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## Critical exponents for Ising-like systems on Sierpinski carpets

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**Résumé.** — Les propriétés critiques du modèle d'Ising sur divers réseaux fractals du type tapis de Sierpinski sont étudiées par simulation numérique. On observe les lois d'échelle et on mesure les exposants  $\gamma$  et  $\nu$  dont les valeurs sont comparées à celles qui ont été récemment obtenues en dimension quelconque par resommation de la série en  $\varepsilon$  de Wilson-Fisher. Il apparaît que pour décrire les propriétés critiques dans le cas général, une dimension effective s'avère nécessaire, en plus de la dimension d'Hausdorff. Lorsque ces deux dimensions sont égales, nos résultats sont compatibles avec la conjecture selon laquelle le réseau fractal interpole les réseaux réguliers en dimension non entière.

**Abstract.** — The critical properties of Ising models on various fractal lattices of the Sierpinski carpet type are studied using numerical simulations. We observe scaling and measure the exponents  $\gamma$  and  $\nu$  which are then compared to the values which have been recently extrapolated from the Wilson-Fisher  $\varepsilon$ -expansion in non integer dimensions. It appears that in the general case an effective dimension, in addition to the Hausdorff dimension, is needed to describe the critical behaviour. When these dimensions are equal, our results are then compatible with the conjecture that the fractal lattice could interpolate regular lattices in non integer dimensions.

### 1. Introduction.

The critical behaviour of Ising-like models on fractal lattices of the Sierpinski carpet type has been studied by means of real space renormalization group methods (RSRG) [1, 2], and more recently by computer simulations on finite lattices [3, 4].

Interest in such systems has been stimulated by the conjecture that they may belong to the universality class of  $\Phi^4$  at some non integer dimension  $d$ , where the fractal thus implements the « analytic continuation » of hypercubical lattices. This hypothesis, suggested by the RSRG results of reference [2] in the range  $d = 1 + \varepsilon$ , requires that a single dimension governs the critical behaviour, which remains poorly known. On one hand, the RSRG study of reference [1] shows for the exponent  $\nu$  a dependence on the Hausdorff dimension  $d_H$ , but also on other topological factors needed to characterize the fractal. On the other hand, the numerical simulation of reference [3] shows that scaling laws between exponents can be fulfilled with a single dimension, which seems, however, to be distinct from  $d_H$ . This point is not investigated in the simulation of reference [4], but it is suggested that

the critical temperature and exponent  $\gamma$  vary, when the fractal parameters change, in a way best described by an effective dimension  $d_S$ . This dimension, defined as the average number of nearest neighbours of an active site, is usually different from  $d_H$ .

On the other hand, an accurate determination of the Ising-like exponents  $\gamma_1(d)$  and  $\nu_1(d)$  in non-integer dimensions  $d$ ,  $1 < d < 4$ , has been recently done [5] in the framework of the  $\varepsilon$ -expansion resummation. This allows a direct comparison between extrapolated  $\Phi^4$  and fractal lattices, such as the one we intend to present here in order to complete our preliminary investigation of reference [6].

We consider various Sierpinski lattices, each choice corresponding to fixed values (below 2) of the pair  $d_S$  and  $d_H$ : these dimensions change with the topology of the fractal or with the way of implementing an Ising model on it. Numerical simulations of the system are performed on such lattices of finite size, corresponding to 2 or 3 iterations of the fractal decimation. A standard analysis of the data (a fit to the temperature dependence of the susceptibility and finite size scaling laws) leads to a scaling law and to the exponents  $\gamma$  and  $\nu$ , which are then compared to the extrapolated values of reference [5]. To summarize our results, we find that two dimensions

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$d_S$  and  $d_H$  are needed to describe the critical behaviour of such models in the general case, which thus appears to be outside the  $\Phi^4$  universality class. However, when the parameters  $d_S$  and  $d_H$  are fixed at some common value  $d$ ,  $d_S = d_H = d$ , the corresponding fractal is a good candidate to extrapolate  $\Phi^4$  at the non-integer dimension  $d$ .

In section 2, we recall the description of the model and the definition of the dimensions  $d_H$  and  $d_S$ . We also give the principles of our analysis in order to find the critical parameters, and illustrate it on an example. Results for the 7 lattices we consider are gathered in section 3, where they are compared with the values extrapolated in reference [5] by Le Guillou and Zinn-Justin.

## 2. The Sierpinski lattices. Definitions and measurements.

A Sierpinski carpet is characterized by two integers  $b$  and  $l$ , with  $1 \leq l \leq b - 2$ . An initial square is divided in  $b^2$  subsquares and a central area of  $l^2$  subsquares is rejected. This procedure is repeated  $k$  times and at each step the carpet is rescaled in order that the smaller cells remain of unit area. We denote by  $(b, l, k)$  a carpet at the  $k$ -th stage of the

subdivision (the fractal is the limit as  $k \rightarrow \infty$ ) which counts  $N_c = (b^2 - l^2)^k$  cells embedded in a square of  $N^2$  area, with  $N = b^k$ . Introducing the fractal (Hausdorff) dimension  $d_H$ ,

$$d_H = \ln c / \ln b \quad \text{with} \quad c = b^2 - l^2 \quad (2.1)$$

one finds

$$N_c = N^{d_H}, \quad d_H \leq 2. \quad (2.2)$$

Two different rules have been already introduced to implement an Ising model on a  $(b, l, k)$  carpet, and we have used both since they allow some flexibility in addition to that which arises from varying the parameters  $b$  and  $l$ . The first rule (Method I, Ref. [4]) is to put an Ising spin at the center of each unit cell of the carpet, and no spin where cells have been deleted. The second one (Method II, Refs. [1-3]) is to put the spins at the corners of the non eliminated cells. In all cases interactions are restricted to nearest neighbours and periodic boundary conditions assumed.

For a given value of  $d_H$ , these two different definitions correspond to different values of the effective dimension  $d_S$ , as can be seen from the following counting rules for the numbers of spins ( $N_S$ ) and links ( $N_L$ ):

$$N_S = \begin{cases} c^k = N^{d_H} & \text{Method I} \\ c^k + 2l(b^k - c^k)/(b - c) - (1 - c^k)/(1 - c) & \text{Method II} \end{cases} \quad (2.3)$$

$$N_L = \begin{cases} 2c^k - 2l(b^k - c^k)/(b - c) & \text{Method I} \\ 2c^k + 2l(b^k - c^k)/(b - c) & \text{Method II} \end{cases} \quad (2.4)$$

and thus, in the  $k$  infinite limit

$$d_S = \begin{cases} 2 - 2l/(c - b) & \leq 2 \quad \text{Method I} \\ 2 \left[ 1 + \frac{(b - l) + c(l - 1)}{(c - 1)(c + l - b)} \right]^{-1} & \leq 2 \quad \text{Method II} \end{cases} \quad (2.5)$$

The critical parameters are determined from the standard observable quantities; defining

$$S_i = \pm 1 \quad M = \sum_{i=1, N_S} S_i$$

and

$$E = -\frac{1}{2} \sum_{\langle ij \rangle} S_i S_j \quad (2.6)$$

and taking statistical averages with respect to the usual partition function  $Z = \text{Tr} \exp(-\beta E)$  we measure:

— the specific heat

$$C = (\langle E^2 \rangle - \langle E \rangle^2) / N_S$$

— the magnetization

$$\mu = \langle |M| \rangle / N_S$$

— the susceptibility

$$\kappa = \langle M^2 \rangle / N_S$$

— its  $\beta$  derivative

$$\kappa' = (\langle E \rangle \langle M^2 \rangle - \langle EM^2 \rangle) / N_S$$

— the 4-site correlation

$$\kappa_4 = \langle M^4 \rangle / N_S - 3 N_S \kappa^2$$

which have, respectively,  $\alpha$ ,  $\beta'$ ,  $\gamma$ ,  $\gamma + 1$  and  $\gamma_4$  as critical exponents. Notice that the correlation length is not measured, since it is a too complicated object on such a lattice, but that its exponent  $\nu$  is expected to appear through finite size scaling laws, as explained later (relation (2.7)).

Our simulation deals with the lattices depicted in table II (the first three examples correspond to the

rule I, the remaining cases 4 to 7 corresponding to the rule II) which have maximal size  $64^2$ . It has been performed on a VAX 11/750 ( $\sim 700$  h) using a multispin coding technique, similar to that of reference [7] but adapted to this kind of lattices where spins have been decimated in a definite way.

We now describe the analysis of the data we perform in order to estimate the values of the critical parameters. The method is essentially standard and follows reference [4], with some modifications implied by two kinds of difficulties which must be stressed. The first difficulty is that of locating precisely the critical coupling  $\beta_c$ , since the usual signals (peaks of the specific heat, sharp rise of the magnetization) are broadened in our case where dimensions are less than 2 ( $\alpha$  is negative and  $\beta'$  small). Moreover, as  $\gamma$  and  $\nu$  increase at low dimension, the « critical slowing down » phenomenon is enhanced, and for practical reasons this study is limited to lattices of maximal sizes  $64^2$ . We thus encounter here the other difficulty, linked to the fractal nature of the lattices : within such a maximal size, only the first few iterations ( $1 \leq k \leq k_M$ ,  $k_M = 2$  or 3) can be numerically simulated. The application of the finite size scaling laws

$$\kappa[b, l, k; \beta_k] \sim b^{k\gamma/\nu} \quad \text{as } \beta_k \rightarrow \beta_c \quad k \rightarrow +\infty \quad (2.7)$$

in the form

$$\ln \left[ \frac{\kappa[b, l, k; \beta_k]}{\kappa[b, l, q; \beta_q]} \right] \frac{1}{(k-q) \ln b} = \rho(\beta_k, \beta_q) \sim \frac{\gamma}{\nu} \quad (2.8)$$

Table I. — Values of some critical parameters as given by the method of Section 2 on some two-dimensional lattices as examples. Numbers in parentheses are absolute errors on the last digit.

	$d_H$ $d_S$	$\beta_c$	$\frac{\gamma}{\nu}(\beta_c)$	$\frac{\gamma}{\nu}(C)$	$\gamma$	$\nu$	$D$
1	2	0.435	1.67		1.63	0.97	1.97
		0.440	1.75	1.81	1.72	0.98	1.99
	2	0.445	1.82	(1)	1.80	0.99	2.01
2	2	0.465	1.67		1.65	0.99	1.98
		0.470	1.74	1.84	1.75	1.00	1.99
	2	0.480	1.84	(3)	1.92	1.04	2.01
3	2	0.440	1.71		1.67	0.98	1.99
			1.77	1.78			
	2	0.445	(3)	(3)	1.87	1.02	2.00

is then quite delicate. We therefore compute this ratio  $\rho$ , for  $k = k_M$  and  $q = k_M - 1$ , in two different ways. First at fixed  $\beta$  ( $\beta_k = \beta_q = \beta_c$ ) and denote it by  $\frac{\gamma}{\nu}(\beta_c)$ , and second at values of  $\beta$  which are  $k$ -dependent and converge to  $\beta_c$ , such as the extrema of  $C$  or  $\kappa'$  (we denote it by  $\frac{\gamma}{\nu}(C)$ ). Assuming for these sequences a behaviour reminiscent of the two dimensional case, as can be seen in the table I, we expect that

$$\frac{\gamma}{\nu}(\beta_c) \leq \frac{\gamma}{\nu} \leq \frac{\gamma}{\nu}(C) \quad (2.9)$$

and thus proceed in the following way :

i) The critical coupling  $\beta_c$  must lie in the range

$$\beta_1 \leq \beta_c \leq \beta_2 \quad (2.10)$$

where  $\beta_1$  is the location of the maximum of  $\kappa'$  for the  $(b, l, k_M)$  iteration (this is a clear signal,  $\beta_1$  increases with  $k$  and reaches  $\beta_c$  as  $k \rightarrow +\infty$ ) and  $\beta_2$  saturates equation (2.9), i.e.

$$\frac{\gamma}{\nu}(\beta = \beta_2) = \frac{\gamma}{\nu}(C). \quad (2.11)$$

ii) The exponent  $\gamma$  is given by a linear fit to  $\ln \kappa[b, l, k_M; \beta]$  in the variable  $\ln(1 - \beta/\beta_c)$ , over the range  $\beta < \beta_c$ . This gives  $\gamma$  as a function of  $\beta_c$ . Inserting this value in relation (2.9) yields in turn an estimate for the range of variation of  $\nu$ . We want to stress that in that way the true  $\gamma$  cannot be smaller than the smallest value we give : this comes from the fact that  $\gamma(\beta_c)$  increases with  $\beta_c$ , and  $\beta_c \geq \beta_1$ . This is relevant for the analysis of section 3, because a lower bound on  $\gamma$  indicates an upper bound on the value of the dimension.

iii) The scaling law we consider is linked to the 4-site correlation and is written so as to define a dimension  $D$  through

$$D = \frac{\gamma_4}{\nu}(\beta_c) - 2 \frac{\gamma}{\nu}(\beta_c) \quad (2.12)$$

where  $\frac{\gamma_4}{\nu}(\beta_c)$  is computed as  $\frac{\gamma}{\nu}(\beta_c)$  and where  $\kappa$  is replaced by  $\kappa_4$  in relation (2.8). The other scaling law involving the magnetization exponent  $\beta'$  gives a dimension compatible with  $D$  within numerical uncertainties.

Proceeding along these lines, the analysis has been performed on 7 kinds of lattices, which are depicted in the table II, where results can be found. In column 3 we give the admissible range (Eq. (2.10)) for  $\beta_c$ , in columns 4 and 5 the values of  $\frac{\gamma}{\nu}(\beta_c)$  and  $\frac{\gamma}{\nu}(C)$ , in column 6 the fitted  $\gamma(\beta_c)$ , the corresponding range for  $\nu$  in column 7 and  $D$ , equation (2.12),

Table II. — Values of some critical parameters for 7 examples of Ising systems on Sierpinski lattices with dimensions  $d_H$  and  $d_S$  (column 2). The geometrical parameter ( $b, l$ ) and the maximal iteration  $k_M$  are listed in the first column. Numbers in parentheses are absolute errors on the last digit.

	$b, l$ $k_M$	$d_H$ $d_S$	$\beta_c$	$\frac{\gamma}{\nu}(\beta_c)$	$\frac{\gamma}{\nu}(C)$	$\gamma$	$\nu$	$D$
1	4, 2 3	1.793 1.500	0.76 0.78 0.80	1.32 1.49 1.57 (5)	1.56 (5)	2.35 2.45 2.55 (5)	1.4 -1.8 1.5 -1.7 1.6 -1.7	1.72 1.77 1.78 (15)
2	3, 1 3	1.893 1.600	0.62 0.64 0.65	1.45 1.63 1.68 (2)	1.68 (2)	1.90 2.05 2.20 (5)	1.1 -1.35 1.2 -1.35 1.3 -1.35	1.81 1.87 1.87 (8)
3	5, 1 2	1.975 1.895	0.480 0.485	1.72 1.77 (2)	1.77 (2)	1.81 1.95	1.05 1.09 (2)	1.96 1.98 (6)
4	7, 5 2	1.633 1.676	0.65 0.68 0.72	1.40 1.54 1.60 (2)	1.67 (2)	2.1 2.5 3.	1.25-1.50 1.50-1.60 1.80-1.87	1.63 1.66 1.67 (4)
5	5, 3 2	1.723 1.721	0.58 0.61	1.52 1.64 (2)	1.66 (2)	2.0 2.4	1.2 -1.3 1.45	1.80 1.77 (6)
6	7, 3 2	1.896 1.889	0.485 0.495	1.57 1.70 (2)	1.73 (8)	1.85 1.90	1.07-1.14 1.12	1.90 1.90 (10)
7	3, 1 3	1.893 1.909	0.470 0.485	1.60 1.72 (2)	1.72 (2)	1.80 1.95	1.04-1.12 1.13	1.90 1.92 (8)

is given in the last column. The table I, with an analogous content, is devoted to the illustration of this method of analysis on some two-dimensional examples, when only 2 iterates of size  $5^2$  and  $25^2$  are used. Example 1 is simply the homogeneous case. Examples 2 and 3 have the same first iteration (5, 1, 1) with spin rule I but the second iteration is in each case a « false fractal », since it is the union of five first iterations (example 2) and a dilatation by a factor 5 of the first iteration (example 3). These examples show how the dimension 2 can be found ( $D$  and  $\nu$  especially) and they have to be compared with the example 3, table II, where a new scale really appears at each step of the iteration.

### 3. Results and conclusions.

The table II shows our estimates for the critical parameters of various lattices (examples 1 to 7). The

examples 1 and 2 have been already studied in the reference [6]. (In this work, the analysis of example 2 involves preliminary data of the fourth iterate — which have not been confirmed — which lead to slightly higher values of  $\gamma$  [and  $\nu$ , since  $\gamma/\nu$  is unchanged] than those given Table II.) The other cases 3 to 7 have been chosen as to exemplify the role of the parameters  $d_S$  and  $d_H$ . The examples 1, 2 and 3 are such that there is a significant difference between the values of  $d_H$  and  $d_S$ . On the contrary examples 5, 6 and 7 are chosen so as to realize  $d_S \sim d_H$ . It is also instructive to compare pairs of such lattices: four example cases 3 and 7 have almost the same  $d_S$ , with different  $d_H$ . Examples 6 and 7 realize in a different geometrical way comparable values for  $d_S$  and  $d_H$ . Confronting these results among themselves and with the extrapolated values  $\gamma_1(d)$  and  $\nu_1(d)$  of reference [5], we can make the following comments :

i) From the first three examples, if  $D$  agrees with one of the dimensions  $d_S$  or  $d_H$ , one finds

$$D = d_H \quad (3.1)$$

rather than  $D = d_S$ . On the other hand from examples 3 and 7 it seems that the exponents  $\gamma$  and  $\nu$  depend more on  $d_S$  than on  $d_H$ : if the exponents are to be considered as functions of one single dimension, it is more probably  $d_S$ . This is confirmed by comparing  $\gamma$  and  $\nu$  of examples 1 and 2 with  $\gamma_1$  and  $\nu_1$ : they are in agreement with  $\gamma_1(d = d_S)$  and  $\nu_1(d = d_S)$ , but not with the extrapolated values at  $d = d_H = D$ , as shown in the figures 1 and 2. On these figures we plot our estimates for  $\gamma$  and  $\nu$  as functions of  $d_S$ , in all our examples, compared to the extrapolated values from the  $\varepsilon$ -expansion: according to the reference [5], the admissible range lies between the lines  $\gamma_1^+$  and  $\gamma_1^-$ . From this we conclude that in the general case, where  $d_S \neq d_H$  and where both  $d_S$  and  $d_H$  play some role, the universality hypothesis of the Sierpinski fractal is not supported by the data.

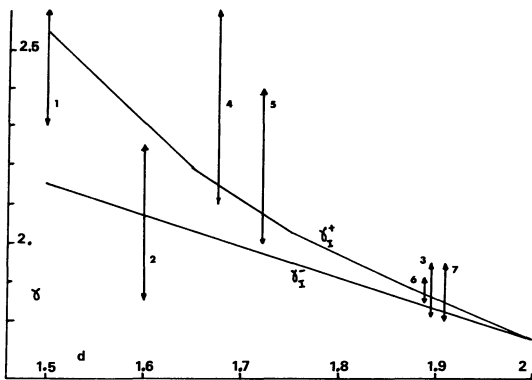


Fig. 1. — Range of values of  $\gamma$  as given by table II for various lattices. They are plotted as functions of  $d_S$ . The lines  $\gamma_1^\pm$  delimit the admissible extrapolated values as given in reference [5].

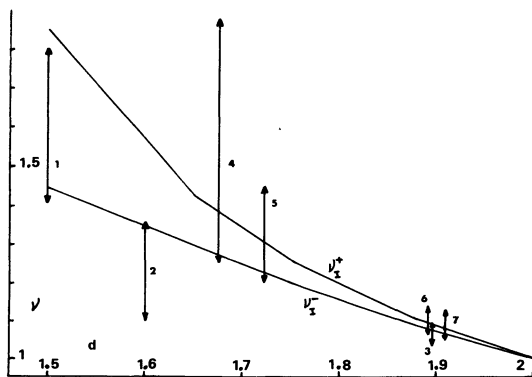


Fig. 2. — Range of values of  $\nu$  as given by table II for various lattices, plotted as functions of  $d_S$ . The lines  $\nu_1^\pm$  delimit the admissible values as given in reference [5].

ii) The conflicting dimensions  $d_S$  and  $d_H$  are adjusted to almost the same value in examples 4 to 7. In fact  $d_H$  is fixed by the geometry and  $d_S$  by the geometry and the spin rule; one can imagine more complicated spin rules which lead to an exact equality  $d_S = d_H$  without changing our semi-quantitative conclusion for such systems: they can be considered as good candidates for interpolating hypercubical lattices at the non integer dimension  $d = D = d_S = d_H$ , as can be checked in the figures 1 and 2.

iii) The value of the critical coupling  $\beta_c$  is often imprecise but nonetheless is sufficient to confirm the results of reference [4]: examples 3, 6 and 7 on one hand, and examples 2 and 6 on the other hand show that  $\beta_c$  depends mainly on  $d_S$ . Although it is not a universal quantity, it is interesting to compare our values to  $\beta_{HT}(d)$  at  $d = d_S$ , where  $\beta_{HT}(d)$  is the analytic continuation in  $d$  of some high temperature series. We have thus constructed  $\beta_{HT}(d)$  in the following way: according to the arguments given in reference [8], we form the [2/3] Padé approximant to  $d/dv \ln G(v)$  where  $v = \tanh \beta$  and  $G(v)$ , linked to the « true range of correlation », is given by a high temperature series for generic hypercubical lattices (as usual the coefficients are polynomials in  $d$ ). The reason for our choice is that such a Padé has the property of being exact for  $d = 1$  and  $d = 2$  (see Ref. [8]), i.e. in our range of interest. The extrapolated  $\beta_{HT}(d)$  is then the pole of this Padé at any value of  $d$ , and these values are drawn in the figure 3, where we have also plotted our estimates of  $\beta_c$  for  $d = d_S$ . The agreement is generally good, in contrast with the values predicted by the RSRG methods of reference [1] which can be applied to examples 4 to 7 and respectively give  $\beta_c = 0.802$ , 0.579, 0.196 and 0.322. These values can thus be

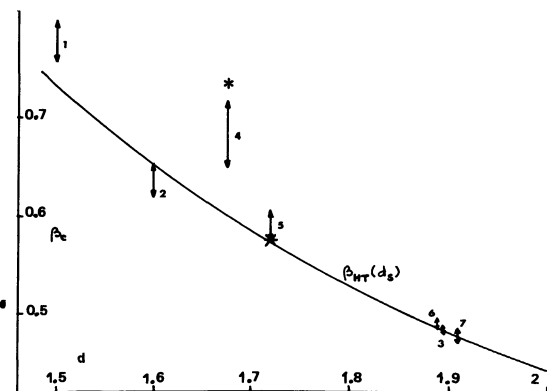


Fig. 3. — Range of values of  $\beta_c$  as given by table II for various lattices, plotted as functions of  $d_S$ . The curve  $\beta_{HT}$  represents the extrapolated value from high-temperature expansions as explained in Section 3. Stars indicate the RSRG values computed from the method of reference [1] (when they can be drawn in the figure).

quite unrealistic and it is then doubtful to infer general information from such methods. A last comment on the values of  $\gamma$  also is suggested by high temperatures series on hypercubical lattices where the relation  $\gamma \sim 2 d\beta_c(d)$  coming from the identification

$$\kappa = 1 + 2 d\beta + \dots \sim \left(1 - \frac{\beta}{\beta_c}\right)^{-\gamma}$$

is a good approximation in integer dimensions : one can check more generally from table II that

$\gamma \sim 2 d_S \beta_{HT}(d_S)$ , which confirms the results of our previous analysis on its  $d_S$  dependence.

As a conclusion, it appears that although our results are not highly accurate, they are sufficient to suggest a minimal condition ( $d_S = d_H$ ) which must be fulfilled by Ising models on Sierpinski lattices in order to be « universal » in the sense already explained. Only if such a constraint is satisfied will improved numerical results (involving larger lattices) have a good chance to confirm this conjecture.

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