation for the probability distribution of  $\varphi$  as function of the time. Using usual techniques for stochastic differential equations one finds the formal equation :

$$\begin{aligned} \partial P[\varphi, t] / \partial t = & \int dx \delta / \delta \varphi(x) [1/2 \partial^2 / \partial x^2 \varphi + 1/2 (\partial \varphi / \partial x)^2] \times \\ & \times P[\varphi, t] + \lambda / 2 \int dx \delta^2 / \delta \varphi(x^2) P[\varphi, t]. \end{aligned} \quad (3.3)$$

If the term  $(\partial \varphi / \partial x)^2$  were absent from equations (3.2), (3.3), an equilibrium, i.e. a time independent, solution of (3.3) would be given by equation (2.6). This result is not surprising as far as equation (3.2) reduces in this case the well studied Langevin equation.

The surprise comes when we reinstall the term  $(\partial \varphi / \partial x)^2$ ; if we use the form (2.6) for  $P[\varphi, t]$  and we compute  $\partial P[\varphi, t] / \partial t$ , we find the extra pieces :

$$\int dx (d^2 \varphi / dx^2 + d^2 \varphi / dx^2 d\varphi / dx) P[\varphi, t]. \quad (3.4)$$

and they both vanishes by integration by part.

If the long time behaviour of  $P[\varphi, t]$  is unique, as it happens in a finite size box, it is given by equation (2.6).

However we have already remarked that the problem we need to solve is to find the solution of equation (3.1) with the following boundary conditions at time 0

$$G(x, 0) = \delta(x). \quad (3.5)$$

The exact computation of the probability distribution of  $G$  is not easy (we shall see some of the difficulties in the appendix II). In absence of exact results we could try to guess an approximate solution. The simplest possibility consists in writing

$$G(x, t) = D(x, t) \exp(\varphi(x, t)). \quad (3.6)$$

where  $D$  is given by (2.14).

In this case we find that  $\varphi$  satisfies the following differential equation

$$\partial \varphi / \partial t = 1/2 \partial^2 / \partial x^2 \varphi + 1/2 (\partial \varphi / \partial x)^2 + \eta(x, t) - x/t \partial \varphi / \partial x, \quad (3.7)$$

and it is not clear if the stationary solution of equation (3.7) is still given by equation (2.6).

In any case the extra term in equation (3.7)  $(x/t \partial \varphi / \partial x)$  is apparently much smaller than the others at large time (we recall that at time 0 we can impose the boundary condition

$\varphi(x, 0) = 0$ ). A perturbative expansion in this extra term seems possible ; a careful analysis may be necessary to understand if the Ansatz (3.6) is correct at large times. However no major inconsistencies are apparent in equation (3.6) and we can assume tentatively to be essentially correct.

We have seen that the direct analysis of the problem, by mapping it on the random stirred Burger equation, seems to reproduce the results of the replica method, with unbroken replica symmetry, however we shall see that replica symmetry is in some sense weakly broken.

**4. Weakly breaking of the replica symmetry.**

The analysis of the previous section seems to confirm the correctness of the unbroken replica symmetry approach, however we shall see in this section that the replica symmetry is weakly broken.

Let us consider the wave function for the  $n$  particle problem in which  $n/m$  groups of  $m$  particles are bound together with the wave function (1.8) (where here  $m$  plays the role of  $n$ ) for each group of particles. In the infinite volume limit, this wave function becomes an eigenvalue of the Hamiltonian (1.7), if we neglect the region of phase space where particles of the two groups are near each other. In other words we consider the case where the  $n$  particles form  $n/m$  bound states of  $m$  particles.

The energy for such wave function can be readily computed and one finds that the energy per particle is given (neglecting proportionality factors) by

$$\tilde{E}_n(m) \equiv E_n(m)/n = (1 - m^2). \tag{4.1}$$

If we assume that a sensible value of  $m$  must satisfy the inequalities

$$1 \leq m \leq n, \tag{4.2}$$

the function  $\tilde{E}_n(m)$  has a minimum for  $m = n$ .

In the strange situation where  $n$  is less than 1, inequalities (4.2) are usually substituted in the replica approach by

$$n \leq m \leq 1. \tag{4.3}$$

When  $n$  is equal to 0 the maximum of  $\tilde{E}_n(m)$  is located at  $m = 0$  and according to the standard folklore in the region  $n < 1$  we must maximize the energy (not minimize !) in order to find the *ground state*.

The ground state seems at  $m = 0$  and the replica symmetry is apparently not broken ; if the minimum of  $\tilde{E}_n(m)$  is located at  $m \neq 0$ , as happens on the Caley tree or in the large dimensions limit [8-11] the replica symmetry would be definitely broken.

However the difference in energy vanishes quadratically with  $m$  and therefore the solution with replica broken symmetry is nearly degenerate with the symmetric one. This fact has serious consequences on the overlap distribution.

The average value of the probability distribution of overlaps  $\text{Pr}(q)$  is given by

$$\int dq \text{Pr}(q) \exp(t\varepsilon q) = Z(\varepsilon) = \langle Z_\eta(\varepsilon) \rangle \tag{4.4}$$

$$Z_\eta(\varepsilon) = \int dx dy d\mu [\omega_1]_{x,t} d\mu [\omega_2]_{y,t} \exp \left[ \varepsilon \int d\tau \delta(\omega_1(\tau) - \omega_2(\tau)) \right] / (G_\eta(t)^2).$$

If the overlap distribution  $\text{Pr}(q)$  is a single delta function, the replica symmetry is usually considered not to be broken ; however it was remarked in reference [12] that, if we want to study the breaking of the replica symmetry in a thermodynamic sense, we must consider the behaviour of the function  $Z(\varepsilon)$  for positive and negative  $\varepsilon$ . If the function

$$F(\varepsilon) = \lim_{t \rightarrow \infty} 1/t \log Z(\varepsilon) \quad (4.5)$$

has a discontinuity in the derivative at  $t = 0$ , i.e. if

$$q_M \equiv \partial F / \partial \varepsilon^+ \neq \partial F / \partial \varepsilon^- \equiv q_m, \quad (4.6)$$

the replica symmetry is broken.

Here the term proportional to  $\varepsilon$  in equation (4.4) corresponds in the replica formalism to adding a new term in the potential between replicas belonging to groups of  $n/2$  elements each ; the Hamiltonian becomes

$$H = H_n - \varepsilon \sum_{i=1, n/2} \delta(x_i - x_{i+n/2}), \quad (4.7)$$

where  $H_n$  is given by equation (1.7).

The new term is an attractive potential for positive  $\varepsilon$  and it is repulsive for negative  $\varepsilon$ . We can now compare, using  $\varepsilon$  as perturbation, the energy of the fully symmetric solution ( $E_S$ ) and the energy we have starting from the situation in which we have two bound states of  $n/2$  particles each ( $E_B$ ), i.d.  $m = n/2$ .

We readily find

$$\begin{aligned} E_S &= \lambda^2/2 n(1 - n^2) - \varepsilon \lambda^2/2 n(1 + n) \\ E_B &= \lambda^2/2 n(1 - n^2/4). \end{aligned} \quad (4.8)$$

For small values of  $n$ , at fixed negative  $\varepsilon$ ,  $E_B$  is smaller than  $E_S$  and the second solution should be preferred. In the limit where  $n$  is going to zero first, we obtain that

$$0 = q_m \neq q_M = \lambda^2, \quad (4.9)$$

and replica symmetry is broken. The breaking is extremely weak because it disappears for finite values of  $n$  when  $\varepsilon$  goes to zero first.

The reader may be puzzled by the fact that the smallest of the two energies ( $E_B$  and  $E_M$ ) is taken, while I have stated before that for  $n < 1$  we should maximize, not minimize the energy. In the same way the choice of taking  $m < n$  was supposed not to be allowed in the region  $n < 1$ . In the replica formalism the difference between minimization and maximization is rather subtle : usually [11] the correct approach consists in verifying the stability of the solution, a task whose meaning is not perfectly clear to me in this context and that I have not yet started to analyze. However the choices I have made seem to me to be the most reasonable ones.

The final predictions are related to the behaviour of the solutions of the Schrodinger equation :

$$\partial G_\eta(x_1, x_2, t) / \partial t = (1/2 \partial^2 / \partial x_1^2 + 1/2 \partial^2 / \partial x_2^2 - \varepsilon \delta(x_1 - x_2) + \eta(x, t)) G_\eta(x_1, x_2, t). \quad (4.10)$$

If  $\varepsilon$  is positive  $G$  should be concentrated in the region where the difference  $x_1 - x_2$  is not large, while for  $\varepsilon$  negative we should find  $x_1 - x_2$  to be of order  $t^{2/3}$ .

In the same way, if the fluctuations in the total free energy

$$F_\eta^{(2)} \equiv -\log \left\{ \int dx_1 dx_2 G(x_1, x_2, t) \right\} \tag{4.11}$$

are proportional to  $C t^{2/3}$  for  $\varepsilon > 0$ , they could be reduced to  $C (t/4)^{2/3}$  for  $\varepsilon < 0$ . This change in the behaviour of the fluctuations is related to the presence of  $n^2/4$  in the formula for  $E_B$ .

The result seems rather reasonable. In the replica symmetric phase configurations with two well separated nearly equal peaks in  $G_\eta(x)$  are possible with a probability which vanishes slowly (as  $1/t^{1/3}$ ); these configurations become dominant as soon as  $\varepsilon$  is negative. They dominate the large time behaviour of  $G_\eta(x_1, x_2, t)$  and the simultaneous presence of two peaks reduces the fluctuation.

The behaviour at  $\varepsilon$  strictly zero is not easily found. Simple minded arguments may suggest that in this case the function  $\text{Pr}(q)$  is asymptotically a single delta function ( $\delta(q - q_M)$ ), while extra contributions vanishes (as  $\delta(q)/t^{1/3}$ ), however one should be very careful in drawing conclusions in this delicate region.

A numerical check of the prediction of this weakly broken replica symmetry would be welcome, especially if we consider that some of the steps done in the derivation are not crystal clear.

**5. Conclusions and outlook.**

We have seen that in the one-dimensional case replica symmetry is nearly broken, because the replica symmetric solution is degenerate with the solution with broken replica symmetry. The behaviour is quite similar to the one found for the directed polymer on a Caley tree [8, 11], with the difference that  $m$  becomes zero on the Caley tree only at zero temperature, while  $m$  is always zero in this two dimensional case.

The most interesting result is the decoding of the information contained in the replica approach in the wave function, in particular the relation between the form of the wave function, its exponential decrease at large distance and the asymptotic increase (with the distance  $x-y$ ) of the expectation value of the ratio  $G_\eta(x, 0)/G_\eta(y, 0)$ . We have already remarked that this result may be the starting point for understanding the superuniversality of some of the critical exponents. The extension of the techniques used in this paper to the higher dimensional case will be hopefully done in the near future.

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**Appendix I.**

In this appendix we present an heuristic derivation of equations (2.22), (2.23), which supplement the more rigorous study of references [6, 7].

It is clear that sizable contributions to the integral (2.21) will come from those configurations of  $\varphi(x)$  in which the function

$$F(x) = \varphi(x) - x^2/(2t) \tag{AI.1}$$

has two distant local maxima which do not differ too much as far as the value of  $F(x)$  is concerned.

In order to simplify the argument it is useful to consider at first the following simpler function

$$P(x, y, t) = \int dP[\varphi] \exp(\varphi(y) + \varphi(x)) / \left\{ \int \int dz \exp(\varphi(z)) \right\}^2, \quad (\text{AI.2})$$

where the function  $\varphi$  is defined in the interval  $0-X$ .

I will argue that the probability of having  $\varphi(x_M + y) = \varphi(x_M) - \delta$  is given for large  $y$  and small  $\delta$  by

$$\text{Pr}(y, \delta) \propto \delta^2 / y^{3/2} \exp(-\delta^2 / y). \quad (\text{AI.3})$$

The crucial point is the presence of  $\delta^2$  in the prefactor of the exponential (the power  $y^{3/2}$  follows from the normalization condition); it can be justified as follows.

The starting point to derive equation (AI.3) consists in estimating the probability  $\text{Pr}(\varphi, x, \varepsilon)$  for having a path  $\varphi(x)$ , which satisfies the initial conditions  $\varphi(0) = -\varepsilon$  and  $\varphi(x) = \varphi$ , such that :

$$\varphi(y) < 0 \quad \text{for} \quad 0 < y < x. \quad (\text{AI.4})$$

Using the method of images one readily finds

$$\begin{aligned} \text{Pr}(\varphi, x, \varepsilon) &= G_0(\varphi + \varepsilon, x) + G_0(\varphi - \varepsilon, x) \\ G_0(\varphi, x) &= 1 / (2 \pi x)^{1/2} \exp(-\varphi^2 / 2x). \end{aligned} \quad (\text{AI.5})$$

In the large  $x$  limit  $\text{Pr}$  simplifies to

$$\text{Pr}(\varphi, x, \varepsilon) = \varepsilon / x G_0(\varphi, x). \quad (\text{AI.6})$$

The total probability for having equation (AI.4) satisfied (independently of the value of  $\varphi(x)$ ) is given (in the large  $x$  limit) by

$$\text{Pr}(x, \varepsilon) \propto \varepsilon / x^{1/2}. \quad (\text{AI.7})$$

$\text{Pr}(x, \varepsilon)$  factorizes in the product of two terms : a term is proportional to  $\varepsilon$ , which is the probability of not crossing the boundary ( $\varphi = 0$ ) for small  $x$ , the other term is proportional to  $x^{1/2}$  and it is the probability for not crossing the boundary in the region  $x \gg \varepsilon^2$ .

If we have a configuration which has a maximum of  $\varphi$  near  $x = 0$  with  $\varphi_M = 0$ , the probability of having  $\varphi(x_0) = \delta$ , with  $\delta = O(1)$  and  $x_0$  large, will be proportional to the normalized probability for the trajectory to satisfy the condition (AI.4) (i.e.  $\delta / x_0^{1/2} G_0(\delta, x_0)$ ) multiplied by the probability that  $\varphi$  remains negative in the region  $x$  just greater than  $x_0$  (i.e.  $\delta$ ). Putting everything together and adjusting the factor  $x$  in order to normalize correctly the total probability we find equation (AI.3).

Equation (AI.3) implies that the expectation value of

$$\exp(-\delta) = \exp(\varphi(x_M) - \varphi(x_M + y)) \quad (\text{AI.8})$$

is proportional to  $1/y^{3/2}$ .

We thus find the probability law (2.23), which in this case implies that

$$\langle (x - y)^2 \rangle \propto X^{3/2}. \quad (\text{AI.9})$$

In the original case the arguments would run quite similarly and the results would be multiplied by smooth functions of  $(X - y)^3/t^2$ . The probability law (2.23) should be modified only in the end of the tail where  $(x - y) \propto t^{2/3}$ . Therefore  $t^{2/3}$  play a role of a cutoff in the same way as  $W$ ; equation (AI.9) is thus replaced by

$$\langle (x - y)^2 \rangle \propto t. \tag{AI.10}$$

The reader who correctly doubts the soundness of this derivation may be comforted by the fact that I have tried to check numerically if the equations derived in the appendix are reasonable. To this end I have extracted  $10^5$  random path ( $\varphi$ ) of  $10^4$  steps and these simulations are in very good agreement with equations (AI.9, AI.10).

The behaviour of the *most likely value* of  $\bar{\Delta}$ , if it is defined as  $\exp(\langle \log \Delta \rangle)$ , is more complex and I have not been able to reach definite conclusions; it seems to increase like  $t^{\beta(t)}$ , where  $\beta(t)$  is a decreasing function of the time in the range 0.5-1.0. The extrapolation of the numerical data at infinite time is ambiguous and preasymptotic terms must be present; if we include corrections to scaling no conclusion can be reached; the data are also consistent with the behaviour:  $\bar{\Delta} = \text{constant} - t^{-\alpha}$ , with  $\alpha \approx 0.15$ .

We could also introduce  $\Delta_M$ , i.e. the value of  $\Delta$  at which the probability distribution  $P(\Delta)$  has a maximum; strictly speaking  $\Delta_M$ , not  $\bar{\Delta}$ , is the most likely value. It turns out that the probability distribution of  $\Delta$  is quite flat, also in logarithmic scale, and  $\bar{\Delta}$  is definitely larger than  $\Delta_M$ . Although it is likely that  $\Delta_M$  and  $\bar{\Delta}$  asymptotically coincide, they may behave in different way for not too large time: indeed in the simulations  $\Delta_M$  behaves as  $t^\gamma$ , with  $\gamma$  weakly dependent on the time ( $\gamma \approx 0.2-0.3$ ).

In principle we could analytically compute the probability distribution of  $\langle \Delta \rangle$ , in the simpler case where  $\varphi$  is defined in the interval  $0-X$ , by evaluating the integrals

$$\int_0^X dz_2 \dots dz_n \psi(x, z_2 \dots z_n), \tag{AI.11}$$

$$\int_0^X dz_3 \dots dz_n \psi(x, y, z_3 \dots z_n).$$

For any given  $n$  it is possible to find a closed form for these integrals. However the number of terms in the computation increases with  $n$  and I have not been able to find enough compact formulae to allow the analytic continuation in  $n$  up to  $n$  equal to 0. It may be possible that using an approach similar to that of reference [14], an exact computation may be done.

**Appendix II.**

In this appendix we will compute the Green function  $G^{(n)}(x_1, x_2, \dots, x_n; t)$ .

We start by looking to a simpler problem, i.e. the computation of the Green function  $G(x; t)$  which is the solution of the equation

$$\partial G / \partial t = -HG, \tag{AII.1}$$

$$H = -1/2 \partial^2 / \partial x^2 + \lambda \delta(x),$$

with the usual boundary condition at time 0:

$$G(x; 0) = \delta(x). \tag{AII.2}$$

The function  $G(x; t)$  can be found for positive  $x$  to be equal to

$$\begin{aligned} & 1/(2\pi) \int dp a(p) D(x, p; t), \\ & a(p) = p/(p - i\lambda) \\ & D(x, p; t) \equiv \exp(ipx) \exp(-tp^2/2). \end{aligned} \quad (\text{AII.3})$$

where the integral over  $p$  is done in the complex plane from  $-\infty + iA$  to  $+\infty + iA$  where  $A$  is large enough in order that the pole of the integrand remains below the integration path, (i.e.  $A > \lambda$ ). For negative  $x$ ,  $G$  can be found by using the relation  $G(x; t) = G(-x; t)$ .

$G$  is obviously a solution of equation (AII.1) at  $x \neq 0$ , because it is a linear combination of  $D(x, p; t)$ ; it satisfies the boundary condition at  $t = 0$ , because for  $x \neq 0$  the integral may be shifted up to  $i_\infty$ . We check by an explicit computation that equation (AII.1) is satisfied also at  $x = 0$ .

In the limit where the time goes to infinity the integral may be deformed on the real axis by peaking the contribution of the pole, which coincide with the bound state contribution.

Similar formulae holds for the  $G^{(n)}(x_1, x_2, \dots, x_n; t)$ . For example in the case  $n = 3$  we can write in the region  $x_1 < x_2 < x_3$ :

$$\begin{aligned} G^{(n)}(x_1, x_2, x_3; t) = & 1/(2\pi)^3 \int dp_1 dp_2 dp_3 D(x_1, p_1; t) D(x_2, p_2; t) D(x_3, p_3; t) \times \\ & \times a(p_1 - p_2) a(p_2 - p_3) a(p_1 - p_3), \end{aligned} \quad (\text{AII.4})$$

where the integral over the  $p$ 's is done along an appropriate path in the complex plane similar to the previous case.

Although it is possible with some work to write down the generalization of (AII.4) to generic  $n$ , the formulae are quite complex (they involve multiple integrations over the  $p$ 's) and I am not able to compute the analytic continuation up to  $n = 0$ .

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