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Two-dimensional dendritic growth at arbitrary Peclet number

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Résumé. — La sélection de vitesse due à l'anisotropie de l'énergie de surface dans la croissance dendritique à deux dimensions est considérée. Dans la limite de petite anisotropie, l'équation intégral-différentielle du spectre de vitesse se réduit à une équation différentielle. La solution de cette équation correspondant à la vitesse maximale est stable alors que les autres solutions présentent des modes instables. Une équation différentielle pour le spectre des taux de croissance est également dérivée. La vitesse de croissance v se comporte comme $\alpha^{3/4} f(p\alpha^{1/2})$ où p est le nombre de Péclet et $\alpha \ll 1$, un petit paramètre d'anisotropie. Pour $p\alpha^{1/2} \sim 1$, la fonction f est calculée numériquement. Les taux de croissance des modes instables Ω se comportent comme $\alpha^{3/2} g(p\alpha^{1/2})$. Pour $p\alpha^{1/2} \ll 1$, g se comporte comme $(p\alpha^{1/2})^3$ et comme $(p\alpha^{1/2})^{-4/5}$ dans le cas contraire.

Abstract. — Velocity selection in two-dimensional dendritic growth caused by anisotropy of the surface energy is considered. In the limit of low anisotropy the integro-differential equation for the velocity spectrum is reduced to a differential equation. The solution of this equation, corresponding to the maximal velocity, is stable, whereas the other solutions have unstable modes. A differential equation for the spectrum of growth rates is also derived. The growth velocity $v \propto \alpha^{3/4} \cdot f(p\alpha^{1/2})$, where p is the Peclet number, $\alpha \ll 1$ is a small anisotropy parameter. At $p\alpha^{1/2} \sim 1$ the function f is calculated numerically. The growth rates of the unstable modes $\Omega \propto \alpha^{3/2} g(p\alpha^{1/2})$. At $p\alpha^{1/2} \ll 1$ the function $g \propto (p\alpha^{1/2})^3$, and in the opposite limit $g \propto (p\alpha^{1/2})^{-4/5}$.

1. Introduction.

Lately the mechanism of velocity selection in dendritic crystallization has been investigated in detail. It has been shown that the crucial role in this selection is played by the anisotropy of the surface energy. The velocity spectrum of a needle crystal is discrete, and the only stable solution is the one with maximal velocity. An analytical approach to this problem is in the general case complicated due to the necessity to solve a nonlinear integro-differential equation. Some progress can be achieved in the low anisotropy limit, when the Ivantsov parabolic solution for the dendritic front [1] can be taken as a starting point. Then the anisotropic surface energy acts as a singular perturbation. To calculate the growth velocity spectrum one should solve an equation near the singular point in the complex plane. Ben-

Amar and Pomeau have been the first to derive a nonlinear differential equation of this kind [2]. From dimensional analysis of this equation the following expression was obtained for the growth velocity

$$v \propto D \alpha^{7/4} p^2 / d_0, \quad (1)$$

where D is the thermal diffusivity, d_0 is the capillary length, p is the Peclet number, and $\alpha \ll 1$ is a small anisotropy parameter. The Peclet number is determined by undercooling and, at small dimensionless undercoolings Δ , we have $p \approx \Delta^2 / \pi$. To derive the above result for the growth velocity, in papers [2-4] the small Peclet number approximation was formally employed. Actually this result is valid in a rather broad interval of p 's [5]. The genuine criterion of validity of equation (1) for the growth velocity follows from estimating the size of the singular region and is given by the inequality $p \ll \alpha^{-1/2}$ [6]. The opposite limit of large Peclet numbers can be considered for arbitrary anisotropy α [7, 8]. The limit of small $\alpha \ll 1$ then gives

$$v \propto (D/d_0) \alpha^{3/4} [1 - \beta (p\alpha^{1/2})^{-2/5}], \quad p\alpha^{1/2} \gg 1, \quad (2)$$

where $\beta \sim 1$. Comparison of limit expressions for the growth velocity shows that their orders of magnitude are comparable at $p\alpha^{1/2} \sim 1$. Thus, it can be conjectured that in a general case at $\alpha \ll 1$ the growth velocity is given by the expression

$$v = (D/d_0) \alpha^{3/4} f(p\alpha^{1/2}). \quad (3)$$

In section 2 of the present paper we consider the problem of selection of the growth velocity of a two-dimensional needle crystal at arbitrary Peclet numbers, exploiting only the smallness of anisotropy α . The equation for the growth velocity spectrum depends only on $p\alpha^{1/2}$. Structurally it is quite similar to that derived previously in reference [2] in the limit of small Peclet numbers. From this equation follow the limit expressions (1) and (2), whereas at $p\alpha^{1/2} \sim 1$ the function f , entering equation (3), is found numerically. In section 3 we derive an equation for the growth rates of unstable modes Ω and show that

$$\Omega = (D/d_0^2) \alpha^{3/2} g(p\alpha^{1/2}),$$

where $g(p\alpha^{1/2}) \sim 1$ at $p\alpha^{1/2} \sim 1$.

2. Selection of the dendritic growth velocity

The dynamics of the crystallization front $y(x, t)$ of a freely growing two-dimensional needle crystal is governed in the symmetric model, by the equation [9]

$$\Delta + \frac{d(\Theta) \kappa(x, t)}{\rho} = \frac{p}{2\pi} \int_0^\infty \frac{d\tau}{\tau} \int_{-\infty}^\infty dx' \dot{y}(x', t - \tau) \times \\ \times \exp \left\{ -\frac{p}{2\tau} [(x - x')^2 + [y(x, t) - y(x', t - \tau)]^2] \right\}. \quad (4)$$

In order to consider in what follows the steady-state growth of a needle crystal of an almost parabolic form with a tip radius ρ and velocity v , and also to analyze the stability of the steady-state growth, we have introduced in equation (9) the following dimensionless parameters. All lengths are measured in units of ρ , time in units ρ/v , $p = v\rho/2D$ is the Peclet number, $\Delta = (T_m - T_0) c_p L^{-1}$ is the dimensionless undercooling, T_m the melting temperature,

T_0 the melt temperature far away from the crystallization front ($T_0 < T_m$), c_p is the specific heat (thermal parameters of the crystal and melt are assumed to be the same), L is the latent heat. The second term in the left-hand-side of equation (4) describes the shift of the melting temperature caused by the curvature of the front $\kappa(x, t)$ (Gibbs-Thomson effect),

$$\kappa(x, t) = (d^2y(x)/dx^2)[1 + (dy/dx)^2]^{-3/2}.$$

The capillary length $d(\Theta) = \gamma_E(\Theta) T_m c_p L^{-2}$, $\gamma_E(\Theta) = \gamma(\Theta) + d^2\gamma(\Theta)/d\Theta^2$, $\gamma(\Theta)$ is the anisotropic surface energy, $\text{tg } \Theta = dy/dx$. Usually for the anisotropy of the capillary length, the simplest model expression is used,

$$d(\Theta) = d_0(1 - \alpha \cos 4\Theta) = d_0 A(\Theta) \quad (5)$$

which corresponds to a four-fold crystal symmetry.

Neglecting the surface energy ($d_0 = 0$) Ivantsov derived a steady-state solution of equation (4) in the shape of a parabola, moving with a constant velocity [1]

$$y = t - x^2/2.$$

The Peclet number p is then related to undercooling by the expression

$$\Delta(p) = 2 p^{1/2} \exp(p) \int_{p^{1/2}}^{\infty} \exp(-x^2) dx. \quad (6)$$

The anisotropic surface energy acts as a singular perturbation. Consequently, the role of the small anisotropy parameter α is two-fold. On the one hand, it gives a small correction to the parabolic front shape which enables one to linearize the integral term in equation (4) about Ivantsov solution. On the other hand, the size of the singular region near the singular point is also small at small α . As a result, derivatives of the correction are not small in the singular region, and the left-hand-side of equation (4) after all simplifications remains nonlinear [2].

To find the corrected steady-state solution, we represent the front shape as

$$y = t - x^2/2 + \zeta(x)$$

and linearize the integral term in equation (4) over ζ . Then we get

$$\sigma A(\Theta) \frac{\zeta'' - 1}{[1 + (\zeta' - x)^2]^{3/2}} = \frac{p}{\pi} \int_{-\infty}^{\infty} dx' [\zeta(x') - \zeta(x)] \exp[p(x^2 - x'^2)/2] \times \\ \times [K_0(pR) - K_1(pR)(x^2 - x'^2)/2R], \quad \sigma = d_0/p\rho, \quad (7)$$

$$R = [(x - x')^2 + (x^2 - x'^2)^2/4]^{1/2}$$

where R is the distance between two points on the parabolic front of crystallization. If we assume that $|\zeta'| \ll 1$, from the left-hand-side of equation (7) it follows that the singular points of this equation are $x = \pm i$. We consider a close vicinity of the singular point $x = i$. In the limit $|x - i| \ll 1$ we have $A(\Theta) \approx 1 + 2\alpha/(x - i)^2$. Therefore, the actual size of the singular region is $|x - i| \sim \alpha^{1/2} \ll 1$. In this region the arguments of the exponential and of the Bessel function in the integrand of equation (7) have different orders of magnitude,

$$p|x^2 - x'^2| \sim p\alpha^{1/2}, \quad p|R| \sim p\alpha^{3/4}.$$

It follows then, that for $p\alpha^{1/2} \leq 1$ the argument of the Bessel function is small. When $p\alpha^{1/2} \gg 1$ in the region where the exponential is non-negligible, the argument of the Bessel functions $p|R| \sim p^{-1/2} \ll 1$. We shall analyze contributions of different terms of the expansion of the Bessel functions.

We start with the term $1/pR$ in the expansion of the function K_1 . Calculating the integral in equation (7) with $\zeta(x')$ an arbitrary smooth function along the real axis of x' , we find that at $x \approx i$ the contribution of the term is of order $\exp(-p/2)$ and should be neglected when $p \gg 1$. The next terms of the expansion contain $\log R$ and have quite different analytical properties. We shall show that at $x \approx i$ the main contribution to the integral comes from the branching point of the logarithm at $x' = x$. From the expression $\ln(pR)$ only $\ln|x - x'|$ should be retained, since the other factors under the logarithm have no singularity on the real axis, and their contributions could also be neglected for the reasons given above for the case of $1/pR$ -term. We write

$$\ln|x - x'| = [\ln(x - x' + i\varepsilon) + \ln(x - x' - i\varepsilon)]/2, \quad \varepsilon \rightarrow 0.$$

The integral with the first term can be directly continued analytically into the upper half of the complex plane of x with the integration contour maintained on the real axis of x' . This contribution can also be neglected at $x \approx i$ for the reasons stated above. The second term corresponds to the integration contour passing above the point x . The shifting of x into the upper half-plane leads unavoidably to the deformation of the integration contour (Fig. 1).

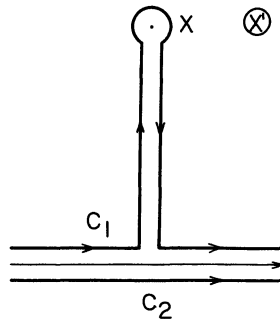


Fig. 1. — The integration contours in the x' complex plane: C_1) for $\ln(x - x' - i\varepsilon)$; C_2) for $\ln(x - x' + i\varepsilon)$. Point x lies in the upper half-plane.

Even for $x = i$ no exponential suppression of this contribution is observed, as it comes mainly from the vicinity of the branching point at $x' \approx x$. To describe the situation completely we take a cut going from the point $x' = x$ vertically down. The values of the logarithm on the different sides of the cut differ by $2\pi i$. Accordingly, to transform the integral term in equation (7) we can use the relation

$$\int_C dx' f(x') \ln|x - x'| = \pi \int_{ix}^{\infty} dy f(-iy). \quad (8)$$

The upper limit of integration in equation (8) is taken to be infinite, which implies the assumption that f decays rapidly along the imaginary axis. Retaining in the Bessel functions only the leading $\ln(pR)$ terms, and setting

$$x = i(1 - \alpha^{1/2}z), \quad x' = i(1 - \alpha^{1/2}z'), \quad \zeta \rightarrow \alpha\zeta \quad (9)$$

in the small α limit we get from (7) the equation

$$\frac{A(\tau)(1 + d^2\zeta/dz^2)}{2^{1/2} \lambda \tau^{3/2}} = \zeta - 2 p \alpha^{1/2} \int_z^\infty dz' \zeta(z') \exp[p \alpha^{1/2}(z - z')] \left[1 - \frac{p \alpha^{1/2}}{2} (z' - z) \right],$$

$$\lambda = \alpha^{7/4} / \sigma, \quad \tau = z + d\zeta/dz, \quad A(\tau) = 1 - 2/\tau^2. \tag{10}$$

Equation (10) is derived under the assumption that $p \gg 1$, but it is valid at arbitrary p . The point is that the coefficient of ζ in the right-hand-side of (10) does not depend on p , whereas the integral term becomes important only at $p \gg 1$ due to the smallness of the anisotropy parameter α .

In the limit $p \alpha^{1/2} \ll 1$ the integral term in equation (10) can be neglected, then the equation is reduced to the differential equation, derived in [2]. It means that this differential equation together with the known scaling relation

$$v \propto (D/d_0) \alpha^{7/4} p^2 \tag{11}$$

are valid in a rather broad interval of Peclet numbers $p \ll \alpha^{-1/2}$.

In the general case of arbitrary Peclet numbers one must solve the integral-differential equation (10). Fortunately, the simple structure of the integral term makes it possible to reduce this equation to a differential equation. Introducing the function

$$F = \frac{A(\tau)(1 + d^2\zeta/dz^2)}{2^{1/2} \lambda \tau^{3/2}},$$

and differentiating twice equation (10), we get a system of differential equations

$$\frac{d^2F}{dz^2} - 2 p \alpha^{1/2} \frac{dF}{dz} + \left[p^2 \alpha - 2^{1/2} \lambda \frac{\tau^{3/2}}{A(\tau)} \right] F = -1$$

$$d\tau/dz = 1 + d^2\zeta/dz^2 = 2^{1/2} \lambda \tau^{3/2} F/A(\tau). \tag{12}$$

These equations have no explicit dependence on z , so they are equivalent to a nonlinear second order differential equation. In order to investigate properties of these equations we consider their asymptotic behavior at large $|z|$. In this limit we can, in the first approximation, neglect the terms with derivatives in equation (12). Then we get

$$\tau \approx z, \quad F \propto z^{-3/2}. \tag{13}$$

After substitution $\tau = z$ we consider the linear differential equation (12) in the limit $|z| \gg 1$. Solutions of the uniform equation have an asymptotic behaviour

$$F_{1,2} \propto \exp\left(\pm \frac{4}{7} 2^{1/4} \lambda^{1/2} z^{7/4}\right)$$

and diverge most rapidly along the rays $\arg(z) = 0, \pm 4 \pi/7$. To ensure an asymptotic behaviour (13) for the function F we need to kill these divergences along the three rays. To fulfill the condition of boundedness of F we have two integration constants and the parameter λ , so that $\lambda = \lambda(p \alpha^{1/2})$ is the eigenvalue of the problem.

To find the function $\lambda(p \alpha^{1/2})$ we have solved the system of equations (12) numerically, taking F to be real and bounded at large real z and requiring that F be bounded at the ray $\arg(z) = 4 \pi/7$. This solution is then automatically bounded on the ray $\arg(z) = -4 \pi/7$ due to the relation $F(z^*) = F^*(z)$. In order to solve equations (12) we have used the

following iterative procedure. At a given $\tau(z)$ the linear equation for the function $F(z)$ was solved and the eigenvalue λ determined. To kill the solutions, growing at large $|z|$, we solve the equation for $F(z)$, going, e.g., from points lying on the rays $\arg(z) = 4\pi/7$ and $\arg(z) = 0$ at $|z| \sim 7$, into the inner region $|z| \sim 1$. The condition of matching of the solutions, originated from different rays, at some z ($|z| \sim 1$) determines the spectrum of λ . Our numerical calculations confirmed that the results for λ do not depend on the location of the matching point within the inner region. The function $F(z)$ found in this way was then used to calculate $\tau(z)$ from the second of equations (12). The convergence of this iteration was fairly fast : the value of λ becomes stable after 6-7 iteration cycles. Hence, we write the growth velocity as

$$v = (D/d_0) \alpha^{3/4} f(p\alpha^{1/2}), \quad f(y) = 2y^2/\lambda_0(y), \quad (14)$$

where λ_0 is minimal value of λ for a given $p\alpha^{1/2}$.

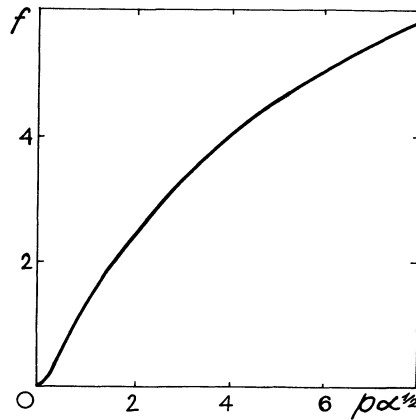


Fig. 2. — Dependence of the function f on $p\alpha^{1/2}$. For $p\alpha^{1/2} > 10$ the function f varies rather slowly, its asymptotic behaviour being given by equation (19).

$f(y)$ is plotted in figure 2. Numerically $\lambda_0(0) \approx 0.42$. At large $y \gg 1$, f saturates to a constant value. This limit was treated in [7, 8]. The asymptotic behaviour of f can be derived from equations (12) in the following way. From the first of equations (12) one can guess that at $p\alpha^{1/2} \gg 1$ the eigenvalue λ is of order $p^2\alpha$. It follows then that almost everywhere in the complex z plane the derivatives in the first of equations (12) can be neglected. In this approximation F is given by

$$F(\tau) = - \left[p^2 \alpha - 2^{1/2} \lambda \frac{\tau^{3/2}}{A(\tau)} \right]^{-1}.$$

The second of equations (12) can then be used to find τ as a function of z .

It can be easily verified that at arbitrary $\lambda \sim p^2\alpha$ there are two points, where the above expression for $F(\tau)$ diverges. We can assume that the eigenvalue of λ is determined by the condition of merging of these points. Hence, we obtain

$$\tau_0^2 = 14/3, \quad \lambda^0 = p^2 \alpha (\tau_0^2 - 2) / (2^{1/2} \tau_0^{7/2}) = p^2 \alpha \frac{1}{7} \left(\frac{6}{7} \right)^{3/4}. \quad (15)$$

To justify the procedure described above and calculate a correction λ^1 to the eigenvalue $\lambda = \lambda^0 + \lambda^1$, we have to take into account the terms with derivatives in equations (12). The second of equations (12) in the vicinity of τ_0 is written as

$$d\tau/dz = p^2 \alpha F. \quad (16)$$

In the first of equations (12) we neglect the term with second derivative compared to the term with first derivative, i.e. we assume that $d^2F/dz^2 \ll p\alpha^{1/2} dF/dz$ (it will be shown below that $dF/dz \sim (p\alpha^{1/2})^{3/5} F$). This corresponds to dropping out the asymptotics $F \propto \exp(p\alpha^{1/2} z)$, which diverges at large real z . The resulting first-order equation for F can be transformed with the use of equation (16) into an equation with the only variable τ ,

$$2(p\alpha^{1/2})^3 F dF/d\tau + F[\lambda^1 p^2 \alpha / \lambda^0 + 2^{-1/2} \lambda^0 d^2/d\tau^2(\tau^{3/2}/A(\tau))_{\tau=\tau_0}(\tau - \tau_0)^2] = -1$$

In writing down this equation use has been made of the fact that $\tau \approx \tau_0$ and $\lambda^1/\lambda^0 \ll 1$. Substituting

$$z \rightarrow z(p\alpha^{1/2})^{-3/5}, \quad F \rightarrow F(p\alpha^{1/2})^{-8/5}, \quad \lambda^1 = \lambda^0 \beta (p\alpha^{1/2})^{-2/5}, \quad (\tau - \tau_0) = x(p\alpha^{1/2})^{-1/5} \quad (17)$$

then gives rise to the dimensionless equation for the spectral parameter β

$$2 F dF/dx + (\beta + 9 x^2/16) F = 1$$

$$dx/dz = F. \quad (18)$$

From equations (14), (15) and (17) we find the final expression for the growth velocity,

$$v = (D/d_0) \alpha^{3/4} (7/4) (56/3)^{3/4} [1 - \beta_0 (p\alpha^{1/2})^{-2/5}], \quad (19)$$

where β_0 is the minimal value of the spectral parameter β which can be found from the solution of equation (18). For a distant region of the spectrum, it follows from equation (18) that $\beta_n \propto n^{2/5}$, where n is an integer, $n \gg 1$.

In conclusion of this section we note that its main results are 1) the reproduction of previously obtained asymptotics of the growth velocity in the limit $p\alpha^{1/2} \gg 1$, and 2) the calculation of the growth velocity dependence on the Peclet number in the intermediate region $p\alpha^{1/2} \sim 1$ (see Fig. 2). These results were obtained in the limit $\alpha \ll 1$ and therefore they only slightly overlap with existing numerical results. The point is that, in order to determine the growth velocity by direct solution of equation (4) at real x , one must take into account singularly small terms with relative magnitude of order of $\exp(-c/\alpha^{7/8})$ (at $p\alpha^{1/2} \ll 1$), where c is number of order unity. So, when α decreases, one should increase the numerical accuracy accordingly, which prevents one from reaching the region of small α . That is why the scaling relation (1) was compared in the literature with numerical results only relevant to the dependence $v \propto p^2$. Our results allow us to go beyond this scaling relation only for $p \geq \alpha^{-1/2}$, i.e., at large Peclet numbers. To our knowledge, up to now no numerical results were obtained for $p \gg 1$.

3. Stability of the steady-state solutions.

In the preceding section we have derived equation (10) and the equivalent system of equations (12) which determine the growth velocity of a needle crystal. In a similar way one can derive an equation which describes the linear stability of this steady-state solution. For

this purpose we set

$$y = t - x^2/2 + \zeta(x) + \zeta_\Omega(x) \exp(\Omega t). \quad (20)$$

The equation for the spectrum of Ω can be found by linearization of (4) in ζ_Ω . It has been pointed above that the steady-state correction $\zeta(x)$ is small, but its derivative near the singular point is not small. Therefore when linearizing equation (4) in ζ_Ω we can also neglect $\zeta(x)$ as compared to $x^2/2$ in the integral term of equation (4). The integral operator in the equation for ζ_Ω differs from that in equation (10) by terms linear in Ω . One of them originates from $\dot{\zeta}$ in equation (4), and the other results from the expansion of the Bessel function K_1 near the singular point, if one takes into account that, in the argument of the Bessel function, p^2 is substituted by $p(p + 2\Omega)$. Omitting details of the derivation of the integral term, which are similar to the derivation of the integral term in equation (10), and linearizing in ζ_Ω the left-hand-side of equation (10), which contain derivatives, we can write the equation for ζ_Ω near the singular point (compare to (10))

$$\frac{d}{dz} \left[B(z) \frac{d\zeta_\Omega}{dz} \right] = \zeta_\Omega - 2 \int_z^\infty dz' \zeta_\Omega(z') \exp[p\alpha^{1/2}(z - z')] \times \left[(p\alpha^{1/2} + \omega) - \frac{p\alpha^{1/2}}{2} (z' - z)(p\alpha^{1/2} + 2\omega) \right], \quad (21)$$

where

$$\Omega = \omega\alpha^{-1/2}, \quad B(z) = A[\tau(z)] / \{2^{1/2} \lambda [\tau(z)]^{3/2}\}. \quad (22)$$

The parameter λ and the function $\tau(z)$ entering equation (21) are determined by the solution of the steady-state problem (see Eqs. (10) and (12)). Note that equation (21) has been derived without the conventional quasistationary approximation. This corresponds to discarding the term with 2ω in (21). It is clear from equation (21) that this is justified only in the limit $p\alpha^{1/2} \ll 1$.

In close analogy with the case of equation (10), the simple structure of the integral term enables one to reduce equation (21) to a third order differential equation. Let us denote the right-hand-side of equation (21) by R . Taking the second derivative of this expression we find that R obeys the equation

$$(d/dz - p\alpha^{1/2})^2 R = (d/dz + 2\omega) d\zeta_\Omega/dz. \quad (23)$$

Substituting

$$d\zeta_\Omega/dz = d^2\varphi/dz^2 \exp(p\alpha^{1/2}z),$$

for the function φ from equations (21) and (23) we obtain the third order equation

$$[(d/dz) + p\alpha^{1/2}][B(z)(d^2\varphi/dz^2) - \varphi] = 2\omega\varphi, \quad (24)$$

where $B(z)$ is determined by expression (22) via the solution of the steady-state problem. For $|z| \gg 1$, $B(z) \approx (2^{1/2} \lambda z^{3/2})^{-1}$, and the three linearly independent solutions of equation (24) are

$$\varphi_{1,2} \propto \exp\left(\pm 2^{1/4} \frac{4}{7} \lambda^{1/2} z^{7/4}\right), \quad \varphi_3 \propto \exp[-(p\alpha^{1/2} + 2\omega)z]. \quad (25)$$

The solution we are looking for behaves asymptotically as φ_3 . To meet this condition one

needs, similarly to the steady-state problem, to suppress the exponential growth of $\varphi_{1,2}$ along the rays $\arg(z) = 0, \pm 4\pi/7$. With these boundary conditions taken into account, equation (24) determines the spectrum of increments ω and the eigenfunctions φ .

It is known that the steady-state solution with the maximal velocity is stable, whereas solutions with smaller velocities have unstable modes ($\omega > 0$) whose number is equal to the sequential number of the unstable solution.

From equation (24) it follows that the growth rates ω depend on the parameter $p\alpha^{1/2}$. At $p\alpha^{1/2} \ll 1$, the growth rates ω take some constant values. In this limit the problem of growth stability in terms of a linear version of the steady-state problem was considered in [10] (see also [11, 12]). Solving equation (24) numerically together with the nonlinear steady-state system (12) at $p\alpha^{1/2} = 0$ we have made sure that for the steady-state solution with maximal velocity there are no positive eigenvalues ω . For the solution next in terms of velocity one unstable mode has been found with $\omega \approx 0.87$.

Similarly to equation (14) for the dimensional growth velocity, we can write an expression for the dimensional growth rates Ω . Having in mind that the time was measured in units of ρ/v , we obtain

$$\Omega = \omega \alpha^{-1/2} v / \rho = (D/d_0^2) \alpha^{3/2} g(p\alpha^{1/2}). \quad (26)$$

The function $g(y)$ is expressed in terms of the function f introduced in equation (14) and of $\omega(y)$,

$$g(y) = f^2(y) \omega(y) / 2y. \quad (27)$$

For $p\alpha^{1/2} \ll 1$, $f \propto y^2$ and $\omega = \text{const}$, so that finally, in the small $p\alpha^{1/2}$ limit,

$$g(y) \propto y^3, \quad y \ll 1. \quad (28)$$

Consider now the limit $p\alpha^{1/2} \gg 1$. Simplifications, which are justified in this case, are quite similar to those introduced in the case of the steady-state problem in the course of the derivation of equation (18). As in section 2, only τ close to τ_0 are important,

$$(\tau - \tau_0) \sim (p\alpha^{1/2})^{-1/5}.$$

The order of equation (24) can be reduced by one, as has already been done in the steady-state problem. After setting

$$\varphi = \Psi \exp(-p\alpha^{1/2} z)$$

we neglect in the equation for Ψ some small terms, taking for granted that

$$(d^n \Psi / dz^n) \ll (d^{n-1} \Psi / dz^{n-1}) p\alpha^{1/2},$$

but noticing that

$$(d^n \Psi / dz^n) \sim (d^{n-1} \Psi / dz^{n-1}) p\alpha^{1/2} (B - 1/p^2 \alpha).$$

Exploiting the expansions of B and λ , after substitutions

$$z \rightarrow z(p\alpha^{1/2})^{-3/5}, \quad (\tau - \tau_0) = x(p\alpha^{1/2})^{-1/5}, \quad \omega = \tilde{\omega}(p\alpha^{1/2})^{1/5}, \quad (29)$$

we obtain the following second order equation, which does not contain the parameter $p\alpha^{1/2}$,

$$d^2 \Psi / dz^2 + (\beta/2 + 9x^2/32) d\Psi / dz + [(9x/16) dx/dz + \tilde{\omega}] \Psi = 0. \quad (30)$$

This equation determines the spectrum of growth rates for the steady-state solution, specified by the value of the spectral parameter β and by the shape function $x(z)$ (see Eq. (18)). In the nearest region of the velocity spectrum, when $\beta \sim 1$, $\tilde{\omega} \sim 1$. In the distant region of the spectrum, when $\beta_n \sim n^{2/5}$, the analysis of equation (30) yields for the most unstable mode

$$\tilde{\omega}_{\max} \sim \beta^2 \sim n^{4/5}. \quad (31)$$

The dependence of the increment ω on the parameter $p\alpha^{1/2}$ is given by relation (29), $\omega \sim (p\alpha^{1/2})^{1/5}$. At $p\alpha^{1/2} \gg 1$ the growth velocity as well as the function $f(p\alpha^{1/2})$ in equation (27) become p -independent. For the function $g(p\alpha^{1/2})$ describing the dependence of the growth rate on the Peclet number, we obtain the following asymptotic behaviour

$$g \sim (p\alpha^{1/2})^{-4/5}. \quad (32)$$

Comparing expressions (28) and (31), one can see that the increment Ω depends on the Peclet number nonmonotonously, exhibiting a maximum at $p\alpha^{1/2} \sim 1$.

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