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Construction of average lattices for quasiperiodic structures by the section method

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Résumé. — La construction d'un réseau moyen avec une modulation bornée pour des pavages quasipériodiques unidimensionnels est considérée du point de vue de l'espace de dimension supérieure \mathbb{R}^2 . Les pavages quasipériodiques sont : a) le pavage canonique 1D obtenu par exemple par la méthode de coupe et projection, b) des pavages engendrés par un algorithme du cercle pour des valeurs particulières des paramètres définissant le modèle. Dans ce dernier cas, la construction comble un vide entre la méthode de coupe et projection ou de section et le modèle de l'algorithme du cercle et apporte une autre preuve de l'ordre quasipériodique : nous construisons des pavages 2D périodiques appropriés donnant les pavages 1D quasipériodiques par section. Cette approche géométrique donne aussi une image intuitive du mécanisme de la disparition du réseau moyen et de l'ordre quasipériodique pour des valeurs génériques des paramètres du modèle. Les considérations données ici peuvent servir de base à la construction de réseaux moyens avec modulation bornée, s'ils existent, pour des pavages de dimensions supérieures.

Abstract. — The construction of an average lattice with bounded modulation, for one dimensional quasiperiodic tilings, is considered from the viewpoint of the higher dimensional space \mathbb{R}^2 . The 1D quasiperiodic tilings are : a) the canonical 1D tiling obtained e.g., by the cut and project method, b) tilings generated by a circle map algorithm, for particular values of the parameters defining the model. In this last case, the construction bridges a gap between the cut and project, or section, methods, and the circle map model, and provides an alternative proof of the quasiperiodic ordering : we build suitable 2D periodic tilings yielding the quasiperiodic ones by section. This geometrical approach gives also an intuitive image of the mechanism of the disappearance of the average lattice, and of the quasiperiodic ordering, for generic values of the parameters of the model. The considerations given here may serve as a basis for the construction of average lattices with bounded modulation, if they exist, of higher dimensional tilings.

1. Introduction.

One of the characterisations of order in one-dimensional incommensurate structures is obtained by considering the question of the existence of an average lattice. In some sense, the existence of an average lattice in a 1D structure gives a measure of its distance to crystalline order, and may help to classify different aperiodic structures, such as conventional

incommensurate structures or other more complex ones, with possibly weaker order. Take a distribution of atoms on a line, for which it is possible to define a mean interatomic distance or inverse density a ; if the deviation of the position u_n of the n -th atom with respect to its average position na is bounded, the structure is said to possess an average lattice [1]. The bounded deviation is called the modulation. For instance, u_n may be given by

$$u_n = na + g(na) \quad (1.1)$$

where $g(x)$ is a b -periodic function, called the modulation (or hull) function. In this case, the Fourier spectrum is discrete with resonant wavevectors at integer combinations of $2\pi/a$ and $2\pi/b$. In other words, the structure is quasiperiodic. Examples of 1D structures which possess an average lattice with bounded modulation are given by : a) the ground states of the Frenkel-Kontorova model [2], b) the so-called « integrable cases » of a circle map model [3, 4], c) structures obtained by the cut and project method [5-7] from 2D to 1D, for suitable values of the width of the strip. In all these cases, the modulation $g(x)$ is a periodic function and the resultant structure is quasiperiodic.

On the other hand, in recent works, two examples of structures without average lattice have been given. These two examples are built from the same quasiperiodic rule, involving a circle map which generates a binary sequence of 0 and 1's. In the first example [1] (« model 1 », or « atoms and vacancies model »), atoms and vacancies are associated to the binary sequence. In the second example [3, 4] (« model 2 », or « circle map tiling »), tiles are associated to the sequence. For both models, in non integrable cases, i.e., for generic values of the parameters of the circle map algorithm, the structure does not possess an average lattice. Yet, in the first example, the structure is quasiperiodic, whereas in the second case, the disappearance of the average lattice has strong consequences on the Fourier spectrum : the intensity spectrum no longer contains any Dirac peak and it was claimed in references [3, 4] that this structure, that may be named « almost quasiperiodic », has a singular continuous Fourier spectrum.

These results show the interplay between two characteristic properties of deterministic structures, namely : (1) a bounded fluctuation of the atomic positions with respect to their average lattice ; (2) a quasiperiodic Fourier spectrum.

The aim of this paper is to give a geometrical interpretation of this interplay in a higher dimensional space, here \mathbb{R}^2 , from which the 1D structure (or tiling) is recovered by a section method. In order to do so, a natural direction to explore consists in trying to cast the circle map algorithm in the section method framework. This seems *a priori* possible, at least for the integrable cases, since they give rise to quasiperiodic structures. The section method gives a direct characterisation of the quasiperiodic ordering : quasiperiodic tilings are sections of periodic ones, just as quasiperiodic functions are sections of periodic functions in several variables. The discrete nature of the Fourier spectrum of quasiperiodic tilings is a direct consequence of this fact.

More precisely :

i) we first show, in section 2, how to construct, by section, the average lattice and the modulation for the canonical 1D tiling. This is of importance, since it is a first step towards the construction of the average lattices, when they exist, of higher dimensional tilings. In particular, the construction given here is easily extendable to all codimension one cases, i.e., projection from a N -dimensional space to a $(N - 1)$ -dimensional one ;

ii) in section 3, this construction is then used, together with a renormalisation procedure, to find periodic tilings of \mathbb{R}^2 that generate, by section, the 1D tilings of model 2 for particular values of its parameters. We end this section by giving an intuitive geometrical image of the mechanism of the disappearance of the average lattice for generic values of the parameters of model 2.

2. From 1D to 2D and back. Average lattice and modulation.

In this section, we first recall model 2, the circle map tiling. For particular values of its parameters, this model is equivalent to the projection method from 2D to 1D, and generates a quasiperiodic tiling which may, as well, be viewed as a standard displacive incommensurate 1D structure, having an average lattice with bounded modulation.

We then show how to construct the average lattice and the modulation of the position of atoms, of this particular tiling, from the higher dimensional space \mathbb{R}^2 .

2.1 MODEL 2, A REMINDER. — This model [3, 4], based on a circle map algorithm, generates a 1D atomic structure or a tiling of the line. The structure is defined by putting atoms on a line, the abscissa of the n th atom being given by

$$u_n - u_{n-1} = \lambda_n \tag{2.1}$$

($u_0 = 0$). The bond lengths λ_n are given by the action of a « window » function of width Δ on the sequence $\{n\omega \bmod 1\}$; ω and Δ are given numbers between 0 and 1 :

$$\lambda_n = \ell_2 + (\ell_1 - \ell_2) \chi_\Delta(n\omega) . \tag{2.2}$$

This is illustrated in figure 1. $\chi_\Delta(x)$ is a 1-periodic function defined by

$$\chi_\Delta(x) = \text{Int}(x) - \text{Int}(x - \Delta) = \begin{cases} 1 & \text{if } 0 \leq \text{Frac}(x) < \Delta \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

$\text{Int}(x)$ and $\text{Frac}(x)$ are the integer and fractional parts of x , respectively. The binary sequence $\{\chi_n \equiv \chi_\Delta(n\omega)\}$ is thus quasiperiodic.

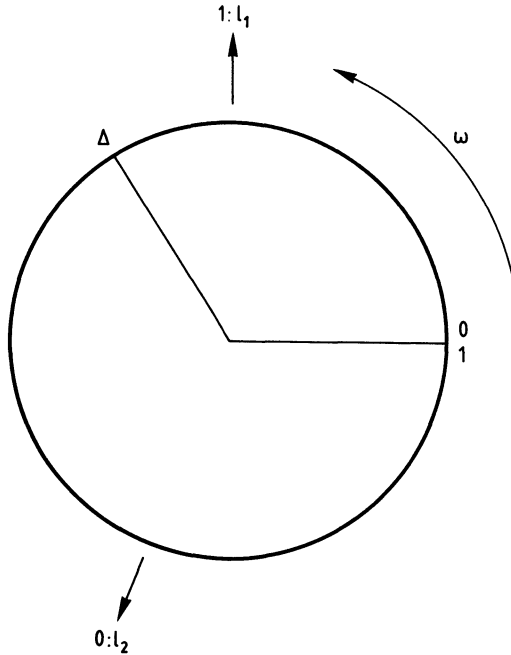


Fig. 1. — Generation of the binary sequence $\{\chi_n\}$, and of a tiling, by a circle map. The sector $(0, \Delta)$ is the window.

We have

$$u_n = n\ell_2 + (\ell_1 - \ell_2) \sum_1^n \chi_{\Delta}(m\omega) \equiv n\ell_2 + (\ell_1 - \ell_2) S_n. \quad (2.4)$$

The interatomic mean distance, or inverse density, of the model is

$$a = \lim_{n \rightarrow \infty} \frac{u_n}{n} = \ell_2 + (\ell_1 - \ell_2) \Delta. \quad (2.5)$$

The modulation of the atomic abscissas with respect to their average lattice (na) may be computed as

$$u_n - na \equiv g_n = (\ell_1 - \ell_2)(S_n - n\Delta) \equiv (\ell_1 - \ell_2) \delta_n \quad (2.6)$$

δ_n is the fluctuation of S_n , the number of letters 1 after n steps, with respect to its average value $n\Delta$.

A theorem, due to Kesten [8], asserts that δ_n is bounded in n , if and only if

$$\Delta = r\omega - s \equiv r\omega \pmod{1} \quad (2.7)$$

where r and s are integers. These cases are called « integrable ». If this condition is not satisfied, the fluctuation δ_n diverges with increasing n . The nature of the divergence depends on the arithmetical nature of ω [9, 10]. This theorem implies that there no longer exists an average lattice for the structure. As a consequence, it was shown that, for generic non integrable cases, the Fourier spectrum of the structure is singular continuous [3, 4], which is the signature of a weaker order than the quasiperiodic one.

For integrable cases, it is possible to compute the fluctuation as

$$\delta_n = -\gamma_r(n\omega) \quad (2.8)$$

where $\gamma_r(x)$ is a r -dependent 1-periodic function [1, 3], therefore a bounded one. The modulation g_n being periodic, the structure is quasiperiodic.

Remark. Model 1 is built on the same sequence $\{\chi_n\}$. Atoms and vacancies are put on the sites of a 1D lattice according to the rule : to 1 corresponds a vacancy, to 0 an atom. Though this structure has no average lattice, it is quasiperiodic [1].

2.2 GEOMETRICAL APPROACH IN \mathbb{R}^2 . — One way to look at the 1D structure from the point of view of the higher dimensional space \mathbb{R}^2 , consists in associating a broken line to the sequence $\{\chi_n\}$, by replacing 1 (or ℓ_1) by ε_1 , and 0 (or ℓ_2) by ε_2 , as shown in figure 2. ε_1 and ε_2 are the unit vectors of \mathbb{Z}^2 . Hence, a point of the broken line is described by

$$U_n = n\varepsilon_2 + (\varepsilon_1 - \varepsilon_2) S_n. \quad (2.9)$$

The average interatomic vector A

$$A = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) \Delta = \Delta\varepsilon_1 + (1 - \Delta) \varepsilon_2 \quad (2.10)$$

gives the average direction of the broken line, denoted E_{\parallel} . Its slope is

$$t = \frac{1 - \Delta}{\Delta} \quad (2.11)$$

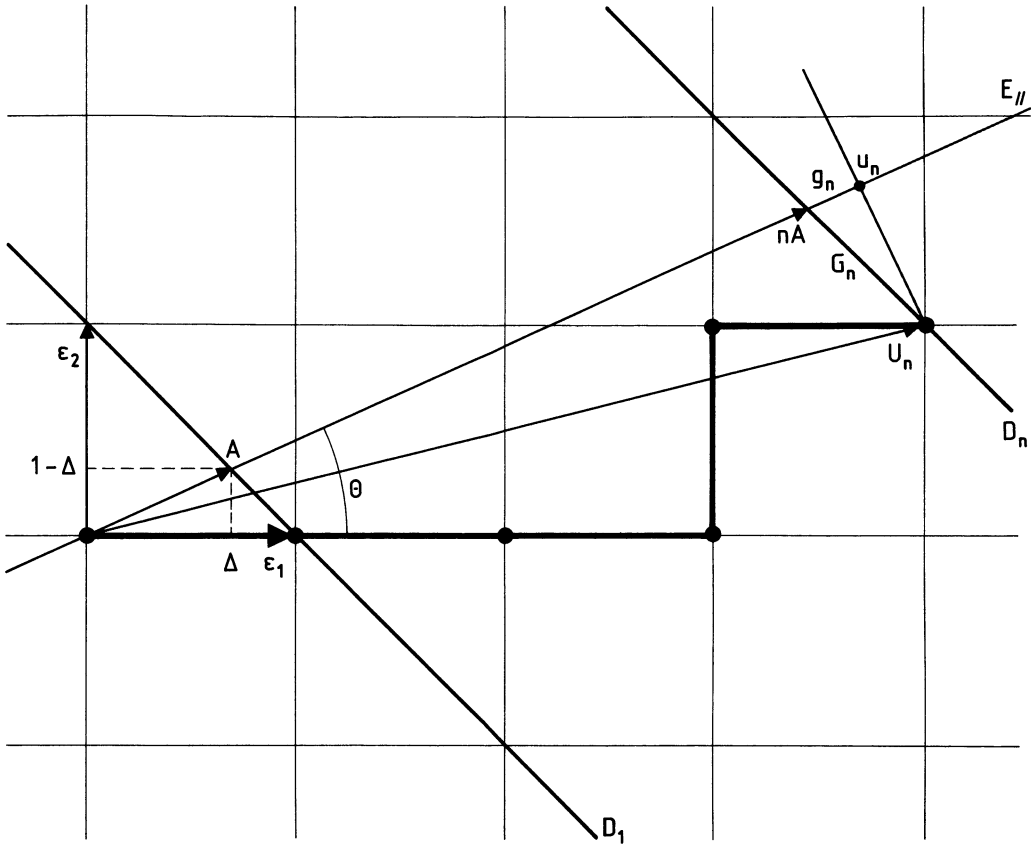


Fig. 2. — Broken line in Z^2 corresponding to the 1D tiling.

where t denotes $\tan \theta$. The 1D tiling is recovered by projection of the broken line on E_1 . Indeed, since it is always possible to rescale the lengths ℓ_1 and ℓ_2 globally, only the ratio ℓ_2/ℓ_1 is a significant parameter. Then, if one restricts the ratio ℓ_2/ℓ_1 to t , ℓ_1 may be identified to c , and ℓ_2 to s , with $c = \cos \theta$, and $s = \sin \theta$. If not, it is possible to project the broken line orthogonally onto a line of slope $\tan \varphi$ such that, now ℓ_1 should be identified to $\cos \varphi$ and ℓ_2 to $\sin \varphi$.

The norm of A is

$$a = \frac{1}{c + s} \tag{2.12}$$

the mean interatomic distance. A has its extremity on the diagonal D_n , where, for any integer n ,

$$D_n = \{z \in \mathbb{R}^2 \mid z_1 + z_2 = n\} . \tag{2.13}$$

The nodes of the average lattice of the 1D tiling, in E_1 , are the intersections of E_1 with the diagonals D_n . The modulation g_n has also an interpretation in \mathbb{R}^2 : one may define a vector $G_n = U_n - nA$ from which g_n is obtained by projection p_1 on E_1 .

$$g_n = p_1(G_n) = (c - s) \delta_n . \tag{2.14}$$

This description will be made more precise below. $p_{\perp}(G_n)$, the projection of G_n onto E_{\perp} ⁽¹⁾, represents the transversal extension of the broken line :

$$p_{\perp}(G_n) = p_{\perp}(U_n - nA) = p_{\perp}(U_n) = p_{\perp}((\varepsilon_1 - \varepsilon_2) \delta_n) = (c + s) \delta_n. \quad (2.15)$$

In particular, in non integrable cases, this perpendicular extension diverges. Note that equations (2.9)-(2.15) are true for any value of Δ .

Let us now restrict our description to the canonical case. This case is defined by the choice $\omega = \Delta$. One has

$$\delta_n = -\text{Frac}(n\omega) \quad (2.16)$$

and therefore,

$$u_n = na - (\ell_1 - \ell_2) \text{Frac}(n\omega). \quad (2.17)$$

This structure, which has an average lattice with periodic modulation, is the same [1, 2] as the generated by the canonical projection method from 2D to 1D. The corresponding tiling is hereafter referred to as the canonical tiling. This is illustrated by figure 3. The broken line is entirely contained in the strip $S = E_{\parallel} + C$ where C is the unit square

$$C = \{(z_1, z_2) \mid 0 \leq z_i \leq 1\}. \quad (2.18)$$

The correspondence between the parameters of the two structures is given by

$$\Delta = \omega = \frac{1}{1+t}. \quad (2.19)$$

This geometrical construction may also be described by a section method [11, 12]. Consider the periodic set of atomic surfaces (here atomic segments), built by attaching an atomic surface \mathcal{A} to each point $\xi \in Z^2$. The set of points u_n of the 1D structure is obtained as the intersection of E_{\parallel} by the set of atomic surfaces

$$\{u_n\} = E_{\parallel} \cap (Z^2 + \mathcal{A}). \quad (2.20)$$

In the canonical case, $\mathcal{A} = -K$ where

$$K = p_{\perp}(C) = E_{\perp} \cap S \quad (2.21)$$

is the profile of the strip.

The periodic set of atomic surfaces may be completed into a 2D tiling [13]. Define the 2D tiles

$$\begin{aligned} T_1 &= p_{\parallel}(\varepsilon_1) \times (-p_{\perp}(\varepsilon_2)) \\ T_2 &= p_{\parallel}(\varepsilon_2) \times (-p_{\perp}(\varepsilon_1)) \end{aligned} \quad (2.22)$$

These two parallelograms give a partition of a fundamental cell of Z^2 . The resulting « oblique » tiling leads to the canonical quasiperiodic tiling of E_{\parallel} by section. The atomic

⁽¹⁾ E_{\perp} is the kernel of p_{\parallel} . It specifies the direction of the atomic segments. This direction is generated by $(-\sin \varphi, \cos \varphi)$ and is held fixed for the various examples we will consider in this paper. This is not the case for E_{\parallel} , the physical space, which will vary from case to case. In particular, E_{\perp} is not necessarily perpendicular to E_{\parallel} .

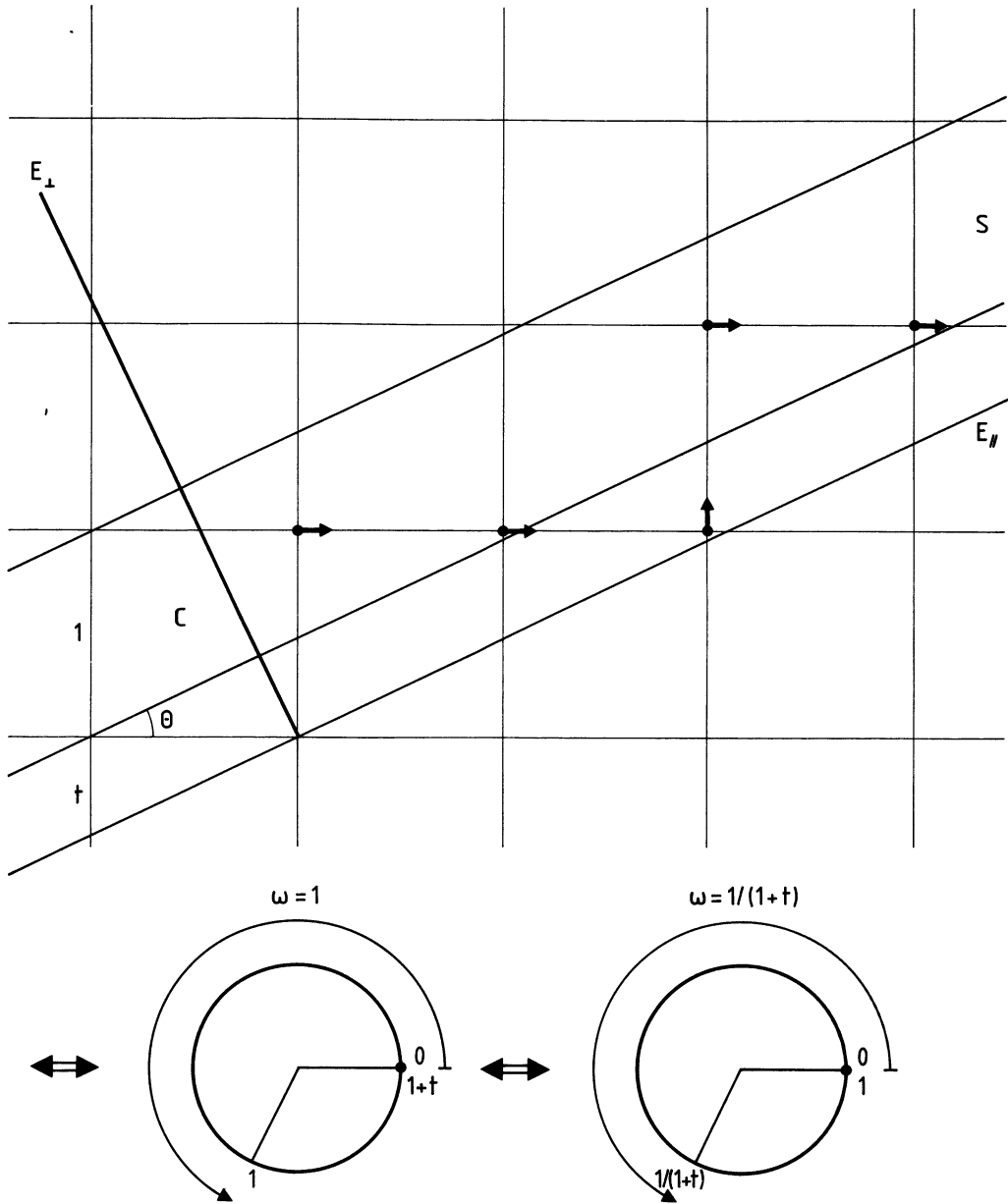


Fig. 3. — Equivalence of the projection method from 2D to 1D, with the circle map algorithm, when $\omega = \Delta$.

surfaces described above, are the boundaries of the oblique tiling, transversal to E_{\perp} , as shown in figure 4.

Let us note that, since the points of the average lattice are the intersections of E_{\perp} by D_n , the diagonals are the atomic surfaces generating the average lattice. In this framework, the modulation g_n also has a simple geometrical interpretation. It is the distance between the point u_n of the structure obtained as a section of the atomic surface \mathcal{A} by

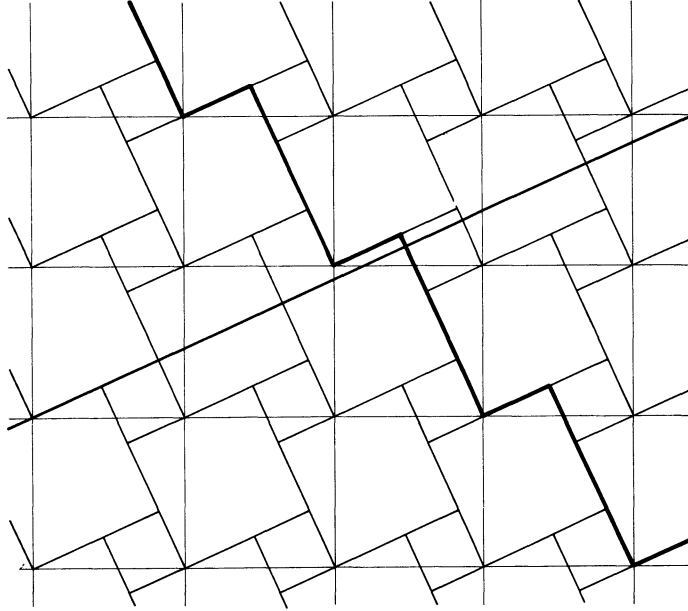


Fig. 4. — Oblique tiling and atomic surface, in the canonical case.

E_{\parallel} , and the point na of the average lattice obtained as a section of D_n (see also Fig. 2). In other terms, the atomic surface, seen along a diagonal, over a period (here one diagonal of the unit square), gives a representation of the hull of the modulation, i.e., of the modulation function $g(x) = -(c-s) \text{Frac}(x)$. Indeed, figure 5b may be obtained from figure 5a, by rescaling AB to 1, and keeping the direction of BC perpendicular to AB, without changing its length.

This correspondence between the modulation function and the atomic surface may also be seen by taking the converse point of view : if the points u_n of the structure are given, and if one knows that they come from a section method, it is possible to get the shape of the atomic surface by folding the plane on a torus, namely, the unit square with parallel edges identified. The parametric equation of the atomic surface \mathcal{A} is given by the hull of the set of points satisfying

$$\begin{cases} z_{1,n} = u_n c \bmod 1 \\ z_{2,n} = u_n s \bmod 1 . \end{cases} \quad (2.23)$$

For instance, if $u_n = na$,

$$\begin{cases} z_{1,n} = \text{Frac}(n\omega) \\ z_{2,n} = 1 - \text{Frac}(n\omega) \end{cases} \quad (2.24)$$

since $a = 1/(c+s)$. Hence, the equation in the torus of the atomic surface generating the average lattice is

$$z_1 + z_2 = 1 \quad (2.25)$$

which is the equation of the diagonal D_1 , as expected. If $u_n = na - (c-s) \text{Frac}(n\omega)$, a

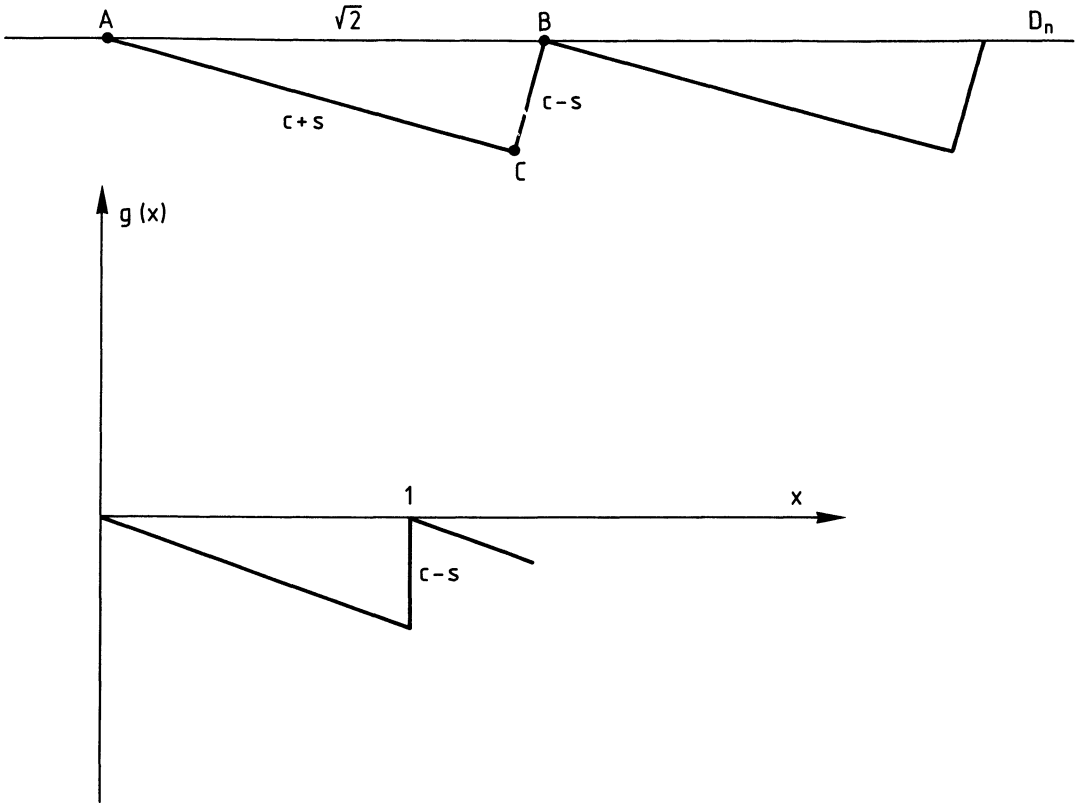


Fig. 5. — Equivalence between the atomic surface seen along a diagonal, and the modulation function, for the canonical case.

simple calculation yields the equation of the atomic surface in the torus, for the canonical tiling

$$z_2 = 1 - \frac{z_1}{t} \quad \text{if } z_1 < t \tag{2.26}$$

$$z_2 = 2 - \frac{z_1}{t} \quad \text{if } z_1 > t . \tag{2.27}$$

As mentioned in the Introduction, this representation, in the extended space, of the average lattice and of its modulation, is easily generalised to any codimension one case. We will now use the framework of this section, to extend the geometrical description given here, to the general integrable cases of model 2.

3. Geometry in higher dimensional space of the general integrable cases.

This section is devoted to the integrable cases with finite fixed $r \neq 1$. As already mentioned, the corresponding tilings are quasiperiodic. It is thus natural to try to describe these structures in higher dimensional space. In particular, we will build a set of atomic surfaces in \mathbb{R}^2 that generate, by section, the 1D structure in E_1 space.

In order to do so, we use a renormalisation method introduced in reference [4], and recalled below. The strategy consists in renormalising the binary sequence $\{\chi \Delta(n\omega)\}$. This

reduces the integrable cases to the canonical case ($\omega', \Delta' = \omega'$), for which the framework of section 2 applies. In some sense, this approach fills in the gap between the projection method $2D \rightarrow 1D$ and the circle map algorithm of model 2, at least for the integrable cases.

We will end this section by considering the first terms of the sequence $\{r_N\}$ of values of r leading, when $N \rightarrow \infty$, to a non integrable case. This will give an intuitive geometrical image of the mechanism of the disappearance of the average lattice (and of quasiperiodicity) in non integrable cases.

3.1 RENORMALISATION, A REMINDER. — Renormalisation consists in changing the scale at which the sequence $\{\chi_n\}$, and consequently the structure that it generates, are looked at. It is an exact decimation procedure acting on $\{\chi_n\}$.

We first need to define an intermediate labelling sequence on the unit circle. A sequence of letters L_n is associated to the sequence of numbers $\alpha_n = \text{Frac}(n\omega)$ by the rule :

$$L_n = \begin{cases} A & \text{if } 0 \leq \alpha_n < \inf(\Delta, \omega) \\ B & \text{if } \inf(\Delta, \omega) \leq \alpha_n < \sup(\Delta, \omega) \\ C & \text{otherwise.} \end{cases} \quad (3.1)$$

Then by choosing

$$A \equiv 1; \quad B \equiv 0 \text{ if } \Delta < \omega \text{ or } B \equiv 1 \text{ if } \omega < \Delta; \quad C \equiv 0 \quad (3.2)$$

the sequence $\{\chi_n\}$ is recovered.

A renormalization operation \mathcal{R} is a product of a combination of four elementary transformations S, T_1, T_2, T_3 , defined in the table, that changes both the values of (ω, Δ) and the sequence $\{L_n\}$:

$$\begin{aligned} (\omega, \Delta) &\rightarrow (\omega', \Delta') \equiv \mathcal{R}(\omega, \Delta) \\ L_n &\rightarrow L'_n \equiv \mathcal{R}L_n. \end{aligned} \quad (3.3)$$

In the last equation, \mathcal{R} is a condensed notation for a substitution $\sigma_{\mathcal{R}}$ on the letters :

$$\begin{aligned} A &\rightarrow A' \equiv \sigma_{\mathcal{R}}(A) \\ B &\rightarrow B' \equiv \sigma_{\mathcal{R}}(B) \\ C &\rightarrow C' \equiv \sigma_{\mathcal{R}}(C). \end{aligned} \quad (3.4)$$

The order in which the elementary operations enter in \mathcal{R} , i.e., the order of a particular sequence of renormalisation operations, is uniquely determined by the continuous fraction expansion of ω and by the ω -expansion of Δ [4]. For example, S always maps $\omega > 1/2$ onto $\omega < 1/2$. In the (ω, Δ) plane, the iteration of the renormalisation transform \mathcal{R} leads in general, to an aperiodic orbit. For some values of (ω, Δ) , it may be a fixed point or a cycle.

Let us give the results of the application of the renormalisation method to the cases of interest for this paper.

a) $(\omega, \Delta = r\omega - s)$. Integrable cases.

S, T_i transform any integrable case into a canonical one ($\omega', \Delta' = \omega'$) in a finite number of steps. This is due to the fact that the elementary renormalisation operations change the values of r and s as

$$\begin{aligned} S(r, s) &= (r', s') = (r, r - s - 1) \\ T_1(r, s) &= (r', s') = (r - s, s) \\ T_{2,3}(r, s) &= (r', s') = (r - s - 1, s) \end{aligned} \quad (3.5)$$

Table I. — Definition of the elementary renormalisation transformations S , T_1 , T_2 , T_3 , T_4 .

Operation	Conditions on (ω, Δ)	ω'	Δ'	σ
S	$\frac{1}{2} < \omega < 1$	$1 - \omega$	$1 - \Delta$	$\begin{cases} A' = A \\ B' = B \\ C' = A \end{cases}$
T_1	$0 < \omega < \frac{1}{2} \quad 0 < \Delta < \omega$	$\frac{\omega}{1 - \omega}$	$\frac{\Delta}{1 - \omega}$	$\begin{cases} A' = AC \\ B' = BC \\ C' = C \end{cases}$
T_2	$0 < \omega < \frac{1}{2} \quad \omega < \Delta < 2\omega$	$\frac{\omega}{1 - \omega}$	$\frac{\Delta - \omega}{1 - \omega}$	$\begin{cases} A' = AB \\ B' = AC \\ C' = C \end{cases}$
T_3	$0 < \omega < \frac{1}{2} \quad 2\omega < \Delta < 1$	$\frac{\omega}{1 - \omega}$	$\frac{\Delta - \omega}{1 - \omega}$	$\begin{cases} A' = AB \\ B' = B \\ C' = C \end{cases}$

$$b) (\omega = \tau^{-2}, \Delta = 1/2), \tau = (1 + \sqrt{5})/2.$$

The 1D structure corresponding to this case was studied analytically in references [3, 4]. In particular, the Fourier spectrum was shown to be singular continuous. This example is given here, since it corresponds to the asymptotic limit of the finite r cases considered in section 3.2, below. In particular, the renormalisation operations needed for the finite r cases and that of the limiting case provided by this example, are the same.

For these values of the parameters, the renormalisation operation

$$\mathcal{R} = ST_1 ST_3 ST_2 \quad (3.6)$$

leaves (ω, Δ) invariant, i.e., $\mathcal{R}(\omega, \Delta) = (\omega, \Delta)$. This fixed point is repulsive, i.e., it is unstable by \mathcal{R} under a small perturbation, e.g., along the $\omega = \text{const.}$ direction :

$$\mathcal{R}(\omega = \tau^{-2}, \Delta = 1/2 + \varepsilon) = (\omega = \tau^{-2}, \Delta = 1/2 - \varepsilon\tau^3). \quad (3.7)$$

Therefore, the infinite sequence $\{L_n\}$ is invariant by a substitution $\sigma_{\mathcal{R}}$

$$\begin{aligned} A' &= \sigma_{\mathcal{R}}(A) = CAC \\ B' &= \sigma_{\mathcal{R}}(B) = ACCAC \\ C' &= \sigma_{\mathcal{R}}(C) = ABCAC \end{aligned} \quad (3.8)$$

$\{L_n\}$ is a fixed point of $\sigma_{\mathcal{R}}$. In other terms, this sequence may be built by « inflation rules », given by equation (3.8). The substitution matrix M gives the number of letters A , B , C in A' , B' , C' .

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}. \quad (3.9)$$

From the n -th power of M , one finds that the number of letters in $\sigma_{\mathcal{R}}^n(A)$, $\sigma_{\mathcal{R}}^n(B)$, $\sigma_{\mathcal{R}}^n(C)$ are F_{3n+1} , F_{3n+2} , F_{3n+2} , respectively, where F_n is the n -th Fibonacci number defined by

$$F_n = F_{n-1} + F_{n-2} \quad F_0 = 0 \quad F_1 = 1. \quad (3.10)$$

Remark. Using equations (3.5)-(3.6), the action of \mathcal{R} on an integrable case reads

$$\mathcal{R}(r, s) = (r - 2s - 1, 2r - 5s - 3). \quad (3.11)$$

c) Approximation of case b) by integrable cases.

The sequence of integers (r_N, s_N) which give the best approximants [4] to $\Delta = 1/2$ by $r_N \omega - s_N$, ($\omega = \tau^{-2}$), is given by (1, 0), (4, 1), (17, 6), (72, 27), ... More generally

$$\begin{aligned} r_N &= \frac{1}{2} F_{3N+3} \\ s_N &= \frac{1}{2} (F_{3N+1} - 1). \end{aligned} \quad (3.12)$$

Hence, the approach to $\Delta = 1/2$ is given by

$$r_N \omega - s_N = \frac{1}{2} + \frac{1}{2} (-\tau^{-3})^{N+1}. \quad (3.13)$$

Remark. Let us note that \mathcal{R} , given by equation (3.6), satisfies

$$\mathcal{R}(r_N, s_N) = (r_{N-1}, s_{N-1}). \quad (3.14)$$

This property is a consequence of the fact that $(\tau^{-2}, 1/2)$ is a fixed point, approached most rapidly by the best approximants [4]. Using equation (3.11), one gets

$$\begin{aligned} r_N &= 5r_{N-1} - 2s_{N-1} - 1 \\ s_N &= 2r_{N-1} - s_{N-1} - 1 \end{aligned} \quad (3.15)$$

from which equation (3.12) is obtained.

Since for any finite r , the 1D structure is quasiperiodic, taking successive values of r means, in some sense, that the limit structure (no longer quasiperiodic, since the Fourier transform does not contain any Dirac peak) is approached by successive quasiperiodic approximant structures, much in the same manner that one may approach a quasiperiodic structure by successive periodic approximants [14]. The geometry in \mathbb{R}^2 of these structures is the subject of section 3.2.

Remark. The canonical cases $(\omega, \Delta = \omega)$ are degenerate, inasmuch as the labelling sequence has only two letters. When $\omega = \tau^{-2}$, $\{\chi_n\}$ is the Fibonacci sequence. Using the same renormalisation technique, one obtains the well-known inflation rules. Note that $(\omega = \tau^{-2}, \Delta = \omega)$ is a fixed point of a renormalisation transform.

To conclude this section, let us stress that a direct consequence of the renormalisation of the sequence $\{\chi_n\}$ is the renormalisation of the 1D tiling itself. This is done by replacing 1 and 0's of the renormalised sequence $\{\chi'_n\}$ by renormalised tiles of lengths ℓ'_2 and ℓ'_1 , respectively.

3.2 GEOMETRICAL APPROACH IN \mathbb{R}^2 . — As seen above, for integrable cases, the renormalised sequence $\{\chi'_n\}$ gives a renormalised tiling with renormalised tiles. We want now to use the renormalisation scheme directly in 2D. We describe the general method first, before illustrating it on specific examples.

We first reduce an integrable case $(\omega, \Delta = r\omega - s)$ to a canonical one $(\omega', \Delta = \omega')$, by a finite number of renormalisation steps. We are thus in position to use the framework of section 2.2. We introduce renormalised coordinates spanned by the vectors β_1 and β_2 , defined as follows. Replace the letters of the labelling sequence by 0 and 1's according to equation (3.2), then 1 by ε_1 and 0 by ε_2 . This yields the finite broken chains $\mathcal{R}\varepsilon_1$ and $\mathcal{R}\varepsilon_2$. β_1 and β_2 are the vectors joining the origin to the end of the broken chain $\mathcal{R}\varepsilon_1$ (resp. $\mathcal{R}\varepsilon_2$). These vectors generate a sublattice of $\mathcal{B}\mathbf{Z}^2$ of \mathbf{Z}^2 , where \mathcal{B} is the matrix, the columns of which are the components of β_1 and β_2 . In these renormalised coordinates the 2D geometry is that of the canonical case. The renormalised oblique tiling built in the coordinates of β_1, β_2 is invariant by the lattice of translations $\mathcal{B}\mathbf{Z}^2$.

Since renormalisation is a decimation, which consists in looking at the structure at scale r , points are missing in the 1D structure, or atomic surfaces in the oblique tiling. In order to get the oblique tiling corresponding to scale 1, with the initial tiles c, s in the section, one has to partition each renormalised tile along E_1 , according to the substitution induced by $\sigma_{\mathcal{R}}$ and acting on β_1, β_2 . This point will be made clearer by the examples below. Four oblique tiles are obtained that way. They read

$$T_{ij} = p_{\parallel}(e_j) \times (-p_{\perp}(\beta_i)) \quad (i, j = 1, 2). \quad (3.16)$$

We illustrate the method by taking the case $(\omega = \tau^{-2}, \Delta = 4\omega - 1)$. The renormalisation transform to apply is given by equations (3.6), (3.8). Following the method described above, the finite broken chains are

$$\begin{aligned} \mathcal{R}\varepsilon_1 &= \varepsilon_2 \varepsilon_1 \varepsilon_2 \\ \mathcal{R}\varepsilon_2 &= \varepsilon_1 \varepsilon_1 \varepsilon_2 \varepsilon_1 \varepsilon_2 \end{aligned} \quad (3.17)$$

and the vectors β_i have the same expressions with + signs between the vectors.

The invariance lattice $\mathcal{B}\mathbf{Z}^2$ is given by

$$\mathcal{B} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad (3.18)$$

$|\det(\mathcal{B})|$, the « volume » of the unit cell, (the index of the lattice in \mathbf{Z}^2) is equal to four.

In figure 6, the fundamental cell of the oblique tiling is made of two oblique tiles, the renormalised tiles, that may be decomposed into the smaller elementary ones. The order in which each smaller tile appears inside the renormalised one is given by the substitution $\sigma_{\mathcal{R}}$ of equation (3.8).

From the oblique tiling, one gets the atomic surfaces. Note that

$$\beta_1 + \beta_2 = 4(\varepsilon_1 + \varepsilon_2) \quad (3.19)$$

and

$$5\beta_1 - 3\beta_2 = 4(\varepsilon_2 - \varepsilon_1). \quad (3.20)$$

Therefore, the periodicity along a diagonal D_n is four, in units of $\varepsilon_2 - \varepsilon_1$ (Fig. 6). Again, it must be noticed that the atomic surface along a diagonal gives a representation of the modulation function. In particular, the atomic surface is made of three different segments (in the E_{\perp} direction). They correspond to the three segments found in the function $\gamma_4(x)$, where

$$\gamma_4(x) = \sum_0^3 -\text{Frac}(x - m\omega) + \text{Frac}(-m\omega). \quad (3.21)$$

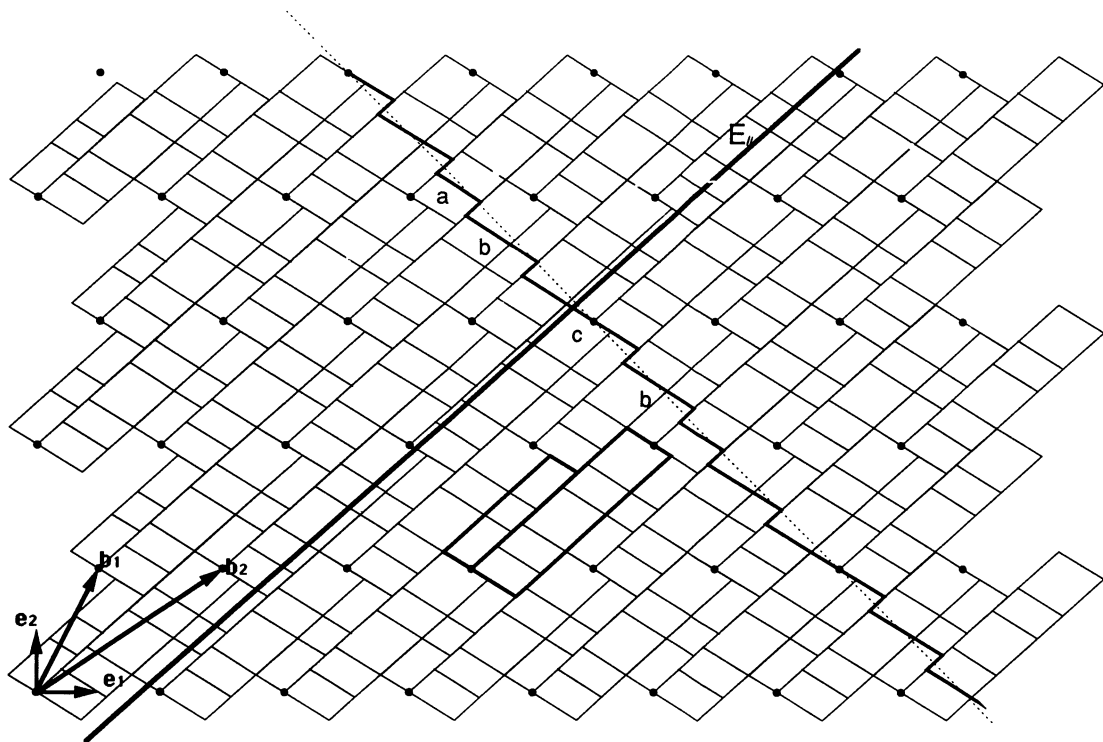


Fig. 6. — 2D geometry of the case ($\omega = \tau^{-2}, \Delta = 4\omega - 1$). In bold : a fundamental cell for the lattice \mathcal{B} ; one of the periodic atomic surfaces.

It is indeed known that $\text{Frac}(n\omega)$ divides the circle in three intervals [8, 15]. The order in which these segments appear in the atomic surface is the same as in the $\gamma_4(x)$ function, namely a, b, c, b , where $a : b : c = \tau^{-4} : \tau^{-3} : \tau^{-2}$. $\gamma_4(x)$ may be obtained by a deformation of the atomic surface (Fig. 7).

Remark. Let us mention that, as in section 2, knowing u_n , we may write the parametric representation of the atomic surfaces in the torus as

$$\begin{cases} z_{1,n} = u_n c \text{ mod } 4 \\ z_{2,n} = u_n s \text{ mod } 4 . \end{cases} \tag{3.22}$$

We end this section by studying the asymptotic behaviour of successive approximants to the limit structure generated by ($\omega = \tau^{-2}, \Delta = 1/2$).

The next best approximant is ($\omega = \tau^{-2}, \Delta = 17\omega - 6$). Applying twice the renormalisation operation equation (3.6), reduces this case to a canonical one. The geometrical scheme in \mathbb{R}^2 is similar to the previous case. Figure 8 illustrates this case.

Let us consider now a general (r_N, s_N) case. β_1 (resp. β_2) is obtained from $\mathcal{R}^N \varepsilon_1$ (resp. $\mathcal{R}^N \varepsilon_2$). The invariance lattice is determined by the matrix

$$\mathcal{B}_N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M^N \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} F_{3N+1} + (-1)^N & F_{3N+2} - (-1)^N \\ F_{3N+1} - (-1)^N & F_{3N+2} + (-1)^N \end{pmatrix} \tag{3.23}$$

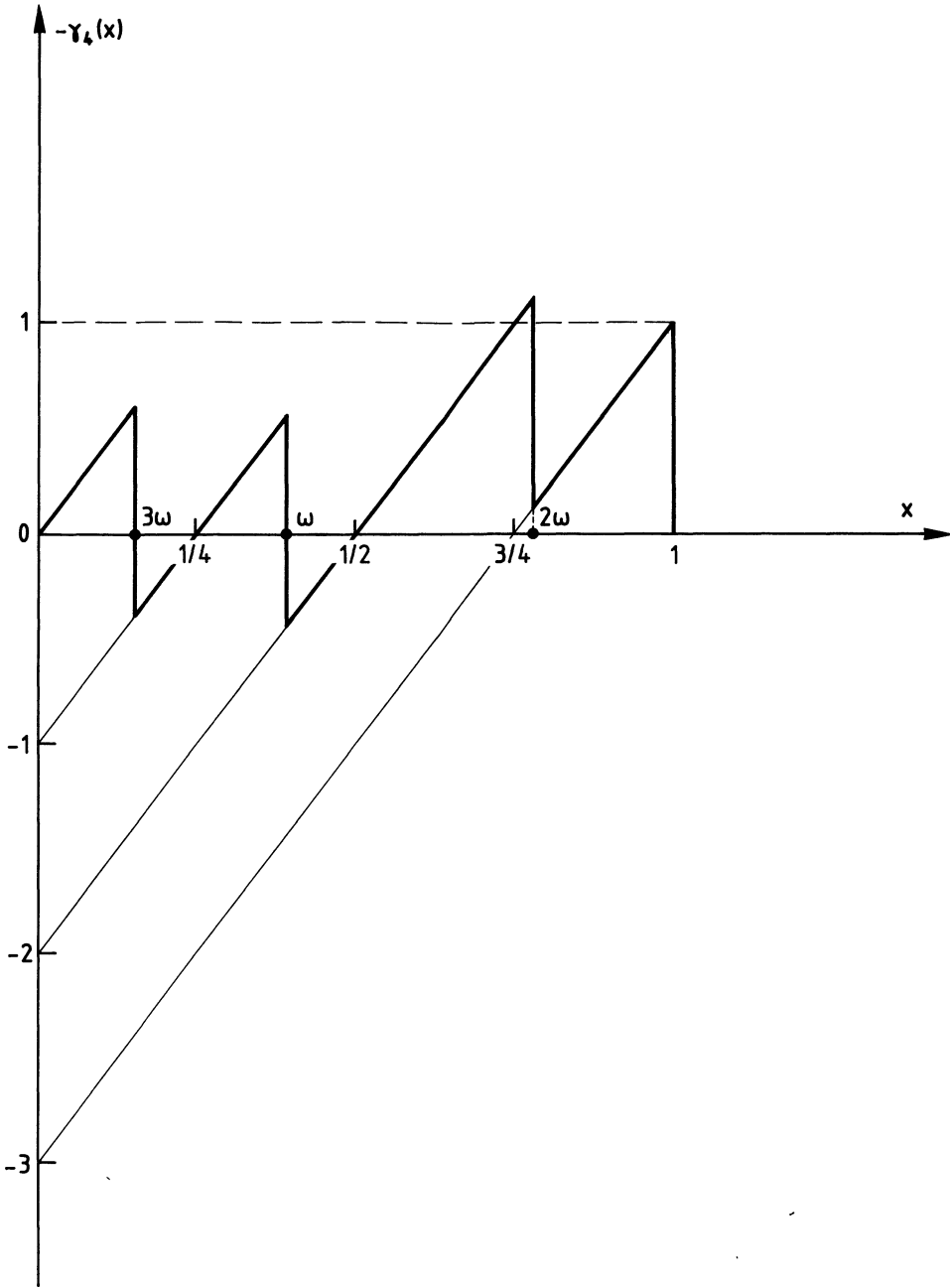


Fig. 7. — The function $\gamma_4(x)$. Comparison with figure 6 shows the equivalence between the atomic surface seen along a diagonal, and the modulation function, for the case ($\omega = \tau^{-2}$, $\Delta = 4\omega - 1$).

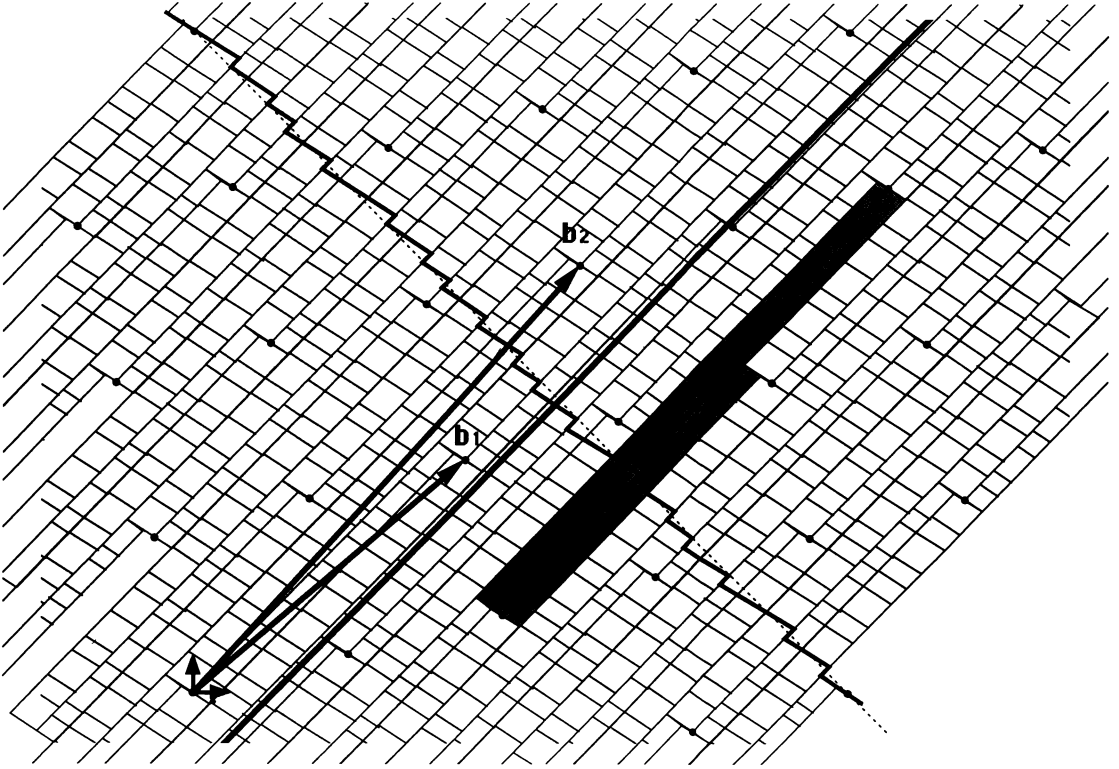


Fig. 8. — 2D geometry of the case ($\omega = \tau^{-2}$, $\Delta = 17\omega - 6$).

where M is given by equation (3.9). The « volume » of the unit cell spanned by β_1 and β_2 , is given by

$$\det \beta_N = \frac{1}{2} F_{3N+3} \equiv r_N. \quad (3.24)$$

Generalising equation (3.19), one has

$$\beta_1 + \beta_2 = r_N(\varepsilon_1 + \varepsilon_2) \quad (3.25)$$

$$F_{3N+2} \beta_1 - F_{3N+1} \beta_2 = (-1)^N r_N(\varepsilon_1 - \varepsilon_2). \quad (3.26)$$

When N increases, the directions of β_1 and β_2 get progressively nearer and nearer the bisectrix, since the number of vectors ε_1 and ε_2 composing β_1 and β_2 are equal in average.

Therefore, the lengths of β_1 and β_2 are well approximated by $F_{3N+2}/\sqrt{2}$ and $F_{3N+1}/\sqrt{2}$, respectively. Using equation (3.24), one deduces that the angle between these vectors goes to zero as τ^{-3N} .

To conclude this section, let us stress that the limit 2D structure is very singular. Indeed, simultaneously, both the lengths of β_1 and β_2 , and the periodicity of the atomic surfaces along the diagonals, become infinitely large.

4. Final remarks.

The description of the integrable cases of the circle map tiling in terms of a section through a 2D structure provides a means to compute the Fourier spectrum of the structure. This may be done along the same lines as in references [16], [17]. The main difference with the canonical tilings is the presence of additional structural units in the renormalised tiles. In analogy with standard crystallography, the Fourier transform of the 2D tiling is the product of the lattice $(\mathcal{BZ}^2)^*$ reciprocal to \mathcal{BZ}^2 by a geometric form factor; the form factor is the Fourier transform of the set of atomic surfaces

$$\mathcal{R}\varepsilon_i + \mathcal{A}_i \quad (i = 1, 2). \quad (4.1)$$

where

$$\begin{aligned} \mathcal{A}_1 &= -p_{\perp}(\beta_2) \\ \mathcal{A}_2 &= -p_{\perp}(\beta_1). \end{aligned} \quad (4.2)$$

In order to get the Fourier spectrum in the physical 1D space, we have to integrate in the transversal direction. The result is a pure point Fourier transform, with support in the projection of $(\mathcal{BZ}^2)^*$ as for a canonical tiling, but with amplitudes modulated by the form factor.

It is worth drawing the attention of the reader to the resemblances between aspects of the work of Frenkel, Henley and Siggia [18], and some considerations given here, e.g., in section 2.

The existence of average lattices for 2D quasiperiodic tilings is a subject of growing interest [19, 20].

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