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## On velocity selection for needle-crystals in a fully non-local model of solidification

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**Résumé.** — Nous montrons que les résultats obtenus, dans le modèle symétrique de croissance dendritique et dans la limite des petits nombres de Péclet  $p$ , par Ben Amar et Pomeau [8] et par Barbieri, Hong et Langer [11] pour la sélection de la vitesse de croissance d'un cristal aiguille sont en fait valables pour tout  $p$ . Ceci implique en particulier que pour des systèmes bidimensionnels ayant une tension de surface faiblement anisotrope, la vitesse sélectionnée varie en  $p^2$  quelle que soit la surfusion.

**Abstract.** — We show that the results on velocity selection for needle-crystals in the symmetric model of dendritic solidification recently derived by Ben Amar-Pomeau [8] and by Barbieri, Hong and Langer [11] in the small Péclet number  $p$  limit, are in fact strictly valid for all  $p$ 's. This means, in particular, that for 2-D systems with a weakly anisotropic surface tension, the selected velocity should scale as  $p^2$  for all undercoolings.

### 1. Introduction.

Considerable progress has been made in the last two years in solving the longstanding puzzle of velocity selection for needle crystals in solidification of a pure liquid from its undercooled melt. As is well known, Ivantsov [1] has shown that, at zero surface tension, this problem has, for a given value of the undercooling  $\Delta$ , an infinite continuous family of exact steady-state solutions. In two dimensions, these correspond to parabolic front shapes characterized by a given value  $p(\Delta)$  of the Péclet number  $p = \rho/l = \rho V/2D$  (where  $\rho$  is the tip radius,  $V$  the constant growth velocity and  $l = 2D/V$  the thermal diffusion length). This is to be contrasted with the experimental result that not only  $p$  (i.e.  $\rho V$ ), but *separately*,  $\rho$  and  $V$ , appear to be sharply selected in dendritic growth.

The study of phenomenological local models [2-4] has generated the idea that, in this problem, surface tension is a singular perturbation the introduction of which leads to a solvability condition for steady-state needle-shaped solutions of the growth equations. This has opened the way to the recent analytical

studies of the fully non-local problem. Up to now, two limiting cases have been analysed, namely :

(i) The large undercooling (or, equivalently,  $p \gg 1$ ) regime has been treated with the help of a generalized WKB method [5, 6] ; this indeed led to a solvability condition, which has no solution for systems with isotropic surface tension.

(ii) The small undercooling (vanishing Péclet number) limit, first studied by Pelcé and Pomeau [7], has been solved, for 2-D systems, by Ben Amar and Pomeau (BA-P) [8] with the help of a mathematical method developed by Kruskal and Segur [9]. They find that, in this limit also, stationary needle crystals must satisfy a solvability condition which has no solution for isotropic systems. In the presence of an anisotropic surface tension such that the capillary length

$$d(\theta) = d_0(1 - \alpha \cos(4\theta)) \quad (1)$$

(where  $\theta$  is the angle between the local normal to the front profile and the growth velocity), they find that the solvability condition has a discrete set of solutions the fastest (and, most likely, physically selected) of which is characterized, at small anisotropies, by  $\sigma = \sigma^*(\alpha) \propto \alpha^{7/4}$  (where  $\sigma = d_0/p\rho = 2d_0D/V\rho^2$  is the Langer-Müller-Krumbhaar [10] parameter), thus leading, at a given undercooling,

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i.e. given Péclet number, to the selection of one particular solution among the initially continuous Ivantsov family.

More recently, Barbieri, Hong and Langer (BHL) [11] have derived the same solvability condition for 2-D systems and  $p \rightarrow 0$  with the help of a different method, closely related to the one set up by Shraiman [12] to treat finger width selection in the Saffman-Taylor problem. They were able to extend their results to the 3-D axisymmetric isotropic case, showing that, as in 2 dimensions, no needle crystal solution is possible in the absence of anisotropy [13].

It has recently been suggested [14] that the  $p = 0$  limit might be singular and that, possibly, the breaking of the Ivantsov family by surface tension would not persist at finite  $p$  values. In this article, we extend the results of references [8, 11, 13] to non-zero Peclet numbers. We show that, in the symmetric model of solidification (identical thermal properties in the solid and liquid phases), in the small- $\sigma$  range where the present analytical methods apply, the  $p = 0$  limit is non-singular: we find that, whatever  $p$ , the steady-state solutions must satisfy a solvability condition. Moreover, we show that this condition is independent of the Péclet number so that, at small anisotropies, the selected  $\sigma$  has its  $p = 0$  value  $\sigma^*(\alpha)$ , and the selected velocity should therefore scale as  $p^2$ .

## 2. Two dimensions.

In the 2-D symmetric model, in the frame moving at velocity  $V$  along the growth direction  $Oz$ , the equation for the stationary front profile  $\zeta(x)$  reads [15]

$$\Delta - \frac{d}{\rho} \mathcal{K}_2\{\zeta(x)\} = \frac{p}{2\pi} \int_0^\infty \frac{d\tau}{\tau} \int_{-\infty}^\infty dy \times \exp\left\{-\frac{p}{2\tau} [(y-x)^2 + (\zeta(y) - \zeta(x) - \tau)^2]\right\} \quad (2)$$

where  $\mathcal{K}_2$  is the two-dimensional curvature

$$\mathcal{K}_2\{\zeta(x)\} = -\frac{\zeta''(x)}{(1 + \zeta'^2(x))^{3/2}}. \quad (3)$$

The length and time units are, respectively,  $\rho$  and  $(\rho l/2D)$ .  $d$  is given by equation (1). The Ivantsov parabola  $\zeta_{iv}(x) = -x^2/2$  solves (2) exactly in the  $d = 0$  limit [7] provided that  $p$  satisfies the Ivantsov relation:

$$\Delta = (\pi p)^{1/2} e^p \operatorname{Erfc}(p^{1/2}). \quad (4)$$

We set  $\zeta(x) = -\frac{x^2}{2} + \zeta_1(x)$ .

At this point, both the BA-P and BHL methods assume that the non-local (r.h.s.) part of equation (2) can be linearized in  $\zeta_1(x)$  — an *a priori* reasonable approximation, at least in the small

surface tension limit. BHL also assume that the anisotropic capillary term  $d$  reduces to its zeroth-order value [16]:

$$d(x) = d_0 A(x) \quad (5)$$

$$A(x) = 1 - \alpha + \frac{8\alpha x^2}{(1+x^2)^2}. \quad (6)$$

The two methods differ in the treatment of the curvature term: BA-P retain its non-linearities in the vicinity of the singular points  $x = \pm i$  of the Ivantsov curvature  $\mathcal{K}_{2,iv} = (1+x^2)^{-3/2}$ . In this « inner region » the front equation is solved by a scaling method. The corresponding solution (constructed so that  $\zeta_1(x)$  be even in  $x$ ), which can be matched asymptotically to the Ivantsov parabola for  $x \rightarrow \pm\infty$ , contains a term small beyond all orders in the small parameter  $\sigma = d_0/p\rho$ , from which the slope  $\zeta'_1(0)$  can be calculated. Imposing that the front profile be smooth ( $\zeta'(0) = 0$ ) then yields the solvability condition.

The BHL method completely linearizes  $\mathcal{K}_2$ . That is, its treatment of the « inner region » is *a priori* less satisfactory. In fact, one can check that this approximation is completely inaccurate in the immediate vicinity of  $x = \pm i$  (the BA-P « more inner » region  $|x-i| \leq \sigma^{1/2}$ ), but that it yields the correct behaviour in the « outer part of the inner region » ( $\sigma^{1/2} \ll |x-i| \ll \sigma^{2/7}$ ), which explains why the two methods finally give the same solvability condition.

So, the BA-P method is clearly more powerful, since it proves that the (otherwise uncontrolled) fully linear result is correct up to a  $\sigma$ -independent multiplicative constant. It therefore proves that both methods are, for all practical purposes, equivalent. From now on, we will follow the BHL approach, which we find slightly more easy to manipulate. However, we have checked that exactly the same results can be obtained with the BA-P one.

Linearizing equation (2), we get:

$$\sigma \zeta_1''(x) - \frac{3\sigma x \zeta_1'(x)}{1+x^2} - \frac{(1+x^2)^{3/2}}{A(x)} \times \int_{-\infty}^\infty dy \frac{\zeta_1(y) - \zeta_1(x)}{y-x} f_p(x, y) = \sigma \quad (7)$$

$f_p(x, y)$  is given by:

$$f_p(x, y) = \frac{p}{2\pi} (y-x) \int_0^\infty \frac{d\tau}{\tau} \left(1 + \frac{y^2 - x^2}{2\tau}\right) \times \exp\left\{-\frac{p}{2\tau} \left[(y-x)^2 + \left(\frac{y^2 - x^2}{2} + \tau\right)^2\right]\right\} \quad (8a)$$

$$= \frac{p}{2\pi} (y-x) [(y^2 - x^2) \beta^{-1} K_1(p\beta) + 2 K_0(p\beta)] \exp\left(\frac{p(x^2 - y^2)}{2}\right) \quad (8b)$$

where

$$\beta = \left\{ (y-x)^2 \left[ 1 + \frac{(x+y)^2}{4} \right] \right\}^{1/2} \quad (8c)$$

and the  $K$ 's are the modified Hankel functions.

Note that the  $1/(y-x)$  factor in the integral term of equation (7) has been separated out so that  $f_p(x, y)$  is finite everywhere on the real  $y$ -axis. In particular :

$$f_p(x, x) = \frac{x}{\pi(1+x^2)}. \quad (9)$$

For  $p \rightarrow 0$ ,

$$f_p(x, y) \rightarrow f_0(x, y) = \frac{1}{2\pi} (y+x) \left[ 1 + \frac{(y+x)^2}{4} \right]^{-1},$$

and equation (7) reduces to the linearized version of the Pelcé-Pomeau equation studied in references [8, 11]. Following [11], we set :

$$\zeta_1(x) = (1+x^2)^{3/4} Z(x). \quad (10)$$

Neglecting non-singular terms of order  $\sigma Z$ , equation (7) then becomes :

$$\begin{aligned} \mathcal{L}_2 Z \equiv \sigma Z''(x) + \frac{(1+x^2)^{3/2}}{A(x)} R_p(x) Z(x) + \\ + \mathcal{P} \int_{-\infty}^{\infty} dy \mathcal{A}_2(x, y) Z(y) = \sigma(1+x^2)^{-3/4} \end{aligned} \quad (11)$$

where  $\mathcal{P}$  stands for the Cauchy principal value, and :

$$R_p(x) = \mathcal{P} \int_{-\infty}^{\infty} dy \frac{1}{y-x} f_p(x, y) \quad (12)$$

$$\mathcal{A}_2(x, y) = \frac{1}{(x-y)} \frac{(1+x^2)^{3/4} (1+y^2)^{3/4}}{A(x)} f_p(x, y). \quad (13)$$

The condition for the linear equation (11) to have a solution is the Fredholm condition :

$$\Lambda_p(\sigma, \alpha) = \int_{-\infty}^{\infty} dx \frac{\tilde{Z}(x)}{(1+x^2)^{3/4}} = 0 \quad (14)$$

where  $\tilde{Z}$  is a null-eigenvector of the adjoint of  $\mathcal{L}_2$  :

$$\begin{aligned} \mathcal{L}_2^+ Z \equiv \sigma Z''(x) + \frac{(1+x^2)^{3/2} R_p(x)}{A(x)} Z(x) + \\ + \mathcal{P} \int_{-\infty}^{\infty} dy \mathcal{A}_2(y, x) Z(y) = 0. \end{aligned} \quad (15)$$

Following BHL, we look for solutions  $\tilde{Z}_{\pm}(x)$  of equation (15) of the form  $\exp[S_{\pm}(x)/\sqrt{\sigma}]$ , where  $S_{\pm}(x)$  has a stationary phase point at  $x_{\pm} \approx \pm i$ , and  $\text{Re}[S_{\pm}(x_{\pm})] < 0$ . We then want to calculate, say,  $\int_{-\infty}^{\infty} dy \mathcal{A}_2(y, x) \tilde{Z}_{+}(y)$ . Folding the contour into the upper half complex  $y$ -plane, one sees that the

part of the contour close to  $x_{+}$  contributes an exponentially small term, of order

$$\exp[\text{Re } S_{+}(x_{+})/\sqrt{\sigma}],$$

which we neglect.

As in the  $p = 0$  limit,  $\mathcal{A}_2(x, y)$  (Eq. (13)) has a pole on the real axis at  $y = x$ . However, for  $p \neq 0$ , due to the presence of the  $K$  functions in equation (8b),  $y = x$  is also a branch point. So, the integral on the deformed contour (Fig. 1) contains not only the pole term, but also a contribution from the discontinuity along the branchcut.

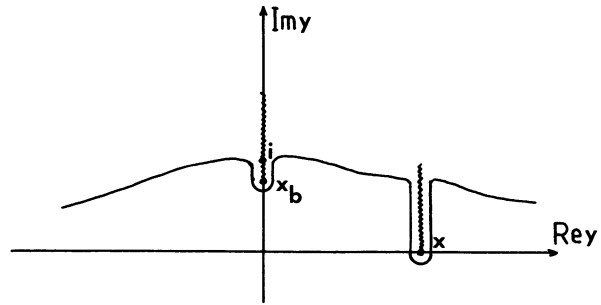


Fig. 1. — Contour in the complex  $y$ -plane for evaluation of  $\int_{-\infty}^{\infty} dy \mathcal{A}_2(y, x) \tilde{Z}_{+}(y)$ . The integrand has a pole at  $y = x$ , a stationary phase point at  $x_{+} = i$ , and a branch point associated with the anisotropy factor at  $x_b \approx i(1 - \sqrt{2}\alpha)$ .

The pole contribution is, from equations (8b, 13),  $x(1+x^2)^{1/2} \tilde{Z}_{+}(x)/A(x)$ . On the other hand, repeating the argument used by BHL to estimate cut contributions in the 3-D case, one easily sees (and can check later using the expression of  $\tilde{Z}_{+}$  which will be found) that the cut term is of order  $C(x, \sigma, p) \times \tilde{Z}_{+}(x)$ , where  $C$  vanishes for small  $\sigma$  as a power of  $\sigma$ . So, we neglect it and only retain the pole term [17]. This, obviously, implies that  $\sigma$  must be small enough. As can be seen from equation (8b),  $C$  is a (regular) function of  $p$ . So, the range of values of  $\sigma$  where this approximation is legitimate must depend on the value of the Péclet number. A more precise determination of the domain of the  $(p, \sigma)$  space where our treatment holds can probably only be provided by comparison with extensive numerical studies.

Equation (15) then reduces to :

$$\begin{aligned} \sigma \frac{d^2 \tilde{Z}_{\pm}}{dx^2} + \frac{(1+x^2)^{1/2}}{A(x)} \times \\ \times [(1+x^2) R_p(x) \pm ix] \tilde{Z}_{\pm}(x) = 0 \end{aligned} \quad (16)$$

and we are left with calculating  $R_p(x)$ , as defined by equation (12).

Using equations (8a) and (12) and performing the following transformation of variables :

$$s = \frac{y-x}{(\tau + (y-x)^2/2)}; \quad t = \sqrt{\tau} + \frac{(y-x)^2}{2\sqrt{\tau}} \quad (17)$$

we obtain :

$$R_p(x) = I_p(x) + I_p(-x) \quad (18a)$$

$$I_p(x) = \pi^{-1} \lim_{\varepsilon \rightarrow 0^+} \int_0^{2/\varepsilon} ds \frac{(1+sx)}{s^2 + (1+sx)^2} \times \exp\left\{ \frac{-p\varepsilon[s^2 + (1+sx)^2]}{s(2-\varepsilon s)} \right\}. \quad (18b)$$

Setting :

$$u = \frac{s^2 + (1+sx)^2}{s(2-\varepsilon s)} \quad (19)$$

and inverting (19) separately on the two  $s$ -intervals  $(0, s_0)$  and  $(s_0, 2/\varepsilon)$ , where  $s_0$  is the  $s$ -value for which  $u$  is minimum, we finally get :

$$I_p(x) = \pi^{-1} \lim_{\varepsilon \rightarrow 0^+} \int_{u_0^+}^{\infty} du e^{-p\varepsilon u} \times \frac{u(x+\varepsilon)+1}{u(1+x^2+\varepsilon u) \sqrt{(u-u_0^+)(u+u_0^-)}} \quad (20)$$

where :

$$u_0^{\pm} = u_0^{\pm}(x, \varepsilon) = \sqrt{1 + (x + \varepsilon/2)^2} \pm (x + \varepsilon/2). \quad (21)$$

We introduce a cutoff  $L$  such that :

$$u_0^+ \ll L \ll 1/p\varepsilon \quad (22)$$

and separate the integral (20) into two terms  $I_p^{(1)}(x)$ ,  $I_p^{(2)}(x)$  corresponding respectively to the two  $u$ -intervals  $(u_0^+, L)$  and  $(L, +\infty)$ . Then, in  $I_p^{(1)}(x)$ , the function  $e^{-p\varepsilon u}(1+x^2+\varepsilon u)^{-1}$  can be expanded in series of  $\varepsilon u < \varepsilon L \ll 1$ , so that :

$$I_p^{(1)}(x) = \pi^{-1} \lim_{\varepsilon \rightarrow 0^+} \sum_{n=0}^{\infty} \varepsilon^n L^{n-1} \frac{b_n}{1+x^2} \times \int_{u_0^+/L}^1 dz \frac{(1+Lxz)z^{n-1}}{\left[ \left( z - \frac{u_0^+}{L} \right) \left( z + \frac{u_0^-}{L} \right) \right]^{1/2}} \quad (23)$$

with  $b_0 = 1$ . Noticing that, in (23), for  $n \geq 1$ ,  $z^{n-1} \leq 1$ , it can then easily be shown that, when  $\varepsilon \rightarrow 0^+$ , the series (23) reduces to its first term.

Analogously, in  $I_p^{(2)}(x) + I_p^{(2)}(-x)$ , the function  $[(u-u_0^+)(u+u_0^-)]^{1/2}$  can be expanded in series of powers of  $u_0^+/u < (u_0^+/L) \ll 1$  and

$$I_p^{(2)}(x) + I_p^{(2)}(-x) = \pi^{-1} \lim_{\varepsilon \rightarrow 0^+} \sum_{n=0}^{\infty} c_n L^{-n-1} \times \int_1^{\infty} dz e^{-p\varepsilon Lz} \frac{(\varepsilon Lz + 1)}{(1+x^2+\varepsilon Lz)z^{n+2}} \quad (24)$$

with  $c_0 = 2$ .

Again, noticing that, in (24),  $z^{-n} \leq 1$ , one shows that :

$$I_p^{(2)}(x) + I_p^{(2)}(-x) = \frac{2}{\pi L(1+x^2)}. \quad (25)$$

Then, taking the  $L \rightarrow \infty$  limit, we find :

$$R_p(x) = \frac{1}{\pi(1+x^2)} \left( \int_{x+\sqrt{(1+x^2)}}^{\infty} dt \times \frac{(1+xt)}{t\sqrt{t^2-2xt-1}} + (x \leftrightarrow -x) \right) \quad (26)$$

$$R_p(x) = (1+x^2)^{-1}.$$

So, contrary to what could have been *a priori* expected when considering the expression (8) of  $f_p(x, y)$ ,  $R(x)$  is strictly  $p$ -independent, and equation (16) for  $Z_{\pm}$  reduces exactly to the one solved by BHL (Eq. (2.17) of Ref. [11]) for the  $p = 0$  limit. This, of course, entails that the results of references [8, 11] can be immediately extended to the whole  $p$ -range.

In particular, this result means that the linearized approximation predicts that, for the symmetric model, the value  $V^*$  of the growth velocity selected by the solvability condition should scale as  $p^2$  at all undercoolings. This prediction can be compared with the results obtained by Kessler, Koplik and Levine [18] by a semi-numerical method. They find that  $V^*/p^2$  is  $p$ -dependent. However, this dependence decreases rapidly with the magnitude of the anisotropy coefficient and is very weak for the smallest  $\alpha$  they consider ( $\alpha = 0.05$ ). Since, in agreement with the BA-P and BHL result, a small  $\alpha$  is associated with a small  $\sigma^*$ , this indicates that, as expected,  $\sigma$  is indeed the relevant parameter whose smallness guarantees that terms non-linear in  $\zeta_1$  are negligible. An extension of the results of reference [18] to smaller values of  $\alpha$  would therefore be of interest to give a quantitative estimate of the value of the linear approximation.

### 3. Three-dimensional isotropic system.

We reduce here to a system with an isotropic capillary length  $d_0$ , and only look for axisymmetric needle crystals. The equation for the stationary front profile  $\zeta(r)$  now reads :

$$\Delta - \frac{d_0}{\rho} \mathcal{K}_3\{\zeta(r)\} = \left( \frac{p}{2\pi} \right)^{3/2} \int_0^{\infty} \frac{d\tau}{\tau^{3/2}} \int d^2r' \times \exp\left\{ -\frac{p}{2\tau} [(\mathbf{r}-\mathbf{r}')^2 + (\zeta(r) - \zeta(r') + \tau)^2] \right\} \quad (27)$$

$\mathbf{r}$ ,  $\mathbf{r}'$  are 2-D vectors in the  $(x, y)$  plane with origin on the growth axis. The 3-D curvature simply reads, for an axisymmetric shape function :

$$\mathcal{K}_3\{\zeta(r)\} = -\frac{\zeta''}{(1+\zeta'^2)^{3/2}} - \frac{\zeta'}{r(1+\zeta'^2)^{1/2}}. \quad (28)$$

For  $d_0 = 0$ , equation (27) has a family of Ivantsov paraboloidal solutions  $\zeta_{iv}(r) = -(r^2/2)$ , provided that  $p$  obeys the relation:

$$\Delta = p e^p \int_p^\infty \frac{e^{-t}}{t} dt. \quad (29)$$

Setting  $\zeta(r) = -(r^2/2) + \zeta_1(r)$ , and linearizing equation (27), we obtain:

$$\sigma \zeta_1''(r) + \sigma \frac{(1-2r^2)\zeta_1'(r)}{r(1+r^2)} - (1+r^2)^{3/2} \times \int d^2r' (\zeta_1(r') - \zeta_1(r)) g_p(\mathbf{r}, \mathbf{r}') = \sigma(2+r^2) \quad (30)$$

with

$$g_p(\mathbf{r}, \mathbf{r}') = \left(\frac{p}{2\pi}\right)^{3/2} \int_0^\infty d\tau \left(\frac{1}{\tau^{3/2}} - \frac{r^2 - r'^2}{2\tau^{5/2}}\right) \times \exp\left\{-\frac{p}{2\tau} \left[(\mathbf{r} - \mathbf{r}')^2 + \left(\frac{r^2 - r'^2}{2} - \tau\right)^2\right]\right\} \quad (31a)$$

$$= \frac{1}{4\pi} \left(\frac{r'^2 - r^2}{\gamma^{3/2}} (1 + p\gamma^{1/2}) + \frac{2p}{\gamma^{1/2}}\right) \times \exp\left\{p\left(\frac{r^2 - r'^2}{2} - \gamma^{1/2}\right)\right\} \quad (31b)$$

and

$$\gamma = r^2 + r'^2 - 2rr' \cos \phi + \frac{(r^2 - r'^2)^2}{4} \quad (32a)$$

$$\cos \phi = \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'}. \quad (32b)$$

Following BHL, we then set:

$$\zeta_1(r) = (1+r^2)^{3/4} r^{-1/2} W(r) \quad (33)$$

so that (up to non-singular corrections of order  $\sigma W$ ):

$$\sigma \left(\frac{d^2}{dr^2} + \frac{1}{4r^2}\right) W(r) + (1+r^2)^{3/2} S_p(r) W(r) + \mathcal{F} \int_0^\infty dr' \mathcal{A}_3(r, r') W(r') = \sigma \frac{(2+r^2)\sqrt{r}}{(1+r^2)^{3/4}} \quad (34)$$

where

$$S_p(r) = \mathcal{F} \int d^2r' g_p(\mathbf{r}, \mathbf{r}') \quad (35)$$

$$\mathcal{A}_3(r, r') = - (1+r^2)^{3/4} (1+r'^2)^{3/4} (rr')^{1/2} \times \int_0^{2\pi} d\phi g_p(\mathbf{r}, \mathbf{r}'). \quad (36)$$

In equations (34), (35)  $\mathcal{F}$  denotes the Cauchy principal value, and we have taken advantage of the fact that

$$h_p(r, r') = \int_0^{2\pi} d\phi g_p(\mathbf{r}, \mathbf{r}') \quad (37)$$

has a (simple) pole at  $r = r'$ , with residue  $[\pi(1+r^2)]^{-1}$ .

As in § 2, one then writes the Fredholm solvability condition for equation (34). The equation for the null eigenvectors,  $\tilde{W}_\pm$  of the adjoint operator now contains the non-local quantity

$$\mathcal{F} \int_0^\infty dr' \mathcal{A}_3(r', r) \tilde{W}_\pm(r'). \quad (38)$$

One therefore must, as in § 2, look for the singularities of  $\mathcal{A}_3(r', r)$  in the complex  $r'$ -plane. One easily shows, by studying the behaviour of  $h_p(r, r')$  that (besides branchpoints with imaginary parts  $\pm 1$  and  $\pm 2$ )  $\mathcal{A}_3(r', r)$  has:

- a pole at  $r = r'$  which is also a branchpoint
- a branchpoint at  $r' = 0$ .

That is, in the 3-D case, the singularities of  $\mathcal{A}_3$  for  $p \neq 0$  are of the same nature as those for  $p = 0$  [11]. We can then follow completely the BHL 3-D analysis and thus reduce (38) to the pole contribution (the only difference being that, as in 2-D, the neglected terms, which vanish as a power of  $\sigma$ , are also  $p$ -dependent).

The equation for  $\tilde{W}_\pm$  then reduces to:

$$\sigma \left(\frac{d^2}{dr^2} + \frac{1}{4r^2}\right) \tilde{W}_\pm(r) + (1+r^2)^{1/2} \times [(1+r^2) S_p(r) \pm ir] \tilde{W}_\pm(r) = 0. \quad (39)$$

In order to calculate  $S_p(r)$  (Eq. (35)), we use expression (31a) for  $g_p(\mathbf{r}, \mathbf{r}')$  and the transformation of variables:

$$\begin{aligned} t &= \sqrt{\tau} + \frac{(\mathbf{r}' - \mathbf{r})^2}{2\sqrt{\tau}} \\ \xi &= \frac{x' - x}{\tau + \frac{1}{2}(\mathbf{r}' - \mathbf{r})^2} \\ \eta &= \frac{y' - y}{\tau + \frac{1}{2}(\mathbf{r}' - \mathbf{r})^2}. \end{aligned} \quad (40)$$

One then gets:

$$\begin{aligned} S_p(r) &= \frac{2}{\pi^{3/2}} \lim_{\varepsilon \rightarrow 0} \int_0^{2/\varepsilon} \rho d\rho \int_0^{2\pi} d\psi \times \\ &\times \frac{1 + \rho r \cos \psi}{[\rho^2 + (1 + \rho r \cos \psi)^2]^{3/2}} \\ &\times \left(\frac{\alpha}{2} e^{-\alpha^2} + \frac{\sqrt{\pi}}{4} (1 - \text{Erfc}(\alpha))\right) \end{aligned} \quad (41)$$

with

$$\alpha = \left(\varepsilon p \frac{\rho^2 + (1 + \rho r \cos \psi)^2}{\rho(2 - \varepsilon \rho)}\right)^{1/2} \quad (42)$$

and

$$\rho = (\xi^2 + \eta^2)^{1/2}; \quad \cos \psi = \frac{\mathbf{r} \cdot \mathbf{p}}{r\rho}.$$

In analogy with the calculation of the 2-D quantity  $R_p(x)$ , we separate the  $\rho$ -integration into three parts with the help of two cut-offs  $\rho_1, \rho_2$  such that

$$\frac{\varepsilon p}{2} \ll \rho_1 \ll \frac{1}{\sqrt{1+r^2}}; \quad 1 \ll \rho_2 \ll \frac{2}{\varepsilon[1+p(1+r^2)]}$$

and find that  $S_p(r)$  is exactly given by :

$$S_p(r) = \frac{1}{1+r^2}. \quad (43)$$

So, as in two dimensions, equation (39) for  $\tilde{W}_{\pm}$  is, in fact, strictly  $p$ -independent, and, following BHL, one can immediately conclude to the absence of axisymmetric needle crystals in the 3-D isotropic case.

This, together with the results of § 2, thus shows that the  $p = 0$  limit studied by BA-P and BHL is indeed non-singular — at least in the range of parameters where the central approximation of their and our calculations, namely the linearization of the non-local term, is legitimate.

Moreover, we find that, in the symmetric model and for 2-D needle crystals, in the presence of anisotropy, the selected value of  $\sigma$  is  $p$ -independent.

Let us also mention that these results agree with those obtained in the  $p \gg 1$  limit with the help of a different (though also linearized) method [5, 6]. In this (large undercooling) regime it could be proved explicitly that the one-sided model exhibited the same behaviour as the symmetric one. This leads us to believe that the non-singularity of the  $p = 0$  limit is most probably a general feature of more realistic non-local models which would take into account the thermal asymmetry between the two phases, though, in such models,  $\sigma^*$  might possibly become  $p$ -dependent.

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