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Some more sandpiles

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Abstract. — For the critical sandpile model of P. Bak *et al.*, we present high statistics results obtained by a fast non-parallel algorithm. In particular, we give results for 2, 3, 4 and 5 dimensional hypercubic lattices, and for Bethe lattices. On the latter, the model is in the same universality class as (dynamic) percolation, but the upper critical dimension seems to be 4 instead of 6 as for percolation. Between $d = 4$ and $d = 6$, the model seems to correspond to branched true SAW's as suggested by Obukhov. But this breaks down definitely below $d = 3$.

1. Introduction.

In a series of papers [1, 2, 3, 4], Bak *et al.* have introduced the concept of « self-organised critically » (SOC).

In conventional critical phenomena (such as the critical point in a gas-liquid transition), one has to fine-tune a control parameter, in order to arrive at the critical point. Thus, if the control parameter is set at random, with probability one the system will not be critical and will thus not show long range fluctuations, neither in time nor in space.

Under these circumstances, it seems hard to understand the ubiquity of $1/f$ -noise (or, more correctly, of $f^{-\alpha}$ -noise, $\alpha \simeq 1$) [5], the basic feature of which is just the long range and self-similarity of temporal fluctuations. One reason for that might be that $1/f$ -noise is simply not related to critical phenomena. A number of alternative explanations have been proposed indeed [6].

The other reason might be that systems are driven into a critical state *without* fine-tuning explicitly. This is the basic idea of SOC. In this case, it is not the experimenter, so to say, but the evolution itself which fine-tunes the control parameter.

This concept is not so new indeed. A well known model with anomalous scaling but without a control parameter is diffusion limited aggregation. A model much closer to the concept of SOC is invasion percolation [7]. In this model, each site i (we discuss only the site version ; there exists of course a completely analogous bond version, too) is attributed a random number $r_i \in [0, 1]$ drawn from some continuous distribution. Starting from a single randomly chosen site, an infinite cluster is built by adding at each time step the perimeter site with the smallest r_i . Here, we call a site a « perimeter site » if it is not itself in the cluster but has a

neighbour which is. In the long time limit, the cluster will have the statistics of an infinite percolation cluster at threshold. Most models discussed in references [1, 2, 3, 4] are indeed similar to invasion percolation, though they seem not to be in the same universality class.

As we said, there are a number of different models which seem to show SOC. All of them involve the flow of some locally conserved order parameter (« sand ») in a medium whose transport properties are determined just by the amount and by the detailed structure of the deposited order parameter ⁽¹⁾. Not all of these models are in the same universality class. For instance, in references [8, 9] versions have been studied in which the flow is directed. Already in the very first papers [1], cases with scalar and vector order parameters were studied (see also [10]). Finally, microscopic aspects such as the discreteness or continuity of the order parameter seem to have an influence on the critical behaviour [10].

The model we study in the present paper is the « height model » studied also in references [1, 2, 11, 12]. Here, the order parameter z is a discrete non-negative scalar, i.e. its value z_i at lattice site i is a non-negative integer. On a uniform lattice with coordination number N , a configuration is called « stable » if for each i one has $z_i \in 0, 1, 2, \dots, N - 1$. The dynamics involves the three rules :

(i) when the configuration is « stable », a site i is chosen random and at this site z_i is increased by 1 unit, $z_i \rightarrow z_i + 1$.

(ii) if the configuration is unstable (e.g. since at the chosen site we had already $z_i = N - 1$), we apply the rule that

$$\begin{aligned} z_i &\rightarrow z_i - N, \\ z_k &\rightarrow z_k + 1 \end{aligned}$$

for all neighbours k of site i . We say in this case that site i « topples ».

(iii) For a boundary site with $< N$ neighbours, rule (ii) is modified such that only the $< N$ neighbouring z_k are increased. In this way, the average $\langle z \rangle$ can decrease. We call these boundary conditions « open » (notice that this terminology does not agree with the one used in [1]).

Alternatively, we have studied also other boundary conditions. In particular, we have used (iii.a) periodic b.c. : here, there is no boundary and hence z can only increase. Thus, after some finite time a state with an « infinite » avalanche is reached which goes on forever. In this case, ergodicity is broken, and it is not clear whether an infinite system is driven into a critical state. A similar problem applies to the « supercritical » simulations described in reference [1] where one starts with large $\langle z \rangle$ and observes the relaxation without ever applying rule (i). Also there it is far from obvious whether the state reached is the same as the one reached with rules (i)-(iii) ; and

(iii.b) cylindrical b.c. : here, open b.c. are used in 1 direction, while periodic b.c. apply to all other directions. This should lead to the same universality class as open b.c. in all directions.

It is the interplay between the increase of z due to rule (i), the transport due to (ii), and the decrease due to (iii) which drives the system into the critical state.

In the sandpile interpretation of the model, z_i is the number of sand grains at site i (the « height of sand »), and rule (i) corresponds to throwing sand grains randomly onto the lattice. Rule (ii) leads then, if the average height has reached a sufficiently large value, to

(¹) Actually, our identification of the order parameter is somewhat problematic. See the end of section 5 for a discussion.

large avalanches where a single toppling makes some neighbouring site unstable, whose toppling renders further sites unstable, and so on. The avalanche is stopped only by sand falling off the edges (rule (iii)), reducing thus the overall height.

Notice that at any given time only a single avalanche exists. The next sand grain is thrown, and the next avalanche can start, only after the present avalanche has stopped.

As noted by several authors, this sandpile interpretation is somewhat unrealistic. In a sandpile, the order parameter would be the slope (a vector), and a different critical behaviour is to be expected. The present model rather applies to cases where z_i is a pressure or a stress, either in a physical sense (e.g. earth quakes [3]), or in a psychological. An amusing (semi-realistic ?) interpretation would be a large office where no employee ever finishes any record and gets a nervous break down if confronted with 4 or more records. In this case the records are distributed among its 4 neighbours, except when the employee was sitting next to a window : then, one of the record is just thrown out of it.

In the next section, we shall describe fast algorithms for simulating this model. In section 3, these are then applied to large hypercubic lattices (with up to 4×10^6 lattice sites) in 2, 3, 4 and 5 dimensions.

Our aim is on the one hand to get just precise critical parameters and critical exponents. On the other hand, we would like to know the upper critical dimension d_c . There have been various speculations on that. For instance, Obukhov [13] has conjectured that $d_c = 4$, based on the analogy of the model with a branched true self-avoiding walk. In contrast, the similarity to percolation (or invasion percolation) might suggest $d_c = 6$.

In order to understand better this problem, we performed also simulations on Bethe lattices (Caley trees) with various coordination numbers. For these lattices, Dhar has announced exact results [11], without giving however any details yet. A number of results on Bethe lattices can be obtained by rather elementary arguments. These and our simulations are described in section 4.

Finally, in section 5 we give our conclusions.

2. Algorithms.

The sandpile model can be formulated as a cellular automaton. It has thus appeared natural to most authors to perform also simulations in parallel as in a typical cellular automaton. The presumably fastest such algorithm was described in [12]. It uses multispin coding with 3 bits/site (updating hence 21 sites each time a computer word is updated), and it fully vectorizes, resulting thus in up to 400 site updates/ μ sec on one processor of a CRAY YMP.

The main drawback of such an algorithm is that most of the lattice is not changing during a typical avalanche. Even though $\approx L^2$ sites topple during the evolution of a typical avalanche of an $L \times L$ lattice, at any given time only a minute fraction of sites is unstable. Thus, the above algorithm during most of the time makes trivial updates. It is thus much better to selectively update only unstable sites (and their neighbours), even if this cannot be vectorized and each single update takes much more time.

In the following, we describe two such algorithms. We do this only for a 2-dimensional square lattice with open boundary conditions. The modifications necessary for other lattices are trivial.

The very simplest algorithm along this line of reasoning exploits the fact, pointed out and used heavily in [11], that the temporal order of the topplings in any avalanche is irrelevant. This is due to the fact that in the considered version of the model the height is an additive variable. It would not be true in models where the order parameter is a slope, such as a realistic sand pile.

Denote by $SAND(I, J)$ a subroutine which describes the effect of adding one sand grain at site (i, j) in a square lattice of size $L \times L$. The simplest algorithm uses recursive calls of $SAND$ itself :

```

LOOP
  I = 1 + RANDOM*L
  J = 1 + RANDOM*L
  SAND (I, J)
END LOOP
PROCEDURE SAND (I, J)
  IF Z(I, J) < 3 THEN
    Z(I, J) = Z(I, J) + 1
  ELSE
    Z(I, J) = 0
    IF I > 1 THEN SAND (I - 1, J)
    IF I < L THEN SAND (I + 1, J)
    IF J > 1 THEN SAND (I, J - 1)
    IF J < L THEN SAND (I, J + 1)
  ENDIF
RETURN

```

We did not use this routine for most of our simulations, through it was indeed nearly as fast (when implemented in BASIC on an ATARI home computer) as the one described below. Its main drawback is that recursive function calls are not implemented in some languages like FORTRAN, are somewhat slow in other implementations, and use an incontrollable amount of stack. The latter became a problem on lattices with $\approx 10^6$ sites. Also, it requires somewhat tricky book-keeping if averages like e.g. the average life time of an avalanche are computed, which then destroys the above simplicity.

The algorithm used for most of the results presented below uses a list of *topple sites*, i.e. sites which have received a sand grain during the last time step and whose value of z_i was already 3. While it applies rules (ii) and (iii), it builds a list of new topple sites for the next time step. When the old list is gone through, it is replaced by the new list and, provided this list is not empty (i.e., the avalanches has not yet stopped), the next time step begins. Running on an IBM 3090 without vector feature, this algorithm was faster than the vectorised multispin-coding algorithm on the CRAY for lattice sizes $L > 130$.

This algorithm is very similar in spirit to the one used e.g. in [14] to build percolation clusters. Important differences to percolation are that the disorder (the height profile $\{z_i\}$) changes slowly in the present case, while it is frozen in percolation. And that any site can only be once a growth site of a percolation cluster, while a single avalanche can pass through a site several times.

3. Sand piles in 2 to 5 dimensions.

Up to now, the most extensive simulations had been performed on 2 dimensional lattices, with sizes up to 672 [12]. We have performed simulations on some even larger lattices (up to size $1\,400 \times 1\,400$), but not with sufficiently high statistics. The high statistics data shown below are based upon lattices of sizes up to 672^2 , 144^3 , 40^4 , and 20^5 .

In these simulations, we started either from empty lattices (all $z_i = 0$), or from equidistributed lattices. In the latter case, the values of z_i were chosen randomly between 0

and $N - 1$. The latter is suggested by the proof of Dhar [11] that in the stationary critical state all configurations which appear at all with non-zero probability are indeed equally probable. Thus, when starting from equidistribution, the evolution towards the stationary state could be just a contraction onto the allowed configurations.

We found that in 2 dimensions both types of initial configurations gave roughly the same speed of convergence. With increasing dimension, starting with equidistributions gave however increasingly better convergence. When starting on an empty lattice, the average height did overshoot and approached the stationary limit slowly from above.

The following results, summarized in figures 1-13, were obtained from runs which we checked carefully to be sufficiently long so that stationarity has been reached before averages were taken.

(a) In figure 1, we show the average cluster size $\langle s \rangle$ versus the lattice size L on a doubly logarithmic scale. The cluster size is here defined as the number of topplings in an avalanche. If a site has toppled n times, it is counted n -fold, irrespectively of whether its z value is the same after the avalanche as before. We see a very clean collapse of the data for all different dimensions on a single curve. It was shown by Dhar [11] (see also [12]) that $\langle s \rangle$ scales on the square lattice as $\langle s \rangle \sim L^2$, and this method allows the same scaling to be deduced for other dimensions too. On heuristic grounds, such a scaling had been predicted in [10]: on a lattice of size L , a newly added sand grain has to travel a distance $\propto L$ before it can fall off an edge. If the transport is essentially diffusive (i.e., by random walks), then it takes $\sim L^2$ topplings to cover this distance. From figure 2, we find that a scaling $\langle s \rangle \sim L^2$ is very nicely satisfied: the local slopes $d\langle s \rangle/dL$ in figure 1 approach a value of 2 when $L \rightarrow \infty$, with scaling corrections $\propto 1/L$.

(b) The average life time per avalanche is plotted, also on a doubly logarithmic scale, in figure 3. While for 2 dimensions we find $\langle t \rangle \sim L$, for higher dimension the increase is definitely less fast than L . At first sight, this is surprising in view of the above heuristic argument which seems to suggest that not only sizes but also life times of avalanches increase as L^2 . The problem is easily resolved by the intermittency of the process. Most of the sand is transported during few very large avalanches. These large avalanches contribute little if one considers the unweighted average as done in figure 3. Weighted average life times increase much faster with L . In particular, we found that $\langle s \cdot t \rangle / \langle s \rangle$ increases for $d = 4$ with a power of ≈ 2 : $\langle s \cdot t \rangle / \langle s \rangle \propto L^{1.85 \pm 0.1}$. This suggests that the power is indeed 2, and that the motion of sand grains is diffusive as expected in [10].

We might add that also the dropping of the sand off the edges of the lattice is a very intermittent process, with very few but strong bursts. This was already observed in 2 dimensions [10]. Though we have not studied it systematically, this intermittency definitely becomes more pronounced when going to higher dimensions.

As seen from figure 4 where we plot again local slopes, we find that $\langle t \rangle$ scaled approximately as $L^{2/d}$ in ≤ 4 dimensions. We should however point out that this scaling sets in very late (as obvious also in Fig. 3), and is correspondingly somewhat uncertain. In particular, for $d = 5$ our data can't distinguish between $L^{2/d}$ and \sqrt{L} .

(c) The next quantity we studied was the average height $\langle z \rangle$ as a function of L . Results are shown in figure 5 only for $d = 4$, but in different dimensions our results were very similar. From the fact that we get straight lines when plotting $\langle z \rangle$ versus $1/L$, we conclude that

$$\langle z \rangle_L = \langle z \rangle_\infty - \text{const.}/L. \quad (1)$$

Such a behaviour is easy to understand: the main effect of the finite lattice size on $\langle z \rangle$ is to create boundaries where $\langle z \rangle$ is smaller than in the bulk of the lattice. The relative fraction of boundary to bulk is just $\propto 1/L$.

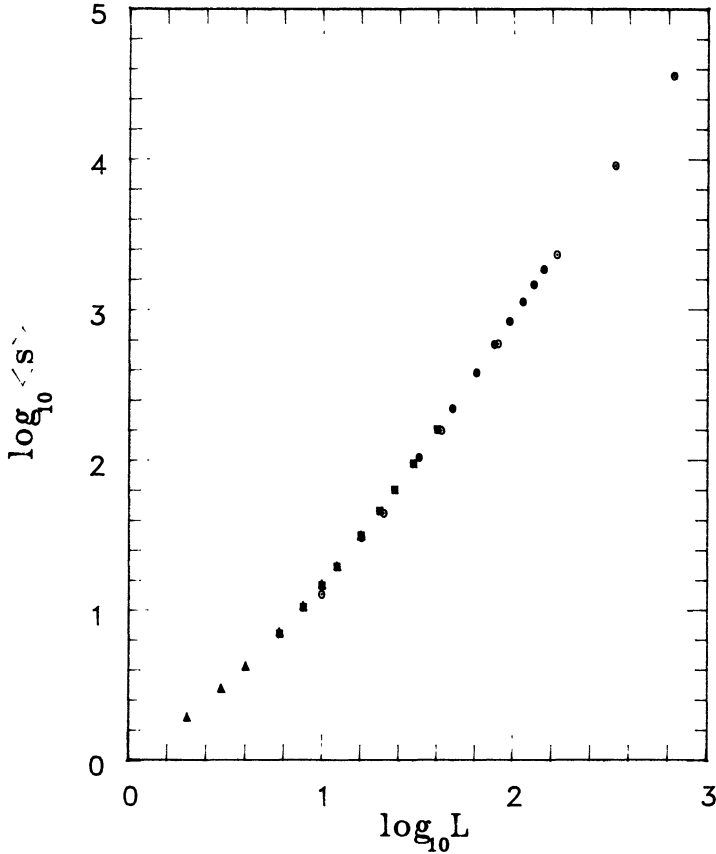


Fig. 1. — Log-log plot of average cluster size $\langle s \rangle$ versus lattice size L for dimensions 2 to 5. Open circles, filled circles, squares and triangles correspond here and in the following figures to $d = 2, 3, 4$ and 5.

To test this, we performed also simulations with different boundary conditions. While the closed dots in figure 5 represent the open boundary conditions discussed above, the open dots are from « cylindrical » boundary conditions where the lattice is open only in one direction, but is periodic in the other 3. This reduces the amount of boundary by a factor 4, an indeed the effect on $\langle z \rangle$ is reduced by the same factor. We should mention that going to cylindrical boundary condition increased also quite considerably the size and the time of the avalanches, and slowed further down the convergence towards the stationary state.

The extrapolated values of $\langle z \rangle$ are given in table I. In addition to the critical height, we also measured the probabilities f_z that $z_i = z$. Values extrapolated to $L = \infty$ are given in table II.

(d) Let us now turn to the fluctuation of z in the critical state,

$$\Delta z = \left[\left\langle \left(\frac{1}{L^d} \sum_i z_i \right)^2 \right\rangle - \left\langle \frac{1}{L^d} \sum_i z_i \right\rangle^2 \right]^{1/2}. \quad (2)$$

In ordinary critical phenomena like e.g. the Ising model, this scales with L as

$$\Delta z \propto L^{-1/\nu}. \quad (3)$$

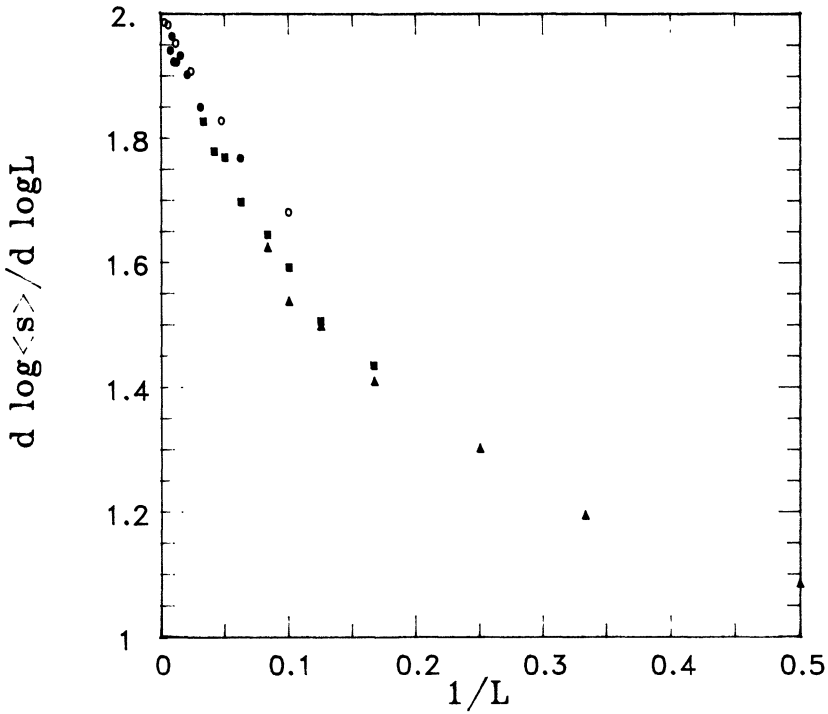


Fig. 2. — Average slopes $d \log \langle s \rangle / d \log L$ versus $1/L$ for dimensions 2 to 5. For the meaning of the symbols, see figure 1.

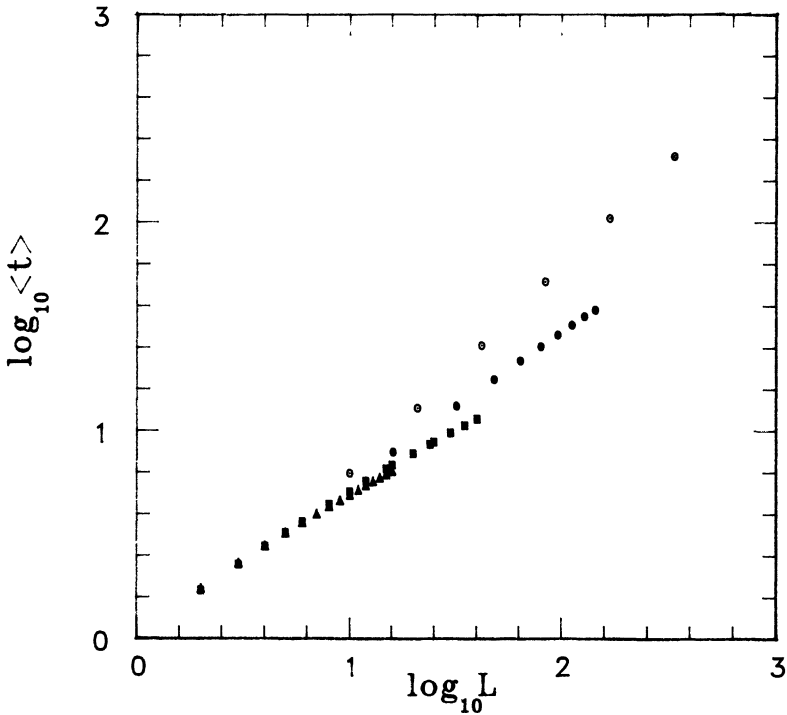


Fig. 3. — Log-log plot of average cluster life time $\langle t \rangle$ versus lattice size L for dimensions 2 to 5. The meaning of the symbols is as in figure 1.

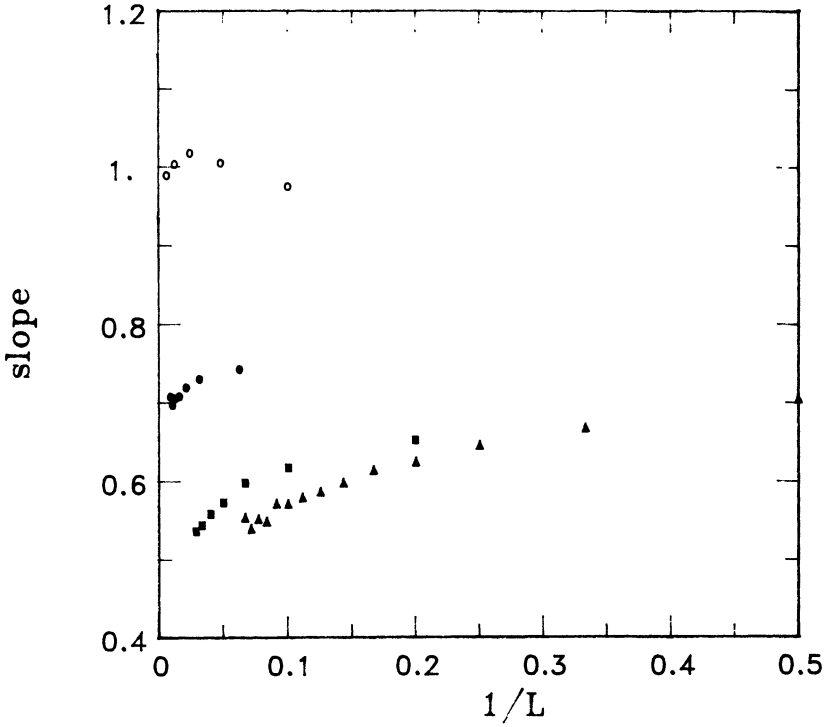


Fig. 4. — Average slopes $d \log \langle t \rangle / d \log L$ versus $1/L$ for dimensions 2 to 5. Symbols as in figure 1.

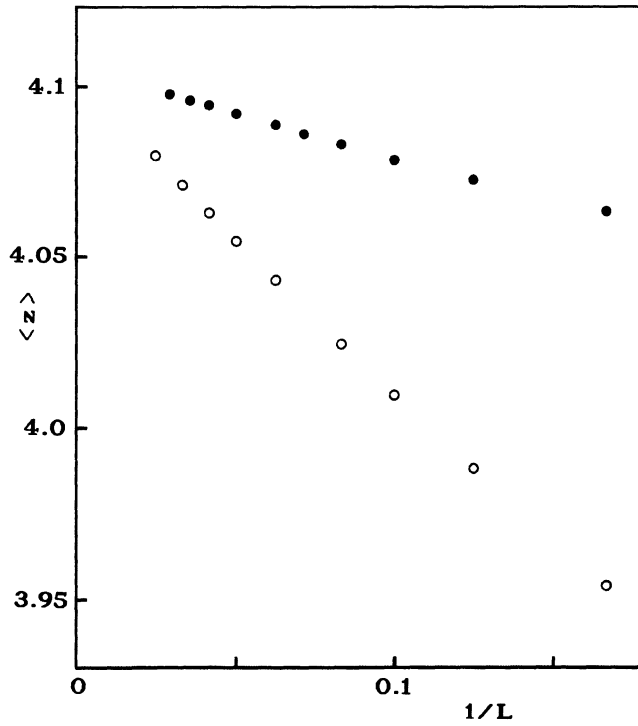


Fig. 5. — Average height $\langle z \rangle$ versus $1/L$, for dimension $d = 4$. Heavy dots represent open boundary conditions, open circles correspond to cylindrical b.c.

Table I. — *Critical density and critical exponents for dimensions 2 to 5.*

d	$\langle z \rangle_c$	ν	x	τ	ξ
2	2.125	0.98	0.36	2.22 ⁽¹⁾	0.37
3	3.135	0.59	0.01	2.33	0.18
4	4.106	0.46	0.0	2.44	0.03
5	5.072	0.385	—	—	—

⁽¹⁾ From reference [12].

Table II. — *Probabilities $f_z = \text{prob}(z_1 = z)$.*

z	$d = 2$	$d = 3$	$d = 4$	$d = 5$
0	0.0736	0.0545	0.0423	0.0341
1	0.1740	0.1172	0.0892	0.0725
2	0.3062	0.1663	0.1210	0.0962
3	0.4462	0.2008	0.1393	0.1080
4		0.2233	0.1481	0.1130
5		0.2380	0.1521	0.1147
6			0.1537	0.1150
7			0.1541	0.1152
8				0.1152
9				0.1153

Figure 6 shows Δz versus L on a doubly logarithmic scale. We see indeed very clean scaling behaviour, with ν given in table I. This table shows that for all dimensions ν is very close to $2/d$, i.e. to very good precision we have $\Delta z \propto \sqrt{L^d}$, as we would get if there were no long range correlations in z . Actually, there is a small but statistically significant deviation from this for all dimensions. The fact that this deviation is independant of d suggests that this deviation might result from corrections to scaling, and that indeed $\Delta z \propto \sqrt{L^d}$.

(e) We know from [11] and from figure 1 that the number s of topples in a typical avalanche in 2 dimensions increases as L^2 . If each site would topple at most once during an avalanche, this would mean that the clusters (defined as those sets of sites which have toppled) are compact. Actually, sites can topple several times during one avalanche. From figure 7 where we plotted the average number of topplings per cluster site, we see that this increases as

$$\frac{\langle s \rangle}{\langle s_{\text{distinct}} \rangle} \sim L^x \tag{4}$$

with $x = 0.36$. Here, s_{distinct} is the number of distinct sites in an avalanche which have toppled. Since the total number of topplings is $\propto L^2$, s_{distinct} increases less fast than L^2 , more precisely [12]

$$\langle s_{\text{distinct}} \rangle \propto L^{1.64} \quad (d = 2) . \tag{5}$$

(f) This might mean that typical clusters in $d = 2$ are fractal. But more careful inspection showed that this is not true. Clusters are compact, but their boundaries are clearly fractal. The increase of the boundary length of clusters which percolate in both directions (which correspond to avalanches which throw off sand on all 4 edges) is shown in figure 8 (we define

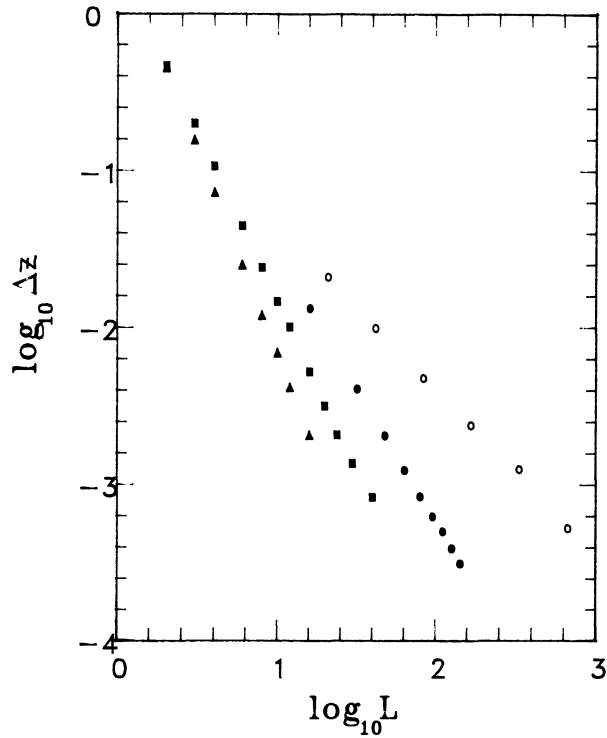


Fig. 6. — Log-log plot of the fluctuation Δz (defined in Eq. (3.2)) versus lattice size L for dimensions 2 to 5. Symbols as in figure 1.

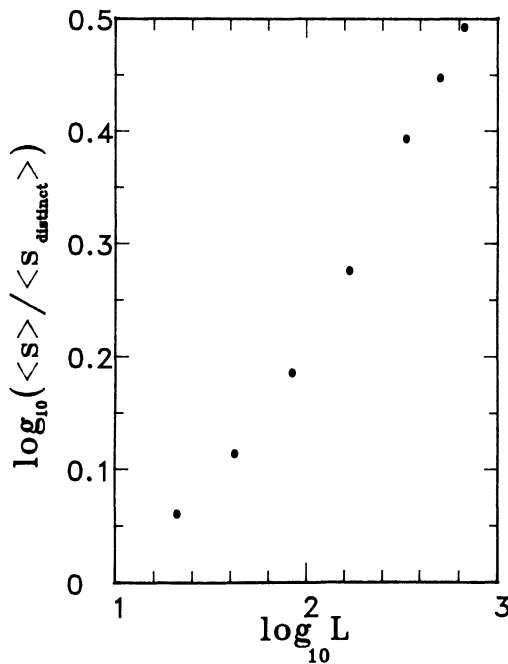


Fig. 7. — Log-log plot of average number of topplings divided by the average number of distinct toppled sites versus L , for dimension $d = 2$.

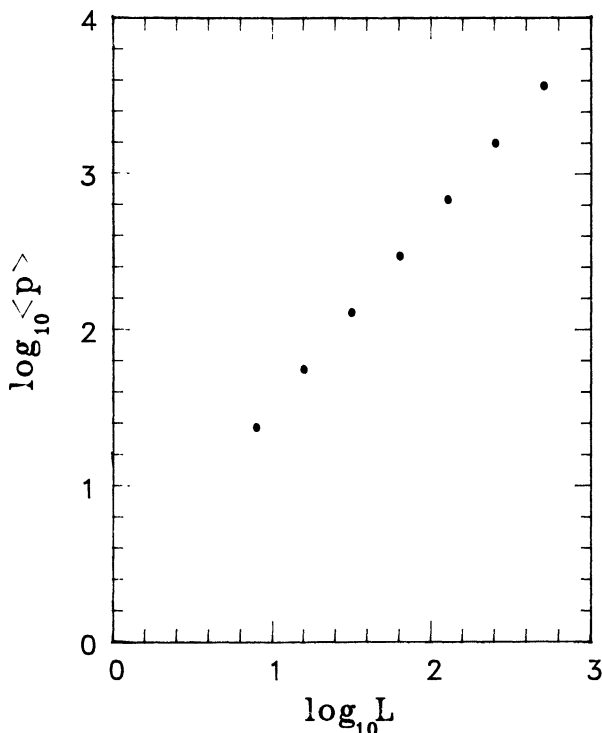


Fig. 8. — Log-log plot of average perimeter length *versus* L for dimension 2.

here the boundary length as the number of sites which are themselves in the cluster but which have at least one neighbour not in the cluster). We find a fractal dimension $d_{\text{boundary}} = 1.21$.

Notice that we did not find any clusters at critically which had holes. Moreover, also the sites which have toppled $\geq n$ times, $n > 1$, form connected clusters without holes. In figure 9a, we show a typical cluster. Sites which toppled 1, 2, 3, ..., 7 times are drawn black, blue, green, ... and red. We see that indeed a nested set of compact clusters, with the innermost having the largest number of topplings. A related structure is seen when looking at those sites which have not returned to their original z value after the avalanche has died out (Fig. 9b). We see that such sites occur only at the boundaries of the above clusters. In avalanches where no site has toppled more than once, all sites except those near the perimeter return to their original z value.

The intriguing structure is essentially observed only for $d = 2$. In higher dimensions, the chance that a site topples twice or even more often is extremely small. For $d = 4$, we hardly found any such site at all, and also for $d = 3$ they formed only a tiny fraction of all cluster sites (see Fig. 10). This fraction increases slowly with L , but we prefer to consider this as a logarithmic effect instead of a power law (in the latter case, the exponent would be $x = 0.002$).

Finally, we found that the clusters in $d = 3$ seemed to be fractal, with fractal dimension $d_F = 2.85$.

(g) In [1] it was suggested that the differential distribution $D(s)$ of the number of topplings scales as

$$D(s) \propto s^{1-\tau}. \quad (6)$$

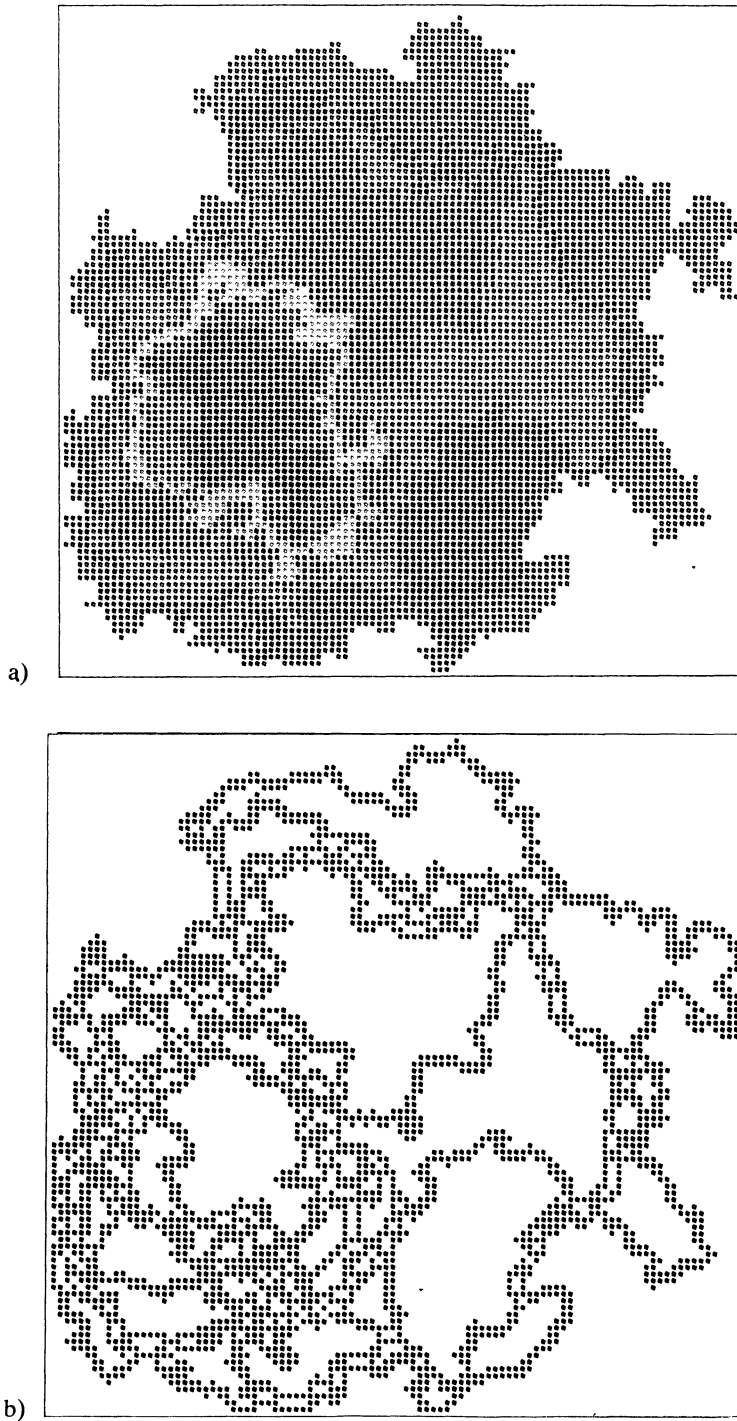


Fig. 9. — Typical large clusters in 2 dimensions. In panel (a), all sites which have toppled n times are given the same color: black for $n = 1$, blue for $n = 2$, light green for $n = 3$, purple for $n = 4$, yellow for $n = 5$, dark green for $n = 6$, and red for $n = 7$. In panel (b), those sites are shown which have not returned to their original z values. Note that these sites are only near the boundaries between different colors in panel (a).

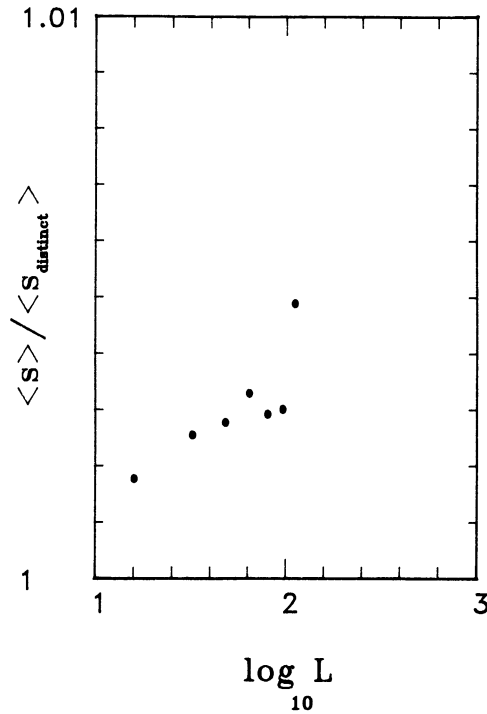


Fig. 10. — Same as figure 7, but for $d = 3$.

In 2 dimensions, this was verified in [12], with an exponent $\tau = 2.22$. Assume now that this holds for $s \leq L^\beta$, with some exponent β yet to be determined. Then we have

$$\langle s \rangle = \frac{\int_0^{L^\beta} ds s D(s)}{\int_0^{L^\beta} ds D(s)} \propto L^{\beta(3-\tau)}. \tag{7}$$

Since we know from [11] and from figure 1 that $\langle s \rangle \propto L^2$, this implies

$$\beta = \frac{2}{3-\tau} = 2.56. \tag{8}$$

Within the error bars, this was indeed verified.

We find thus that equation (6) must hold up to values of s larger than the number L^2 of lattice sites, which obviously works only since sites can topple twice. Let us see how the analogous situation is in $d \geq 3$. There we know from the above that (at least within our error bars) multiple topplings can be neglected, whence $\beta \leq d$. As for $d = 2$, we have still $\langle s \rangle \propto L^2$. Thus, if we assume again the scaling law (6) for $s < L^\beta$, then we must have

$$\tau = 3 - 2/\beta \leq 3 - 2/d. \tag{9}$$

In order to test these predictions, we show in figures 11 and 12 $D(s)$ for $d = 3$ and $d = 4$ on a logarithmically coarse grained scale as used also in [12]. More precisely, the points

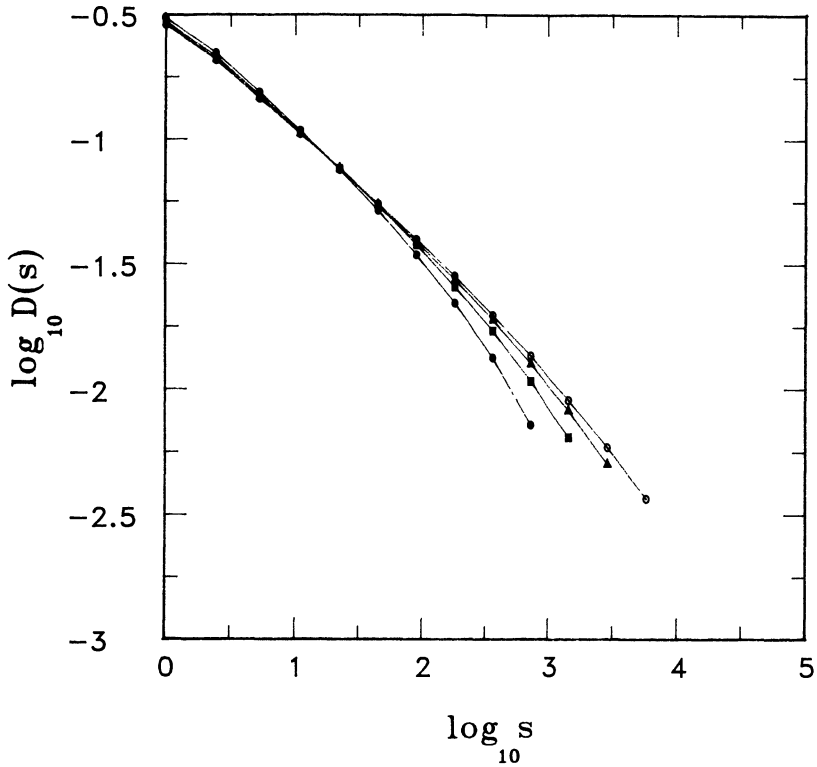


Fig. 11. — Cluster size distribution for $d = 4$ on a log-log scale. The height of the i -th bin shows the fraction of avalanches with $2^i \leq s < 2^{i+1}$. Open circles, triangles, squares and dots represent $L = 144, 96, 64, 32$.

shown in figures 11 and 12 represent $\sum_{s=2^i}^{2^{i+1}-1} D(s)$. Extrapolating these data to $L = \infty$ in the

way described in [12], we obtain the values of τ given in table I. For $d = 3$, τ agrees roughly with the result of [4]. For both $d = 3$ and $d = 4$, the values are very close to the upper bounds in equation (9), suggesting that equation (9) is indeed saturated in $d = 3$ and $d = 4$ [2]. This would then imply that the largest avalanches have $s \sim L^d$ (which agrees indeed with what we found), and explains why the process becomes increasingly intermittent in higher dimensions.

(h) As a last set of data, let us show the average numbers $\langle s \rangle_t$ of toppling sites at fixed times. We know that on a very large lattice this cannot decrease with t , as all sand has to be carried to the boundary. Indeed, if the avalanche represents a pure branching process (see the next section), then in the critical state on an infinite lattice $\langle s \rangle_t \rightarrow \text{const.}$ for $t \rightarrow \infty$. If there are also loops which correspond to branches which try to invalidate regions which have toppled already during the same avalanche, then $\langle s \rangle_t$ has to grow beyond all limits for $t \rightarrow \infty$.

In figure 13, we show $\langle s \rangle_t$ for $d = 2, 3, 4$. In order to minimize boundary effects, we used cylindrical b.c. and used in the averaging only avalanches which had started at a distance $> L/2$ from both open boundaries. For $d = 2$ and $d = 3$, we see a very clear power increase

$$\langle s \rangle_t \sim t^\xi \quad (10)$$

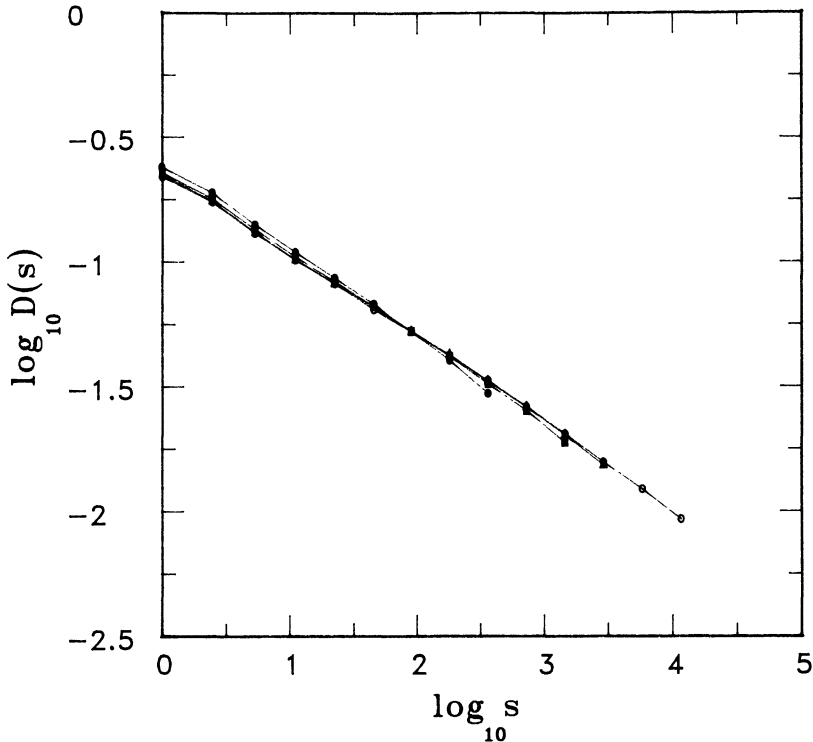


Fig. 12. — Same as figure 11, but for $d = 3$. Lattice sizes are $L = 40$ (circles), 32 (triangles), 24 (squares) and 16 (dots).

with exponents ξ given in table I. For $d = 4$, $\langle s \rangle_t$ either reaches a constant, or at most increases logarithmically. If we would try there a power fit, the exponent would be ≈ 0.03 .

In 2 dimensions, we also looked at those sites which toppled at fixed time t , and which had already toppled before. Their number increases roughly like a power of t , with a poorly determined exponent ≈ 0.6 , somewhat larger than that of $\langle s \rangle_t$.

(j) We have performed runs also for periodic boundary conditions, though we do not want to show detailed results. The reason (alluded to already in Sect. 1) is that we are not sure that we reached in this way the critical state. In 4 and 5 dimensions, we are indeed definite that we did *not* reach it. For instance, on lattices with 50^4 sites, $\langle z \rangle$ overshooted when we started with an empty lattice, and an infinite avalanche occurred only at $\langle z \rangle = 4.151$. This was brought down to 4.110 by using equipartition in the initial state, but this is still significantly higher than the value expected on an infinite lattice. We had started these runs in the hope of seeing scaling laws $\langle s \rangle$, $\langle t \rangle \propto (z_{\text{crit}} - \langle z \rangle)^{\text{const}}$. For $d = 2$ (and, to a less clear extent, for $d = 3$) we found indeed such scaling laws, but we are not sure that they present universal features. Even when there is no overshooting, we do not have an ergodic system, and we do not know whether the results obtained depend on the initial state. For $d = 3$, either the latter is indeed the case, or there are unusually large finite size effects.

We shall discuss the results of this section, and what they suggest about upper critical dimensions, in the last section.

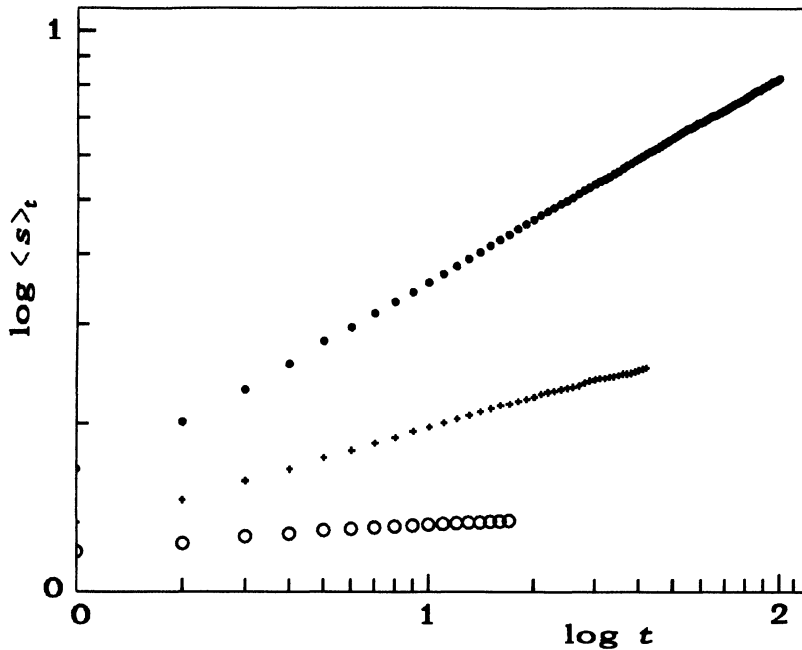


Fig. 13. — Log-log plots of numbers of topplings at fixed time t versus t . Data are shown for $d = 2$ ($L = 360$; dots), $d = 3$ ($L = 84$; crosses), and $d = 4$ ($L = 34$; circles). Boundary conditions are cylindrical. In order to reduce finite size effects, the statistics includes only avalanches starting a distance $\geq L/4$ from the boundary.

4. Sand piles on Bethe Lattices.

(a) When going to very large dimensions, the most important effect is that loops become less and less important. In order to simulate thus the limit of very large dimensions (and in particular the behaviour above any upper critical dimension), the most obvious thing to do is to study the model on a Bethe lattice.

We used lattices with various coordination numbers (up to 5), and with various boundaries. Below, we shall mainly present results for coordination number $N = 3$. Any site on such a lattice can be given a « generation number » g according to its distance from the boundary (see fig. 14). The leaves of the lattice are generation 1, the nodes leading up to the leaves are generation 2, etc. By generation of the lattice G we mean the generation of the center node, $G = \max_{\text{nodes}} g$.

In contrast to any hypercubic lattice, in a Bethe lattice of any generation most sites are close to the boundary. Indeed, for $G \rightarrow \infty$ half of all nodes have $g = 1$, a quarter has $g = 2$, etc. Thus, averages such as $\langle s \rangle$ or $\langle t \rangle$ are not very meaningful, as they measure essentially properties of the boundary layer.

In addition, if we want to measure only « bulk » properties, it is more important than in a hypercubic lattice to keep the boundary layer thin. We found that (at least for $N \leq 5$) there exists a possibility to keep it to exactly the first generation. This is achieved by modifying rule (iii) to

(iii.c) If site i is a leaf, we decrease z_i not to $z_i = 0$ but rather to $z_i = 1$. (Notice that we could formulate this also in an alternative way, by modifying the range of z for leaves to $z_i \in \{0, 1, \dots, N - 2\}$.)

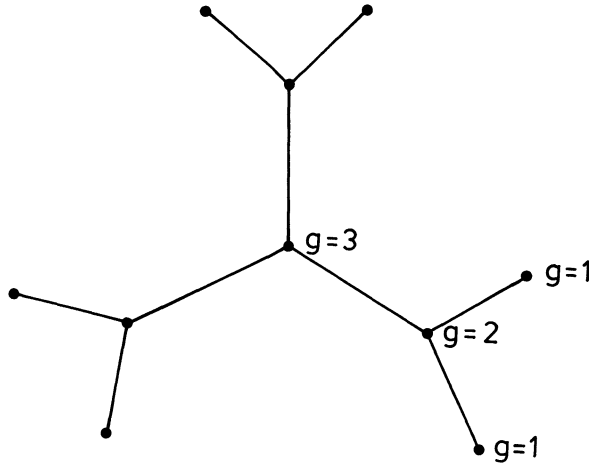


Fig. 14. — Bethe lattice with coordination number $N = 3$ and with $G = 3$ generations. In our terminology, the generation g of a node increases from the leaves ($g = 1$) to the center ($g = G$).

With rule (iii.c) instead of (iii), we found e.g. for $N = 3$ that $f_0 = 1/12$, $f_1 = 1/3$, and $f_2 = 7/12$ (giving $\langle z \rangle = 3/2$) for all sites i with $g \geq 2$, while $f_1 = 1/3$ and $f_2 = 2/3$ for leaves. This was verified numerically for lattices with $4 \leq G \leq 20$, with errors fully within the statistically expected ones. With rule (iii), instead, we obtained a roughly exponential convergence to $\langle z \rangle \approx 1.5$ when $g \rightarrow \infty$. Similar simple ratios were obtained also for lattices with larger coordination numbers, see table III.

In the following, we thus present only results obtained with rule (iii.c.).

Table III. — Probabilities $f_z = \text{prob}(z_i = z)$ on Bethe lattices with coordination number N and with b.c. according to (iii.c).

z	$N = 3$		$N = 4$		$N = 5$	
	$g = 1$	$g > 1$	$g = 1$	$g > 1$	$g = 1$	$g > 1$
0	0	1/12	0	2/27	0	81/1280
1	1/3	1/3	1/4	2/9	1/5	27/160
2	2/3	7/12	3/8	1/3	4/15	153/640
3			3/8	10/27	4/15	21/80
4					4/15	341/1280

(b) Let us next prove the following

Theorem : Sites on a Bethe lattice can only topple $n \geq 2$ times during one avalanche if they are the origins from where the avalanche started, or if the neighbour which initiated the first toppling has itself toppled at least n times.

For an inductive proof, let us first assume that the site i is a leaf, and that it is was not the origin of the avalanche. Then it can topple n times only if its neighbour toppled $\geq n$ times. Let us now consider a site i (with generation g) which is not the starting point. This site divides the lattice into 3 branches (we assume here and in the following $N = 3$; for $N > 3$ the proof is completely analogous). The starting point is in one of the branches (call it

branch 1) and we make the inductive hypothesis that the theorem applies to all sites in the other two branches (called 2 and 3). In that latter case, we easily see that the theorem indeed applies also to site i : in order to topple n times, site n has to receive $3n$ sand grains; from branches 2 and 3, it can at most receive $2n$, whence it must have received at least n from branch 1. In this way, we can move by induction into the interior of the lattice until only the starting point of the avalanche is left, QED.

This theorem tells us that the set of sites which topple n times must form a connected cluster which contains the starting point of the avalanche, for all $n \geq 1$. Furthermore, the starting point (and hence any other site) can topple twice only if all its 3 neighbours have $z = 2$. In the same way, one sees that in addition all 6 next-nearest neighbours must have $z = 2$ if the starting point (and thus any other point, too) is to topple 3 times, and so on. We see thus that multiple topplings are very rare, and can in particular not significantly influence the properties of large avalanches — unless there were very strong long range correlations.

(c) According to the above, and to the discussion in section (2.h), we expect $\langle s \rangle_t$ to tend towards a constant for $t \rightarrow \infty$, for avalanches started at infinitely large g . Indeed, this constant should be exactly $12/7$, since the average amount of sand transported away from the site where the first grain has been thrown should be 1, and the number of avalanches per grain thrown is just f_2 . We show the corresponding data in figure 15. We see there that indeed

$$\langle s \rangle_t \rightarrow \frac{12}{7} = 1.714, \quad t \rightarrow \infty. \quad (11)$$

We also see that $\langle s \rangle_t > 12/7$ for finite $t > 1$. This is due to multiple topplings.

(d) We present next the fluctuation of z in the critical state. As we see from figure 16, a very good fit is obtained with

$$\Delta z \propto \sqrt{2^G}, \quad (12)$$

as we would have expected from the absence of long range correlations.

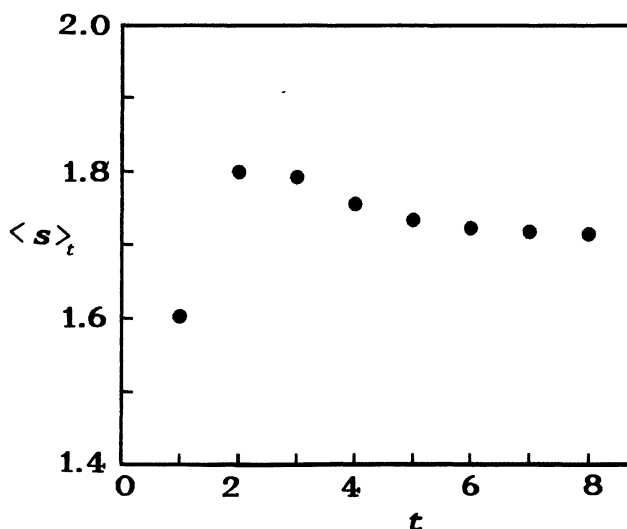


Fig. 15. — Numbers of topplings at fixed time t versus t for Bethe lattices with coordination number 3. Data are averaged over a range of lattice sizes, $G = 13$ to 19.

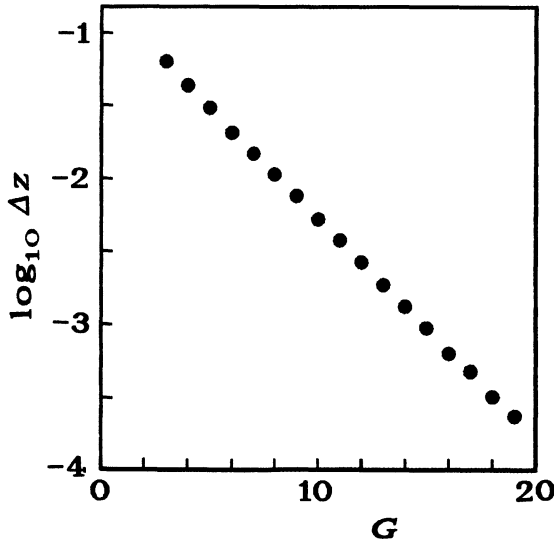


Fig. 16. — Logarithmic plot of the fluctuation Δz (defined in Eq. (3.2)) versus lattice size G for Bethe lattices with coordination number $N = 3$.

There do however exist non-vanishing short range correlations. The nearest neighbour correlations in the interior of the lattice are for instance obtained from $\langle s \rangle_{t=1}$ since

$$\langle s \rangle_{t=1} = 3 f_{22}/f_2, \tag{13}$$

where f_{22} is the probability to find $z = 2$ on two neighbouring sites. This gives numerically

$$f_{22}/f_2^2 - 1 = -0.085, \tag{14}$$

i.e. a small anticorrelation. For the correlations between next-nearest sites we found $f_{222} f_2/f_{22}^2 = 0.801$, and for the chance to find $z = 2$ on all 4 sites of a Mercedes star we got $f_{2222} = 0.915 f_{22}^2/f_{22}$.

5. Discussion and conclusions.

(a) The first conclusion we can draw from the last section is that on a Bethe lattice the sandpile model is in the universality class of critical percolation. More precisely, it is *dynamic* site percolation in the sense of [15], or a *general epidemic process without recovery* exactly as the critical point.

The analogy between this process and the present one is the following. While percolation can proceed only if a site is open, an avalanche can proceed only if $z = N - 1$ (we assume that the coordination number is N). The differences are

- (i) in percolation, the open sites are usually considered uncorrelated, while they are correlated in SOC. This should however be irrelevant since the correlation is short range ;
- (ii) in percolation, any site can be wetted (« infected ») only once, while a site can topple more than once in SOC. Again, this should be irrelevant since multiple topplings are rare and become negligible for large avalanches.

There is of course the very important difference that in SOC the state is driven into the critical state, while nothing like that is true for percolation. But this means just that there are important correlations between *successive* avalanches (if for some time the avalanches have been too small, the following avalanches will tend to be larger to carry away the accumulated sand); it is not reflected in the statistics of individual avalanches.

We might add that f_{N-1} — which corresponds to the critical percolation probability p_c — is rather close to what we would have expected in the absence of correlations. On a Bethe lattice, the latter is just $1/(N-1)$. For SOC we found numerically that $f_{N-1} - 1/(N-1) = 1/6, 1/27, \text{ resp. } 21/1280$ for $N = 3, 4$ and 5 . The fact that $f_{N-1} > 1/(N-1)$ is related to the negative correlations expressed e.g. in equation (14).

(b) In 4 dimensions, the sand pile model shows the same behaviour as on a Bethe lattice: we have seen that $\langle s \rangle_t$ increases less fast than a power of t , and we have seen that multiple topplings are practically absent.

The only result which contradicts this is that ν seems to be slightly less than $1/2$. We tend to take this as a finite size effect, and it is a very small effect anyhow.

Notice however that between $d = 4$ and $d = 6$ the sandpile model cannot be percolation-like, since the latter has $d_c = 6$. Also, for percolation the number of growth sites increases [14] with time with the exponent $\xi_{\text{percol}} = \hat{d} - 1 - \beta/\nu$, where \hat{d} is the spreading dimension. For $d = 4$, this gives $\xi_{\text{percol}} = 0.29$, in clear contrast to the sandpile model.

The above suggests that some exponents are, for $d \geq 4$, those of mean-field percolation [2] (see below for caveats). This applies in particular to τ which for percolation above $d = 6$ is $2.5 \ll 16$. Our result $\tau = 2.44$ (for $d = 4$) is compatible with this within the errors.

c) In [13], it was conjectured that the sandpile model is a branched true self-avoiding walk (SAW). Notice that dynamic percolation is just a branching ordinary SAW. The reason why branches of a sandpile should be self avoiding is clear: after an avalanche has passed and the sites have toppled, a second avalanche is hindered from passing again the same region. The difference between a SAW and a true SAW is that the former is — when seen as a real walk — not self-avoiding but self-killing. In a true SAW, a walker is not killed when he comes back to his own traces, but he gently turns away. Now it is clear that the same happens here to *some* degree, as the z values near a branch of an avalanche have been enhanced by the topplings, and there is thus some incentive for a next branch to turn away. But it is by no means clear that both the tendency of self-avoidance and of compensating for the tendency to kill are strong enough.

If sandpiles are indeed branched true SAW's, this would give an upper critical dimension $d_c = 4$. This is consistent with the above, but it faces another problem: in $d = 2$, the avalanches are clearly not self avoiding. Instead, the average number a site is visited increases like a power of L . This should definitely have an influence on the universality class the model is in.

In $d = 3$, the situation is similar to the one in $d = 4$ in that the avalanches seem self avoiding. But — unless we have been deceived by very strong logarithmic corrections — $\langle s \rangle_t$ increases like a power of t . This means that not only the self-avoidance is effective, but it is also self-killing. It is only since most of the branches are killed when forming loops, that $\langle s \rangle_t$ has to increase at the critical point.

Our conclusion is thus that $d_c = 4$, and that above d_c the sandpile avalanches are essentially branched true SAW's as suggested in [13]. But below $d_c = 4$, the sandpile model seems to be in a new universality class, not equivalent to branched true SAW's.

Since the sandpile model in $d < 3$ dimensions corresponds to branched walks which are not strictly self-avoiding, one might suggest it to be branched ordinary random walks. This would

mean an epidemic process without immunization, or directed percolation in $d + 1$ dimensions [17]. That this is not correct is seen by comparing critical exponents : for $d = 2$, directed percolation would give $\xi = 0.540 \pm 0.006$ [17], clearly different from the value 0.37 found in the present paper. Also, the life times would be distributed as $D(t) \approx t^{-1.46}$ [17], while for the present model $D(t) \approx t^{-1.38}$ [12].

(d) Let us next make some comments on mean field approximations. Our results suggest that the most natural approximation is that there are no correlations among the z_i . Notice that this is not exactly true even on Bethe lattices with finite N , since even there the chance for falling back (and thus for multiple topplings) is not zero. Otherwise said, in the mean field approximation we neglect the effect of all loops, including the trivial two-step loops involving a single link. It is easily seen that in such an approximation, the above sandpile model on a lattice with coordination number N is equivalent to an *exactly solved* directed model (as studied by Dhar and Ramaswamy [9]) on a Bethe lattice with coordination number $2N$. In the latter model, the nodes form an infinite stack of copies of the original sites, and each node is connected by N outgoing links to the following copy, and by N incoming links to the previous one. As shown in [9], the directed model is in turn equivalent to a voter model which can be solved exactly. Indeed, as we had already said, critical exponents in this mean field approximation (like $\tau = 5/2$ and $\xi = 0$) are those of mean field percolation [2]. Notice, however, that not all exponents are those of percolation, since we do allow multiple topplings. In 2 dimensions, the average number of sites which topple at time t though they had already toppled at some earlier time scales roughly as $t^{0.6}$, while it would be zero in a mean field theory of percolation.

(e) In the present paper, we have only studied the behaviour exactly at the critical point. In [1, 2], also the situation off the critical point has been discussed. In particular, the exponent β has been defined *via* the average density of topplings at given $\langle z \rangle$. Denoting it as j , they propose

$$j \sim (\langle z \rangle - \langle z \rangle_c)^\beta . \quad (15)$$

in the supercritical regime $\langle z \rangle > \langle z \rangle_c$. As we have said already, we must be very careful in interpreting such an ansatz. In [1, 2] it was tested by starting with a randomly chosen configuration with $\langle z \rangle \gg \langle z \rangle_c$, and letting the system relax towards a stationary state. It is easily seen that this can lead to wrong results, since the system can (and in general will!) become stationary in a non-critical state.

One way to render equation (15) meaningful consists in *first* driving the system into the critical state, then adding randomly m sand grains at one time (i.e., increasing $z_i \rightarrow z_i + 1$ for m randomly chosen sites, without letting unstable sites topple) with $1 \ll m \ll L^d$, and then letting the system relax. It is easily seen that this gives $\beta = 1$ *exactly*, due to the additivity of the heights and the resulting independence of avalanches [11]. The value $\beta = 1$ was found in [2] as a mean field result, but was clearly violated in the numerical simulations of [1, 2]. This wrong result is most likely due to the fact that the systems in [1, 2] did not relax towards critical states.

An alternative way to go off the critical state consists in not waiting until avalanches are completed before adding new sand grains. Instead, one sets $z_i \rightarrow z_i + 1$ at a fixed rate σ . Again due to additivity, the resulting j and $\langle z \rangle - \langle z \rangle_c$ are both $\sim \sigma$ for small σ , and $\beta = 1$ exactly (values $\beta \neq 1$ could be obtained in models where the scalar z is replaced e.g. by a vector, as in a more realistic sandpile models where topplings are not triggered when the height becomes too large, but the slope).

It might seem [2] that the density of topplings j is a better candidate for an order parameter than z since the latter is not singular at the critical point. We see however that j does not

behave as a usual order parameter either. For a system with constant spontaneous growth rate $\sigma < L^2$, we have

$$j \sim \sigma L^2 \quad (16)$$

since the number of topplings per avalanche is $\sim L^2$, independently of the eventual presence of other avalanches. For $\sigma > L^2$, the behaviour of j is more complicated since the time scales are changed when avalanches overlap. But in no case j is a density of an extensive quantity. Thus neither j nor z are completely analogous to order parameters in conventional critical phenomena. For convenience of presentation, we stuck to calling z an order parameter.

(f) Finally, we should point out that all densities on Bethe lattices seemed to be simple rationals for all N studied. This suggests that a simple closed solution should exist, as has indeed been announced in [11].

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Note added in proof:

After submitting the paper, we were able to prove exactly, on any hypercubic lattice, that clusters of sites which have toppled $\geq k$ times ($k \geq 1$) form simply connected clusters (see Sect. 3f).

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