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**Short Communication**

## Accurate critical exponents from field theory

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**Résumé.** — Ceci est le troisième article d'une série dans laquelle le calcul des exposants critiques des modèles avec symétrie  $O(N)$  à partir de la théorie des champs est réexaminé. Dans un premier article nous avons appliqué des méthodes de sommation aux nouveaux termes calculés du développement en  $\varepsilon$  (connu maintenant jusqu'à l'ordre  $\varepsilon^5$ ). Dans un deuxième article nous avons incorporé dans la méthode de sommation du développement en  $\varepsilon$ , dans le cas des systèmes du type Ising ( $N = 1$ ), la connaissance des exposants critiques à deux dimensions. Ici nous utilisons la même méthode dans le cas de la statistique des polymères ( $N = 0$ ) pour laquelle les exposants sont maintenant également connus en dimension deux. Par ailleurs comme nous avons constaté une légère différence dans le cas de l'exposant  $\eta$  pour  $N = 1$  entre les valeurs venant des développements en  $\varepsilon$  et à dimension fixée, nous avons appliqué une variante plus raffinée de la méthode de sommation à ce dernier cas. Les résultats changent très peu indiquant à la fois la fiabilité de la méthode et ses limites.

**Abstract.** — This is the third article in a sequence in which we reexamine the calculation of the critical exponents of the  $N$ -vector model from field theory. In the first article we have applied summation methods to the  $\varepsilon$ -expansion for which new terms have been calculated (up to order  $\varepsilon^5$ ). In the second article we have incorporated in the summation method, in the case of Ising-like systems ( $N = 1$ ), the knowledge of the exact values of exponents in  $d = 2$  dimensions. Since exact values are now also known in the case of polymers ( $N = 0$ ), we apply here the same method to this case. Moreover, since for the exponent  $\eta$  for  $N = 1$  the results coming from the  $\varepsilon$ -expansion and perturbation series at fixed dimension are slightly different, we have recalculated the exponents in the latter case using the variant of the summation method developed for the  $\varepsilon$ -expansion. The results do not change significantly showing both the reliability and the limits of the method.

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## 1. Introduction.

Wilson-Fisher's  $\varepsilon$ -expansion [1] ( $\varepsilon = 4 - d$ ) has provided the first semi-quantitative predictions coming from field theory for the critical exponents of the  $N$ -vector model. However, the first accurate estimates [2, 3] were based on the summation of perturbation series at fixed dimension  $d$  because it was possible to calculate more terms of the expansion. Recently the  $\varepsilon$ -expansion has been extended up to order  $\varepsilon^5$  [4]. We have therefore applied a variant of the summation method, based on Borel transformation and mapping, used successfully in the previous case, to the  $\varepsilon$ -expansion [5]. In three dimensions the apparent errors of the new results are slightly larger than those coming from the series at fixed dimension and this is consistent with the fact that the latter series are still longer (order  $g^6$ ). One advantage of the  $\varepsilon$ -expansion is that the same summation methods can yield estimates in two dimensions where exact results are known. Indeed we have verified that the estimates for critical exponents were consistent for  $N = 1$  and  $d = 2$  with the exact values of the Ising model. Subsequently this has led us quite naturally to constrain the summation method in such a way that it reproduces automatically these known results. We have first applied this idea to Ising-like systems [6]. The accuracy of the results in three dimensions is then comparable to those coming from the series at fixed dimension. Both sets of results are still consistent. The main difference arises in the case of the exponent  $\eta$  and this will prompt us, in the second part of the article, to reanalyze the series at fixed dimension for  $N = 1$  with a variant of the summation method, first introduced for the  $\varepsilon$ -expansion, to understand whether the difference comes from the series (i.e. the fact that we know only a finite number of terms) or from the summation method. However, the main purpose of this article is to take into account the results derived for the exponents of the self-avoiding walk ( $N = 0$ ) in two dimensions [7] to improve the field theory predictions in three dimensions. Since the summation method has already been thoroughly discussed in previous publications [2,3,5,6], we shall in next section just recall the main steps and then present our results.

## 2. The summation method.

Let us first recall the method as applied in the case of the series at fixed dimension [2]. We want to calculate a function  $F(g)$  which is given by a divergent series :

$$F(g) = \sum_{k=0}^{\infty} F_k g^k, \quad (1)$$

for which the large order behavior is known :

$$F_k \underset{k \rightarrow \infty}{\sim} k! (-a)^k k^b c. \quad (2)$$

In the field theory cases this information has been derived from instanton considerations [8]. For the method only the  $k!$  behavior and the constant  $a$  are essential. The constant  $b$ , which is unknown in the case of the  $\varepsilon$ -expansion [9], is only marginally useful.

We introduce the Borel-Leroy transform of the function  $F(g)$  :

$$F(g) = \int_0^{\infty} t^\rho e^{-t} B(gt) dt. \quad (3)$$

The expansion of  $B(z)$  is then given by :

$$B(z) = \sum_0^{\infty} \frac{F_k}{\Gamma(k + \rho + 1)} z^k. \tag{4}$$

The asymptotic form (2) implies that  $B(z)$  is analytic in a circle of radius  $1/a$ . However, to be able to calculate  $F(g)$  from the representation (3) we need the function  $B(z)$  on the real positive axis. In the case of the  $\phi^4$  field theory various considerations suggest that correlation functions are actually analytic in a cut-plane. To generate an expansion which converges in a neighborhood of the real positive axis, we therefore map the cut-plane onto a circle, leaving the origin invariant :

$$z = \frac{4}{a} \frac{u}{(1 - u)^2}. \tag{5}$$

We have found it useful to write  $B(z(u))$  in the form :

$$B(z(u)) = \tilde{B}(u)(1 - u)^{-\sigma}, \tag{6}$$

$\tilde{B}(u)$  being the function which is expanded in a power series in  $u$ .  $\rho$  and  $\sigma$  are two parameters which are adjusted to optimize the convergence of the method.

Finally we do not expand the physical quantities directly in powers of the coupling constant  $g$  (or  $\varepsilon$  in case of the  $\varepsilon$ -expansion) but instead first make a change of variable :

$$g \mapsto g' : g' = g/(1 - \tau g), \tag{7}$$

in which  $\tau$  is also an adjustable parameter. This modification was essential in the case of the  $\varepsilon$ -expansion to map away expected singularities on the real positive axis too close to the physical values.

### 3. The self-avoiding walk ( $N = 0$ ).

We have calculated critical exponents in  $d = 3$  dimensions from the  $\varepsilon$ -expansion for the  $N = 0$  vector model, taking into account the exactly known values in two dimensions. The method is very simple. Let  $f(\varepsilon)$  be a function given by an expansion in powers of  $\varepsilon$  for which we know the value for  $\varepsilon = 2$  exactly. We then set :

$$f(\varepsilon) = f(2) + (2 - \varepsilon) \tilde{f}(\varepsilon), \tag{8}$$

and expand instead  $f(\varepsilon)$  in a power series which we sum, using the method recalled in previous section. Table I gives the results coming from the perturbation series at fixed dimension (1), the unconstrained  $\varepsilon$ -expansion (2), and the constrained  $\varepsilon$ -expansion (3).

Table I.— *Self-avoiding walk exponents.*

	$\gamma$	$\nu$	$\beta$	$\eta$
(1)	$1.1615 \pm 20$	$0.5880 \pm 15$	$0.3020 \pm 15$	$0.027 \pm 4$
(2)	$1.160 \pm 4$	$0.5885 \pm 25$	$0.3025 \pm 25$	$0.031 \pm 3$
(3)	$1.157 \pm 3$	$0.5880 \pm 15$	$0.3035 \pm 20$	$0.0320 \pm 25$

We remark that, as in the case of Ising-like systems, the apparent errors decrease and globally the results (3) are as good as the results (1) derived from perturbation series at fixed dimension. The stablest exponent is  $\nu$  for which the two determinations are almost identical. All determinations of  $\beta$  agree well within error bars. However, the estimates for  $\gamma$  and  $\eta$  are only marginally consistent.

For  $\gamma$  the unconstrained  $\varepsilon$ -expansion (2) is more consistent with the result (1). The reason to this deviation can be related to the value of the two dimensional estimate for  $\gamma$  coming from the  $\varepsilon$ -expansion which is too large compared to the exact result :

$$\gamma(\varepsilon = 2) = 1.39 \pm 0.04, \quad \gamma_{\text{exact}} = 1.34375.$$

This means that for  $\gamma$  the direct results coming from the  $\varepsilon$ -expansion cannot be really improved by imposing  $d = 2$  results.

For  $\eta$  the situation is different ; the  $\varepsilon$ -expansion leads to satisfactory estimates in two dimensions :

$$\eta(\varepsilon = 2) = 0.21 \pm 0.05, \quad \eta_{\text{exact}} = 0.2083\dots$$

and the new results are more reliable. They confirm a trend already observed for Ising-like systems, i.e. that the  $\varepsilon$ -expansion leads to estimates for  $\eta$  systematically larger than those coming from the fixed dimension series. The new result (3) is in addition more accurate than the old estimate (1). We think that the new result is likely to be closer to the exact value.

Let us note that the various scaling relations among exponents (for instance  $\beta = (d\nu - \gamma)/2$  and  $\eta = 2 - \gamma/\nu$ ) are satisfied within the apparent errors for the three methods and therefore do not discriminate between them.

Let us finally indicate that new determinations coming from high temperature series [10] give :

$$\gamma = 1.161 \pm 0.002, \quad \nu = 0.592 \pm 0.003,$$

and that Monte Carlo simulations yield for  $\nu$  :

$$\nu = 0.592 \pm 0.002 \text{ [11]} \quad \text{and} \quad \nu = 0.592 \pm 0.003 \text{ [12]}.$$

While the estimate for  $\gamma$  compares well with the field theory result,  $\nu$  is somewhat large. We can

only emphasize again that all field theory estimates for  $\nu$  are extremely consistent, and that the  $\varepsilon$ -expansion leads to good estimates in two dimensions :

$$\nu(\varepsilon = 2) = 0.76 \pm 0.03, \quad \nu_{\text{exact}} = 0.75.$$

We also recall that the most accurate experimental values coming from polymers are [13] :

$$\nu = 0.586 \pm 0.004.$$

#### 4. Ising-like systems ( $N = 1$ ).

To try to understand the small discrepancy between the two estimates for  $\eta$  coming from the series at fixed dimension and the  $\varepsilon$ -expansion, we have reanalyzed the old fixed dimension series in three dimensions with the method described in section 2 and which differs from the original method by the change of variable (7). In the case of the  $\varepsilon$ -expansion this modification plays a very important role. In addition we have realized that it is more efficient to consider the series for  $\sqrt{\eta}$  than  $\eta$  itself because  $\eta$  is of order  $g^2$ . The new results given in table II : line (1) corresponds to the  $\varepsilon$ -expansion, line (2) gives the early fixed dimension estimates, and (3) the new results.

The obvious conclusion is that the discrepancy for  $\eta$  has not disappeared. Applying the same method to the unfortunately shorter fixed dimension series in two dimensions we obtain :

$$\gamma = 1.79 \pm 4, \quad \nu = 0.96 \pm 4, \quad \eta = 0.18 \pm 4.$$

We see that the result for  $\eta$  is smaller than the exact Ising model value. If we assume that the same tendency is observed in the three-dimensional series we would conclude that the results (2) and (3) underestimate  $\eta$  and that the  $\varepsilon$ -expansion result is more reliable. In addition this  $\varepsilon$ -expansion value is in better agreement with high-temperature results [14].

Table II.— *Ising-like system exponents.*

	$\gamma$	$\nu$	$\beta$	$\eta$
(1)	$1.2390 \pm 25$	$0.6310 \pm 15$	$0.3270 \pm 15$	$0.0375 \pm 25$
(2)	$1.2410 \pm 20$	$0.6300 \pm 15$	$0.3250 \pm 15$	$0.031 \pm 4$
(3)	$1.2405 \pm 15$	$0.6300 \pm 15$	$0.3250 \pm 15$	$0.032 \pm 3$

Note finally that the same method applied to the  $N$ -vector model for  $N = 2$  also yields for example  $\nu = 0.6695 \pm 20$  showing again that the old fixed dimension estimates are hardly affected by the modification of the summation method.

A last result which we give here because it has not been reported in a previous publication is the following : for  $N = 1$  in two dimensions no correction terms associated with the correc-

tion exponent  $\omega$  have been found. However results coming from conformal field theories yield a correction exponent for multicritical points which extrapolated to the ordinary critical point is :

$$\omega = 4/3.$$

If we believe this prediction we can incorporate it in the summation method of the corresponding  $\varepsilon$ -expansion to obtain a new result for  $\omega$  in three dimensions :  $\omega_1$ . We have called  $\omega_2$  the unconstrained value derived from the  $\varepsilon$ -expansion. We can then compare the result with the fixed dimension summation ( $\omega_3$ ). The results are :

$$\omega_1 = 0.78 \pm 4, \quad \omega_2 = 0.81 \pm 4, \quad \omega_3 = 0.79 \pm 3.$$

The modification brings the results closer but the apparent error is not reduced.

## 5. Conclusion.

By incorporating into the  $\varepsilon$ -expansion the knowledge of the exact values of exponents in two dimensions we obtain estimates in three dimensions which are essentially as accurate as those derived from perturbation series in three dimensions. The agreement between results coming from both methods is quite satisfactory. The main discrepancy occurs for the exponent  $\eta$ . It seems that in the absence of any new information it is not possible to further improve the accuracy of the determination. More terms in the series or a better analytic understanding are required. Finally it would be worthwhile completing the whole analysis by also considering the  $N$ -vector model for  $N = 2$  and  $N = 3$  since there too exact results in and near two dimensions are available.

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