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## On a mechanism for explicit replica symmetry breaking

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**Résumé.** — On montre qu'une brisure explicite de la symétrie des répliques peut être réalisée en moyennant sur un petit champ magnétique aléatoire une puissance convenable de la fonction de partition. Ces résultats jettent un nouvel éclairage sur la signification physique de la brisure de symétrie des répliques et donnent une nouvelle manière de calculer le paramètre d'ordre des verres de spin  $q(x)$ .

**Abstract.** — We show that explicit replica symmetry breaking can be implemented by averaging over a small random magnetic field the partition function to an appropriate power. These results give a new insight on the physical meaning of replica symmetry breaking and a new way to compute the spin glass order parameter  $q(x)$ .

### 1. Introduction.

At the present moment there is a general agreement that the approach based on replica symmetry breaking gives the correct treatment of the mean field theory for spin glasses. The peculiarities of this approach have been elucidated using the cavity method, where it is clear that the main approximation is to neglect fluctuations (or equivalently correlations) [1].

As usual mean field results can be directly applied to the infinite range model and to the infinite dimensional short range model. Otherwise, in more general cases, the approximation should be extended by a systematic expansion that includes the effects of fluctuations. This extension should allow both the construction of an expansion in inverse powers of the dimensions and the study of the low temperature behavior in all dimensions. In particular it could give information on the value of the lower critical dimension, i.e. the dimension below which the replica symmetry breaking transition disappears.

Unfortunately this expansion is extremely difficult to construct, due to the complexity of the method [2]. At the one loop level it has been shown that in order to do a clean computation, replica symmetry should be explicitly broken by introducing an external field coupled to the matrix  $q$  defined in replica space [3]. This adds a mass to all the Goldstone bosons. At the end of the computation, as usual, the external field must be removed.

The physical interpretation of such a field in replica space is not transparent. Nevertheless it is important to study its effect because the implications of spontaneous replica symmetry breaking become clearer when one considers the behavior of the system under an explicit replica symmetry breaking.

In this note we will study carefully the behavior of the system when replica symmetry is explicitly broken. In section 2 we show that there is an alternative definition of this mechanism in terms of a random external magnetic field acting on the partition function elevated to a fractional power. As a bonus we derive a new definition of the order parameter  $q(x)$ .

In section 3 we study the implications of these results on the structure of the states (in the simplified case where there is only one level of replica symmetry breaking).

In section 4 we derive the results in the cavity approach by doing an explicit (non trivial) computation using the information on the structure of the states. In doing this work we have been once more amazed by the difference in style and complexity between the compact replica approach and the explicit probabilistic approach.

Finally in section 5 we discuss the implications of our finding on the breaking of replica symmetry.

## 2. Some basic observations.

We start by considering  $m$  real replicas of a usual spin glass Hamiltonian of  $N$  spins which are coupled together with strength  $\varepsilon$  [4] :

$$\begin{aligned} H_m(\varepsilon) &= \sum_{a=1,m} \left( H(\sigma^a) - \varepsilon \sum_i \sum_{b=1,m; b \neq a} \sigma_i^a \sigma_i^b \right) \\ H(\sigma^a) &= \sum_{i,k} J_{i,k} \sigma_i^a \sigma_k^a. \end{aligned} \quad (1)$$

Here  $i$  and  $k$  labels the  $N$  sites,  $a$  and  $b$  the  $m$  replicas and the  $J$ 's are the usual quenched randomly distributed coupling constants. In the following we will not use any detailed property of the Hamiltonian  $H$ : we will only assume that replica symmetry breaking is spontaneously broken in the usual way for  $\varepsilon = 0$ .

We also define the corresponding free energy  $F_m(\varepsilon)$  as

$$F_m(\varepsilon) = -1/(\beta m N) \ln \left[ \sum_{\{\sigma\}} \exp(-\beta H_m(\varepsilon)) \right]. \quad (2)$$

$F_m(\varepsilon)$  is obviously well defined for integer  $m$ . An interpolation at non integer  $m$ 's can be explicitly constructed as :

$$\begin{aligned} F_m(\varepsilon) &= -1/(\beta m N) \ln \left\{ \int \prod_i [dh_i \exp(-(\beta/4) h_i^2)] [Z(\varepsilon^{1/2} h)]^m \right\} \\ Z(\varepsilon^{1/2} h) &= \sum_{\{\sigma\}} \exp[-\beta H(\sigma, \varepsilon^{1/2} h)] \\ H(\sigma, \varepsilon^{1/2} h) &= H(\sigma) + \sum_i [\varepsilon - \varepsilon^{1/2} h_i \sigma_i]. \end{aligned} \quad (3)$$

By performing the Gaussian integration over the  $h$ 's we can see that equation (3) coincide with equation (2) for integer  $m$ , while equation (3) is well defined for any  $m$ .

The field  $h$  can be considered as a dynamical field with a peculiar Hamiltonian :  $(1/4) \sum_i h_i^2 + ((m - 1)/\beta) \ln Z(\varepsilon^{1/2} h)$  or alternatively as a quenched field similar to the  $J_{ik}$  with an *a priori* distribution :

$$\prod_i [dh_i \exp(- (\beta/4) h_i^2)] [Z(\varepsilon^{1/2} h)]^m \tag{4}$$

Accordingly when measurements on two real replicas are considered we can imagine replicas with the same  $J_{ik}$  but different, independent evolution in the fields  $h_i, \sigma_i$  or they may have the same  $J_{ik}, h_i$  and differ only in the  $\sigma$ 's.

Particularly interesting is the behavior of  $F_m(\varepsilon)$  for small  $\varepsilon$  and non integer  $m$ . We define two functions  $Q^+(m)$  and  $Q^-(m)$  as

$$(m - 1) Q^\pm(m) = \lim_{\varepsilon \rightarrow 0^\pm} dF_m/d\varepsilon . \tag{5}$$

In the next section we will compute the two functions  $Q^\pm$  in the replica approach. We will see that :

$$Q^+(m) \neq Q^-(m) \tag{6}$$

in the phase where replica symmetry is spontaneously broken.

In section 4 we will compute  $Q^+(m)$  by a direct approach which makes use of the information on the probability distribution of the states and their different weights, as implied by the theory of replica symmetry breaking. The computation will be much more involved (technical details are shown in the appendix), but the results coincide at the end. In principle one should be able to compute also  $Q^-(m)$  with the same approach, but we have been unable to do it.

### 3. The replica approach.

In the replica approach it is convenient to introduce  $n$  replicas with  $n$  multiple of  $m$ , where  $n$  will eventually go to 0. We can label these  $n$  replicas by two indices  $s$  and  $a$  which span the range

$$\begin{aligned} s &= 1, \dots, n/m \\ a &= 1, \dots, m . \end{aligned} \tag{7}$$

Using the usual trick one can write that :

$$\begin{aligned} NF_m(\varepsilon) &= - \lim_{n \rightarrow 0} 1/(\beta n) \left[ \sum_{\{\sigma^{s,a}\}} \exp(-\beta H_m(\varepsilon)) \right]^{n/m} = \\ &= \lim_{n \rightarrow 0} 1/(\beta n) \sum_{\{\sigma^{s,a}\}} \exp \left[ -\beta H_n(\sigma^{s,a}) - \varepsilon \sum_{i=1, N} \sum_{s=1, n/m} \sum_{a,b=1, m; b \neq a} \sigma^{s,a_i} \sigma^{s,b_i} \right] , \end{aligned} \tag{8}$$

where  $H_n(\sigma^{s,a})$  is the usual replicated Hamiltonian.

If we introduce the usual order parameter  $q^{\{s,a\} \{s,b\}} = \langle \sigma^{s,a} \sigma^{s,b} \rangle$ , we have that

$$F_m(\varepsilon) = \left\{ F(q) - \varepsilon/n \sum_{s=1, n/m} \sum_{a,b=1, m; b \neq a} q^{\{s,a\} \{s,b\}} \right\}_{\text{stationary point}} \tag{9}$$

where equation (9) holds in the limit  $n \rightarrow 0$  and  $F(q)$  is the free energy as function of the matrix  $q$ . The term proportional to  $\varepsilon$  breaks explicitly the  $S_n$  replica symmetry to the semidirect product of  $S_{n/m}$  with  $S_m$  to the power  $n/m$ , which is exactly the first term of the

usual hierarchical breaking. A symmetry breaking term of this kind was first introduced in reference [4].

The free energy  $F(q)$  is invariant under replica symmetry, so one finds that

$$(m-1) Q^{\pm}(m) = 1/n \sum_{s=1, n/m} \sum_{a,b=1, m; b \neq a} q_{\pm}^{\{s, a\}, \{s, b\}} \quad (10)$$

where  $q_{\pm}$  is the matrix  $q$  which is the stationary point of equation (9) in the limit  $\varepsilon \rightarrow 0^{\pm}$ . The matrix  $q_{\pm}$  will coincide apart from a permutation in replica space with the matrix  $q$  derived without any explicit breaking ( $\varepsilon = 0$ ). If more than one matrix  $q_{\pm}$  can be found, their average should be used. The term proportional to  $\varepsilon$  has the role of picking one (or more) particular solution among all the possible ones.

The computation of  $Q^{\pm}(m)$  can be done in the usual hierarchical model without difficulties if  $m$  is in the interval 0-1 :

$$\begin{aligned} (1-m) Q^{+}(m) &= \int_m^1 q(x) dx \\ (1-m) Q^{-}(m) &= \int_0^{1-m} q(x) dx, \end{aligned} \quad (11)$$

where we have used the fact that the function  $q(x)$ , which parametrizes the breaking of replica symmetry is a monotonous function as usual.

When  $m$  is greater than 1, equation (11) involves the function  $q(x)$  outside the interval 0-1. We can extend the definition of  $q(x)$  :

$$\begin{aligned} q(x) &= q(1) & \text{for } x > 1 \\ q(x) &= q(0) & \text{for } x < 0, \end{aligned} \quad (12)$$

to find the very reasonable result for  $m > 1$  :

$$\begin{aligned} Q^{+}(m) &= q(1) \\ Q^{-}(m) &= q(0). \end{aligned} \quad (13)$$

Equation (11) is very interesting because it gives an explicit (although quite difficult from the numerical point of view) way of computing the order parameter  $q(x)$  without introducing a non integer number of replicas or doing the decomposition of the symmetric equilibrium state in pure clustering states. Unfortunately the meaning of  $Q^{\pm}(m)$  is clear only for integer  $m$ , where the representation (2) applies ; for  $m$  smaller than 1, where the non linearity of  $Q^{\pm}(m)$  signals the breaking of replica symmetry, we must use the representation (3) and the physical meaning of the results is less transparent.

Indeed in the very simple case where

$$\begin{aligned} q(x) &= q_0 & \text{for } x < z \\ q(x) &= q_1 & \text{for } x > z, \end{aligned} \quad (14)$$

we wind the baffling result :

$$\begin{aligned} Q^{+}(m) &= \frac{(z-m)q_0 + (1-z)q_1}{1-m} & \text{for } m < z \\ Q^{+}(m) &= q_1 & \text{for } m > z \\ Q^{-}(m) &= \frac{(1-z-m)q_1 + zq_0}{1-m} & \text{for } 1-m > z \\ Q^{-}(m) &= q_0 & \text{for } 1-m < z. \end{aligned} \quad (15)$$

$Q^\pm(m)$  is the average value of  $q_\pm^{\{s,a\},\{s',b\}}$ . It corresponds to measurements done on replicas with the same field  $h$ . Similarly we can define quantities that measure the overlaps between replicas with different  $h$ 's. They will be the average value of  $q_\pm^{\{s,a\},\{s',b\}}$  with  $s \neq s'$  :

$$R^\pm(m) = \frac{1}{m^2} \sum_{a,b=1,m} q_\pm^{\{s,a\},\{s',b\}} \quad \text{for } s \neq s' . \tag{16}$$

One finally finds that

$$mR^\pm(m) + (1 - m) Q^\pm(m) = \int_0^1 q(x) dx = (1 - z) q_1 + zq_0 . \tag{17}$$

These calculations can be generalized to the case in which we have more than one field  $h$  acting on the system. For instance we could add a second field  $h'$  acting on the partition function to the power  $m'$ .

$$F_{m,m'}(\varepsilon, \varepsilon') = -1/(\beta m' N) \ln \left\{ \int \prod_i [dh'_i \exp(-(\beta/4) h_i'^2)] \times \right. \\ \left. \times \left[ \int \prod_i [dh_i \exp(-(\beta/4) h_i^2)] [Z(\varepsilon^{1/2} h + \varepsilon'^{1/2} h')]^m \right]^{m'/m} \right\} . \tag{18}$$

In any of the following cases : a)  $m > 1$ , b)  $m' > 1$ , c)  $m < 0$ , d)  $m' < 0$ , e)  $m < m'$  the function  $x(q)$  cannot be monotonous. Therefore the equation :

$$P(q) = dx/dq \tag{19}$$

with  $P(q)$  a probability cannot be correct in the whole interval. This is reasonable because the probabilistic interpretation will have to take into account the possible different experimental arrangements : two real replicas with the same  $h$  and  $h'$  (and a corresponding probability distribution  $P_1(q)$ ) ; two real replicas with equal  $h'$  but different  $h$  (with probability distribution  $P_2(q)$ ) and two real replicas with different  $h$  and different  $h'$  (with probability distribution  $P_3(q)$ ). Then instead of (19) we will have :

$$P_1(q) = \frac{1}{1 - m} \frac{dx}{dq} \quad \text{in the region } [m, 1] \\ P_2(q) = \frac{1}{m - m'} \frac{dx}{dq} \quad \text{in the region } [m', m] \\ P_3(q) = \frac{1}{m'} \frac{dx}{dq} \quad \text{in the region } [0, m'] . \tag{20}$$

The new consistency conditions on  $x(q)$  can now be derived. For instance if  $m' > m$  but both  $m$  and  $m'$  are between 0 and 1 ; then  $x(q)$  grows from 0 to  $m'$  then decreases to  $m$  and finally grows again up to 1.

The computations of this section are quite straightforward, but the physical meaning of the results is mysterious. This mystery will be unveiled in the next section.

#### 4. The probabilistic approach.

We now compute directly  $Q^+(m)$  and  $R^+(m)$  by using the representation of equation (3) and the information on the probabilistic distribution of the states and of their overlaps that has been shown to be equivalent to the usual replica approach. This computation is illuminating

for understanding the physical interpretation of the mechanism proposed. It is clear from equation (3) that the field  $h$  is partially quenched and partially annealed. In this section we will clarify what is the dynamical reaction to such a field.

Let us first compute  $F_m(\varepsilon)$ . We will work out in full detail the case of one level of replica symmetry breaking anticipating that the generalization to the more general case will be straightforward. There are many equilibrium states labeled by  $\alpha$ . Each state carries a Gibbs-Boltzmann weight  $W_\alpha$  and has local magnetizations :

$$\langle \sigma_i^\alpha \rangle = m_i + m_i^\alpha \quad (21)$$

where

$$\begin{aligned} 1/N \sum_i (m_i)^2 &= q_0 \\ 1/N \sum_i (m_i^\alpha)^2 &= q_1 - q_0, \quad 1/N \sum_i m_i^\alpha m_i^\beta = 0, \quad 1/N \sum_i m_i m_i^\beta = 0. \end{aligned} \quad (22)$$

The local susceptibility has in average the following value :

$$\chi = \beta (1 - q_1). \quad (23)$$

Apart from a multiplicative normalization factor we can write

$$W_\alpha \propto \exp(-\beta F_\alpha) \quad (24)$$

and the average number of states with free energy  $F$  is given by

$$C \exp(\beta z F). \quad (25)$$

Equations (21-25) summarize the usual results of the mean field theory at this level of spontaneous replica symmetry breaking.

If we put everything together we obtain for small  $\varepsilon$  :

$$\begin{aligned} F_m(\varepsilon) = F_0 - 1/(\beta m N) \left[ \ln \int \prod_i [dh_i / \sqrt{4\pi/\beta} \exp(-\beta h_i^2/4 + m\beta \sqrt{\varepsilon} h_i m_i)] \right] \times \\ \times \left[ \sum_\alpha W_\alpha \exp\left(\beta \sqrt{\varepsilon} \sum_i h_i m_i^\alpha - \beta \varepsilon \chi\right) \right]^m, \end{aligned} \quad (26)$$

where  $F_0$  is a normalization constant. We are interested in the terms proportional to  $\varepsilon$  so that we can neglect the fluctuations in the susceptibility. The term  $\sqrt{\varepsilon} h_i m_i^\alpha$  is crucial because it gives a result of order  $\varepsilon$  after integration over the  $h$ 's. This last integration simplifies if we recall that the  $m_i^\alpha$  are approximately orthonormal vectors so that we can introduce the  $h^\alpha$ , which are the component of  $h$  in the direction of  $m_i^\alpha$ .

Integrating over the other components and collecting the terms proportional to  $\varepsilon$  we obtain :

$$\begin{aligned} F_m(\varepsilon) = F_0 + \varepsilon \chi - \varepsilon q_0 m + D_m(\varepsilon) \\ D_m(\varepsilon) = -1/(\beta m N) \ln \left[ \int \prod_\alpha [dh_\alpha / \sqrt{4\pi/\beta} \times \right. \\ \left. \times \exp(-\beta h_\alpha^2/4)] \left\{ \sum_\alpha W_\alpha \exp[\beta (\varepsilon N q)^{1/2} h_\alpha] \right\}^m \right], \end{aligned} \quad (27)$$

where for semplicity we have defined :

$$q_1 - q_0 \equiv q . \tag{28}$$

The computation of  $D_m$  is tricky and will be the subject of the rest of this section.

Let us first assume that the sum over the different states is dominated by just one state (say the  $\alpha_0$ ). Then the leading contribution to the integral will come from the region  $h_{\alpha_0} \approx O(\sqrt{N})$ ,  $h_\alpha \approx 1$   $\alpha \neq \alpha_0$  where the corresponding Gaussian measure will be small  $O(\exp(-N))$ . If instead the sum is dominated by a finite number of states ( $\alpha_1, \dots, \alpha_p$ ) then the leading contribution comes from the union of the  $p$  regions labeled by  $k = 1, \dots, p$  :

$$h_{\alpha_k} = O(\sqrt{N}) ; h_\alpha \approx 1, \alpha \neq \alpha_k . \tag{29}$$

In these two cases we find that :

$$D_m(\varepsilon) = -1/(\beta m N) \ln \left[ \sum_\alpha (W_\alpha)^m \times \int dh_\alpha / \sqrt{4 \pi / \beta} \exp(-\beta h_\alpha^2 / 4) \exp[m\beta (\varepsilon N q)^{1/2} h^\alpha] \right] . \tag{30}$$

The integral over the  $h$ 's can be readily done :

$$G_m(\varepsilon) = -\varepsilon q m - (1/\beta m N) \ln \left[ \sum_\alpha (W_\alpha)^m \right] . \tag{31}$$

The prefactor proportional to  $1/N$  has been already calculated in the literature [5] :

$$\sum_\alpha (W_\alpha)^m = \Gamma(m - z) / (\Gamma(m) \Gamma(1 - z)) , \tag{32}$$

where  $\Gamma$  is the Euler gamma function. The sum is obviously divergent for  $m < z$  and this divergence indicates that the result (31-32) is not correct in this region.

Finally the sum may be dominated by an infinite number (of order  $O(\exp(\lambda N))$ ) of terms. This may happen only where  $F$  is large. Let us consider the contribution to the sum in equation (27) of the states at fixed  $F$ . Their number,  $K(F)$  is given by

$$K(F) = C \exp(N\beta z F) . \tag{33}$$

Knowing that the  $h$ 's are Gaussian variables, we would like to estimate :

$$H(F) \equiv \max (h_\alpha) , \tag{34}$$

where the max is done over all the states at a given  $F$ .

Now a simple computation shows that the max of  $K$  Gaussian random variables is for large  $K$  given by

$$H(F) = \sqrt{(4/\beta) \log K} \tag{35}$$

with probability 1. The contribution of these states to the sum is

$$\exp(-\beta N F) \exp(\beta \sqrt{\varepsilon N q (4/\beta) \log K}) = \exp(-\beta N F + 2 N \beta (\varepsilon q z F)^{1/2}) . \tag{36}$$

The leading contribution comes from :

$$F = \varepsilon q z \tag{37}$$

and its value is

$$\exp(\varepsilon N \beta q z m), \quad (38)$$

which dominates on the result of the previous computation

$$\exp(\varepsilon N \beta q m^2), \quad (39)$$

in the region  $m < z$ .

It is important to notice that in this last case the random fields  $h_\alpha$  that contribute are such that their gaussian measure is of order 1. Another instructive way to derive (38) explicitly uses this fact. Let us consider again the distribution of states before turning on the random field :

$$dN(F) = C \exp(\beta z N F) dF. \quad (40)$$

We now couple the  $h$  field :

$$F \rightarrow F' = F + \varepsilon \chi - \sqrt{\varepsilon}/N \sum_i h_i (m_i^\alpha + m_i). \quad (41)$$

Each state  $\alpha$  among the  $\exp(\beta z N F)$  will be displaced in free energy by a constant amount  $(\varepsilon \chi - \sqrt{\varepsilon} \sum_i h_i m_i)$  plus a quantity that fluctuates from state to state :  $h_\alpha = \sum_i h_i m_i^\alpha$ . For every field  $h_i$  such that the gaussian measure is of order 1,  $h_\alpha$  can be seen as a random variable  $H$  with zero mean and variance equal to :

$$\sum_i h_i^2 (m_i^\alpha)^2 = \sqrt{2/\beta} (q_1 - q_0) N \quad (42)$$

therefore from (40) we derive :

$$dN_\varepsilon(F', H) = C \exp[\beta z (N F' - \varepsilon N \chi + \sqrt{\varepsilon} \sum_i h_i m_i + \sqrt{\varepsilon} H) - (\beta/4) H^2 / (qN)] dF' dH. \quad (43)$$

We integrate over  $H$  to derive :

$$dN_\varepsilon(F') = C \exp\left[\beta z \left(N F' - \varepsilon N \chi + \sqrt{\varepsilon} \sum_i h_i m_i + \varepsilon z q N\right)\right] dF'. \quad (44)$$

As in the cavity method [1] the constant  $C$  in (40) may be related to the « free energy » of the ancestor (unique in this level of approximation) while (44) may be used to investigate the shift in this « free energy » due to the interaction with the random magnetic field. Taking into account that :

$$C = \exp(-\beta z N G) \quad (45)$$

we obtain :

$$G' = G + \varepsilon (1 - q_1(1 - z) - q_0 z) - (\sqrt{\varepsilon}/N) \sum_i h_i m_i. \quad (46)$$

We can now use the fact that there is only one ancestor to express  $Z$  as the  $\exp(-\beta G')$  so as to derive that :

$$F_m = F_0 + \varepsilon (1 - q_1(1 - z) - q_0(z - m)) \quad (47)$$

which coincides with the results obtained with the replica method.

We want now to compute the equivalent of the matrix element of  $q_{\alpha, \beta}$ . Previous arguments show that :

$$Q(m) \equiv q_S = \frac{\int d\rho_m(h_i) \mu_i^2(\varepsilon^{1/2} h)}{\int d\rho_m(h_i)}$$

$$R(m) \equiv q_D = \frac{\int d\rho_m(h_i) d\rho_m(\tilde{h}_i) \mu_i(\varepsilon^{1/2} h) \mu_i(\varepsilon^{1/2} \tilde{h})}{\int d\rho_m(h_i) \int d\rho_m(\tilde{h}_i)} \tag{48}$$

where we have used the simplified notation

$$d\rho_m(h_i) = \prod_i (\exp(-(\beta/4) h_i^2) dh_i) [Z(\varepsilon^{1/2} h)]^m \tag{49}$$

Roughly speaking  $q_S(q_D)$  is the average value of the overlap between two real replicas of the system coupled to the same (different) random magnetic field.

Let us first do the computation for  $m > z$ . In this case we have assumed that only one state dominates the sum so that we have

$$q_S = q_1 \tag{50}$$

The computation for  $q_D$  is less trivial. If we do the same approximation as before, assuming that only one term dominates the sum, we find that the integral over the magnetic fields can be done so that we are left with :

$$q_D = \frac{\sum_{\alpha} W_{\alpha}^m \sum_{\beta} W_{\beta}^m \mu_i^{\alpha} \mu_i^{\beta}}{\left(\sum_{\alpha} W_{\alpha}^m\right)^2} = \frac{\sum_{\alpha} W_{\alpha}^{2m} q + \sum_{\alpha, \beta} W_{\alpha}^m W_{\beta}^m q_0}{\left(\sum_{\alpha} W_{\alpha}^m\right)^2} = q f(m) + q_0 \tag{51}$$

where

$$f(m) = \sum_{\alpha} W_{\alpha}^{2m} / \left(\sum_{\alpha} W_{\alpha}^m\right)^2 \tag{52}$$

The evaluation of  $f(m)$  can be done by using the usual techniques (the details can be found in the appendix) and we finally derive :

$$f(m) = 1 - x/m \tag{53}$$

in perfect agreement with the result obtained with the replica trick.

In the other case,  $m < x$ , the computation is quite simple for  $q_D$ . In this case different equilibrium state contributes for the different configuration of the fields  $h$ 's so that we find

$$q_D = q_0 \tag{54}$$

But this time the computation of  $q_S$  is more involved. By definition :

$$q_S = \int dr(h) Z(\varepsilon, h)^m [\mu_i(\varepsilon, h)]^2 / \int dr(h) Z(\varepsilon, h)^m \tag{55}$$

where

$$\begin{aligned}
 Z(\varepsilon, h) &= \sum_{\alpha} \exp[\beta (\varepsilon N q)^{1/2} h_{\alpha}] W_{\alpha} \\
 \mu_i(\varepsilon, h) &= \sum_{\alpha} \exp[\beta (\varepsilon N q)^{1/2} h_{\alpha}] W_{\alpha} \mu_i^{\alpha} / Z(\varepsilon, h) \\
 dr(h) &= \prod_{\alpha} [\exp(-\beta h_{\alpha}^2/4) dh_{\alpha}]. \quad (56)
 \end{aligned}$$

Both the denominator and the numerator will be dominated by the same value of the free energy. In the same way as for the previous computation we can write :

$$\begin{aligned}
 q_D &= q_0 + qf(m) \\
 f(m) &= \frac{\int dr(h) \left\{ \sum_{\alpha} \exp[\beta \sqrt{\varepsilon N q} h_{\alpha}] W_{\alpha} \right\}^{m-2} \sum_{\alpha} \left\{ \exp[\beta \sqrt{\varepsilon N q} h_{\alpha}] W_{\alpha} \right\}^2}{\int dr(h) \left\{ \sum_{\alpha} \exp[\beta \sqrt{\varepsilon N q} h_{\alpha}] W_{\alpha} \right\}^m}. \quad (57)
 \end{aligned}$$

In the evaluation of  $f(m)$  the dominating contribution comes from a region where the  $F$  is given by equation (37). We have already remarked that the value of the largest field  $h_{\alpha}$  when  $\alpha$  spans over  $K$  values is already fixed. We need to evaluate the integrals in equation (57) under the assumption that the integral is dominated by the largest  $h$ 's.

This computation can be done as follows. In the region of large  $h$ 's we can approximate the Gaussian distribution by writing :

$$h_{\alpha} = H(F) - 2 t_{\alpha} / \beta H(F) \quad \text{with} \quad t_{\alpha} \ll \beta H^2, \quad (58)$$

so that the probability distribution of the  $t$ 's becomes approximately

$$dt_{\alpha} \exp(t_{\alpha}). \quad (59)$$

If we substitute back in equation (57) we find that we have to evaluate the integral

$$\frac{\int \prod_{\alpha} [dt_{\alpha} \exp(t_{\alpha})] \left[ \sum_{\alpha} \exp(-t_{\alpha}/z) \right]^{m-2} \sum_{\alpha} \exp(-2 t_{\alpha}/z)}{\int \prod_{\alpha} [dt_{\alpha} \exp(t_{\alpha})] \left[ \sum_{\alpha} \exp(-t_{\alpha}/z) \right]^m}. \quad (60)$$

The evaluation of the integral can be found in the appendix, the final result is

$$(1 - z)/(1 - m), \quad (61)$$

as it was derived with replicas.

## 5. Conclusions.

We have seen that it is possible to obtain some information on the spin glass order parameter by introducing a random infinitesimal magnetic field and by averaging the partition function at the appropriate power. In a similar way we can get the values of the function

$q(x)$  if we introduce an infinite number of different levels of random magnetic fields ; only in this case the replica symmetry is completely broken in an explicit way.

This procedure is just the one that has been followed in reference [2] (in replica space) in order to construct a sensible perturbative expansion. In absence of a symmetry breaking term severe infrared divergences appear.

These results suggest that the equations written in this paper are the appropriate definition of the order parameter in a short range model. Indeed it is well known that usually the spontaneous breaking of a symmetry can be unambiguously described only when we apply an infinitesimal external field.

The following example may be illuminating. Let us consider a three dimensional Ising model at low temperature with the spins up at the boundary in the infinite volume limit at strictly zero external magnetic field. The boundary conditions are sufficient to push the magnetization of the model to a positive value for the whole system in the thermodynamic limit and only one phase is present. Still the existence of two phases may be demonstrated by adding a positive, or negative magnetic field and performing first the infinite volume limit and sending the magnetic field to zero only at the end. (A more subtle example is given by four dimensional gauge theory with the  $Z_2$  group at the self dual point, where periodic boundary condition break the self duality condition.) In other words the breaking of a symmetry should be observed by looking to effects proportional to  $N$  in the free energy in the thermodynamic limit.

This suggestion may have deep physical implications. In our computation in the case  $m < z$ , the integrals were dominated by states having a free energy larger than that of the equilibrium state by terms proportional to  $\varepsilon N$ , while usually replica symmetry breaking is interpreted as stating the existence of states with quite similar free energies, i.e. the differences are of order 1, not  $N$ .

Explicit breaking of replica symmetry tests the probability distributions of the states only in the region where the free energy is larger than that of the equilibrium state by terms proportional to  $\varepsilon N$  and corrections to the mean field approximation can be computed (i.e. are small) only when replica symmetry is explicitly broken. These facts suggest that the predictions of replica symmetry breaking and the definition of the order parameter can be taken seriously only in the case where replica symmetry is explicitly broken and that the probability distribution of states and overlaps are strictly valid only for the states of high free energy. The definition of the order parameter in terms of the function  $P(q)$  (i.e. the probability of having two states with self overlap  $q$ ), is too narrow for short range model and is probably valid only in the infinite range model. The function  $P(q)$  may be a delta function and still replica symmetry be spontaneously broken in the sense that the order parameter defined by equation (1-3) and by their generalizations is non trivial. The relation among  $P(q)$  and the function  $q(x)$

$$dx/dq = P(q) \quad (62)$$

has been proved only in the mean field approximation and even in the infinite range model it has never been checked to survive fluctuations (although it is quite likely that this happens in this case).

Numerical simulations looking for the breaking of replica symmetry should be done in a careful way, without paying too much attention to the function  $P(q)$ .

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### Appendix.

The aim of this appendix is to compute

$$f(m) = \left\langle \frac{\sum_{\alpha} W_{\alpha}^{2m}}{\left(\sum_{\alpha} W_{\alpha}^m\right)^2} \right\rangle \quad (\text{A1})$$

knowing that the probability distribution of the  $W$ 's is given by equation (25). The computation is done in a similar way to those of reference [5].

Equation (A1) can be written more precisely as

$$\begin{aligned} f(m) &= \int \prod_{\alpha} [(1/K) dy_{\alpha} \exp(zy_{\alpha})] \sum_{\alpha} \exp(-2my_{\alpha}) \left/ \left( \sum_{\alpha} \exp(-my_{\alpha}) \right)^2 \right. \\ K &= \int dy_{\alpha} \exp(zy_{\alpha}) \propto \exp(zy_c) \end{aligned} \quad (\text{A2})$$

where we have set  $\beta F_{\alpha} = y_{\alpha}$ .

Introducing the integral representation of a negative power we have

$$\begin{aligned} f(m) &= \int d\lambda \lambda \int \prod_{\alpha} [(1/K) dy_{\alpha} \exp(zy_{\alpha})] \sum_{\alpha} \exp(-2my_{\alpha}) \exp\left(-\lambda \sum_{\alpha} \exp(-my_{\alpha})\right) \\ &= M \int d\lambda \lambda A(\lambda) B(\lambda)^{M-1} / K^M \\ A(\lambda) &= \int dy \exp(zy) \exp(-2my) \exp(-\lambda \exp(-my)) \\ B(\lambda) &= \int dy \exp(zy) \exp(-\lambda \exp(-my)) \end{aligned} \quad (\text{A3})$$

where  $M$  is the number of states on which the sum over  $\alpha$  is supposed to run and the integral is done from  $-\infty$  to  $y_c$  and we send  $M$  to infinity keeping  $M \exp(-zy_c) = v$  ( $v$  being an arbitrary number).

We find

$$\begin{aligned} A(\lambda) &= \int dt t^{-z/m+1} \exp(-\lambda t) = \Gamma(-z/m+2) \lambda^{z/m-2} \\ B(\lambda) &= \int dt t^{-z/m-1} \exp(-\lambda t) = \exp(zy_c) - \Gamma(1-z/m) \lambda^{z/m} + O(\lambda). \end{aligned} \quad (\text{A4})$$

We thus find

$$\begin{aligned} f(m) &= M/K \int d\lambda \lambda \Gamma(-z/m+2) \lambda^{z/m-2} \exp(-M/K \Gamma(1-z/m) \lambda^{z/m}) \\ &= \Gamma(-z/m+2) / \Gamma(1-z/m) \int d\lambda / \lambda \lambda^{z/m} \exp(-\lambda^{z/m}) \\ &= (1-z/m) m/z = 1-z/m, \end{aligned} \quad (\text{A5})$$

as we have stated in the text.

In a similar way we can treat the two integrals in equation (60). Indeed we find that the denominator and the numerator are proportional respectively to

$$\frac{1/\Gamma(2-m) \Gamma(2-z) \Gamma(1-m/z) [\Gamma(1-z)/z]^{(-1+m/z)},}{1/\Gamma(-m) \Gamma(-m/z) [\Gamma(1-z)/z]^{(m/z)}} \quad (\text{A6})$$

After some cancellations we find that the ratio is equal to

$$\frac{\Gamma(-m)/\Gamma(2-m) \Gamma(2-z)/\Gamma(1-z) \Gamma(1-m/z)/\Gamma(-m/z) z}{= (1-z)/(1-m)}. \quad (\text{A7})$$

Also in this case we obtain the result derived with the replica formalism.

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