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Thermodynamical behavior of polymerized membranes

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Résumé. — Nous analysons par des techniques de théorie des champs le comportement thermodynamique de membranes polymérisées fluctuantes, en l'absence de répulsion stérique, et soumises à des conditions aux limites libres ou contraintes. La nature de la transition de froissement est précisée en montrant que la tension engendrée par des conditions aux limites contraintes peut être considérée comme le champ conjugué au paramètre d'ordre correspondant à la transition. La phase « plate » de basse température, existant pour des membranes avec conditions aux limites libres, correspond à la phase critique associée à une transition de flambage. Nous présentons la solution explicite, dans la limite de grande dimensionnalité d de l'espace, du modèle élastique des membranes fluctuantes, et nous présentons un traitement complet de la renormalisation des fluctuations dans la phase plate.

Abstract. — We analyze by field theoretical methods the thermodynamical behavior of polymerized membranes, fluctuating without excluded volume interactions and in presence of either free or constrained boundary conditions. We highlight the nature of the crumpling transition, by showing how the tension arising in the presence of constrained boundary conditions may be considered as the field conjugate to the corresponding order parameter. The low temperature flat phase of membranes with free boundary conditions is viewed as a critical phase corresponding to the buckling transition. We present the explicit solution, in the large d limit, of the elastic model of fluctuating elastic membranes and we complete the renormalization of the fluctuations in the flat phase.

1. Introduction.

The thermodynamical behavior of membranes is strongly influenced by their internal structure. Indeed, recent theoretical studies have shown that *polymerized* membranes, in contrast to linear polymers, remain *flat* at sufficiently low temperatures [1-5]. Thus there exists a finite temperature *crumpling transition* between this flat phase and the high-temperature, crumpled phase. The presence of this transition makes the behavior of

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polymerized and fluid membranes qualitatively different. Although the notion of the crumpling transition was in fact first introduced for *fluid membranes* [6] one can show that in these systems the flat phase could be stabilized only in the presence of the long-range forces (or for abstract, theoretical membranes, whose intrinsic dimension D exceeds two). In a sense, the coupling of bending or « undulation » modes with the elastic « phonon » modes, present for polymerized membranes, induced such effective long-range interactions.

The very existence of a flat phase at $D = 2$ is surprising. In fact, it is possible to consider the flat phase to be one, in which the Euclidean symmetry with respect to the space in which the membrane is embedded is broken. Since this symmetry is continuous, one would expect the Mermin-Wagner theorem to forbid such a spontaneous symmetry breaking for bidimensional systems [7]. This paradox can be lifted in two ways : on the one hand one might argue, as we have just mentioned, that the effective phonon-mediated interaction among undulations is long-range, and does not fall therefore within the scope of the Mermin-Wagner theorem ; on the other hand one may, perhaps more interestingly, draw the conclusion that the elastic coefficients are nontrivially renormalized, in contrary to the regularity assumptions usually made in the elastic theory of membranes [8-10]. From both points of view the nature of the flat phase is worth investigating.

The up-to-date studies always considered a fluctuating membrane with free boundary conditions. We find that the nature of the crumpling transition and of the flat phase is made much clearer, if one considers constrained boundary conditions, in which the boundary of the membrane is attached to a rigid frame [11]. With a suitable choice of the frame, this induces a homogeneous tension or compression on the membrane. The tension applied to the frame can be considered as the field f conjugate to the order parameter describing the crumpling transition. Thus we consider as the parameters of the model both the temperature and the field f . The case of free boundary conditions, considered by the previous authors [3, 4, 9, 10], corresponds to the line $f = 0$. The consideration of new directions in this space, beyond allowing for the introduction of new critical exponents for the crumpling transition, allows us to consider the flat phase from a different point of view. Indeed, when a homogeneous tension f is applied, the membrane is stretched and flat at all temperatures. However, if the temperature T is lower than the crumpling temperature T_c , the membrane remains flat also when $f \rightarrow 0$, and the membrane relaxes to its equilibrium size. If we now imagine to attempt to reduce further the size of the membrane by acting on the frame, the membrane *buckles*, assuming an inhomogeneous state and exerts a pressure on the frame. The flat phase at $f = 0$ can be thus considered as describing the *buckling transition* which separates stretched from buckled membranes. The buckled state can be considered as a thermodynamical mixture of flat states with different orientations.

The resulting phase diagram is similar to that of $O(n)$ symmetric magnetic systems, with f playing the role of the magnetic field, and T_c that of the critical temperature. The flat phase lies on the « coexistence curve » corresponding to $f = 0$, $T < T_c$. It is different from the corresponding line of magnetic systems since it is described by an *interacting effective theory*, which implies a nontrivial critical behavior. As a consequence, classical elasticity theory breaks down on the coexistence curve.

We have investigated the phase diagram of polymerized membranes and the nature of the flat phase by two approaches :

- (i) we have solved a model of fluctuating polymerized membranes with inner dimension D in the limit in which the dimensionality d of ambient space goes to infinity ; we have found a crumpling transition for $D > 2$ and a non classical behavior, both at the crumpling and at the buckling transition, for $D < 4$;
- (ii) we have renormalized the effective theory for a stretched membrane for small, but not

necessarily vanishing, « tension » f for D near the upper critical dimension four. We are therefore able to give the values of all the most relevant critical exponents of the buckling transition, to first order in an ε -expansion, where $\varepsilon = 4 - D$. In addition we argue that one of the unstable fixed points found in the ε expansion describes the buckling transition for fluid membranes.

The plan of the paper is the following : the continuum elastic model which we adopt is introduced in section 2 ; the known results on the crumpling transition for membranes with free boundary conditions are briefly reviewed in section 3. The conjugate field f is introduced, by means of constrained boundary conditions, in section 4, where the phase diagram is discussed. Section 5 contains the derivation of the effective Hamiltonian for flat membranes in the general case. The results of the renormalization group calculations on this effective Hamiltonian are reported in section 6. Section 7 contains conclusions and perspectives. Appendix A contains the $d = \infty$ treatment of the elastic continuum model. Appendix B contains the renormalization scheme for the effective Hamiltonian of flat membranes, and the derivation of several scaling laws. Appendix C contains the calculation of the buckling transition exponents to first order in $\varepsilon = 4 - D$.

2. Model.

We define here the continuum model [3, 4] of the elasticity of polymerized membranes we adopt and we discuss the relevant boundary conditions. As mentioned in the introduction, it is convenient to consider at once the general case of our elastic manifold, whose internal dimension D may be different from two.

The configuration of a polymerized manifold is given, once the location in the d -dimensional ambient space of each of its molecules is known. We identify the molecules by means of a D -dimensional coordinate system

$$\sigma \equiv (\sigma^i), \quad (i = 1, 2, \dots, D). \tag{2.1}$$

The configuration of the membrane is therefore identified by the embedding $\sigma \rightarrow \mathbf{X}(\sigma)$, where

$$\mathbf{X}(\sigma) \equiv (X^\alpha(\sigma)), \quad (\alpha = 1, 2, \dots, d). \tag{2.2}$$

We assume that the configuration $\mathbf{X}^0(\sigma)$ of minimal energy (« at rest ») is flat. It is therefore possible to choose the coordinate system σ in such a way that

$$\mathbf{X}^0(\sigma) = (\sigma^1, \dots, \sigma^D, 0, \dots, 0). \tag{2.3}$$

The induced metric tensor g_{ij} is defined by

$$g_{ij} = \partial_i \mathbf{X} \cdot \partial_j \mathbf{X}. \tag{2.4}$$

For the minimal energy configuration $\mathbf{X}^0(\sigma)$ one has, in this set of coordinates,

$$g_{ij}^0 = \partial_i \mathbf{X}^0 \cdot \partial_j \mathbf{X}^0 = \delta_{ij}. \tag{2.5}$$

The curvature tensor \mathbf{K}_{ij} is defined by

$$\mathbf{K}_{ij} = D_i D_j \mathbf{X}, \tag{2.6}$$

where D_i denotes the covariant derivative. One has at rest

$$\mathbf{K}_{ij}^0 = \partial_i \partial_j \mathbf{X}^0 = 0. \tag{2.7}$$

The elastic energy density \mathcal{H} of an arbitrary configuration $\mathbf{X}(\sigma)$ can be expressed, in the spirit of elasticity theory, as a Taylor series in $\partial_i \mathbf{X}$ and its derivatives. In this expansion only terms which are Euclidean invariant in the ambient space \mathbb{R}^d and scalar in the manifold space \mathbb{R}^D may appear. We have therefore

$$H = \int d^D \sigma \mathcal{H} \quad (2.8)$$

where

$$\begin{aligned} \mathcal{H} = & \mathcal{H}_0 + \nu_0 \partial_i \mathbf{X} \cdot \partial_i \mathbf{X} + \frac{1}{8} \lambda_0 (\partial_i \mathbf{X} \cdot \partial_i \mathbf{X})^2 + \\ & + \frac{1}{4} \mu_0 (\partial_i \mathbf{X} \cdot \partial_j \mathbf{X})(\partial_i \mathbf{X} \cdot \partial_j \mathbf{X}) \\ & + \frac{1}{2} \kappa_0 \partial_i \partial_j \mathbf{X} \cdot \partial_j \partial_i \mathbf{X} + \dots \end{aligned} \quad (2.9)$$

The terms neglected here are of higher order in \mathbf{X} or involve higher derivatives, and may be shown to be irrelevant. Other terms may be reduced to the above ones by partial integration. The fact that $\mathbf{X}^0(\sigma)$ corresponds to an energy minimum imposes the following relation :

$$4 \nu_0 + \lambda_0 D + 2 \mu_0 = 0. \quad (2.10)$$

If we now choose \mathcal{H}_0 so that the elastic energy vanishes at rest, we can write equation (2.9) in a more compact form. We introduce the strain tensor u_{ij} :

$$u_{ij} = \frac{1}{2} (\partial_i \mathbf{X} \cdot \partial_j \mathbf{X} - g_{ij}^0), \quad (2.11)$$

measuring the local stretching of the membrane. We then have :

$$\mathcal{H} = \frac{1}{2} \lambda_0 u_{ii}^2 + \mu_0 u_{ij}^2 + \frac{1}{2} \kappa_0 (\Delta \mathbf{X})^2, \quad (2.12)$$

where Δ is the ordinary Laplacian. The coefficient κ_0 is the (bare) rigidity, and λ_0 and μ_0 are the bare Lamé coefficients. The first two terms represent the stretching elasticity, while the third one corresponds to the bending elasticity. Remark that since we use a set of coordinates satisfying equation (2.5) we do not distinguish between covariant and contravariant indices. The case of manifold with internal constraints, introduced e.g. by disclinations, could be handled by considering a metric at rest g_{ij}^0 which is not flat. In this case it may be helpful to consider more general coordinate sets. The expression of the elastic energy H then becomes

$$H = \int d^D \sigma \sqrt{g^0} \mathcal{H}. \quad (2.13)$$

where $g^0 = \det(g_{ij}^0)$ and \mathcal{H} is given by

$$\mathcal{H} = \frac{1}{2} \lambda_0 u_i^i u_j^j + \mu_0 \mu_j^i u_j^i + \frac{1}{2} \kappa_0 (\Delta_0 \mathbf{X})^2. \quad (2.14)$$

All indices are raised or lowered according to the metric g_{ij}^0 , and Δ_0 is the corresponding scalar Laplacian.

We consider the D -dimensional manifold of linear size L . We assume two types of boundary conditions :

- (a) *free boundary conditions* : the sides of the fluctuating manifold are free to move about ;
- (b) *constrained boundary conditions* : the sides are instead attached to a D -dimensional frame, which is assumed to be a hypercube of linear size ζL .

The factor ζ is called the extension factor. When it exceeds 1 the membrane is stretched. Thus the equilibrium configuration $\mathbf{X}_{\text{eq}}(\sigma)$ is no more a minimum of H , and linear terms appear in its expansion around \mathbf{X}_{eq} . These terms represent the internal tension introduced by the boundary conditions.

3. Crumpling transition.

In this section we review some results concerning the thermal behavior of elastic membranes, with *free boundary conditions*. We discuss the nature of the crumpling transition, which separates a regime in which the membrane is flat from one in which it is crumpled and highly folded.

The property which distinguishes elastic membranes from shells is the value of their elastic constants, e.g. of the bending rigidity κ_0 . In shells, κ_0 is large and thermal fluctuations can be neglected. For real two-dimensional molecular membranes κ_0 is of the order of $k_B T$, and thermal fluctuations play an important role in their behavior. They have two important consequences, namely to renormalize the elastic constants, and thus produce a breakdown of classical elasticity theory [8-10] or to completely suppress the average planar shape of the membrane and to induce a *crumpling transition* [3, 4, 10]. The notion of such a transition was introduced in the context of the thermal behavior of *fluid* membranes [6]. It was shown that a model of fluid membranes, whose inner dimension D is larger than two, exhibits a crumpling transition at a finite temperature T_c . This temperature vanishes for the realistic case of two dimensional membranes, which are therefore crumpled at any nonzero temperature.

It was soon realised, however, that two-dimensional *polymerized* membranes may remain flat at finite temperatures, yielding a finite T_c [1]. This is a consequence of the interplay between shape fluctuations (« undulations ») and elastic in-plane degrees of freedom (« phonons »). Integrating out the phonons introduces an effective *long-range* interaction among undulations, which stabilizes the flat phase even for $D = 2$.

Although a real, physical system exhibiting a crumpling transition has not yet been built, it has been possible to observe it in a computer simulation. A Monte-Carlo study of « tethered membrane » (without excluded volume) showed a finite temperature transformation, with a pronounced peak in the specific heat [2]. This suggests that for $D = 2$, $d = 3$, the transition is continuous or weakly first order. Monte Carlo simulations done on similar models [5] suggest either a third order crumpling transition, or continuously varying critical exponents below the critical temperature T_c . These discrepancies may be the effect of the discretized nature of the models (finite size effects), or of crossover effects. Clearly more detailed investigation of larger systems are needed. Thus in the following we shall assume that a crumpling transition takes place in $D = 2$, $d = 3$ according to the mechanism discussed in references [1, 3, 4].

The crumpling transition can be investigated by means of the elastic continuum models described in the previous section [3, 4]. In the presence of fluctuations the average configuration of the manifold will be different from the one at rest, with free boundary conditions the manifold will in general shrink from its configuration at rest. This effect can be aptly described by introducing the vectors

$$\mathbf{t}_i = \langle \partial_i \mathbf{X} \rangle, \quad i = 1, \dots, D, \quad (3.1)$$

where the average is taken with respect to the Boltzmann weight defined by H (Eq. (2.8)) :

$$\langle \dots \rangle = \int D\mathbf{X} \dots e^{-H/T} / \int D\mathbf{X} e^{-H/T}$$

we use units in which the Boltzmann constant is equal to 1. The average extension factor, i.e. the ratio between the actual linear size of the fluctuating manifold and its size at rest, is given by

$$\zeta_{\text{sp}}^2(T) = \frac{1}{D} (\mathbf{t}_i \cdot \mathbf{t}_i). \quad (3.2)$$

At low temperatures, $\zeta_{\text{sp}}(T)$ is nearly equal to one. As T increases, $\zeta_{\text{sp}}(T)$ becomes smaller and smaller. Above a certain temperature T_c , $\zeta_{\text{sp}}(T)$ vanishes : this means that the actual size of the membrane is no more proportional to its size at rest. This identifies T_c as the crumpling transition temperature, and $\zeta_{\text{sp}}(T)$ as the corresponding order parameter. Above the crumpling transition, the effective Hamiltonian describing the manifold reduces to

$$H_{\text{eff}} = \int d^D\sigma \frac{1}{2} \nu_{\text{eff}} (\partial_i \mathbf{X} \cdot \partial_i \mathbf{X}). \quad (3.3)$$

The behavior of such Gaussian elastic manifolds has already been thoroughly investigated [12, 13]. In the absence of excluded volume interactions, they fold into very convoluted configurations [12]. A way to describe them is to define their fractal dimension d_F , which measures the way the size of the embedded manifold increases with the increasing linear size L of the membrane at rest. The size of the fluctuating membrane can be estimated by the radius of gyration R_G , defined by

$$L^{2D} R_G^2 = \left\langle \int d^D\sigma \int d^D\sigma' |\mathbf{X}(\sigma) - \mathbf{X}(\sigma')|^2 \right\rangle. \quad (3.4)$$

The fractal dimension d_F is defined by

$$R_G^{d_F} \sim L^D. \quad (3.5)$$

One obtains

$$d_F = \frac{2D}{2-D}, \quad (3.6)$$

which is compatible with the well known result $d_F = 2$, valid for linear polymers ($D = 1$). On the other hand, one obtains $d_F = \infty$ for $D = 2$, which corresponds in fact to

$$R_G^2 \sim \ln L, \quad (D = 2). \quad (3.7)$$

If the crumpling transition is continuous ⁽¹⁾ critical exponents can be defined in the usual way. Most of them involve the consideration of constrained boundary conditions and will be discussed later. One can however define in a straightforward way

$$\zeta_{\text{sp}}(T) \sim |T - T_c|^\beta, \quad T \leq T_c. \quad (3.8)$$

⁽¹⁾ The arguments of reference [4] suggest that the crumpling transition could be weakly first order for small dimensions of ambient space. We shall assume, however, that the transition is continuous and not preempted by any discontinuous transition.

In a similar way, one can introduce the correlation length ξ , which measures the range of the correlation function

$$G(\sigma - \sigma') = \langle \mathbf{X}(\sigma) \mathbf{X}(\sigma') \rangle - \langle \mathbf{X}(\sigma) \rangle \langle \mathbf{X}(\sigma') \rangle . \tag{3.9}$$

Note that this range is measured in the coordinate system at rest. One sets by definition

$$\xi \sim |T - T_c|^{-\nu} . \tag{3.10}$$

The behavior of the correlation function G at $T = T_c$ allows one to define the exponent η . If $\Gamma^{(2)}(p)$ is the inverse Fourier transform of G with respect to $\sigma - \sigma'$, one has

$$\Gamma^{(2)}(p) \sim p^{(4-\eta)} . \tag{3.11}$$

Actually the fractal dimension d_F and the exponent η are related by

$$\frac{D}{d_F} = \frac{4 - D - \eta}{2} . \tag{3.12}$$

Below the crumpling temperature, the membrane is flat on average, and its extension factor equals $\zeta_{sp}(T)$. This phase has been investigated in references [9, 10]. It is remarkable, since it may be described at all temperatures below T_c as a critical phase. In fact, it is possible to conceive the crumpling transition as one, below which the Euclidean symmetry in ambient space is spontaneously broken.

The deformations :

$$\delta \mathbf{X}(\sigma) = \mathbf{X}(\sigma) - \langle \mathbf{X}(\sigma) \rangle \tag{3.13}$$

can be therefore decomposed into parallel deformations u^i (« phonons ») and transversal deformations \mathbf{h} (« undulations ») by means of :

$$\delta \mathbf{X} = u^i \mathbf{t}'_i + \mathbf{h} ; \tag{3.14}$$

where

$$\mathbf{t}'_i = \mathbf{t}_i / \zeta_{sp}(T) . \tag{3.15}$$

The fields $(\partial_i \mathbf{h})$ play the role of Goldstone modes and are thus « massless » (the kinetic energy of \mathbf{h} is proportional to k^4). In contrary, the fields $(\partial_i u^i)$ get a « mass » (the kinetic energy of u^i is proportional to k^2). Equation (3.14) is analogous to the decomposition of the spin field into σ and π fields in the low temperature phase of $O(n)$ symmetric magnetic models. In that case, the effective Hamiltonian for the $(n - 1)$ Goldstone modes π , which governs the infrared behavior of the model, is the free one :

$$H_{\text{eff}} \propto \int d^D \sigma (\nabla \pi)^2 \tag{3.16}$$

and the corresponding exponents can be obtained by power counting. This is not the case for the rigid phase we are discussing. The Goldstone modes $(\partial \mathbf{h})$ are now interacting in this phase, i.e. the effective Hamiltonian at large distances is no more the free one. One can define the exponents η' , η'_u by means of the behavior of the inverse propagators $\Gamma_{hh}^{(2)}$, $\Gamma_{uu}^{(2)}$ of h and u respectively :

$$\Gamma_{hh}^{(2)}(q) \sim q^{4-\eta'} , \tag{3.17}$$

$$\Gamma_{uu}^{(2)}(q) \sim q^{2+\eta'_u} . \tag{3.18}$$

These exponents can be also interpreted in the following way. Since the fields \mathbf{h} , u_i are interacting, the elastic constants κ , λ , μ , are nontrivially renormalized and turn out to be dependent on the wave vector q . One has therefore

$$\kappa(q) \sim q^{-\eta'}, \quad (3.19)$$

$$\lambda(q), \quad \mu(q) \sim q^{\eta''}. \quad (3.20)$$

other exponents will be introduced in the next section.

The crumpling transition has been investigated :

(i) for $D = 2$, to first order in a $1/d$ expansion, by means of a nonlinear version of the elastic model (2.12) [3]. This model is obtained by taking the limit $\lambda_0, \mu_0 \rightarrow \infty$ in equation (2.12) and is analogous to the nonlinear σ -model for $O(n)$ symmetric magnetic systems. In this limit, one introduces the constraint that the induced metric of the fluctuating manifold be equal to the rest metric g_{ij}^0 . One obtains therefore the Hamiltonian

$$H_{\text{NL}} = \int d^D \sigma \frac{1}{2} \kappa_0 (\Delta \mathbf{X})^2, \quad (3.21)$$

with the constraint

$$\partial_i \mathbf{X} \cdot \partial_j \mathbf{X} = \delta_{ij}. \quad (3.22)$$

The model exhibits, to first order in a $1/d$ expansion, an ultraviolet stable fixed point describing a continuous crumpling transition. The Hausdorff dimension d_F is given by

$$d_F = \frac{2d}{d-1} + O\left(\frac{1}{d^2}\right), \quad (3.23)$$

and the exponents β and ν are respectively given by

$$\beta = \frac{1}{2} + O\left(\frac{1}{d}\right), \quad \nu = \frac{d}{2} + O(1). \quad (3.24)$$

It is possible to exploit this calculation to show that the lower critical dimension D_1 , below which the crumpling transition occurs at $T = 0$, is equal to

$$D_1(d) = 2 - \frac{2}{d} + O\left(\frac{1}{d^2}\right). \quad (3.25)$$

(ii) for general d , to first order in an ε -expansion, where

$$\varepsilon = 4 - D. \quad (3.26)$$

It turns out that, to this order, the crumpling transition is continuous for $d \geq 219$, and is first order otherwise [4] ⁽²⁾.

(iii) in $d = 3$, $D = 2$ a real-space renormalization group calculation has been performed [14] which suggests that the crumpling transition remains continuous.

⁽²⁾ In this same reference [4] the authors also give the results for the $d \rightarrow \infty$, $2 \leq D \leq 4$ calculation which we shall discuss in detail below.

The nature of the low temperature flat phase has been investigated :

(i) by a self-consistent approach, which assumed no renormalization of the phonon elastic constants λ, μ [1]. One obtained for $D = 2, d = 3$:

$$\eta' = 1 . \tag{3.27}$$

(ii) in an ε -expansion, with ε given by (3.26) [9]. It has been possible to identify a nontrivial stable fixed point describing the flat phase, yielding the exponent values

$$\eta' = \frac{\varepsilon}{2 + d_c/12} ; \quad \eta'_u = \frac{\varepsilon}{1 + 24/d_c} ; \tag{3.28}$$

where

$$d_c = d - D . \tag{3.29}$$

Let us remark that the results of the $1/d$ expansion (Ref. [3]) imply for $D = 2$

$$\begin{aligned} \eta' &= \frac{2}{d} + O\left(\frac{1}{d^2}\right) \\ &\simeq \frac{2}{3} \quad (d = 3) . \end{aligned} \tag{3.30}$$

To investigate further the nature of the fixed point describing the flat phase it is convenient to introduce constrained boundary conditions.

4. The phase diagram.

To make the nature of the crumpling transition clearer, it is convenient to consider *constrained boundary conditions*, in which the extension factor ζ may be different from its spontaneous value $\zeta_{sp}(T)$. We introduce therefore the (T, ζ) plane, where we draw the curve $(T, \zeta_{sp}(T))$. We obtain therefore the diagram of figure 1.

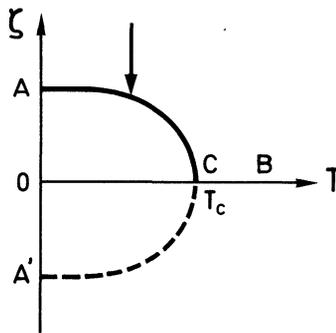


Fig. 1. — Phase diagram in the (ζ, T) plane.

The curve joining A to C corresponds to $\zeta = \zeta_{sp}(T)$ and describes the « flat » phase. We have also drawn its symmetrical one, joining A' to C. Negative values of ζ correspond to situations in which the orientation of the manifold is reversed with respect to the rest configuration.

Fluctuating membranes with free boundary conditions are described by points on the AC curve, if $T < T_c$, and on the $\zeta = 0$ axis, if $T \geq T_c$. We can thus call the curve ACA' « the coexistence curve ». But any point in the (T, ζ) plane can be obtained, if we consider constrained boundary conditions. In this case, however, a tension (or a compression) is exerted on the frame. It is convenient to characterize it by the quantity

$$f = \frac{1}{L^D} \frac{\partial F}{\partial \zeta}, \quad (4.1)$$

where F is the Helmholtz free energy of the membrane. Although we shall call f the « tension » it is useful to keep in mind that the physically measurable tension is given by

$$f_p = \frac{\partial F}{\partial \Omega}, \quad (4.2)$$

where $\Omega = (\zeta L)^D$ is the actual volume of the membrane. One has of course

$$f = D\zeta^{D-1} f_p. \quad (4.3)$$

We can thus consider the phase diagram in the (T, f) plane. It is drawn in figure 2.

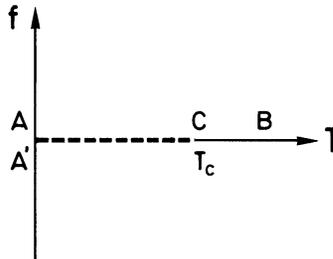


Fig. 2. — Phase diagram in the (f, T) plane.

The « coexistence curve » ACA' reduces to the segment $0 < T \leq T_c$ of the $f = 0$ axis. The only points realizable with free boundary conditions lie on the $f = 0$ axis.

The diagrams shown in figures 1, 2 closely resemble to those of ordinary critical phenomena, with ζ playing the role of the order parameter, and f that of its conjugate field. It is known that in this case it is possible to produce states inside the coexistence curve, by considering mixtures of thermodynamical phases. Physically this correspond e.g. to magnetic domains, in which the order parameter is oriented in different directions in the sample. By the same token, the points of the (T, ζ) plane inside the ACA' curve correspond to a mixture of flat phases oriented in different directions. Physically this corresponds to a buckled manifold, whose equilibrium shape is no more planar. We can thus view the ACA' curve in a different way. As we approach this line, e.g., along the arrow in figure 1, the « tension » f becomes smaller and smaller, and eventually vanishes when $\zeta = \zeta_{sp}(T)$. If we keep on reducing ζ , we are actually *compressing* the membrane, which has therefore to buckle. We expect that in this state the membrane is made of regions relatively flat and unstrained, separated by « domain walls » with high stress. The detailed nature of the buckled state may depend on microscopic details as well as on the way boundary conditions are imposed. We can thus consider the coexistence curve in figure 1, on the ACA' line in either figure 1 or figure 2, as describing the *buckling transition*.

Consideration of the enlarged phase diagram allows us to define new critical properties and exponents, both for the buckling and for the crumpling transitions.

For the *crumpling transition* we may consider the relations between the « tension » f and the order parameter ζ . At $T = T_c$ we have in fact

$$\zeta \sim f^{1/\delta}, \tag{4.4}$$

which defines the new exponent δ . We may also introduce the susceptibility χ

$$\chi = \frac{\partial \zeta}{\partial f}. \tag{4.5}$$

We have, for $T \sim T_c$, $f = 0$

$$\chi \sim |T - T_c|^{-\gamma}. \tag{4.6}$$

On the other hand, for the *buckling transition*, consideration of a nonzero f allows us to move away from criticality. Since $\zeta_{sp}(T)$ is a regular curve (at least as long as $T < T_c$) the distance from the buckling transition can be aptly measured by $\zeta - \zeta_{sp}(T)$. We can thus define the exponent δ' by

$$f \sim |\zeta - \zeta_{sp}(T)|^{\delta'}. \tag{4.7}$$

As soon as $\zeta \neq \zeta_{sp}(T)$, the correlation lengths ξ_u and ξ_h , which describe the range of the correlations of phonons and undulations respectively, are finite. We define therefore the exponents ν'_u , ν'_h by

$$\xi_u \sim |\zeta - \zeta_{sp}(T)|^{-\nu'_u}, \tag{4.8}$$

$$\xi_h \sim |\zeta - \zeta_{sp}(T)|^{-\nu'_h}. \tag{4.9}$$

The exponents for the *crumpling transition* can be easily read, in an ε -expression, off the results of reference [4], since ordinary critical scaling laws are valid. In appendix A we perform a $d = \infty$ calculation on the model defined by equation (2.12) for $2 < D < 4$. We are able to obtain the results ⁽³⁾ :

$$\beta = \frac{1}{2}; \quad \gamma = \frac{2}{D-2}; \quad \delta = \frac{D+2}{D-2}; \quad \eta = 0; \quad d_F = \frac{2D}{4-D}; \quad \nu = \frac{1}{D-2}. \tag{4.10}$$

Other exponents can be obtained by the usual scaling laws in D dimensions.

The *properties of the buckling transition* will be investigated below in the framework of the ε -expansion (Sects. 5 and 6). In appendix A we also obtain the exponents for the buckling transition in the limit $d \rightarrow \infty$

$$\nu'_h = \frac{1}{D-2}; \quad \nu'_u = \frac{1}{D-2}; \quad \eta' = 0; \quad \eta'_u = 4 - D; \quad \delta' = \frac{2}{D-2}. \tag{4.11}$$

They satisfy a set of scaling laws which will be made explicit in the framework of the ε -expansion.

⁽³⁾ These results already given in reference [4] for the crumpling transition are here extended to the whole phase diagram (see Appendix A).

5. The effective theory of stretched membranes.

We now derive the effective Hamiltonian of a stretched membrane. Let us assume that the membrane is subject to constrained boundary conditions which impose an extension factor ζ different in general from $\zeta_{sp}(T)$. We can thus consider small fluctuations around the stretched configuration

$$\mathbf{X}_s(\sigma) = \zeta \mathbf{X}^0(\sigma). \quad (5.1)$$

We rescale the coordinates σ by ζ in such a way that

$$\mathbf{X}_s(\sigma) = (\sigma^1, \dots, \sigma^D, 0, \dots, 0). \quad (5.2)$$

We now consider the effective Hamiltonian governing the small fluctuations $\delta \mathbf{X}$ around \mathbf{X}_s :

$$\delta \mathbf{X}(\sigma) = \mathbf{X}(\sigma) - \mathbf{X}_s(\sigma). \quad (5.3)$$

In the spirit of elasticity theory we assume that this Hamiltonian allows for an expansion analogous to equation (2.9). However, since \mathbf{X}_s is not necessarily an extremum of the effective Hamiltonian, no condition analogous to equation (2.10) should be imposed. If we now define the strain tensor u_{ij} by means of

$$u_{ij} = \frac{1}{2} (\partial_i \mathbf{X} \cdot \partial_j \mathbf{X} - g_{ij}^s), \quad (5.4)$$

where $g_{ij}^s = \partial_i \mathbf{X}_s \cdot \partial_j \mathbf{X}_s = \delta_{ij}$, we obtain the following expression for the effective Hamiltonian density \mathcal{H}_{eff} :

$$\mathcal{H}_{\text{eff}} = \tau_0 u_{ii} + \frac{1}{2} \lambda_0 u_{ii}^2 + \mu_0 u_{ij}^2 + \frac{1}{2} \kappa_0 (\Delta \mathbf{X})^2. \quad (5.5)$$

This expression is different from equation (2.12) because of the $\tau_0 u_{ii}$ term. This term corresponds to local isotropic tension or compression of the membrane, which endeavors to move away from the reference configuration \mathbf{X}_s . In general, the case $\tau_0 > 0$ corresponds to a membrane under tension, which would spontaneously assume an extension factor smaller than that imposed by \mathbf{X}_s . For $\tau_0 = 0$, the reference configuration \mathbf{X}_s is an extremum of the effective Hamiltonian. This corresponds to the case of vanishing (bare) tension. For $\tau_0 < 0$, one applies a *compression* on the membrane. In this case the planar stretched configuration \mathbf{X}_s is unstable and the membrane takes on a buckled state. The identification of the stable buckled configuration is a complex problem, whose investigation lies beyond the scope of this paper.

We can now parametrize the fluctuations $\delta \mathbf{X}$ in terms of the phonon modes u_i and the undulation modes \mathbf{h} :

$$\delta \mathbf{X} = u_i \mathbf{t}_i + \mathbf{h}, \quad (5.6)$$

where

$$\mathbf{t}_i = \partial_i \mathbf{X}_s; \quad \mathbf{h} \cdot \mathbf{t}_i = 0, \quad i = 1, \dots, D. \quad (5.7)$$

Assuming that the fluctuations are small, we can drop terms quadratic in u_i in the expression of the strain tensor u_{ij} and of the bending energy. It can be shown in fact that these terms are

irrelevant for the behavior of small fluctuations in the flat phase. We obtain therefore the following truncated expression $\mathcal{H}_{\text{flat}}$ for \mathcal{H}_{eff} :

$$\mathcal{H}_{\text{flat}} = \tau_0 u_{ii} + \frac{1}{2} \lambda_0 u_{ii}^2 + \mu_0 u_{ij}^2 + \frac{1}{2} \kappa_0 (\Delta \mathbf{h})^2, \tag{5.8}$$

where

$$u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i + \partial_i \mathbf{h} \cdot \partial_j \mathbf{h}). \tag{5.9}$$

We now show that equations (5.8), (5.9) define a class of field theoretical models which renormalizes onto itself near the upper critical dimension $D = D_u = 4$. Let us remark that by dropping higher order terms in u_i we have explicitly broken the rotation invariance in d -dimensional space still possessed by equation (5.5). On the other hand, equations (5.8), (5.9) are still invariant with respect to the following symmetry groups : (i) translations in d -dimensional space ; (ii) rotations in the $(d - D)$ -dimensional space orthogonal to X_s ; (iii) isometries in the D -dimensional space spanned by the internal coordinates of the membrane. Moreover, although full rotational symmetry has been explicitly broken, one may check that these expressions are invariant with respect to the transformations defined, for any set of D vectors \mathbf{A}_i with $(d - D)$ components, by

$$\begin{aligned} \mathbf{h}(\sigma) &\rightarrow \mathbf{h}(\sigma) + \mathbf{A}_i \sigma^i, \\ u_i(\sigma) &\rightarrow u_i(\sigma) - (\mathbf{A}_i \cdot \mathbf{h}(\sigma)) - \frac{1}{2} (\mathbf{A}_i \cdot \mathbf{A}_j) \sigma^j. \end{aligned} \tag{5.10}$$

These transformations are linearized versions of d -dimensional rotations, represented in the variables u_i, \mathbf{h} . The associated Ward identities for the effective potential $\Gamma[u_i, \mathbf{h}]$ are

$$\int d^D \sigma \left[\sigma^i \frac{\delta \Gamma}{\delta \mathbf{h}} - \mathbf{h} \frac{\delta \Gamma}{\delta u_i} \right] = 0. \tag{5.11}$$

The general solution (involving only terms relevant by power counting for $D = 4$) of these Ward identities, satisfying the additional symmetries mentioned above, is given by equations (5.8), (5.9), with arbitrary values of the coefficients $\tau, \kappa, \lambda, \mu$. This proves the renormalizability of the model we had anticipated.

The presence of a term τu_{ii} in the general solution of the Ward Identities implies that such a term will in general be generated by the renormalization, even if τ_0 is set to zero in the bare Hamiltonian (2.9). This is an expression of the physical fact that even if the size of the frame is equal to the size of the membrane at rest (at $T = 0$), the membrane will in general shrink because of thermal fluctuations and an effective tension τ will thus be generated. This phenomenon is actually a consequence of the imposed boundary conditions. With free boundary conditions, it is indeed possible to reset $\tau = 0$ by a suitable isotropic shift of u^i ($u^i \rightarrow u^i + (1 - \zeta_{\text{sp}}) \sigma^i$). In particular, this is automatically performed if one uses a dimensional regularization scheme. (It is a property of the dimensional regularization scheme that if a strongly relevant field is set to zero in the bare Hamiltonian, it remains zero in the renormalized one). Such a procedure is however only consistent when free boundary conditions are adopted. The introduction of a frame implies imposing fixed boundary conditions on the displacement u^i ($u^i = 0$ on the boundary of an hypercube of side ζL) and thus forbids us to perform any shift on u^i . In that case, a non-vanishing tension coefficient τ must be considered and its renormalization has to be investigated.

6. ε expansion for the buckling transition.

The effective Hamiltonian H_{flat} for flat stretched membranes was derived in the previous section (Eq. (5.8)). It allows to predict the properties of the buckling transition within mean field theory. For this purpose, it is convenient to decompose the strain tensor u_{ij} into its traceless part v_{ij} and its trace v

$$v_{ij} = u_{ij} - \frac{u_{kk}}{D} \delta_{ij}, \quad (6.1)$$

$$v = u_{kk}. \quad (6.2)$$

$u_{ij} = v_{ij}$ and $u_{ij} = \delta_{ij} v/D$ correspond to pure shear and to pure compression (or dilation) deformations of the membrane respectively. The Hamiltonian H_{flat} is given by :

$$H_{\text{flat}} = \int d^D \sigma \left\{ \frac{\kappa_0}{2} (\Delta \mathbf{h})^2 + \mu_0 (v_{ij})^2 + \frac{K_0}{2} v^2 + \tau_0 v \right\} \quad (6.3)$$

where K_0 is the compression modulus.

$$K_0 = \lambda_0 + \frac{2}{D} \mu_0. \quad (6.4)$$

This Hamiltonian is bounded from below provided that

$$\begin{aligned} \kappa_0 \geq 0, \quad \mu_0 \geq 0, \quad K_0 > 0 \\ \text{or} \\ K_0 = \tau_0 = 0. \end{aligned} \quad (6.5)$$

These conditions define the domain of stability for flat membranes with mean field theory. For fixed $\kappa_0 > 0$ the boundary lines $\mu_0 = 0$ and $K_0 = 0$ correspond to isotropic elastic plates with zero shear modulus (« liquid » state) and with zero compression modulus (« conformal » plates) respectively. The mean field theory predicts the buckling transition at $\tau_0 = 0$. Equation (6.3) allows to obtain the classical results of the theory of elastic plates. For instance, Hooke's law with $\delta' = 1$ (where δ' is defined by (4.7)) can be obtained. In the mean field approximation, the exponents for the buckling transition are :

$$\eta'_u = 0, \quad \delta' = 1, \quad \eta' = 0, \quad \nu' = \frac{1}{2}. \quad (6.6)$$

The mean field theory breaks down below the upper critical dimension D_u whose value can be obtained easily from the canonical dimension of the coupling constants which appear in (6.3). After rescaling of the fields, the Hamiltonian depends on a bare tension parameter $\hat{\tau}_0 = \tau_0/\kappa_0$ with canonical dimension 2 (in units of mass) ⁽⁴⁾ and on two coupling constants $\hat{\mu}_0 = \mu_0/\kappa_0^2$ and $\hat{K}_0 = K_0/\kappa_0^2$, with dimension

$$[\mu_0/\kappa_0^2] = [K_0/\kappa_0^2] = 4 - D. \quad (6.7)$$

At the upper critical dimension D_u , $\hat{\mu}_0$ and \hat{K}_0 become relevant and therefore $D_u = 4$, as for the crumpling transition. Below D_u , the renormalization of the coupling constants may be

(2) $(\hat{\tau}_0)^{1/2}$ plays the role of a mass for the Goldstone bosons $\partial \mathbf{h}$.

studied in the standard $\epsilon = 4 - D$ expansion. This study, including the renormalization of the tension $\hat{\tau}_0$, is detailed in appendix B. Here we present only the main results.

Let us first discuss some general features of the renormalization, which are valid to every order in ϵ . As discussed in section 5, the Ward identities (5.11) ensures that it involves only four independent renormalization factors, (see appendix B, Eqs. (B9) to (B13))

- a wave function renormalization Z for the fields \mathbf{h} and u_i ;
- two renormalization $Z_{\hat{\mu}}$ and $Z_{\hat{K}}$ for the coupling constants μ_0 and K_0 ;
- a multiplicative renormalization $Z_{\hat{\tau}}$ for $\hat{\tau}_0$ (its multiplicative nature is a feature of the ϵ expansion scheme).

The corresponding Wilson functions γ , $\beta_{\hat{\mu}}$, $\beta_{\hat{K}}$ and $\gamma_{\hat{\tau}}$ (defined by Eqs. (B15), (B16)) permit to study the renormalization group flow for the renormalized coupling constants $\hat{\mu}_R$, \hat{K}_R and $\hat{\tau}_R$. The general features of this flow are :

- the critical surface (corresponding to a flat membrane without tension) is as expected defined by $\hat{\tau}_R = 0$ (vanishing renormalized tension) ;
- on the critical surface the R.G. flow has the following properties depicted in figure 3.

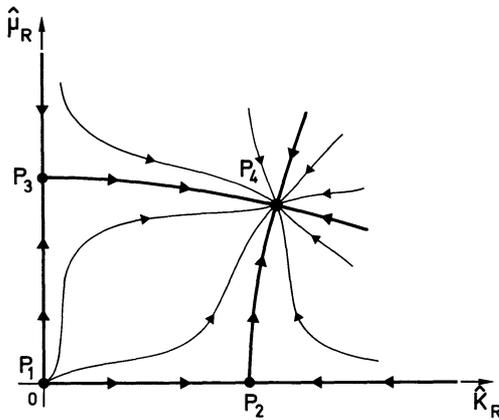


Fig. 3. — Renormalization group flow in the $\hat{\tau}_R = 0$ plane.

(i) The lines $\hat{\mu}_R = 0$ and $\hat{K}_R = 0$ are « fixed lines », i.e. they are renormalized into themselves. Thus they define the boundary of the domain of stability, which coincides with the mean field domain of stability, and will be referred as the « boundary lines ».

(ii) There are four fixed points :

- one trivial infrared unstable fixed point P_1 at $\hat{\mu}_R = \hat{K}_R = 0$;
- two partially unstable fixed point P_2 and P_3 on each boundary line ;
- one infrared stable fixed point P_4 inside the domain of stability $\hat{\mu}_R > 0, \hat{K}_R > 0$, which is its domain of attraction.

For $\hat{K}_R > 0$, away from the critical surface ($\hat{\tau}_R > 0$), the coupling constants flow away from the critical surface in the infrared. On the boundary line $\hat{K}_R = 0$ (and in particular at the fixed point P_3), one cannot induce any tension by putting the membrane on a frame. It is thus impossible to move away from the critical surface $\tau_R = 0$.

For $\tilde{K}_R > 0$ the critical surface $\hat{\tau}_R = 0$ corresponds to the buckling transition. The stable fixed point P_4 describes the generic large distance behaviour of isotropic elastic membranes in the flat phase. The boundary fixed points P_2 and P_3 are somewhat special. P_3 should describes the large distance behaviour of an elastic membrane with no compression modulus. Such a membrane has only shear modulus and arbitrary large dilations (and more generally conformal transformations) do not cost any energy. The fixed point P_3 describes a « conformal membrane » which seems a somewhat abstract object. For such a membrane, there is no buckling transition since it can always adjust its size to that of the frame. The fixed point P_2 is more interesting. It describes the large distance behavior of an isotropic elastic membrane with no shear modulus. Such an object is called a « fixed connectivity fluid » in [9]. However in the effective theory described by (6.3) no reference is made to the connectivity of the underlying lattice. One assumes only that the membrane is isotropic. In our opinion an isotropic elastic medium with no shear modulus is nothing but a liquid. Hence we conjecture that the fixed point P_2 describes nothing but the large distance properties of D -dimensional *fluid membranes* in their flat phase for $2 < D < 4$. (The existence of a flat phase and of a crumpling transition for $D > 2$ for fluid membranes was first predicted in [6]. This phase disappears at $D = 2$.)

The critical exponents η' , n'_u , δ' , ν'_u and ν'_h which characterize the buckling transition make sense only for $\tilde{K}_R > 0$ but may be also associated formally to the fixed point P_3 (conformal membrane).

In appendix B the scaling laws are derived, which provide the relation for the anomalous dimensions of the fields, valid at the three nontrivial fixed points P_2 , P_3 and P_4 .

The linearized rotational invariance (5.10) implies [9]

$$\eta'_u = \varepsilon - 2 \eta' , \quad (6.8)$$

which relates the anomalous dimensions of u^i and \mathbf{h} .

The exponents δ' , ν'_u and ν'_h associated to the buckling transition are in fact not independent from η' . Indeed one has

$$\delta' = \frac{2 - \eta'}{2 - \varepsilon + \eta'} , \quad (6.9)$$

$$\nu'_u = \nu'_h = \frac{1}{2 - \varepsilon + \eta'} . \quad (6.10)$$

These relations which have no physical meaning at P_3 , are however formally true at that point.

The scaling relation (6.9) connecting δ' to η' was first derived in [10] in the following way. One can introduce a tension by setting $\hat{\tau}_R = 0$ (thus considering the critical theory) and by introducing a linear term δH which breaks explicitly the symmetry (5.10)

$$\delta H = - f \int d^D \sigma (\partial_i u^i) . \quad (6.11)$$

In (6.11) the tension f appears as the conjugate of the field u^i . Thus the anomalous dimension of f is related to that of u^i and, using (6.8), (6.9) can be easily obtained.

In our approach, where the tension has been introduced through the relevant coupling constant $\hat{\tau}_0$, the relation (6.9) follows from the fact that δ' does not involve the anomalous dimension of $\hat{\tau}_0$ (given by γ_τ at the fixed point), but only the wave function renormalization (given by γ). However in general the wave function renormalization (given by γ) and the renormalization of $\hat{\tau}_0$ (given by γ_τ) are independent and in principle they should lead to two

independent critical exponents for the buckling transition. Actually one cannot find any other independent exponent. One can indeed show (see Appendix B) that, as a consequence of the equation of motion, the wave function Wilson function γ and the τ_0 Wilson function γ_τ stop being independent at the two non trivial fixed points P_2 and P_4 corresponding to $\hat{K}_R > 0$, and satisfy the relation :

$$\gamma^* + \gamma_\tau^* - \varepsilon = 0. \quad (6.12)$$

Thus, both for flat elastic and fluid membranes, the buckling transition should be characterized by only one independent critical exponent (for instance η'). (6.12) does not hold at P_3 ($\hat{K}_R = 0$) in contrast with equations (6.9) and (6.10), but the critical exponents have no physical meaning since there is no buckling transition in that case.

Let us end this section by giving explicit results for the critical exponents computed to first order in ε . These values are computed in appendix C, where the explicit form of the Wilson functions β_μ , β_κ , γ and γ_τ in the minimal subtraction scheme are also given. The position of the four fixed points depicted in figure 3 are given in table I to first order in ε .

Table I. — Fixed points at first order in ($\varepsilon = 4 - D$, $d_c = d - D$).

	$\hat{\mu}_R$	\hat{K}_R
P_1	0	0
P_2	0	$\frac{16 \pi^2}{d_c} \cdot \varepsilon$
P_3	$\frac{96 \pi^2}{d_c + 20} \cdot \varepsilon$	0
P_4	$\frac{96 \pi^2}{d_c + 20} \cdot \varepsilon$	$\frac{16 \pi^2}{d_c} \cdot \varepsilon$

The corresponding exponents for the nontrivial fixed points P_2 , P_3 and P_4 are shown in table II.

Let us finally discuss the case of fluid membrane. It is worth mentioning that at order ε the critical exponents corresponding to the fixed point P_2 are in agreement with the predictions of the model of fluid membranes [6] for $D > 2$. Indeed for $D > 2$ this model predicts a flat phase described by a Gaussian fixed point, and thus $\eta' = 0$ for fluid membranes. As discussed in appendix C, we expect that, in the model of elastic membranes, there should be no wave function and $\hat{\mu}$ renormalizations on the line $\hat{\mu}_R = 0$. This implies that the critical exponents for the fixed point P_2 given by table II should be exact for $0 < \varepsilon < 2$. This corroborates our conjecture that P_2 describes the flat phase of liquid membranes.

Another interesting point can be raised on the structure of the R.G flow for the rotationally invariant model of elastic membranes obtained in [4]. On the critical surface, which corresponds then to the *crumpling transition*, the R.G flow has, at first order in $\varepsilon = 4 - D$ and for $d > 219$, the same global structure as the R.G flow for the *buckling transition* depicted in figure 3. A fixed point P'_4 analogous to P_4 describes the crumpling

Table II. — *Critical exponents for the buckling transition at first-order in ε ($\varepsilon = 4 - D$, $d_c = d - D$).*

	P_2	P_3	P_4
η'	0	$\frac{10}{d_c + 20} \varepsilon$	$\frac{12}{d_c + 24} \varepsilon$
η'_u	ε	$\frac{d_c}{d_c + 20} \varepsilon$	$\frac{d_c}{d_c + 24} \varepsilon$
$\delta' - 1$	$\frac{\varepsilon}{2}$	$\frac{d_c}{d_c + 20} \frac{\varepsilon}{2}$	$\frac{d_c}{d_c + 24} \frac{\varepsilon}{2}$
$2 - \frac{1}{\nu'}$	ε	$\frac{d_c + 10}{d_c + 20} \varepsilon$	$\frac{d_c + 12}{d_c + 24} \varepsilon$

transition for elastic membranes. Its domain of attraction is bounded by two lines which are attracted toward two unstable fixed points P'_2 and P'_3 analogous to P_2 and P_3 . For $d < 219$, P'_3 and P'_4 merge and disappear, leaving the unstable fixed point P'_2 alone [4]. We suggest that, in analogy with P_2 , P'_2 describes the crumpling transition for fluid membranes for $2 < D < 4$. Although we have no other evidence for this conjecture than this analogy and the fact that in the large d limit P'_2 should give the correct exponents (which are those of the spherical model), we think that it is not completely unrealistic. In that case, at $D = 2$ (which is the lower critical dimension for fluid membranes), P_2 and P'_2 should give identical critical exponents.

7. Conclusion and perspectives.

Elastic (polymerized) membranes have recently attracted a lot of attention both from theoretical and experimental points of view [15]. The theory of such objects predicts for instance a non-trivial crumpling transition between the low-temperature rigid phase and the high-temperature crumpled phase. Although only few polymerized membranes have been created up to now in a laboratory [16, 17], further theoretical investigations of the thermodynamic properties of these systems seem important in view of future experiments.

In this paper we have generalized the theory of a fluctuating elastic membrane to the case where a nonzero tension is exerted on its boundary. In the rigid phase this tension will increase the lateral extension of the membrane (beyond its « spontaneous » value corresponding to the free boundary conditions). An interesting phenomenon occurs if one decreases then the tension so that the membrane relaxes to its spontaneous size. Such a relaxation can be viewed as a critical phenomenon, with some characteristic non-trivial exponents. The fact that the exponents do not have usual, « mechanical » values (e.g. like the linear Hooke's law between the tension and the extension) is one of the consequences of the breaking down of the classical theory of elasticity. Indeed, thermal fluctuations do modify the classical behavior of the elastic membranes. More interestingly, if the lateral tension was decreased further the membrane would transform to a buckled state with coexisting rigid regions of different orientations. Therefore, it is natural to call the critical phenomenon introduced above the *buckling transition*.

The field theoretical calculations presented here try to quantitatively describe the nature of this transition. The simplest way to verify our results, and in particular the values of the critical exponents, is to perform a Monte Carlo simulation of tethered surfaces similar the simulations which observed the crumpling transition [2, 5]. Obviously, the experimental situation is much more complex. It is probably too early to suggest the way in which the lateral tension of the elastic membranes could be controlled. This would depend on the detailed nature of the system under study : a « theoretical rigid frame » which we introduced in our calculations cannot easily be created in the laboratory. Let us, however, attract the reader's attention to the case of polymerized phospholipid vesicles recently studied by Sackmann, Ringsdorf and their collaborators [17, 18]. In such closed objects the tension can be introduced by varying both the osmotic pressure difference Δp (between the interior and the exterior of the vesicles) and the temperature. For instance, by decreasing the temperature one can contract the polymerized network of the phospholipids by solidifying the membrane components⁽⁵⁾. One could also imagine that the polymerized vesicles will buckle if one decreases their interior volume, V (e.g. by changing Δp) [19]. This cannot happen in fluid membranes, nor even in erythrocytes (note that the network of spectrins in the erythrocytes is *not* polymerized but only forms a ionic gel [21]) since in these systems the changes in V will simply provoke the global shape transformations [19]. Such global transformations, however, are in general hindered if the membrane is covalently polymerized.

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Appendix A

The large d limit.

A.1 THE EFFECTIVE POTENTIAL. — We report in this appendix the calculations concerning the large d limit of the linear model defined by equation (2.12). This limit is obtained, as usual, by taking the coupling constants λ_0, μ_0 to be of order $1/d$. The Hamiltonian (2.12) takes therefore the form

$$H[\mathbf{X}] = \int d^D\sigma \left[\frac{\tilde{\lambda}_0}{8d} (\partial_i \mathbf{X} \cdot \partial_i \mathbf{X} - \delta_{ii})^2 + \frac{\tilde{\mu}_0}{4d} (\partial_i \mathbf{X} \cdot \partial_j \mathbf{X} - \delta_{ij})^2 + \frac{1}{2} \kappa_0 (\Delta \mathbf{X})^2 \right]. \quad (\text{A1})$$

We now introduce a dummy integration over the auxiliary variable λ^{ij} . Absorbing a suitable constant into the definition of the functional integral we obtain

$$Z = \int \mathbf{D}\mathbf{X} \exp \{-H[\mathbf{X}]/T\} = \int \mathbf{D}\mathbf{X} \int \mathbf{D}\lambda^{ij} \exp \left\{ -\frac{d}{T} \tilde{H}[\mathbf{X}, \lambda^{ij}] \right\}, \quad (\text{A2})$$

⁽⁵⁾ Note : Some photographs of reference [17] show indeed the creation of « buckled » vesicles, with coexisting smooth regions separated by the network of « defects » (with the linear extension $\lambda = 100 \text{ \AA}$). Whether this phenomenon has anything to do with the described here buckling transition needs to be proven by further experimental and theoretical studies.

where

$$\tilde{H}[\mathbf{X}, \lambda^{ij}] = \int d^D \sigma \left[\frac{1}{2} \tilde{\kappa}_0 (\Delta \mathbf{X})^2 + \lambda^{ij} (\partial_i \mathbf{X} \cdot \partial_j \mathbf{X} - \delta_{ij}) - \alpha_0 (\lambda^{ij})^2 - \beta_0 (\lambda^{ii})^2 \right], \quad (\text{A3})$$

in which we have introduced the notations

$$\alpha_0 = \frac{1}{\tilde{\mu}_0}; \quad \beta_0 = - \frac{\tilde{\lambda}_0}{\tilde{\mu}_0 (2 \tilde{\mu}_0 + D \tilde{\lambda}_0)}; \quad \tilde{\kappa}_0 = \frac{\kappa_0}{d}. \quad (\text{A4})$$

The nonlinear model considered in reference [3] corresponds to the case $\alpha_0 = \beta_0 = 0$, $\tilde{\kappa}_0$ fixed.

The effective potential can be now computed in the standard way. We split $\mathbf{X}(\sigma)$ into its average $\mathbf{X}_{\text{av}}(\sigma)$ and fluctuations,

$$\mathbf{X}(\sigma) = \mathbf{X}_{\text{av}}(\sigma) + \mathbf{X}_{\text{fl}}(\sigma), \quad (\text{A5})$$

and we perform explicitly the Gaussian integration over \mathbf{X}_{fl} . In the large d limit, the remaining integral over λ^{ij} can be performed by the saddle point method. The result for the effective potential $\Gamma[\mathbf{X}_{\text{av}}]$ reads

$$\frac{1}{d} \Gamma[\mathbf{X}_{\text{av}}] = \left\{ \tilde{H}[\mathbf{X}_{\text{av}}, \lambda^{ij}] + \frac{T}{2} \text{Tr} \ln (\tilde{\kappa}_0 \Delta^2 - 2 \partial_i \lambda^{ij} \partial_j) \right\}_{\text{s.p.}} \quad (\text{A6})$$

The notation s.p. means that one has to evaluate the expression within curly brackets at its saddle point with respect to λ^{ij} . An ultraviolet cutoff Λ is needed to regularize the trace. The detailed cutoff procedure will be made explicit later.

A.2 PLANAR CONFIGURATIONS. — When the membrane is subject to isotropic stretching it is natural to expect $\mathbf{X}_{\text{av}}(\sigma)$ to correspond to an isotropic planar configuration :

$$\mathbf{X}_{\text{av}}(\sigma) = \zeta \sigma^i \mathbf{e}_i, \quad (\text{A7})$$

where

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \quad (\text{A8})$$

We thus expect λ^{ij} , at the saddle point, to be of the form

$$\lambda^{ij} = \lambda_c \delta^{ij}. \quad (\text{A9})$$

Looking for the extremum of the expression in curly brackets with respect to λ_c we obtain an equation relating ζ to λ_c :

$$(1 - \zeta^2) + 2 \lambda_c (\alpha_0 + \beta_0 D) = \frac{T}{D \tilde{\kappa}_0} \int^\Lambda \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + (2 \lambda_c / \tilde{\kappa}_0)}. \quad (\text{A10})$$

We choose a simple regularization procedure, cutting off wavenumbers whose modulus exceeds Λ . The « tension » f , conjugate to ζ , is defined by

$$f = \frac{1}{L^D} \frac{\partial \Gamma}{\partial \zeta}. \quad (\text{A11})$$

One has at the saddle point

$$f = 2 D \lambda_c \zeta . \tag{A12}$$

We now expand the effective potential for small deformations from the planar configuration (A7). We define

$$\mathbf{X}_{av}(\sigma) = \zeta \sigma^i \mathbf{e}_i + u^i \mathbf{e}_i + \mathbf{h} , \tag{A13}$$

$$\lambda^{ij} = \lambda_c \delta^{ij} + \tilde{\lambda}^{ij} , \tag{A14}$$

where $\mathbf{h} \cdot \mathbf{e}_i = 0$ ($i = 1, \dots, D$), and $u^i, \mathbf{h}, \tilde{\lambda}^{ij}$ are small. The saddle point condition on λ^{ij} allows us to express $\tilde{\lambda}^{ij}$ as a function of u^i and \mathbf{h} . We can thus derive the following expression for the inverse propagators $\Gamma_{hh}^{(2)}$ and $\Gamma_{uu}^{(2)}$ of \mathbf{h} and u^i respectively :

$$\Gamma_{hh}^{(2)} = \tilde{\kappa}_0 \left(p^4 + \frac{2 \lambda_c}{\tilde{\kappa}_0} p^2 \right) , \tag{A15}$$

$$\Gamma_{uu}^{(2)} = \tilde{\kappa}_0 \delta_{ij} \left(p^4 + \frac{2 \lambda_c}{\tilde{\kappa}_0} p^2 \right) + 4 \zeta^2 G_{ij,kl}(p) p^k p^l , \tag{A16}$$

where $G_{ij,kl}(p)$ is the symmetrized inverse of the matrix $F_{ij,kl}(p)$ defined by

$$F_{ij,kl}(p) = \frac{1}{2} \alpha_0 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \beta_0 \delta_{ij} \delta_{kl} + \frac{T}{\tilde{\kappa}_0^2} \int \frac{d^D k}{(2 \pi)^D} \frac{k^i q^j k^k q^l}{(k^4 + r k^2)(q^4 + r q^2)} . \tag{A17}$$

In the last expression, we have used the shorthands

$$r = \frac{2 \lambda_c}{\tilde{\kappa}_0} ; \quad q = k + p . \tag{A18}$$

A.3 FREE BOUNDARIES : THE CRUMPLING TRANSITION. — If the boundaries are free, no tension is exerted on the frame. In view of equation (A12), this implies either $\zeta = 0$, or $\lambda_c = 0$. Now, if we let $\lambda_c = 0$ in equation (A10) we obtain

$$1 - \zeta^2 = \frac{T}{T_c} , \tag{A19}$$

where we have defined T_c by

$$\frac{1}{T_c} = \frac{1}{D \tilde{\kappa}_0} \int \frac{d^D k}{(2 \pi)^D} \frac{1}{k^2} . \tag{A20}$$

Of course, equation (A19) has real solutions only if $T \leq T_c$. Let us denote by $\zeta_{sp}(T) > 0$ the solution of equation (A19) for $T < T_c$. In this case, the equilibrium configuration $\mathbf{X}_{cq}(\sigma)$ is given by equation (A7), with $\zeta = \zeta_{sp}(T)$. This describes the low temperature, flat phase with free boundaries. As T increases, $\zeta_{sp}(T)$ decreases, and eventually vanishes for $T = T_c$. Above T_c , equation (A19) cannot be satisfied, and we are forced to take $\zeta = 0, \lambda_c \neq 0$. This introduces a term $\lambda_c \partial_i \mathbf{X} \cdot \partial_i \mathbf{X}$ in the effective potential, which dominates the large distance behavior of the membrane.

We can thus identify T_c with the crumpling transition temperature. The integral defining it in equation (A20) diverges for $D \leq 2$, yielding therefore a vanishing T_c . This identifies two as the lower critical dimension D_1 for $d = \infty$, in agreement with the results obtained in reference [3] for the nonlinear model.

A.4 CONSTRAINED BOUNDARIES: ÉQUATIONS OF STATE. — It is convenient to rewrite equation (A10) with the help of the definitions of T_c and f . One obtains

$$\zeta^2 - \left(1 - \frac{T}{T_c}\right) = A \cdot \left(\frac{f}{\zeta}\right) + T \cdot B \left(\frac{f}{\zeta}\right) \cdot \left(\frac{f}{\zeta}\right)^{\frac{D-2}{2}}, \quad (\text{A21})$$

where

$$A = \frac{\alpha_0 + D\beta_0}{D}, \quad (\text{A22})$$

$$B(t) = \frac{1}{D\bar{\kappa}_0} \int^{\wedge/t} \frac{d^D k}{(2\pi)^D} \left[\frac{1}{k^2 + (1/D\bar{\kappa}_0)} - \frac{1}{k^2} \right].$$

The function $B(t)$ approaches a nonzero constant B when $t \rightarrow 0$ for $D < 4$. If one is interested in small tensions $(f/\zeta) \ll 1$, the equation of state (A21) simplifies to

$$\zeta^2 = 1 - \frac{T}{T_c} + TB \left(\frac{f}{\zeta}\right)^{\frac{D-2}{2}}. \quad (\text{A23})$$

When $D > 4$, the third term in equation (A21) becomes proportional to (f/ζ) and may be absorbed into the regular terms. Classical values of the exponents are thus obtained. This identifies four as the upper critical dimension D_u . We assume in the following $2 < D < 4$.

A.5 CRITICAL EXPONENTS. — The values of the critical exponents can be read off the equation of state (A23) and the expressions (A15, 16) of the inverse propagator, keeping in mind the relation between f and λ_c (Eq. (A12)).

(i) *Crumpling transition.* — The crumpling transition point is given by $T = T_c$, $\zeta = f = 0$. Letting $f = 0$, $T < T_c$ in equation (A23) we obtain $\zeta \sim |T_c - T|^\beta$,

where

$$\beta = \frac{1}{2}. \quad (\text{A24})$$

Letting $T = T_c$, we obtain $\zeta \sim f^{1/\delta}$, where, for $D < 4$,

$$\delta = \frac{D+2}{D-2}. \quad (\text{A25})$$

Since δ is larger than one the hypothesis that $(f/\zeta) \ll 1$ for small f is justified *a posteriori*. We can also define the susceptibility $\chi = \partial\zeta/\partial f$. We have for $T > T_c$, $f = 0$, $\chi \sim |T - T_c|^{-\gamma}$, where

$$\gamma = \frac{2}{D-2}. \quad (\text{A26})$$

For $T \geq T_c$, $f = 0$, we have $\zeta = \zeta_{sp}(T) = 0$. Letting $\zeta = 0$ into equation (A16), we see that the second term vanishes, and that rotational symmetry is restored. We have therefore

$$\Gamma^{(2)}(p) = \tilde{\kappa}_0(p^4 + \xi^{-2} p^2), \tag{A27}$$

where

$$\xi^{-2} = \frac{2 \lambda_c}{\tilde{\kappa}_0}. \tag{A28}$$

At $T = T_c$, we have $\lambda_c = 0$. Hence $\Gamma^{(2)} \sim p^4$. This yields $\eta = 0$, or

$$d_F = \frac{2 D}{4 - D}. \tag{A29}$$

For $T > T_c$, we can set $\zeta = 0$ in equation (A10), obtaining an equation relating λ_c , and therefore ξ , to T . Keeping only the most singular term we obtain

$$1 = \frac{T}{D \tilde{\kappa}_0} \int \frac{d^D k}{(2 \pi)^D} \frac{1}{k^2 + \xi^{-2}}.$$

By means of the definition (A20) of T_c this equation may be written in a more transparent way :

$$\frac{1}{D \tilde{\kappa}_0} \int \frac{d^D k}{(2 \pi)^D} \left[\frac{1}{k^2 + \xi^{-2}} - \frac{1}{k^2} \right] = \frac{1}{T} - \frac{1}{T_c}. \tag{A30}$$

The l.h.s. behaves as $\xi^{-(D-2)}$ for $\xi \rightarrow \infty$. We have therefore $\xi \sim (T - T_c)^{-\nu}$, where

$$\nu = \frac{1}{D - 2}. \tag{A31}$$

Remark that the scaling law $\gamma = (2 - \eta) \nu$ is satisfied.

(ii) *Buckling transition.* — We now consider $T < T_c$. First of all we obtain from equation (A23) $\zeta - \zeta_{sp}(T) \sim f^{1/\delta'}$, with

$$\delta' = \frac{2}{D - 2}. \tag{A32}$$

Since $\zeta \neq 0$, we see from equation (A12) that $\lambda_c \sim f$. On the other hand, equation (A15) implies that the correlation length ξ_h of h is given by

$$\xi_h^{-2} = \frac{2 \lambda_c}{\tilde{\kappa}_0}. \tag{A33}$$

we obtain therefore $\xi_h^{-2} \sim f \sim (\zeta - \zeta_{sp})^{\delta'}$. Hence

$$\nu'_h = \frac{1}{D - 2}. \tag{A34}$$

On the other hand, when $f = 0$ we have $\Gamma_{hh}^{(2)} \sim p^4$, which yields

$$\eta' = 0. \tag{A35}$$

For $D < 4$, the second term dominates the first in equation (A16). One has in fact $F_{ij,kl}(p) \sim p^{D-4}$ as $p \rightarrow 0$. We obtain therefore, as $f = 0$, $\Gamma_{uu}^{(2)} \sim p^{6-D}$, hence

$$\eta'_u = 4 - D. \quad (\text{A36})$$

(If $D > 4$, $F_{ij,kl}(p)$ has a finite limit as $p \rightarrow 0$ and one recovers $\eta'_u = 0$). The correlation length ξ_u is defined by

$$\Gamma_{uu}(p) = \xi_u^{-(6-D)} f(p\xi_u). \quad (\text{A37})$$

Equation (A16) implies

$$\xi_u^{-2} \propto \frac{2\lambda_c}{\tilde{\kappa}_0} = \xi_h^{-2}. \quad (\text{A38})$$

This confirms the validity of scaling near the buckling transition and yields

$$\nu'_u = \frac{1}{D-2}. \quad (\text{A39})$$

Appendix B

Renormalization and scaling relations for the model of flat membranes.

We start with the Hamiltonian (5.8) for the model of flat membranes

$$H_{\text{Flat}} = \int d^D\sigma \left\{ \frac{1}{2} (\partial^2 \tilde{\mathbf{h}})^2 + \hat{\mu}_0 (u_{ij})^2 + \frac{\hat{\lambda}_0}{2} (u_{ii})^2 + \hat{\tau}_0 (u_{ii}) \right\} \quad (\text{B1})$$

with the stress tensor

$$u_{ij} = \frac{1}{2} (\partial_i \tilde{u}^j + \partial_j \tilde{u}^i + \partial_i \tilde{\mathbf{h}} \partial_j \tilde{\mathbf{h}}). \quad (\text{B2})$$

where

$$\begin{aligned} \tilde{u}^i &= \kappa_0 u^i, & \tilde{\mathbf{h}} &= \kappa_0^{1/2} \mathbf{h}, \\ \hat{\mu}_0 &= \kappa_0^{-2} \mu_0, & \hat{\lambda}_0 &= \kappa_0^{-2} \lambda_0, \\ \hat{\tau}_0 &= \kappa_0^{-1} \tau_0. \end{aligned} \quad (\text{B3})$$

It is in fact convenient to decompose u_{ij} into its traceless part

$$v_{ij} = u_{ij} - \frac{u_{kk}}{D} \delta_{ij} \quad (\text{B4})$$

and its trace

$$v = u_{kk}, \quad (\text{B5})$$

so that (B1) becomes

$$H_{\text{Flat}} = \int d^D\sigma \left\{ \frac{1}{2} (\partial^2 \tilde{\mathbf{h}})^2 + \hat{\mu}_0 (v_{ij})^2 + \frac{\hat{K}_0}{2} v^2 + \hat{\tau}_0 v \right\} \quad (\text{B6})$$

with the compression modulus

$$\hat{K}_0 = \hat{\lambda}_0 + \frac{2}{D} \hat{\mu}_0. \tag{B7}$$

In mean field, the classical domain of stability, where the Hamiltonian H_{Flat} is bounded from below, is defined by

$$\hat{\mu}_0 > 0, \quad \hat{K}_0 > 0 \tag{B8}$$

and is bounded by the two lines $\hat{\mu}_0 = 0$ (which describes an isotropic solid with vanishing shear modulus) and $\hat{K}_0 = 0$ (which describes an isotropic solid with vanishing compression modulus). Thanks to the Ward identities (5.11) associated to the linearized rotational symmetry (5.10), H_{Flat} keeps its form under renormalization and may be written in terms of renormalized fields \tilde{u}_R , \tilde{h}_R and of renormalized coupling $\hat{\mu}_R$, \hat{K}_R and $\hat{\tau}_R$

$$H_{\text{Flat}} = \int d^D \sigma \left[\frac{Z}{2} (\partial^2 \tilde{h}_R)^2 + M^\varepsilon Z_\mu \hat{\mu}_R (v_{ij}^R)^2 + M^\varepsilon Z_K \frac{\hat{K}_R}{2} (v^R)^2 + Z_\tau \tau_R v^R \right] \tag{B9}$$

with

$$u_{ij}^R = \frac{1}{2} (\partial_i \tilde{u}_R^j + \partial_j \tilde{u}_R^i + \partial_i \tilde{h}_R \partial_j \tilde{h}_R), \tag{B10}$$

$$v_{ij}^R = v_{ij}^R - \frac{1}{D} u_{kk}^R \delta_{ij}, \quad v^R = u_{kk}^R, \tag{B11}$$

where the renormalization factors Z are defined by

$$\tilde{u}_R^i = Z^{-1} \tilde{u}^i, \quad \tilde{h}_R = Z^{-1/2} \tilde{h}, \tag{B12}$$

and

$$\begin{aligned} \tilde{\mu}_R &= M^{-\varepsilon} Z^2 Z_\mu^{-1} \hat{\mu}_0, & K_R &= M^{-\varepsilon} Z^2 Z_K^{-1} \hat{K}_0, \\ \hat{\tau}_R &= Z Z_\tau^{-1} \hat{\tau}_0. \end{aligned} \tag{B13}$$

By ignoring a possible additive term in equation (B13) we have assumed that we have used dimensional regularization. M is the renormalization mass scale and

$$\varepsilon = 4 - D, \tag{B14}$$

$\hat{\tau}_R$ and \hat{K}_R are dimensionless coupling constants, $\hat{\tau}_R$ has the dimension of a squared mass. When using the minimal subtraction scheme the renormalization factors Z_μ , Z_K , Z and Z_τ depend only on the dimensionless coupling constants $\hat{\lambda}_R$ and \hat{K}_R , and not on $\hat{\tau}_R$. The Wilson functions are defined in the standard way.

$$\beta_\mu = M \frac{\partial}{\partial M} \hat{\mu}_R \Big|_0, \quad \beta_K = M \frac{\partial}{\partial M} \hat{K}_R \Big|_0 \tag{B15}$$

$$\gamma = M \frac{\partial}{\partial M} \ln Z \Big|_0, \quad \gamma_\tau = M \frac{\partial}{\partial M} \ln \hat{\tau}_R \Big|_0, \tag{B16}$$

where $|_0$ means that the bare quantities $\hat{\mu}_0, \hat{K}_0, \hat{\tau}_0$ are kept fixed. Using (B13) we get

$$\begin{bmatrix} \beta_\mu \\ \beta_K \end{bmatrix} = -\varepsilon \begin{bmatrix} \frac{\partial}{\partial \hat{\mu}_R} \log(\hat{\mu}_R Z_\mu Z^{-2}) \frac{\partial}{\partial \hat{K}_R} \log(\hat{\mu}_R Z_\mu Z^{-2}) \\ \frac{\partial}{\partial \hat{\mu}_R} \log(\hat{K}_R Z_K Z^{-2}) \frac{\partial}{\partial \hat{K}_R} \log(\hat{K}_R Z_K Z^{-2}) \end{bmatrix} \cdot \begin{bmatrix} \hat{\mu}_R \\ \hat{K}_R \end{bmatrix}, \quad (B17)$$

and

$$\gamma = \beta_\mu \frac{\partial}{\partial \hat{\mu}_R} \ln Z + \beta_K \frac{\partial}{\partial \hat{K}_R} \log Z, \quad (B18)$$

$$\gamma_\tau = -\beta_\mu \frac{\partial}{\partial \hat{\mu}_R} \log(Z_\tau Z^{-1}) - \beta_K \frac{\partial}{\partial \hat{K}_R} \log(Z_\tau Z^{-1}). \quad (B19)$$

Let us first show a general feature of the renormalization of the model. On the critical surface $\hat{\tau}_0 = 0$, the two instability lines $\hat{\mu}_0 = 0$ and $\hat{K}_0 = 0$ are preserved by renormalization. In other words

$$\hat{\mu}_0 = 0 \Leftrightarrow \hat{\mu}_R = 0 \quad (B20)$$

$$\hat{K}_0 = 0 \Leftrightarrow \hat{K}_R = 0 \quad (B21)$$

and the classical stability domain (B8) is globally preserved by the renormalization group flow. This remarkable property is a simple consequence of the additional symmetries that the action (B1) possess on the two lines. Indeed, if $\hat{K}_0 = 0$ (and $\hat{\tau}_0 = 0$), the action is invariant under the global dilation.

$$\tilde{u}^i(\sigma) \rightarrow \tilde{u}^i(\sigma) + A\sigma^i; \quad \tilde{\mathbf{h}}(\sigma) \rightarrow \tilde{\mathbf{h}}(\sigma). \quad (B22)$$

In the standard way, the Ward identities associated to this symmetry are sufficient to prove that \hat{K}_0 and $\hat{\tau}_0$ are not generated by renormalization. The same argument holds if $\hat{\mu}_0 = 0$ by considering the global shear transformations

$$\tilde{u}^i(\sigma) \rightarrow \tilde{u}^i(\sigma) + A^{ij}\sigma^j; \quad \tilde{\mathbf{h}}(\sigma) \rightarrow \tilde{\mathbf{h}}(\sigma) \quad (B23)$$

where A^{ij} is a traceless matrix.

Scaling relations and critical exponents. — We denote by $\Gamma_R^{(M,N)}$ the renormalized 1-particle irreducible function with M external $\tilde{\mathbf{h}}_R$ legs and N external \tilde{u}_R legs. It obeys the renormalization group equation

$$\left[M \frac{\partial}{\partial M} + \beta_\mu \frac{\partial}{\partial \hat{\mu}_R} + \beta_K \frac{\partial}{\partial \hat{K}_R} - \left(\frac{M}{2} + N \right) \gamma + \gamma_\tau \hat{\tau}_R \frac{\partial}{\partial \hat{\tau}_R} \right] \Gamma_R^{(M,N)} = 0 \quad (B24)$$

and from scaling, p_i denoting the D -dimensional momenta carried by the $M + N$ external legs,

$$\left[\sum_i p_i \frac{\partial}{\partial p_i} + \frac{M}{2} (D - 4 + \gamma) + N (D - 3 + \gamma) - D + (2 - \gamma_\tau) \hat{\tau}_R \frac{\partial}{\partial \hat{\tau}_R} \right] \Gamma_R^{(M,N)} = \left[\beta_\mu \frac{\partial}{\partial \hat{\mu}_R} + \beta_K \frac{\partial}{\partial \hat{K}_R} \right] \Gamma_R^{(M,N)}. \quad (B25)$$

Thus for the free membrane with vanishing tension ($\hat{\tau}_R = 0$), at a fixed point ($\hat{\mu}_R^*, \hat{K}_R^*$) where $\beta_\rho^* = \beta_\rho(\hat{\mu}_R^*, \hat{K}_R^*) = 0$ and $\beta_\kappa^* = \beta_\kappa(\hat{\mu}_R^*, \hat{K}_R^*) = 0$, the two point functions (provided that they are not singular), behave as

$$\Gamma_{hh}(\mathbf{p}) = \Gamma^{(2,0)}(\mathbf{p}) \sim |p|^{4-\gamma^*} \tag{B26}$$

$$\Gamma_{uu}(\mathbf{p}) = \Gamma^{(0,2)}(\mathbf{p}) \sim |p|^{2+\varepsilon-2\gamma^*}. \tag{B27}$$

And we obtain the scaling relation

$$\eta'_u = \varepsilon - 2\eta' \tag{B28}$$

where

$$\eta' = \gamma^*. \tag{B29}$$

For a stretched membrane ($\hat{\tau}_R > 0$) the effective potential $\Gamma(\tilde{\mathbf{h}}_R, \tilde{u}_R^i)$ is now minimal at $\partial^2 \tilde{\mathbf{h}}_R = 0$, $\partial_i \tilde{u}_R^j = (\zeta - 1) \delta_{ij}$ with $\zeta > 1$. The « tension » f defined by (4.1) is, using the Ward identities (5.11)

$$\begin{aligned} f &= \frac{1}{L^D} \int d^D \sigma \sigma^i \frac{\partial}{\partial \sigma^i} \Gamma \Big|_{\zeta=0} \\ &= -D \frac{\partial}{\partial p^2} \Gamma^{(2,0)}(\mathbf{p}) \Big|_{p^2=0}. \end{aligned} \tag{B30}$$

From the R.G. equation (2.25) we expect that at any fixed point

$$\Gamma^{(2,0)}(p^2) \sim (\hat{\tau}_R)^{\frac{4-\gamma^*}{2-\gamma^*}} g \left(p^2 \hat{\tau}_R^{\frac{-2}{2-\gamma^*}} \right) \tag{B31}$$

where g is some scaling function, regular at $p^2 = 0$. On the other hand $\zeta - 1$ should vanish with $\hat{\tau}_R$. $\zeta - 1$ is obtained by minimizing $\Gamma(\zeta)$, where from (2.23) we expect the scaling form at any fixed point (Y is some scaling function)

$$\Gamma(\zeta) = Y[(\zeta - 1)^{2-\gamma^*} \hat{\tau}_R^{2-D-\gamma^*}]. \tag{B32}$$

Minimizing $\Gamma(\zeta)$ with respects to ζ we obtain

$$(\zeta - 1) \propto \hat{\tau}_R \frac{D - 2 + \gamma^*}{2 - \gamma^*} \tag{B33}$$

and thus we get the scaling relation between δ' and η'

$$\delta' = \frac{2 - \gamma^*}{D - 2 + \gamma^*} = \frac{2 - \eta'}{D - 2 + \eta'}. \tag{B34}$$

Finally from (2.25) for $\hat{\tau}_R > 0$ at a fixed point the correlation lengths ξ_n and $\xi_{\tilde{h}}$ for \tilde{u}_R and \tilde{h}_R are finite and should both scale as

$$\xi \sim (\hat{\tau}_R)^{\frac{1}{\gamma^*-2}}. \tag{B35}$$

Thus we obtain the scaling relations for ν'_u and ν'_h

$$(\nu'_u)^{-1} = (\nu'_h)^{-1} = D - 2 + \eta'. \quad (\text{B36})$$

Relation between γ_τ and γ . — One may notice that in the scaling relations (B28), (B34) and (B36), γ_τ^* does not appear. Indeed, in [10] it was shown that the role of the tension may be studied by considering free boundary conditions, instead of fixed one, and by adding an external source of the form $-f \int d^D \sigma (\partial_i \tilde{u}_i)$ to the Hamiltonian (B1) instead of a tension term $\hat{\tau}_0$. It follows that critical exponents involving the tension f should depend on the anomalous dimension of u (related to η) only, and not on the renormalization of the tension $\hat{\tau}_0$. The question is whether the exponent γ_τ^* represents a new critical exponent at the buckling transition or if there is a scaling relation between γ_τ^* and γ^* .

We show here that in fact, for any fixed point corresponding to a compressible membrane ($\hat{K}_R \neq 0$), the following scaling relation holds :

$$\gamma_\tau^* + \gamma^* = \varepsilon. \quad (\text{B37})$$

From (B33) this implies that $\zeta - 1 \sim \hat{\tau}_R$. This relation is a consequence of the fact that starting from the Hamiltonian (B1) with $\hat{\tau}_0 \neq 0$, one can always perform a global dilation of the form (B22) in order to make $\hat{\tau}_0$ equal to zero (of course such a dilation changes the boundary conditions on the field \tilde{u}^i). Indeed we have the relation

$$H_{\text{Flat}}(\tilde{\mathbf{h}}, \tilde{u}^i; \hat{\mu}_0, \hat{K}_0, \hat{\tau}_0) = H_{\text{flat}}(\tilde{\mathbf{h}}, \tilde{u}^i; \tilde{\mu}_0, \hat{K}_0, 0) + L^D \frac{\hat{\tau}_0^2}{2 \hat{K}_0} \quad (\text{B38})$$

with

$$\tilde{u}^i(\sigma) = \tilde{u}^i(\sigma) + \sigma^i (\hat{\tau}_0 / D \hat{K}_0). \quad (\text{B39})$$

The same relation (B38) holds if we replace the Hamiltonian H_{flat} by the full effective potential Γ . In terms of renormalized quantities it yields (using (B12) and (B13))

$$\Gamma_R(\tilde{\mathbf{h}}_R, \tilde{\mu}_R^i; \hat{\mu}_R, \hat{K}_R, \hat{\tau}_R) = \Gamma_R(\tilde{\mathbf{h}}_R, \tilde{\mu}_R^i; \hat{\mu}_R, \hat{K}_R, 0) + \dots \quad (\text{B40})$$

with

$$\tilde{u}_R^i(\sigma) = \tilde{u}_R^i(\sigma) + \sigma^i \frac{\hat{\tau}_R}{\hat{K}_R} \cdot \frac{Z_\tau}{M^\varepsilon Z_K} \cdot \frac{1}{D} \quad (\text{B41})$$

where ... denotes a constant which does not depend on the fields $\tilde{\mathbf{h}}_R$ and \tilde{u}_R . Since Γ_R must be a regular function of the renormalized fields and couplings, we deduce immediately that the function

$$a(\hat{\mu}_R, \hat{K}_R) = \frac{1}{D} \frac{Z_\tau}{\hat{K}_R M^\varepsilon Z_K} = \frac{1}{D} \frac{\hat{\tau}_0}{\hat{K}_0} \cdot \frac{1}{Z \hat{\tau}_R} = \frac{1}{D} \frac{Z_\tau}{Z^2 \hat{K}_0} \quad (\text{B42})$$

is a finite function of the renormalized couplings even when $\varepsilon \rightarrow 0$. Using (B15) and (B16) we have

$$\begin{aligned} M \frac{d}{dM} \ln a \Big|_0 &= -\varepsilon + \beta_\rho \frac{\partial}{\partial \rho_R} \ln a + \beta_\kappa \frac{\partial}{\partial \kappa_R} \ln a \\ &= -\gamma - \gamma_\tau. \end{aligned} \quad (\text{B43})$$

Hence the relation between the γ functions is

$$\gamma + \gamma_\tau - \varepsilon = -\beta_\mu \frac{\partial}{\partial \rho_R} \ln a - \beta_\kappa \frac{\partial}{\partial \kappa_R} \ln a . \tag{B44}$$

At a fixed point ($\beta_\mu^* = \hat{\beta}_\kappa = 0$), the scaling relation (B37) follows, provided that the function a is not singular. As we shall see in appendix C, where the renormalization factors Z are computed to one loop in the minimal subtraction scheme, we expect that the factors Z and Z_τ are singular only on the line $\hat{K}_0 + 2 \left(\frac{D-1}{D} \right) \hat{\mu}_0 = 0$ which is outside the stability domain for $D > 1$. From (B42) the function a will be singular only on the line $\hat{K}_0 = 0$, that is $\hat{K}_R = 0$! Thus the scaling relation (B37) should hold only for fixed points inside the stability domain ($\hat{\mu}_R \geq 0, \hat{K}_R > 0$).

Appendix C

ε expansion for flat membranes.

In this appendix we compute the renormalization factors Z, Z_μ, Z_K and Z_τ at one loop in the minimal subtraction scheme and give the renormalization group flows to first order in $\varepsilon = 4 - D$.

From the Hamiltonian (B1) we see that the diagrammatic rules of the perturbative expansion for flat membranes are the following.

The $\langle \tilde{h}^\alpha \tilde{h}^\beta \rangle$ propagator is

$$\alpha \xrightarrow[p]{} \beta = \delta_{\alpha\beta} (p^4 + \hat{\tau}_0 p^2)^{-1}, \quad \alpha, \beta = 1, \dots, d_c = d - D . \tag{C1}$$

The $\langle \tilde{u}^i \tilde{u}^j \rangle$ propagator is

$$\begin{matrix} i \\ \text{~~~~~} \\ j \end{matrix} \xrightarrow[p]{} = \frac{1}{p^2} \left[\frac{1}{\hat{\mu}_0} \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) + \frac{1}{2 \hat{\mu}_0 + \hat{\lambda}_0} \frac{p_i p_j}{p^2} \right] \tag{C2}$$

$$i, j = 1, D ,$$

and is singular if $\hat{\mu}_0 = 0$ or $2 \hat{\mu}_0 + \hat{\lambda}_0 = 0$. The vertices are of two types. One vertex involves four \tilde{h} external legs

$$\begin{matrix} 1_\alpha & & \beta 2 \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ 4_\delta & & \gamma 3 \end{matrix} = \delta_{\alpha\delta} \delta_{\beta\gamma} (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{p}_3 \cdot \mathbf{p}_4) \frac{\hat{\mu}_0}{4} + \delta_{\alpha\gamma} \delta_{\delta\beta} (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{p}_3 \cdot \mathbf{p}_4) \frac{\hat{\lambda}_0}{8} . \tag{C3}$$

One vertex involves two \tilde{h} and one \tilde{u} external legs (C5)

$$\begin{matrix} 1_\alpha & & \\ & \diagdown & \\ & \text{~~~~~} & \\ & / & \\ 2_\beta & & \end{matrix} \ell_3 = -\frac{i}{2} \delta_{\alpha\beta} \{ [(\mathbf{p}_3 \cdot \mathbf{p}_1) p_2^\ell + (\mathbf{p}_3 \cdot \mathbf{p}_2) p_1^\ell] \tilde{\mu}_0 + (\mathbf{p}_1 \cdot \mathbf{p}_2) p_3^\ell \tilde{\lambda}_0 \} . \tag{C4}$$

$$Z_{\hat{\lambda}} = 1 + \frac{d_c}{96 \pi^2 \varepsilon} \frac{\hat{\mu}_0^2 + 6 \hat{\lambda}_R \hat{\mu}_R + 6 \hat{\lambda}_R^2}{\hat{\lambda}_R} \tag{C7}$$

or equivalently

$$Z_{\hat{K}} = 1 + \frac{d_c}{16 \pi^2 \varepsilon} \cdot \hat{K}_R. \tag{C8}$$

Z_{τ} and Z are obtained by estimating the pole at $\varepsilon = 0$ of the one loop contribution to $\Gamma^{(2,0)}$, given by (g + h)

$$\begin{aligned} & \alpha \text{---} \text{---} \text{---} \text{---} \text{---} \beta \quad + \quad \alpha \text{---} \text{---} \text{---} \text{---} \text{---} \beta = \\ & = \frac{1}{\varepsilon} \frac{\delta_{\alpha\beta}}{\hat{\tau}_0^{\varepsilon/2}} \cdot \frac{1}{32 \pi^2} \left[-p^4 \cdot \frac{10 \hat{\mu}_0 (\hat{\lambda}_0 + \hat{\mu}_0)}{\hat{\lambda}_0 + 2 \hat{\mu}_0} + \hat{\tau}_0 p^2 d_c (\hat{\mu}_0 + 2 \hat{\lambda}_0) \right]. \end{aligned} \tag{C9}$$

This pole is cancelled by the counterterms

$$Z_{\tau} = 1 + \frac{d_c}{32 \pi^2 \varepsilon} (\hat{\mu}_R + 2 \hat{\lambda}_R), \tag{C10}$$

$$Z = 1 - \frac{5}{16 \pi^2 \varepsilon} \frac{\hat{\mu}_R (\hat{\lambda}_R + \hat{\mu}_R)}{(\hat{\lambda}_R + 2 \hat{\mu}_R)}, \tag{C11}$$

or equivalently

$$Z_{\tau} = 1 + \frac{d_c}{16 \pi^2 \varepsilon} \hat{K}_R. \tag{C12}$$

$$Z = 1 - \frac{5}{16 \pi^2 \varepsilon} \frac{\hat{\mu}_R (2 \hat{K}_R + \hat{\mu}_R)}{(2 \hat{K}_R + 3 \hat{\mu}_R)}. \tag{C13}$$

Let us stress that although the propagator (C2) diverges at $\hat{\mu}_0 = 0$, the counterterms are regular at $\hat{\mu}_R = 0$. In fact, starting from the Hamiltonian (B1), one can integrate explicitly over the field \tilde{u}_i to get an effective potential for \tilde{h}^{α} of the form

$$\begin{aligned} \Gamma_{\text{eff}}[\tilde{\mathbf{h}}] &= \int d^D \sigma \frac{1}{2} (\partial^2 \tilde{\mathbf{h}})^2 + \int d^D \sigma \int d^D \sigma' (\partial_i \tilde{\mathbf{h}} \partial_j \tilde{\mathbf{h}}) (\sigma) \\ &\quad \times G^{ij,kl} (\sigma - \sigma') (\partial_k \tilde{\mathbf{h}} \partial_l \tilde{\mathbf{h}}) (\sigma') \end{aligned} \tag{C14a}$$

$$G^{ij,kl} = \frac{\hat{\mu}_0}{2} \left\{ (P_{ik} P_{j\ell} + P_{i\ell} P_{jk}) + \frac{2 \hat{\lambda}_0}{\hat{\lambda}_0 + 2 \hat{\mu}_0} P_{ij} P_{kl} \right\} \tag{C14b}$$

where P_{ij} is the projector on transverse modes

$$P_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \tag{C15}$$

$G^{ij,kl}$ is only singular on the line $\hat{\lambda}_0 + 2 \hat{\mu}_0 = 0$, and vanishes on the line $\hat{\mu}_0 = 0$. From the Ward identities (5.11) all renormalization factors can be deduced from $\tilde{\mathbf{h}}$ correlation functions. Since for $\hat{\mu}_0 = 0$ the effective theory for $\tilde{\mathbf{h}}$ is free we expect that $Z = Z_\rho = 1$. It is however a nontrivial result that for $\hat{\mu}_0 = 0$, $Z_{\hat{K}}$ and Z_τ are still non singular and non trivial.

We now compute the renormalization group functions. Using (B17-19) they read

$$\beta_\rho = -\varepsilon \hat{\mu}_R + \frac{1}{16 \pi^2} \hat{\mu}_R^2 \left[\frac{d_c}{6} + 10 \frac{\hat{K}_R + 2 \hat{\mu}_R}{\hat{K}_R + 6 \hat{\mu}_R} \right] \quad (\text{C16})$$

$$\beta_{\hat{K}} = -\varepsilon \hat{K}_R + \frac{1}{16 \pi^2} \hat{K}_R \left[\frac{d_c}{4} \hat{K}_R + 10 \frac{\hat{K}_R + 2 \hat{\mu}_R}{\hat{K}_R + 6 \hat{\mu}_R} \cdot \hat{\mu}_R \right] \quad (\text{C17})$$

$$\gamma = \frac{5}{16 \pi^2} \hat{\mu}_R \frac{\hat{K}_R + 2 \hat{\mu}_R}{\hat{K}_R + 6 \hat{\mu}_R} \quad (\text{C18})$$

$$\gamma_\tau = \frac{1}{16 \pi^2} \left[\frac{d_c}{4} \hat{K}_R + 5 \hat{\mu}_R \cdot \frac{\hat{K}_R + 2 \hat{\mu}_R}{\hat{K}_R + 6 \hat{\mu}_R} \right]. \quad (\text{C19})$$

The corresponding renormalization group flow in the $\hat{\tau}_R = 0$ plane has four fixed points

$$\begin{aligned} P_1 \quad & \hat{\mu}_R = 0, \quad \hat{K}_R = 0; \\ P_2 \quad & \hat{\mu}_R = 0, \quad \hat{K}_R = \varepsilon \cdot \frac{16 \pi^2}{d_c}; \\ P_3 \quad & \hat{\mu}_R = \varepsilon \cdot \frac{96 \pi^2}{d_c + 20}, \quad \hat{K}_R = 0; \\ P_4 \quad & \hat{\mu}_R = \varepsilon \cdot \frac{96 \pi^2}{d_c + 24}, \quad \hat{K}_R = \varepsilon \cdot \frac{16 \pi^2}{d_c + 24}. \end{aligned} \quad (\text{C20})$$

P_1 is the Gaussian fixed point and has two unstable directions, P_2 and P_3 have one IR unstable direction and are located on the two boundaries of the domain of stability. P_4 is IR stable and inside the domain of stability. It governs the behavior of flat membranes. The anomalous dimensions γ^* and γ_τ^* are given for the three nontrivial fixed points by

	P_2	P_3	P_4
γ	0	$\frac{10 \varepsilon}{d_c + 20}$	$\frac{12 \varepsilon}{d_c + 24}$
γ_τ	ε	$\frac{10 \varepsilon}{d_c + 20}$	$\frac{(d_c + 12) \varepsilon}{d_c + 24}$

(C21)

One checks that the scaling relation (B37) holds only for the two fixed points P_2 and P_4 which are not located on the line $\hat{K}_R = 0$, as argued in appendix B. Indeed, from the explicit expressions for Z_τ and $Z_{\hat{K}}$, the function a defined by (B42) is at one loop.

$$a = \frac{1}{M^\varepsilon} \frac{1}{\hat{K}_R D} \quad (C22)$$

In a is singular and the (r.h.s.) of (B44) is indeed nonzero for $\hat{K}_R = 0$.

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