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Invariant-imbedding approach to localization. II. Non-linear random media

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Résumé. — On utilise une méthode de plongement invariant pour écrire une équation aux dérivées partielles décrivant le coefficient de réflexion par un milieu désordonné non linéaire de longueur L. La méthode des caractéristiques réduit cette équation à un système dynamique. On moyenne l'effet du désordre sur les orbites afin d'étudier la loi de probabilité de $R(L)$. Deux cas sont considérés : sortie fixée w_0 (problème A) et entrée fixée (problème B). Pour une grande classe de non-linéarités, le comportement générique pour le problème A est comme suit : i) pour une faible non-linéarité, un changement de régime entre une décroissance exponentielle de la transmission $\langle t \rangle \sim \exp(-L/4\,\xi)$ à faible L, et une décroissance en loi de puissance à grand L, se produit à une échelle de longueur $L^* = \xi \ln (1/w_0)$, ii) pour une forte non-linéarité, le comportement est une loi de puissance en L. L'origine physique est attribuée au renforcement des non-linéarités par le désordre. Pour le problème B, le comportement asymptotique est toujours une décroissance exponentielle. Les fluctuations associées aux différents régimes sont étudiées. Le cas d'une non-linéarité aléatoire est considéré aussi, où un phénomène d'auto-répulsion est mis en évidence, à une distance finie. L'importance de nos résultats pour des situations expérimentales est brièvement discutée.

Abstract. — By employing an invariant-imbedding method a partial differential equation is derived for the complex reflection amplitude $R(L)$ of a one-dimensional non-linear random medium of length L. The method of characteristics reduces this equation to a dynamical system. Averaging of the perturbation of orbits by weak disorder is used to investigate the probability distribution of $R(L)$. Two different situations are considered : fixed output w_0 (Problem A) and fixed input (Problem B). For a large class of non-linearities the generic behaviour for Problem A is as follows : i) For weak non-linearities, a crossover between an exponential decay of transmission $\langle t \rangle \sim \exp(-L/4\,\xi)$ at short L and a power law decay at large L is shown to take place at a length scale $L^* = \xi \ln (1/w_0)$, ii) Strong non-linearities dominate the full behaviour and give rise to a power law decay. The physical origin of this behaviour is traced back to the enhancement of non-linearities by disorder. For Problem B, the asymptotic behaviour is shown to be always an exponential decay. The fluctuations associated with both regimes are obtained. Random non-linearities are also investigated and shown to lead to a self-repelling phenomenon at finite distances. The relevance of our results to experimental situations is briefly discussed.

1. Introduction.

Up to now Anderson localization phenomena have been studied mainly in linear random media [1]. A natural question then arises : what happens in a medium where both randomness and non-linearities are relevant ? In the first paper (hereafter I) of this series [2], the simplest form of this problem (wave transmission across a non-linear medium) has been addressed where the basic equations have been derived using an invariant-imbedding method [3]. In this paper, we analyze this problem, for a large class of non-linearities, by looking at the possible modifications due to the presence of non-linear terms in the wave field equations. Our approach will be limited here to 1D problems and we leave the extension of the results to other cases for a future work.

We describe the system which we have studied by the generalized non-linear Schrödinger equation, for the complex amplitude of the field $\psi(x, t)$:

$$i\,\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} - V(x)\,\psi + \alpha \psi f(|\psi|^2) \qquad (1.1)$$

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where $f(|\psi|^2)$ is an arbitrary function of the intensity $|\psi|^2$ with $f(0) = 0$; t is time and $V(x)$ is a real (or complex) regular (or random) potential. Examples of physical phenomena described by equation (1.1) have been discussed in I. We consider equation (1.1) for a one-dimensional case, corresponding to a plane-layered medium, which occupies the region $0 \leqslant x \leqslant L$. Outside this region, $V(x) = 0$ and $f = 0$. As in I, we only consider the stationary regime, corresponding to solutions of (1.1) of the form : $\psi(x, t) = e^{ik^2 t}\, \varphi(x)$, where $k^2 = E$ is the wave energy. This leads to the following equation for $\varphi(x)$:

$$d^2\varphi/dx^2 + k^2[1 + \varepsilon(x, |\varphi|^2)]\, \varphi = 0 . \quad (1.2)$$

Here, the function ε is given by

$$\varepsilon(x, |\varphi|^2) = (-V(x) + \alpha f(|\varphi|^2))/k^2 .$$

Let now a plane wave $\varphi_0(x) = A\, e^{ik(L-x)}$ be incident on the layer from the right. Then the solution of (1.2) can be represented in the form $\varphi(x) = Au(x)$, where $u(x)$ satisfies a similar equation, involving the intensity $w = |A|^2$ of the incident wave and $|u(x)|^2$. To the right and left of the layer, the wave field has the form :

$$\begin{aligned} u(x) &= \exp[ik(L-x)] + \\ &\qquad + R(L, w) \exp[ik(x-L)], \quad x \geqslant L \\ u(x) &= T(L, w) \exp(ikx), \quad x \leqslant 0 \end{aligned} \quad (1.3)$$

where $R(L, w)$ and $T(L, w)$ are the complex reflection and transmission coefficients respectively. The wave number k is assumed to be the same inside and outside the layer. This restriction corresponds to an approximation where the creation of other harmonics is neglected.

The main object of this paper is to answer the following question : how can non-linearities modify the usual [1] exponential decay of transmission $\langle t \rangle \sim \exp(-L/4\,\xi)$ due to the enhanced backscattering mechanism induced by disorder ? To our knowledge, such a question has never been studied before, except in the recent work reported in [4, 5]. Due to the non-linear terms in (1.2) all the techniques used in linear problems (i.e. at $f = 0$) seem to fail completely in the present case. Furthermore, in contrast with linear problems where the superposition principle holds, the transmission problem is no longer uniquely defined : in a large number of cases, including the non-random limit, ambiguities arise with possible bistability and hysteresis phenomena [6]. Despite these intrinsic difficulties, there is one method to obtain a closed solution for reflected field (i.e. reflection coefficient). For real potentials $V(x)$, $|R|^2 + |T|^2 = 1$ holds and then information on $t \equiv |T|^2$ can be deduced from $|R|^2 \equiv r = 1 - t$. Making use of the invariant imbedding idea, we have shown in I the possibility of reducing the calculation of $R(L, w)$ to an initial-value Cauchy equation, relative to the imbedding parameters, namely the position L of the right-hand layer boundary and the intensity w of the incident wave. The field at the boundary, $1 + R(L, w)$ is described by a closed non-linear partial differential equation (Eq. (2.1) below). This is the basic equation of our approach and the paper is organized as follows. In section 2, the main steps of our approach : the method of characteristics and the Hamiltonian formulation are described. We show in particular that two problems are to be distinguished : transmission at fixed output (problem A) and transmission at fixed input (problem B). Using a dynamical system point of view, these two problems are investigated in the weak disorder limit. In both cases, the method of averaging the perturbation of orbits [7] is used to obtain the behaviour of transmission as a function of L and w. Problem A is worked out in section 3 and problem B in section 4. In section 5, the case of non-linear randomness is investigated. In the final section 6, some remarks will be given about the generality of our results and the remaining open problems.

2. Formalism and basic properties

In the previous paper [2], the following equation

$$\begin{aligned} \frac{\partial R(L, w)}{\partial L} &= 2\, ikR + \\ &\quad + \frac{ik}{2}\left[(1+R)^2 + (R - R^*)\, w \frac{\partial R}{\partial w}\right] \times \\ &\quad \times \varepsilon(L, w(1+R)(1+R^*)) \end{aligned} \quad (2.1)$$

has been derived for the complex reflection coefficient $R(L, w)$. Equation (2.1) is a quasi linear partial differential equation, subject to the boundary condition $R(L = 0, w) = 0$, which allows for a complete solution $R(L, w)$. It is interesting to notice that (2.1) reduces to a Riccati equation at $w = 0$. In this limit one recovers the linear case, previously investigated [8] following the same line of approach.

For a random potential $V(x)$, (2.1) can be viewed as a stochastic equation, describing the evolution of the « stochastic process » $R(L, w)$ under the action of the « multiplicative noise » $V(L)$. We recall that in (2.1), ε refers to the function

$$\varepsilon(x, |\varphi|^2) = [-V(x) + \alpha f(|\varphi|^2)]/k^2 \quad (2.2)$$

where $f(|\varphi|^2)$ describes the non-linearities in the wave field. Examples of $f(u)$ which are of physical interest are :

i) $f(u) = u^n$ (n = integer)

ii) $f(u) = \ln u$

iii) $f(u) = a - b\, e^{-u}$, $b > 0$

iv) $f(u) = a/(1+u)$, $a < 0$.

In the following sections, we limit our attention to exemple i) and this for the sake of clarity. The

remaining examples can be worked out following the same lines of approach.

In contrast to an ordinary differential equation no simple Liouville-like theorem can be used here as usually done [9] for the probability density of the process. Nevertheless, one can proceed otherwise, by using the method of characteristics [10]. In this section, we describe this approach in some detail.

2.1 LIOUVILLE THEOREM AND PARTIAL DIFFERENTIAL EQUATIONS. — Let us first show that no simple Liouville-like theorem can be written for the calculation of the probability density of a dynamical variable described by a partial differential equation (PDE). For the sake of simplicity, we consider the case of one dynamical variable $\psi(x, y)$ and assume that $\psi(x, y)$ is the solution of the following quasi-linear PDE

$$\frac{\partial \psi}{\partial x} + a(x, y\,;\psi)\frac{\partial \psi}{\partial y} = f(x, y\,;\psi)\,. \tag{2.3}$$

In (2.3), x and y can be viewed as generalized « times », a and f are arbitrary functions and the initial values of ψ are known (Cauchy data). Let us denote by $Q(\psi\,; x, y)$ the (probability) measure associated with the evolution of $\psi(x, y)$. For each instant (x, y) and point ψ of the phase space, one can define two currents : j_x and j_y in x and y directions. Then (2.3) implies

$$j_x(x, y\,;\psi) + a(x, y\,;\psi)\, j_y(x, y\,;\psi) = \\ = Q(x, y\,;\psi)\, f(x, y\,;\psi)\,. \tag{2.4}$$

One has :

$$\frac{\partial Q}{\partial x} = -\frac{\partial j_x}{\partial \psi}\,; \quad \frac{\partial Q}{\partial y} = -\frac{\partial j_y}{\partial \psi} \tag{2.5}$$

and

$$\frac{\partial}{\partial \psi} j_y = \frac{1}{a}\frac{\partial}{\partial \psi}(a j_y) - \frac{1}{a}\left(\frac{\partial a}{\partial \psi}\right) j_y \tag{2.6}$$

which in comparison with (2.4)-(2.5) leads to

$$\frac{\partial Q}{\partial x} + a\frac{\partial Q}{\partial y} = -\frac{\partial}{\partial \psi}(Qf) + \frac{\partial a}{\partial \psi} j_y\,. \tag{2.7}$$

The last equation reduces, in the case where y is absent for instance, to the well known Liouville equation [9]. However in the general case, where more than one « time axis » is present, there are additional drift terms, coming from the flow in the other directions. In fact the current cannot be related to the density in a simple way. Therefore in contrast to dynamical systems described by ordinary differential equations, (2.7) cannot be closed and then no Liouville-like theorem can be used here.

2.2 THE METHOD OF CHARACTERISTICS. — Equation (2.1) is actually a quasi-linear PDE for the complex amplitude of reflection $R(L, w)$. Taking the real and imaginary parts ($V(x)$ is real) and $R = R_1 + iR_2$, one obtains a system of two PDEs :

$$\frac{\partial R_1}{\partial L} + kR_2\,\varepsilon[L, w((1+R_1)^2 + R_2^2)]\,.\,w\frac{\partial R_1}{\partial w} = \\ = -2\,kR_2 - kR_2(1+R_1)\times \\ \times\,\varepsilon[L, w((1+R_1)^2 + R_2^2)] \tag{2.8a}$$

$$\frac{\partial R_2}{\partial L} + kR_2\,\varepsilon[L, w((1+R_1)^2 + R_2^2)]\cdot w\frac{\partial R_2}{\partial w} = \\ = 2\,kR_1 + \frac{k}{2}((1+R_1)^2 - R_2^2)\times \\ \times\,\varepsilon[L, w((1+R_1)^2 + R_2^2)] \tag{2.8b}$$

This differential system is subject to the boundary conditions : $R_1(L=0, w) = R_2(L=0, w) = 0$. The set of equations (2.8) can be shown [10] to be equivalent to a set of three ordinary differential equations :

$$\frac{\mathrm{d}w}{\mathrm{d}L} = w\,.\,kR_2\,\varepsilon[L, w((1+R_1)^2 + R_2^2)] \tag{2.9a}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}L} = -2\,kR_2 - kR_2(1+R_1)\times \\ \times\,\varepsilon[L, w((1+R_1)^2 + R_2^2)] \tag{2.9b}$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}L} = 2\,kR_1 + \frac{k}{2}((1+R_1)^2 - R_2^2)\times \\ \times\,\varepsilon[L, w((1+R_1)^2 + R_2^2)]\,. \tag{2.9c}$$

The first (2.9a) gives the equation of the characteristics $w(L)$ and (2.9b, c) are the so-called relations on the characteristics. The boundary conditions are (Cauchy data) : $R_1(L=0, w) = R_2(L=0, w) = 0$. The above equations allow for a complete solution $R(w, L)$ for a given form of the function ε. In the following, we focus our study to the cases described by (2.2). In these cases, α and $\eta(L) = -V(L)/k^2$ will be scaled so that $k = 1$. Note however that more general situations including the case of random α can also be treated following the same approach as that described below. For the sake of simplicity, the random potential $V(x)$ used in the next sections is assumed to be a Gaussian white-noise :

$$\langle V(x)\rangle = 0 \quad \text{and} \quad \langle V(x)\,V(x')\rangle = g\delta(x - x')\,, \\ g > 0\,.$$

Here $\langle\ldots\rangle$ denotes the average of the random potential. Within this class of models, the expression of the function ε in (2.9) becomes :

$$\alpha f[w((1+R_1)^2 + R_2^2)] + \eta(L)\,.$$

A straightforward algebra shows that (2.9) imply a remarkable equation for the characteristic curves :

$$w(L) = w(0)/(1 - r(L)) \tag{2.10}$$

where $r = R_1^2 + R_2^2$. Here $w(0) = w_0$ refers to the initial

value of w at the origin $L = 0$. As will become clear below, (2.10) is the key to understand the qualitative behaviour of transmission. Note further that for random $\eta(L)$, $w(L)$ are actually random curves (Fig. 1).

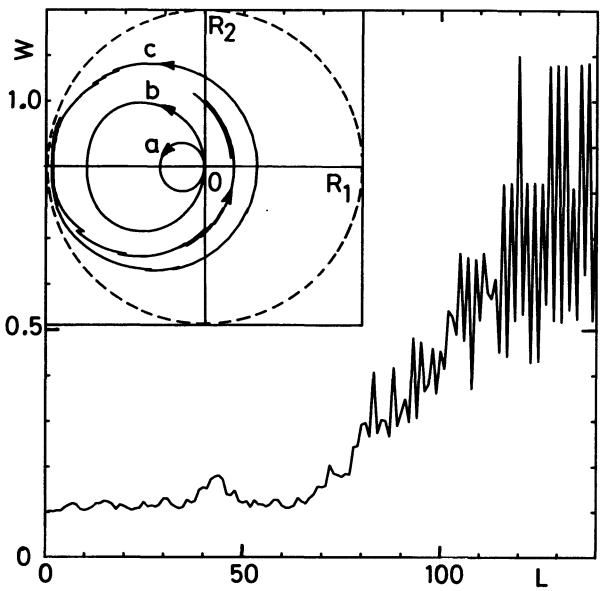

Fig. 1. — A typical characteristic (w, L), obtained by numerical integration of (2.9) ($w_0 = 0.1$) for $f(u) = u$ and with a white-noise potential $V(x)$, corresponding to a localization length $\xi \simeq 45$. The insert shows some orbits of the dynamical system : (a) $w_0 = 1$, (b) $w_0 = 10$ in the absence of disorder and (c) a perturbed orbit by randomness.

Using polar coordinates, $R_1 = r^{1/2} \cos\theta$, $R_2 = r^{1/2} \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, $r \geqslant 0$, one deduces from (2.9) :

$$\frac{dr}{dL} = r^{1/2}(1-r)\sin\theta \times \times \{\alpha f[w(1+r+2r^{1/2}\cos\theta)] + \eta(L)\}$$

$$\frac{d\theta}{dL} = 2 + \left[1 + \frac{1}{2}(r^{1/2} + r^{-1/2})\cos\theta\right] \times \times \{\alpha f[w(1+r+2r^{1/2}\cos\theta)] + \eta(L)\} . \quad (2.11)$$

Another interesting property of (2.11) is provided by the existence of an invariant of motion. Indeed, let us consider the « time » evolution of r and θ, viewed as dynamical variables (L being the « time »), on a given characteristics $w(L)$ of origin $w(0) \equiv w_0$. Let us denote by $\mathcal{F}(u)$ a primitive of the function $f(u)$: $f(u) = d\mathcal{F}(u)/du$. Then, using (2.10)-(2.11), it is easy to verify that

$$F(r, \theta) = \frac{2}{1-r} + \frac{\alpha}{2w_0}\mathcal{F}\left[w_0 \frac{1+r+2r^{1/2}\cos\theta}{1-r}\right] \quad (2.12)$$

is actually an invariant of the motion, i.e. $dF/dL = 0$, in the absence of the potential $\eta(L)$. More generally, for a constant potential $\eta(L) = \eta$,

$$F(r, \theta) + \frac{\eta}{2}\frac{1+r+2r^{1/2}\cos\theta}{1-r}$$

is an invariant of the motion.

The existence of the invariant $F(r, \theta)$ has a simple physical meaning (see below) and allows for a simple description of the dynamics [11]. Remembering that for real $\eta(L)$, $1 - r = t$ is nothing else than the transmission coefficient, two different problems are to be distinguished. The first (problem A) corresponds to the transmission at *fixed output* w_0, which arises if one needs a fixed power at the end $x = 0$ of the layer. In that case, one follows the influence of the potential $\eta(L)$ on just one characteristics. The linear problems belong to this category : the characteristic curve being $w(L) = 0$. The second one (problem B) corresponds to a *fixed input* w and one has to perform the calculation of $R(L, w)$ by considering all the characteristics passing through the chosen point (L, w). In the next sections, both problems will be investigated and the corresponding results are different.

2.3 INVARIANTS AND HAMILTONIAN FORMULATION. — The existence of the invariant F can be traced back to the second-order differential equation describing the wave field inside the medium. The wave field is actually a complex number, and there are two first integrals. The first one is the current and the second is simply the energy : F is actually just the ratio of these two quantities. To see that, it is useful to use the impedance [12]

$$Z = \frac{1}{\psi(x)}\frac{d\psi}{dx} \equiv a + i/b .$$

In terms of Z, the reflection coefficient $R = R_1 + iR_2$ is given by $R = (i - Z(L))/(i + Z(L))$. The real (a) and imaginary $(1/b)$ parts of Z are given by

$$a = 2r^{1/2}\sin\theta/(1+r+2r^{1/2}\cos\theta)$$
$$b = (1+r+2r^{1/2}\cos\theta)/(1-r) \quad (2.13)$$

It is easy to see that the current expression (w_0) is :

$$j \equiv w_0 = \frac{1}{2i}\left(\psi^* \frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right) \equiv |\psi|^2 \operatorname{Im} Z = = |\psi|^2/b . \quad (2.14)$$

In particular this implies $|\psi|^2 \equiv bw_0$ and similarly one has $|\partial\psi/\partial x|^2 = |\psi|^2 |Z|^2 = bw_0(a^2 + 1/b^2)$. For the class of potentials considered here, the wave field equation can actually be derived from a « Hamiltonian », which assumes the following expression :

$$\left|\frac{\partial\psi}{\partial x}\right|^2 + \alpha\mathcal{F}(|\psi|^2) + |\psi|^2 .$$

Using the above notation, it is very easy to check that $F(r, \theta)$ is nothing else than the ratio of this invariant to the current, up to a factor 1/2 and an additive constant term.

The previous remarks allow us to write down a Hamiltonian formulation in terms of the variables a and b. Indeed, in the absence of the potential η, the equations of motion for a and b are given by :

$$\begin{aligned} \frac{da}{dx} &= -\frac{\partial}{\partial b}(2F) \\ \frac{db}{dx} &= \frac{\partial}{\partial a}(2F) \end{aligned} \tag{2.15}$$

where $2F$ is given by (2.12). For instance, for $f(u) = u^n$, $F(a, b)$ assumes the following simple form

$$2F(a, b) = (a^2 + 1)b + \frac{1}{b} + \frac{\alpha w_0^n}{(n+1)} b^{n+1}. \tag{2.16}$$

More generally, in the presence of the potential $V(x)$, Hamilton's equations (2.15) become

$$\begin{aligned} \frac{da}{dx} &= -\frac{\partial}{\partial b}(2F) + V(x) \\ \frac{db}{dx} &= \frac{\partial}{\partial a}(2F). \end{aligned} \tag{2.17}$$

The equation of motion of F, in the presence of $V(x)$, is also simple and can be written as :

$$\frac{d}{dx}(2F) = 2abV(x). \tag{2.18}$$

It is interesting to notice that the use of the conjugate variables a and b leads to a simplified dynamical system, rather than the apparently complicated set of initial equations. However, (2.17), (2.18) are still stochastic differential equations, with an additive noise for (2.17) and a multiplicative noise for (2.18).

3. Transmission at fixed output.

In this section we investigate the behaviour of the transmission coefficient for a fixed output w_0. A familiar example of this problem is given by the fixed point $w_0 = 0$, which corresponds to the linear case [1] : $\langle t \rangle \sim \exp(-L/4\xi)$, where ξ refers to the localization length. The approach used here is that of dynamical systems. Indeed, in the absence of disorder, the representative point (r, θ) exhibits a periodic motion on closed orbits. The effect of a small disorder can then be considered as a perturbation of the orbital motion. This results in a random perturbation calculation of the orbits and the net effect can be followed on the Poincaré [7] sections, on the real axis for instance, $r(0)$ and $r(\pi)$. It turns out that at large L, $r(\theta)$ approaches the attractive point $r = 1$, $\theta = \pi$. In this limit, the orbital motion exhibits a slowing down near $\theta \simeq \pi$ and a rapid revolution elsewhere. The main contribution of disorder comes however from the regions where $\theta \neq \pi$. The appropriate method to handle this perturbed orbital motion is provided by the well known [7] averaging of the perturbation over each period. The main object of the next subsections is a rather detailed exposition of this approach.

3.1 ANALYSIS OF THE UNPERTURBED MOTION. — The analysis of the phase space portrait is facilitated by the existence, in the absence of disorder, of the following invariant of motion

$$F(r, \theta) = \frac{2}{(1-r)} + \frac{\alpha w_0^n}{2(n+1)} \times \left[\frac{1 + r + 2r^{1/2}\cos\theta}{1-r}\right]^{n+1}. \tag{3.1}$$

In the present case, the boundary condition $r(L = 0) = 0$ gives : $F(r, \theta) = 2 + \alpha w_0^n/2(n+1)$ which allows for a detailed description of the orbits $r(\theta)$.

Consider, for instance, the limit $w_0 = \infty$. The orbital motion reduces to : $r^{1/2} = -\cos\theta$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, and the corresponding trajectory is given by the circle of radius 1/2, centred at $(-1/2, 0)$ (see Fig. 2).

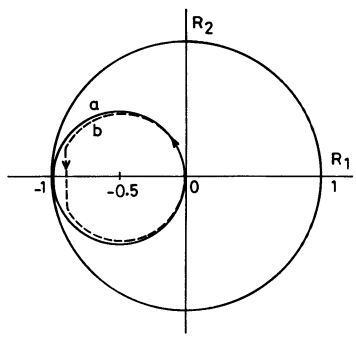

Fig. 2. — Typical orbits (R_1, R_2) in the absence of the random potential, at $w_0 = \infty$ (a) and $w_0 \gg 1$ (b) respectively.

3.1.1 *Large w_0 limit.* — For large $w_0 \gg 1$, the trajectories are strongly modified near $r = 1$ and $\theta = \pi$. In fact, in that region, $2/(1-r)$ in (3.1) becomes dominant, and a net deviation from the circular orbit results. Indeed, for large w_0, the circular motion is followed in a large angular sector : $\sqrt{r} = -\cos\theta$, $\pi/2 \leq \theta < \pi - \bar{\varphi}$, but in the remaining sector $\pi - \bar{\varphi} < \theta \leq \pi$, the first term in (3.1) dominates, and the trajectory is described by : $1 - r = 4(n+1)/w_0^n$. Here $\bar{\varphi}$ denotes a small number given by

$$\bar{\varphi} = 2\left[\frac{2(n+1)}{w_0^n}\right]^{1/2} \simeq w_0^{-n/2}, \quad w_0 \gg 1.$$

The fixed points associated to the dynamical system are located on the negative real axis $\theta = 0$ and are given by the solutions of $d\theta/dL = 0$:

$$2 + \left(1 - \frac{1}{2}(r^{1/2} + r^{-1/2})\right)\left(\frac{1 - r^{1/2}}{1 + r^{1/2}}\right)^n \alpha w_0^n = 0 .$$

In particular, for $w_0 \gg 1$: $r^{1/2} \simeq 1 - 2(1/w_0^n)^{1/(n+2)}$, $\theta = 0$ gives the fixed points of the dynamics. In terms of $w = w_0/(1 - r)$, this corresponds to: $1 - r^{1/2} \simeq w^{-n/2(n+1)}$. Some examples of the orbital motion are shown in figure 3.

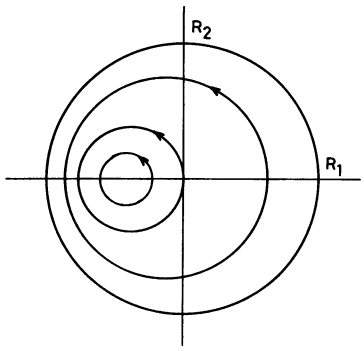

Fig. 3. — Orbital motion in the phase space (r, θ), $0 \leq r \leq 1$.

The period of motion L_p on the closed trajectories can be calculated from the angular velocity expression $(k = 1)$.

$$\frac{d\theta}{dL} = 2k + k\left[1 + \frac{1}{2}(r^{1/2} + r^{-1/2})\cos\theta\right] \times$$
$$\times \alpha w_0^n \left[\frac{1 + r + 2r^{1/2}\cos\theta}{1 - r}\right]^n . \quad (3.2)$$

Therefore

$$L_p = \int_0^{2\pi} d\theta\,(d\theta/dL)^{-1} . \quad (3.3)$$

The main contribution to L_p is given by the sector $\pi - \bar{\varphi} \leq \theta \leq \pi$ and then $L_p \simeq \frac{1}{k} w_0^{-n/2}$. In the remaining sector, the contribution to L_p is comparatively small and can then be neglected. Note that L_p becomes smaller and smaller at large w_0 and this in contrast with the other limit $w_0 \ll 1$, discussed below. In particular this implies a rapid revolution of (r, θ) at large w_0.

The above analysis of the orbital motion can be used to follow the pattern of characteristics and then the surface $r(L, w)$. A typical characteristic (w, L) is shown in figure 4, where the periodic motion on the orbits is shown to induce a periodic oscillation of the characteristics. The slowing down of the orbital motion at $\theta \simeq \pi$ results in a flat behaviour, at $w = w_1 \equiv$

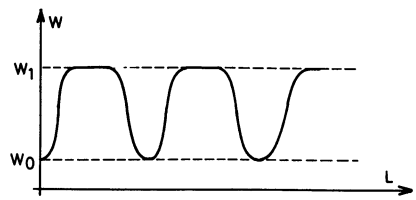

Fig. 4. — Oscillatory behaviour of the characteristic (w, L) of origin w_0.

$w_0^{(n+1)}/4(n+1)$. In that region $r(\theta = \pi)$ is given by $1 - r(\pi) = w_0/w_1$, i.e.

$$r(\pi) = 1 - (4(n+1))^{1/(n+1)} w_1^{-n/(n+1)} . \quad (3.4)$$

These remarks are actually very useful in solving problem B (see Sect. 4). In fact, in this case, w is fixed and we have to solve the implicit equation $w = w_0/(1 - r(L, w_0))$ for w_0. This means that, for a fixed level w, all the characteristics starting at w_0 and satisfying this equation must be taken into account. From the above analysis, the relevant w_0 are given by: $[4(n+1)w]^{1/(n+1)} \leq w_0 \leq w$. Furthermore, for fixed w, $0 \leq r \leq r_{max}$, where

$$r_{max} = 1 - (4(n+1))^{1/(n+1)} w^{-n/(n+1)} .$$

Therefore, the procedure to be used for the calculation of $r(L, w)$ at fixed w is the following one. Each characteristic of origin w_0, $w_0 < w < w_1$ generates a sequence of points $r(L, w)$ which is periodic in L. The union of all these sequences provides the desired solution $r(L, w)$, which defines the reflection coefficient at the chosen w and L. It is important to notice that the period in L is not the same for all the sequences. In fact each characteristic exhibits a large number of folds (see Fig. 5) which becomes important at large L. The period of the maxima $(\theta = \pi)$ is given by: $\lambda[4(n+1)w]^{-n/2(n+1)}$ whereas the period of minima is: $\lambda w^{-n/2}$. This folding leads in particular to the following multiplicity of folds:

$$L[w^{n/2} - (4(n+1)w)^{n/2(n+1)}]/\lambda ,$$

which increases with both L and w. As will be shown in section 4, this behaviour of the degeneracy of folds is not altered by the presence of disorder.

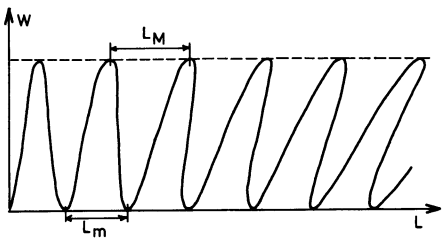

Fig. 5. — Folding of the characteristic (w, L) as a function of L. L_m and L_M are the periods of minima and maxima: $L_m \sim \alpha w^{-n/2}$, $L_M \sim w^{-n/(2n+2)}$.

3.1.2 *Small w_0 limit.* — In this limit, r remains in the vicinity of the origin and $F \simeq 2 + \alpha w_0^n/2(n+1)$. This leads to the following equation for the orbits : $r^{1/2} = -\alpha w_0^n \cos\theta/2$. The corresponding trajectories are simple circles of radius : $\alpha w_0^n/4$. The fixed points are given by the centres of these circles. The angular velocity on the orbits becomes

$$\frac{d\theta}{dL} = k + \frac{1}{2}\alpha k w_0^n + 0(k w_0^{2n}) . \tag{3.5}$$

In particular, (3.5) yields the following expression for the period ($k = 1$)

$$L_p = \frac{1}{k}\frac{\pi}{1 + \alpha w_0^n/2} . \tag{3.6}$$

As before, the characteristics are periodic in L : $w(L)$ oscillates between w_0 and $w_1 = w_0/(1 - w_0^{2n}/4)$. Therefore, at fixed w, r oscillates between 0 and $w^{2n}/4$. The period of minima is associated with the characteristics starting at $w_0 = w$: the corresponding period in L is given by $\dfrac{\pi/k}{1 + w^n/2}$. Similarly, the period of maxima corresponds to the characteristics starting at $w_0 = w(1 - w^{2n}/4)$: the corresponding period is

$$\left(\frac{\pi}{k}\right) \Big/ \left[1 + \frac{1}{2}w^n\left(1 - n\frac{w^{2n}}{4}\right)\right].$$

Finally, the proliferation of folds can be measured with the multiplicity of folding :

$$L \cdot \left[\left(1 + \frac{1}{2}w^n\right) - \left(1 + \frac{1}{2}w^n\left(1 - n\frac{w^{2n}}{4}\right)\right)\right] \simeq \simeq L\frac{k}{\pi}\frac{n}{8}w^{3n} .$$

The degeneracy increases linearly with L, but much more slowly with w in comparison with the large w limit.

3.2 PERTURBATION OF THE ORBITAL MOTION BY DISORDER. — In this section, the effect of a small disorder (i.e. potential $\eta(L)$) on the orbital motion is investigated. A more systematic study of the influence of disorder will be found in the next section. Here we limit our investigation to the large L limit, where $1 - r = t$ is vanishing.

Let us first consider the behaviour of the orbits. To the lowest order in t, (3.1) gives the following expression for $F(r, \theta)$

$$F(r, \theta) \simeq \frac{2}{t} + \frac{1}{2(n+1)w_0}\left[\frac{2w_0(1 + \cos\theta)}{t}\right]^{n+1} . \tag{3.7}$$

Similarly, the angular velocity becomes

$$\frac{d\theta}{dL} \simeq 2k + k\left[\frac{2w_0(1 + \cos\theta)}{t}\right]^n (1 + \cos\theta) . \tag{3.8}$$

Assuming that $t = t_0$ at $\theta = \pi$ before the random perturbation is turned on, (3.7) shows that the equation of motion can be then simplified as follows ($\theta = \pi - \varphi$).

$$t(L) = t_0 , \quad 0 \leqslant \theta \leqslant \theta^* \tag{3.9}$$

and

$$\frac{2w_0(1 + \cos\theta)}{t(L)} = \left[\frac{4w_0}{t_0}(n+1)\right]^{1/(n+1)} , \quad \pi - \varphi^* \leqslant \theta \leqslant \pi .$$

Here, φ^* denotes the extension of the angular sector where the second term in (3.7) dominates :

$$\varphi^* = [4(n+1)]^{1/2(n+1)}\left(\frac{t_0}{w_0}\right)^{n/2(n+1)} . \tag{3.10}$$

Performing the same sort of analysis for (3.8), one obtains the following behaviour for the angular velocity :

$$\frac{d\theta}{dL} = 2k + 2k(n+1)\left(\frac{\varphi}{\varphi^*}\right)^{2(n+1)} , \quad \pi - \varphi^* \leqslant \theta \leqslant \pi \tag{3.11}$$

and

$$\frac{d\theta}{dL} = 2k + k\left[4(n+1)\frac{w_0}{t_0}\right]^{n/(n+1)}(1 + \cos\theta) , \quad 0 \leqslant \theta \leqslant \theta^* .$$

The two angular sectors are matched at $\theta = \pi - \varphi^*$. Equation (3.11) shows that, as long as θ is close to π, the angular velocity can be expressed as a function of the reduced variable φ/φ^*. The contribution $\simeq \left(\dfrac{t_0}{w_0}\right)^{n/2(n+1)}$ of this sector to the period in L dominates that of the second sector, which is proportional to $(t_0/w_0)^{n/(n+1)}$. This leads in particular to the following expression of the period : Const. $(t_0/w_0)^{n/2(n+1)}$. Here the constant in front refers to an n-dependent prefactor.

3.2.1 *Averaged perturbation method.* — In order to follow the effect of a small disorder on the transmission coefficient, we consider the dynamical system giving the « time » evolution of r and θ :

$$\frac{dr}{dL} = kr^{1/2}(1 - r)\sin\theta \times \left[\eta(L) + \alpha w_0^n\left(\frac{1 + r + 2r^{1/2}\cos\theta}{1 - r}\right)^n\right] \tag{3.12}$$

$$\frac{d\theta}{dL} = 2k + k\left[1 + \frac{1}{2}(r^{1/2} + r^{-1/2})\cos\theta\right] \times \left[\eta(L) + \alpha w_0^n\left(\frac{1 + r + 2r^{1/2}\cos\theta}{1 - r}\right)^n\right]. \tag{3.13}$$

In the absence of $\eta(L)$, the orbital motion corresponding to these equations has been studied in the previous

section 3.1. In the limit considered here : $1 - r \ll 1$, (3.12) and (3.13) reduce to :

$$\frac{\mathrm{d}r}{\mathrm{d}L} \simeq k(1-r)\sin\theta\left[\eta(L) + \alpha w_0^n\left(\frac{1+\cos\theta}{(1-r)/2}\right)^n\right] \tag{3.14}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}L} \simeq 2k + k(1+\cos\theta)\times \times\left[\eta(L) + \alpha w_0^n\left(\frac{1+\cos\theta}{(1-r)/2}\right)^n\right]. \tag{3.15}$$

The presence of $\eta(L)$ generates fluctuations around the closed orbits described above. The main contribution of $\eta(L)$ comes from the sector where $\theta \neq \pi$, because $\eta(L)$ enters the equations of motion *via* the combinations $(1+\cos\theta)\,\eta(L)$ and $\sin\theta\,.\,\eta(L)$. This results in a diffusion-like motion of the Poincaré section points. To the first order in η, the average of the perturbation Δr, integrated over each revolution period $(0 \leq \theta \leq 2\pi)$, vanishes. In fact, using (3.14) and (3.15), one deduces

$$\Delta^{(1)}\ln(1-r) = -k\int^L \eta(L')\sin\theta(L')\,\mathrm{d}L'$$

$$\Delta^{(1)}\theta(L) = k\int^L \eta(L')(1+\cos\theta(L'))\,\mathrm{d}L'.$$

The average over η ($\langle\eta\rangle = 0$) gives no contribution to $(1-r)$. To the second order, one obtains

$$\Delta^{(2)}\ln(1-r) = -k^2\int^L \eta(L')\cos\theta(L')\,\mathrm{d}L' \times \times\int^{L'} \eta(L'')(1+\cos\theta(L''))\,\mathrm{d}L''. \tag{3.16}$$

The average over η leads to :

$$\langle\Delta^{(2)}\ln(1-r)\rangle = -gk^2\int^L \mathrm{d}L'\cos\theta(L')\times \times[1+\cos\theta(L')]. \tag{3.17}$$

Therefore, the net effect of disorder can be cast as follows

$$\langle\Delta^{(2)}\ln(1-r)\rangle\,|_{\text{period}} = = -gk^2\int_0^{2\pi}\frac{\cos\theta(1+\cos\theta)\,\mathrm{d}\theta}{\mathrm{d}\theta/\mathrm{d}L}. \tag{3.18}$$

The different contributions to the integral in (3.18) can be separated and the final result is $\simeq \varphi^{*3}$. However, the period of motion is proportional to φ^* and this leads to

$$\left\langle\frac{\Delta\ln(1-r)}{\Delta L}\right\rangle_{\text{period}} = -\text{Const.}\left(\frac{t_0}{w_0}\right)^{n/(n+1)}. \tag{3.19}$$

Due to the persistence of a quasi-orbital motion in the presence of the perturbation, a non-monotonic behaviour of $t \equiv 1 - r$ is obtained on each characteristics $w(L)$. The typical values as given by (3.19) are respectively : $t_{\min}(L) \simeq L^{-(n+1)/n}$ (at $\theta = \pi$) and $t_{\max}(L) \simeq L^{-1/n}$ (at $\theta = 0$). The average value of t : $t_{\min}(L) \leq t(L) \leq t_{\max}(L)$ is therefore no longer an exponential function of L, but a power law. This behaviour contrasts with the well known exponential decay in the linear case (i.e. at $w_0 = 0$). The power laws $t_{\min}(L)$ and $t_{\max}(L)$ must be viewed actually as the lower and upper envelops of the true $\langle t(L)\rangle$. A similar behaviour has been obtained in [4], for $n = 1$ and another derivation of this result will be given below.

3.2.2 *Fluctuations and self-averaging properties.* — It is well known [1] that at $w_0 = 0$, i.e. in the linear case, the fluctuations of t, r, ..., etc., are very significant and the full probability distribution of t is of importance. In order to follow the influence of non-linearities on the statistical distribution of t, we shall use the approximated equations for the unperturbed orbital motion ($\varphi = \pi - \theta$).

$$\begin{aligned}\frac{1}{t} &= \frac{1}{t_0}, && \text{for } \varphi \leq \varphi^* \\ &= \frac{1}{t_0}\left(\frac{\varphi^*}{\varphi}\right)^2, && \text{for } \varphi^* \leq \varphi \ll 1 \\ &= \left[\frac{4w_0}{t_0}(n+1)\right]^{1/(n+1)}/2w_0(1+\cos\theta) && \text{elsewhere}.\end{aligned} \tag{3.20}$$

The corresponding angular velocity in the three angular sectors is given by

$$\begin{aligned}\mathrm{d}\theta/\mathrm{d}L &= 2k + 2k(n+1)(\varphi/\varphi^*)^{2(n+1)}, \\ &\quad \text{for } \varphi \leq \varphi^* \\ &= 2k + 2k(n+1)(\varphi/\varphi^*)^2, \\ &\quad \text{for } \varphi^* \leq \varphi \ll 1 \\ &= 2k + k\left[4\frac{w_0}{t_0}(n+1)\right]^{n/(n+1)}(1+\cos\theta) \\ &\quad \text{elsewhere}.\end{aligned} \tag{3.21}$$

As will be shown below, there are two sources of fluctuations : the first is due of course to disorder, and the second comes from the orbital motion. Actually, the averaging procedure over the closed orbits generates « fluctuations » which dominate at large w_0, where non-linearities govern the behaviour of t. Here we shall investigate this second source of fluctuations.

3.2.2.1 Average and fluctuation of $1/t(L)$. — Following the definition of the average over each period, one has ([...] denotes the average over orbits) :

$$\left[\frac{1}{t}\right]_{\text{period}} = = \int_0^\pi t^{-1}(\theta)\,\mathrm{d}\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}L}\right)^{-1}\Big/\int_0^\pi \mathrm{d}\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}L}\right)^{-1}. \tag{3.22}$$

The sector $\theta \neq \pi$ does not contribute to the integrals, because t and also $(\mathrm{d}\theta/\mathrm{d}L)$ are large in this sector. The main contribution to the numerator is $\simeq \varphi^*/t_0$, whereas the period (denominator) is $\simeq \varphi^*$. This implies in particular that $[1/t] = \text{Const.}\ 1/t_0$.

The same argument also holds for $1/t^2$ and $[1/t^2] = \text{Const.}\ 1/t_0^2$. This leads in particular to the relative fluctuation

$$\lim_{t_0 \to 0} \frac{[t^{-2}] - [t^{-1}]^2}{[t^{-1}]^2} = \text{Const.} \quad \text{at fixed } w_0 . \tag{3.23}$$

This « self-similar » behaviour of the fluctuation of $1/t$ originates from the following fact. The angular sector contributing to the average of $1/t$ has the same scale φ^* as that contributing to the period.

3.2.2.2 Average and fluctuation of t. — According to (3.20) and (3.21), t increases in the sectors where $\mathrm{d}\theta/\mathrm{d}L$ increases. Therefore the fluctuations of t should be large. Using a similar expression as (3.22) for t and taking the ratio $t/\left(\frac{\mathrm{d}\theta}{\mathrm{d}L}\right)$ from (3.20), (3.21), one obtains $(k = 1)$

$$[t] \simeq t_0 \left(\frac{w_0}{t_0}\right)^{n/2(n+1)} . \tag{3.24}$$

Note that for $\theta = \pi$: $t = t_0$ and for $\theta = 0$:

$$t = 4\, w_0 \Big/ \left[\frac{4\, w_0}{t_0}(n+1)\right]^{1/(n+1)} .$$

Equation (3.24) shows that $[t]/w_0$ is nothing else than the geometrical mean of these two extreme values of t/w_0. This implies in particular a large fluctuation of t. Indeed, the expression of $[t^2]$ assumes the following form :

$$[t^2] = \text{Const.}\ t_0^2 \left(\frac{w_0}{t_0}\right)^{\frac{3}{2}\frac{n}{n+1}} . \tag{3.25}$$

The resulting relative fluctuation behaves as

$$\frac{[t^2] - [t]^2}{[t]^2} \simeq \left(\frac{w_0}{t_0}\right)^{n/2(n+1)} \tag{3.26}$$

which diverges as $t_0 \to 0$, at fixed w_0.

3.2.2.3 Average and fluctuation of $\ln t$. — The average of $\ln t$ is dominated by the sector $\theta \simeq \pi$, because of the slowing down of the angular velocity in this region. Taking the main contributions to $\ln 1/t$ and $(\ln 1/t)^2$ into account one deduces that the relative fluctuation of $\ln 1/t$, due to the orbital motion, vanishes at $t_0 \to 0$.

Anticipating the results of the next section, the above results can then be summarized as follows :

1) $\langle 1/t \rangle$ is dominated by $t_{\min}(L)$, $\langle 1/t \rangle \sim L^{1+1/n}$ and the distribution of $1/t$ becomes self-similar at large L.

2) $\langle \ln t \rangle$ is similarly dominated by $t_{\min}(L)$ and the relative fluctuation goes to zero at $L = \infty$.

3) $\langle t \rangle$ is given by the geometrical mean of $t_{\min}(L)$ and $t_{\max}(L)$ and decays to zero as a power law of L.

The physical origin of the power law decay comes from the equation $w = w_0/(1-r)$ of the characteristics, which implies that non-linearities dominate definitely at large L. In fact, due to disorder, $r(L)$ approaches its asymptotic value $(= 1)$ and this enhances in a sensitive way the role of non-linear terms.

3.3 HAMILTONIAN APPROACH. — An equivalent approach to the transmission problem A is provided by the Hamiltonian formalism described in section 2. This approach is facilitated by the existence of the invariant F. Making the change of variables

$$a = 2\, r^{1/2} \sin\theta / (1 + r + 2\, r^{1/2} \cos\theta) ,$$
$$b = [1 + r + 2\, r^{1/2} \cos\theta]/(1 - r) ,$$

the corresponding Hamiltonian is $\mathcal{H} = 2\,F$. The associated Hamilton's equations, when $\eta(L)$ is present, are given by

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}L} &= -\frac{\partial \mathcal{H}}{\partial b} + \eta(L) \\ \frac{\mathrm{d}b}{\mathrm{d}L} &= \frac{\partial \mathcal{H}}{\partial a} . \end{aligned} \tag{3.27}$$

It is important to notice that the perturbation $\eta(L)$ is « time » dependent. The Hamiltonian formalism allows us to write down a Fokker-Planck equation for $F(r, \theta)$ and then to follow its probability distribution. Recall that $\mathcal{H}$ is given by the following expression $(k = 1)$

$$\mathcal{H} = 2\,F = (a^2 + 1)\, b + \frac{1}{b} + \alpha w_0^n \frac{b^{n+1}}{n+1} . \tag{3.28}$$

In the presence of the potential $\eta(L,)$ the « time » evolution of F, as deduced from (3.27)-(3.28) is given by :

$$\frac{\mathrm{d}}{\mathrm{d}L}(2\,F) = 2\, ab\eta(L) . \tag{3.29}$$

Using Novikov's theorem [9], the « time » evolution of the moments $\langle F^m \rangle$ of F can be deduced from (3.29). For instance, $\langle F \rangle$ and $\langle F^2 \rangle$ are the solutions of the ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}\langle F \rangle}{\mathrm{d}L} &= g\langle b \rangle \\ \frac{\mathrm{d}\langle F^2 \rangle}{\mathrm{d}L} &= 2\, g\langle a^2 b^2 \rangle + 2\, g\langle Fb \rangle . \end{aligned} \tag{3.30}$$

Equation (3.30) permits us in particular to follow the variation of F in the general case. In the weak disorder limit, $\langle b \rangle$, $\langle a^2 b^2 \rangle$, $\langle Fb \rangle$, etc., can be approximated

by their values corresponding to the motion on an unperturbed orbit.

3.3.1 *Large w_0 limit.* — Before going to the calculation of the moments of F and its probability distribution, let us first consider the unperturbed orbital motion in the phase space (a, b). In the new variables, the closed orbits are invariant under $a \to -a$ and the ranges of variations are : $-\infty \leqslant a \leqslant \infty$, $b \geqslant 0$. In the limit $r \to 1$, $\theta = 0$ corresponds to $b = \infty$ and $\theta = \pi$ is reached at $b = 0$.

In what follows, $\mathcal{H}$ will be written as

$$\mathcal{H} \equiv 2F = a^2 b + G(b) \tag{3.31}$$

where $G(b) \equiv b + 1/b + \alpha w_0^n \dfrac{b^{n+1}}{n+1}$ denotes the positive function shown in figure 6. The minimum of

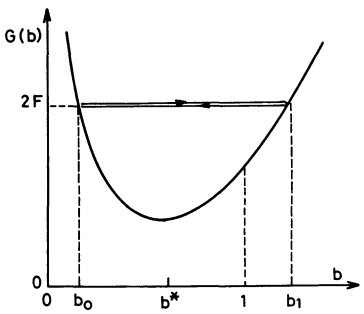

Fig. 6. — Schematic plot of the function $G(b)$ used in the text. The values b_0 and b_1 of b are associated with the position of the turning points, of the periodic motion at a fixed level $2F$.

$G(b)$, which occurs at $b = b^*$ is actually the fixed point of the dynamical system. In the limit of large w_0,

$$b^* \simeq \left(\frac{1}{\alpha w_0^n}\right)^{1/(n+2)} \ll 1 .$$

The position of b^* is close to b_c corresponding to the value of b, where the main terms: $1/b$ and $\alpha w_0^n b^{n+1}/n+1$ become comparable :

$$b_c \sim \left[\frac{n+1}{\alpha w_0^n}\right]^{1/(n+2)}$$

at large w_0. Two other special values of b are given by the solutions of : $\mathcal{H} = G(b)$. In the vicinity of the first $b = b_0 \equiv 1/2F$, $G(b) \sim 1/b$ and

$$a = \pm \left[\frac{1}{b}\left(\frac{1}{b_0} - \frac{1}{b}\right)\right]^{1/2} \tag{3.32}$$

whereas for

$$b \sim b_1 \equiv \left(\frac{2F(n+1)}{\alpha w_0^n}\right)^{1/(n+1)} ,$$

$$G(b) \sim \alpha w_0^n b^{n+1}/(n+1)$$

and

$$a = \pm (2F)^{1/2} \left[\frac{1}{b}\left(1 - \left(\frac{b}{b_1}\right)^{n+1}\right)\right] . \tag{3.31}$$

In terms of the above notation, the period L_p in L takes the following simple form

$$L_p = 2\int_{b_0}^{b_1} db/(db/dL) = \int_{b_0}^{b_1} db/(ab) . \tag{3.32}$$

Actually, b_0 and b_1 can be viewed as the turning points for the periodic motion on the orbit corresponding to the fixed level $2F$ in figure 6. For an initial condition $r(L = 0) = 0$, i.e. $b_1 = 1$, one has $2F = 2 + \alpha w_0^n/(n+1) \gg 1$ at large w_0. This implies in particular that in (3.32), the dominant contribution is given by the interval $[b_c, b_1]$, $b_c \ll 1 \lesssim b_1$ and then

$$L_p = C\left[\frac{n+1}{\alpha (2Fw_0)^n}\right]^{1/2(n+1)} . \tag{3.33}$$

The constant factor C is given by

$$C(n) = \frac{1}{n+1} B\left(\frac{1}{2}, \frac{1}{2(n+1)}\right).$$

It is interesting to notice that (3.33) reproduces the result $L_p \sim (t_0/w_0)^{n/2(n+1)}$ obtained previously at $t_0 \sim 0$, where $F \sim 2/t_0$. Note however that equation (3.33) is more general and no assumption on t_0 is used here.

Following the procedure described before, we consider now the « time » evolution of $\langle F \rangle$, $\langle F^2 \rangle$, ... using the average perturbation, due to disorder, on each orbit. All the calculations are performed to the second order in $\eta(L)$.

First consider the « drift » of F, due to disorder. Using (3.30), one obtains for weak disorder

$$\langle \Delta F \rangle_{\text{period}} = g \,.\, 2\int_{b_0}^{b_1} b\, db/(2\, ab) . \tag{3.34}$$

The integral in (3.34) is dominated, at large w_0, by the contribution of the interval $[b_c, b_1]$. This yields in particular :

$$\langle \Delta F \rangle_{\text{period}} = \text{Const.}\ g(2F)^{-1/2}\left[\frac{2(n+1)}{\alpha w_0^n} F\right]^{3/2(n+1)} \tag{3.35}$$

and then, when $\langle \Delta F \rangle$ is divided by L_p :

$$\frac{d\langle F \rangle}{dL} = \lambda(n)\, g\left[\frac{2(n+1)}{\alpha w_0^n} F\right]^{1/(n+1)} . \tag{3.36}$$

Here $\lambda(n) = B\left(1/2, \dfrac{3/2}{n+1}\right) \Big/ B\left(1/2, \dfrac{1/2}{n+1}\right)$, where

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} ,$$

$\Gamma(x) =$ gamma function. The integration of (3.36), with the initial condition

$$2\,F_0 \equiv 2\,F(L=0) = 2 + \frac{\alpha w_0^n}{n+1}$$

at $b = 1$, yields

$$\langle F(L)\rangle = \frac{\alpha w_0^n}{2(n+1)}\left[1 + 2\,\lambda\,(n)\,g\,\frac{n}{\alpha w_0^n}L\right]^{(1+n)/n} \tag{3.37}$$

Note that (3.37) is valid for all values of L, at large w_0. The result $\langle F(L)\rangle \sim 1/w_0$ is obtained at large L. Furthermore, w_0/t_0 appears as the scaling variable in this regime. Under the action of disorder, $F(L)$ diffuses with a systematic « drift » towards larger values. The associated diffusion in phase space is illustrated in figure 7.

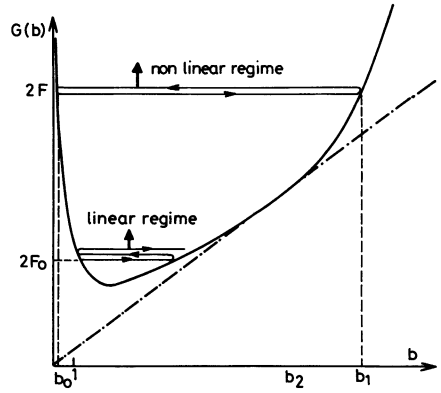

Fig. 7. — Diffusion-like motion, towards large F, under the action of the random potential.

The same calculation can be repeated for $\langle F^2\rangle$, starting from (3.30). For weak disorder $\langle a^2 b^2\rangle$ and $\langle bF\rangle$ can be replaced by their unperturbed values. This leads to

$$\Delta[\langle F^2\rangle - \langle F\rangle^2]_{\text{period}} = 2\,g\int_{b_0}^{b_1} ab\,.\,\mathrm{d}b\,. \tag{3.38}$$

Here again, $[b_c, b_1]$ dominates the integral in (3.38), and then

$$\Delta[\langle F^2\rangle - \langle F\rangle^2] = \\ = \text{Const.}\; g(2\,F)^{1/2}\left[F\,\frac{2(n+1)}{\alpha w_0^n}\right]^{3/2(n+1)}. \tag{3.39}$$

Comparing (3.39) and (3.37), one arrives at the following conclusion. The relative fluctuation of F: $\dfrac{\langle F^2\rangle - \langle F\rangle^2}{\langle F\rangle^2}$ stabilizes at a constant value for large F, i.e. large L. This behaviour of F will be termed « self-similar » in the following.

The results so obtained for the moments of F are confirmed by the « time » evolution of the probability distribution $W(F, L)$ of the stochastic process F. The Fokker-Planck equation satisfied by $W(F, L)$ can be written as [9]

$$\frac{\partial W}{\partial L} = -\frac{\partial}{\partial F}(\langle\Delta F\rangle W) + \frac{1}{2}\frac{\partial^2}{\partial F^2}(\langle(\Delta F)^2\rangle W)\,. \tag{3.40}$$

The « drift » coefficient, in (3.40) is given by

$$\langle\Delta F\rangle = \langle\Delta F\rangle_{\text{period}}/L_p = \\ = \lambda\,(n)\,g\left[\frac{2(n+1)}{\alpha w_0^n}F\right]^{1/(n+1)}. \tag{3.41}$$

Similarly, the diffusion coefficient is given, to first order in g, by

$$\langle(\Delta F)^2\rangle = 2\,\mu\,(n)\,g(2\,F)\left[\frac{2(n+1)}{\alpha w_0^n}F\right]^{3/2(n+1)}. \tag{3.42}$$

Here

$$\mu\,(n) = B\left(3/2, \frac{3/2}{n+1}\right)\Big/B\left(1/2, \frac{1/2}{n+1}\right).$$

The resulting Fokker-Planck equation can then be written :

$$\frac{\partial W}{\partial L} = -\,\lambda\,(n)\,g\,\frac{\partial}{\partial F}\left[\left(\frac{2(n+1)}{\alpha w_0^n}F\right)^{1/(n+1)}W\right] \\ + \mu\,(n)\,g\,\frac{\partial^2}{\partial F^2} \\ \times\left[2\,F\left(\frac{2(n+1)}{\alpha w_0}F\right)^{1/(n+1)}W\right]. \tag{3.43}$$

A more familiar form for (3.43) can be obtained, by making the following change of variables :

$$I = \left[\frac{2(n+1)}{\alpha w_0^n}F\right]^{n/(1+n)}, \quad u = \frac{2\,ng}{\alpha w_0^n}\cdot L \\ W(F, L) = \overline{W}(I, L)\cdot\frac{\mathrm{d}I}{\mathrm{d}F}\,. \tag{3.44}$$

This change of scales leads to :

$$\frac{\partial\overline{W}}{\partial u} = -\,(\lambda\,(n) - 2\,\mu\,(n))\frac{\partial\overline{W}}{\partial I} + \\ + \mu\,(n)\frac{2\,n}{n+1}\frac{\partial}{\partial I}\left(I\,\frac{\partial\overline{W}}{\partial I}\right). \tag{3.45}$$

The normalized solution of (3.45) is

$$(\overline{W}(I, u=0) = \delta\,(I-1)):$$

$$\overline{W}(I, u) = \frac{1}{d(n)\,\Gamma\left(1 + \dfrac{c(n)}{d(n)}\right)\cdot u}\times \\ \times\left(\frac{I}{d(n)\,u}\right)^{c(n)/d(n)}\exp(-\,I/d(n)\,u) \tag{3.46}$$

for $I > 0$ and $\overline{W}(I, u) = 0$ for $I \leqslant 0$, where $c(n) = \lambda(n) - 2\,\mu(n)$ and $d(n) = \mu(n)\dfrac{2n}{n+1}$.

The obtained solution for $\overline{W}(I, u)$ permits the calculation of the moment $\langle F^p \rangle$ of F:

$$\langle I^p \rangle = (d(n)\,u)^p \frac{\Gamma(p + 1 + c(n)/d(n))}{\Gamma(1 + c(n)/d(n))}. \quad (3.47)$$

In particular, the previous results for $\langle F \rangle$, $\langle F^2 \rangle$ and the relative fluctuation

$$\frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} = \frac{d(n)}{c(n) + d(n)} = \text{Const.}$$

are recovered.

The solution (3.46) is actually valid for large L, such that $\langle F(L) \rangle \gg F(L = 0) \equiv F_0$ is fulfilled. In order to describe the other limit, we make the change of variable : $I' = I - I_0$, where $I_0 = 1$ corresponds to the initial value $F_0 = \dfrac{1}{2}\dfrac{(n+1)}{\alpha w_0^n}$ of F. The probability distribution $\overline{W}'$ for I' is now the solution of the following equation (valid for $|I'| \ll 1$).

$$\frac{\partial \overline{W}'}{\partial u} = -\,(c(n) - d(n))\frac{\partial \overline{W}'}{\partial I'} + d(n)\frac{\partial^2 \overline{W}'}{\partial I'^2}. \quad (3.48)$$

Equation (3.48) shows that (3.46) is no longer valid for $|I'| \lesssim 1$. Using (3.48), one deduces the Gaussian distribution for I' :

$$\overline{W}'(I', u) = (4\,\pi d(n)\,u)^{-1/2} \times \\ \times \exp\{-\,[I' - (c(n) - d(n))\,u]^2/4\,d(n)\,.\,u\}\,. \quad (3.49)$$

Equation (3.49) describes the region where $|I'| \lesssim 1$, whereas (3.46) is valid at $I' \gtrsim 1$. The difference between these two distributions can be seen on the behaviour of the moments of I. For the exponential distribution :

$$\langle I \rangle = [c(n) + d(n)]\,u\,,$$
$$\langle I^2 \rangle - \langle I \rangle^2 = d(n)(c(n) + d(n))\,u^2\,.$$

For the Gaussian distribution :

$$\langle I \rangle = [c(n) + d(n)]\,u + 1$$
$$\langle I^2 \rangle - \langle I \rangle^2 = [c(n) + d(n)]\,d(n)\,u^2 + 2\,d(n)\,u\,.$$

Note that $\mu(n) < \lambda(n)$ and then

$$d(n) + c(n) = \lambda(n) - 2\,\mu(n)/(n+1) > 0$$

as it should be. Furthermore the only dependence of the relative fluctuation of F on n is *via* $\lambda(n)$ and $\mu(n)$.

Let us conclude this section by noting that (3.43) can be used to support the approximation, used in the derivation of (3.37), where the fluctuation of ΔF (Eq. (3.35)) has been neglected. Indeed starting from (3.43), one deduces, for large L :

$$\frac{\mathrm{d}}{\mathrm{d}L}\langle F \rangle = \lambda(n)\,g\langle I^{1/n} \rangle = \\ = \lambda(n)\,g[d(n)\,u]^{1/n}\frac{\Gamma\left(1 + \dfrac{1}{n} + \dfrac{c(n)}{d(n)}\right)}{\Gamma\left(1 + \dfrac{c(n)}{d(n)}\right)} \quad (3.50)$$

and then

$$\left\langle \frac{2(n+1)}{\alpha w_0^n} F \right\rangle = \lambda(n)(d(n))^{1/n} \times \\ \times \frac{\Gamma\left(\dfrac{1}{n} + 1 + c(n)/d(n)\right)}{\Gamma(1 + c(n)/d(n))}\,u^{\frac{n+1}{n}}. \quad (3.51)$$

Repeating the same calculation for $\langle F^2 \rangle$, one obtains in a similar fashion :

$$\left\langle \left(\frac{2(n+1)}{w_0^n} F \right)^2 \right\rangle = (\lambda(n) + 2\,\mu(n))[d(n)]^{\frac{n+2}{n}} \times \\ \times \frac{\Gamma\left(2 + \dfrac{2}{n} + c(n)/d(n)\right)}{\Gamma(1 + c(n)/d(n))}\,u^{2(n+1)/n}. \quad (3.52)$$

The results so obtained support the previous ones, and show that $2(n+1)\,F/w_0^n$ is actually the appropriate scaling variable.

3.3.2 *Small w_0 limit and linear-non linear crossover.* — In the previous section, we have considered the case of strong non-linearities, i.e. large w_0, where already at $L \sim 0$, the whole behaviour is dominated by the non linear terms. However, there are characteristics, starting at vanishing w_0, where disorder dominates first at short length scales, and non-linearities govern at large L. Such a crossover between a linear and a non-linear behaviour takes place (for small w_0) at a length scale L^*.

From the form of $G(b)$ as given by (3.31), it is clear that three regimes can appear : $G(b) \simeq 1/b$, $G(b) \simeq b$ and $G(b) \simeq \dfrac{\alpha w_0^n}{n+1} b^{n+1}$, according to the value of b. The crossover between the first two ones is at $b \simeq 1$, whereas the two other ones match at $b_2 \simeq \left(\dfrac{n+1}{\alpha}\right)^{1/n}/w_0$. Therefore a necessary condition for the dynamics to be dominated by non-linearities is given by : $g(b_2) \ll 2\,F$. For small w_0, $b_2 \gg 1$ and this condition can be written : $F w_0 \gg \left(\dfrac{n+1}{\alpha}\right)^{1/n}$. In what follows we show that this condition is actually a sufficient one. For the values of w_0 which violate this requirement, the exponential decay of transmission will be shown to take place.

Let us first consider the orbital motion in the absence

of disorder. The dynamics can be approximated as follows. In the interval, $b_0 < b < 1$ where $b_0 = 1/2\,F$ one has $G(b) \sim 1/b$ and then $a = \left[\frac{1}{b}\left(\frac{1}{b_0} - \frac{1}{b}\right)\right]^{1/2}$. In the second interval, $1 < b < b_2$, $G(b) \sim b$ is linear in b and $a = \left[\frac{2\,F}{b} - 1\right]^{1/2}$. Finally for $b_2 < b < b_1$ where

$$b_1 = \left[\frac{2(n+1)}{\alpha w_0^n} F\right]^{1/(n+1)},$$

$G(b)$ can be written as

$$G(b) = 2\,F(b/b_1)^{n+1}$$

and

$$a = \left[\frac{2\,F}{b}\left(1 - \left(\frac{b}{b_1}\right)^{n+1}\right)\right]^{1/2}.$$

This leads to the following expression for the oscillation period :

$$L_p = \int_{b_0}^{b_1} db/ab\,.$$

For small w_0, $b_2 \ll b_1$ and the integral is dominated by the contribution of $[b_2, b_1]$ and then $L_p \simeq [b_1/2\,F]^{1/2}$, which is the same behaviour as for large w_0. Note however that in the quasi-linear regime where $G(b_2) \gtrsim 2\,F$, the interval $[b_2, b_1]$ is no longer present. In this limit, the period becomes independent of F as it should be.

Following the same procedure as that used at large w_0, the « drift » of F, due to disorder (Eq. (3.34)) is given by

$$\langle \Delta F \rangle_{\text{period}} = g \int_{b_0}^{b_2} db/a\,.$$

The main contribution comes from $[b_2, b_1]$ and then

$$\langle \Delta F \rangle_{\text{period}} \simeq (2\,F)^{-1/2}\, b_1^{+3/2}. \tag{3.53}$$

Similarly, using (3.38), one obtains :

$$\Delta(\langle F^2 \rangle - \langle F \rangle^2) \simeq (2\,F)^{1/2}\, b_1^{+3/2}. \tag{3.54}$$

Note that in the quasi linear regime $G(b_2) \gtrsim 2\,F$, $\langle \Delta F \rangle_{\text{period}} \simeq F$. However, L_p is independent of F in that region. Therefore, an exponential decay of transmission is expected. Equations (3.53) and (3.54) show that beyond a crossover value of F, the large w_0 behaviour is recovered. This crossover is given by

$$F w_0 = \left(\frac{n+1}{\alpha}\right)^{1/n}. \tag{3.55}$$

In particular for $\theta = \pi$, where $F \simeq 2/t$, this corresponds to a precise value of t and w :

$$t = 2\,w_0(\alpha/(n+1))^{1/n}, \quad w = \frac{1}{2}\left[\frac{n+1}{\alpha}\right]^{1/n}.$$

Similarly, at $\theta = 0$ where $F \simeq \frac{\alpha w_0^n}{2(n+1)}\left(\frac{4}{t}\right)^{n+1}$, the corresponding values are :

$$t = \frac{4}{2^{1/(n+1)}} w_0 \left(\frac{\alpha}{n+1}\right)^{1/n}$$

and

$$w = \frac{2^{1/(n+1)}}{4}\left(\frac{n+1}{\alpha}\right)^{1/n}$$

respectively, i.e. nearly the same values as $\theta = \pi$. One therefore deduces the expression of the crossover length ($\langle t \rangle \sim e^{-L/4\xi}$ in the linear regime)

$$L^* \simeq \frac{4\,\xi}{n} \ln[(n+1)/\alpha w_0^n]\,. \tag{3.56}$$

The exponential decay regime ends up at $L \sim L^*$ and this agrees with the intuitive picture described in the introduction. Beyond L^*, t decays as a power law of L because of the enhancement of the non-linearities. Note that L^* diverges as $w_0 \to 0$ or $\alpha \to 0$. This means that for vanishing non-linearities, the exponential decay extends on a larger and larger length scale.

Beyond L^*, F is given by an expression similar to (3.37) :

$$F w_0 = \left(\frac{n+1}{\alpha}\right)^{1/n} \times \left[1 + 2^{1/(n+1)}\,\lambda(n)\, g\, \frac{n}{n+1}(L - L^*)\right]^{(n+1)/n}, \quad L \geqslant L^*\,. \tag{3.57}$$

This is consistent with the fact that

$$[2(n+1)\,F/\alpha w_0^n]^{n/(1+n)}$$

is the appropriate scaling variable.

In figure 7, the different behaviours are clearly shown. Starting at small w_0, we have periodic oscillations between two turning points in the « well » $G(b)$ *vs.* b. Under the action of disorder, a diffusion-like motion takes place, towards large F. As far as $G(b) \sim b$, we have an exponential decay of transmission. However for larger F, $G(b)$ changes over towards $G(b) \sim b^{n+1}$, where non-linearities dominate. This picture explains in a rather simple way the crossover described above. For more general forms of non-linearities, such a crossover always takes place and this because of the modification of the form of $G(b)$ at large b.

4. Transmission at fixed input.

In this case (Problem B) one is interested in transmission at fixed w and L. As explained above this requires performing the averages over all characteristics. In the limit of large L, the function $w(L)$ along a characteristic curve oscillates strongly as a consequence (Eq. (2.10))

of the oscillating behaviour of $r(L)$. Furthermore, the oscillation period depends on the initial value w_0. As a result, there is in general a large number of characteristics which satisfy $w(L) = w$ (for a given w). This multiplicity of solutions is indeed an important feature of the propagation of waves in non-linear media, and this occurs for each individual realization of the potential $\eta(L)$. In the following we shall limit our attention to the large w limit. In this case, the small L behaviour is dominated by non-linearities. However, at large L the wave intensity becomes smaller and smaller and an exponential decay is recovered. The other limit, $w \to 0$, is definitely dominated by disorder and non-linearities are irrelevant. This behaviour at fixed input contrasts with the previous one (Sect. 3) at fixed output.

Following the same procedure as that used in the previous section, it is useful to follow the behaviour of transmission at $\theta = 0$ and π. For this, we recall the expression of F:

$$F = \frac{\alpha w_0^n}{2(n+1)} \left[1 + 2\, g\lambda(n) \frac{n}{\alpha w_0^n} L \right]^{(n+1)/n} \quad \text{at} \quad \frac{\alpha w_0^n}{n+1} \gg 1 \qquad (4.1.a)$$

and

$$F = \frac{1}{w_0} \left(\frac{n+1}{\alpha} \right)^{1/n} \times \left[1 + 2^{1/(n+1)} \lambda(n)\, g \frac{n}{n+1} (L - L^*) \right]^{(n+1)/n} \quad \text{at} \quad \frac{\alpha w_0^n}{n+1} \ll 1 \quad \text{and} \quad L > L^* \qquad (4.1.b)$$

where $L^* = \frac{4\xi}{n} \ln((n+1)/\alpha w_0^n)$. Assuming $w \gg 1$, we shall first find the characteristics leading to $w(L) = w$ at $\theta = 0$ and $\theta = \pi$ respectively. The possible values of $r(L) \equiv 1 - t(L)$ are lying between two limits $r_1(L) \geq r(L) \geq r_2(L)$. Here $r_1(L)$ (resp. $r_2(L)$) are given by the characteristics of origin w_0, such that $w(L) = w$ at $\theta = \pi$ (resp. $\theta = 0$).

4.1 Recovery of an exponential decay. — Let us consider first the case $\theta = \pi$. In the limit $w \gg 1$, $F \simeq 2/t$ and $w = Fw_0/2$. Using the expression of F (Eq. (4.1.a), one has to solve for w_0 the following equation

$$w = \frac{\alpha w_0^{n+1}}{4(n+1)} \left[1 + 2\lambda(n)\, g \frac{n}{\alpha w_0^n} L \right]^{(n+1)/n} . \qquad (4.2)$$

The corresponding solution can be written as

$$w_0 = \left[\frac{4(n+1)\, w}{\alpha} \right]^{1/(n+1)} (1 - L/L_1)^{1/n} \qquad (4.3)$$

where we have defined the following length scale L_1:

$$L_1(w) \equiv \frac{\alpha}{2\lambda(n)\, gn} \left[\frac{4(n+1)\, w}{\alpha} \right]^{n/(n+1)} . \qquad (4.4)$$

Equation (4.3) leads in particular to the following expression for the transmission coefficient $t_1(L)$:

$$t_1(L) = \frac{1}{w} \left[\frac{2\lambda(n)\, gn}{\alpha} \right]^{1/n} (L_1 - L)^{1/n} \qquad (4.5)$$

which, at $L \simeq 0$, reproduces the known result of a pure non-linear medium:

$$t_1(L) = \left(\frac{4(n+1)}{\alpha w^n} \right)^{1/(n+1)} .$$

The range of validity of the above expression is actually much shorter than $L_1(w)$, because we have used the expression of F corresponding to $\frac{\alpha w_0^n}{n+1} \gg 1$. Indeed, as L approaches $L_1(w)$, w_0 decreases, and the above expression for $t_1(L)$ is no longer valid. Using the obtained solution for w_0, $\alpha w_0^n/(n+1) \simeq 1$ gives the following range of validity:

$$(1 - L/L_1(w)) \geq \left(\frac{n+1}{\alpha} \right)^{1/(n+1)} \left(\frac{1}{4w} \right)^{n/(n+1)} \qquad (4.6)$$

and for large w, this range extends up to $L_1(w)$.

For length scales larger than $L_1(w)$, the relevant characteristics originate at vanishing w_0, where disorder effects dominate. In this limit, the origin w_0 is given by the solution of $w = Fw_0/2$, F being given by (4.1.b):

$$w = \frac{1}{2} \left(\frac{n+1}{\alpha} \right)^{1/n} \times \left[1 + 2^{1/(n+1)} \lambda(n)\, g \frac{n}{n+1} (L - L^*) \right]^{(1+n)/n} . \qquad (4.7)$$

The solutions w_0 are given by the implicit equation ($w \gg 1$):

$$L^* = L - L_1(w) = \frac{4\xi}{n} \ln \frac{n+1}{\alpha w_0^n} \qquad (4.8)$$

i.e.

$$w_0 = \left(\frac{n+1}{\alpha} \right)^{1/n} \exp\left(- \frac{L - L_1(w)}{4\xi} \right) . \qquad (4.9)$$

Therefore, one obtains:

$$t_1(L) = \frac{w_0}{w} = \left(\frac{n+1}{\alpha w^n} \right)^{1/n} \exp\left(- \frac{L - L_1(w)}{4\xi} \right), \quad L > L_1(w) . \qquad (4.10)$$

We now turn to the other limit $\theta = 0$, where for large w the following expression for F

$$F \simeq \frac{\alpha w_0^n}{2(n+1)} \left(\frac{1 + r + 2\, r^{1/2}}{1 - r} \right)^{n+1} \tag{4.11}$$

holds. Using (4.1.a) one deduces the following equation for w_0 :

$$w^n = w_0^n + \frac{2\, \lambda\, (n)\, gn}{\alpha} L \tag{4.12}$$

and the corresponding solution leads to

$$t_2(w) = (1 - L/L_2)^{1/n} \quad \text{at} \quad t_2(w) \lesssim 1 \tag{4.13.a}$$

$$t_2(w) = 4(1 - L/L_2'(w))^{1/n} \quad \text{at} \quad t_2(w) \sim 0\,. \tag{4.13.b}$$

Here,

$$L_2(w) = \frac{\alpha w^n}{2\, \lambda\, (n)\, gn}$$

and $L_2'(w) = \alpha\, (4\, w)^n / 2\, \lambda\, (n)\, gn$ respectively.

The range of validity of the above expressions for $t_2(w)$ is fixed, as for $t_1(w)$, by $\dfrac{\alpha w_0^n}{n+1} \sim 1$:

$$(1 - L/L_2'(w)) \geqslant \frac{n+1}{\alpha} (4\, w)^{-n}\,. \tag{4.14}$$

For length scales beyond $L_2(w)$, one has to use (4.1.b) for the calculation of w_0. As for $\theta = \pi$, one finds the following implicit equation for w_0 :

$$L_2' = L - L^* \tag{4.15}$$

and then

$$w_0 = \left(\frac{n+1}{\alpha} \right)^{1/n} \exp\left\{ - \frac{L - L_2'(w)}{4\, \xi} \right\}.$$

The corresponding expression for the transmission coefficient is therefore given by

$$t_2(w) = \frac{w_0}{w} = \frac{1}{w} \left(\frac{n+1}{\alpha} \right)^{1/n} \exp\left\{ - \frac{L - L_2'(w)}{4\, \xi} \right\}. \tag{4.16}$$

It is important to notice that $\xi \ll L_1(w) \ll L_2(w)$ holds at large w. This means that these three length scales are well separated at large (fixed) input w. The curves $r_1(L) = 1 - t_1(L)$ and $r_2(L) = 1 - t_2(L)$, as shown in figure 8, are actually the envelops of $r(L)$, which oscillate between them. The real $r(L)$ exhibits a large number of folds as L increases and this degeneracy will be discussed below. It is important to realize that an exponential decay is obtained here at large L. In fact as L increases the origin w_0 of the relevant characteristics decreases and then a crossover towards an exponential decay is obtained for the transmission. The corresponding crossover lengths are $L_1(w) \sim (\alpha w^n)^{1/(n+1)}\, \xi$ and $L_2(w) \sim \alpha w^n . \xi$ respectively.

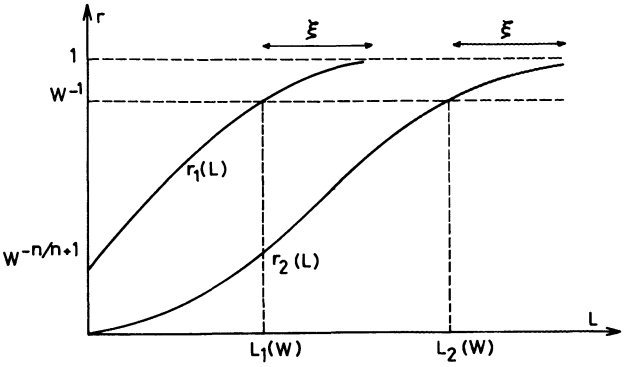

Fig. 8. — Schematic behaviour of the upper and lower bounds $r_1(L)$ and $r_2(L)$ of the reflection coefficient.

This behaviour contrasts with problem A, where an algebraic decay has been shown to take place. Physically, this originates in the initial damping of the wave intensity, on a length scale given by $L_1(w)$ (or $L_2(w)$) which are larger than ξ at large w. This damping allows the recovery of the linear-like behaviour, corresponding to the fixed point $w = 0$.

The obtained behaviour for $r(L)$ can actually be understood by noticing that at $\theta = 0$ as well as $\theta = \pi$, w is a function of the scaling variable Fw_0. However, Fw_0 becomes independent of w_0 at large L on each characteristics. This leads in particular to the behaviour of the asymptotic envelops, shown in figure 9. In fact, using (4.2) and (4.12) it is easy to see that, at large w, $w \sim L^{1+1/n}$ at $\theta = \pi$ and $w \sim L^{1/n}$ at $\theta = 0$ and then w_0 does not appear in the equations of these asymptotic envelops.

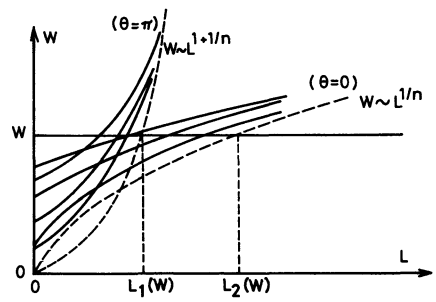

Fig. 9. — The two dashed lines : $w \sim L^{1+1/n}$ and $w \sim L^{1/n}$ show the asymptotic envelops for the characteristics (w, L).

Note that the existence of such asymptotic envelops (towards which the characteristic envelops converge) is a very general property which holds for a large class of non-linear models. This remarkable property is at the origin of the existence of two well defined length scales $L_1(w)$ and $L_2(w)$. To see this, we recall that the transmission across a non-linear medium described by

$$-\frac{d^2\psi}{dx^2} - \alpha f(|\psi|^2)\, \psi = \psi$$

can be described in terms of the « Hamiltonian » $\mathcal{H} = 2F$, where

$$2F = (a^2+1)\,b + \frac{1}{b} + \frac{\alpha}{w_0}\,\mathcal{F}(w_0\,b) \qquad (4.17)$$

where $\mathcal{F}$ denotes a primitive of the function f. $\mathcal{H}$ governs the evolution of a and b in the absence of disorder. In the presence of disorder, one has the following evolution equation for F (Eq. (3.10))

$$\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}x} = g\,\langle b\rangle\,.$$

Furthermore, when non-linearities dominate,

$$\langle b\rangle \sim b_{\max} = \frac{1}{w_0}\,\mathcal{F}^{-1}\left(\frac{2\,Fw_0}{\alpha}\right)$$

and then

$$\frac{\mathrm{d}\langle Fw_0\rangle}{\mathrm{d}x} = g\mathcal{F}^{-1}\left(\frac{2\,Fw_0}{\alpha}\right). \qquad (4.18)$$

This equation for $\langle Fw_0\rangle$ shows that Fw_0 is actually the appropriate scaling variable in the problem described by (4.17). For both $\theta = 0$ and $\theta = \pi$, $w(L)$ on a given characteristics, becomes independent of w_0. Note that this remarkable result is a very general one and holds for any non-saturating ($f(\infty) = \infty$) positive function f.

4.2 FOLDING AND DEGENERACY. — For a given realization of $\eta(L)$, the reflection coefficient $r(L)$ oscillates between $r_1(L)$ and $r_2(L)$ as calculated above. Furthermore, $r(L)$ exhibits a large number of folds, which are responsible for the hysteresis of $r(L)$ (Fig. 10). It appears that this degeneracy increases with both w and L, and is independent of the strength of disorder g.

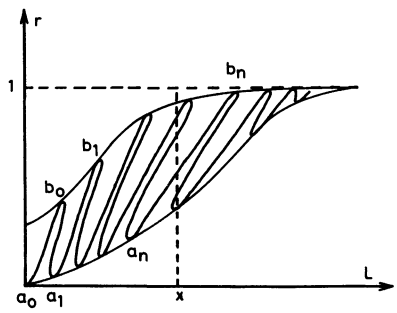

Fig. 10. — Folds of a typical $r(L)$.

To see this, let us define the degeneracy as follows. The abscissas of the points where $r(L)$ hits the lower (resp. upper) envelop are ordered as follows: $a_0 < a_1 < \cdots < a_n < \cdots$ (resp. $b_0 < b_1 < \cdots < b_m < \cdots$). For a fixed abscissa x, there are two different intervals containing this value: $b_m < x < b_{m+1}$, $a_n < x < a_{n+1}$, where $m \ll n$. The successive arcs of $r(L)$, crossing the vertical line at x, are of number $2(n-m)-1$:

$$[a_{m+1}, b_{m+1}],\ [b_{m+1}, a_{m+2}],\ \ldots,\ [a_n, b_n]\,.$$

The degeneracy of folding is then defined by

$$\deg(L) \simeq 2\int_0^L \mathrm{d}L'(1/\pi_<(L') - 1/\pi_>(L')) \qquad (4.19)$$

where $\pi_<(L')$ (resp. $\pi_>(L')$) refers to the short (resp. large) period of oscillation at L', corresponding to the lower (respectively upper) envelop.

It appears that, at large L, $\pi_<$ and $\pi_>$ become independent of L. More precisely,

$$\pi(L) = \gamma\,(Fw_0)^{-n/2(n+1)} \qquad (4.20)$$

where γ denotes a constant number.

In fact, on the upper envelop, $Fw_0 = 2\,w$ and then

$$\pi_>(L) = \gamma\,(2\,w)^{-n/2(n+1)}\,. \qquad (4.21)$$

Similarly, on the lower envelop,

$$Fw_0 = \frac{\alpha}{2(n+1)}\,(\beta w)^{n+1}$$

with $\beta = 1$ at $r(L) \sim 0$ and $\beta = 4$ at $r \sim 1$. Therefore,

$$\pi_<(L) = \gamma\left[\frac{\alpha}{2(n+1)}\,(\beta w)^{n+1}\right]^{-n/2(n+1)}. \qquad (4.22)$$

Using (4.21) and (4.22), one deduces

$$\deg(L) \simeq \frac{2}{\gamma}\left[\frac{\alpha}{2(n+1)}\right]^{n/2(n+1)}(\beta w)^{n/2}\,L\,. \qquad (4.23)$$

This result shows that the degeneracy of folding increases linearly with L (at large L) and is independent of g: only w is involved in the expression of deg (L).

5. Non-linear randomness.

In the previous sections we have focused our attention to the cases where the random terms enter the field equations additively. In this section, we shall consider another case where $\alpha(x)$ is a random variable. This implies that the non-linear term is a random variable with a prescribed probability distribution. We only consider the case where the potential $V(x)$ is absent and only random non-linearities are present. In such a situation, it is useful to consider first the case of a negative constant α before going to the random case.

5.1 CASE OF NEGATIVE α. — Using the same form of non-linearities, one is interested in a wave field described by

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha\,|\psi|^{2n}\,\psi = k^2\,\psi \qquad (5.1)$$

where α is a negative real number.

The invariant of motion is still defined, as before, by

$$\mathcal{K} \equiv 2F = (a^2+1)\,b + \frac{1}{b} + \alpha\,\frac{w_0^n}{(n+1)}\,b^{n+1} \equiv$$
$$\equiv a^2 b + G(b) \quad (5.2)$$

and the equations of motion are

$$\frac{\mathrm{d}a}{\mathrm{d}x} = -\frac{\partial}{\partial b}(2F) \quad \text{and} \quad \frac{\mathrm{d}b}{\mathrm{d}x} = \frac{\partial}{\partial a}(2F)\,. \quad (5.3)$$

The function $G(b) = b + \frac{1}{b} + \alpha\,\frac{w_0^n}{n+1}\,b^{n+1}$ is shown in figure 11 for $|\alpha| \ll 1$ and $|\alpha| \gg 1$ respectively. In the limit of small non-linearities, there is a non-trivial fixed point b^*, which is unstable. Qualitatively, the dynamics

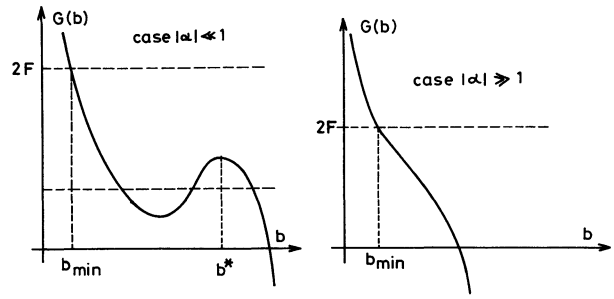

Fig. 11. — Schematic picture of the function $G(b)$ (as in Fig. 6) for small ($|\alpha| \ll 1$) and large ($|\alpha| \gg 1$) non linearities. Here b^* refers to the unstable fixed point and $b_{\min}$ is the initial value of b.

in the phase space is modified : the trajectory reaches the point $b = \infty$ in a finite « time » L_∞. Indeed L_∞ is given by ($\mathrm{d}b/\mathrm{d}L = 2\,ab$)

$$L_\infty = \int_{b_{\min}}^{\infty} \mathrm{d}b/2\,ab\,. \quad (5.4)$$

On the other hand, one has

$$a = \left(\frac{\alpha w_0^n}{n+1}\right)^{1/2} [(b^{n+1} - b_{\min}^{n+1})/b]^{1/2} \quad (5.5)$$

and then

$$L_\infty \simeq \frac{1}{2}\,b_{\min}^{-n/2}\left(\frac{n+1}{\alpha w_0^n}\right)^{1/2} \int_1^\infty \mathrm{d}z\,[z(z^{n+1}-1)]^{-1/2}$$
$$= \frac{1}{2}\,b_{\min}^{-n/2}\left(\frac{n+1}{\alpha w_0^n}\right)^{1/2} \cdot B\left(\frac{1}{2}, \frac{n}{2(n+1)}\right) < \infty\,. \quad (5.6)$$

The limiting point $a = b = \infty$ corresponds to $r = 1$ and $\theta = \pi$. This point is not a fixed point however : the phase is still running over the interval $[0, 2\pi]$. The phase trajectories are shown in figure 12, for weak and strong non-linearities respectively. A qualitative modification of the dynamics occurs at a critical value of

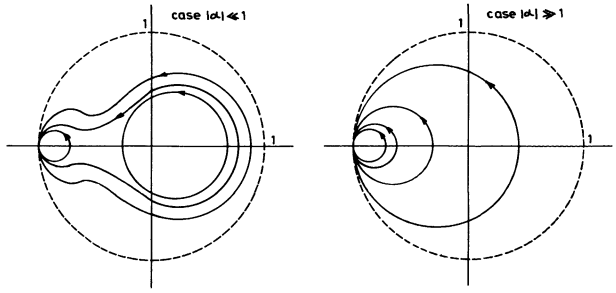

Fig. 12. — Orbital motion in the phase space (r, θ), $0 \leq r \leq 1$, for negative values of α (compare with Fig. 3).

(αw_0^n) corresponding to the disappearance of the unstable fixed point b^*.

In the present case, the reflection coefficient reaches the value $r = 1$ at a finite distance L_∞ and this corresponds to a « self-repelling » of the incident wave. This phenomenon occurs at F, large enough, thus allowing the maximum of $G(b)$ to be reached at small α ; and for any value of F when there is no maximum (large α).

5.2 RANDOM $\alpha(x)$ CASE. — Assume that α is the sum of two terms : a positive constant α and a random term $\beta(x)$. In the following, $\beta(x)$ is assumed to be a Gaussian white-noise : $\langle\beta(x)\rangle = 0$, $\langle\beta(x)\,\beta(x')\rangle = g\delta(x-x')$. The equations of motion are now given by

$$\frac{\mathrm{d}a}{\mathrm{d}x} = -\frac{\partial}{\partial b}(2F) - \beta(x)\,w_0^n\,b^n\,, \quad (5.7)$$

$$\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{\partial}{\partial a}(2F) \quad (5.8)$$

and differ from the pure case by the presence of an additional term for $\mathrm{d}a/\mathrm{d}x$. Here F is given, as above, by (5.2).

As for the additive noise, the equation of motion of F can be used to follow the influence of disorder. In the present case, we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} = -\beta(x)\,w_0^n\,ab^{n+1}\,. \quad (5.9)$$

Using Novikov's [9] theorem, one deduces

$$\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}x} = g w_0^{2n}\,\langle b^{2n+1}\rangle \quad (5.10)$$

$$\frac{\mathrm{d}\langle F^2\rangle}{\mathrm{d}x} = 2\,g w_0^{2n}\,\langle a^2 b^{2n+2}\rangle + 2\,g w_0^{2n}\,\langle F b^{2n+1}\rangle\,. \quad (5.11)$$

In particular, for weak disorder one obtains :

$$\frac{\mathrm{d}}{\mathrm{d}x}[\langle F^2\rangle - \langle F\rangle^2] = 2\,g w_0^{2n}\,\langle a^2 b^{2n+2}\rangle\,. \quad (5.12)$$

In comparison with the additive noise case, there is an additional term $w_0^{2n} b^{2n}$.

The same analysis can actually be repeated and the following results are obtained. The average rate of variation over one period becomes

$$\left.\frac{\mathrm{d}\langle F w_0\rangle}{\mathrm{d}L}\right|_{\text{period}} = \\ = \lambda'(n)\, g \left(\frac{2(n+1)}{\alpha}\right)^{2-1/(n+1)} (F w_0)^{2-1/(n+1)} \tag{5.13}$$

where $\lambda'(n)$ denotes a constant given by

$$\lambda'(n) = \\ = \int_0^1 \mathrm{d}z\, z^{2n}[z/(1-z^{n+1})]^{1/2} \Big/ \int_0^1 \mathrm{d}z\,[z(1-z^{n+1})]^{-1/2} \\ = B\left(\frac{1}{2}, \frac{4n+3}{2n+2}\right) \Big/ B\left(\frac{1}{2}, \frac{1}{2n+2}\right).$$

Neglecting the fluctuations of F, one obtains

$$\frac{\mathrm{d}(F w_0)}{(F w_0)^{1+n/(n+1)}} = \lambda'(n)\, g \left[\frac{2(n+1)}{\alpha}\right]^{1+n/(n+1)} \mathrm{d}L \tag{5.14}$$

and then for $F(L=0) = F_0$:

$$\langle F(L)\rangle = F_0\left[1 - g\frac{n}{n+1}(F_0 w_0)^{n/(n+1)} \times \right. \\ \left. \times \lambda'(n)\left(\frac{2n+2}{\alpha}\right)^{1+n/(n+1)} . L\right]^{-(1+1/n)} . \tag{5.15}$$

Equation (5.15) shows in particular that $\langle F\rangle$ diverges at a finite length and this expression makes sense below this length scale. The above approximation is actually valid either at large w_0 or at small w_0 but at values of L such that $f(L)$ has reached large values (beyond the region where $G(b) \sim b$). This behaviour of $F(L)$ can be explained as follows. The random term $\alpha + \beta(x)$ is actually a white-noise, so that this sum can assume negative values and this situation has been analysed in the previous section. In fact for negative $\alpha + \beta(x)$ and large F, the point $r = 1$, $\theta = \pi$ is reached by all orbits. This implies that F increases very rapidly. Therefore, $\langle F\rangle$ can diverge at a finite length. Beyond this length, the random variable F is described by a singular probability distribution, for which $\langle F\rangle, \langle F^2\rangle, \ldots$ diverge. Such a behaviour would correspond to a complete self-repelling of the incident wave.

The above picture becomes more precise by working out the Fokker-Planck equation describing the evolution of F. Using (5.9), the rate of variation of the fluctuations of F, calculated over one period, is given by

$$\frac{\mathrm{d}}{\mathrm{d}L}[\langle F^2\rangle - \langle F\rangle^2] = 2\,\mu'(n)\, g \times \\ \times 2F\left[\frac{2F}{\alpha w_0^n}(n+1)\right]^{1/(n+1)} \\ \times w_0^{2n}\left[\frac{2F}{\alpha w_0^n}(n+1)\right]^{2n/(n+1)} \tag{5.16}$$

where

$$\mu'(n) = \int_0^1 \mathrm{d}z\, z^{2n}[z(1-z^{n+1})]^{1/2} \Big/ \int_0^1 \mathrm{d}z \times \\ \times [z(1-z^{n+1})]^{-1/2} \\ = B\left(\frac{3}{2}, \frac{4n+3}{2n+2}\right) \Big/ B\left(\frac{1}{2}, \frac{1}{2n+2}\right).$$

The probability distribution $W(F\,;L)$ of F is then the solution of the following Fokker-Planck equation

$$\frac{\partial W}{\partial L} = -\lambda'(n)\, g w_0^{2n} \frac{\partial}{\partial F} \times \\ \times \left\{\left[\frac{2F}{\alpha w_0^n}(n+1)\right]^{(2n+1)/(n+1)} W\right\} + \mu'(n)\, g w_0^{2n} \\ \times \frac{\partial^2}{\partial F^2} \cdot \left\{2F\left[\frac{2F}{\alpha w_0^n}(n+1)\right]^{(2n+1)/(n+1)} W\right\}. \tag{5.17}$$

From this equation, one deduces

$$\frac{\mathrm{d}}{\mathrm{d}L}\langle F\rangle = \\ = \lambda'(n)\, g w_0^{2n}\left\langle\left(\frac{2F}{\alpha w_0^n}(n+1)\right)^{(2n+1)/(n+1)}\right\rangle \\ \geqslant \lambda'(n)\, g w_0^{2n}\left\langle\frac{2F}{\alpha w_0^n}(n+1)\right\rangle^{(2n+1)/(n+1)} . \tag{5.18}$$

This result shows that (5.15) gives actually a lower bound for $\langle F\rangle$. In particular this implies that $\langle F\rangle$ diverges at a length $L \leqslant L^{>}$, where $L^{>}$ is given by

$$L^{>} = \frac{1}{g}\left[\left(1+\frac{1}{n}\right)/\lambda'(n)\right]\left(\frac{\alpha}{2(n+1}\right)^{(2n+1)/(n+1)} \times \\ \times (w_0 F_0)^{-n/(n+1)} . \tag{5.19}$$

In particular, at large w_0, where $F_0 = \alpha w_0^n/2(n+1)$, $L^{>}$ assumes the following simple form

$$L^{>} = \frac{\alpha}{2n}[\lambda'(n)\, g w_0^n]^{-1} . \tag{5.20}$$

This means that $\langle F\rangle$ diverges more and more rapidly either at large g (strong disorder) or at large w_0 (large power). Furthermore, $\langle F^m\rangle$ diverges at finite L for every integer $m > 0$, and this because

$$\frac{\mathrm{d}}{\mathrm{d}L}\langle F^m\rangle \simeq \langle F^{m+n/(n+1)}\rangle .$$

Let us now assume that beyond this length $L^{>}$, $W(F, L)$ reaches (if any) a stationary distribution. Then (5.17) leads to the following behaviour for this distribution

$$W(F, L) \simeq A . F^{-(2n+1)/(n+1)} + B . F^{-n/(2n+2)} \tag{5.21}$$

at large F and large L. This means that the stationary distribution does not exist. All the moments (as well as their derivatives) diverge : in particular such a distribution is not normalizable.

6. Conclusion.

The main results of this paper have been summarized in the abstract and the introduction. In this final section we discuss some general features of our results and a number of open problems.

1. It is clear that the formalism we have used here permits us to investigate a large class of problems, where non-linearities enter through the intensity of the wave field. We have limited our attention to a simple form $f(u) = u^n$ of non-linear terms in the wave field equation, but the derived results are actually very general and hold for different distributions of disorder. The generic behaviour of transmission can be understood from (2.10) : $w(L) = w(0)/(1 - r(L))$. Due to the random potential, the reflection coefficient increases and this results in a sensible *enhancement* of the non-linear terms. For instance, in the case of fixed output, the usual exponential decay will dominate only at short distances or for very weak non-linearities. At large length scales non-linear terms dominate the transmission behaviour and this can be viewed as a breakdown of the backscattering mechanism. The absence of a superposition principle is actually at the origin of this behaviour.

2. Neglecting the generation of harmonics is not a serious limitation to our results : a higher group velocity corresponds to a larger localization length ξ and then a weaker sensitivity to disorder. This approximation can be expressed in terms of a length scale condition for the range of applications of the results.

3. The problems worked out in this paper are relative to the stationary regime. The stability of such a regime has not been addressed here and is still a completely open question.

4. Transmission through optical fibers are probably potential candidates for an experimental realization of the systems investigated in this paper.

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