



HAL
open science

Persistent currents in mesoscopic rings : ensemble averages and half-flux-quantum periodicity

Hélène Bouchiat, Gilles Montambaux

► **To cite this version:**

Hélène Bouchiat, Gilles Montambaux. Persistent currents in mesoscopic rings: ensemble averages and half-flux-quantum periodicity. *Journal de Physique*, 1989, 50 (18), pp.2695-2707. <10.1051/jphys:0198900500180269500>. <jpa-00211094>

HAL Id: jpa-00211094

<https://hal.science/jpa-00211094v1>

Submitted on 4 Feb 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Classification

Physics Abstracts

72.10B — 71.30 — 73.50Bk — 73.60Aq

Persistent currents in mesoscopic rings : ensemble averages and half-flux-quantum periodicity

Hélène Bouchiat and Gilles Montambaux

AT & T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.
and Laboratoire de Physique des Solides, Associé au CNRS,
Université Paris-Sud, 91405 Orsay Cedex, France

(Reçu le 17 février 1989, accepté le 17 avril 1989)

Résumé. — Dans un anneau métallique isolé, dont les dimensions sont inférieures à la longueur de cohérence de phase des électrons, a été prédite l'existence d'un courant permanent en présence d'un champ magnétique à travers la boucle. Pour un seul anneau, il a été montré que ce courant est une fonction périodique du flux qui le traverse, avec la période $\phi_0 = h/e$. C'est une conséquence de la sensibilité des niveaux d'énergie aux conditions aux limites déterminées par le champ magnétique. Nous nous intéressons ici au moment magnétique d'un ensemble d'anneaux multicanaux indépendants. *Nous montrons que cette moyenne d'ensemble est périodique en $\phi_0/2$.* Nos résultats reposent à la fois sur des arguments analytiques et des simulations numériques sur le modèle d'Anderson. Dans la limite de très faible désordre, la moyenne d'ensemble $\langle I \rangle$ du courant est indépendante du nombre de canaux. Dans la limite de fort désordre, $\langle I \rangle / I_0$ est de l'ordre du carré du courant typique $\langle I^2 \rangle / I_0^2$ (avec $I_0 = ev_t/L$). Nous insistons sur le fait que ces résultats sont obtenus en fixant le nombre d'électrons dans chaque anneau (ce nombre pouvant varier d'un anneau à l'autre). Le changement de périodicité n'est pas obtenu si on effectue un calcul à potentiel chimique fixé.

Abstract. — An isolated metallic loop, whose dimensions are smaller than the electronic phase coherence length, is expected to present a permanent current in the presence of a magnetic field through the loop. For a single ring, it has been shown that this permanent current (which is a consequence of the sensitivity of the energy levels to the change of boundary conditions determined by the magnetic field) is a periodic function of the magnetic flux ϕ with a period $\phi_0 = h/e$. We are interested here in the magnetic moment of an assembly of independent multichannel rings. *We show that this ensemble average is periodic in $\phi_0/2$.* Our findings rely on both analytical arguments and numerical simulations on the Anderson model. In the zero disorder limit, the ensemble average $\langle I \rangle$ is independent of the number of channels. In the strong disorder limit, $\langle I \rangle / I_0$ is of the order of the square of the typical current $\langle I^2 \rangle / I_0^2$ (expressed in $I_0 = ev_F/L$ units). We emphasize the fact that these results are obtained with a fixed number of electrons in each ring (this number being different from one ring to the other). Calculations performed when fixing the chemical potential do not yield this $\phi_0/2$ periodicity.

1. Introduction.

The transport properties of micron size metallic samples at low temperature have been shown to exhibit features characteristic of the quantum coherence of the electronic wave function along the whole sample. One of the most striking has been the detection of Aharonov Bohm oscillations in the resistance of a loop pierced by a magnetic field perpendicular to its plane [1]. In such resistivity measurements, the electric probes induce a coupling of the sample with reservoirs of electrons in which dissipation occurs.

Isolated metallic rings are predicted to present an even more spectacular behavior : Büttiker, Imry and Landauer [2] suggested the existence of persistent currents in the presence of a magnetic field. These currents, reminiscent of the diamagnetism of aromatic molecules, are a consequence of the sensitivity of the eigenstates to the boundary conditions along the ring. In the presence of a magnetic field, the periodic boundary conditions are indeed modified into :

$$\psi(x + L) = \psi(x) e^{i\varphi} \quad (1.1)$$

where $\varphi = 2\pi\phi/\phi_0$. ϕ is the magnetic field through the loop. ϕ_0 is the flux quantum h/e . The permanent current is related to the flux derivative of the eigen energies by :

$$I(\phi) = \sum_n \partial E_n / \partial \phi \quad (1.2)$$

where n labels the energy levels. The current is a periodic function of ϕ with period ϕ_0 . It reflects the periodicity of the electronic motion along the ring. These permanent currents have been mainly studied for a purely one dimensional system [3-5]. In this simple limit, their dependence of the disorder, the number of electrons, the temperature and the inelastic scattering is now well understood. However the computation of these currents for a multichannel ring (which corresponds to the physical description of a real metallic ring) is much more tricky [6-9].

Experimentally, two kinds of investigations are possible. One is the study of a single loop [1]. It needs an extremely sensitive detector to measure the magnetic moment (of the order of 10^4 Bohr magnetons at most) induced by the permanent current. The other one is the investigation of a great number of loops (typically 10^7). This *a priori* needs a less sensitive detector but does not provide the response of a single object.

In this paper, we address the question of the ensemble average of the permanent currents for different realisations (i.e. disorder *and* number of electrons) of the ring, a question which is very relevant for the many rings experiment. The current of one single ring has *a priori* a random sign depending on the microscopic realisation of the ring. As a result, one would expect that the magnetic response of N rings ($N \gg 1$) is of the order of \sqrt{N} . However, as pointed out by Cheung *et al.*, this is not the case for one dimensional rings [4, 5] : the odd harmonics of the current response $I(\phi)$ for one single ring have a sign which alternates with the parity of the number of electrons but the even harmonics are always positive. This shows up as an asymmetry in the $I(\phi)$ curve. For a system of N rings, the odd harmonics of the total response scale like \sqrt{N} but the even harmonics are proportional to N . In the case of multichannel rings, this question of the ensemble average has not been completely addressed yet. So far, authors in references [6-9] have only considered averages on different configurations of disorder, at a fixed value of the chemical potential. However they suggest that the period halving phenomenon depicted above should not survive in multichannel rings. In the diffusive regime, which is relevant to the experiments, they find that the typical current is of

order of $\sqrt{\langle I^2 \rangle_D} \approx I_0 \ell_c/L$. (I_0 is the zero disorder 1D current, $I_0 = ev_F/L$, v_F being the Fermi velocity and L the perimeter of the ring, ℓ_c is the mean free path). But these authors find that the average current $\langle I \rangle_D$ is exponentially small : the first harmonics depends on L as $\langle I_1 \rangle_D \approx I_0 \exp(-L/2\ell_c)$ and the second as $\langle I_2 \rangle_D \approx I_0 \exp(-L/\ell_c)$. In systems with slightly different chemical potentials, all harmonics should scale as \sqrt{N} because of the randomness of the sign of the current. If this is true, the search for permanent currents in multiring systems becomes desperate.

However, we want to emphasize that it is essential, in procedures of averaging, to keep fixed the number of electrons N in each ring instead of the chemical potential. We are going to show that it leads to different physical results, namely that period halving should survive as in the 1D case, with a $\langle I_2 \rangle_{D,N}$ which, for a multichannel ring, is much larger than $\exp(-L/\ell_c)$. Our findings rely on both numerical simulations on the Anderson model and analytical arguments.

2. The model.

We have used the Anderson model [10] to simulate the disorder in a ring of length L and finite width, so that the hamiltonian in a field is given by :

$$H = \sum_{\ell mn} \varepsilon_{\ell mn} |\ell mn\rangle \langle \ell mn| + \sum_{\substack{\ell mn \\ \ell' m' n'}} t_{\ell mn \ell' m' n'} |\ell mn\rangle \langle \ell' m' n'| \quad (2.1)$$

((ℓ, m, n) label the sites coordinates in the ring.)

The transfer term is taken as a constant t between first neighbours. The field effect is simply to change the boundary condition along the ring so that

$$t_{L, m, n_1, m, n} = t e^{2i\pi\phi/\phi_0} \quad (2.2)$$

Open boundary conditions are taken in the two other directions. The disorder is given by a random choice of the on-site energy $\varepsilon_{\ell mn}$ between $-W/2$ and $W/2$. We are interested in the flux dependence of the total current $I(\phi)$ as a function of L , the number of channels M and the number of electrons N . For this purpose, we have computed the whole spectrum $E_n(\phi)$ for 50 equally spaced values of ϕ between 0 and $\phi_0/2$. The current carried by each band is $i_n = \partial E_n / \partial \phi$. At $T = 0$ K, the total current in a ring with N electrons is

$$I_N(\phi) = \sum_{n=1}^N i_n(\phi) \quad (2.3)$$

We have studied 2D ($L \times M$) and 3D ($L \times \sqrt{M} \times \sqrt{M}$) systems. In the following, we only discuss the currents at zero temperature.

3. Weak disorder.

We first discuss the situation of very weak disorder. In this case, the only averaging possibility is on the number of electrons which may vary in different rings. We want to show that, in this case, the even harmonics survive to averaging as in the 1D case. The energy levels can be deduced from the zero disorder levels $E(k_x(\phi), k_y, k_z)$ where $k_x(\phi)$, k_y and k_z are determined by the above boundary conditions. The effect of a weak disorder is to open small gaps at each crossing point of two energy levels $E[k_x(\phi), k_y, k_z]$ and $E[k'_x(\phi), k'_y, k'_z]$ (See Fig. 1). The state n , defined as the n^{th} highest energy level, is thus related to different values

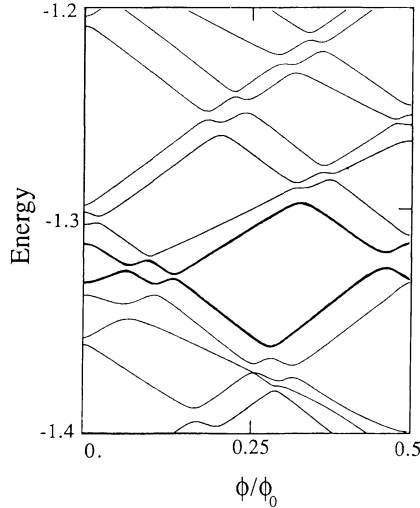


Fig. 1. — The effect of a weak disorder is to open small gaps at each crossing point of two energy levels $E(k_x(\phi), k_y, k_z)$ and $E(k'_x(\phi), k_y, k_z)$. $L = 50$, $M = 10$, $W = 0.2 t$ in this figure. The currents carried by the levels 130 and 131 (shown in thick curves) are depicted in figure 2.

of k_y and k_z when the flux varies. The current contribution of such a state $i_n(\phi) = -\partial E_n(\phi)/\partial\phi$ presents a number of discontinuities $\Delta_i = \Delta(\phi_i)$ at the positions ϕ_i where level crossings occur, as shown in figure 2a. The maximum number of discontinuities is $2M$ in one period $[-\phi_0/2, \phi_0/2]$. One now considers the sum of the currents i_{n-1} and i_n carried by two neighbouring bands. It is straightforward to notice that all the *negative* discontinuities in $i_n(\phi)$ are exactly compensated by all the *positive* discontinuities in $i_{n-1}(\phi)$ (Fig. 2a). When repeating this argument to each couple of energy levels, starting from the lowest energy level, it is possible to justify the shape of $I_n(\phi)$, the total current. There is no negative discontinuity left in $I_n(\phi)$. The amplitude and position of the discontinuities in $I_n(\phi)$ are identical to those of the positive ones in $i_n(\phi)$. A typical current $I_N(\phi)$ is shown in figure 2b for two values of N , the number of electrons in the ring. The function $I_N(\phi)$ strongly fluctuates with N .

We have analyzed the Fourier spectrum of these currents. Figure 3 shows the amplitude of the first and second harmonics of the current *versus* N . A quick analysis shows that although the sign of the first harmonics is a random function of N , *the second harmonics remains positive in average*. We have computed the average of $I_N(\phi)$ on a range ΔN of the number of electrons

$$I(\phi) = \frac{1}{\Delta N} \sum_{N-\Delta N/2}^{N+\Delta N/2} I_N(\phi), \quad (3.1)$$

with $1 \ll \Delta N < N$. This quantity is shown in figure 4. This is a periodic function of ϕ with a period $\phi_0/2$ instead of ϕ_0 (and whose shape and amplitude are independent of M). This is the key result of our paper which we want to discuss now.

In this limit of very weak disorder, it is easy to prove this result analytically using a free electrons picture. In this case, the current of a state $i_n(\phi)$ is linear in ϕ between the discontinuities and can be written, in $[-\phi_0/2, \phi_0/2]$:

$$i_n(\phi) = -\frac{e^2}{mL^2} \phi + \sum_i \frac{\Delta_i}{2} \text{sgn}(\phi - \phi_i) \quad (3.2)$$

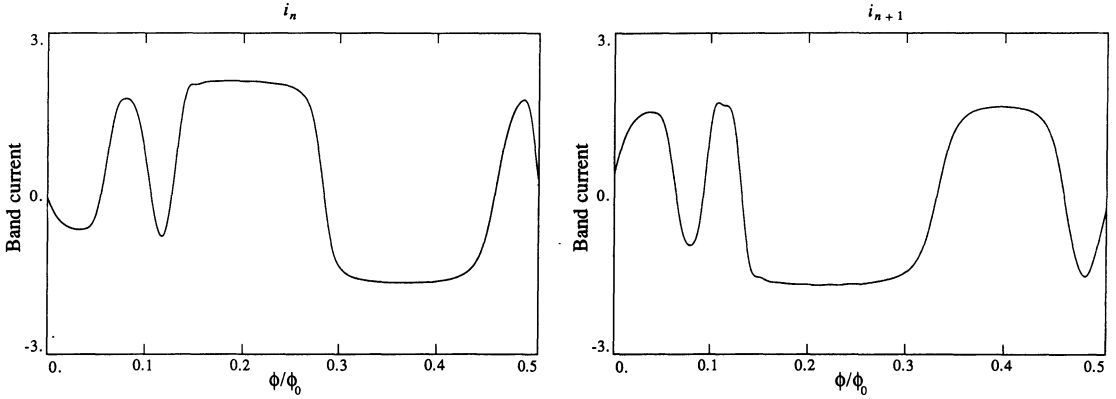


Fig. 2a

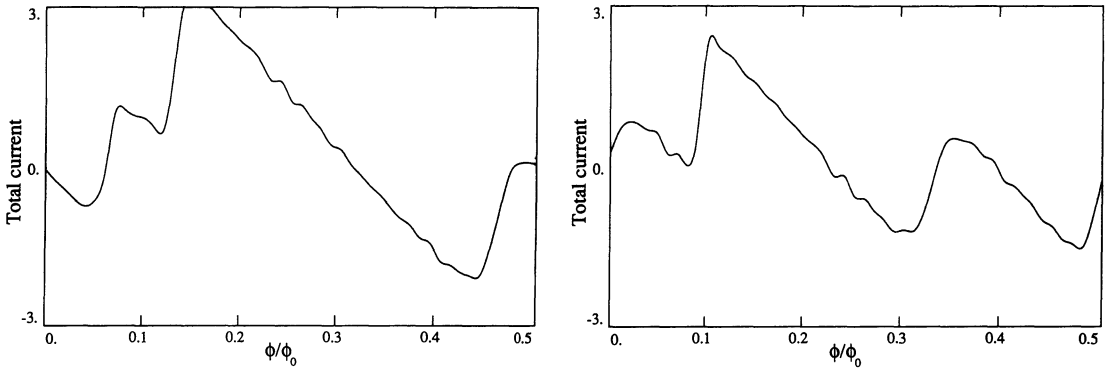


Fig. 2b

Fig. 2. — (a) Currents carried by the levels $n = 130$ and $n = 131$ shown in thick curves in figure 1. Negative discontinuities in $i_n(\phi)$ are equal to positive discontinuities in $i_{n-1}(\phi)$. (b) Total currents $I_n(\phi)$ and $I_{n+1}(\phi)$ for $n = 130$. $I_N(\phi)$ strongly fluctuates with N but has only positive discontinuities.

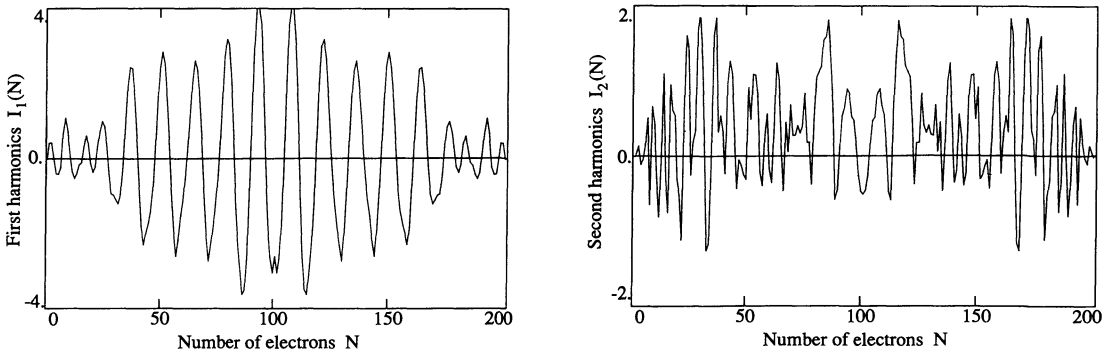


Fig. 3. — First and second harmonics of the current $I_N(\phi)$ versus N , for one configuration of disorder $L = 20$, $M = 10$, $W = 0.2 t$. The average value of the second harmonics is positive, in average over N .

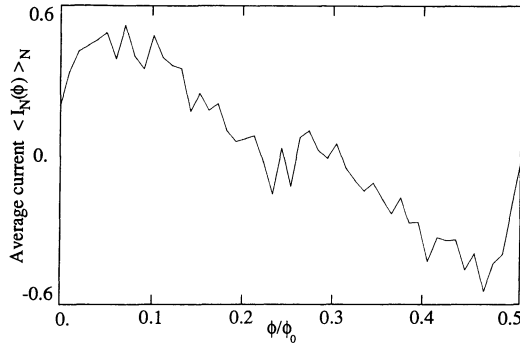


Fig. 4. — This curve is an average of $I_N(\phi)$ over $150 \leq N \leq 250$, in the range $[0, \phi_0/2]$. It has the period $\phi_0/2$. $L = 50$, $M = 10$, $W = 0.2 t$.

where m is the electron mass. One now considers the sum of all the current contributions up to the level N . The total current $I_N(\phi)$ has only positive discontinuities because of the compensation phenomenon depicted above. It has the following form in $[-\phi_0/2, \phi_0/2]$

$$I_N(\phi) = -\frac{Ne^2}{mL^2} \phi + \sum_i \frac{\Delta_i}{2} \text{sgn}(\phi - \phi_i) \tag{3.3}$$

where ϕ_i, Δ_i label now only *positive* discontinuities. This current can be expanded in a Fourier series :

$$I_N(\phi) = \sum_{\ell} x_{\ell} \sin(2\ell\pi\phi/\phi_0) \tag{3.4}$$

where

$$x_{\ell} = \frac{2}{\ell\pi} \sum_i \Delta_i \cos(2\ell\pi\phi_i/\phi_0) . \tag{3.5}$$

The ϕ_i 's depends on N, M and are *randomly* distributed excepted $\phi_i = 0$ and $\phi_i = \pm \phi_0/2$, because there is always a crossing point between levels of the same channel (same k_y and k_z), either at the center of zone or at the zone edge. This simply reflects the periodicity of the spectrum $E(\phi + \phi_0) = E(\phi)$. The ensemble average of the x_{ℓ} , when N varies on a range $\Delta n \gg 1$, can be deduced :

$$\langle x_{\ell} \rangle = \frac{2}{\ell\pi} \langle \Delta(0) + (-1)^{\ell} \Delta(\phi_0/2) \rangle . \tag{3.6}$$

In average, $\langle \Delta(0) \rangle = \langle \Delta(\phi_0/2) \rangle$ so that

$$\langle x_1 \rangle = \frac{1 + (-1)^{\ell}}{\ell\pi} \langle \Delta(0) \rangle . \tag{3.7}$$

The ensemble average of $I_N(\phi)$ is thus a periodic function of ϕ with period $\phi_0/2$. It is written as :

$$\begin{aligned} \langle I_N(\phi) \rangle &= \langle \Delta(0) \rangle \sum_{p=1}^{\infty} \frac{2}{\pi p} \sin(4p\pi\phi/\phi_0) \\ &= \langle \Delta(0) \rangle (\text{sgn}(\phi) - 4\phi/\phi_0) \end{aligned} \tag{3.8}$$

in $(-\phi_0/2, \phi_0/2)$.

The average discontinuity $\langle \Delta(0) \rangle$ is proportional to the average velocity along the x axis at the Fermi level :

$$\langle \Delta(0) \rangle = 2 \frac{e}{L} \langle |v_x| \rangle \approx e v_F / L . \quad (3.9)$$

As a result, the ensemble average of the current response in a multichannel ring is just identical to the response of a single channel ring (with the same average Fermi energy). On the other hand, the average square of the current response harmonics involve the quantities :

$$\begin{aligned} \langle x_{\ell}^2 \rangle &= \frac{4}{\ell^2 \pi^2} \sum_i \langle \Delta_i^2 \rangle \cos^2 (2 \pi \ell \phi_i / \phi_0) \\ &\approx M \langle \Delta^2(0) \rangle = M \frac{e^2 \langle v_x^2 \rangle}{L^2} = \frac{M}{3} \langle v_F^2 \rangle \frac{e^2}{L^2} \end{aligned} \quad (3.10)$$

and the typical response is proportional to \sqrt{M} . This last result is the same the one obtained by Cheung *et al.* [6, 7]. But it is important to note that these authors have assumed a fixed value μ of the chemical potential instead of fixing constant the number of electrons in each ring. The quantity they compute $I_{\mu}(\phi)$ is thus very different from $I_N(\phi)$ we have discussed. It is very easy to figure out that the average of $I_{\mu}(\phi)$ over the chemical potential is zero. Indeed, the level crossings occurring at $\phi = 0$ or $\phi = \pm \phi_0/2$ (see Fig. 2a) do not contribute to $I_{\mu}(\phi)$. This contribution is precisely at the origin of the non zero average of the even harmonics of $I_N(\phi)$. Note that its amplitude corresponds to the current induced by one single electron at the Fermi energy. It is precisely of the order of magnitude of the difference between $\langle I_{\mu}(\phi) \rangle$ and $\langle I_N(\phi) \rangle$. Figure 5 shows the second harmonics *versus* chemical potential. It is seen that, in this case, it averages to zero.

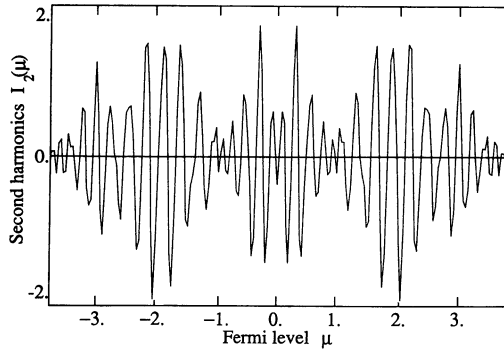


Fig. 5. — Second harmonics of the current $I_{\mu}(\phi)$ *versus* μ for one configuration of disorder. The parameters are those of figure 3. It is zero in average over μ .

This description of a great number of rings in terms of a « modified canonical ensemble » [11] (in which one averages over a great number of samples with a *fixed* number of electrons in each of them, this number being possibly different from one sample to another) instead of a grand canonical ensemble (in which the number of particles in each sample is not fixed) is at the origin of the effect described in this paper. This difference between the quantities $\langle I_N(\phi) \rangle_N$ and $\langle I_{\mu}(\phi) \rangle_{\mu}$ was already pointed out in reference [4], for the analysis of ensemble averages in one dimensional rings.

4. Increasing disorder.

It is straightforward to extend the result obtained above (in the limit of zero disorder), in the presence of a disordered potential whose variance W is smaller than the level spacing. When the disorder can be treated in first order of perturbation, the energy levels are only modified in the very vicinity of free electron level crossings in a symmetric way which preserves the compensation effect between current contributions of adjacent bands. The overall effect of the disorder is just to give a finite width to the current discontinuities discussed above and the conclusions concerning the ensemble average of the even harmonics are not modified.

When increasing the disorder, it is not possible to extend the above arguments in a straightforward way and one has to rely on the results of numerical simulations. One can distinguish three different regimes : the ballistic regime where the length L of the ring is smaller than the elastic mean free path ℓ_e , the localised regime where the length of the ring is longer than the localisation length ξ and, between them, the diffusive regime (which only exists in a multichannel ring since, in 1D, $\xi = \ell_e$). The increase of ξ as a function of M has been computed by Pichard *et al.* [12] on strips of various width. Now, we discuss our results obtained on the system 20×10 studied in the range of disorder $W = 0$ to $W = 8$. From reference [12], we deduce that the diffusive regime corresponds approximately to $2.2 \leq W \leq 4.5$. The energy spectra depicted in figure 6 illustrate the difference already pointed out in reference [13], between the diffusive regime, where the sensitivity of the energy levels to the boundary conditions E_c is larger than the level spacing η , and the localised regime, where E_c is smaller than η . As already pointed out in [6, 7], when W increases, the levels repel each other, become flatter, and their current contributions decrease. The highest Fourier components vanish faster than the lower ones. The two first harmonics of $I_N(\phi)$ are shown in figure 7 as a function of N , for $W = 4$ and $W = 6$. We note that E_c , which is also of the order of the characteristic energy range between two successive maxima of the function $I_1(N)$, strongly decreases with W . It has indeed been shown by Thouless that E_c is proportional to the conductivity of the system [13, 14]. One should also notice that E_c measures the amplitude of the variation of the chemical potential $\mu(\phi)$ at fixed N .

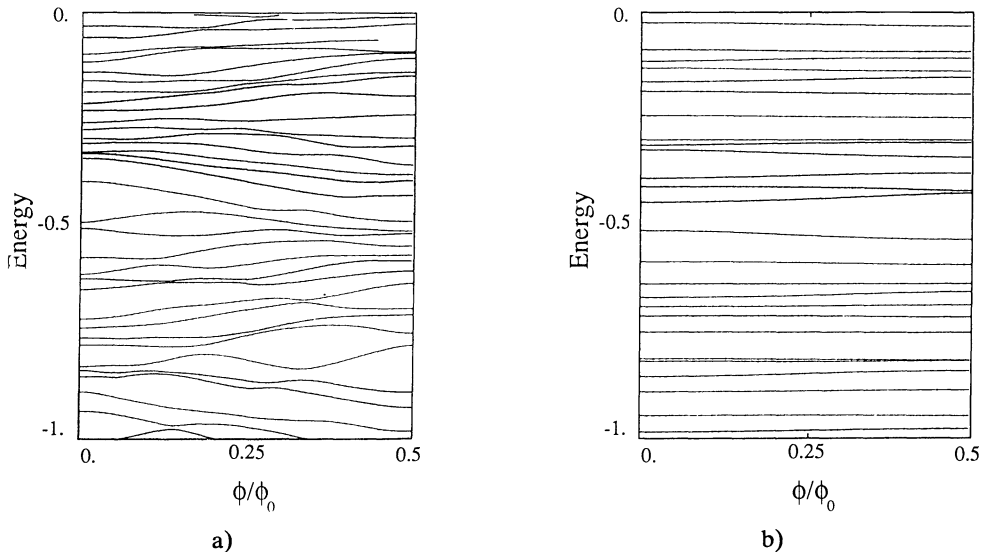


Fig. 6. — Evolution of the spectrum with increasing disorder. $L = 20$, $M = 10$, (a) : $W = t$ (b) : $W = 4t$.

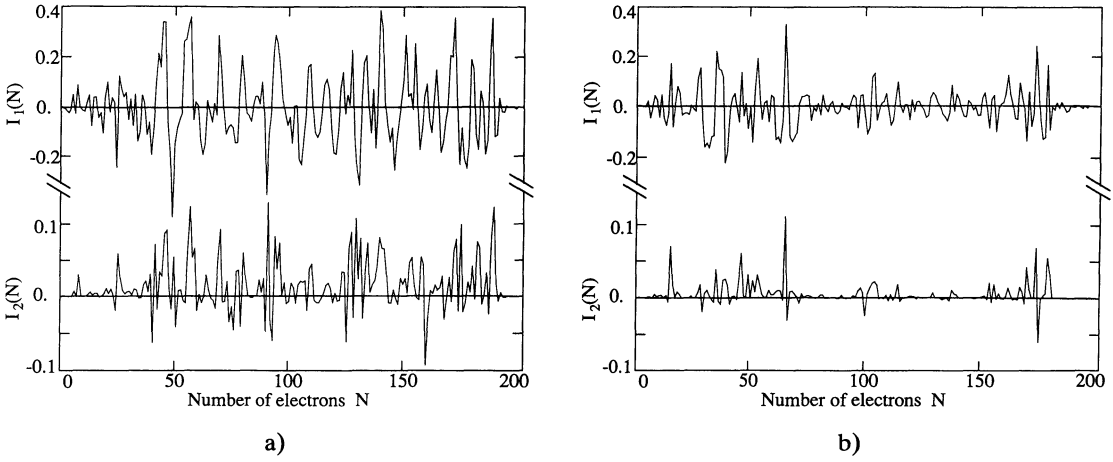


Fig. 7. — First and second harmonics, $I_1(N)$ and $I_2(N)$, of the total current *versus* N for one configuration of disorder and $L = 20$, $M = 10$. a) $W = 4t$. b) $W = 6t$. $I_1(N)$ oscillates with a typical period which is the Thouless energy E_c .

The difference between the first and the second harmonics is even more striking than in the $W = 0$ case since the second harmonics presents barely exclusively positive values, whereas the first harmonics are still randomly positive and negative. The average over N , $\langle I_N(\phi) \rangle_N$, is depicted in figure 8 for two different configurations of disorder $W = 4$. It is well described by a simple sinus wave with periodicity $\phi_0/2$.

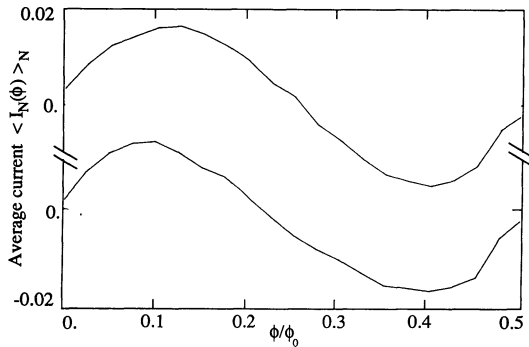


Fig. 8. — Average of the total current $I_N(\phi)$ over the number of electrons : $50 \leq N \leq 150$, in the range $[0, \phi_0/2]$, for two configurations of disorder $L = 20$, $M = 10$, $W = 4t$. Note that $\langle I_N(\phi) \rangle$ is independent of the configuration of disorder.

There is now another possibility of doing an ensemble average which consists in averaging over different configurations of the disorder at a fixed value of N . The two first harmonics of this quantity $\langle I_N(\phi) \rangle_D$ are depicted in figure 9 as a function of N . The sign of the first harmonics is a very fluctuating function of N and its amplitude is considerably reduced compared to the typical amplitude obtained for one configuration of the disorder. On the other hand, the second harmonics is always positive and is only slightly reduced compared to

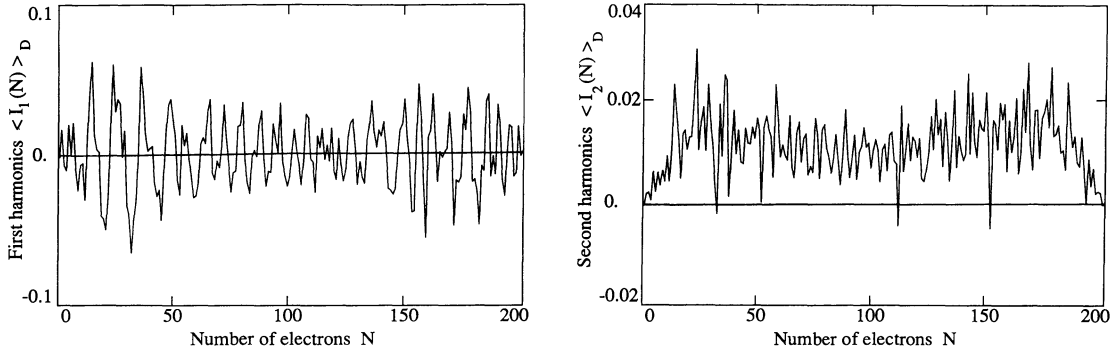


Fig. 9. — Averages of the first and second harmonics $\langle I_1(\phi) \rangle_D$ and $\langle I_2(\phi) \rangle_D$ over 50 configurations of disorder, versus N for $W = 4t$. The first harmonics is a fluctuating function of N which is 0 in average over N . The second harmonics is always positive and is only slightly reduced compared to its typical value.

its typical value. It is of the order of the average with respect to N $\langle I_N(\phi) \rangle_N$ described above.

At this stage, it is difficult to give a precise estimate of the order of magnitude of this ensemble average of the persistent currents in the diffusive regime since, for the small sample sizes we have investigated, the range of disorder relevant for the study of this regime is too much restricted. Our preliminary results however suggest that the second harmonics of $\langle I_N(\phi) \rangle$ averaged on both N and disorder configurations is a lot larger than $\exp(-L/\ell_c)$ as predicted in references [7-8] for the second harmonics of $\langle I_\mu(\phi) \rangle_D$ computed for a fixed value of the chemical potential and averaged on different disorder configurations. This prediction implies that the average current in the localized regime should not depend on the number of channels and should be identical to its value for a single channel ring. On the other hand, we find that the averaged second harmonics is 4 orders of magnitude greater in the 20×10 system than in the 20×1 system for $W = 6$. This result rules out an $\exp(-L/\ell_c)$ behavior. Moreover, it is shown in figure 10 that the second harmonics $I_2(N)$ is of the order of magnitude of the square of the first harmonics $I_1^2(N)$ (in I_0 units) and both behave, as a function of N , in a striking similar way. This suggest that, in average, the second harmonics of $\langle I_N(\phi) \rangle_D$ is of the order of the square typical current $\langle I_N^2(\phi) \rangle_D$. These results are compatible with an $\exp(-L/\xi)$ dependence for $\langle I_2(N) \rangle$ in the localized regime.

In strong disorder, figure 11 shows that only a few levels contribute to the current these levels being strongly correlated in pairs $(n, n + 1)$: $i_2(n)$ is positive, $i_2(n + 1) = -i_2(n)$ and thus $I_2(n)$ is always positive or zero. One can understand this structure in the following way. The eigen energies can be developed in successive powers of t using the Brillouin-Wigner expansion. The perturbed energy of a state is given by :

$$E_p = E_p^0 + \sum' t_{pq} t_{qr} \dots t_{zp} / (E_p - E_q^0) \dots (E_p - E_z^0) \quad (4.1)$$

where the summation \sum' runs over all the paths starting from p and returning to p (without visiting p). We limit the following discussion to the case where $t < \min |E_j^0 - E_{j+1}^0|$. The products $t_{pq} \dots t_{zp}$ depend on the magnetic field through the phase factor $e^{2i\pi\phi/\phi_0}$ if the path $pq \dots zp$ encloses the flux ϕ . Otherwise they do not have any flux

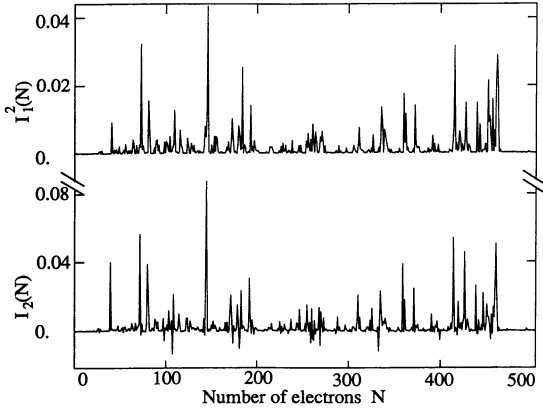


Fig. 10.

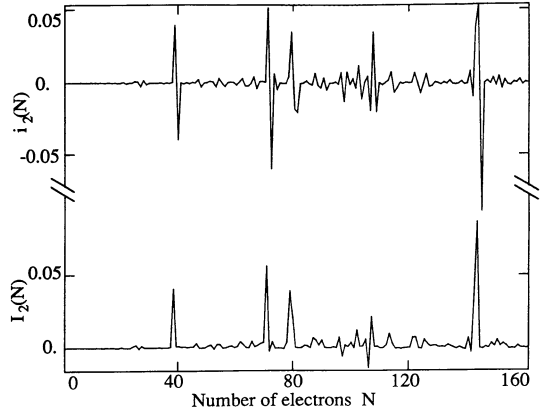


Fig. 11.

Fig. 10. — Second harmonic of the total current compared to the square of the first harmonic for $L = 50$, $M = 10$ and $W = 4 t$.

Fig. 11. — Second harmonics $i_2(N)$ and $I_2(N)$ of the band current and of the total current for $L = 50$, $M = 10$, $W = 4 t$.

dependence. As a result, the first harmonics of $E_p(\phi) = \lambda_p^{(0)} + \lambda_p^{(1)} \cos(2\pi\phi/\phi_0) + \lambda_p^{(2)} \cos(4\pi\phi/\phi_0) + \dots$ is of order t^L .

We first consider a one dimensional system for which the first harmonics can be simply written as :

$$\lambda_p^{(1)} = \frac{t^L}{\prod_{q \neq p} (E_p^0 - E_q^0)} \tag{4.2}$$

The second harmonics $\lambda_p^{(2)}$ is obtained by expanding E_p in the denominator of equation (4.1).

$$\lambda_p^{(2)} = -(\lambda_p^{(1)})^2 \cdot \left(\sum_q \frac{1}{E_p^0 - E_q^0} \right) \tag{4.3}$$

$\lambda_p^{(2)}$ is proportional to the square of $\lambda_p^{(1)}$. From expression (4.2), it is clear that the contribution to the persistent current is essentially related to pairs of levels p and $p + 1$ which are close in energy. Assuming for all q distinct from p and $p + 1$,

$$|E_q - E_p| \gg |E_p - E_{p+1}| \quad \text{and} \quad |E_q - E_{p+1}| \gg |E_p - E_{p+1}|,$$

in that case $\lambda_{p+1}^{(1)} \approx -\lambda_p^{(1)}$ and $\lambda_{p+1}^{(2)} = -\lambda_p^{(2)} = -(\lambda_p^{(1)})^2 / (E_{p+1} - E_p) < 0$.

The current contributions of such a pair of levels nearly cancel one another. The sign of their first harmonics depends on p . On the other hand, the sign of the second harmonics is always positive for the lowest energy level. As a result, the second harmonics of the total current I_N is positive for $N = p$ and nearly equal to zero for $N = p + 1$.

This behaviour of pairs of energy levels close in energy is illustrated by the results of numerical simulations in figure 11.

This result can be easily extended to a multichannel ring. In that case, there are two kind of contributions in $\lambda_p^{(2)}$. The first one is related to paths enclosing two times the flux ϕ , the second one is obtained when expanding the energy denominators in the graphs C_{1p} enclosing only one time the flux ϕ .

The first term has a random sign whereas the second one contains the sum of the squares of the diagrams contributing to $\lambda_p^{(1)}$:

$$\Delta\lambda_p^{(2)} = \sum_{C_{1p}} \left\{ \left(\prod_{r \in C_{1p}} \frac{t^2}{(E_p^0 - E_r^0)^2} \right) \left(\sum_{q \in C_{1p}} \frac{1}{E_q^0 - E_p^0} \right) \right\} \quad (4.4)$$

where $\sum_{C_{1p}}$ is a sum over all the graphs C_{1p} enclosing one flux ϕ .

This sum gives rise to a positive contribution in the second harmonics of the total current I_2 for the same reasons than for the one dimensional case. the averaged second harmonics (with respect to N) is thus positive but presents more fluctuations due to the multiplicity of diagrams with a random sign which also contribute to $\lambda^{(2)}$. The order of magnitude of $\langle I_2 \rangle_N$ is

$$\langle I_2 \rangle_N \sim \langle i_2 \rangle_N \sim \langle \lambda_p^{(2)} \rangle / \phi_0 \sim (\langle \lambda_p^{(1)} \rangle)^2 / \phi_0 \eta \quad (4.5)$$

where η is the level spacing. Since $I_0 \sim \eta M / \phi_0$, then :

$$\langle I_2 \rangle_N \sim \frac{M \langle i_1^2 \rangle_N}{I_0} \sim \frac{M \langle I_1^2 \rangle_N}{I_0}. \quad (4.6)$$

in agreement with our numerical results shown in figure 10. Since it is admitted [6] that the typical value of the first harmonics varies like $\exp(-L_x/2\xi)$ where ξ is the localisation length, this gives $\langle I_2 \rangle_N \propto \exp(-L_x/\xi)$ which is very different from the behavior in $\exp(-L_x/\ell_c)$ predicted in [6-9].

5. Conclusion.

We have studied the ensemble average of persistent currents in multichannel rings at zero temperature and shown that it is periodic in $\phi_0/2$, i.e. it can be expanded in terms of the successive harmonics of $\sin(\pi\phi/\phi_0)$, all of them being positive. This result can be understood physically as a direct consequence of the increase of the sensitivity of the eigenstates to a change of boundary conditions (i.e. the typical current) with increasing energy. The current $\langle I_N(\phi) \rangle$ has been computed by fixing the number of electrons N in each sample and averaging with respect to N . Our results do not describe the same physical situation than the study of other authors [6-9] who have computed the current $\langle I_\mu(\phi) \rangle$ within the assumption of a fixed value of the chemical potential instead of maintaining constant the number of electrons. One also could have used a grand canonical ensemble with a field dependent chemical potential $\mu(\phi)$ in order to fix the number of electrons N . The typical variation of $\mu(\phi)$ is then of the order of E_c , the Thouless energy.

In the limit of weak disorder $\langle I_N(\phi) \rangle_N$ has been computed exactly and is found to depend only on the Fermi velocity and to be equal to the value in the one dimensional ring independently of the number of channels. When increasing the disorder $\langle I_N(\phi) \rangle_N$ is still independent of the disorder realisation and $\langle I_N(\phi) \rangle_N \sim \langle I_N(\phi) \rangle_{N,D}$. In the limit of strong disorder (i.e. in the localised regime) $\langle I_N(\phi) \rangle_{N,D}$ drastically depends on the number of channels and is proportional to the square of the typical current $\langle I_N(\phi) \rangle_{N,D} \propto \langle I_N(\phi)^2 \rangle_{N,D}$. This result rules out an $\exp(-L_x/\ell_c)$ dependence (where ℓ_c is the elastic mean

free path) like the one obtained by other authors [6-9] for $I_{\mu}(\phi)$. On the other hand, our results are compatible with :

$$\langle I_N(\phi) \rangle \propto I_0 e^{-L_x/\xi} \sin(\pi\phi/\phi_0) + \text{higher harmonics}$$

where ξ is the localisation length of the multichannel system ($\xi \approx M\ell_e$) in 3D. Work is still needed concerning a precise estimation of the average persistent current in the diffusive regime.

Our result is very important from the experimental point of view since it encourages the detection of the permanent currents in an assembly of many independent rings which should be easier than their detection in one single ring. Although our result is reminiscent of the Aharonov-Bohm resistance oscillations with period $\phi_0/2$ in cylinders or array of connected rings [1, 15, 16], we have not clearly determined the relationship between these two effects.

Acknowledgements.

We are very indebted to L. Lévy for very suitable comments and suggestions on this work. We have also benefited of many fruitful discussions with B. Alsthuler, M. Azbel, G. Dolan, B. Douçot, Y. Imry, J. L. Pichard, R. Rammal, E. Riedel, M. Sanquer and N. Trivedi.

References

- [1] WASHBURN S. and WEBB R. A., *Adv. Phys.* **35** (1986) 395.
- [2] BÜTTIKER M., IMRY Y. and LANDAUER R., *Phys. Rev.* **96A** (1983) 365.
- [3] LANDAUER R. and BÜTTIKER M., *Phys. Rev. Lett.* **54** (1985) 2049 ;
BÜTTIKER M., *Phys. Rev. B* **32** (1985) 1846.
- [4] CHEUNG H. F., GEFEN Y., RIEDEL E. K. and SHIH W. H., *Phys. Rev. B* **37** (1988) 6050.
- [5] TRIVEDI N. and BROWNE D. A., *Phys. Rev. B* **38** (1988) 9581.
- [6] CHEUNG H. F., GEFEN Y. and RIEDEL E. K., *IBM J. Res. Develop.* **32** (1988) 359.
- [7] RIEDEL E. K., CHEUNG H. F. and GEFEN Y., *Phys. Scr.* (to be published).
- [8] CHEUNG H. F., RIEDEL E. and GEFEN Y., *Phys. Rev. Lett.* **62** (1989) 587.
- [9] ENTIN WOHLMANN O. and GEFEN Y., *Europhys. Lett.* **5** (1989) 447.
- [10] ANDERSON P. W., *Phys. Rev.* **109** (1958) 1492.
- [11] We are very indebted to Y. Imry to have emphasised this point to us.
- [12] PICHARD J. L. and SARMA G. J., *J. Phys. C* **14** (1981) L 127, L 617.
- [13] EDWARDS J. T. and THOULESS D. J., *J. Phys. C* **5** (1972) 807.
- [14] THOULESS D. J., *Phys. Rep.* **132** (1974) 93.
THOULESS D. J., *Phys. Rev. Lett.* **18** (1977) 1167.
- [15] ARONOV A. G. and SHARVIN Yu V., *Rev. Mod. Phys.* (1987) 755.
- [16] PANNETIER B., CHAUSSY J., RAMMAL R. and VILLEGIER J., *Phys. Rev. Lett.* **53** (1984) 1845.