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Walk inside Hofstadter's butterfly

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Résumé. — Cet article est une invitation à comprendre à l'aide de dessins calculés numériquement les propriétés du spectre de l'équation de Harper dans l'esprit du célèbre dessin dû à Hofstadter. On cherchera ainsi à expliquer certains résultats récents de mathématiciens et de physiciens sur la structure cantorienne de ce spectre.

Abstract. This paper describes with help of numerically computed pictures the fascinating properties of the spectrum of Harper's equation in the spirit of celebrated Hofstadter's picture. We hope to explain recent results obtained by mathematicians and physicists about the Cantor structure of this spectrum.

Introduction.

In 1976, the physicist Hofstadter [1] produced in his thesis a beautiful butterfly in order to describe the dependence of the spectrum of a finite-difference equation discovered by Harper [2] with respect to some parameter α . This equation appears in the study of the Schrödinger equation with constant magnetic field (or more generally periodic magnetic field) in dimension 2. More precisely, if we consider the following operator in \mathbb{R}^2 :

$$P_{B,V} = \left(hD_{x_1} - \left(\frac{eB}{2c} \right) x_2 \right)^2 + \left(hD_{x_2} + \left(\frac{eB}{2c} \right) x_1 \right)^2 + V_R(x_1, x_2)$$

where $B > 0$, $h > 0$, $R > 0$ and V_R is C^∞ periodic on a lattice $R\Gamma$, the reduction to Harper's equation appears in many asymptotic situations according to the different relative values of B , h , R . Let us recall briefly some of these contexts :

- the weak magnetic field limit (treated by Peierls approximation) : $B \rightarrow 0$
- the strong magnetic field limit : $B \rightarrow \infty$
- the semi-classical limit : $h \rightarrow 0$
- the tight binding approximation : $R \rightarrow \infty$.

In all these cases, the parameter α appears to be equal (or the inverse of) the product of the magnetic flux through a lattice cell $\phi_R(B)$ divided by a flux quantum (hc/e). It is not possible to give here a complete list of references but let us mention some works of physicists or mathematicians : Chambers [3], Kohn [4], Landau [5], Luttinger [6], Novikov [7], Peierls [8], Lyskova [9], Bellissard [10], [11], Wilkinson [12], Helffer-Sjöstrand [13-15], Nenciu [16].

Let us now define Harper's equation more precisely ; we look at the following family of bounded operators on $\ell^2(\mathbb{Z})$ (where $\ell^2(\mathbb{Z})$ is the set of the square summable sequences) :

$$\ell^2(\mathbb{Z}) \ni u_n \Rightarrow (H_\theta^{\alpha, \mu} u)_n = u(n+1) + u(n-1) + 2\mu \cos(2\pi\alpha n + \theta) u(n)$$

and we study the spectrum $\Sigma_\theta^{\alpha, \mu}$ of $H_\theta^{\alpha, \mu}$ and more precisely the set :

$$\Sigma^{\alpha, \mu} = \bigcup_{\theta} \Sigma_\theta^{\alpha, \mu} .$$

By definition the spectrum $\Sigma_\theta^{\alpha, \mu}$ is the complementary of the set of the real E such that the operator $(H_\theta^{\alpha, \mu} - E)$ has a bounded inverse in $\mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}))$. For α rational, a way to determine the spectrum is to look for the generalized eigenvectors, that is to determine the value of E for which there exists a bounded sequence (i.e. in $\ell^\infty(\mathbb{Z})$) $(u(n))_{n \in \mathbb{Z}}$ of :

$$(0.1) u(n+1) + u(n-1) + 2\mu \cos(2\pi\alpha n + \theta) u(n) = Eu(n) .$$

In the case where α is rational the set $\Sigma^{\alpha, \mu}$ is the union of q bands and the idea of Hofstadter was just to show how the repartition of these bands depends on the expansion of α as a continuous fraction (which is of course finite in the case where α is rational). This picture permits to imagine the structure of the spectrum in the irrational case which is usually assumed to be a Cantor set (for the mathematicians, this property is a conjecture of Kac and Simon [17] called it the 10 Martinis conjecture). From a careful analysis of the butterfly, Hofstadter gave some precise rules for the repartition of the bands in the rational case. Another complementary description was given by Wannier [18] (see also Claro-Wannier [19]) ; it is related to considerations on the integrated density of states (see the discussion following picture 12). In the last ten years, this problem has inspired many contributions from mathematicians who tried to explain different properties which appear on Hofstadter's butterfly ; let us mention : Bellissard-Simon [20], Helffer-Sjöstrand [13, 14, 21], Van Mouche [22], Choi-Elliott-Yui [23] ; these mathematicians were quite often inspired by earlier partially heuristic but fundamental results by physicists as for example : Claro-Wannier [19], Wilkinson [12], Azbel [24]... It is indeed in these papers that appear the basic ideas of renormalization, microlocal techniques, role of the density of states which will be described later.

What follows is neither a mathematical article nor a physical article. We just intend to present some properties of the butterfly and to expose what is, to our knowledge, actually proved. All the results we are speaking about are illustrated by pictures computed numerically. We have computed many zooms inside classical Hofstadter's butterfly and we hope that this article could be an introduction through pictures to all above mentioned articles. All the pictures are obtained by combination of adapted zooms, variation of the parameter μ and particular choices of sequences of rationals :

$$1 - 1/(q+1), 1/(2+1/q), 1/(3+1/(2+1/q)), \text{ et } c .$$

When mathematical proofs exist, we try to give precise references.

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1. Butterfly's programm.

We want to study the spectrum $\Sigma_\theta^{\alpha, \mu}$ of Harper's operator :

$$\ell^2(\mathbb{Z}) \ni u_n \Rightarrow (H_\theta^{\alpha, \mu} u)_n = u(n+1) + u(n-1) + 2\mu \cos(2\pi\alpha n + \theta) u(n)$$

and more precisely we look at the set :

$$\Sigma^{\alpha, \mu} = \bigcup_\theta \Sigma_\theta^{\alpha, \mu}.$$

μ is a real parameter in $[0, +\infty[$ and α is in $[0, 1]$; θ belongs to $[0, 2\pi]$.

Let us recall the following properties of Σ :

- (1.1) If $\alpha = p/q$ is rational, the spectrum is the union of q disjoint bands (the only exception is for the even q 's for which the two central bands touch at the middle of $\Sigma^{\alpha, \mu}$) (see [20, 22, 23]). Moreover the length of the gap between two bands is minorized by 8^{-q} [23].
- (1.2) If α is irrational the spectrum $\Sigma_\theta^{\alpha, \mu}$ is independent of θ (see for example a proof in [13]).

This is no longer true in the rational case ; because we want to understand $\Sigma_\theta^{\alpha, \mu}$ for α irrational from numerical computations for α rational, we have to use some continuity properties of the spectrum with respect to α which are only true for $\Sigma^{\alpha, \mu}$.

- (1.3) The spectrum is symmetric by the map : $E \rightarrow -E$.
- (1.4) The spectrum is contained in $[-2 - 2\mu, 2 + 2\mu]$.
- (1.5) $\Sigma^\mu = \mu \Sigma^{1/\mu}$ (this is Aubry's duality, see [25, 17, 20]).
- (1.6) For some families of irrational α , the spectrum Σ is a cantor set :
 - [1.6a] Liouville numbers (see (23)), i.e. such that :

$$\forall C > 0, \exists p/q \text{ such that } |\alpha - p/q| < C^{-q}$$

(more intuitively, the Liouville numbers are transcendental numbers which can be very well approximated by rational numbers)

or [1.6b] (see [13, 14, 21]) numbers whose expansion as a continuous fraction satisfies the following condition :

$$\alpha = 1/(q_0 + 1/(q_1 + 1/(q_2 + \dots))) \text{ with } |q_i| \geq C,$$

(for some C large enough) (this is only proved in the case $\mu = 1$)

or [1.6c] for a G_δ -set which can not be explicitated of α (cf. [20] in relation with the more recent result by Van Mouche [22]). (Recall that a G_δ -set is a countable intersection of dense open sets and that this set is dense ; this is a useful concept to describe the instability of the spectrum).

- (1.7) $\Sigma^{\alpha, \mu} = \Sigma^{1-\alpha, \mu}$.

THE BUTTERFLY. — For each rational value of α , we present on a vertical line of abscissa α the spectrum Σ downwards. Up to some rotation, this is classical Hofstadter's butterfly. After Hofstadter, many physicists or mathematicians have produced similar pictures (let us mention Claro-Wannier [19], Wilkinson [12], Van Mouche [22], Bellissard [10, 11]). We have reproduced some of these pictures here. But the originality of some other pictures presented here is perhaps in the introduction of particular zooms and subpictures corresponding to special sequences of rationals in order to follow the dependence of the spectrum with respect to the expansion of α as a continuous fraction.

REMARKS ON THE NUMERICAL METHOD. — In a concrete way, we start from a given $\alpha \in \mathbb{Q} \cap]0,1[$, and we are looking for values of E and θ such that :

$$(*) \quad u(n+1) + u(n-1) + (2\mu \cos(2\pi\alpha n + \theta) - E)u(n) = 0$$

has a solution in $\ell^\infty(\mathbb{Z})$ (cf. [20]). To realize this, we write (*) on the form :

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = B_n(\theta, E, \mu) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

with :

$$B_n(\theta, E, \mu) = \begin{pmatrix} b_n(\theta, E, \mu) & -1 \\ 1 & 0 \end{pmatrix}$$

$$b_n(\theta, E, \mu) = E - 2\mu \cos(2\pi\alpha n + \theta).$$

If $\alpha = p/q$, we remark that with :

$$M(\theta, E, \mu) = B_{q-1}(\theta, E, \mu) \cdot B_{q-2}(\theta, E, \mu) \dots B_0(\theta, E, \mu),$$

we have :

$$\begin{pmatrix} u_{n+q} \\ u_{n+q-1} \end{pmatrix} = M(\theta, E, \mu) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

and we are reduced to the study of :

$$\begin{pmatrix} v_{n+1} \\ v_n \end{pmatrix} = M(\theta, E, \mu) \begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix}.$$

If one remarks that $\det M = 1$, we shall get a solution in $\ell^\infty(\mathbb{Z})$ for the problem (*) if :

$$(**) \quad |\text{Tr } M(\theta, E, \mu)| \leq 2.$$

Consequently E will belong to Σ if and only if there exists θ such that (**) is satisfied. Actually, the following relation has been observed by Chambers [26] (see also Butler-Brown [27] and Obermair [28])

$$\text{Tr } M(\theta, E, \mu) = P_{\mu, p, q}(E) + 2(-1)^q \mu^q \cos(q\theta).$$

So we can transform the criterion of belonging to Σ into the following simpler criterion :

$$(***) \quad |\text{Tr } M(\pi/2q, E, \mu)| \leq 2 + 2\mu^q.$$

We use this criterion for the numerical computations.

SEMI-CLASSICAL APPROACH AND RENORMALIZATION. — Wilkinson's work [12] is based on a WKB analysis with infinitely many « potential » wells in the space $T^*\mathbb{R}$. It is first observed that the spectrum of $H_{\beta}^{\alpha, \mu}$ is the same as the spectrum of the pseudo-differential operator :

$$p(x, hD_x) = 2 \cos x + 2 \mu \cos hD_x \quad \text{with} \quad h = 2 \pi \alpha$$

associated by the Weyl quantification with the Hamiltonian $p(x, \xi) = 2 \cos x + 2 \mu \cos \xi$. This means that for each $u \in \mathcal{S}(\mathbb{R})$ (where $\mathcal{S}(\mathbb{R})$ is the space of the C^∞ rapidly decreasing functions), we define $p(x, hD_x) u$ by :

$$(p(x, hD_x) u)(x) = (2 \pi h)^{-n} \iint e^{i \langle x-y | \xi \rangle / h} p((x+y)/2, \xi) u(y) dy d\xi .$$

If we have $E \neq 0$, $-2 - 2 \mu \leq E \leq 2 + 2 \mu$, the WKB technique consists in looking first at the set : $p^{-1}(E)$ in $T^*\mathbb{R}$. $p^{-1}(E)$ is a union of infinitely many compact components $C_\beta(E)$ ($\beta \in \mathbb{Z}^2$) (called the wells) where $C_\beta(E)$ is obtained from $C_0(E)$ by translation of $2 \pi \beta$. For each of these wells, there is a standard way to determine near E semi-classical eigenvalues modulo $O(h^\infty)$ $\lambda_j(h)$ determined by a Bohr-Sommerfeld type condition and a corresponding quasimode $\phi_j^0(x; h)$ living essentially in $C_0(\lambda_j(h))$. In doing this construction we forget the tunneling interactions between the wells which modify completely the nature of the spectrum but not the localization modulo $O(h^\infty)$ (actually modulo an exponentially small term of order $\exp(-S/h)$). Of course, because of the translation invariance in x and ξ , we can associate with each well $C_\beta(\lambda_j(h))$ a quasimode :

$$\phi_j^\beta(x; h) = e^{-i(2\pi)x\beta_2/h} \phi_j^0(x - (2\pi)\beta_1; h) \quad \text{where} \quad \beta = (\beta_1, \beta_2) .$$

Let us now observe that the spectrum is localized in small intervals $I_j(h)$ around the $\lambda_j(h)$. In each of these intervals (of length less than Ch^2), we can understand the spectrum as follows ; let $\Sigma_j(h)$ the spectrum inside $I_j(h)$ and $E_j(h)$ the corresponding spectral space. Then starting from the $\phi_j^\beta(x; h)$, we can construct an orthonormal basis $e_j^\beta(x; h)$ with essentially the same properties as the $\phi_j^\beta(x; h)$ (localization in $C_\beta(\lambda_j(h))$, invariance by the translations : $e_j^\beta(x; h) = e^{-i(2\pi)x\beta_2/h} e_j^0(x - (2\pi)\beta_1; h)$) and such that $(e_j^\beta(x; h) - \phi_j^\beta(x; h))$ is quite small in a sense which can be specified. The study of the spectrum is then reduced to the study of the infinite matrix :

$$M_{\gamma\beta}^j(h) = (P(x, hD_x) e_j^\beta | e_j^\gamma) .$$

This infinite matrix has the following invariance properties :

$$M_{\gamma\beta}^j = e^{i\omega(\gamma, \beta)h'} f_j(\gamma - \beta, h)$$

where h' is related to h by the following relation :

$$(2 \pi / h) = k + (h' / 2 \pi)$$

and :

$$\omega(\gamma, \beta) = \gamma_1 \beta_2 - \gamma_2 \beta_1 .$$

The principal term is the diagonal term $f_j(0, h)$ which is not far of $\lambda_j(h)$. The other terms corresponding to $\beta \neq \gamma$ correspond to the interactions between the different wells and Wilkinson suspects that, in the case where $\mu = 1$, the most important terms are obtained for $|\gamma - \beta| = 1$ (the mathematical proof is given in [13]).

The important remark is now the following ; if $\mu = 1$ and if h is small enough, the study of the spectrum of the infinite matrix $M_{\gamma\beta}$ is equivalent to the study of a pseudodifferential operator with parameter h' . Moreover, this pseudodifferential operator is given by :

$$f(0 ; h) + f((1, 0) ; h) (\cos h' D_x + \cos x) + O(e^{-S'/h})$$

where $f((1, 0) ; h) = h^\nu (1 + O(h)) e^{-(S(E)/h)}$, $S'(E) > S(E)$, $\nu \in \mathbb{R}$, $E = \lambda_j(h)$.

$S(E)$ is an action measuring the minimal « distance » between two different components of $p^{-1}(E)$.

What we find in this correspondence is just a perturbation of Harper's equation *with the same symmetry properties* and if h' is small we can iterate the procedure. This is the semiclassical version of the renormalization procedure. This approach was proposed by Wilkinson and was mathematically justified in [13].

We omit here to explain the problem occurring for E near 0 and which was treated heuristically in Azbel's paper [24]. The new problems are that, for $E = 0$, $p^{-1}(E)$ is not a union of compact components and the traditional WKB construction fails. This problem is treated mathematically in [21].

FURTHER REMARKS ON THE NUMERICAL ASPECTS. — Let us first remark that we observe (for example in picture n° 1) artificial white vertical bands near $\alpha = 0, 1, 1/2, \dots$, due to the repartition of the rationals α with bounded denominator (for example we take the denominator less than 50).

As proved in [13] (which is inspired by the heuristical semi-classical approach given by Wilkinson in [12]) and explained before, most of the bands have an exponentially small length as α tends to 0 (i.e. comparable with $\exp(-2\pi S/\alpha)$). This is due to the tunneling effect. This phenomenon is quite visible on Hofstadter's butterfly ; this is however no longer true when you approach the middle of the spectrum i.e. for the bands near the 0 energy, because the tunneling between the different connected compact components of the energy level (for $E \neq 0$) is not small when E approaches 0. This exponential length of the bands leads to numerical difficulties if we want to visualize the picture without missing too many bands and if we want to avoid exponentially increasing time of computation (when we consider rationals with large denominators). To get good pictures with a reasonable time of computation, we have added to the criterion (***) , a criterion of change of sign (which detects the presence of a band when $P_{\mu,p,q}(E)$ changes sign). This criterion is apparently not used in [22] whose butterflies are amputated from many bands. The improved criterion is quite efficient for $\alpha = 1/q$. But for α in the form $1/(q + 1/2n)$ ($n \in \mathbb{N}$), the criterion is again bad because the spectrum appears as a union of tiny packets of an even numbers of bands (in fact, $q - 1$ packets of $2n$ bands + in the middle one packet of $2n + 1$ bands) (see pictures n° 26 and 27).

2. Description of the pictures.

Picture n° 1. This is the reproduction of the famous Hofstadter's butterfly. μ is chosen equal to 1. We consider all the rationals α between 0 and 1 whose denominator q is less than 50 (we have written $\alpha = p/q$ as an irreducible fraction).

Pictures 2 to 6 are inspired by those of Van Mouche [22]. The idea is to follow the dependence of the spectrum with respect to μ as μ goes from 1 to 0. According to property 1.5 in section 1 this is sufficient. The spectra are represented in the rectangle $[0, 1] \times [-2 - \mu, 2 + \mu]$ (the picture corresponds to the couple (α, E) with $E \in \Sigma^\alpha$).

Picture n° 2. We take $\mu = 0.75$ and we consider all the rationals α between 0 et 1 whose denominators are less than 30.

Picture n° 3. We take $\mu = 0.5$ and we consider all the rationals α between 0 et 1 whose denominators are less than 30. Outside large gaps around the diagonals of the rectangle, the other gaps are narrower.

Picture n° 4. We take $\mu = 0.25$.

Picture n° 5. We take $\mu = 0.125$.

Picture n° 6. We take $\mu = 0.0625$. We consider all the rationals of denominators ≤ 40 .

Pictures 7 to 12 show corresponding negatives. On each vertical line of abscissa α , we put the complementary of the spectrum α . To allow a better visualization, we have amplified the gaps by modifying the numerical procedure. Consequently, one can see in the picture an artificial horizontal line of gaps with ordinate 0 corresponding to the points where the central bands touch in the even case (see [20]).

Picture n° 7. We take $\mu = 0.5$.

Picture n° 8. We take $\mu = 0.25$.

Picture n° 9. We take $\mu = 0.0625$.

Picture n° 10. We take $\mu = 0.001$.

Pictures 2 to 10 show the evolution of the spectrum with respect to μ . The set of the gaps is distorted continuously and at the limit we get the graphs of $E = \cos \pi \ell \alpha$ ($\ell \in \mathbb{N}^*$) et $E = \cos \pi (1 - \ell \alpha)$ ($\ell \in \mathbb{N}^*$) which are reproduced in :

Picture n° 11. Graphs of $E = \cos \pi \ell \alpha$ ($\ell \in \mathbb{N}$) and $E = \cos \pi (1 - \ell \alpha)$ ($\ell \in \mathbb{N}$) with $1 \leq \ell \leq 12$.

This is related to the computation of the integrated density of states. The structure of this graph is homeomorphic to the following picture introduced by Claro-Wannier [19] :

Picture n° 12. $x = E[\ell \alpha + m]$ ($E[y]$ is the fractional part of y ; $\ell, m \in \mathbb{Z}$). This picture is obtained by drawing in the square $[(0, 0), (1, 0), (1, 1), (0, 1)]$ the lines starting from the points $(0, 0)$ and $(0, 1)$ of equation $x = \ell \alpha$ and $x = 1 - \ell \alpha$ ($1 \leq \ell \leq 12$) and the reflected rays in the square according to the billiard reflection rule.

In order to understand the previous pictures better, let us recall :

Some properties of the density of states.

The integrated density of states D_α was introduced a long time ago by physicists (cf. [17, 29, 30]... for references). If \mathfrak{X}_ℓ is the operator of multiplication (operating on $\ell^2(\mathbb{Z})$) by the characteristic function of $\{n \mid -\ell \leq n \leq \ell\}$, we can define it by :

$$D_{\alpha, \theta}(E) = \lim_{\ell \rightarrow \infty} (2\ell + 1)^{-1} \text{Tr} (\mathfrak{X}_\ell P_{(-\infty, E)}(H_\theta^{\alpha, \mu}))$$

where $P_\Omega(H)$ is the spectral projection of H on the interval Ω . It has the following properties :

P0 $E \rightarrow D_{\alpha, \theta}(E)$ is independent of E in $\mathbb{R}/\Sigma^{\alpha, \mu}$. It does not either depend on θ if α is irrational.

P1 $E \rightarrow D_{\alpha, \theta}(E)$ is continuous, constant in each gap of the spectrum.

According to P0 and P1, we shall denote it by D_α when it does not depend of θ (i.e. for α irrational or for E in $\mathbb{R}/\Sigma^{\alpha, \mu}$). In the other cases you can think to D_α as the mean value over θ of $D_{\alpha, \theta}$.

- P2 For E in $\mathbb{R}/\Sigma^{\alpha, \mu}$, there exists 2 integers m and n in \mathbb{Z} such that : $D_{\alpha}(E) = m\alpha + n$ (cf. [10, 11] for a presentation and references). Moreover, if α varies, one can find m and n independent of α if E remains in $\mathbb{R}/\Sigma^{\alpha, \mu}$.
- P3 For E in $\mathbb{R}/\Sigma^{\alpha, \mu}$ and $\alpha = p/q$, $D_{\alpha}(E)$ can be computed as the number of bands in $\{y \leq E\}$ divided by the total number of bands ($= q$).
- P4 $D_{\alpha}(E)$ is independent of μ as E remains in $\mathbb{R}/\Sigma^{\alpha, \mu}$.

All these properties are fundamental. However the conjunction of P2 and P3 does not permit to compute m and n exactly. The following result is announced in [20] and used in [23] (but the proof is not detailed). The convenient m is determined for α rational by choosing in all the pairs (m, n) such that : $D_{\alpha}(E) = m\alpha + n$ the m with minimal module. The deformation from $\mu = 1$ to $\mu = 0$ respects all the gaps of $\mathbb{R}/\Sigma^{\alpha, \mu}$ for α rational and consequently one can use P4 for the computation of the integrated density of states. We tend to the situation where the gaps are concentrated around the points of intersection of the line : $\alpha = p/q$ and of the graphs represented in picture n° 11. This picture is sent into picture n° 12 by the map $A \cos$. At this point of intersection the pairs (m, n) such that : $D_{\alpha}(E) = m + n\alpha$, are represented by the lines : $x = m\alpha + n$ and the canonical pair corresponds to the line with minimal slope. According to P3 the knowledge of the canonical pairs (m, n) corresponding to the different gaps in $\mathbb{R}/\Sigma^{\alpha, \mu}$ for some $\alpha = \alpha_0 = 1/(q_0 + 1/(q_1 + (q_{n-1} + 1/q_n)))$ allows to describe in a precise way how each of these bands is split when we consider : $\alpha = 1/(q_0 + 1/(q_1 + (q_{n-1} + 1/(q_n + 1/q))))$ at least for large $|q|$. We get the following rule.

Rule 1. Each band of $\Sigma^{\alpha_0, \mu}$ splits into $|q| + \varepsilon_{\ell}$ bands where :

$$\varepsilon_{\ell} = (q/|q|) \nu_{\ell} \quad \text{and} \quad \nu_{\ell} = 0, 1 \quad \text{ou} \quad -1.$$

ν_{ℓ} (which is independent of q) can be easily computed from the pair (m, n) . It can also be expressed as an integer related to the computation of a Chern class (cf. [11] or [31]). This explains the integer character and many properties of stability by deformation but does not seem to give a way to compute the integrated density of states explicitly. We shall see many examples in the different pictures which we shall present later.

Picture n° 13. Spectrum for α near 0.

We make a left zoom of the spectrum near 0 (see picture n° 5). μ is equal to 0.125 but we stay in a region where α is small with respect to μ and this is the semi-classical behavior with respect to α which permits to understand the picture. Here we have considered the part of the spectrum which is between -2.25 and -1.5 and the rationals α whose denominators q satisfies : $10 \leq q \leq 55$ and $0 \leq \alpha \leq 0.1$.

Picture n° 14. Spectrum for α near 1.

We take $\mu = 1$, $\alpha = 1/(1 + 1/q)$ with $2 \leq q \leq 80$ and $0.93750 \leq \alpha \leq 1$. We observe the spectrum in the intervall $(-0.5, 0.5)$. This picture describes partly the structure of the spectrum in the middle. The reader can see the difference between the case q odd (corresponding to an even denominator !) where the central band must be thought as a union of two touching bands and the case q even (odd denominator), where the central band is an isolated band which is symmetric around 0.

The study of the spectrum near 0, which corresponds to a saddle point of the hamiltonian $\cos x + \cos \xi$ associated to the Harper's operator (whose spectrum is the same as the spectrum of the pseudodifferential operator on $L^2(\mathbb{R})$: $\cos x + \cos(2\pi\alpha D_x)$) is realized carefully in [21] (see also the old paper by Azbel [24]). One of the difficulties is that the energy surface corresponding to $E = 0$ is no more a union of connected compact components

(in the opposite of the case when $E \neq 0$), and locally we have to solve some microlocal branching problem.

Picture n° 15. Spectrum for α near 1 (bis).

It has the same characteristics as the picture n° 14 but we consider all the α with : $\alpha = p/q$ with $2 \leq q \leq 80$ and $0.9375 \leq \alpha \leq 1$.

Picture n° 16. Spectrum for α near $1/2$.

In this picture we intend to show the asymptotic behavior as α tends to $1/2$. For that we consider the following sequence of $\alpha : \alpha = 1/(2 + 1/q)$ with $-30 \leq q \leq 30$, $0.39 \leq \alpha \leq 0.61$. μ is equal to 1. The picture is symmetric with respect to the line $E = 0$ and $\alpha = 1/2$. For $\alpha = 1/2$, the spectrum is the union of two touching bands. Each of the two bands is the set generated by one of the eigenvalues of a family of 2×2 matrices depending periodically of the two Floquet parameters : x, ξ :

$$\begin{pmatrix} \cos \xi & \cos x \\ \cos x & -\cos \xi \end{pmatrix} .$$

For α rational near $1/2$, the bands constituting Σ^α are concentrated around lines or parabolic paths tending as α tends to $1/2$ to the ends of the two bands corresponding to $\alpha = 1/2$. In fact, the problem is reduced in this case to the semi-classical analysis of the pseudo-differential 2×2 system :

$$\begin{pmatrix} \cos hD_x & \cos x \\ \cos x & -\cos hD_x \end{pmatrix}$$

with $h/2 \pi = \alpha - \frac{1}{2}$.

All these properties are rigorously studied in [13] and [11].

Picture n° 17. Spectrum for α between $1/2$ and $2/3$.

We take $\mu = 1$, α rational with denominator less than 50 and $1/2 \leq \alpha \leq 2/3$. This is a zoom corresponding to the left part of the spectrum that is $-3 \leq E \leq -1$. The bands corresponding to $1/2$ and $2/3$ are easily guessed. If one compares with picture n° 1, one observes the recursive structure of the picture. One observes also the same type of asymptotic behavior of the bands near $1/2, 2/3$ and $3/5$ (where two bands are visible). This is studied in detail in [14] and [11].

Picture n° 18. Spectrum for α around $1/3$.

$\mu = 1$. We consider the rationals of denominator less than 55 with $0.266 \leq \alpha \leq 0.401$. Once again the asymptotic behavior of the spectrum as α tends to $1/3$ is organized around the three bands. The bands are concentrated around half-lines starting from the ends of the bands corresponding to $\alpha = 1/3$. The slope of these half lines (corresponding to $\alpha < 1/3$ and $\alpha > 1/3$) are not necessarily symmetric with respect to the line $\alpha = 1/3$. This effect is studied in [11] and [14] in relation with the Rammal-Wilkinson formula (see [12]).

The next three pictures try to explain the structure around $1/(3 + 1/3)$.

Picture n° 19. Spectrum for α around $1/(3 + 1/3)$: overall view.

We consider the α of the form $1/(3 + 1/(3 + 1/q))$ with $\{-12 \leq q \leq -3\} \cup \{q = +\infty\} \cup \{2 \leq q \leq 12\}$. We can describe the spectrum corresponding to α by analyzing its expansion as a continuous fraction. The spectrum corresponding to $1/3$ has 3 bands. The spectrum corresponding to $1/(3 + 1/3)$ has ten bands corresponding to the repartition $3 + 4 + 3$ (downwards). We have (with the notation preceding the picture n° 13) : $\nu_1 = 0, \nu_2 = 1, \nu_3 = 0$.

As $q = -3$ (corresponding to the last vertical line on the left), the counting of the bands

downwards, which can be made with help of pictures n° 20 and 21 gives that the corresponding indices $\nu_{j,k}$ associated to $1/(3 + 1/3)$ are given by :

$$\nu_{1,1} = 0, \quad \nu_{1,2} = 1, \quad \nu_{1,3} = 0; \quad \nu_{2,1} = 0, \quad \nu_{2,2} = 1, \quad \nu_{2,4} = 0; \\ \nu_{3,1} = 0, \quad \nu_{3,2} = 1, \quad \nu_{3,3} = 0$$

$\nu_{2,2}$ is an index corresponding to the double band. This gives the following repartition :

$$((3 + 2 + 3) + (3 + 5 + 3) + (3 + 2 + 3)) = 10|q| - 3 = 27$$

according to the rule preceding picture n° 13.

Picture n° 20. Spectrum for α around $1/(3 + 1/3)$, upper zoom.

We amplify picture n° 19 for : $-3 < E < -1.5$. For $q = -3$, the band (1,2) splits in two subbands which seem to touch together. By examination of $q = -4$, where you can see 3 subbands, we can be sure that the two subbands are distinct for $q = -3$.

Picture n° 21. Spectrum for α around $1/(3 + 1/3)$, central zoom.

We amplify picture n° 19 for : $-1 < E < +1$.

Let us now consider the spectrum for α around $1/(3 + 1/2)$.

Picture n° 22. Spectrum for α around $1/(3 + 1/2)$, global view.

We consider the α of the form $1/(3 + 1/(2 + 1/q))$ with $\{-12 \leq q \leq -5\} \cup \{q = +\infty\} \cup \{5 \leq q \leq 12\}$. We can describe the spectrum corresponding to α by analyzing its expansion as a continuous fraction. The spectrum corresponding to $1/3$ has 3 bands. The spectrum corresponding to $1/(3 + 1/2)$ has 7 bands corresponding to the repartition $2 + 3 + 2$ (downwards). We have (with the notation preceding the picture n° 13) : $\nu_1 = 0, \nu_2 = 1, \nu_3 = 0$.

As $q = -5$ (corresponding to the last vertical line on the left), the counting of the bands downwards, which can be made with help of pictures 23 and 24 gives that the corresponding indices $\nu_{j,k}$ associated with $1/(3 + 1/2)$ are given by :

$$\nu_{1,1} = 0, \quad \nu_{1,2} = 1; \quad \nu_{2,1} = 0, \quad \nu_{2,2} = 1, \quad \nu_{2,3} = 0; \quad \nu_{3,1} = 0, \quad \nu_{3,2} = 1.$$

This corresponds to the repartition : $((5 + 4) + (5 + 4 + 5) + (4 + 5)) = 7|q| - 3 = 32$ according to the rule preceding picture n° 13. Let us observe that the two bands in the middle touch actually because the denominator of α is even for $q = -5$.

Picture n° 23. Spectrum for α around $1/(3 + 1/2)$, upper zoom.

We amplify the picture n° 22 for : $-2.75 < E < 2.4$. For $q = \infty$, one can see two bands almost touching (this is the beginning of the semiclassical behavior for α of the form $\alpha = 1/(n + 1/2)$ with large n , here $n = 3$).

Picture n° 24. Spectrum for α around $1/(3 + 1/2)$, central zoom.

We amplify the picture n° 22 for : $-1 < E < +1$.

Picture n° 25. The transition between $1/(3 + 1/2)$ and $1/(3 + 1/3)$.

We present the spectrum for $0.285 \leq \alpha \leq 0.305$, $\alpha = 1/(3 + 1/(2 + 1/q))$ or $1/(3 + 1/(3 - 1/q))$ with $1 \leq q \leq 20$.

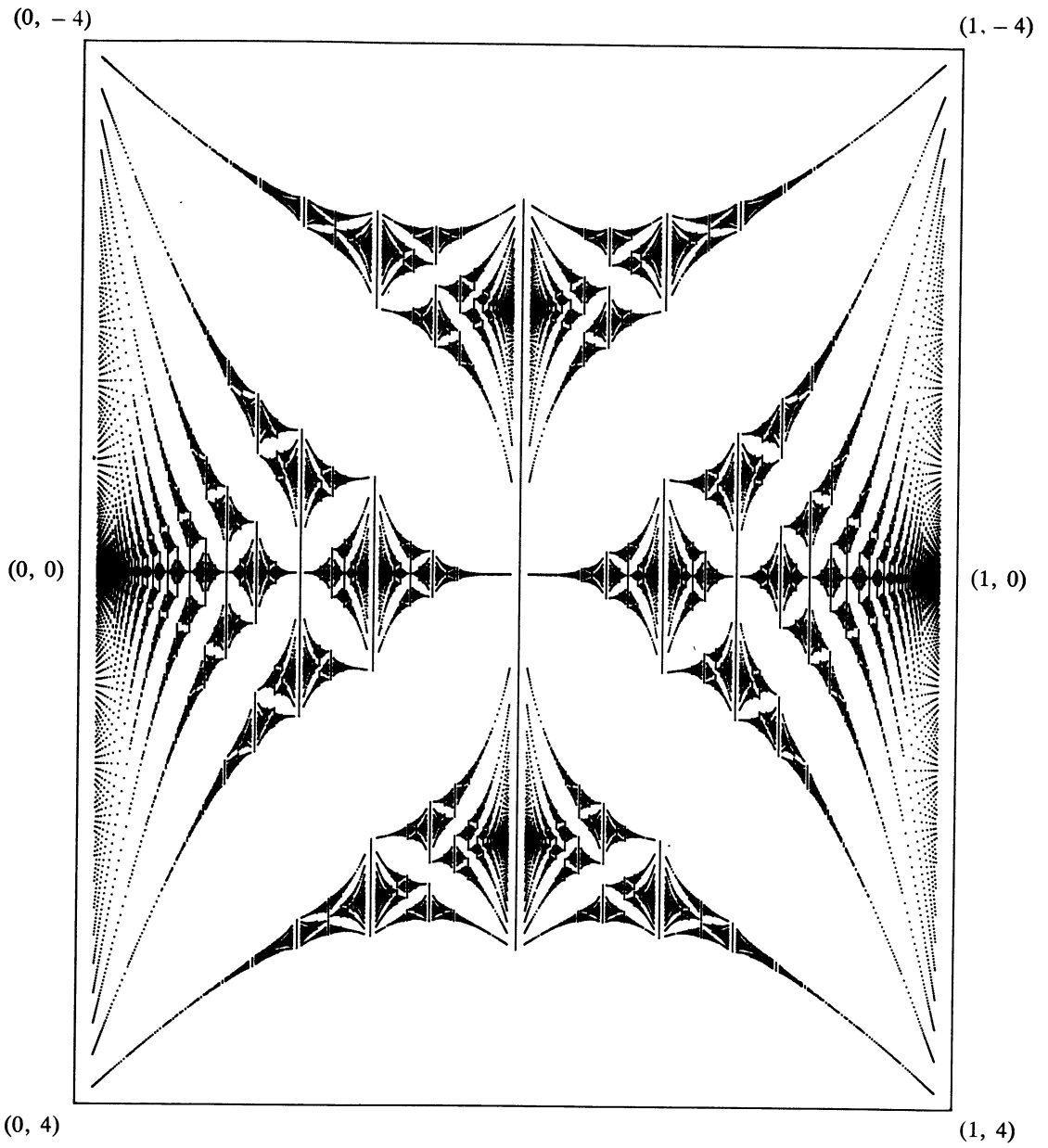
Picture n° 26. Study of the sequence $\alpha = 1/(q + 1/2)$.

We present the spectrum for $0 < \alpha < 0.3$, $\alpha = 1/(q + 1/2)$. The spectrum appears as a union of packets of two subbands which almost touch. This corresponds to the fact that, in the renormalization procedure described (and proved) in [13], the spectrum is given near each semiclassical eigenvalue corresponding to $\alpha_0 = 1/q$ (and far from $E = 0$), up to some dilation and translation, by the spectrum of (a tiny perturbation of) an Harper's operator,

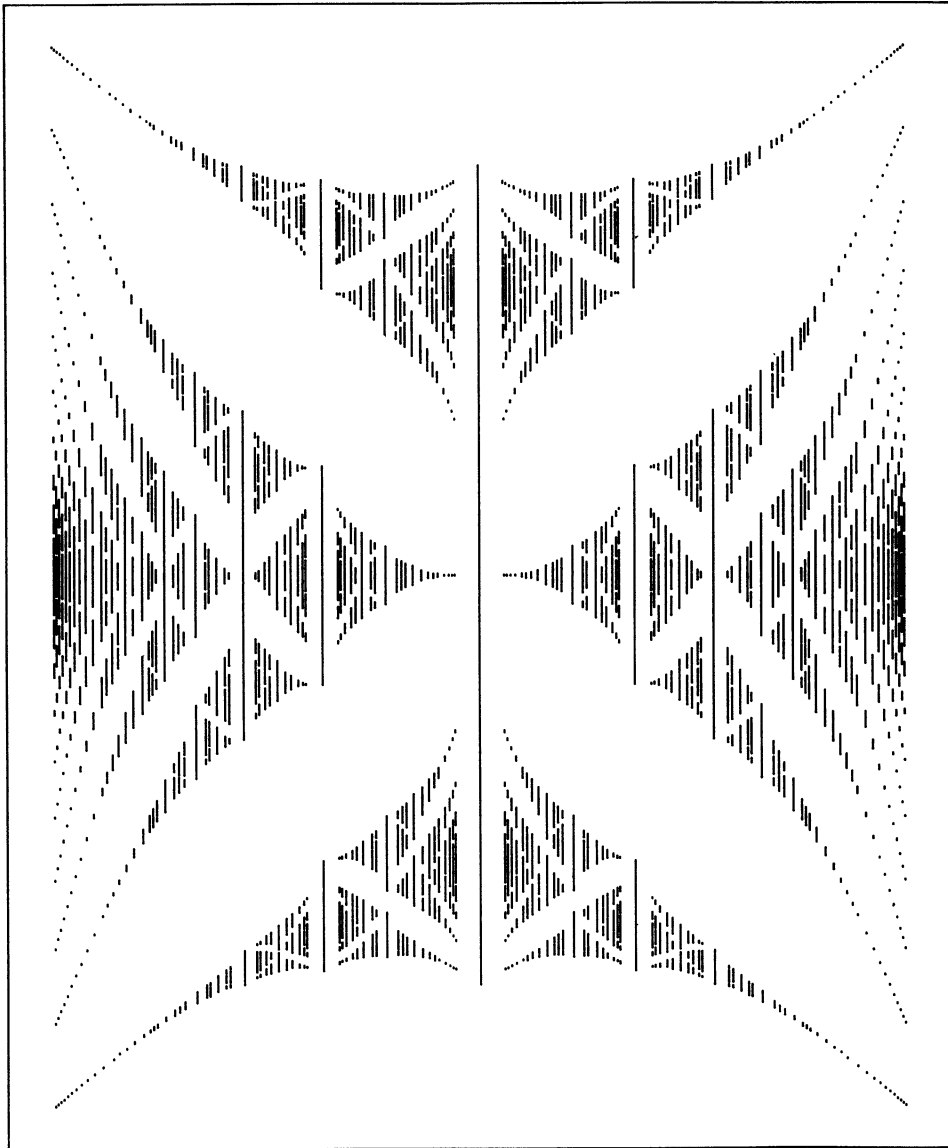
corresponding to a new parameter $\alpha' = 1/2$. This « tiny perturbation » is quite important because according to [22] and [23], there exists effectively $2q + 1$ disjoint bands. Without the perturbation, we should have observed $q - 1$ double bands. Let us observe that according to [23] the distance between these two subbands is minorized by $8^{-(2q+1)}$, which is compatible as q tends to ∞ with the majoration of the perturbation (related to a measure of the tunneling effect) deduced from the results of [13]. It could be interesting to try to optimize the two different methods to get simultaneously better minoration and majoration. We have limited the numerical computations to $2 \leq q \leq 20$ and we loose some bands as explained in section 1. The picture is limited to $E < 0$.

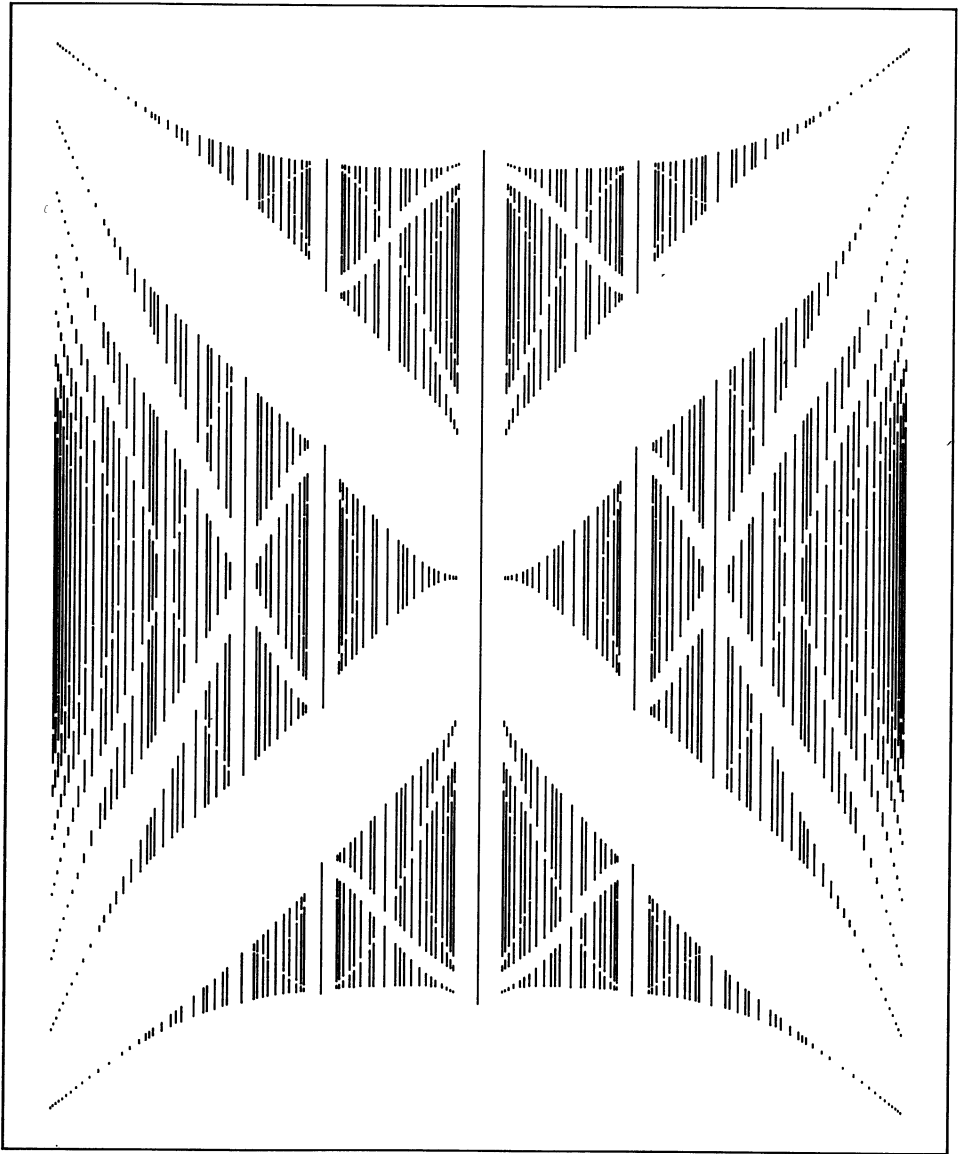
Picture n° 27. Study of the sequence $\alpha = 1/(q + 1/3)$.

We present the spectrum for $0 < \alpha < 0.33$, $\alpha = 1/(q + 1/3)$. The spectrum appears as a union of packets of 3 subbands which almost touch, but now the size of the gaps is of the same order as the length of the subbands. This corresponds to the fact that, in the renormalization procedure described in [13] and recalled in section 1, the spectrum is given near each semiclassical eigenvalue corresponding to $\alpha_0 = 1/q$ (and far from $E = 0$), up to some dilation and translation, by the spectrum of (a tiny perturbation of) an Harper's operator, corresponding to a new parameter $\alpha' = 1/3$. But this operator has now 3 distinct bands. Here we have $3 \leq q \leq 30$ and we present $E < 0$.

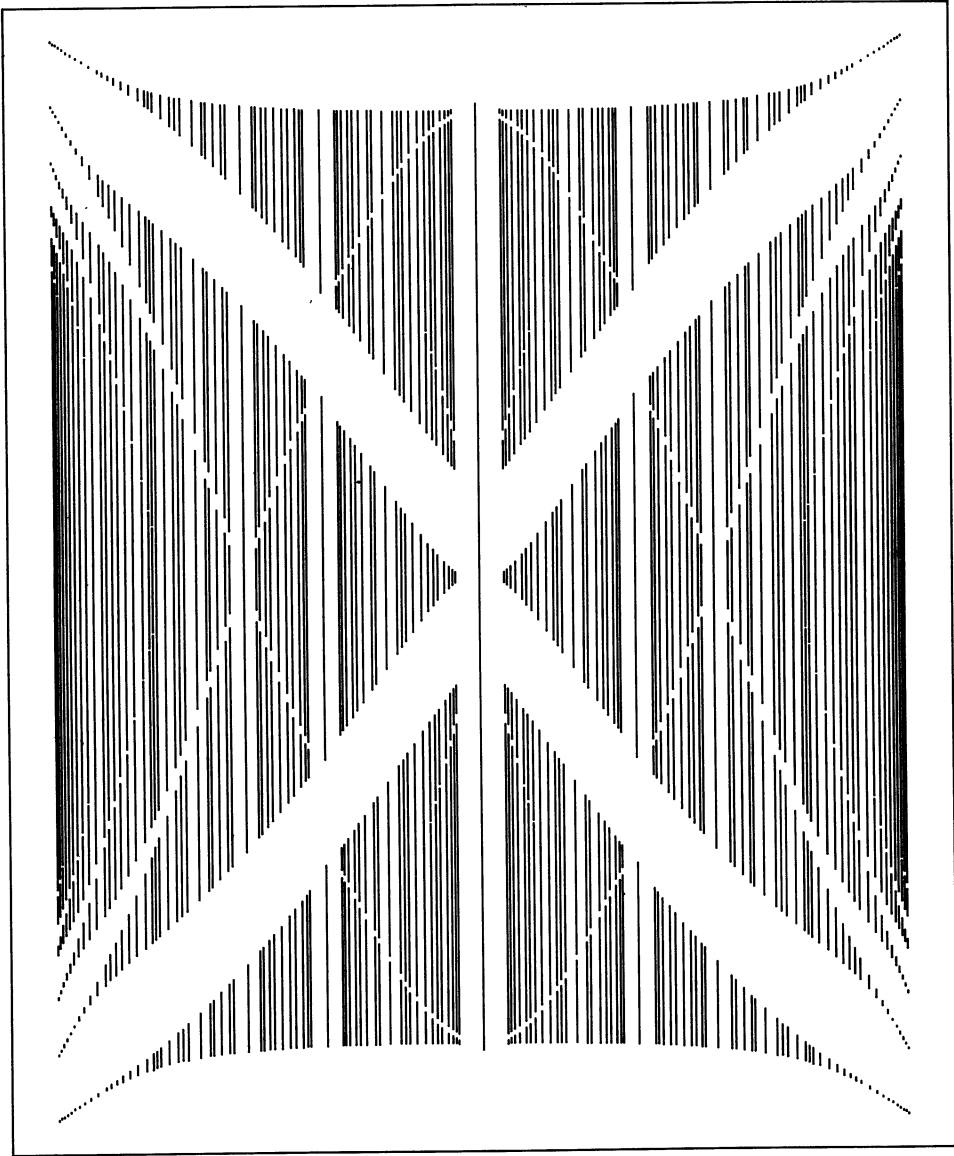


Picture 1. — $\mu = 1$; the Hofstadter's butterfly.

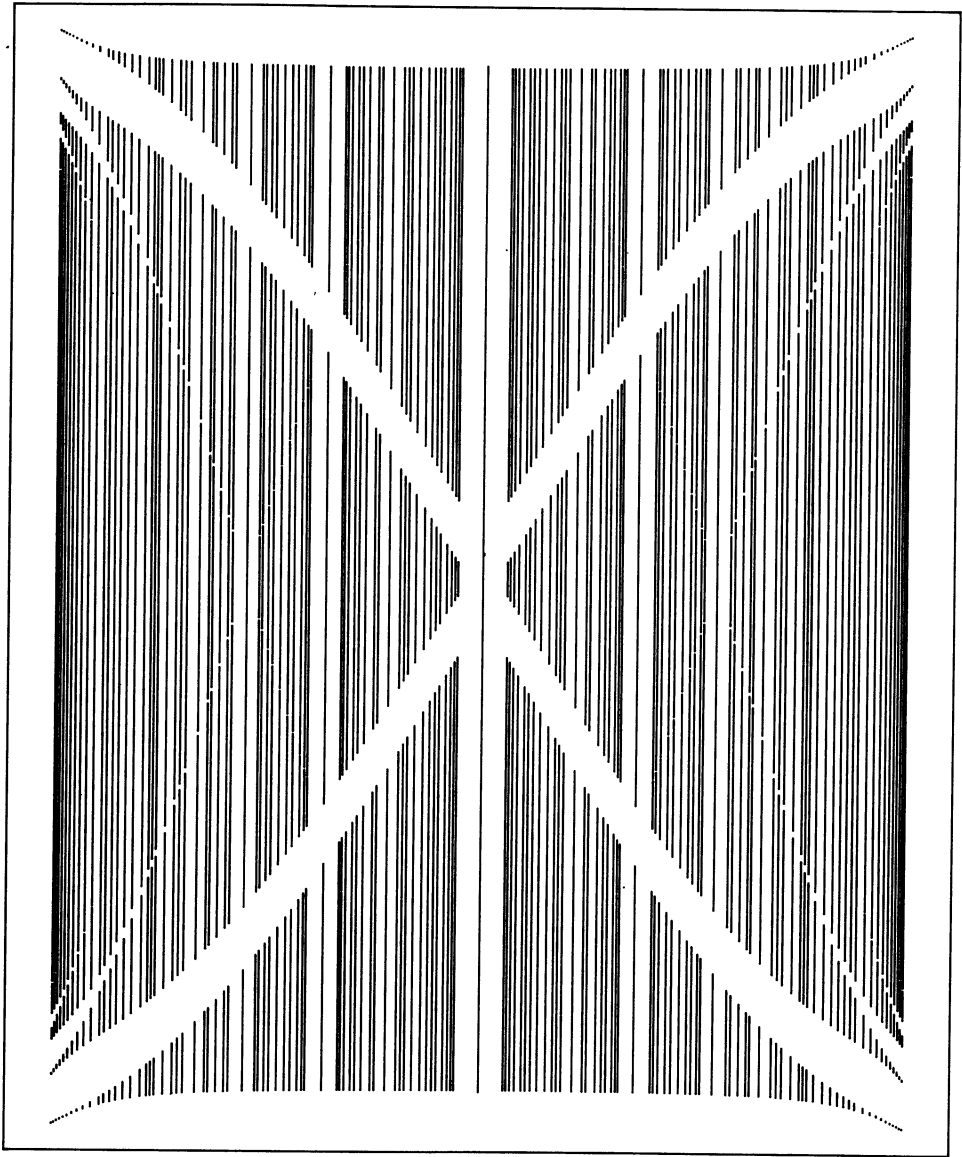
$(0, -3.5)$ $(0, 0)$ $(0, 3.5)$ Picture 2. — $\mu = 0.75$.



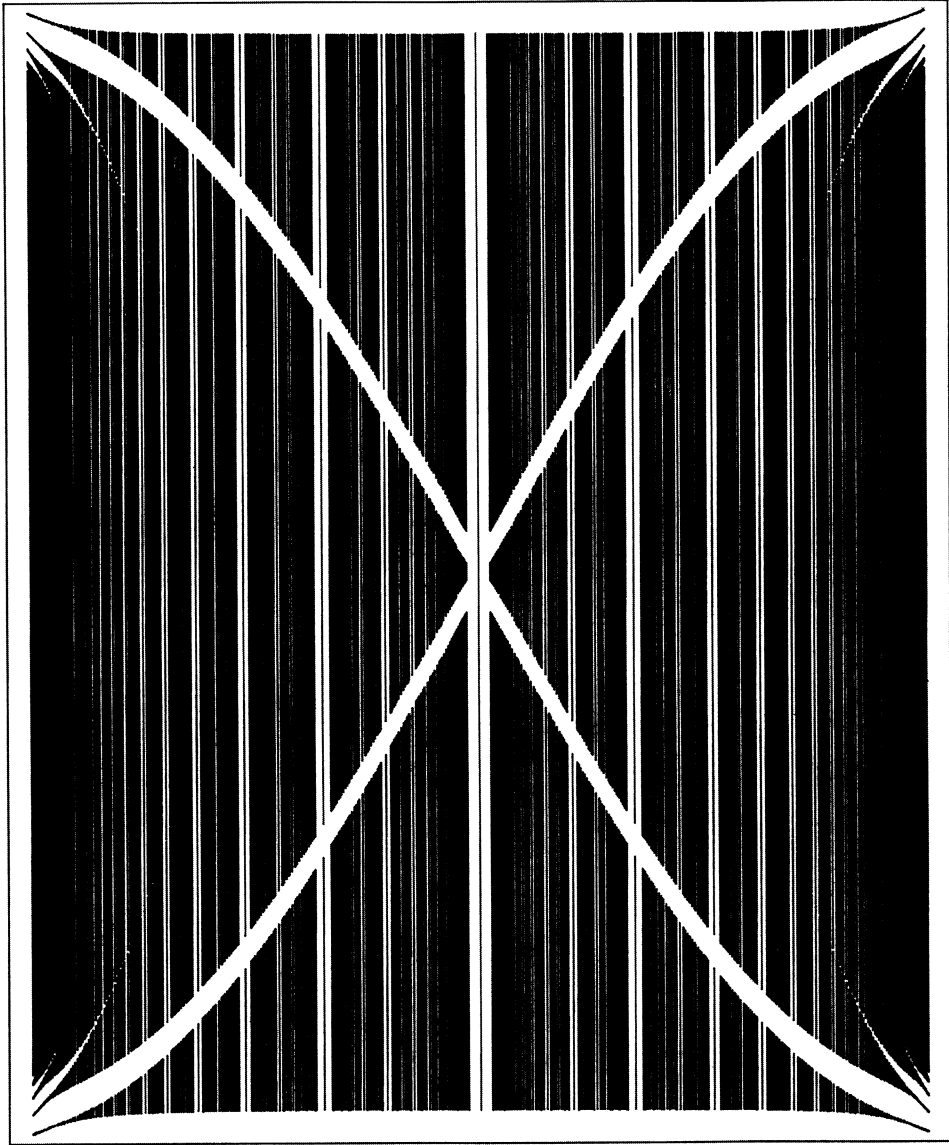
Picture 3. — $\mu = 0.5$.



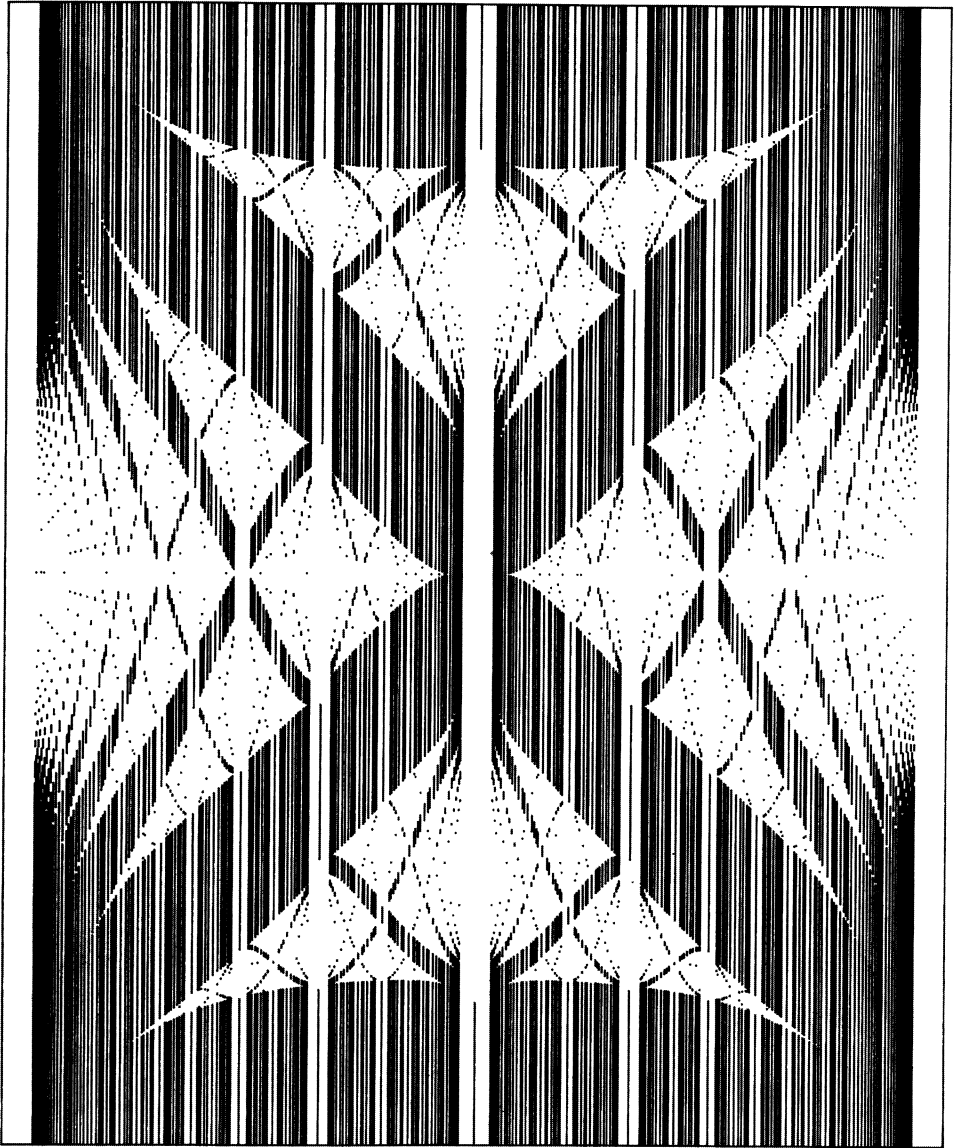
Picture 4. — $\mu = 0.25$.



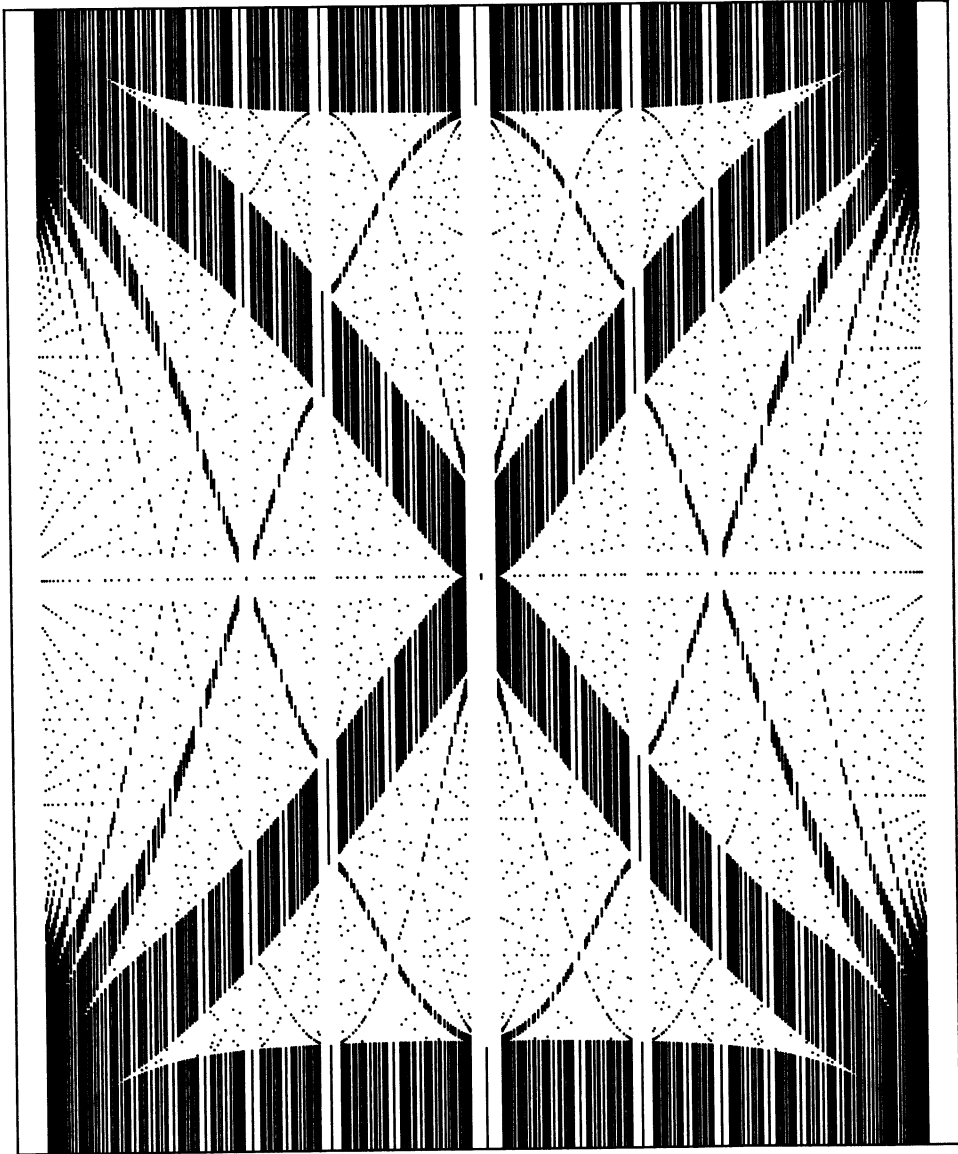
Picture 5. — $\mu = 0.125$.



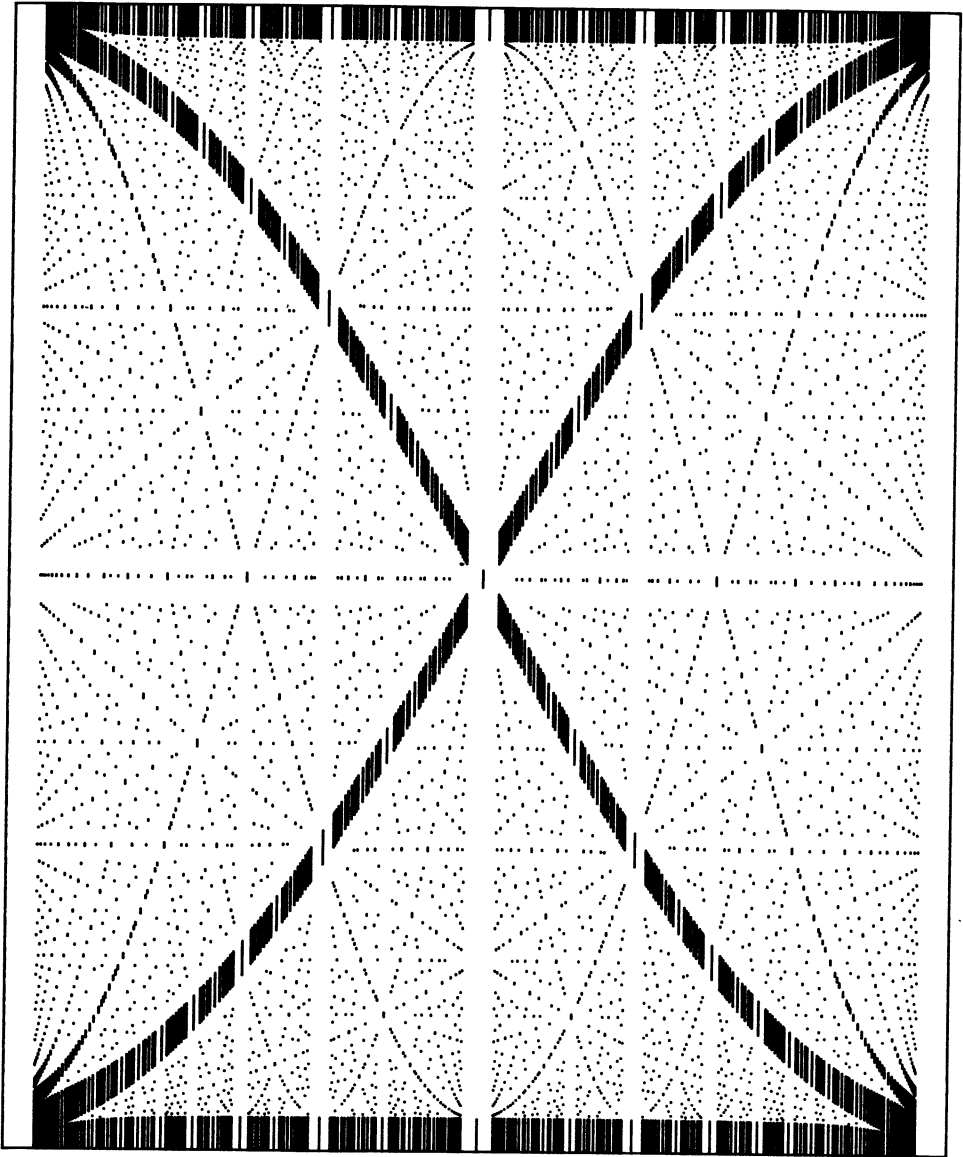
Picture 6. — $\mu = 0.0625$.



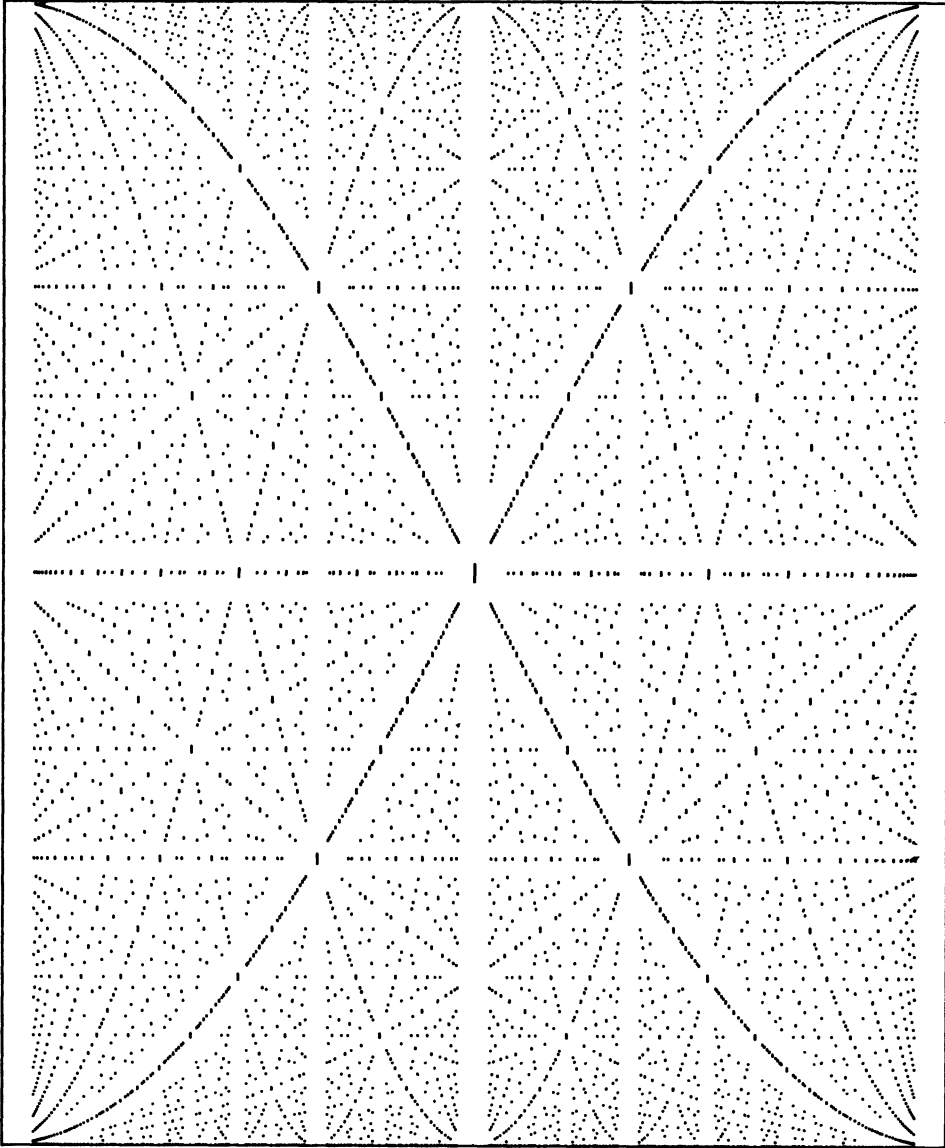
Picture 7. — $\mu = 0.5$, negative.



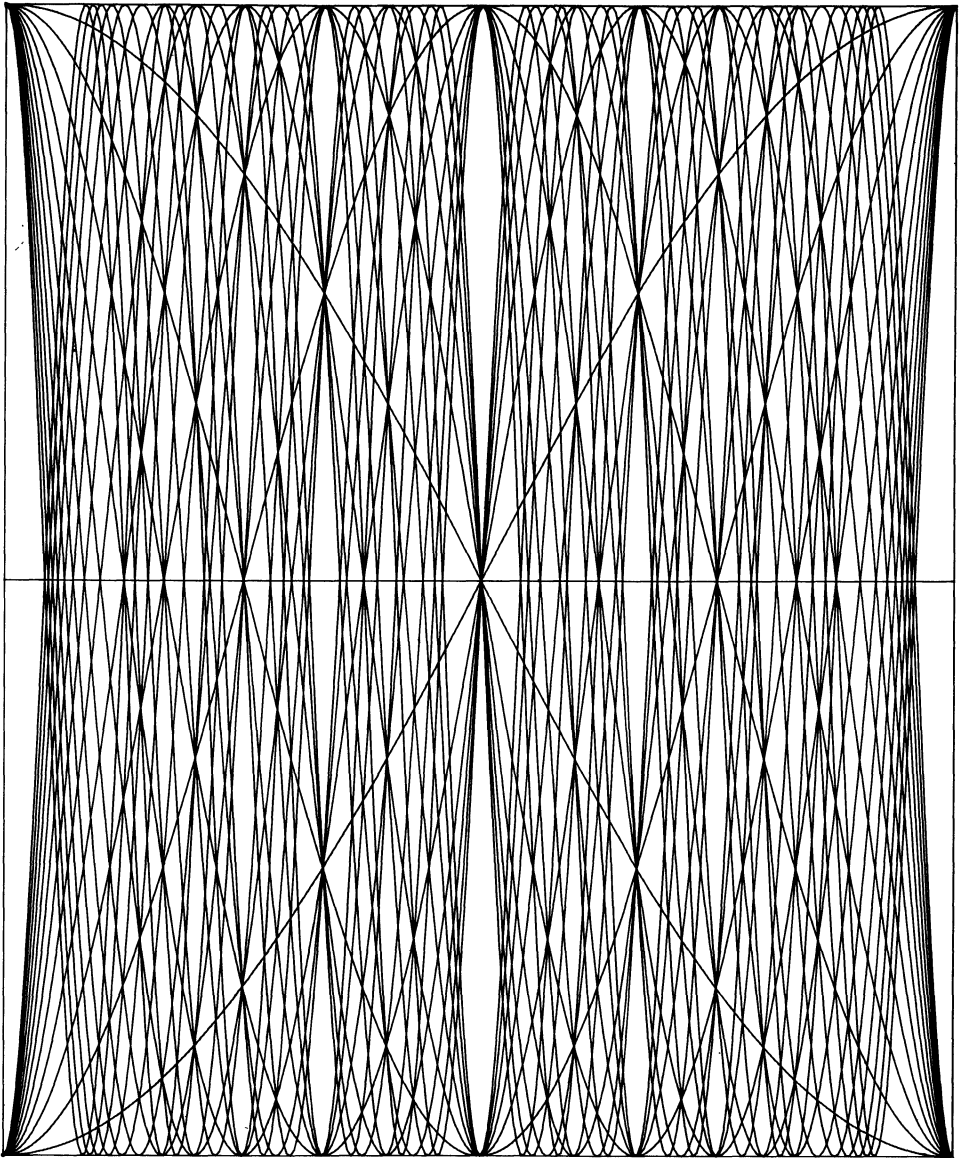
Picture 8. — $\mu = 0.25$, negative.



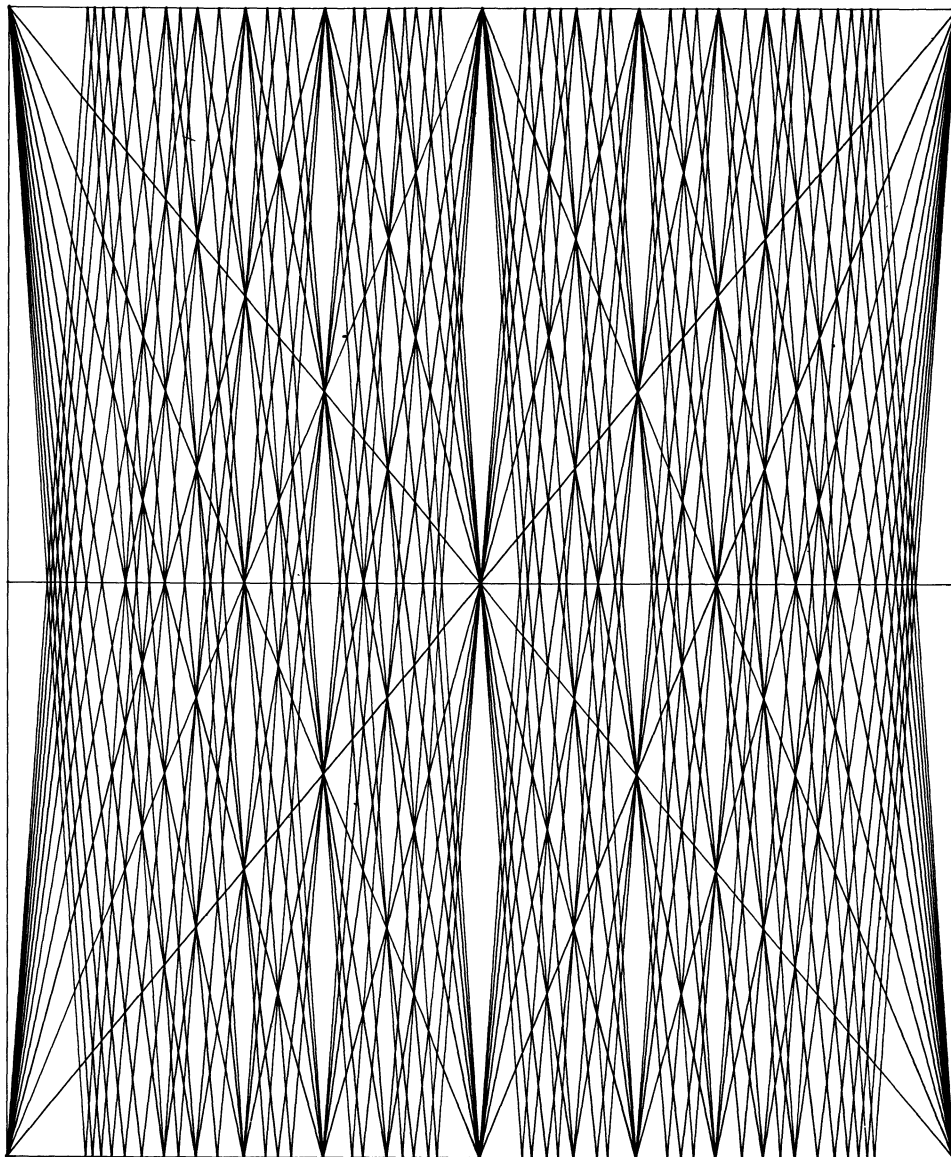
Picture 9. — $\mu = 0.0625$, negative.



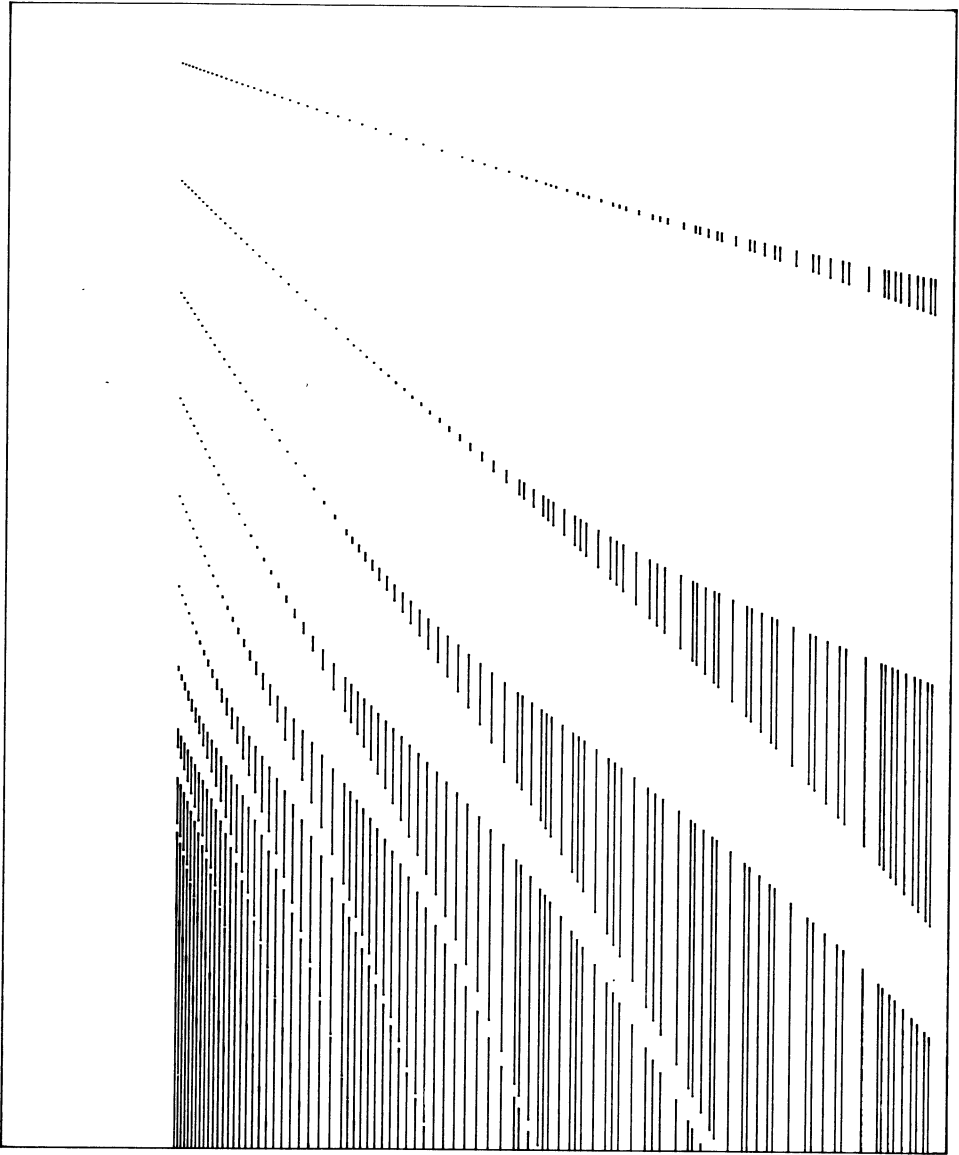
Picture 10. — $\mu = 0.001$, negative.



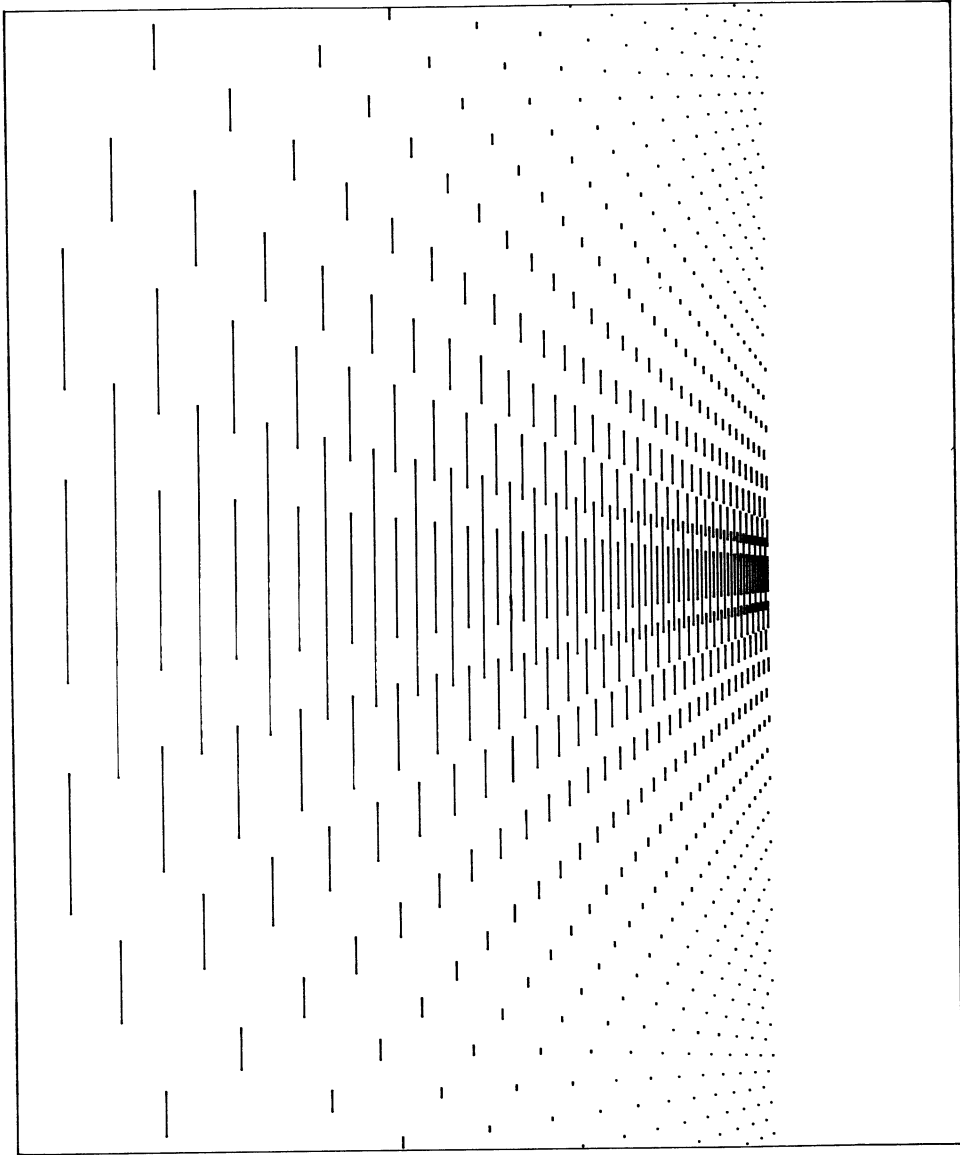
Picture 11. — The limit model, $\mu = 0^+$, $E = \cos(\pi\ell\alpha)$ or $\cos(\pi(1-\ell\alpha))$.



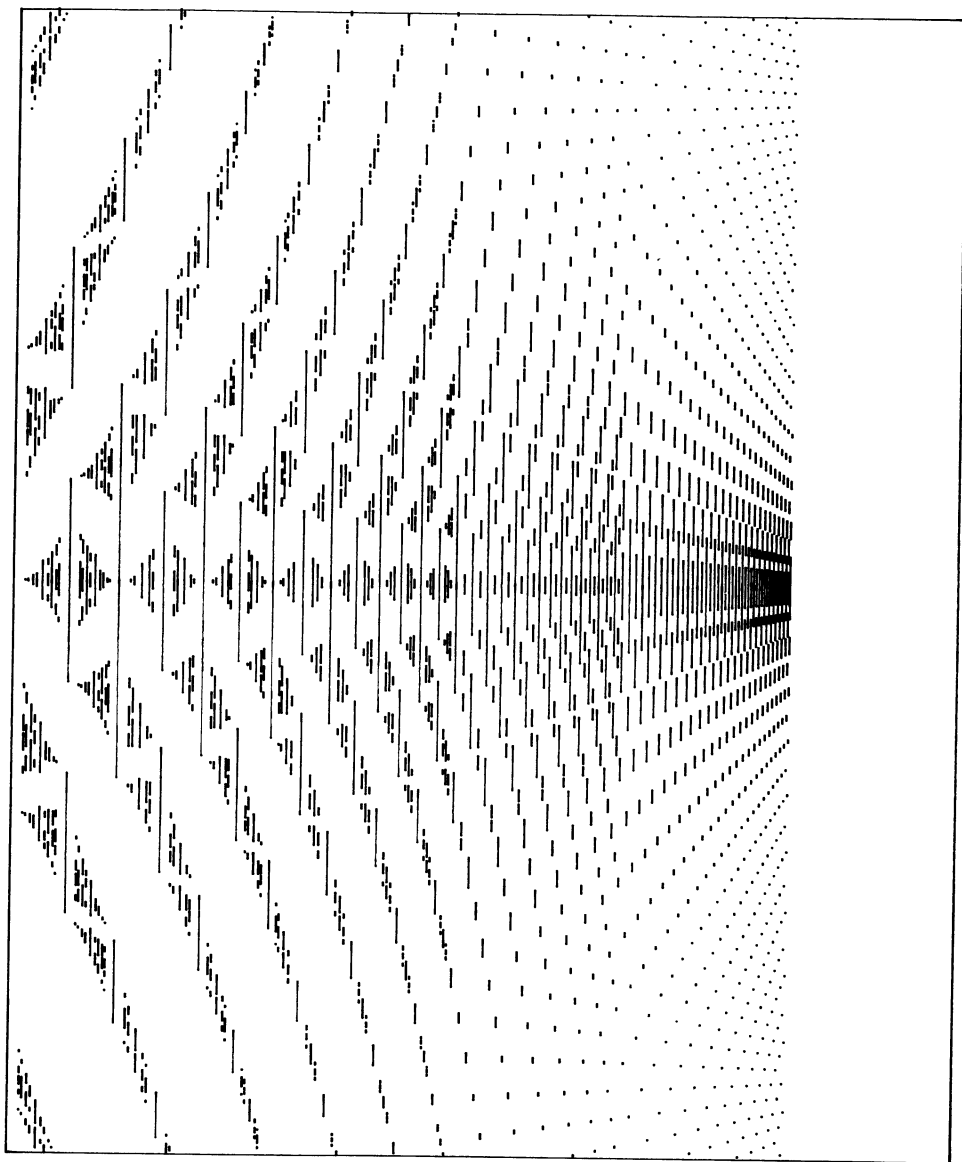
Picture 12. — The Claro-Wannier representation of the integrated density of states.



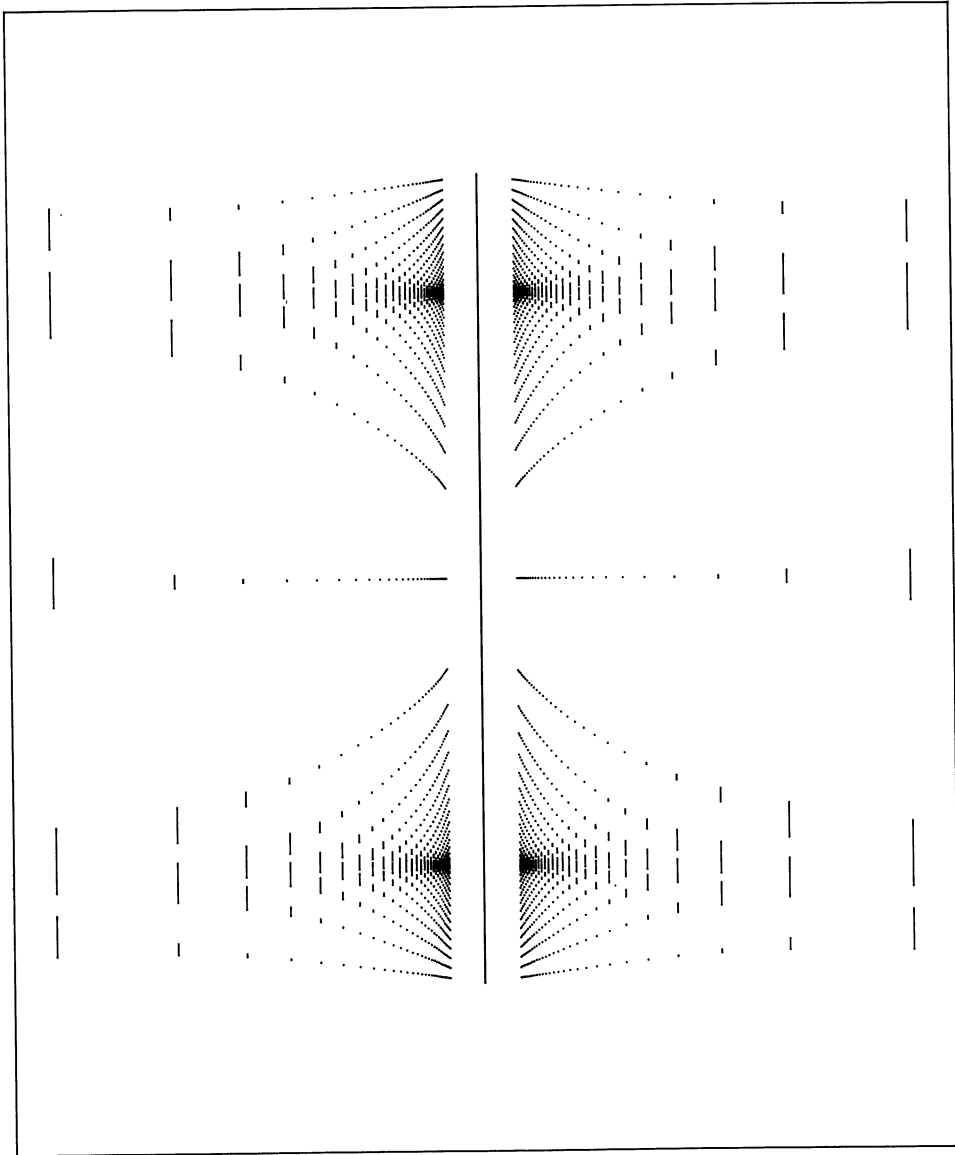
Picture 13. — Spectrum for α near 0, $\mu = 0.125$, zoom in $[0, 0.1] \times [-2.25, -1.5]$.



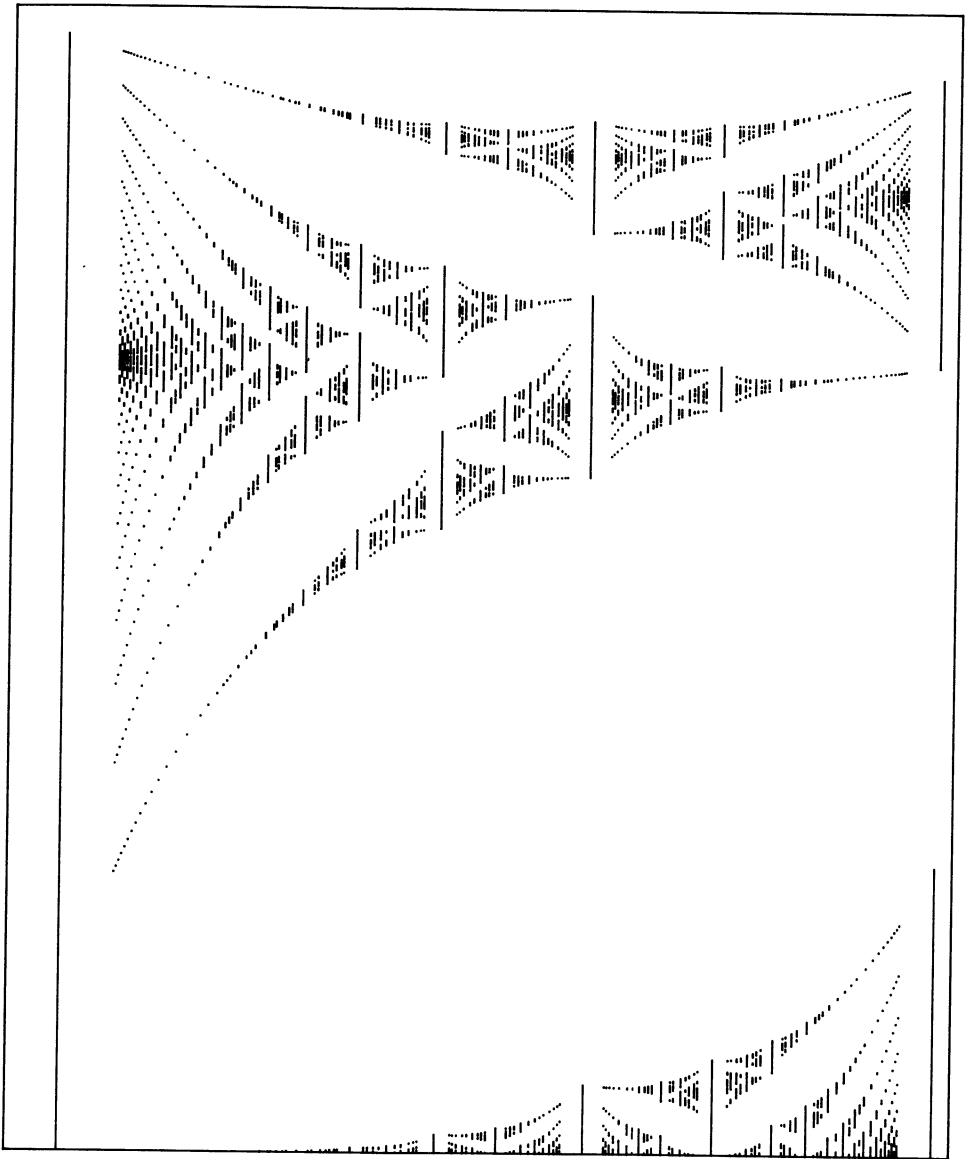
Picture 14. — Spectrum for α near 1, $\mu = 1$, partial sequence $\alpha = q/q + 1$, zoom in $[0.9375, 1] \times [-0.5, 0.5]$.



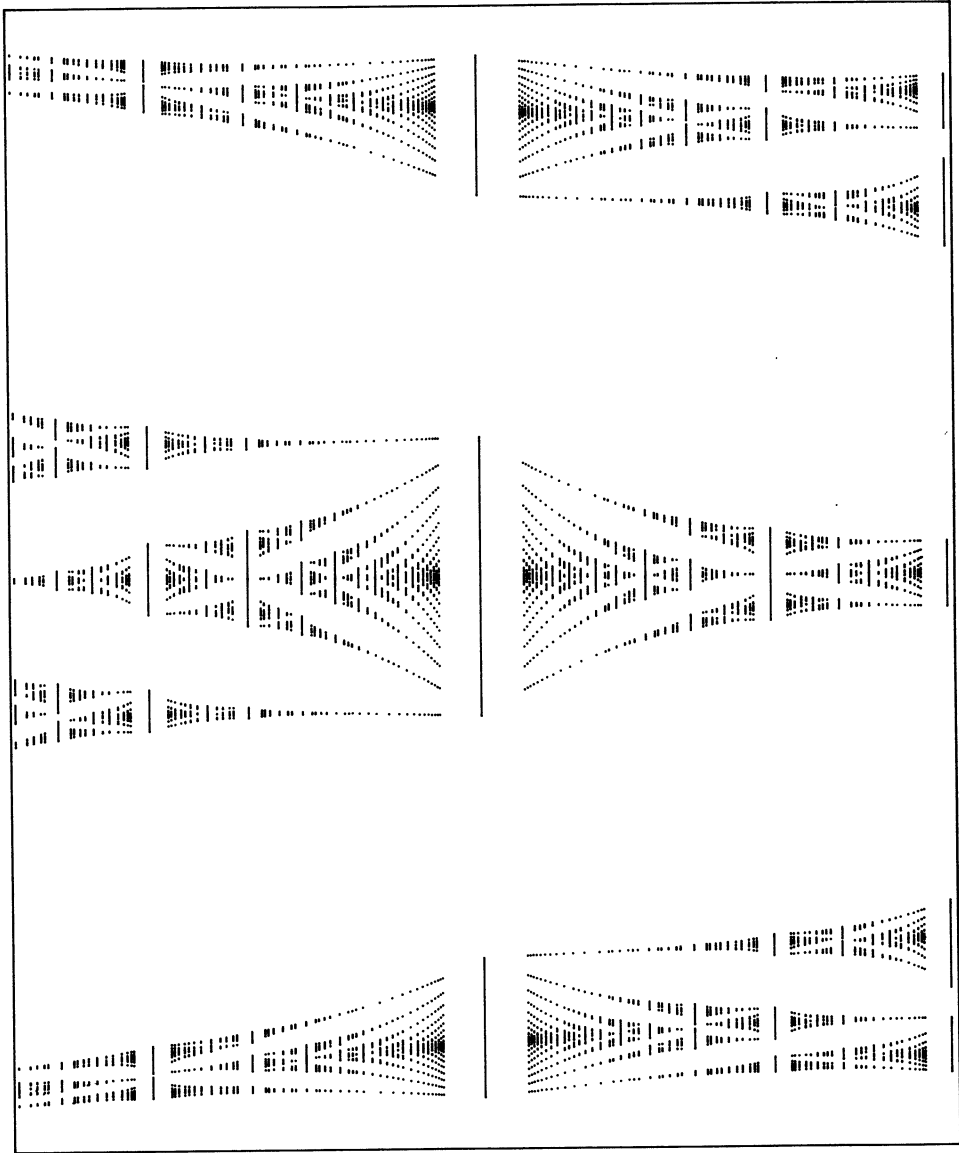
Picture 15. — Spectrum for α near 1, $\mu = 1$, zoom in $[0.9375, 1] \times [-0.5, 0.5]$.



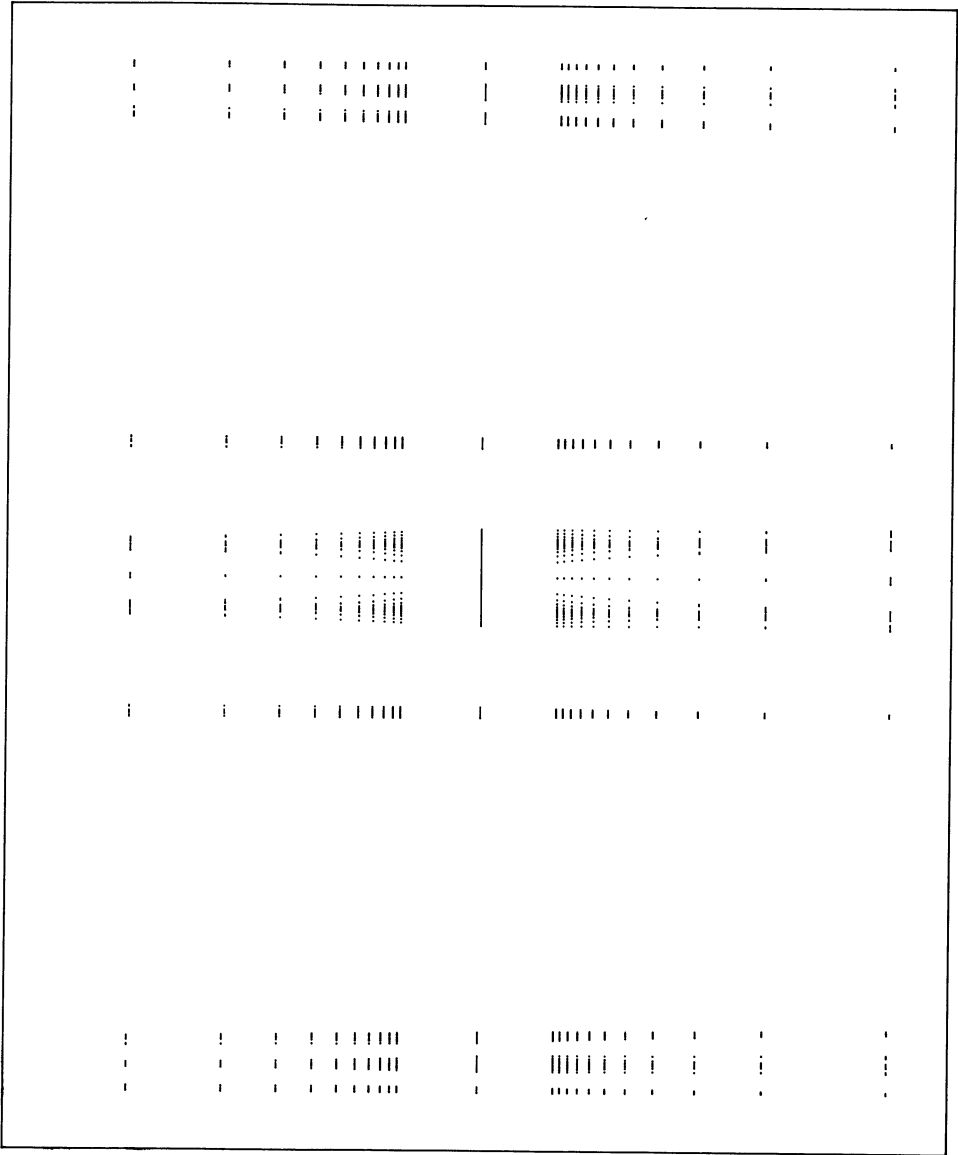
Picture 16. — Spectrum for α near $1/2$, $\mu = 1$, partial sequence $\alpha = q/2q + 1$, zoom in $[0.39, 0.61] \times [-4, 4]$.



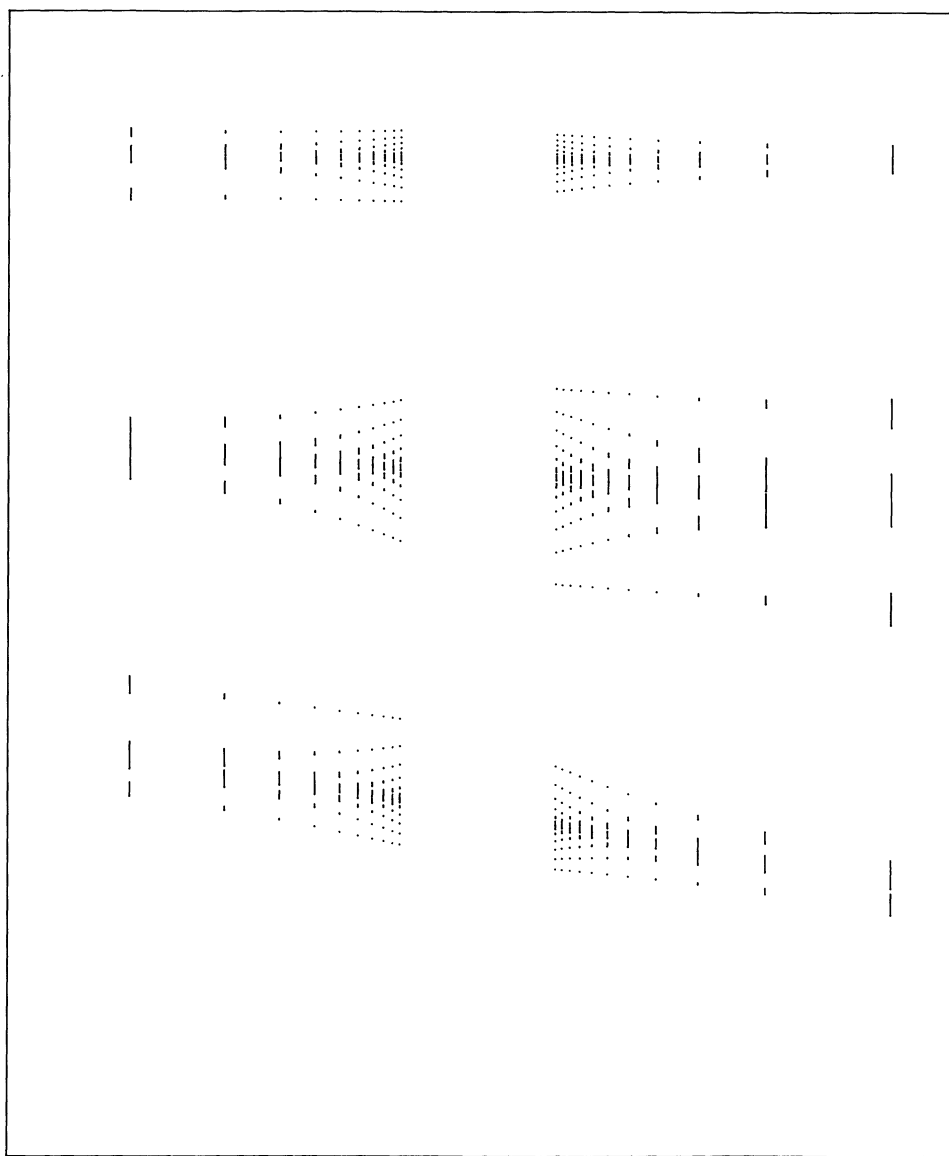
Picture 17. — Spectrum for α between $1/2$ and $2/3$, $\mu = 1$, zoom in $[0.49, 0.667] \times [-2.9, 0]$.



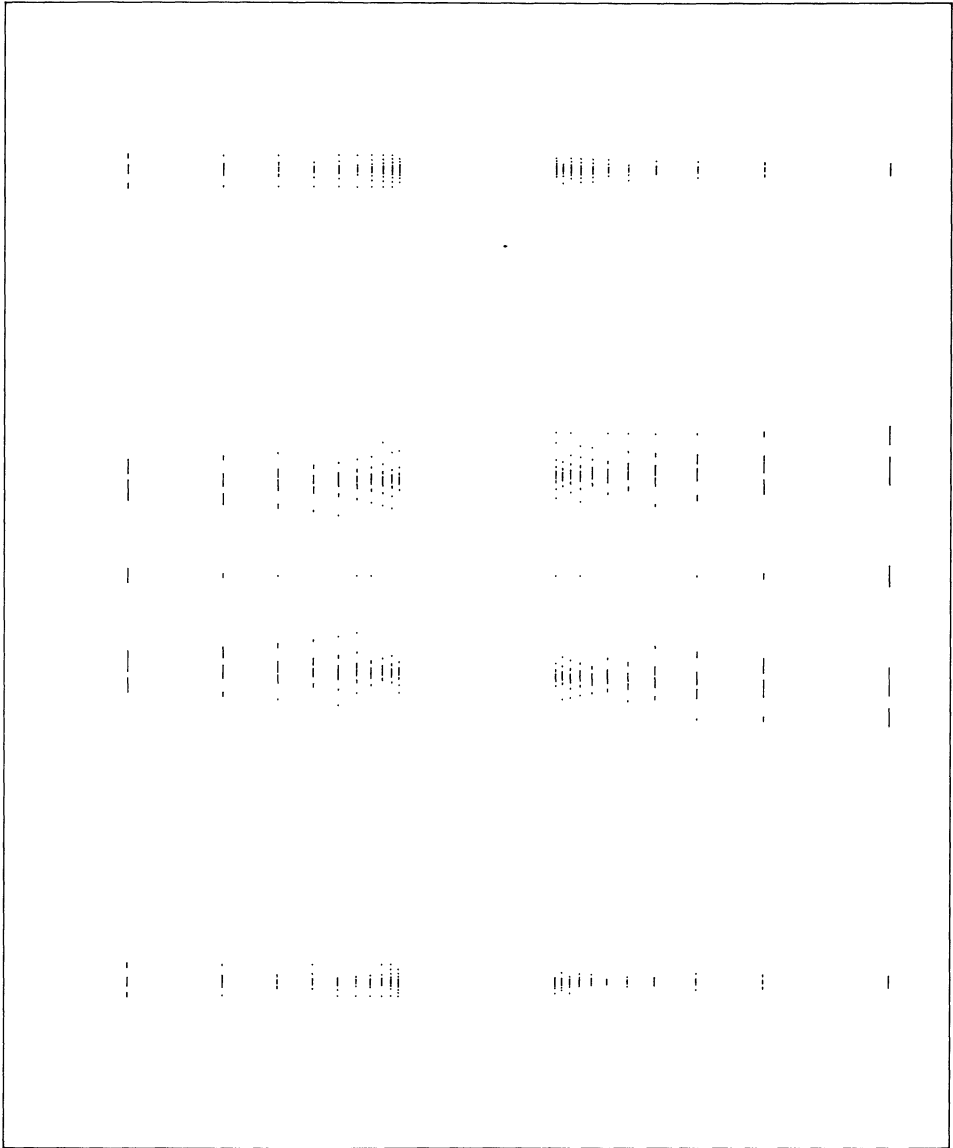
Picture 18. — Spectrum for α near $1/3$, $\mu = 1$, zoom in $[0.266, 0.401] \times [-3, 3]$.



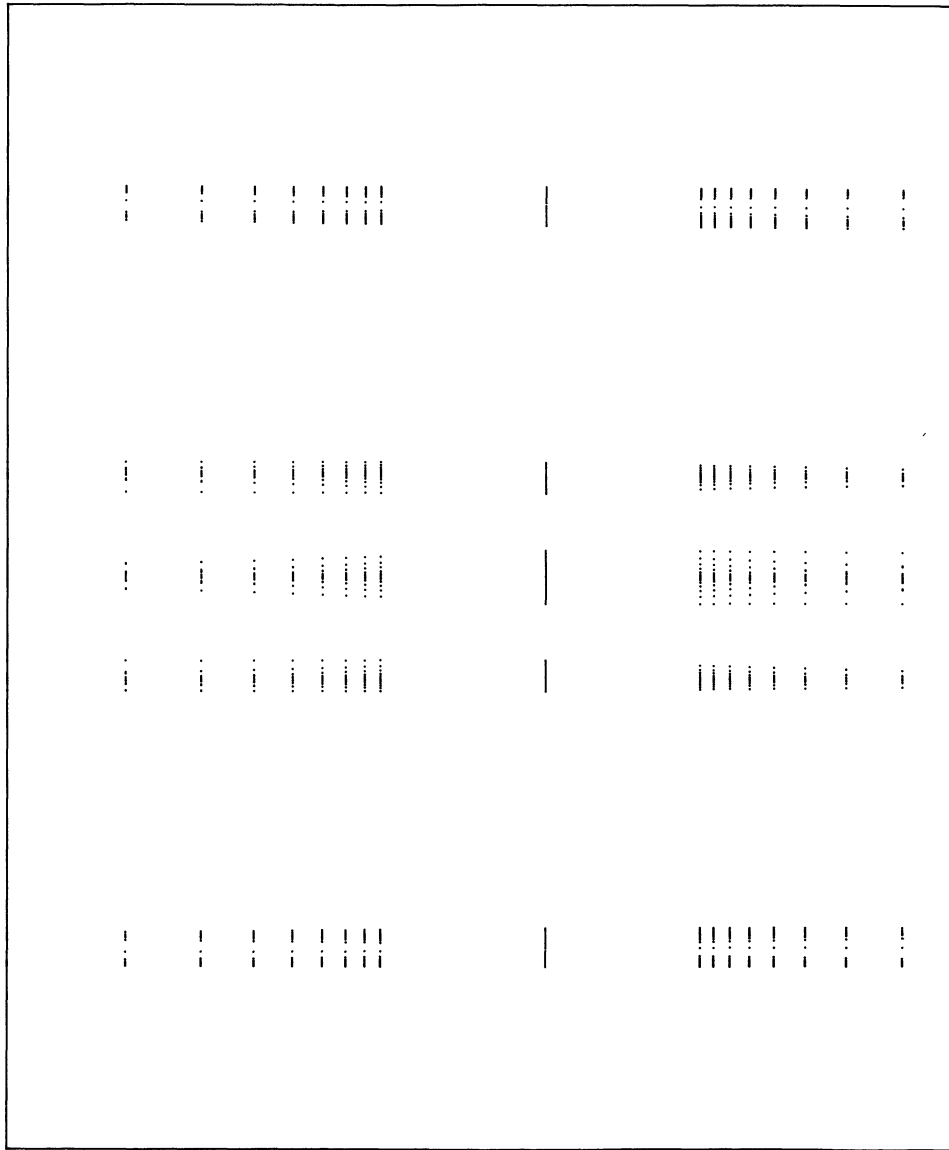
Picture 19. — Spectrum for α near $1/(3 + 1/3)$, global view, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(3 + 1/q))$.



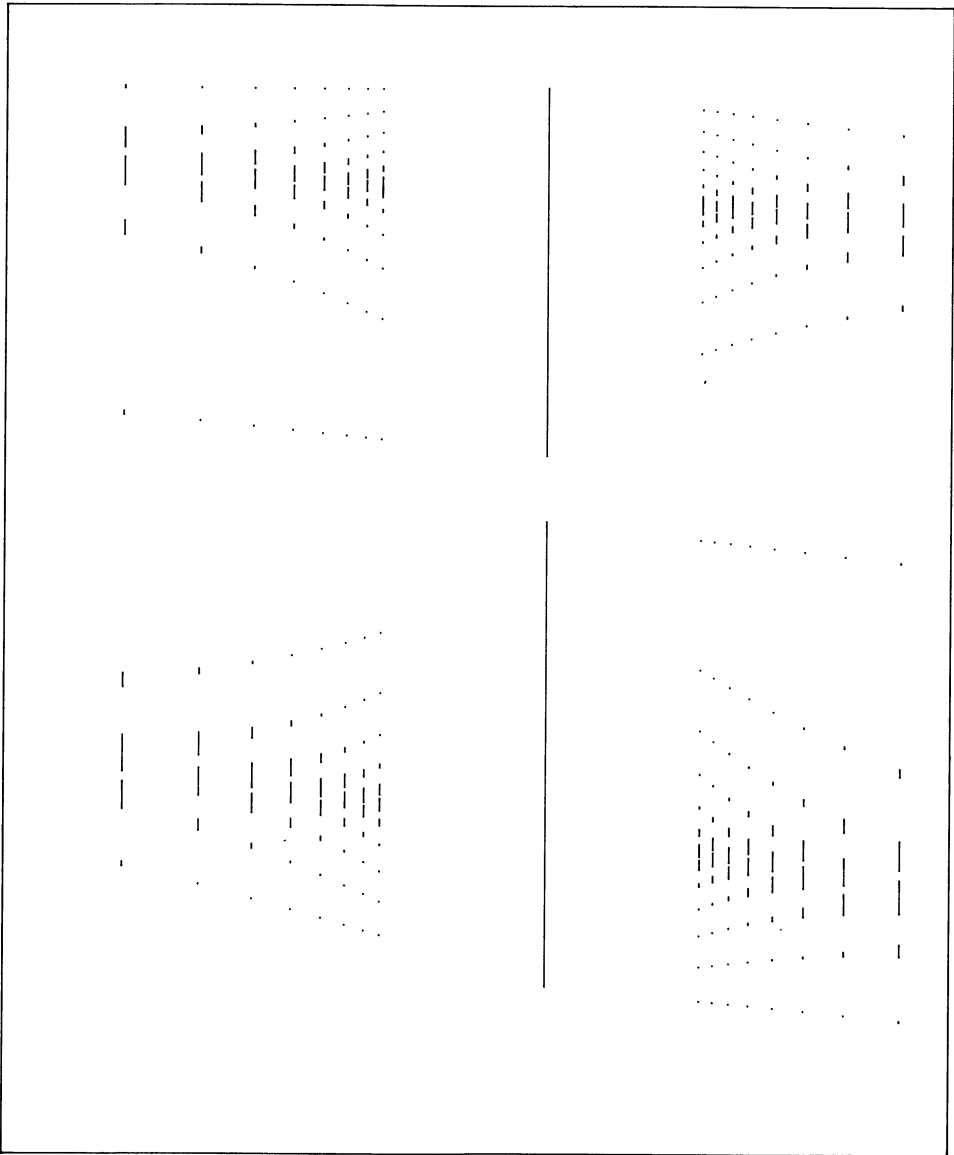
Picture 20. — Spectrum for α near $1/(3 + 1/3)$, upper zoom, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(3 + 1/q))$.



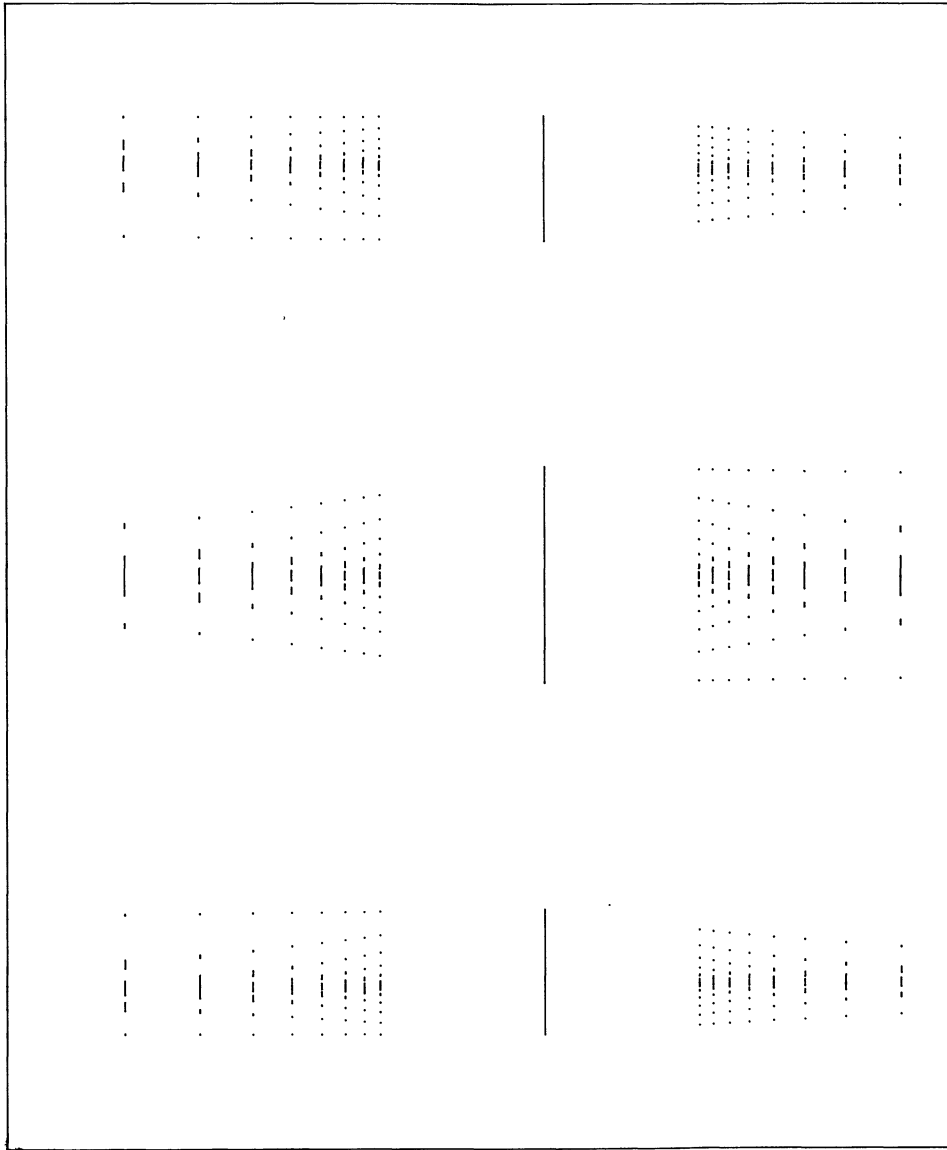
Picture 21. — Spectrum for α near $1/(3 + 1/3)$, central zoom, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(3 + 1/q))$.



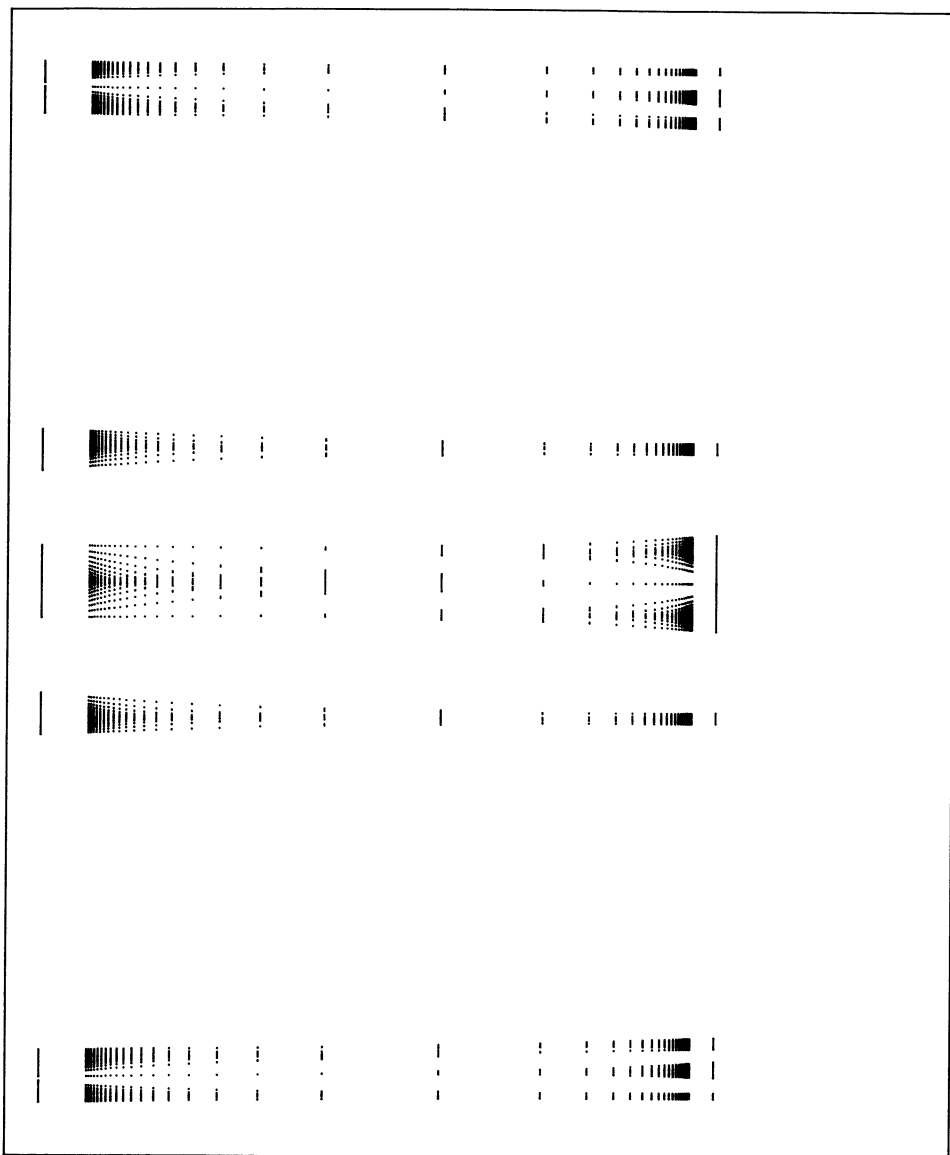
Picture 22. — Spectrum for α near $1/(3 + 1/2)$, global view, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(2 + 1/q))$.



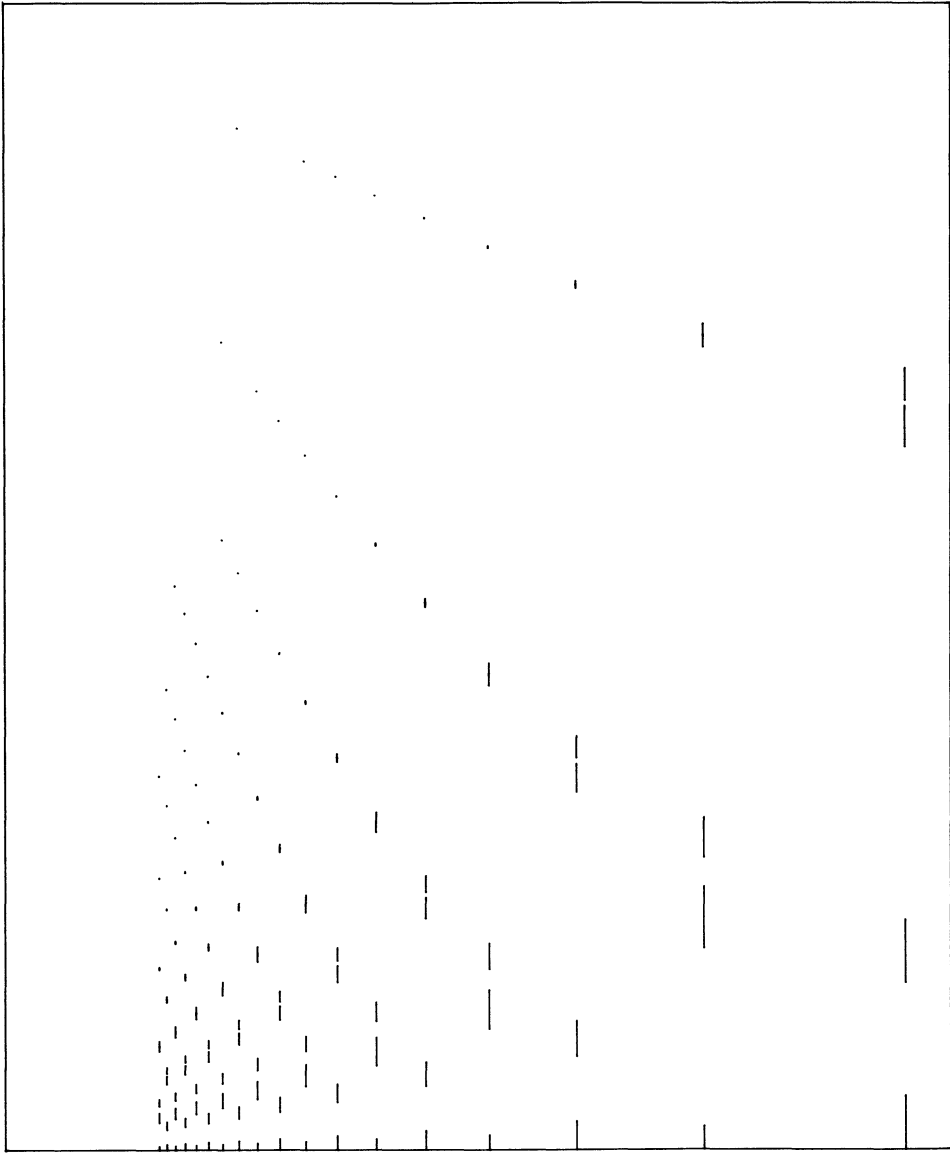
Picture 23. — Spectrum for α near $1/(3 + 1/2)$, upper zoom, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(2 + 1/q))$.



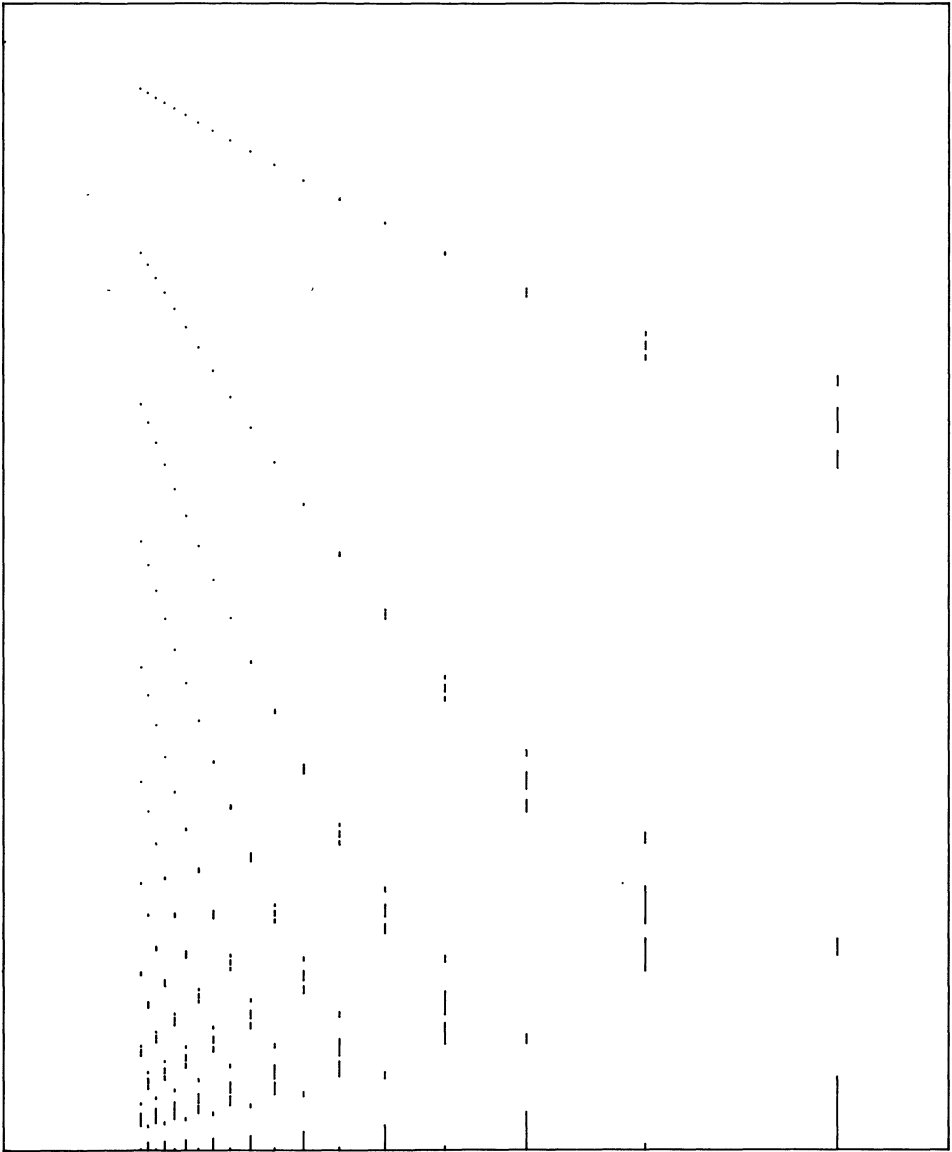
Picture 24. — Spectrum for α near $1/(3 + 1/2)$, central zoom, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(2 + 1/q))$.



Picture 25. — Spectrum for α near $1/(3 + 1/2)$ and $1/(3 + 1/3)$, $\mu = 1$, partial sequence $\alpha = 1/(3 + 1/(2 + 1/q))$ or $1/(3 + 1/(3 - 1/q))$.



Picture 26. — Spectrum for α near 0, $\mu = 1$, partial sequence $\alpha = 1/(q + 1/2)$.



Picture 27. — Spectrum for α near 0, $\mu = 1$, partial sequence $\alpha = 1/(q + 1/3)$.

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