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Critical properties of fluid membranes with hexatic order

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Résumé. — Les propriétés à grande distance d'un modèle de membranes fluides présentant un ordre hexatique d'orientation sont étudiées. Il est montré que la raideur hexatique K_A n'est pas renormalisée par les fluctuations thermiques tant que les défauts d'orientation (disinclinations) peuvent être négligés. Pour des grandes valeurs de K_A , les trajectoires du groupe de renormalisation pour le module de rigidité κ sont attirées vers un point fixe infrarouge non trivial. Dans cette situation, lorsque la tension de surface s'annule, une membrane hexatique est un objet fractal caractérisé par une dimension fractale $d_F > 2$ et une dimension d'étalement $d_s < 2$, qui varient continûment avec K_A .

Abstract. — The long distance behaviour of a model of fluid membranes with orientational (hexatic) order and small surface tension is investigated. It is shown that, if orientational defects (disclinations) are neglected, the hexatic stiffness K_A is not renormalized by thermal fluctuations. The renormalization flow of the rigidity modulus κ goes, at large K_A , to a nontrivial infrared stable fixed point. In this situation, hexatic membranes with vanishing effective surface tension are smooth critical objects with a finite fractal dimension $d_F > 2$ and a spreading dimension $d_s < 2$ which depend continuously on K_A , in contrast with the case of fluid membranes.

1. Introduction.

Models of fluid two-dimensional membranes with small surface tension have recently been the object of extensive research. They are of interest both as idealizations of amphiphilic layers and of interfaces in microemulsions [1] and as toy models of strings in quantum field theory [2]. In a fluid membrane, molecules can flow freely to adapt themselves to any particular shape of the surface. Hence the elastic free energy depends only on the membrane shape : i.e. it cannot depend on the particular coordinate system chosen to represent the surface. If the surface tension is small the dominant contribution to the free energy is the bending elasticity, which depends on the extrinsic (mean) curvature of the membrane. Large transverse fluctuations (undulations) take place, which make the effective rigidity decrease at large distances [2-6]. Normals to the membrane are

thus expected to be correlated only up to a persistence length ξ . This can be explicitly checked in the limit where the dimension D of bulk space in which the membrane is embedded goes to infinity [7-10]. At distances larger than ξ the membrane is crumpled, i.e. bending rigidity is ineffective. It is then reasonable to expect that at such distances the effective action describing its behaviour coincides with Polyakov's string model [2].

This picture changes drastically if correlations among the positions of the molecules forming the membrane exist [11]. The molecules may exhibit in plane crystalline order, and form a kind of two dimensional solid. The stretching elasticity associated with this order conspires with the bending elasticity to make the membrane rigid and flat at long distances [11]. One may thus observe a *crumpling transition* [4], separating a rigid phase from a crumpled phase. Such a suggestion has been corroborated by numerical simulations [12]. On the other hand, the molecules may exhibit a weaker order in

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which the orientations (but not the lengths) of bonds connecting neighbouring particles are correlated at long distances [13]. This *hexatic* phase is analogous to a two-dimensional nematic liquid crystal. A field theoretical model describing a hexatic membrane susceptible to undulations was introduced in reference [11]. A first order calculation showed that the associated *hexatic stiffness* tends to increase the effective rigidity at long distances [11], and to counteract the effect of undulations. One can thus expect that the combined effects of bending elasticity and hexatic stiffness may lead to a novel behaviour. These effects are analysed by field theoretical techniques in the present paper. We find that the long distance behaviour of hexatic membranes is determined by a nontrivial infrared stable fixed point at least in the limit of large hexatic stiffness. We identify and study the new phase corresponding to this fixed point. This phase is reminiscent of the low temperature phase of the two dimensional *XY* model, being characterized by nontrivial exponents which depend continuously on the hexatic stiffness.

The model is described and discussed in section 2, and its renormalization is derived in section 3. Section 4 is dedicated to the analysis of the critical exponents characterizing the new phase. Conclusions and perspectives of further study are contained in section 5.

2. The model for hexatic membrane.

2.1 THE ACTION. — We first recall the covariant formulation of membrane models. Defining locally a system of coordinate $\underline{\sigma} = (\sigma^i; i = 1, 2)$ on the membrane and denoting by $\mathbf{X}(\underline{\sigma})$ the position of the point $\underline{\sigma}$ in bulk D -dimensional Euclidean space, the metric tensor $g_{ij}(\underline{\sigma})$ induced by the embedding is

$$g_{ij}(\underline{\sigma}) = \mathbf{t}_i(\underline{\sigma}) \cdot \mathbf{t}_j(\underline{\sigma}); \quad \mathbf{t}_i(\underline{\sigma}) = \frac{\partial \mathbf{X}}{\partial \sigma^i}(\underline{\sigma}). \quad (2.1)$$

The extrinsic curvature tensor \mathbf{K}_{ij} is

$$\mathbf{K}_{ij}(\underline{\sigma}) = D_i D_j \mathbf{X}(\underline{\sigma}) \quad (2.2)$$

where D_i are covariant derivatives with respects to the metric g_{ij} . The element of area reads

$$d^2s = d^2\underline{\sigma} \sqrt{|g|} \quad |g| = \det(g_{ij}). \quad (2.3)$$

The action is then

$$F_1 = \kappa_0 \frac{1}{2} \int d^2s \mathbf{K}_i^i \mathbf{K}_j^j + r_0 \int d^2s \quad (2.4)$$

κ_0 and r_0 are respectively the microscopic rigidity modulus and the microscopic surface tension.

We now want to describe in a reparametrization invariant way a membrane with orientational order.

This means that to each point on the membrane is associated a preferred direction within the tangent plane of the membrane. (In an hexatic phase this direction will correspond to the bonds orientation at that point.) Let us associate to each point with coordinate $\underline{\sigma}$ an order parameter $\mathbf{n}(\underline{\sigma})$ which is a D -dimensional unit vector tangent to the membrane.

It writes in the basis of the tangent plane given by the \mathbf{t} 's (2.1) :

$$\mathbf{n}(\underline{\sigma}) = n^i(\underline{\sigma}) \mathbf{t}_i(\underline{\sigma}) \quad (2.5)$$

with the constraint

$$|\mathbf{n}|^2 = n^i(\underline{\sigma}) n^j(\underline{\sigma}) g_{ij}(\underline{\sigma}) = 1. \quad (2.6)$$

If \mathbf{n} is the order parameter corresponding to an hexatic phase, the action must be invariant under global rotations of \mathbf{n} by $\pm \pi/3$. This symmetry is the remnant of the rotational symmetry of the triangular crystalline phase in the hexatic phase [13]. As we shall see in next subsection, the only dimensionless action for \mathbf{n} which respects this \mathbb{Z}_6 symmetry is the one proposed in [11]

$$F_2 = \frac{K_A}{2} \int d^2s g^{ij}(\underline{\sigma}) g_{kl}(\underline{\sigma}) D_i n^k(\underline{\sigma}) D_j n^l(\underline{\sigma}). \quad (2.7)$$

The dimensionless coupling constant K_A is the hexatic stiffness. It may be rewritten in terms of an angle variable θ by introducing at each point $\underline{\sigma}$ two orthonormal vectors $\mathbf{v}_a(\underline{\sigma})$ ($a = 1, 2$) tangent to the membrane. This is equivalent to introduce a 2-bein $\mathbf{e}_a^i(\underline{\sigma})$ compatible with the induced metric $g_{ij}(\underline{\sigma})$

$$\mathbf{e}_a^i \mathbf{e}_b^j g_{ij} = \delta_{ab} \quad (2.8)$$

if we write

$$\mathbf{v}_a = \mathbf{e}_a^i \mathbf{t}_i. \quad (2.9)$$

In this basis, the order parameter \mathbf{n} is written

$$\mathbf{n}(\underline{\sigma}) = n^a(\underline{\sigma}) \mathbf{v}_a(\underline{\sigma}) \quad (2.10)$$

$$n^a(\underline{\sigma}) n^b(\underline{\sigma}) \delta_{ab} = 1.$$

The angle $\theta(\underline{\sigma})$ is then defined locally by

$$\begin{cases} n^1(\underline{\sigma}) = \cos \theta(\underline{\sigma}) \\ n^2(\underline{\sigma}) = \sin \theta(\underline{\sigma}). \end{cases} \quad (2.11)$$

We introduce the spin connection [14]

$$\omega_{ib}^a = \mathbf{v}_b \cdot (\partial_i \mathbf{v}_a) = g_{jk} \mathbf{e}_a^j D_i (\mathbf{e}_b^k). \quad (2.12)$$

Since it is a 2×2 antisymmetric matrix with respect to a and b , it may be written as

$$\omega_{ib}^a(\underline{\sigma}) = \varepsilon_b^a \Omega_i(\underline{\sigma}) \quad (2.13)$$

(ε_b^a is the rank 2 antisymmetric tensor). We have also

$$\partial_i n^a(\underline{\sigma}) = \varepsilon_b^a \partial_i \theta(\underline{\sigma}). \quad (2.14)$$

In term of the angle variable $\theta(\underline{\sigma})$, F_2 finally reads :

$$F_2 = \frac{K_A}{2} \int d^2s g^{ij} (\partial_i \theta + \Omega_i) (\partial_j \theta + \Omega_j). \quad (2.15)$$

This form of the action is invariant under local transformations $\theta(\underline{\sigma}) \rightarrow \theta(\underline{\sigma}) + \Lambda(\underline{\sigma})$, $\Omega_i(\underline{\sigma}) \rightarrow \Omega_i(\underline{\sigma}) - \partial_i \Lambda(\underline{\sigma})$. This gauge invariance has no physical meaning and corresponds to a local rotation of the reference frame (\mathbf{v}_a).

Let us note that the scalar curvature R is :

$$R = \frac{2}{\sqrt{|g|}} \varepsilon^{ij} \partial_i \Omega_j. \quad (2.16)$$

Thus, the action F_2 corresponds to a XY model on the membrane. Indeed on a flat surface, one can impose $\Omega_j = 0$, which leads to the usual XY model. If the curvature R is not zero, Ω_j cannot be set to zero and it induces a nontrivial coupling between the order parameter and the intrinsic geometry of the membrane. Such a coupling describes the frustration induced by the curvature : it is not possible to find a vector field \mathbf{n} which is mapped on itself by parallel transport along a closed curve.

2.2 DERIVATION OF THE HEXATIC ACTION. — We now show that the action (2.7) is the only one allowed by dimensional considerations, by the hexatic rotational symmetry, and by reparametrization invariance. The action must be a functional of \mathbf{n} , of its covariant derivatives, and of the membrane shape *via* its metric and extrinsic curvature tensor. Since the action density must have (inverse length) dimension two, at most two derivatives must appear in each of its terms. Moreover, we will show that it is not possible to build up terms invariant upon rotations of \mathbf{n} by $\pi/3$ in the tangent plane, as dictated by hexatic symmetry, without introducing derivatives of \mathbf{n} . In fact imposing this symmetry is tantamount to impose symmetry upon rotations by an arbitrary angle. The vector field \mathbf{n} should only appear *via* its covariant derivatives :

$$\nabla_a n^b = e_a^i (\partial_i n^b + \omega_{ic}^b n^c). \quad (2.17)$$

The only scalar of dimension two which can be built with the covariant derivatives of \mathbf{n} , up to total derivatives, is given by :

$$(\nabla_a n^b)(\nabla^a n_b). \quad (2.18)$$

One can substantiate this conclusion by considering all possible scalars of dimension two which can be build from \mathbf{n} and the extrinsic curvature tensor

$$\mathbf{K}_{ab} = e_a^i e_b^j \mathbf{K}_{ij}. \quad (2.19)$$

Beyond (2.18) and the bending energy density \mathbf{K}^2 , we have :

$$[(n_a \nabla^a) n_b][(n_c \nabla^c) n^b]; \quad (2.20a)$$

$$\mathbf{K}_a^a \cdot \mathbf{K}_{cd} n^c n^d; \quad (2.20b)$$

$$\mathbf{K}_{ac} \cdot \mathbf{K}_d^a n^c n^d; \quad (2.20c)$$

$$\mathbf{K}_{ab} \cdot \mathbf{K}_{cd} n^a n^b n^c n^d. \quad (2.20d)$$

Other terms can be reduced to these, up to total derivatives (in particular the scalar curvature R).

It is easy to see that each term in equations (2.20) breaks the invariance upon global rotations of \mathbf{n} . For example, the term (2.20a) tends to align \mathbf{n} along the direction of its gradient.

The physical meaning of the other terms in equations (2.20) is especially simple in three dimensions. Indeed, \mathbf{K}_{ij} is proportional to the normal vector \mathbf{m}

$$\mathbf{K}_{ij} = K_{ij} \mathbf{m} \quad (2.21)$$

and along the principal axis the metric tensor g_{ij} and the second fundamental form K_{ij} read

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{ij} = \begin{pmatrix} 1/r_1 & 0 \\ 0 & 1/r_2 \end{pmatrix} \quad (2.22)$$

r_1 and r_2 are the two curvature radii. Then (2.20b, c, d) respectively reduces to

$$\frac{1}{2} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \cos 2\theta + \frac{1}{2} \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \quad (2.23a)$$

$$\frac{1}{2} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \cos 2\theta + \frac{1}{2} \left(\frac{1}{r_1} + \frac{1}{r_2} \right)^2 \quad (2.23b)$$

$$\left[\frac{1}{2} \left(\frac{1}{r_2} + \frac{1}{r_1} \right) + \frac{1}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \cos 2\theta \right]^2. \quad (2.23c)$$

Those terms tend to align the vector \mathbf{n} along one of the principal axes. Hence, the terms contained in equations (2.20) are not invariant under global rotations of \mathbf{n} :

$$\theta(\underline{\sigma}) \rightarrow \theta(\underline{\sigma}) + \theta_0. \quad (2.24)$$

Only the kinetic term (2.18) is invariant under such a global $O(2)$ symmetry group. This ensures that the other terms will not be generated by the renormalization if they are set to zero in the bare action for \mathbf{n} .

This last requirement is satisfied because those terms also break explicitly the discrete subgroup \mathbb{Z}_6 of rotation by $\pi/3$ corresponding to the hexatic rotational symmetry.

3. Renormalization group properties at large hexatic stiffness K_A .

In order to understand the long distance properties of the system described by the action $F_1 + F_2$, the

renormalization group flow for the couplings κ and K_A has to be determined. Let us first show that as long as topological excitations for θ are neglected (vortices in the XY language or disclinations in the hexatic language), K_A gets only a finite renormalization at all orders in perturbation theory.

Indeed, fixing the shape $\mathbf{X}(\sigma)$ of the membrane and minimizing F_2 with respect to $\theta(\sigma)$, the classical equation for θ is ($\Delta = D^i D_i$ is the scalar laplacian) :

$$\Delta \theta_{\text{cl}} + D^i \Omega_i = 0. \quad (3.1)$$

Writing $\theta = \tilde{\theta} + \theta_{\text{cl}}$ and using (2.15) and (2.16) we get

$$F_2(\theta) = \frac{K_A}{2} \int d^2s g^{ij} \partial_i \tilde{\theta} \partial_j \tilde{\theta} + \frac{K_A}{8} S_{\text{Liouville}}[g_{ij}] \quad (3.2)$$

$S_{\text{Liouville}}$ is the celebrated Liouville action which is known to play an important role in Polyakov's string model [15], and which appears here at the classical level

$$S_{\text{Liouville}}[g_{ij}] = \int d^2\sigma \sqrt{|g(\sigma)|} \times \int d^2\sigma' \sqrt{|g(\sigma')|} R(\sigma) \left(\frac{1}{-\Delta} \right)_{\sigma\sigma'} R(\sigma'). \quad (3.3)$$

If the fugacity of topological excitations for the hexatic order parameter is very small, we may forget the fact that $\tilde{\theta}$ is defined modulo 2π and treat it as a real scalar field. A fundamental result by Polyakov [15] shows that for a one component real scalar field ϕ in a two dimensional metric g_{ij} , the effective action may be computed explicitly *via* the conformal anomaly with the result :

$$\int \mathcal{D}[\phi] e^{-1/2 \int \sqrt{g} g^{ij} \partial_i \phi \partial_j \phi} = e^{-\Gamma[g]} \quad (3.4)$$

$$\Gamma(g) = -\frac{1}{96\pi} S_{\text{Liouville}}(g) \quad (3.5)$$

(up to a renormalization of the surface tension r_0 and an additional piece depending on the modular parameters which characterize the conformal class of the metric g ; this last piece does not play any role in what follows). This result has two consequences. First the integration over the fluctuations $\tilde{\theta}$ leads to the effective action

$$S_{\text{eff}}(g) = \frac{1}{96\pi} (12\pi K_A - 1) S_{\text{Liouville}}(g) \quad (3.6)$$

which contains all the effect of hexatic order (if topological excitations are neglected) and has to be added to F_1 before integrating over the position of

the membrane. Second the same effective action (3.6) can be obtained from a N -component massless free scalar field $\Phi = (\phi^\alpha; \alpha = 1, N)$ coupled to the metric of the membrane by the $0(N)$ invariant action

$$F'_2 = \frac{1}{2} \int d^2s g^{ij} \partial_i \phi^\alpha \partial_j \phi^\beta \delta_{\alpha\beta} \quad (3.7)$$

provided that $N = 1 - 12\pi K_A$. Considering now the model given by the action $F_1 + F'_2$, it is quite obvious that fluctuations in the shape of the membrane cannot renormalize N . Since the two models $F_1 + F_2$ and $F_1 + F'_2$ give the same effective action (up to a change in r_0) K_A does not get renormalized either. Thus as long as topological excitations are not taken into account, K_A gets only a finite renormalization

$$K_A^{\text{ren}} = K_A - \frac{1}{12\pi} \quad (3.8)$$

and the β function associated to K_A vanishes at all orders

$$\beta_{K_A} = -\Lambda \frac{\partial}{\partial \Lambda} K_A^{\text{ren}} = 0. \quad (3.9)$$

Once we know the renormalization properties of K_A , we may set up a systematic perturbative expansion in $1/K_A$ valid for large K_A . Indeed, rescaling

$$\kappa_0 = \bar{\kappa}_0 \cdot K_A, \quad r_0 = K_A \bar{r}_0 \quad (3.10)$$

the total action becomes proportional to K_A and one can study within the $1/K_A$ expansion the renormalization of $\bar{\kappa}_0$ and \bar{r}_0 . For that purpose it is convenient to use the Monge form for almost planar membranes, i.e. to project the membrane onto the reference plane (x_1, x_2) by writing

$$\mathbf{X}(\sigma) = (x^1 = \sigma^1, x^2 = \sigma^2, \mathbf{x}_\perp(\sigma)) \quad (3.11)$$

where the transverse displacement \mathbf{x}_\perp has $D-2$ components, and to expand around the classical solution $\mathbf{x}_\perp = 0$. In such a coordinate system there is no need to introduce a Faddeev-Popov determinant in order to take into account the gauge fixing.

As long as we do not compute correlation functions of the \mathbf{n} field, it is convenient to integrate out θ explicitly and to use the action $F_1 + S_{\text{eff}}$ as given by (2.4) and (3.6). Expanding S_{eff} to leading order in \mathbf{x}_\perp we get the non local interaction term

$$\frac{K_A}{8} \int d^2\sigma \int d^2\sigma' R(\sigma) R(\sigma') G(\sigma, \sigma') \quad (3.12)$$

where

$$G(\sigma, \sigma') = (1/ -\partial^2)_{\sigma\sigma'} = \frac{1}{4\pi} \log |\sigma - \sigma'|^2 \quad (3.13)$$

is the two dimensional massless propagator and $R(\sigma)$ is the scalar curvature at first order

$$R(\sigma) = \varepsilon_{ab} \varepsilon_{cd} \partial_a \partial_c \mathbf{x}_\perp(\sigma) \partial_b \partial_d \mathbf{x}_\perp(\sigma) \quad (3.14)$$

(3.14) agrees with the previous result of [11].

Renormalization of the bending rigidity $\bar{\kappa}$ is most easily studied by looking at the \mathbf{x}_\perp two point function. One can check that at first order in $1/K_A$, no other contributions occur than the one loop contributions of the ordinary membrane model and the contribution coming from (3.12) already computed in [11]. The effective bending rigidity $\bar{\kappa}$ is found to be

$$\bar{\kappa}_{\text{eff}}(p) = \bar{\kappa}_0 + \frac{1}{K_A} \frac{1}{8\pi} \left(-D + \frac{3}{4\bar{\kappa}_0} \right) \times \ln \left(\frac{\Lambda}{p} \right)^2 + 0 \left(\frac{1}{K_A^2} \right) \quad (3.15)$$

where Λ is some momentum cut-off. Introducing as in [2] the inverse rigidity $\bar{\alpha} = 1/\bar{\kappa}$, the β function for $\bar{\alpha}$ reads to first order

$$\beta_{\bar{\alpha}}(\bar{\alpha}, K_A) = \frac{1}{K_A} \frac{1}{4\pi} \left(-D\bar{\alpha}^2 + \frac{3}{4}\bar{\alpha}^3 \right) + 0 \left(\frac{1}{K_A^2} \right). \quad (3.16)$$

Thus the β function has a non trivial zero at

$$\bar{\alpha}^* = \frac{4D}{3} + 0 \left(\frac{1}{K_A} \right) \quad (3.17)$$

which corresponds to a non trivial infrared stable fixed point, as depicted in figure 1. As suggested in [11], if the membrane is rigid at short distance, it will become softer at large distance ; on the contrary, if soft enough at short distance, it will become stiffer at large distance.

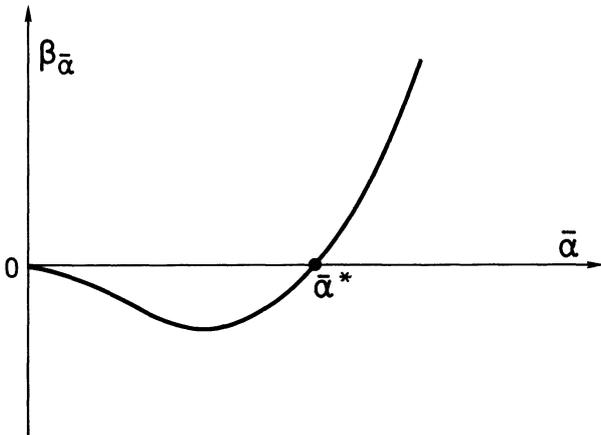


Fig. 1. — The β function for the inverse rigidity $\bar{\alpha}$ at large hexatic stiffness.

4. Critical properties at large hexatic stiffness.

The existence of a non trivial infrared fixed point for large hexatic stiffness K_A has drastic consequences on the large distance properties of the membrane. At the critical value r_0^{crit} of the microscopic surface tension r_0 where the effective surface tension vanishes in perturbation theory and where the membrane becomes a critical object with infinite area, the membrane will be in a self similar « crinkled » phase, instead of the « crumpled » phase described in [2, 4]. This new phase will be characterized by non trivial critical indices.

As r_0 goes to r_0^{crit} the persistence length ξ_p (correlation length for the normals to the surface) should diverge as

$$\xi_p \sim |r_0 - r_0^{\text{crit}}|^{-\nu} \quad (4.1)$$

and the effective surface tension should vanish as

$$\tau \sim |r_0 - r_0^{\text{crit}}|^\mu. \quad (4.2)$$

At the critical point, the two point correlation function C between normals to the surface should decay only with a power law at large distance (in bulk space) R

$$C(R) \sim |R|^{-\eta} \quad (4.3)$$

and the membrane will have non trivial scaling dimensions such as its fractal dimension d_F and its spreading dimension d_s .

In order to compute those indices it is convenient to use observables invariant under the Euclidean group of displacement in bulk D dimensional space [16]. Indeed such observables are free from infrared divergences [17] and allow direct calculations at the critical point where the surface tension vanishes in perturbation theory. For instance the area of the membrane situated within distance R from some point σ_0 on it is given by

$$A(R) = \int d^2\sigma \sqrt{|g(\sigma)|} \times \theta [R^2 - (\mathbf{X}(\sigma) - \mathbf{X}(\sigma_0))^2]. \quad (4.4)$$

Expanding (4.4) to second order in \mathbf{x}_\perp by using (3.11) and computing its expectation value with the action (2.4) (the term (3.12) will enter only at fourth order in \mathbf{x}_\perp) gives the explicit first order term of the $1/K_A$ expansion for A :

$$A(R) = \pi R^2 + \frac{1}{K_A} \bar{\alpha}_0 \frac{(D-2)}{8} R^2 \ln (\Lambda^2 R^2) + \dots \quad (4.5)$$

Since, at the fixed point $\bar{\alpha}^*$, $A(R)$ should behave as R^{d_F} the fractal dimension is

$$d_F = 2 + \frac{1}{K_A} \frac{D(D-2)}{3\pi} + 0 \left(\frac{1}{K_A^2} \right). \quad (4.6)$$

Moreover, we expect that the Euclidean distance $[\mathbf{X}(\underline{\sigma}) - \mathbf{X}(\underline{\sigma}_0)]^2$ will not be renormalized, since it defines the physical distance in bulk space. Then, the only renormalized operator in (4.4) is the composite operator $\sqrt{|g|}$ and d_F is nothing but its scaling dimension. Since $\sqrt{|g|}$ is the operator conjugate to the surface tension r , whose scaling dimension defines ν^{-1} , we have the scaling relation

$$\nu^{-1} = d_F. \tag{4.7}$$

The results (3.16), (4.6) and (4.7) are corroborated by looking at a general Euclidean invariant observable

$$\hat{F}(\mathbf{X}(\underline{\sigma})) = \int d^2\sigma \sqrt{|g(\underline{\sigma})|} F([\mathbf{X}(\underline{\sigma}) - \mathbf{X}(\underline{\sigma}_0)]^2) \tag{4.8}$$

where F is an arbitrary (regular enough) function of the Euclidean distance between the points labelled by $\underline{\sigma}$ and $\underline{\sigma}_0$. Using the Monge form (3.11), we can expand to increasing orders in $\frac{1}{K_A}$ the expectation value

$$\begin{aligned} \langle \hat{F} \rangle_{\Lambda, \bar{\alpha}_0, \bar{r}_0} &= \\ &= \frac{\int [\mathcal{D}\mathbf{X}] \hat{F}[\mathbf{X}(\underline{\sigma})] \exp[-(F_1 + S_{\text{eff}})]}{\int [\mathcal{D}\mathbf{X}] \exp[-(F_1 + S_{\text{eff}})]}. \end{aligned} \tag{4.9}$$

(We suppose that $\bar{\alpha}_0 = K_A \alpha_0$ and $\bar{r}_0 = \frac{r_0}{K_A}$ are less than or of order unity). The « Λ » subscript indicates that fluctuations of $\mathbf{x}_\perp(\sigma)$ with momenta higher than Λ have been ignored. An explicit calculation of $\langle \hat{F} \rangle_{\Lambda, \bar{\alpha}_0, \bar{r}_0}$ to second order in $\frac{1}{K_A}$ shows that the divergences which occur in the limit $\Lambda \rightarrow \infty$ are removed by the following redefinitions :

$$\begin{aligned} \bar{\alpha}_R &= \bar{\alpha}_0 + \frac{\bar{\alpha}_0^2}{4 \pi K_A} \left(D - \frac{3}{4} \bar{\alpha}_0 \right) \ln \left(\frac{\Lambda}{\bar{\mu}} \right) + \\ &\quad + 0 \left(\frac{1}{K_A^2} \right) \\ \bar{r}_R &= \bar{r}_0 - \frac{1}{K_A} \left(\frac{\Lambda^2 D}{8 \pi} - \bar{r}_0 \frac{(D-2)}{4 \pi} \bar{\alpha}_R \ln \left(\frac{\Lambda}{\bar{\mu}} \right) \right) + \\ &\quad + 0 \left(\frac{1}{K_A^2} \right) \end{aligned} \tag{4.10}$$

$$\hat{F}_R = Z_F \hat{F};$$

$$Z_F = 1 - \frac{\bar{\alpha}_R}{4 \pi K_A} (D-2) \ln \left(\frac{\Lambda}{\bar{\mu}} \right) + 0 \left(\frac{1}{K_A^2} \right)$$

$\bar{\mu}$ is the renormalization scale. The renormalized

quantities $\bar{\alpha}_R$, \bar{r}_R and \hat{F}_R are defined by requiring the existence of

$$\begin{aligned} \mathcal{F}_R(\bar{\alpha}_R, \bar{r}_R, \bar{\mu}) &= \\ &= \lim_{\Lambda \rightarrow \infty} \langle \hat{F}_R \rangle_{\Lambda, \bar{\alpha}_0(\bar{\alpha}_R, \bar{r}_R, \bar{\mu}, \Lambda), \bar{r}_0(\bar{\alpha}_R, \bar{r}_R, \bar{\mu}, \Lambda)}. \end{aligned} \tag{4.11}$$

This condition for an arbitrary F fixes uniquely (up to finite parts) the three renormalizations (4.10). Since the quantity

$$\begin{aligned} \langle \hat{F} \rangle_{\Lambda, \bar{\alpha}_0, \bar{r}_0} &= Z_F^{-1}(\Lambda, \bar{\mu}, \bar{\alpha}_0, \bar{r}_0) \times \\ &\times \mathcal{F}_R(\bar{\alpha}_R(\Lambda, \bar{\mu}, \bar{\alpha}_0, \bar{r}_0), \bar{r}_R(\Lambda, \bar{\mu}, \bar{\alpha}_0, \bar{r}_0), \bar{\mu}) \end{aligned} \tag{4.12}$$

is independent of $\bar{\mu}$, we obtain the renormalization group equation for \mathcal{F}_R

$$\begin{aligned} \left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta_{\bar{\alpha}}(\bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} - \gamma_F(\bar{\alpha}) + \tilde{\gamma}(\bar{\alpha}) \bar{r} \frac{\partial}{\partial \bar{r}} \right) \times \\ \times \mathcal{F}_R(\bar{\alpha}, \bar{r}, \bar{\mu}) = 0, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \beta_{\bar{\alpha}} &= \bar{\mu} \frac{\partial}{\partial \bar{\mu}} \bar{\alpha}_R \Big|_{\bar{\alpha}_0, \bar{r}_0, \Lambda} = \\ &= \frac{1}{4 \pi K_A} \left(-D \bar{\alpha}^2 + \frac{3}{4} \bar{\alpha}^3 \right) + 0 \left(\frac{1}{K_A^2} \right) \end{aligned} \tag{4.14a}$$

$$\begin{aligned} \gamma_F(\bar{\alpha}) &= \bar{\mu} \frac{\partial}{\partial \bar{\mu}} \log Z_F \Big|_{\bar{\alpha}_0, \bar{r}_0, \Lambda} = \frac{(D-2)}{4 \pi K_A} \bar{\alpha} + \\ &\quad + 0 \left(\frac{1}{K_A^2} \right) \end{aligned} \tag{4.14b}$$

$$\begin{aligned} \tilde{\gamma}(\bar{\alpha}) &= \frac{\bar{\mu}}{\bar{r}_R} \frac{\partial \bar{r}_R}{\partial \bar{\mu}} \Big|_{\bar{\alpha}_0, \bar{r}_0, \Lambda} = \\ &= \frac{-(D-2)}{4 \pi K_A} \bar{\alpha} + 0 \left(\frac{1}{K_A^2} \right) \end{aligned} \tag{4.14c}$$

(4.14c) is the same as the anomalous dimension for the surface tension in fluid membranes without hexatic order. There is an algebraic error in the result of [4].

The $\beta_{\bar{\alpha}}$ function confirms the result (3.16). From (4.13) standard arguments lead to

$$d_F = 2 + \gamma_F(\bar{\alpha}^*); \quad \nu^{-1} = 2 - \tilde{\gamma}(\bar{\alpha}^*). \tag{4.15}$$

Thus our explicit calculation confirms the direct calculation (4.6) as well as the scaling relation (4.7). We also have the standard scaling relation between ν and the critical exponent for the string tension

$$\mu = 2 \nu. \tag{4.16}$$

The critical exponent η is obtained in a similar way by computing invariant correlation functions be-

tween tangent planes to the membrane. It is found to be

$$\eta = \frac{1}{K_A} \frac{2D(D-2)}{3\pi} = 0 \left(\frac{1}{K_A^2} \right). \quad (4.17)$$

We now consider the spreading dimension d_s [18]. One can obtain it by replacing in equation (4.4) the Euclidean distance $|\mathbf{X}(\underline{\sigma}) - \mathbf{X}(\underline{0})|$ between the points $\underline{\sigma}$ and $\underline{0}$ by the geodesic distance $d(\underline{\sigma}, \underline{0})$ measured along the membrane. The expression of $d(\underline{\sigma}, \underline{0})$ is not simple, because it involves the determination of the geodesic $\tilde{\sigma}(s)$ ($\tilde{\sigma}(0) = 0$, $\tilde{\sigma}(1) = \underline{\sigma}$). For an almost flat membrane, the geodesic can be expanded in powers of the deviation h_{ij} of the metric from a flat one. In the Monge form one has :

$$h_{ij} = \partial_i \mathbf{x}_\perp \cdot \partial_j \mathbf{x}_\perp, \quad (4.18)$$

and the expansion in h_{ij} becomes an expansion in powers of \mathbf{x}_\perp . The expansion of the geodesic is then substituted in the expression of the arc length to yield an expansion of $d(\underline{\sigma}, \underline{0})$, which is then renormalized. It turns out that the first nontrivial term appears only at order $1/K_A^2$ and involves the term of fourth order in \mathbf{x}_\perp in the expansion of $d(\underline{\sigma}, \underline{0})$. The final result for the spreading dimension d_s reads :

$$d_s = 2 - \frac{1}{K_A^2} \frac{2D^2(D-2)}{9\pi^2} + 0 \left(\frac{1}{K_A^3} \right). \quad (4.19)$$

It is interesting to remark that it is smaller than two.

The spreading dimension d_s gives the « intrinsic » dimensionality of the membrane. Since it is different from 2, it is not clear whether the critical theory at the non trivial fixed point may be studied by the techniques of two dimensional conformal field theory. Moreover, it is not possible to define in a natural way a 2-dimensional stress energy tensor on the membrane, since the model does not depend on a classical two dimensional background metric. Indeed, the intrinsic metric on the surface is defined only *via* the embedding in term of the metric properties of bulk space, described by the bulk metric tensor $G_{\mu\nu}(\mathbf{X})$:

$$g_{ij}(\underline{\sigma}) = \partial_i X^\mu(\underline{\sigma}) \partial_j X^\nu(\underline{\sigma}) G_{\mu\nu}(\mathbf{X}(\underline{\sigma})). \quad (4.20)$$

The stress tensor in this kind of models is obtained as the functional derivative of the action with respect to the *bulk* metric tensor $G_{\mu\nu}$, and is not therefore a local object from the membrane point of view.

Let us finally remark that the scale transformations which define the renormalization group correspond to dilations in *bulk* and not in internal space :

$$G_{\mu\nu} \rightarrow \Lambda G_{\mu\nu}. \quad (4.21)$$

This is also reasonable from a physical point of view, since renormalization corresponds to summing over

fluctuations which modify the shape of the membrane at small scale, leaving its average position at large scales unchanged.

5. Conclusion.

We have analysed the long distance behaviour of hexatic membranes at large hexatic stiffness K_A , and we have found that it is determined by a non trivial infrared stable fixed point, depending on K_A . In this « crinkled » phase both the fractal and spreading dimensions of the membrane depend continuously on K_A . This behaviour is reminiscent of the low temperature phase of the two-dimensional XY model. One can indeed recover this model by going to the $\kappa \rightarrow \infty$ limit at fixed K_A . Let us consider the plane ($\bar{\alpha} = K_A/\kappa, K_A^{-1}$) (Fig. 3). The model appears

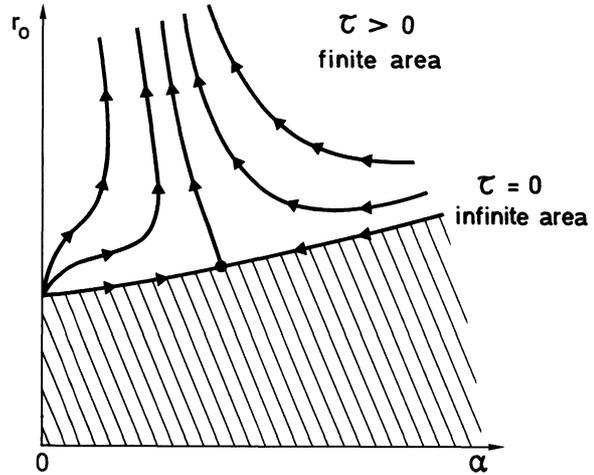


Fig. 2. — Phase diagram and renormalization group flow in the (α, r_0) plane for large hexatic stiffness. Above the critical line the mean area of the membrane and the surface tension τ are finite. The phase below the critical line cannot be described without taking into account self avoidance effects.

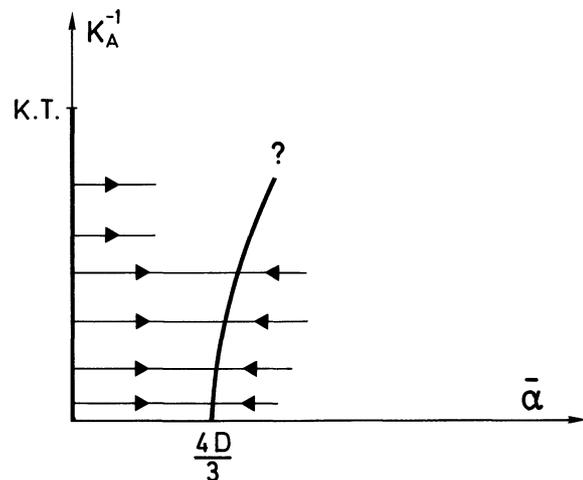


Fig. 3. — The lines of fixed points in the $(\bar{\alpha}, K_A^{-1})$ plane. The form and location of the stable curve is conjectural.

to have two lines of fixed points in this plane : one at $\bar{\alpha} = 0$, is the XY line ; the other, starting at ($\bar{\alpha} = 4D/3$, $K_A^{-1} = 0$), determining the behaviour of crinkled membranes. A family of renormalization group trajectories ($\bar{\alpha} = \bar{\alpha}(\rho)$, $K_A^{-1} = \text{Const.}$) connects the two lines, at least at large values of K_A . It is well known that the first line terminates at the Kosterlitz-Thouless transition point, corresponding to the unbinding of vortices (disclinations in the original, hexatic system). We expect that disclination unbinding also terminates the second line, but are unable to locate this transition with respect to the trajectory which leaves off the Kosterlitz-Thouless transition point.

Other features of the phase diagram are worth of a closer look. The region we have investigated corresponds to K_A and κ both large, and of the same order of magnitude. Actually K_A cannot be very small, since in this case hexatic order would be disrupted by disclination unbinding, and the membrane would behave like an ordinary, fluid one. On the other hand, if K_A is large, but κ is of order one, the intrinsic metric of the membrane will be almost flat, since this minimizes the Liouville action. Hence the

model should be related, in this limit, to the models of flat surfaces with bending elasticity considered by Pisarski [19]. If these models have a different behaviour, one may infer the existence of a further transition line in the $(\bar{\alpha}, K_A^{-1})$ plane.

All these issues are worth of closer investigations.

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Note added in proof: The reader should not mistake the rescaled bending rigidity $\bar{\kappa}_0$ introduced in (3.10) for the Gaussian curvature rigidity $\bar{\kappa}$ considered for instance in [6] and which has not been considered in this paper.

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